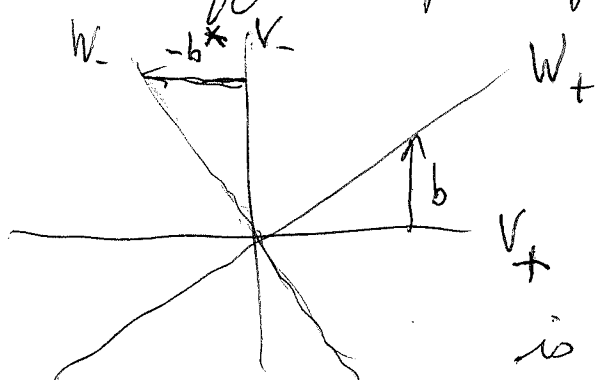
Summary: Some progress was made on the symmetric space $Sp(2n)/U(n)$. Specifically you examined polarizations $H = W_+ \oplus W_-$ close to the basepoint polarization $H = V_+ \oplus V_-$. Such $W$ polarizations is parametrized by a symmetric complex $n \times n$ matrix $b$. It's probably clearer to say that the space of Lagrangian subspaces $W_+$ such that $W_+ \cap V_- = 0$, i.e. transversal to $V_-$, is an affine space parametrized by such $b$.




First look at these subspaces from the viewpt of the positive herm. inner product. Then $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ is skewadjoint and its C.T. is

$$g = F\varepsilon = \frac{1+X}{1-X}, \quad \text{also } \underbrace{u}_{} = g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

has the property that $u\varepsilon u^{-1} = u^2\varepsilon = F\varepsilon\varepsilon = F$.

Somehow this means that $u$ should yield a midpoint between the polarizations $V_+$, $W_+$. You've checked that when $b^t = b$, i.e. $W_+$ Lagrangian, then $g, u \in Sp(2n)$.

---

Other things to work on. Spectral theory for symmetric bilinear forms on a complex v.s. equipped with pos. herm. inner product. (also skew symm. bilinear forms).

Consider then $b : V \xrightarrow[\text{symm.}]{} V^t$, $* : V^t \xrightarrow{\sim} V$. You analyze as follows. Choose an orthonormal basis for $V$ and use the dual basis for $V^t$.

$$V \xrightarrow{b} V^t \xrightarrow{*} V \qquad\qquad \text{better} \qquad V \xrightarrow{b} V \xrightarrow{t} V^t \xrightarrow{*} V$$

$$x \longmapsto x^t b \longmapsto \underset{\bar{b}\,\bar{x}}{\overset{}{b^*\bar{x}}} \qquad\qquad ? \qquad x \longmapsto bx \xrightarrow{t} x^t b^t \longmapsto \bar{b}\bar{x}$$

$$\text{square} \qquad x \longmapsto \bar{b}\bar{x} \longmapsto \bar{b}(\bar{b}\bar{x})^{-} = (\bar{b}b)x$$

You have: $V \xrightarrow{b} V^t \xrightarrow{*} V$

given by $(*b)x = \bar{b}\bar{x}$ with square $(*b)^2 x = (\bar{b}b)x$.
Now $\bar{b}b = b^*b$ (when $b^t = b$) so $V$ can be split into eigenspaces for $b^*b$ corresp. to eigenvalues $\geq 0$.
One has a polar decomposition of $*b$ into its abs. value $(b^*b)^{1/2}$ and a phase which should be an anti linear operator on $V$ of square $1$. (ignoring $\lambda = 0$)

Explain this. The antilinear operator $*b$ commutes with its square $(*b)(*b) = b^*b$ which is $\geq 0$. $V$ splits into eigenspaces $V_\lambda$, $\lambda \geq 0$ on which $b^*b = \lambda^2$. Suppose $V_\lambda \neq 0$ with $\lambda > 0$. Look at $*b$ restricted to $V_\lambda$: it's antilinear with square $= \lambda^2$, so $(*b)\lambda^{-1}$ is antilinear with square $= 1$. Therefore you get a real structure on $V_\lambda$. Putting these together (assuming $V_\lambda \neq 0 \Rightarrow \lambda > 0$) you get a polar decomposition of $*b$ into a pos. op $(\bar{b}b)^{1/2}$ times a phase which is a antilinear op of square $+1$.

Now there's another approach to symmetric bilinear forms under unitary equivalence, namely the infinitesimal symmetric space arising from $Sp(2n)/U(n)$. Here you have Cartan splitting

$$\mathcal{L}Sp(2n) = \mathfrak{K} \oplus \mathfrak{p} = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$$

$K = \mathcal{L}\left\{ \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} : u \in U(n) \right\}$. One has conjugation action of $K$ on $\mathfrak{p}$:

$$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ -\bar{u}\bar{b}u^* & 0 \end{bmatrix}$$

$$-\bar{u}\bar{b}u^* = (-ubu^t)^-$$

The problem is how to reconcile these two approaches to a spectral theory for $\mathbb{C}$-linear symmetric bilinear forms on a positive herm. vector space/$\mathbb{C}$.

Consider $b^t = b$, $b$ nondeg. You want to relate antilinear maps occurring twice.

First place $\quad V \xrightarrow{b} V^t \xrightarrow{*} V$ $\quad$ | here use $b^t = b$

$\quad x \longmapsto x^t b \longmapsto b^* \bar{x} = \bar{b}\bar{x}$

$\boxed{\text{Idea}}$ For any $n \times n$ matrix $b$ you get antilinear with square

$$V \xrightarrow{b} V \xrightarrow{t} V^t \xrightarrow{*} V \qquad \bar{b}\,\overline{\bar{b}\bar{x}} = (\bar{b}b)x$$

$$x \longmapsto bx \longmapsto x^t b^t \longmapsto \bar{b}\bar{x}$$

Maybe you should restrict to just $x \mapsto \overline{bx}$.
When does this have square $= 1$? $\quad \overline{b\overline{bx}} = (\bar{b}b)x$.

So you need $\bar{b}b = 1$, in which case $\bar{b}, b$ commute

Discuss. General case $V \xrightarrow{b} V^t$ $\mathbb{C}$-bilinear form

$$V \xrightarrow{b} V^t \xrightarrow{*} V$$

$$x \longmapsto x^t b \longmapsto b^* \bar{x}$$

So you hope that any antilinear map $V \xrightarrow{\alpha} V$ should have the form $x \longmapsto a\bar{x}$

$$\mathbb{C}^n \xleftarrow{a} \mathbb{C}^n \xleftarrow{-} \mathbb{C}^n \qquad \text{clear.}$$

square of $\quad x \mapsto b\bar{x}$ is $b\overline{b\bar{x}} = (b\bar{b})x$

$\qquad\qquad x \mapsto \overline{bx}$ is $\overline{b\overline{bx}} = (\bar{b}b)x$

So the requirement that $b^t = \pm b$ serves to make $b\bar{b}$ and $\bar{b}b$ diagonalizable

Recalculate. $V \xrightarrow{b} V^* \xrightarrow{*} V$

$x \longmapsto x^t b \longmapsto b^* \bar{x}$

$x \mapsto bx \mapsto (bx)^t \overset{*}{\mapsto} \overline{bx}$

so the choices you had before for an antilinear operator are included in the general form $x \longmapsto b\bar{x}$ which has square $b\,\overline{b\bar{x}} = (b\bar{b})x$. You would like $b\bar{b}$ to be diagonalizable, which should be true if either $b^t = b$ or $b^t = -b$.

$$b = b^t \implies \bar{b} = b^* \implies \begin{cases} b\bar{b} = bb^* \geq 0 \\ \bar{b}b = b^*b \geq 0 \end{cases}$$

$$b = -b^t \implies \bar{b} = -b^* \implies \begin{cases} b\bar{b} = -bb^* \leq 0 \\ \bar{b}b = -b^*b \leq 0. \end{cases}$$

---

You still need to link antilinear maps to

$$X = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix}$$

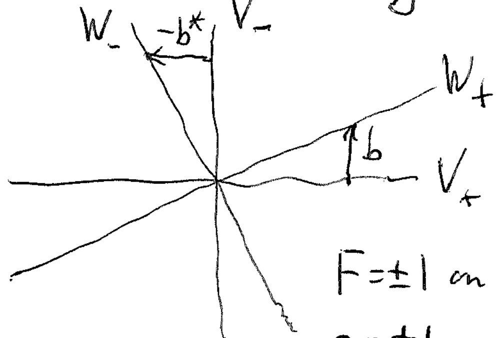
The latter says $X$ is skew adjoint — former — $X \in \mathcal{J}$.

You know that $u = g^{1/2} = \dfrac{1+X}{(1-X^2)^{1/2}} \in Sp(2n)$ and that

$u \varepsilon u^{-1} = u^2 \varepsilon = F\varepsilon\varepsilon = F.$ ??

It seems that the way to get an antilinear operator out of $X$ is to form $u_t = \dfrac{1+tX}{(1-t^2X^2)^{1/2}}$ and to let $t \to \infty$ in which case it seems that you get $\dfrac{X}{|X|}$. You have to assume $b$ invertible

Idea: Show that the space of polarizations is a symmetric space, more precisely, that there is a reflection through each point. This should be clear locally at least. Assume $V_- \cap W_+ = 0$




$$V_+ \xrightarrow[\sim]{\begin{bmatrix} 1 \\ b \end{bmatrix}} W_+ , \quad V_- \xrightarrow[\sim]{\begin{bmatrix} -b^* \\ 1 \end{bmatrix}} W_-$$

$$X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} \qquad F(1+X) = (1+X)\varepsilon$$

$F = \pm 1$ on $W_\pm$

$\varepsilon = \pm 1$ on $V_\pm$

$$g = F\varepsilon = \frac{1+X}{1-X}$$

$$u = g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

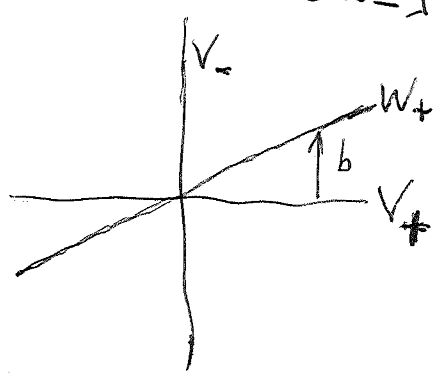
These calculations take place inside a Grassmannian. You need to understand why a Grassmannian: $U(p+q)/U(p)\times U(q)$ is a symmetric space, ~~at~~ least what is the reflection through a point.

Consider $\mathbb{P}V$, a point of $\mathbb{P}V$ is an orth. splitting $V = V_+ \oplus V_-$ with $\dim V_+ = 1$. The tangent space to $\mathbb{P}V$ at this point is $\mathrm{Hom}(V_+, V_-)$.

---

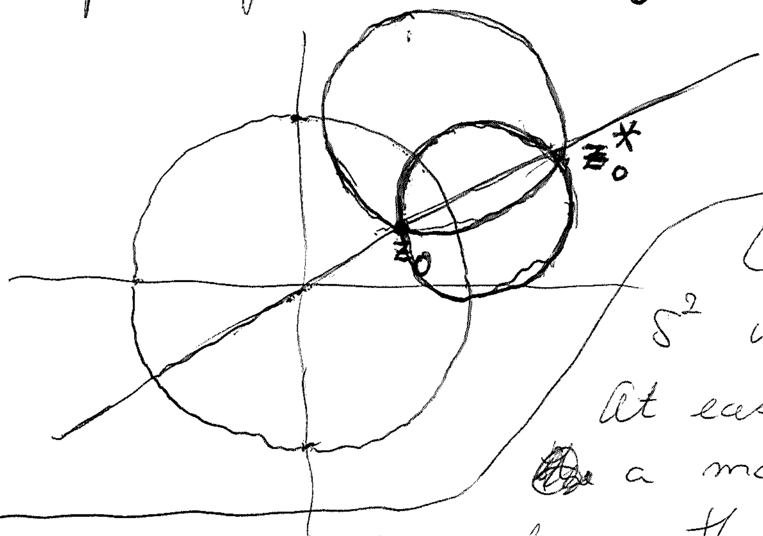
Let's start with a polarized complex v.s. $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ where $V$ has inner product compatible with splitting. Consider also the Grass variety of polarizations $W = \begin{bmatrix} W_+ \\ W_- \end{bmatrix}$ with $\dim W_+ = \dim V_+$.

