Elliptic functions

1 Expository lecture by Pavel Etingof.

Def. A function f of a complex variable $z \in \mathbb{C}$ is said to be analytic at $z_0 \in \mathbb{C}$ if it expands near z_0 in a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n.$$

One says that z_0 is a zero of f of order k if $a_k \neq 0$ but $a_i = 0$, $i < k$

An entire function is a function analytic for all $z \in \mathbb{C}$. Such a function expands in a series $\sum_{n=0}^{\infty} a_n z^n$ which converges for all z.

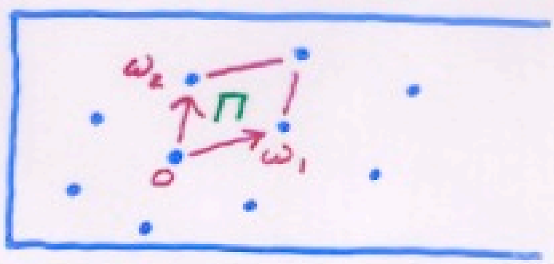
Thus entire functions are generalizations of polynomials

Ex. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is entire.

In real analysis, an important class of functions is the class of periodic functions, i.e. such that $f(x+T) = f(x)$. A useful property of such functions is that it can be expanded in a Fourier series: $f(x) = \sum_{n \in \mathbb{Z}} b_n e^{\frac{2\pi i n x}{T}}$

So one may ask: what is the complex analog of periodic functions? Since the complex plane has two real dimensions, a natural analog would be entire functions which are doubly periodic, i.e. have two $\mathbb{R}$-independent periods, ω_1 and ω_2: $f(z+\omega_1) = f(z+\omega_2) = f(z)$

Bad news: Any such function must be a constant.



Proof. Any such function is determined by its values in the parallelogram Π. Since it's continuous, it is bounded. So the result follows from

Liouville's theorem: A bounded entire function is constant.

Proof of Liouville's theorem.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be such a function, $|f(z)| \leq A$.

Take $R > 0$, and consider the integral

$$I_n = \int_0^{2\pi} f(Re^{i\varphi}) e^{-in\varphi}\, d\varphi = \int_0^{2\pi} \sum_k a_k R^k e^{i(k-n)\varphi}\, d\varphi$$

Since $\int_0^{2\pi} e^{ik\varphi}\, d\varphi = \begin{cases} 0, & k \neq 0 \\ 2\pi, & k = 0, \end{cases}$

we find : $\boxed{I_n = 2\pi a_n R^n.}$

On the other hand the absolute value estimate yields $\boxed{|I_n| \leq 2\pi A}$

So $|a_n| \leq A/R^n$.

Since R is arbitrary, we get $a_n = 0$, $n \geq 1$, so $\boxed{f(z) = a_0.}$

Remark. The Liouville theorem easily implies the fundamental theorem of algebra: If P_{x_0} is a polynomial without roots then $f = \frac{1}{P}$ is a bounded entire function, hence P is a constant.

Thus we see that to develop an interesting theory of complex periodic functions, we must relax one of the two conditions:

1) Double periodicity;

2) entirety.

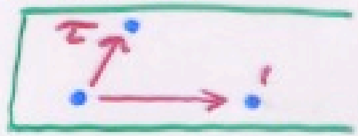
In fact, relaxing (1) leads to the remarkable theory of theta functions, and relaxing (2) – to the just as remarkable and closely related theory of elliptic functions. These are the theories we will now consider

2. <u>Theta functions</u>. We relax Condition 1 in the following way:

<u>Def.</u> A theta function of degree n with parameter $b \neq 0$ is an entire function $f(z)$ such that

$$\boxed{f(z+\omega_1) = f(z), \quad f(z+\omega_2) = b\, e^{-2\pi i n z/\omega_1} f(z)}.$$

Let us classify all such functions. For simplicity assume that $\omega_1 = 1$, $\omega_2 = \tau$, $\operatorname{Im}\tau > 0$.



(this can always be achieved by rescaling z and permuting ω_1, ω_2). Since $f(z+1) = f(z)$, look for f in the form of a Fourier series:

$$f(z) = \sum_{k \in \mathbb{Z}} a_k e^{2\pi i k z}.$$

Then the equation $f(z+\tau) = b e^{-2\pi i n z} f(z)$ translates into the recursive relation

$$a_k q^k = b\, a_{k+n}, \quad q \overset{def}{=} e^{2\pi i \tau}.$$

These recursive equations are easy to solve. Consider separately three cases.