You want to understand reflection.




Reflection on the tangent space at the point $V_+$ should be $\times(-1)$

Let's aim to understand the Riemann sphere as a symmetric space. More precisely you want to find the reflection through any point. Any point of $S^2$ determines an antipodal point: the point $z$ corresps to the line $\begin{bmatrix} 1 \\ z \end{bmatrix}\mathbb{C}$, the orthogonal line is $\begin{bmatrix} -\bar{z} \\ 1 \end{bmatrix}\mathbb{C}$, the antipodal map is $z \longmapsto \frac{1}{-\bar{z}} = -\bar{z}^{-1}$. Geodesics on $S^2$ are great circles. You should be able to visualize the family of great circles passing through a given point $z_0$, which also should pass through the antipodal point $z_0^* = -\bar{z}_0^{-1}$.

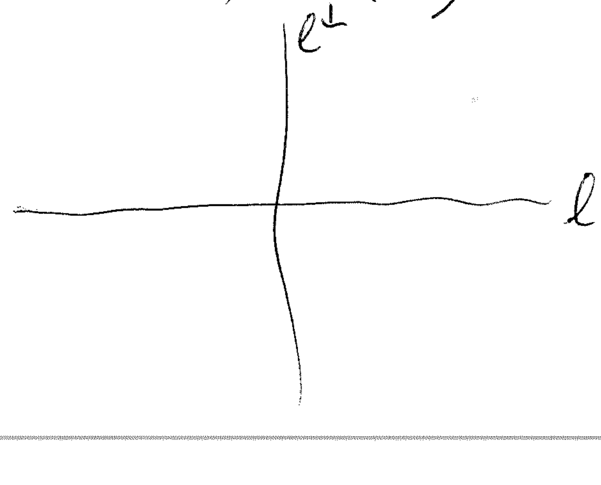



So what do you want? You pick a point $z_0$

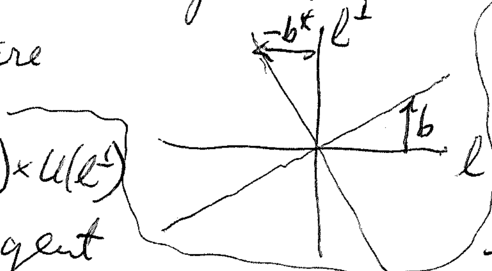
Consider the Riemann sphere $S^2$ with action of $SU(2)$. At each point $z$ the stabilizer is a maximal torus which is the group of rotations fixing the point $z$ and its antipodal pt. Rotation through $\pi$ gives the reflection you want.

Consider $\mathbb{P}^n = U(n+1)/U(1) \times U(n)$, more generally consider $U(V)$ acting on $\mathbb{P}(V)$. Stabilizer of a line $l$ is $U(l) \times U(l^\perp)$.

You want the operator $= \begin{cases} +1 & \text{on } l \\ -1 & \text{on } l^\perp \end{cases}$



(376) $U(V)$ acting on $\mathbb{P}(V)$. The isotropy group at $\ell \in \mathbb{P}(V)$ is $U(\ell) \times U(\ell^\perp)$, this acts on the tangent space at the point $\ell$ which is $\simeq \mathrm{Hom}(\ell, \ell^\perp)$, have picture




You want an elt. of $U(\ell) \times U(\ell^\perp)$ which induces $-1$ on the tangent space.

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You are trying to understand why $\mathbb{P}V$ is a symmetric space. Given a point of $\mathbb{P}V$, i.e. a line $\ell \subset V$, you have an orthogonal splitting $V = \ell \oplus \ell^\perp$. You would like to construct a symmetry of $\mathbb{P}V$, more precisely, a unitary transf $g \in U(V)$ such that $g(\ell) = \ell$, whence $g(\ell^\perp) = \ell^\perp$. Try taking $g = i\varepsilon = \begin{cases} i & \text{on } \ell \\ -i & \text{on } \ell^\perp \end{cases}$

Since $(i\varepsilon)^2 = -1$ is a scalar, it's clear that $g^2 = \mathrm{id}$ on $\mathbb{P}V$. Find the fixpoints of $g$ on $\mathbb{P}V$.

Choose coord system $V = \mathbb{C}^n$ $\quad \ell = \begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}$, $\ell^\perp = \begin{bmatrix} 0 \\ \mathbb{C}^{n-1} \end{bmatrix}$

Take another line $\begin{bmatrix} \lambda \\ v \end{bmatrix}\mathbb{C}$ in $\mathbb{P}V$. Apply $g = i\varepsilon$ to get $\begin{bmatrix} i\lambda \\ -iv \end{bmatrix}\mathbb{C}$, when is $\begin{bmatrix} i\lambda \\ -iv \end{bmatrix}\mathbb{C} = \begin{bmatrix} \lambda \\ v \end{bmatrix}\mathbb{C}$?

If $\lambda \neq 0$, then $\begin{bmatrix} i\lambda \\ -iv \end{bmatrix} = \begin{bmatrix} i\lambda \\ iv \end{bmatrix} \Rightarrow -iv = iv \Rightarrow v = 0$.

If $\lambda = 0$, then $v$ can be any nonzero vector in $\ell^\perp$ so it looks like $\mathbb{P}(\ell^\perp)$ is the set of $g$ fixpts other than $\begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbb{C}$

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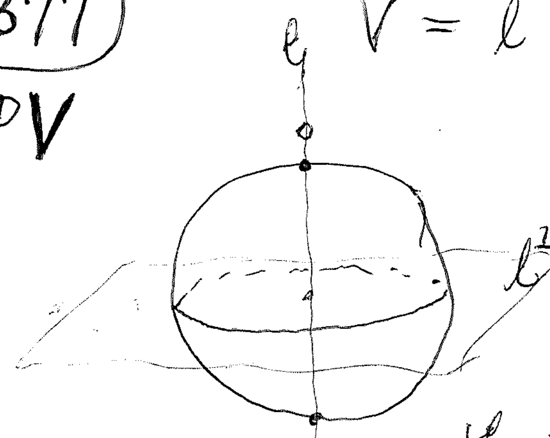
**PV**

$V = \ell \oplus \ell^\perp$ $\quad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ Describe




$U(V)$ acts on $\mathbb{P}V$, the stab of the base point $\ell = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\mathbb{C}$ is $U(\ell) \times U(\ell^\perp)$. The tangent space at the basept $\ell$ is $\mathrm{Hom}(\ell, \ell^\perp)$. It seems that the action of $\iota\varepsilon$ on $\mathbb{P}V$ yields $-1$ on $\mathrm{Hom}(\ell, \ell^\perp)$. ~~Now this action should be~~ You should describe $\mathbb{P}V = U(V)/U(\ell) \times U(\ell^\perp)$ and then the action of $g \in U(V)$ on $\mathbb{P}V$ is left mult by $g$ on this homog. space, and it is given by conjugation by $g$ when $g \in$ the isotropy group

---

Consider $\mathrm{Grass}_p(V)$ the space of $p$-diml subspaces of $V$. $U(V)$ acts transitively on $\mathrm{Grass}_p(V)$, if we pick a basept $V_+ \subset V$, let $V_- = V_+^\perp$ then the isotropy group is $U(V_+) \times U(V_-)$, so $U(V)/U(V_+) \times U(V_-) \xrightarrow{\sim} \mathrm{Grass}_p(V)$.

You can also identify a $p$ diml subspace $V_+$ of $V$ with the self-adjoint involutions $F$ such that $F = \pm 1$ on $V_\pm$. If $g \in U(V)$, then $gV_+$ is the action of $g$ on $V_+$; in terms of $F$ the action of $g$ is $gFg^{-1}$.

The Grassmannian is supposed to be a symmetric spaces, which means that at each point there is an appropriate reflection. ~~Now you want to take~~ You have to construct this reflection. Start with a polarization point $V = V_+ \oplus V_-$, let $\varepsilon = \pm 1$ on $V_\pm$. The isotropy group of the polarization is the

[redacted] centralizer $U(V_+)\times U(V_-)$ of $\varepsilon$.
You have an isom.

$$U(V)/U(V_+)\times U(V_-) \xrightarrow{\sim} \mathrm{Grass}_p(V)$$

$$g(\quad " \quad) \longmapsto g\varepsilon g^{-1}$$

You can identify $T_\varepsilon(\mathrm{Gr})$ with
$\mathcal{L}U(V)/\mathcal{L}U(V_+)\oplus\mathcal{L}U(V_-) \simeq \mathrm{Hom}(V_+,V_-)$

[redacted] Alternate: $\mathcal{L}U(V) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{matrix} a^*+a=0 \\ c+b^*=0 \\ d^*+d=0 \end{matrix} \right\}$

So $T_\varepsilon(\mathrm{Gr}) = \left\{ \begin{bmatrix} 0 & b \\ -b^* & 0 \end{bmatrix} \right\}$. So on this tangent space you have conjugation by $\varepsilon$ which $= -1$. But conjugation by $\varepsilon$ is defined [redacted] on the manifold level.

Let's look at this from the abstract viewpoint
$G$ group, $\sigma : G \to G$ an autom of order 2, get homog space $G/G^\sigma$. Now $\sigma$ should operate on $G/G^\sigma$ via $\sigma(gG^\sigma) = \sigma(g)G^\sigma$. Check comp. with left $G$-action

$$\sigma(h(gG^\sigma)) \cancel{= \sigma(h)\sigma(gG^\sigma)} \quad \cancel{\sigma(h)\sigma(g)G^\sigma}$$

$$= \sigma(hgG^\sigma) = \sigma(hg)G^\sigma = \sigma(h)\sigma(g)G^\sigma$$

Still confused. Go back to the Grass case.

$$\mathrm{Grass}_p(V) = \{V_+ ; \dim V_+ = p\} = \left\{ \begin{bmatrix} V_+ \\ V_- \end{bmatrix} : V = V_+ \overset{\perp}{\oplus} V_- \right\}$$

(379) $U(V)$ acts transitively on $Gr_p(V)$, stabilizer of polarization $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ is $\begin{bmatrix} U(V_+) & 0 \\ 0 & U(V_-) \end{bmatrix} \subset U(V)$

$$\therefore \quad U(V)/U(V_+)\times U(V_-) \xrightarrow{\sim} Gr_p(V), \qquad F = \pm 1 \text{ on } V_\pm$$

---

Start again with the space of polarizations $V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ where $\dim(V_+) = p$. These are the same as operators $F = F^* = F^{-1}$ on $V$ where $F_\pm = 1$ on $V_\pm$, where $F_+ = 1$ has mult $p$. You know that $U(V)$ acts transitively on the space $Gr_p(V)$ of polarizations, and that the stab. of $F$ is the centralizer of $F$. $= \begin{bmatrix} U(V_+) & 0 \\ 0 & U(V_-) \end{bmatrix}$

$$U(V)/U(V_+)\times U(V_-) \xrightarrow{\sim} Gr_p(V)$$
$$g \longmapsto gFg^{-1}$$

Conjugation by $F$ is an autom of $U(V)$ of order 2, fixpt subgroup is $U(V_+)\times U(V_-)$:

$$FgF^{-1} = g \iff Fg = gF. \iff g \in \mathrm{Cent}(F).$$

What is mult. by $iF$ on $Gr_p(V)$

$$iF \longmapsto iFF(-i)F^{-1} = \quad ?$$

---

$U(V)$ acts on $Gr_p(V)$ via $(g, V_+) \longmapsto gV_+$

Using $F = \pm 1$ on $V_\pm$ this becomes

$(g, F) \longmapsto gFg^{-1}$. yields $U(V)/U(V)^F \xrightarrow{\sim} Gr_p(V)$

Now you also have conjugation by $F$ acting $U(V)/U(V)^F$

380 This time you would like to define all objects in terms of the v.s $V$, $\mathrm{End}(V)$ etc. Manifolds should be submanifolds, ideally you should exhibit a local nbd retraction.

Start with $V$ a complex v.s. with pos. hermitian inner product. $G_p$ = Grassmannian of $p$ diml subspaces can be identified with the space $\{F \in \mathrm{End}(V) \mid F = F^* = F^{-1},$ dim $\mathrm{Ker}(F-1) = p\}$. Actually it's better to define $\mathcal{G} = \{F \in \mathrm{End}(V) : F = F^* = F^{-1}\}$, such an $F$ same as a polarization: orth splitting $V = W_+ \oplus W_-$ where $W_\pm$ is the $\pm 1$ eigenspace of $F$.

Tangent space to $\mathcal{G}$ at $F$:

$$T_F(\mathcal{G}) = \{\delta F \in \mathrm{End}(V) \mid F\delta F + \delta F F = 0, \ \delta F^* = \delta F\}$$

Why: $F = F^* \Longrightarrow \delta F = \delta F^*$, $F^2 = 1 \Longrightarrow \delta F F + F \delta F = 0\}$.

Restrict to $\mathcal{G}_p = \{F ;$ multiplicity of $F = 1$ is $p\}$.

$U(V)$ acts transitively on $\mathcal{G}_p$. Choose a basepoint polarization $V = V_+ \oplus V_-$ and let $\varepsilon$ be the corresponding involution. Let $W = W_+ \oplus W_-$, $F$ be any polarization with $p = \dim W_+$. Choosing orth bases for $V_+$ and $W_+$, and also for the orthog complements $V_-$ and $W_-$, yields a unitary $g \in U(V)$ such that $gV_\pm = W_\pm$, equiv. $g\varepsilon g^{-1} = F$.

More generally the action of $g$ on $F \in \mathcal{G}_p$ is $gFg^{-1}$. The isotropy gp is the centralizer

$$U(V)^F = U(V_+) \times U(V_-) \quad \therefore \quad \boxed{U(V)/U(V)^F \xrightarrow{\sim} \mathcal{G}_p, \quad g \longmapsto gFg^{-1}}$$

(381) $V$ complex v.s. equipped with herm. inner product
polarization of $V$ is a splitting $V = V_+ \oplus V_-$ into orthog subspaces. Equiv. assoc. to such a polarization the operator $F = \pm 1$ on $V_\pm$. Then $F = F^* = F^{-1}$, and one has a bijection between polarizations and such operators $F$.
Put $\mathcal{G} = \{F \in \mathrm{End}(V) : F = F^* = F^{-1}\}$, $\mathcal{G}_p = \{F \in \mathcal{G} : \underbrace{\dim \mathrm{Ker}(F-1)}_{=p}\}$.

Natural action of $g \in U(V)$ sends $V = V_+ \oplus V_-$ into $V = gV_+ \oplus gV_-$, equiv. to sends $F$ to $gFg^{-1}$. This action transitive on $\mathcal{G}_p$ (Why: Given $V = W_+ \oplus W_-$ with $\dim(W_+) = p$ choose orth bases for $V_+, W_+$ and orth bases for $V_-, W_-$)

Fix basept $V = V_+ \oplus V_-$, let $\varepsilon$ be corresp. involution
Transitivity $\Rightarrow$ $U(V)/U(V)^{\varepsilon} \xrightarrow{\sim} \mathcal{G}_p$, where $U(V)^{\varepsilon} =$ cent. of $\varepsilon$ in $U(V)$, which is $U(V_+) \times U(V_-)$.

---

What would you like to accomplish? Given a point $F$ of $\mathcal{G}_p$ you would like to construct a global symmetry of $\mathcal{G}_p$ which is a "reflection" through the point $F$. By "global symmetry of $\mathcal{G}_p$" you would like the action of an element $g$ of $U(V)$, and by "reflection" you mean that the global symmetry $g$ induces $-1$ on $T_F(\mathcal{G}_p)$ and that it respects 1-parameter subgroups.

By transitivity one can assume the point $F$ is the basepoint $\varepsilon$ of $\mathcal{G}_p$. The global symmetry should be conjugation by $\varepsilon$ on $\mathcal{G}_p$, which is the same as conjugation by $g = i\varepsilon$, since $i$ is a scalar operator. It's clear that this $g$ fixes $\varepsilon$, also $g$ fixes $-\varepsilon$ the opposite polarization.

 On the tangent space $T_\varepsilon(\mathcal{G}_p)$ conjugation by $\varepsilon$ has order 2, so eigenvalues are $\pm 1$, hence all eigenvalues of $\varepsilon_*$ on $T_\varepsilon(\mathcal{G}_p)$ are $-1$, otherwise ?? Something's missing.

What you might do to calculate the action of the isotropy group $U(V)^\varepsilon = U(V_+) \times U(V_-)$ on $T_\varepsilon(\mathcal{G}_p)$. Recall $\mathcal{G}_p = \{F \in \mathrm{End}(V) : F = F^* = F^{-1}\}$. A tangent vector to $\mathcal{G}_p$ at $\varepsilon$ is a 1st order variation $\delta\varepsilon$ preserving the conditions $F = F^*$ and $F^2 = 1$, yielding $\delta F = \delta F^*$ and $F\delta F + (\delta F)F = 0$; thus $T_\varepsilon(\mathcal{G}_p)$ is the space of self adjoint operators in $\mathrm{End}(V)$ which anticommute with $F$. Clearly conjugation by $\varepsilon$ on $\underline{\mathrm{End}(V)}$ is $-1$ on $T_\varepsilon(\mathcal{G}_p)$.

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Let's try to summarize: polarization of $V$
= orthogonal splitting $V = V_+ \oplus V_-$
= operator $F \in \mathrm{End}(V)$ s.t. $F = F^* = F^{-1}$. The space of polarizations $\mathcal{G}$ is the space of self adjoint involutions. $U(V)$ acts on $\mathcal{G}$, ~~the orbits~~ by $g : F \longmapsto gFg^{-1}$, stabilizer of $F$ $U(V)^F = U(V_+) \times U(V_-)$. The orbits of $U(V)$ are $\mathcal{G}_p = \{F \in \mathcal{G} : +1 \text{ mult } p\}$.

Problem: $\mathcal{G} \subset \mathrm{End}(V)$, identify $T_F(\mathcal{G}) \subset \mathrm{End}(V)$. You know the answer

$$T_F(\mathcal{G}) = \{X \in \mathrm{End}(V) \mid FX = -XF,\ X^* = X\}$$

So you have ~~described~~ the space $\mathcal{G}$ of polarizations of $V$ consisting of $F \in \mathrm{End}(V)$ s.t. $F = F^* = F^{-1}$. $U(V)$ acts trans. on $\underbrace{\mathcal{G}_p}_{\text{the conn. comp.}}$

stabilizer $U(V)^F = U(V_+) \times U(V_-)$

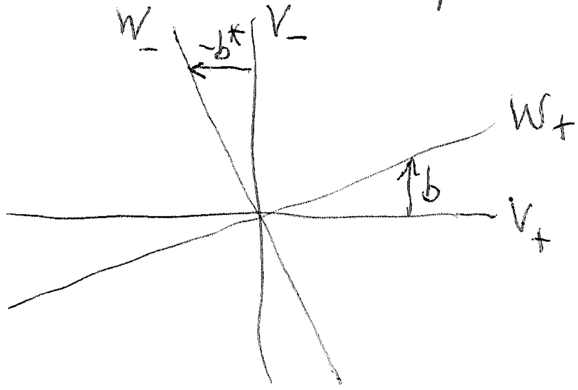
tangent space at $F$ : $\left\{ \delta F \in \mathrm{End}(V) : \begin{array}{l} \delta F = \delta F^* \\ F\delta F + \delta F F = 0 \end{array} \right\}$,

$$\mathrm{End}(V) = \begin{bmatrix} V_+ \otimes V_+^* & V_+ \otimes V_-^t \\ V_- \otimes V_+^t & V_- \otimes V_-^* \end{bmatrix}$$

Notationally things might be simpler if you worked with

$$\mathcal{L}U(n) = \{ X \in \mathrm{End}(V) : X + X^* = 0 \}.$$

Let's now work locally near the basepoint $\varepsilon$ of $\mathcal{G}_p$. Standard picture




You are assuming that $V_- \cap W_+ = 0$ so that

$W_+$ is the graph $\begin{bmatrix} 1 \\ b \end{bmatrix} V_+$

$W_-$ ———— $\begin{bmatrix} -b^* \\ 1 \end{bmatrix} V_-$

$$W_+ \oplus W_- = \underbrace{\begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix}}_{} \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$$

invertible since $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ skew adj

Then $F(1+X) = (1+X)\varepsilon$, $\quad F\varepsilon = \dfrac{1+X}{1-X}$

(384) Recap. Space $\mathcal{G}$ of polarizations of $V$,

$\mathcal{G} = \{F \in End(V) : F = F^* = F^{-1}\}$, tangent space

$$T_F(\mathcal{G}) = \{\delta F \in End(V) : \delta F = \delta F^*,\ F(\delta F) + (\delta F)F = 0\}.$$

Can analyze 2nd condition as follows

$$End(V) = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \otimes \begin{bmatrix} V_+^* & V_-^* \end{bmatrix}$$

$$= \begin{bmatrix} V_+ \otimes V_+^* & V_+ \otimes V_-^* \\ V_- \otimes V_+^* & V_- \otimes V_-^* \end{bmatrix} \qquad V_+ \otimes V_-^* = Hom(V_-, V_+)$$

condition $\delta F$ anticommutes with $F$ means diagonal operators must be zero.

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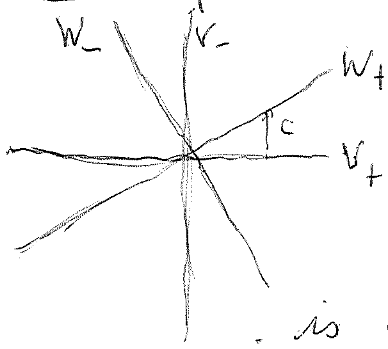
$$\mathcal{L}\,U(V) = \{X \in End(V) : X + X^* = 0\} \quad \{U(V) : g^*g = 1\}$$

$$V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \qquad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad End(V) = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \begin{bmatrix} M_{pp} & M_{pq} \\ M_{qp} & M_{qq} \end{bmatrix}\right\}$$

$$\mathcal{L}\,U(V) = \left\{X = \begin{bmatrix} a & b \\ c & d \end{bmatrix},\ \begin{matrix} a^*+a=0,\ d^*+d=0 \\ b = -c^* \end{matrix}\right\}$$

$$\mathcal{L}\left(U(V)/U(V_+)\times U(V_-)\right) = \left\{X = \begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix};\ c : V_+ \longrightarrow V_-\right\}$$

Restrict to open set of $\mathcal{G}_p$ of $V = \begin{bmatrix} W_+ \\ W_- \end{bmatrix}$ such that $V_- \cap W_+ = 0$.




$$X = \begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix} \qquad X \text{ skew adj.}$$

Point $U(V)$ submani. of $M_n\mathbb{C} = End(V)$, $\mathcal{G}_p$ also, action of $g \in U(V)$ is conjugation $gFg^{-1}$. Perhaps the idea needed is that $T_F(\mathcal{G}_p)$ consists of

$$\delta(gFg^{-1}) = \delta g\, F + F(-\delta g) = XF - FX.$$

An inf unitary acts by bracket. So

$T_F(\mathcal{G}_p) = \{ XF - FX : X + X^* = 0 \}$, $\quad XF - FX = F(FXF - X)$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix} = - \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix}$$

Again $\quad XF - FX = (X - FXF)F$

$$= \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

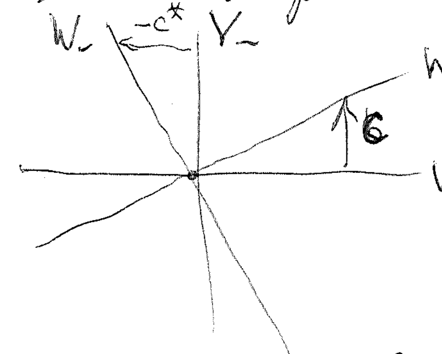
$$= \begin{bmatrix} 0 & 2b \\ 2c & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2b \\ 2c & 0 \end{bmatrix} = 2 \begin{bmatrix} 0 & c^* \\ c & 0 \end{bmatrix}$$

What does this mean? You have $U(V)$ acting on $\mathcal{G}_p$, and a point $F$ of $\mathcal{G}_p$. The tangent space $T_F(\mathcal{G}_p)$ is the image of the infinitesimal action $X \longmapsto [X, F]$ by $\mathcal{L}\, U(V)$ at $F$.