<u>Case 1</u>. $n<0$. Suppose $a_m \neq 0$ for some m. Then $a_{m+n} = b^{-1} q^m a_m$, $a_{m+2n} = b^{-1} q^{m+n} b^{-1} q^m a_m, \dots,$ $a_{m+pn} = b^{-p} q^{mp + np(p-1)/2} a_m$.

Since $n<0$ and $|q|<1$, the sequence $\{q^{np(p-1)/2}\}$, and hence $\{|a_{m+pn}|, p \geq 1\}$, very rapidly goes to ∞. Thus the series $\sum a_k e^{2\pi i k z}$ is divergent. We conclude that any theta function of negative degree is zero.

<u>Case 2</u>. $n=0$. If $b=q^k$ for some k, then $f = Ce^{2\pi i k z}$. Otherwise $f=0$.

So the most interesting case is Case 3. $n>0$. Let us write k as $m+pn$, $0\le m\le n-1$. Like in Case 1, we have

$$a_{m+pn} = b^{-p} q^{mp+np(p-1)/2} a_m.$$

So the solution is determined by $a_0, \dots, a_{n-1}$.

But now $q^{np(p-1)/2}$ rapidly tends to 0, and the series $\sum a_k e^{2\pi i k z}$ converges for all z (check it!). Thus we conclude that the space of theta functions of degree n with parameter b has dimension n.

Let's look more closely at the most important special case $n=1$. In this case, the recursive equation is $a_{k+1} = b^{-1} q^k a_k$.

Clearly, b can be changed by shifts $z \to z+\alpha$. So it suffices to look at the case $b=-1$.

Thus, $a_{k+1} = -q^k a_k$, so $a_k = (-1)^k q^{k(k-1)/2} a_0$. We'll take $a_0 = 1$ and call the corresponding function $\theta(z,q)$. Thus, $\theta(z,q) =$
$$= \theta(z) = \sum_{k\in\mathbb{Z}} (-1)^k q^{k(k-1)/2} e^{2\pi i k z}.$$

Let us study zeros of the theta function $\theta(z)$. It is easy to see that $\theta(0) = 0$, so by (modified) double periodicity, $\theta(j+k\tau) = 0 \quad \forall j,k \in \mathbb{Z}$.

<u>Prop</u>. The points $j+k\tau$ are the only zeros of θ, and they are simple, i.e. $\theta'(j+k\tau) \neq 0$.

<u>Proof</u>. Consider the infinite product
$$\varphi(z) = \prod_{n=0}^{\infty} \left(1 - q^n e^{2\pi i z}\right)\left(1 - q^{n+1} e^{-2\pi i z}\right).$$

This product converges to an entire function, which has simple zeros at points z such that $\boxed{e^{2\pi i z} = q^n}$, $n \in \mathbb{Z}$, i.e. exactly the points $j + k\tau$.

Thus, the function $g(z) = \dfrac{\theta(z)}{\varphi(z)}$ is <u>entire</u>.

Now, from the definition of φ it is easy to see that $\varphi(z+1) = \varphi(z)$. Further, we have

$$\varphi(z+\tau) = \prod_{n=0}^{\infty} \left(1 - q^n q e^{2\pi i z}\right)\left(1 - q^n e^{-2\pi i z}\right)$$

$$= -e^{-2\pi i z} \prod_{n=0}^{\infty} \left(1 - q^n e^{2\pi i z}\right)\left(1 - q^{n+1} e^{-2\pi i z}\right)$$

$$= -e^{-2\pi i z}\varphi(z).$$

Thus, θ/φ is doubly periodic and hence, by Liouville theorem, is constant.

Thus we have proved the proposition and also obtained the following

Corollary 1

$$\theta(z) = C \prod_{n \geq 0} (1 - q^n e^{2\pi i z})(1 - q^{n+1} e^{-2\pi i z}),$$

where $C = C(q)$ is a nonzero constant.

We also have the following analog of the fundamental thm of algebra

Corollary 2. The general nonzero theta function of degree n is of the form

$$\boxed{f(z) = K\theta(z - z_1) \cdots \theta(z - z_n) e^{2\pi i r z}}$$

for some $z_1, \dots, z_n \in \mathbb{C}$, $r \in \mathbb{Z}$.

Proof. Let $z_1, \dots, z_p$ be the zeros of f in the parallelogram $\Pi = \{a + b\tau, 0 \leq a, b < 1\}$ with multiplicities.