Still not very clear. You should be able to show $\mathcal{G} = \{ F \in \mathrm{End}(V) : F = F^* = F^{-1} \}$ is a submanifold of $\mathrm{End}(V)$ by exhibiting local retractions at the same time finding the tangent space (bundle).

Pick a point of $\mathcal{G}$: $\quad V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \quad \varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

An open nbd of $\varepsilon$ is given by any $V = \begin{bmatrix} W_+ \\ W_- \end{bmatrix}$




such that the orthogonal projection onto $V_+$ restricted to $W_+$ is invertible, which means $V_- \cap W_+ = 0$ and $V_+ \cap W_- = 0$, equivalently that $g = F\varepsilon$ does not $-1$ as an eigenvalue. In this good case $W_+ = \begin{bmatrix} 1 \\ c \end{bmatrix} V_+$ is the graph of $c : V_+ \to V_-$ and $W_- = \begin{bmatrix} -c^* \\ 1 \end{bmatrix} V_-$. The involution

F corresponding to the W polarization is given by $F(1+X) = (1+X)\varepsilon$ where $X = \begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix}$. As $X$ is skewadjoint $1 \pm X$ is invertible (simpler might be $-X^2 = \begin{bmatrix} c^*c & 0 \\ 0 & cc^* \end{bmatrix} \geq 0$, so $1 - X^2 = (1+X)(1-X)$ is invertible hence also $1 \pm X$. Then $F\varepsilon = \frac{1+X}{1-X}$ and $u = \frac{1+X}{(1-X^2)^{1/2}}$ satisfies $\varepsilon u = u^{-1}\varepsilon$, $u^2 = F\varepsilon$.

You have parametrized all polarizations $F$ in the open nbd of $\varepsilon$ given by $1 + F\varepsilon$ invertible using $\mathbb{C}$ linear maps $c : V_+ \longrightarrow V_-$ : $F = \frac{1+X}{1-X}\varepsilon$ or $F = u\varepsilon u^{-1}$. You can now find the action of $i\varepsilon$ on $\mathcal{G}$; the $i$ occurs so that $i\varepsilon \in U(V)$, but since the action is by conjugation on $F$, the scalar $i$ drops out. Clearly $\varepsilon X = -X\varepsilon$, so conjugation by $\varepsilon$ changes $c$ to $-c$.

summarize. You define the space of polarizations and show that at each point there's a globally defined reflection, also a complex affine space nbd. Next project is to handle the $Sp(2n)$ case.

Start with a complex vector space with symplectic form and also a positive hermitian product. There's a compatibility condition: the "ratio" of these two forms is an antilinear operator whose square should be $-1$.

$J^* = -\bar{J}$

$-\bar{J}J = J^*J$

$$V \xrightarrow{J} V^t \xrightarrow{*} V$$

$$x \longmapsto x^t J \longmapsto J^* \bar{x} \rightsquigarrow J^* \overline{J^* x} = (J^* \bar{J}^t)x$$

or $$x \longmapsto Jx \longmapsto x^t J^t \xrightarrow{*} \bar{J}\bar{x} \rightsquigarrow \bar{J}\,\overline{\bar{J}x} = \bar{J}Jx$$

Consider $Sp(2n)$ situation. Here you have a $2n$ diml $\mathbb{C}$-v.s. $V$ equipped with a pos. hermitian form and a $\mathbb{C}$-bilinear symplectic form subject to a compatibility condition. The pos herm. form gives rise to an antilinear isom $V \xrightarrow{*} V^t$ which we denote $*$, because if you choose an orth. basis for $V$ and the dual basis for $V^t$, regard elts of $V$ as column vectors and elts of $V^t$ as row vector, then the map $*$ which represents linear functionals via the hermitian form, is given by $x \longmapsto x^*$ the transpose conjugate row vector corresponding to the column vector $x$.

The herm. form $(x,y) \in V \longmapsto x^* y$

Let the symplectic form be denoted $(x,y) \longmapsto x^t J y$ where $J : V \longrightarrow V^t$. Now consider more generally any skew-symm. $J : V \to V^t$. Composing with $* : V^t \to V$ yields an antilinear transf.

$$V \xrightarrow{J} V^t \xrightarrow{*} V$$

1st choice: $x \longmapsto x^t J \xmapsto{*} (x^t J)^* = J^* \bar{x}$

2nd choice: $x \longmapsto Jx \mapsto (Jx)^t \longmapsto ((Jx)^t)^* = \overline{Jx} = \bar{J}\bar{x}$

Compute the squares of these antilinear transf.

1st: $J^* \overline{J^* \bar{x}} = (J^* J^t) x$

2nd: $\bar{J}\, \overline{\bar{J} \bar{x}} = (\bar{J} J) x$

Note $(J^* J^t)^* = \bar{J} J$ so these two squares are related by $*$.

So far you haven't used $J^t = -J$. This implies $J^* = -\bar{J}$ so that the 1st + 2nd choices for the antilinear map differ by sign, and the square coincide: $J^* J^t = (-\bar{J})(-J) = \bar{J} J$

The compatibility condition relating a pos herm $* : V \longrightarrow V^t$ and a symplectic $J : V \to V^t$ [redacted] is that the square of the antilinear operator is $-1$. Let $\sigma_J : x \longmapsto J^* \bar{x}$ be the first choice, so that $\sigma_J^2 x = (\bar{J} J) x$. Recall that $J^t = -J \Rightarrow J^* = -\bar{J}$ $\Rightarrow J^* J = -\bar{J} J$ so you want $J^* J = 1$ so that $\bar{J} J = -1$.

Somehow you must link this to polarizations. Say you've started with $V \xrightarrow{J} V^t \xrightarrow{*} V$, where $J$ is a skew-symmetric matrix such that $\bar{J} J = -1$, equivalently $J^* J = 1$, which means that $J$ is a unitary skew-symmetric matrix. Look at $n = 1$. Then

$$J = \begin{bmatrix} 0 & \xi \\ -\xi & 0 \end{bmatrix}, \quad |\xi| = 1 \qquad J^* J = \begin{bmatrix} 0 & -\bar{\xi} \\ \bar{\xi} & 0 \end{bmatrix} \begin{bmatrix} 0 & \xi \\ -\xi & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V = \mathbb{C}^2 \qquad \sigma_J \begin{bmatrix} x \\ y \end{bmatrix} = J^* \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} 0 & -\bar{\xi} \\ \bar{\xi} & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} -\bar{\xi} \bar{y} \\ \bar{\xi} \bar{x} \end{bmatrix}$$

$$\sigma_J \sigma_J \begin{bmatrix} x \\ y \end{bmatrix} = \text{[redacted]} \begin{bmatrix} 0 & -\bar{\xi} \\ \bar{\xi} & 0 \end{bmatrix} \begin{bmatrix} -\xi y \\ \xi x \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

Because $J$ describes a bilinear form, $u \in \mathcal{U}(V)$ acts on such a $J$ $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ via $u^t J u$ so

$$\begin{bmatrix} \bar{\xi} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \xi \\ -\xi & 0 \end{bmatrix} \begin{bmatrix} \bar{\xi} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\xi & 0 \end{bmatrix} \begin{bmatrix} \bar{\xi} & 0 \\ 0 & 1 \end{bmatrix}$$

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ideas, to replace antilinear transformations [redacted] by doubling. [redacted] you should be able to translate the calculations involving a positive herm. form and a symmetric bilinear form on $V$ into calculations within $\mathcal{L}(Sp(2n))$, where $n = \dim V$. [redacted] Maybe even within $\mathcal{L}(Sp(2n)/\varphi\,\mathcal{U}(n))$.

You want an [redacted] intrinsic interpretation of [redacted] symmetric unitary, and skew-symmetric unitary matrices.

Your aim today will be to link your spectral theory for symmetric bilinear forms up to unitary equivalence (involving antilinear operators) to the infinitesimal symmetric space $\mathcal{L}(Sp(2n)/\varphi U(n))$.

Recall that you ~~~~ start with a ~~~~ complex pos herm. v.s. $V$ and a symmetric form $b : V \longrightarrow V^t$ from which you get an antilinear op.

$$\begin{array}{ccccc} V & \xrightarrow{b} & V^t & \xleftarrow{\simeq} & \bar{V} \\ x & \longmapsto & x^t b & \longmapsto & b^* \bar{x} = \overline{bx} \\ & & b(x,-) & & \end{array}$$

Having worked on the Grassmannian yesterday you should try to generalize to $Sp(2n)$.

Thus you should focus on $\mathcal{L}(Sp(2n)/\varphi U(n))$

$$= \left\{ X = \begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix} : c^t = c \right\} \qquad \varphi(u) = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$$

It now becomes clear that working ~~~~ concretely with matrices is a good tool, perhaps ~~~~ it justifies the way you handle $V$.

Point: Because you invoke duality in the $Sp(2n)$ and $O(2n)$ cases, the basic repn for $Sp(2n)$ is $\begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ equipped with the symplectic form $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. ??

It seems that you are continually fighting with $V, V^t, \bar{V}, V^*$ which are 4 different vector spaces ~~~~ from some general viewpoint: $V, \bar{V}$ covariant $V^t, V^*$ contravariant. A pos. herm. product identifies $\bar{V} = V^t$, $V = V^*$.

(390) $V$ pos. herm. v.s. $\underbrace{V = V^*}_{V_+}$, $\underbrace{\bar V = V^t}_{V_-}$

Form $H(V) = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ basic repn.

$$\mathcal{L}\, Sp(2n) \longrightarrow End(H(V)) = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \otimes [V_+^* \ V_-^*] \quad ??$$

$$\Vert$$

$$\left\{ X = \begin{bmatrix} a & -\bar c \\ c & \bar a \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ c^t = c \end{matrix} \right\}$$

$$\overset{a}{n^2} + \overset{c}{\frac{n(n+1)}{2}}\, 2 = 2n^2 + n = \frac{2n(2n+1)}{2}$$

You need to understand how the operator $a^* = -a$ on $V_+$ gives rise to an operator $\bar a$ on $V_-$ also [redacted] skew adjoint.

Go back to original picture of $\mathcal{L}(Sp(2n))$

$$Sp(2n, \mathbb{C}) = \left\{ g \in GL(2n, \mathbb{C}) \;\middle|\; g^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$U(2n) = \{ g \in GL(2n, \mathbb{C}) \mid g^* g = 1 \}.$$

$n = 1$. $\quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\qquad \det(g)^2 = 1.$

$$g^t = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} g^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Case 1. $\det(g) = 1$. $\qquad \det(g) = -1$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -c & a \\ -d & b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$g = \begin{bmatrix} a & b \\ -\bar b & \bar a \end{bmatrix}$ $\quad |a|^2 + |b|^2 = 1$

$\therefore$ $Sp(2, \mathbb{C}) = SL(2, \mathbb{C})$

$$g^* = g^{-1} \qquad \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\therefore$ $d = \bar a$, $c = -\bar b$

So you do all these matrix calculations for the nth time. $\boxed{\text{Idea}}$ In the theory of $C^*$ algebras the basic objects are algebras + alg. morphisms. There isn't always a basic repn whose endos give the algebra.

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You want to define $Sp(2n)$ as the Lie group of symmetries of a certain linear algebra type structure

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Interested in harmon. oscillators & glueing. Viewpoint is to examine $n=1$; $g=\begin{bmatrix} a & b \\ c & d\end{bmatrix} \in GL(2,\mathbb{C})$, two conditions $g^*g=1$, i.e. $g \in U(2)$, $g^t\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}g=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$, i.e. $g \in Sp(2,\mathbb{C})$. Analyze 2nd condition $\Longrightarrow \det(g)=\pm 1$, if $-1$ no solutions, if $\det(g)=1$ get all $g \in SL(2,\mathbb{C})$.

Combining 2 conditions $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ $\quad a\bar{b}+c\bar{d}=0$

$\begin{bmatrix} a \\ c\end{bmatrix}$ prop. to $\begin{bmatrix} -\bar{b} \\ \bar{d}\end{bmatrix}$ ?

$$a\bar{b}+c\bar{d}=0 \qquad \begin{vmatrix} a & \bar{d} \\ c & -\bar{b}\end{vmatrix}=0$$

take $\bar{d}=a$, $\bar{b}=-c$ $\quad \begin{bmatrix} a & -\bar{c} \\ c & \bar{a}\end{bmatrix}$

$$Sp(2)=\left\{\underbrace{\begin{bmatrix} a & -\bar{c} \\ c & \bar{a}\end{bmatrix}}_{g} : |a|^2+|b|^2=1\right\} \qquad \mathcal{L}\,Sp(2)=\left\{\underbrace{\begin{bmatrix} a & -\bar{c} \\ c & \bar{a}\end{bmatrix}}_{X} : a+\bar{a}=0\right\}$$

OK what is a good viewpoint; you want to pass from $n=1$ to higher $n$ by the same construction which should be very simple. You've done this for

$$\mathcal{L}\,Sp(2n)=\left\{X=\begin{bmatrix} a & -\bar{b} \\ c & \bar{a}\end{bmatrix} \in M_{2n}\mathbb{C} : \begin{matrix} a^*+a=0 \\ b^t=c\end{matrix}\right\}$$

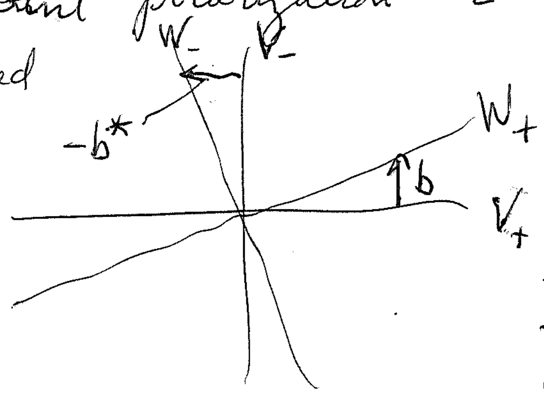
(392) Problem: [redacted] Is there an intrinsic way to handle [redacted] $\mathcal{L}\,Sp(2n)$? This is the idea you have been missing. You [redacted] want to start with a complex vector space $V$ equipped with a pos hermitian form, then form $H(V)$ with its 3 structures, any two $\Rightarrow$ third. What's intrinsic about a [redacted] a unitary skew-symmetric matrix, or a unitary symmetric matrix?[2]

For the moment you want to use $\mathcal{L}(Sp(2n)/qU(n))$ to study polarizations. $H(V) = \begin{bmatrix} V_+ = \mathbb{C}^n \\ V_- = \mathbb{C}^n \end{bmatrix}$ basepoint polarization. [redacted] First you need to understand the space or variety of polarzns = space of Lag subsp.

It seems worthwhile trying to embed the space of Lagrangian subspaces / polarizations into the Grassmannian $U(2n)/U(n)^2$.

[redacted]

It seems clear that the space of Lagrangian polarizations [redacted] can be identified with the space of self-adjoint involutions $F$ such that the $F = +1, -1$ eigenspaces are Lagrangian. Locally around the basepoint polarization $\varepsilon$ a point $F$ of $G_n$ can be pictured




You should know that $W_+$ is Lagrangian iff $b^t = b$, and this implies that $-b^* = -\bar{b}$ is also symmetric so that $W_-$ is Lagrangian.

You were supposed to use $c$ instead of $b$.

Laura Ashley

What [redacted] you need to do next is to study the infinitesimal symmetric space $\mathcal{L}(Sp(2n)/\mathcal{U}(n))$ which should yield [redacted] tangent spaces to the flag manifold of polarizations. A polarization of $H$ is [redacted] a $\mathbb{C}$-linear operator $F$ on $H$ with certain properties; $F = F^* = F^{-1}$. There's another property, which should imply that the eigenspaces of $F$ are Lagrangian. There are various things to try. Let's try the local picture near the basepoint where you get $F$ [redacted] $= \frac{1+X}{1-X}\varepsilon \quad X = \begin{bmatrix} 0 & -c^* \\ c & 0 \end{bmatrix}$.

Recall the three properties [redacted] for $X \in \mathcal{L}(Sp(2n))$:

(i) $X^* + X = 0$, (ii) $X^t J + JX = 0$, (iii) $JX = \bar{X} J$

$$\bar{X} = \begin{bmatrix} \bar a & \bar b \\ \bar c & \bar d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ -a & -b \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

(iii) means $X = \begin{bmatrix} a & -\bar c \\ c & \bar a \end{bmatrix}$. [redacted]

Notice that $\varepsilon = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ anticommute and they are real. The operator $X = \begin{bmatrix} 0 & -\bar c \\ c & 0 \end{bmatrix}$ where $c^t = c$ [redacted] satisfies (i) and (iii) so it satisfies (ii).

Let's check that the C.T. $g = \frac{1+X}{1-X}$ is a symplectic autom: $g^t J g = J$? [redacted]

[redacted] $g^t = J g^{-1} J^{-1} = \bar{g}^{-1} = \frac{1-\bar X}{1+\bar X}$

$$g^t = \left(\frac{1+X}{1-X}\right)^t = \frac{1+X^t}{1-X^t}$$

$$X = \begin{bmatrix} 0 & -\bar c \\ c & 0 \end{bmatrix} \Rightarrow \bar X = \begin{bmatrix} 0 & -c \\ \bar c & 0 \end{bmatrix} = -X^t$$

$$X^t = \begin{bmatrix} 0 & c \\ -\bar c & 0 \end{bmatrix} = -\bar X$$

(394) What have you learned? You seem to be able to use self-adjoint involutions $F$ satisfying $JFJ^{-1} = -F$ to describe polarizations. Go back to $H \doteq \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \right\}$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad \text{polar}$$

---

Let's go over this stuff again. You have symp. form: $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, herm form. $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

The (mult. difference) of these form is an antilinear of.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \longmapsto \begin{bmatrix} x_1^t & y_1^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -y_1^t & x_1^t \end{bmatrix} = \begin{bmatrix} x_0^* & y_0^* \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} -y_1^t & x_1^t \end{bmatrix}^* = \begin{bmatrix} -\bar{y}_1 \\ \bar{x}_1 \end{bmatrix}$$

---

$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{C}^n \right\}$ two forms. | symp, pos herm.

define polarization $H = V_+ \overset{\perp}{\oplus} V_-$, $V_+, V_-$ isotropic.

$F$ is the involution with $F = \pm 1$ on $V_\pm$.

Using $F$ and the symplectic form you want to express that $V_+, V_-$ isotropic.

$$\Omega(\xi_+ + \xi_-, \eta_+ + \eta_-)$$

---

Go back to $V \xrightarrow{c} V^t \xrightarrow{\sim} \bar{V} \xrightarrow{c} V^* \xrightarrow{\sim} V$

$$x \longmapsto x^t c \longmapsto c^* \bar{x} \longmapsto c c^* \bar{x} \qquad c^* \overline{c^* \bar{x}} \quad (c^* c^t) x$$

$$V \xrightarrow{c} V^t \xrightarrow{*} V \xrightarrow{c} V^t \xrightarrow{*} V$$

$$x \qquad x^t c \qquad c^* \bar{x} \qquad (c^* \bar{x})^t c = x^* \bar{c} c \qquad (\bar{c} c)^* x = (c^* c^t) x$$

$HV = \mathbb{H} \otimes_{\mathbb{C}} V$. You want a good picture of a polarization of $HV$. Let's recall possibilities. Question: Is there a quaternionic picture? What might you mean by this? Start with $V$ a $\mathbb{C}$ vector space of dim $n$ equipped with pos hermitian form.

Then it should be possible to construct a canon. isom.

$$\mathbb{H} \otimes_{\mathbb{C}} V \xrightarrow{\sim} HV$$

which respects a lot of the structure. Natural question is which embedding $\mathbb{C} \hookrightarrow \mathbb{H}$ do you use? Possible embeddings are elements of $\mathbb{H}$ of square $= -1$.

$$(t + x\hat{i} + y\hat{j} + z\hat{k})^2 \qquad t, x, y, z \in \mathbb{R}$$

$$= t^2 + 2t(x\hat{i} + y\hat{j} + z\hat{k}) - (x^2 + y^2 + z^2)$$

If this $\in \mathbb{R}$, then either $t = 0$ or $x\hat{i} + y\hat{j} + z\hat{k} = 0$

$t = 0$ square is $-(x^2 + y^2 + z^2)$ so you get $S^2$ possibilities, If $t \neq 0$ then $x = y = z = 0$ and $t^2 > 0$, not $= -1$.

Conclude $S^2 =$ embeddings $\mathbb{C} \hookrightarrow \mathbb{H}$

$S^2/\text{antipodal map} =$ complex subfields of $\mathbb{H}$.

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The problem is still to understand polarizations. It should be easy because you have control of the relevant structures. Maybe you should introduce K-theory picture where you take limits as $n \to \infty$

Let's try linking the spectral theory of [redacted] symmetric forms [redacted] modulo unitary equivalence to the [redacted] infinitesimal symmetric space $\mathcal{L}(Sp(2n)/\varphi U(n))$. [redacted]

$\mathcal{L}(Sp(2n)/\varphi U(n))$ is the groupoid [redacted] whose objects are matrices $\begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix}$ where $c$ is a complex symmetric $n \times n$ matrix, and where the morphisms arise from the conjugation action of

$$\varphi(u) = \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}, \text{ for } u \in U(n).$$

[redacted] e.g.

$$\varphi(u)^{-1} \begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix} \varphi(u) = \begin{bmatrix} u^{-1} & 0 \\ 0 & \bar{u}^{-1} \end{bmatrix} \begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$$

$$= \begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} \begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -u^* \bar{c} \bar{u} \\ u^t c u & 0 \end{bmatrix}$$

Next go back to "multiplicative difference" between pos herm & symmetric forms.

$$V \xrightarrow{c} V^t \xrightarrow{\sim} V \xrightarrow{c} V^t \xrightarrow{\sim} V$$

$$x \longmapsto (cx)^t \overset{*}{\longmapsto} \overline{cx} \longmapsto (c\overline{cx})^t \overset{*}{\longmapsto} \overline{c\overline{cx}} = (\bar{c}c)x$$

antilinear operator $\quad \sigma_c(x) = \overline{cx}$, $\quad \sigma_c \sigma_c(x) = \overline{c\overline{cx}} = (\bar{c}c)x$

You have to square $\begin{bmatrix} 0 & -\bar{c} \\ c & 0 \end{bmatrix}$ ?

---

Let's go over the general pattern. Let $V$ be a complex v.s. with pos. herm. form. There should be a spectral theory for symmetric $\mathbb{C}$-bilinear forms and also one for skew-symmetric $\mathbb{C}$-bilinear forms. The former should be linked to $Sp(2n)/\varphi U(n)$, the latter to $SO(2n)/\varphi U(n)$. These symmetric spaces are the homogeneous spaces consisting of all polarizations in the hyperbolic symplectic and orthogonal spaces associated to $V$.

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Review polarizations close to the base point. | You want the F picture.

To set up $HV$ properly. $V$ complex v.s. with pos herm inner product. $V \xrightarrow{\sim} V^*$, $\bar{V} \xrightarrow{\sim} V^t$. ~~The best~~

$$HV = \begin{bmatrix} V \\ \bar{V} \end{bmatrix}$$

What is $HV$?

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Go back to the beginning: $V$ cx v.s. w. pos herm form, no take $V = \mathbb{C}^n$.