Then $g(z) = \dfrac{f(z)}{\theta(z - z_1) \cdots \theta(z - z_p)}$ is a nowhere vanishing theta function of degree $n - p$.

Thus $n-p \geqslant 0$. Also, $n-p$ cannot be >0, since then $\frac{1}{g(z)}$ will be a theta function of negative degree. So $p=n$, and $g(z)$ is a theta function of degree 0, i.e. $g(z) = Ke^{2\pi i r z}$. ■

③ Elliptic functions.

Another way to obtain an interesting theory of doubly periodic functions is to relax the entirety condition.

Def. Let f be a function defined near $z_0 \in \mathbb{C}$ (but not necessarily at z_0). We say that f is meromorphic at z_0 if it is the ratio of two analytic functions near z_0. (so it's a generaliz. of rational func.)

It is easy to show that f is meromorphic near z_0

if and only if it expands near z_0 in a "Laurent series" $f(z)=\sum_{n\geq -N}^{\infty} a_n(z-z_0)^n$ (the difference with analytic functions is that negative powers of $z-z_0$ are allowed). If $a_{-N}\neq 0$, one says f has a pole of order N at z_0.

Def. An *elliptic function* with periods ω_1, ω_2 is a doubly periodic (with periods ω_1, ω_2) meromorphic function on $\mathbb{C}$.

Theorem. A general non-constant elliptic function has the form

$$f(z)= K\,\frac{\theta(z-a_1)\cdots\theta(z-a_K)}{\theta(z-b_1)\cdots\theta(z-b_K)},$$

where $\sum_i a_i = \sum_i b_i$.

Proof. Let f be an elliptic function, and let $a_1,\dots,a_\ell$, $b_1,\dots,b_K$ be its zeros and poles with multipticities in the parallelogram Π'. Then the function

$$g(z) = f(z)\prod_{i=1}^{k}\theta(z-b_i)\prod_{j=1}^{\ell}\frac{1}{\theta(z-a_j)}$$

is entire and nonvanishing, and it's a theta function of degree $k-\ell$. So $k=\ell$ and $g(z)=Ke^{2\pi i r z}$. The factor $e^{2\pi i r z}$ can be killed by shifting a_1 by $r\tau$. Thus, after possible replacement of a_1, we have

$$f(z) = K\prod_{i=1}^{k}\frac{\theta(z-a_i)}{\theta(z-b_i)}.$$

The identity $\sum a_i = \sum b_i$ easily follows from the condition $f(z+\tau)=f(z)$. ∎

So there is no elliptic function with only one zero and one pole.

Def. The Weierstrass function $\wp(z)$ is the elliptic function whose only singularity in the parallelogram Π' is a double pole at 0, and such that
$\wp(z) - \frac{1}{z^2} \to 0, \quad z \to 0.$

Clearly, such a function is unique. Indeed, if f_1, f_2 are two such functions then $f_1 - f_2$ is an entire doubly periodic function (i.e. a constant) which vanishes at 0. So $f_1 - f_2 = 0$.

To show that $\wp$ exists, pick any $a \in \Pi'$, $a \neq 0$, and set

$$f_a(z) = \frac{\theta(z+a)\theta(z-a)}{\theta(z)^2} \cdot \frac{\theta'(0)^2}{\theta(a)\theta(-a)}$$

This function is elliptic and even (check it).

Near $z=0$, this function has Laurent expansion $\frac{1}{z^2} + c_0(a) + \dots$

Thus the function

$$\wp(z) = f_a(z) - c_0(a)$$

satisfies the required conditions (and by the above arguments, does not depend on a).

Thus, we have shown the existence and uniqueness of $\wp$, and also proved the identity

$$\wp(z) - \wp(a) = \frac{\theta(z+a)\theta(z-a)}{\theta(z)^2} \cdot \frac{\theta'(0)^2}{\theta(a)\theta(-a)}$$

Especially useful special cases of this formula are obtained when $a_{1,2,3} = \frac{1}{2}, \frac{1}{2} + \frac{\tau}{2}$, and $\frac{\tau}{2}$. The values of $\wp$ at these points are called e_1, e_2, e_3, respectively.

So we get

$$\wp(z) - e_j = \frac{\theta(z+a_j)\theta(z-a_j)\theta'^2(0)}{\theta^2(z)\,\theta(a_j)\theta(-a_j)}$$

Thus $\wp(z) - e_j$ has a double zero at a_j.