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Here some ideas: $H = \begin{bmatrix} V \\ V^t \end{bmatrix}$ $\quad \mathrm{End}(H) = \begin{bmatrix} V \\ V^t \end{bmatrix} \otimes \begin{bmatrix} V \\ V^t \end{bmatrix}^t$

$$\mathrm{End}(H) = \begin{bmatrix} V \\ V^t \end{bmatrix} \otimes \begin{bmatrix} V \\ V^t \end{bmatrix}^t = \begin{bmatrix} V \\ V^t \end{bmatrix} \otimes [V^t \; V] = \begin{bmatrix} V \otimes V^t & V \otimes V \\ V^t \otimes V^t & V^t \otimes V \end{bmatrix}$$

---

$$HV = \mathbb{H} \otimes_{\mathbb{C}} V \qquad \mathbb{H} = (\underbrace{\mathbb{R} + \mathbb{R}i}_{\mathbb{C}} + \mathbb{R}j + \mathbb{R}k) \qquad ji = -k$$

$$= V \oplus jV \qquad \mathbb{C}j = j(\mathbb{R} + \mathbb{R}i) = j\mathbb{C}$$

for left mult by $\mathbb{C}$ $\quad jV = \bar{V}$.

$$\mathbb{H} \otimes_{\mathbb{C}} V = (\mathbb{R} + \mathbb{R}i + \mathbb{R}j \oplus \mathbb{R}k) \otimes_{\mathbb{C}} V$$

Now $\mathbb{R}j + \mathbb{R}k = (\mathbb{R}1 + \mathbb{R}i)j = j(\mathbb{R} + \mathbb{R}i)$
$= \mathbb{C}j = j\mathbb{C}$ $\qquad j(-i) = k$

$\therefore\ \mathbb{H} \otimes_{\mathbb{C}} V = V \oplus jV = \begin{bmatrix} V \\ jV \end{bmatrix}$. You are still missing the good viewpt. There are probably 3 viewpoints, at least 2, namely (i) $\mathbb{H} \otimes_{\mathbb{C}} V$ the $\mathbb{H}$-module generated by the $\mathbb{C}$-module $V$, (ii) the hyperbolic space $\begin{bmatrix} V \\ V^t \end{bmatrix}$ associated to $V$.

---

You want $HV$ explained clearly, the important topic seems to be polarizations.

Where's the problem. One picture of $HV$ is $\begin{bmatrix} V \\ V^t \end{bmatrix}$ with skew-symmetric form
$$\begin{bmatrix} x \\ \xi \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}$$

Another picture of $HV$ depends on a positive herm. form given on $V$. Then you get $V \xrightarrow{\sim} V^t$ invertible but antilinear. It might be better to say that you have $(x, y) \longmapsto \langle x | y \rangle$, $\bar{V} \xrightarrow{\sim} V^t$

Also you have $V \xrightarrow{\sim} V^*$, $y \longmapsto \langle x | y \rangle$

You still want a clear picture of $H(V)$. Let's agree to start with the hyperbolic symplectic form

$$HV = \begin{bmatrix} V \\ V^t \end{bmatrix}, \quad \begin{bmatrix} x \\ \xi \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}.$$ then use pos. herm. form

$\bar{V} \longrightarrow V^t \qquad x \longmapsto (y \longmapsto \langle x | y \rangle)$.

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~~[scribbled over]~~

What is $HV$? $\quad HV = \begin{bmatrix} V \\ V^t \end{bmatrix}, \begin{bmatrix} x \\ \xi \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}$

What $End(HV)$? $\quad End(HV) = \begin{bmatrix} V \\ V^t \end{bmatrix} \otimes [V^t \ \ V]$

$$= \begin{bmatrix} V\otimes V^t & V\otimes V \\ V^t\otimes V^t & V^t \otimes V \end{bmatrix}$$

Here's something you've not looked at enough.

$Hom(V \leftarrow W) = V \otimes_{\mathbb{C}} W^t = V \otimes_{\mathbb{C}} [\mathbb{C} \leftarrow W]$. Maybe the problem is how to reconcile $W^t = \overline{W}$. Just what is meant by $V \otimes \overline{W}$? Maybe involves adjoints.

$[V_+ \leftarrow V_+]$ $\quad [V_+ \leftarrow V_-]$

$[V_- \leftarrow V_+]$ $\quad [V_- \leftarrow V_-]$

~~Hom(V, W)~~ $\quad [V, W] = V \otimes W^t$. ~~[illegible]~~

intrinsic description. $\quad$ category of f.d. Hilb spaces

$a(v, w) = v \otimes ?$ $\quad$ Rank 1 operators

$vw^*$ $\quad$ ~~[illegible]~~ pattern should be that the space of ~~[illegible]~~ operators is dual to the space of sesquilinear forms $f(w, v)$ linear in $v$, antilinear in $w$.

clarify. Given $f : \overline{W} \longrightarrow V^t$, $vw^*$

Def $(f, vw^*) = f(w, v)$. Another idea might be ~~[illegible] adjoints~~ $tr(fg)$

On vector level have $x, \bar{x}, x^t, x^*$

Discuss the problem. You want an intrinsic picture, description of the hyperbolic symplectic space $HV$, where $V$ is a f.d. Hilb space.

As a vector space / $\mathbb{C}$ one has $HV = V \oplus V^t$. Because of the inner product on $V$ one has $V^t = V^\sigma$. $V^\sigma$ seems simpler as it's covariant.

So fix $HV = \begin{bmatrix} V \\ V^\sigma \end{bmatrix}$.

Now you can focus on End(HV) the space of $\mathbb{C}$-linear operators on $HV$.

First you should make precise $V^\sigma$ and the isom. $V^\sigma \xrightarrow{\sim} V^t$. What is a v.s. – it's a set with +, scalar mult. define $V^\sigma$ to be the set $V$ with given addition operation. Maybe better would be to define $V^\sigma$ to be a complex v.s. equipped with a bijection $V \xrightarrow{\sigma} V^\sigma$ s.t. comp. with + and s.t. $\sigma(zv) = \bar{z}\,\sigma(v)$. Notice $\sigma^{-1} : V^\sigma \longrightarrow V$ is a bijection, $\sigma v \longmapsto v$,
satisfying $\sigma^{-1}(z\,\sigma(v)) = \sigma^{-1}(\sigma(\bar{z}v)) = \bar{z}v = \bar{z}\,\sigma^{-1}(\sigma v)$

Have exhibited $\sigma^{-1} : V^\sigma \longrightarrow V$ which is bijective, comp with +, $\sigma^{-1}(z\sigma v) = \bar{z}\,\sigma^{-1}(\sigma v)$. Therefore $(V^\sigma)^\sigma \xrightarrow{\sim} V$. Maybe a functorial approach is better, namely you define an antilinear map $f : V \longrightarrow W$ by $f(\bar{z}v) = \bar{z} f(v)$, and then show

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \sigma\searrow & & \nearrow \exists!\ \text{linear map} \\ & V^\sigma & \end{array}$$

Next look at $V^t$ or $V^*$. Let's use $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$, then $End(HV) = \begin{bmatrix} V \\ V^t \end{bmatrix} \otimes [V^t \ V]$

$$\cong \begin{bmatrix} V\otimes V^t & V\otimes V \\ V^t\otimes V^t & V^t\otimes V \end{bmatrix}$$

The problem is still to understand intrinsically the hyperbolic symplectic space $HV$

You just noticed that $End(HV)$ is the Lie alg of infinitesimal symmetries of the vector space $\begin{bmatrix} V \\ V^t \end{bmatrix}$. The sub Lie alg [redacted] preserving the symplectic form is probably $\begin{bmatrix} V\otimes V^t & S^2V \\ S^2(V^t) & V^t\otimes V \end{bmatrix}$.

[redacted] Next you want to equip $V$ with pos herm inner product, extend [redacted] the inner product to $V^t$ in the standard way. You ~~expect~~ know the Lie sub alg preserving both forms using matrix notation should be

$$\mathcal{L}(Sp(2n)) = \left\{ X = \begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix} : \begin{array}{l} a^* + a = 0 \\ \text{[redacted]}\ c^t = c \end{array} \right\}$$

You seem to have something [redacted] that you did not notice before, namely, an intrinsic form for $\mathcal{L}U(V) \subset V\otimes V^t \cong \mathcal{L}GL(V)$.

What do you hope to do next? $V$ f.d Hilb sp.

$$V^{\sigma} \xrightarrow[\sim]{} V^t$$

$$x \longmapsto \langle x|$$

or $x \longmapsto x^*$

How does this relate to $End(V) = V\otimes V^t \cong V\otimes V^{\sigma}$ ?

Let's review the preceding. There seems to be something strange occurring. Let $V, W$ be finite diml Hilb spaces. Consider linear transfs $T: V \longrightarrow W$. This should be equiv. to an elt of $W \otimes V^t$

$$\mathrm{Hom}(V, \mathbb{C}) \otimes_{\mathbb{C}} W = V^t \otimes_{\mathbb{C}} W.$$

On the other hand you have $V^\sigma \xrightarrow{\sim} V^t$.

---

To understand linear transf and their adjoints

$$\begin{tikzcd} W \arrow[d, "s"'] & V \arrow[l, "T"'] \arrow[d, "s"'] \\ W^* \arrow[r, "T^*"] & V^* \end{tikzcd}$$

$$\langle w | Tv \rangle = \langle w^* T | v \rangle$$

Do it in words. $T$ is a linear transf from $V$ to $W$. $\forall w$ one has a lin. transf. $w^*: W \longrightarrow \mathbb{C}$ given by the herm. form. Compose to get a linear ~~transf~~ functional $w^*T : V \longrightarrow \mathbb{C}$. Use the herm. form to write $w^*T = (T^*w)^*$.

$$\langle w, Tv \rangle = \langle T^* w, v \rangle$$

Yesterday you ran into trouble ~~when handling operators~~ with $\mathrm{End}(V) = V \otimes V^t = V \otimes V^\sigma$

An orthonormal basis yields a partition of 1 whose operators are self-adjoint $\geq 0$. There is no phase to worry about. If $V$ is a line then $\mathrm{End}(V) = V \otimes V^t$ is canonically $\simeq \mathbb{C}$. What happens to $V \otimes V^\sigma$?

You continue struggling with an intrinsic picture of the hyperbolic symplectic space $HV$ where $V$ is equipped with pos herm inner product. Let's consider the simpler situation of just $V$. In this case the symmetry group is the unitary group $U(V)$ and $\mathcal{L}\,U(V)$ is the Lie algebra of skew adjoint operators on $V$.

There is also the contragredient representation of $U(V)$ on $V^t$, the dual.

Review the ideas. Given $V$ with pos. herm. form. Then given any $\mathbb{C}$-linear op $T$ on $V$ you get two sesquilinear forms $\langle x | Ty \rangle$ and ??

Better given $V, W$ with pos herm. forms and $T : W \longrightarrow V \to V^*$ $\quad \langle v | Tw \rangle$ is a lin fnl on $W$

$$\langle v | Tw \rangle = \langle T^* v | w \rangle$$

$$\begin{tikzcd} V \arrow[d, "s"'] & W \arrow[l, "T"'] \arrow[d, "s"] \\ V^* \arrow[r, "T^*"] & W^* \end{tikzcd}$$

---

Start again. Define $W^* = \{ f : W \to \mathbb{C} \mid f(zw) = \bar{z} f(w),\ f\text{ add.} \}$

$V \xrightarrow{T} W$ linear transf., $f \in W^* \Rightarrow fT \in V^*$

$fT(zw) = f(zTw) = \bar{z}\, fT(w)$. Now use $\exists!$ $w$ s.t. $f = \langle w|$, then $fT = \langle w|T \in V^*$ then $\exists!\ v$ s.t. $\langle v| = \langle w|T$ which means $\langle v|v'\rangle = \langle w|T|v'\rangle$ $\forall v'$.

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$V, W$ pos herm. $T: V \to W$ $\mathbb{C}$ linear.