Weierstrass differential equation.

Theorem One has

$$\wp'(z)^2 = 4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

Proof. Since $\wp(z)-e_i$ has a double zero at a_i, $i=1,2,3$, the function $\wp'(z)$ has simple zeros at a_1, a_2, a_3. It also has a triple pole at 0: $\wp'(z) = -\frac{2}{z^3} + c_2 z + \cdots$

Thus the function $g(z) = \dfrac{\wp'(z)^2}{(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)}$

is an elliptic function without zeros or poles (both numerator and denominator have a pole of degree 6 at 0 and double zeros at a_1, a_2, a_3) Thus $g(z) = \text{const}$. But $g(0)=4$, so $g(z)=4$. ∎

Corollary. $e_1 + e_2 + e_3 = 0$.

Proof. This is obtained by comparing the coefficients of $\boxed{\frac{1}{z^4}}$ in the Laurent expansion of both sides of the Weierstrass equation

Elliptic curves and elliptic integrals

Using the last corollary, the Weierstrass equation may be written as

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = -4(e_1e_2 + e_1e_3 + e_2e_3)$,

$g_3 = 4e_1e_2e_3$.

Thus the formulas $x = \wp(z)$, $y = \wp'(z)$ can be used to parametrize the algebraic curve

$$y^2 = 4x^3 - g_2x - g_3.$$

One may show that any curve of the form $\boxed{y^2 = 4x^3 - ax - b}$

(for which the polynomial on the right hand side has distinct roots) is obtained in this way, after a possible rescaling $x \to \lambda^2 x$, $y \to \lambda^3 y$.

Such curves are called elliptic.

They cannot be parametrized as $x = x(t)$, $y = y(t)$, where x, y are rational functions

Now we'll explain how the term "elliptic function" appeared.

Consider an ellipse $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1, A\neq B$

This ellipse is parametrized by $x=A\cos t$, $y=B\sin t$, and the length of its arc is thus given by

$$L=\int\sqrt{x'^2+y'^2}\,dt=\int\sqrt{A^2\sin^2 t+B^2\cos^2 t}\,dt$$

Setting $\sin t=v$, we get

$$L=\int\sqrt{\frac{1+\lambda v^2}{1-v^2}}\,B\,dv,\quad \lambda=\frac{A^2}{B^2}-1.$$

The integral $E=\int\sqrt{\frac{1+\lambda v^2}{1-v^2}}\,dv$ is called an elliptic integral. It is not computable in elementary functions, but we will show it can be computed using elliptic functions

Indeed, consider the elliptic sine

$$\operatorname{sn} z \overset{def}{=} \frac{1}{\sqrt{\wp(z)-e_3}}.$$

It is easy to show that (for appropriate ω_1, ω_2)

this function satisfies the differential equation

$$\left(\frac{d\,\mathrm{sn}\,z}{dz}\right)^2=(1-\mathrm{sn}^2 z)\,(1+\lambda\,\mathrm{sn}^2 z).$$

(this equation follows from the Weierstrass equation for $\wp$).

Thus, substituting in the elliptic integral $v=\mathrm{sn}\,z$, we get

$$E=\int(1+\lambda\,\mathrm{sn}^2 z)\,dz=$$

$$=\int\left(1+\frac{\lambda}{\wp(z)-e_3}\right)dz=\int\left(1+\lambda'\wp\left(z-\frac{\tau}{2}\right)\right)dz$$

This integral is easy to express via θ and θ' (how?).

The Jacobi triple product identity:

$$\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}} w^n = \prod_{n\geq 1}(1-q^n)(1-q^{n-\frac{1}{2}}w)(1-q^{n-\frac{1}{2}}w^{-1})$$

Proof. Consider the function

$$f(z) = \theta\left(z+\tfrac{\tau}{2}\right) = \sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{n^2}{2}} e^{2\pi i n z}.$$

We know that

$$f(z) = C(q)\prod_{n\geq 1}\left(1-q^{n-1/2}e^{2\pi i z}\right)\left(1-q^{n-1/2}e^{-2\pi i z}\right)$$

So all we need to show is that $C(q) = \prod_{n\geq 1}(1-q^n)$.

To prove this, notice that f satisfies the heat equation:

$$-2\pi^2 q\,\frac{\partial f(z,q)}{\partial q} = \frac{\partial^2 f(z,q)}{\partial z^2}.$$

Therefore, $-2\pi^2 q\,\partial_q \ln f\Big|_{z=0} = \dfrac{f_{zz}}{f}\Big|_{z=0}$

$$= \left[\frac{f_{zz}}{f} - \left(\frac{f_z}{f}\right)^2\right]\Bigg|_{z=0} = (\ln f)_{zz}\big|_{z=0}.$$

(as $f'(0)=0$).

Let $D(q) = C(q) \prod_{n \geq 1} (1-q^{n-\frac{1}{2}})^2 = f(0,q)$

The last equation gives

$$-2\pi^2 q \frac{\partial_q D}{D} = \partial_{zz}\Big|_{z=0} \ln \prod_{n \geq 1} (1-q^{n-\frac{1}{2}} e^{2\pi i z}) \cdot$$

$$\cdot (1-q^{n-\frac{1}{2}} e^{-2\pi i z}).$$

After a direct computation this gives (check it!)

$$q\frac{\partial_q D}{D} = -2 \sum_{n \geq 1} \frac{q^{n-\frac{1}{2}}}{(1-q^{n-\frac{1}{2}})^2}$$

Expanding each of these fractions we get

$$q\frac{\partial_q D}{D} = -2 \sum_{n,s \geq 1} q^{(n-\frac{1}{2})s} \cdot s$$

Integrating this differential equation, we obtain

$$\ln D = -2 \sum_{n,s \geq 1} \frac{1}{n-\frac{1}{2}} q^{(n-\frac{1}{2})s} =$$

$$= \ln \prod_{s\geq 1} \frac{1-q^{s/2}}{1+q^{s/2}}.$$

Thus, $D(q) = \prod_{s\geq 1}\left(\frac{1-q^{s/2}}{1+q^{s/2}}\right)$,

and hence

$$C(q) = \prod_{s\geq 1}\left(\frac{1-q^{s/2}}{1+q^{s/2}}\right)\frac{1}{(1-q^{s-\frac{1}{2}})^2}$$

$$= \prod_{s\geq 1}\frac{1-q^{s/2}}{1-q^{s-\frac{1}{2}}} = \prod_{n\geq 1}(1-q^n)$$

Corollary 1 (Euler's pentagonal theore

$$\prod_{n\geq 1}(1-q^n) = \sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{3m^2+m}{2}}.$$

Proof. In the Jacobi identity, put $w = q^{1/6}$.

Corollary 2. (Gauss identity)

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{m\geq 0}(-1)^m(2m+1)q^{\frac{m(m+1)}{2}}$$

Proof. Differentiate Jacobi identity at $z=-\frac{\tau}{2}$

Jacobi four square theorem.

The number of representations of a positive integer m as a sum of four squares is $8N$, where N is the sum of positive divisors of m which are not divisible by 4.

Proof (sketch) Putting $z = \omega_1$ in the formula for $\wp(z) - e_3$, and using the Jacobi triple product identity, one finds that

$$\boxed{\frac{e_1 - e_3}{\pi^2} = \psi(q)^4, \text{ where } \psi(q) = \sum_{n \in \mathbb{Z}} q^{n^2/2}.}$$

Thus $\frac{e_1 - e_3}{\pi^2} = \sum_{m \geq 0} b_m q^{m/2}$, where b_m is the desired number of representations of m as a sum of four squares.

On the other hand, e_i can be computed as the 0-th Laurent coefficient of the function

$$-\frac{\theta(z+a_i)\theta(z-a_i)}{\theta(z)^2}\cdot\frac{\theta'(0)^2}{\theta(a_i)\theta(-a_i)}.$$

This allows to compute e_1-e_3 in a different way. Namely, using the Heat equation as above one can get (using Jacobi again)

$$\frac{i(e_1-e_3)}{4\pi}=\frac{d}{d\tau}\ln q^{-\frac{1}{8}}\prod_{n\geq 1}\frac{(1-q^{n/2})^2}{(1-q^{2n})^2}$$

Equating the right hand sides of these two equations, one obtains Jacobi's theorem.

Corollary. (Lagrange). Any number $n\in\mathbb{N}$ is a sum of four squares.

In a similar manner one proves the following:

<u>Thm</u>. The number of representations of a positive integer n as a sum of two squares is $N_1 - N_2$ where N_1 is the four times the number of divisors of m of the form $4k+1$, and N_2 the number of divisors of m of the form $4k+3$.

<u>Corollary</u> (Fermat) any prime $p = 4k+1$ has a unique representation as $p = m^2 + n^2$, $m, n \in \mathbb{N}$.