To define the adjoint $T^*: W \to V$. Formula

$$(w, Tv) = (T^*w, v) \qquad \forall w, v$$

equiv $\underbrace{(Tv, w)}$ $= (v, T^*w)$.

left side is an antilinear fnl on $V$, hence rep by a unique elt. Define $T^*w$ to be this unique rep. elt. Then check $T^*w$ is $\mathbb{C}$-linear

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\sigma_V"', "\wr"] & W \arrow[d, "\wr", "\sigma_W"'] \\ V^* & W^* \arrow[l, "T^t"'] \end{tikzcd}$$

$w \in W$ $\quad w^* \in W^*$

$v \longmapsto w \mapsto w^*Tv$

$w^*Tv$

You would really like to define $T^*$ as $T^t$ conjugated via $\sigma_V, \sigma_W$ $\qquad T^* = \sigma_V^{-1} T^t \sigma_W$

---

$V$ pos herm v.s. $\qquad h_V(v, v')$ $\mathbb{C}$-linear in $v'$, antilinear in $v$.

$$\begin{array}{c} V \\ \sigma_V \downarrow \\ V^t \end{array}$$

An important point is that $\sigma_{V^t}\sigma_V = 1_V$

Why? because

Somehow a hermitian form has a Galois significance. ~~It should~~ It leads to maximal compact subgrps.

---

choose an orthonormal basis for $V$, there's a corresp. dual orthonormal basis for $V^t$, which yields a pos herm. form for $V^t$. Can view $V = \mathbb{C}^n$ column vectors, $V^t$ as row vectors, $\sigma_V = *$, Next you need a way to transport the herm. form on $V$ to a herm form on $V^t$.

Result:

(405) Yesterday 29 Jun 03 you found a way around the difficulties with hermitian forms. Background: Your aim is to construct an intrinsic picture of [redacted] $\mathcal{L}\,Sp(2n)$, the Lie alg of infinitesimal symmetries of the hyperbolic symplectic space $H\mathbb{C}^n$. You want to replace $H\mathbb{C}^n$ by $HV$, where $V$ is a positive hermitian vector space. Since $HV = V \oplus V^t$, $\mathcal{L}\,Sp(HV)$ is contained in (a [redacted] Lie subalg) of

$$\mathrm{End}(HV) = \begin{bmatrix} V \\ V^t \end{bmatrix} \otimes \begin{bmatrix} V^t & V \end{bmatrix} = \begin{bmatrix} V\otimes V^t & V\otimes V \\ V^t\otimes V^t & V^t\otimes V \end{bmatrix}.$$

This means that elts of $\mathcal{L}\,Sp(2n)$ are described by $2\times 2$ blocks of $n\times n$ complex matrices. In the intrinsic picture you seek the diagonal blocks are Endo rings, and the odd blocks are [redacted] spaces of symmetric bilinear forms. Let's now [redacted] try to understand the effect of a pos herm. form on $V$. Recall the definition: It is a sesquilinear form $\langle x|y\rangle$, [redacted] $x, y \in V$ which is $\mathbb{C}$-antilinear in $x$ and $\mathbb{C}$-linear in $y$. [redacted] A sesquilinear form is equivalent to [redacted] an antilinear map $V \longrightarrow V^t$, $x \longmapsto \langle x|-$. Next one has hermitian symmetry $\overline{\langle x|y\rangle} = \langle y|x\rangle$, which is equivalent to $\langle x|x\rangle \in \mathbb{R}$ for all $x \in V$. Finally there's the positivity condition $\langle x|x\rangle > 0$ for $x \neq 0$.

Ex [redacted] $V = \mathbb{C}^n$ (column vectors).

$$\langle x|y\rangle = x^* y = \sum_a \bar{x}_a y_a$$

(406) If $V$ is equipped with pos. herm. $\langle x|y\rangle$ then the anti-linear map $V \longrightarrow V^t$, $x \longmapsto \langle x|-$ is injective by positivity and so is bijective. Denote this map by $\sigma_V : V \xrightarrow{\sim} V^t$ [redacted] Thus a pos herm form on $V$ determines an antilinear isomorphism $\sigma_V : V \xrightarrow{\sim} V^t$. Conversely an antilinear transf $V \xrightarrow{h} V^t$ [redacted] is equivalent to a sesquilinear form $h(x,y)$, [redacted]

Now discuss what it means for $h$ to be hermitian symmetric

$\boxed{\text{Idea}}$: You may need the polarization identity which expresses $\langle x|y\rangle$ in terms of 4 diagonal terms: $\langle z|z\rangle$

---

~~Digression. Let $b : V \longrightarrow V^t$ be a bilinear form on $V$. [redacted] What is $x \longmapsto b(x,-)$ the transpose of the map $b$? $b^t : (V^t)^t \longrightarrow V^t$. $(V^t)^t \overset{s}{\cong} V$. Let $\varphi_x : V^t \longrightarrow \mathbb{C}$ be $\lambda \mapsto \lambda(x)$.~~

~~What is $b^t(\varphi_x)$? In general it is $\varphi_x b$~~

[redacted]

---

$b(x,y)$ bilinear on $V \times V$. From $b$ get a map $V \longrightarrow V^t$, $x \mapsto b(x,-)$

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$b(x,y) \in L$ bilinear on $V \times V$

same as a linear map $V \otimes V \longrightarrow L$

same as the linear map $x \longmapsto (y \mapsto b(x,y))$

$V \longrightarrow \mathrm{Hom}(V, L)$

$b' : V \longrightarrow V^t$, $x \mapsto (y \mapsto b(x,y))$

What is $b'^t$? It's defined on $(V^t)^t = V$. For $\forall y \in V$ let $\varphi_y \in (V^t)^t$ be defd by $\varphi_y(\lambda) = \lambda(y)$.

Then $\varphi_y : V^t \longrightarrow \mathbb{C}$ $\quad (b')^t \varphi_y = (\varphi_y \circ b') = \{x \mapsto \underbrace{\varphi_y b' x}_{b(x,y)}\}$

TRY again. $b(x,y)$ bilinear on $V \times V$.

Let $b' : V \longrightarrow V^t$ $\quad b'(x) = \{z \mapsto b(x,z)\}$.

What is $b'(x)^t : (V^t)^t \longrightarrow V^t$ ?

$b'(x)^t(\varphi_y) = \varphi_y \circ b'(x) = \varphi_y \{z \mapsto b(x,z)\}$

$b(x,y)$ bilinear on $V \times V$

Define $b' : V \to V^t$ by $b'(x) = \{y \mapsto b(x,y)\}$

To calculate $b'^t : (V^t)^t \longrightarrow V^t$. $b'^t$ is defined $(b'^t(\varphi_y))(x) = \varphi_y b'(x) = \varphi_y$

$b(x,y)$ bilinear on $V \times W$

$b' : V \longrightarrow W^t$ defined by $b'(x) = \{y \mapsto b(x,y)\}$

$(b')^t : \underbrace{(W^t)^t}_{\varphi_y} \longrightarrow V^t$ defined by $(b')^t \varphi_y = \varphi_y \circ b'$

$= \{x \mapsto b(x,y)\}$.

Something is still funny.

$$\mathrm{Hom}(V \otimes W, \mathbb{C}) = \mathrm{Hom}(V, \mathrm{Hom}(W, \mathbb{C}))$$

also $= \mathrm{Hom}(W, \mathrm{Hom}(V, \mathbb{C}))$

$$(V \otimes W)^t = \mathrm{Hom}(V, W^t) = \mathrm{Hom}(W, V^t)$$

This might be interesting, [redacted] because [redacted] transpose should relate the latter two Hom's, even though the transpose of $V \longrightarrow W^t$ is a ~~map~~ $W^{tt} \longrightarrow V^t$.

---

~~[illegible]~~ Let's spend time on the identification $\sigma_V : V \xrightarrow{\sim} V^t$ (antilinear) for a pos hermitian vector space. Question. What is a hermitian forms on $V$ a complex vector space? If $V, W$ are $\mathbb{C}$-vector spaces, there is the notation of a sesquilinear form $f(v, w)$, linear in the variable $w$, antilinear in the variable $v$. Same as linear functionals on $V^\circ \otimes_{\mathbb{C}} W$. What happens if $V, W$ are complex lines.

---

Let $V$ be a pos. herm. v.s. Then the herm. form yields an ~~[illegible]~~ antilinear isom. $\sigma_V : V \longrightarrow V^t$. Question: Is $\sigma_V^2 = 1$ in some sense? ~~Probably~~ ~~[illegible]~~ There should be a way s.t. the dual $V^t$ has an induced pos. herm. form.

$\boxed{\text{Idea}}$ A basis for a $\mathbb{C}$-vector space $V$ yields the strongest identification of $V$ with $V^t$. A pos herm form gives a weaker identification:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "s"] & W \arrow[d, "s"] \\ V^* & W^* \arrow[l, "T^*"'] \end{tikzcd}$$

This defines the adjoint operator for $T$

You want the endos of $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$ explained intrinsically

$$\begin{bmatrix} V \otimes V^t & V \otimes V \\ V^t \otimes V^t & V^t \otimes V \end{bmatrix}$$

You seem to have two intrinsic pictures for End(V)

Problem. $\mathrm{End}(V) = V \otimes V^t \quad , \quad V^t = V^\sigma$

$\therefore \quad V \otimes V^t = V \otimes V^\sigma$

$x \otimes y^*$ There's a real problem here in understanding

Maybe there are two tensor pictures for End(V). Is it possible that the product in the picture $V \otimes V^\sigma$ invokes the hermitian form on $V$.

symplectic form on $HV = \begin{bmatrix} V \\ V^t \end{bmatrix} \ni \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

$\mathbb{C} \xrightarrow{y} V^t \Rightarrow V \xrightarrow{y^t} \mathbb{C}$

$$\begin{bmatrix} x_1^t & y_1^t \end{bmatrix} \begin{bmatrix} y_2 \\ -x_2 \end{bmatrix} = \begin{matrix} x_1^t y_2 \\ -y_1^t x_2 \end{matrix}$$

go over the difficulties.

The vreiwpoint you need might be ideas from the Schwartz kernel theorem. Explain. Your problem is to find an intrinsic picture of operators on $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$, where $V$ is a hermitian vector space. By operator on $HV$ you mean an infinitesimal symmetry preserving the relevant structure: the inner product and the symplectic form. When $V = \mathbb{C}^n$ with usual hermitian inner product $x^*y$ (or $x^*\xi$), you get the Lie algebra $\mathcal{L}(Sp(2n)) = \left\{ X = \begin{bmatrix} a & -\bar{c} \\ c & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ c^t = c \end{matrix} \right\}$.

So you need intrinsic pictures for $a, \bar{a}, c, \bar{c}$.

Now $a$ is a skew adjoint operator on $V$.

---

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\wr"'] & W \arrow[d, "\wr"] \\ V^* & W^* \arrow[l] \end{tikzcd}$$

$V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$

---

$\mathcal{L}(U(V)) = \{X \in \mathrm{End}(V) \text{ s.t. } X + X^* = 0\}$

---

basic representation $V$, contragredient repn. $V^t = \bar{V}$

all you have to do is to ~~show how~~ use

$\bar{X} = -X^t$. So you can write

$$\mathrm{End}(V) \cong V \otimes V^t \cong V \otimes \bar{V}$$

---

Look at notation for elts of $V$ and $V^t$

An elt $x \in V$ is equivalent to the linear map $\mathbb{C} \longrightarrow V$ sending $1$ to $x$. ~~the transpose~~

An elt of $V^t$ is ~~equiv. to the~~ a linear map $V \xrightarrow{\xi} \mathbb{C}$ ~~sending 1 to $\xi$~~, which is equiv. by transpose to $\xi^t : \mathbb{C} \longrightarrow V^t$

fin. dim.

$V$ vector space, $V^t$ the dual vector space

elt of $V$ same as map $\mathbb{C} \xrightarrow{x} V$ same as $V^t \xrightarrow{x^t} \mathbb{C}$

elt of $V^t$ ———— $\mathbb{C} \xrightarrow{\xi} V^t$ " " $V \xrightarrow{\xi^t} \mathbb{C}$

$$\mathbb{C} \xrightarrow{x} V \xrightarrow{\xi^t} \mathbb{C} \qquad \xi^t x$$

$$\mathbb{C} \xrightarrow{\xi} V^t \xrightarrow{x^t} \mathbb{C} \qquad x^t \xi$$

[redacted] Maybe the point is that if we interpret elements $x \in V$ and $\xi \in V^t$ as maps $\mathbb{C} \xrightarrow{x} V$ and $\mathbb{C} \xrightarrow{\xi} V^t$, then the transposes of these maps

$$V^t \xrightarrow{x^t} \mathbb{C} \qquad \text{and} \qquad (V^t)^t \xrightarrow{\xi^t} \mathbb{C}$$

$$\eta \longmapsto \eta(x) \qquad\qquad V \ni y$$

what is $y(\eta)$? Ans $\eta(y)$

so $y \longmapsto (\eta \longmapsto \eta(y))$

$\qquad\qquad \xi \longmapsto \xi(y)$

In words you can identify $y \in V$ with the linear fnl $\eta \longmapsto \eta(y)$ on $\{\eta \in V^t\}$. Next you have $\xi \in V^t$ yielding $\mathbb{C} \xrightarrow{\xi} V^t$, and $\xi^t$ takes $\eta \in V^t$ to $\mathbb{C} \xrightarrow{\xi} V^t \longrightarrow$

Review Start with $\mathbb{C} \xrightarrow{\xi} V^t$ and compute its transpose $(V^t)^t \xrightarrow{\xi^t} \mathbb{C}$

$$\begin{array}{c} (V^t)^t \\ \uparrow\wr \\ V \ni y \end{array}$$

$y$ gives the elt $ev_y \in (V^t)^t$ defined by $ev_y(\eta) = \eta(y)$. $\therefore$ $y$ becomes $\eta \mapsto \eta(y)$.

What is $\xi^t$ then? You take any linear fnl on $V^t$ – it has the form $\eta \mapsto \eta(y)$

$\mathbb{C} \xrightarrow{\xi} V^t \xrightarrow{ev_y} \mathbb{C}$ so you get $ev_y(\xi) = \xi(y)$.

So you have calculated that the transpose of the map $\mathbb{C} \xrightarrow{\xi} V^t$ is the linear map on $V$ sending each $y \in V$ to the ~~comp~~ composite map

$$\mathbb{C} \xrightarrow{\xi} V^t \xrightarrow{ev_y} \mathbb{C}$$

$$\eta \longmapsto \eta(y)$$

$$\Rightarrow \quad \xi \longmapsto \xi(y).$$

$$\begin{array}{l} \xi^t : (V^t)^t \longrightarrow \mathbb{C} \\ \quad\quad\;\; \parallel \\ \quad\quad\;\; V \end{array}$$

maybe you can ~~make~~ distinguish between $\xi^t : (V^t)^t \to \mathbb{C}$ and $\xi : V \longrightarrow \mathbb{C}$

~~Recap. Let $V$ be a fin. dim v.s, $V^t$ its dual~~
~~An $x \in V$ is equivalent to the linear map~~
~~$\mathbb{C} \longrightarrow V$, $z \longmapsto zx$~~

$V$ fin. dim'l, $V^t$ the dual space

An element $x \in V$ same as a map $\mathbb{C} \xrightarrow{x} V$

same as transpose map $V^t \xrightarrow{x^t} \mathbb{C}$

(413) $V$ fin diml v.s., $V^t$ dual space

TFAE: (i) ~~the~~ elt $x \in V$

(ii) ~~the~~ linear map $\mathbb{C} \xrightarrow{x} V$ sending $1$ to $x$

(iii) the transposed map $V^t \xrightarrow{x^t} \mathbb{C}$ to (ii).

Similarly TFAE: (i) the elt $\xi \in V^t$

(ii) the linear map $\mathbb{C} \xrightarrow{\xi} V^t$

(iii) the transposed map $V \xrightarrow{\xi^t} \mathbb{C}$ to (ii)

(Remark that defined the transposed map requires the canon isom $V \xrightarrow{\sim} (V^t)^t$. Proof. Injective because $x \neq 0 \Rightarrow \exists\, \xi \in V^t$ s.t. $\xi(x) \neq 0$. Then get iso as both spaces have the same finite dimension).

Review the computation of the transpose of the map (ii): One has

$$\begin{tikzcd} (V^t)^t \arrow[r, "\xi^t"] & \mathbb{C} \\ V \arrow[u, "ev", "\wr"'] \end{tikzcd}$$

canon. map: For $y \in V$, define $ev(y)$ to be the linear fnl on the dual space $\{\eta \in V^t\}$ given by $ev(y)(\eta) = \eta(y)$.

Calculate $\xi^t \circ ev$. Given $y \in V$, one has $ev(y)$ is the linear fnl on $\{\eta \in V^t\}$ given by $ev(y)(\eta) = \eta(y)$. $\xi^t$ is the transpose of $\mathbb{C} \xrightarrow{\xi} V^t$, which means that for any linear fnl on $V^t$, equivalently for any $y \in V$, you compose (or pull back) the linear functional $ev(y)(\eta) = \eta(y)$ by means of the map $\xi$, getting $ev(y)(\xi) = \xi(y)$.

(414) So it seems that your calculation of the composition $V \xrightarrow{\sim} (V^t)^t \xrightarrow{\xi^t} \mathbb{C}$ is

$$y \longmapsto ev(y) \longmapsto \xi(y).$$

$\xi^t \circ ev$ is the actual linear fnl $y \mapsto \xi(y)$.

(The background here is that you had some ambiguity involved before with the notation for a linear functional $\xi$. On one hand $\xi$ is a map $\xi : V \longrightarrow \mathbb{C}$, which has a transpose $\xi^t : \mathbb{C} \longrightarrow V^t$ which sends $1 \in \mathbb{C}$ to the element $\xi \in V^t$.

---

$\mathrm{End}(V) = V \otimes V^t$ Didn't you look at elements of a tensor product $X \otimes Y$ of two vector spaces, Separable algebras being quasi-free?

Consider $\xi = \sum_{i=1}^{n} x_i \otimes y_i$. Suppose the elts $x_1, \dots, x_n$ are lin. dep., say $x_1 = \sum_{i=2}^{n} c_i x_i$

$$\xi = \sum_{i=2}^{n} x_i \otimes y_i + \sum_{i=2}^{n} c_i x_i \otimes y_1$$

$$\begin{aligned}
\xi &= x_1 \otimes y_1 + \sum_{i=2}^{n} x_i \otimes y_i \\
&= \left(\sum_{i=2}^{n} c_i x_i\right) \otimes y_1 + \sum_{2}^{n} x_i \otimes y_i \\
&= \sum_{i=2}^{n} x_i \otimes c_i y_1 + \sum_{i=2}^{n} x_i \otimes y_i = \sum_{i=2}^{n} x_i \otimes (c_i y_1 + y_i)
\end{aligned}$$

Idea: Let $\xi \in X \otimes Y$. Claim $\xi$ has a rank $r$ and $\exists \left| \begin{array}{l} x_1, \ldots, x_r \\ y_1, \ldots, y_r \end{array} \right.$ linearly indep.

s.t. $\xi = \sum_{i=1}^{r} x_i \otimes y_i$. Proof. Can suppose $Y$ is finite dimensional, whence $X \otimes Y = \mathrm{Hom}(Y^t, X)$, so $\xi$ can be interpreted as a linear map from $Y^t$ to $X$. Look at the image: $X \hookleftarrow W \twoheadleftarrow Y^t$, choose a basis $w_1, \ldots, w_r$ for $W$, let $v_1, \ldots, v_r$ ??

You need notation for basis and dual basis for a finite dim'l v.s. $W$. Basis $w_1, \ldots, w_r \in W$ and dual basis $\omega_1, \ldots, \omega_r \in W^t$. Is $\omega_i = w_i^t$? What does it mean for $w_1, \ldots, w_r$ to be a basis for $W$?

$$W \xleftarrow{[w_1, \ldots, w_n]} \mathbb{C}^n \xleftarrow{\begin{bmatrix} w_1^t \\ w_n^t \end{bmatrix}} W$$

get $w_i^t w_j = \delta_{ij}$, $\sum_i^r w_i w_i^t = 1$

Let's be careful. Suppose that $e_1, \ldots, e_r$ is a basis for $W$. This means that the map

$$W \xleftarrow{[e_1, \ldots, e_n]} \mathbb{C}^n$$

$$\sum_{i=1}^{r} e_i z_i \longleftarrow \begin{bmatrix} z_1 \\ z_n \end{bmatrix}$$

is an isomorphism. The inverse will take any $w$ into its coords relative to the basis. So you get $\varepsilon_1, \ldots, \varepsilon_r \in W^t$ such that

$$\varepsilon_j(w) = z_j \quad \text{where } w = \sum e_i z_i$$

$$= \varepsilon_j(w) = \sum_i \varepsilon_j(e_i) z_i$$

$\forall w \in W \quad \exists! \quad \varepsilon_i(w) \in \mathbb{C}$ s.t.

$$w = \sum_i e_i \, \varepsilon_i(w)$$

$$\varepsilon_j(w) = \sum_i \varepsilon_j(e_i)\, \varepsilon_i(w)$$

---

$\forall w \in W \quad \exists! \quad \varepsilon_i(w) \in \mathbb{C} \quad 1 \le i \le n$ s.t.

$w = \sum_i e_i \, \varepsilon_i(w)$ by uniqueness $\varepsilon_i \in W^t$ and $\varepsilon_i(e_j) = \delta_{ij}$

$$\left.\begin{aligned} zw &= \sum_i e_i \, \varepsilon_i(zw) \\ zw &= \sum_i e_i \, z\, \varepsilon_i(w) \end{aligned}\right\} \Rightarrow \varepsilon_i(zw) = z\varepsilon_i(w)$$

What is $e_i^t$

$$\mathbb{C}e_i \subset W$$

$$W^t \longrightarrow (\mathbb{C}e_i)^t$$

$$\eta \longmapsto \eta(e_i)$$

---

~~Let W have basis $e_i$. What is $e_i^t \in W^t$? You know that $\forall w \in W$ there is $w^t \in W^t$ such that $w^t w_1$? $\mathbb{C} \xrightarrow{w} W \Rightarrow W^t \xrightarrow{w^t} \mathbb{C}$~~

---

$$\mathbb{C} \xrightarrow{x} X \Longrightarrow X^t \xrightarrow{x^t} \mathbb{C} \qquad x^t \in (X^t)^t$$

$$X \xrightarrow{\xi} \mathbb{C} \Longrightarrow \mathbb{C} \xrightarrow{\xi^t} X^t$$

---

Review confusion $X \xrightarrow{u} Y^t \iff Y = (Y^t)^t \xrightarrow{u^t} X^t$

417 Let's try to do something using tensor product pictures. $End(V) = V \otimes V^t \simeq V \otimes V^\sigma$

Let $a \in \mathcal{L}U(V)$, it's the same as $a + a^* = 0$. You want to obtain $\bar{a}$ somehow.

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Problem: $V$ pos herm. v.s. You want to understand $U(V)$. Look at $\mathcal{L}U(V)$ first. ~~$\mathcal{L}U(V)$ is~~ $\mathcal{L}U(V) \subset End(V)$ is the space of skew-adjoint operators; it's a Lie algebra under bracket.

On $End(V)$ you have $T \mapsto T^t$. No.

$$V \xrightarrow{T} V \rightsquigarrow V^t \xrightarrow{T^t} V^t$$

This is an anti autom. You also have the adjoint $T \mapsto T^*$.

$$V \xrightarrow{T} V \rightsquigarrow V \xrightarrow{T^*} V$$

$x \mapsto y^* T x$ linear functional on $V$. repn. thm. for linear fnls says $\forall y, \exists\, Fy \in V$ s.t.

$(Fy)^* = y^* T$ i.e. $(Fy)^* x = y^* T x$. Then ~~denote~~ replace $F$ by $T^*$. $(T^* y)^* = y^* T$.

Is there something interesting here you've overlooked. Idea — Schwartz kernel stuff.

(418) If $T \in \mathrm{End}(V)$, then $y^*Tx \in \bar{V} \otimes V$.

Given $T \in \mathrm{End}(V)$ you get this "kernel" $y^*Tx$ which is sesquilinear. Clearly there should be a 1-1 correspondence between operators and sesquilinear forms. Next apply $\sigma$ to the sesquilinear form, you get $\overline{y^*Tx}$ which is linear in $y$, antilinear in $x$. Now interchange $x, y$ to get $\overline{x^*Ty}$ which is sesquilinear in the right way, so that it corresponds to an operator $T^*$ satisfying $y^*Tx = (T^*y)^*x$

Program. You want to see if your observation about $T \in V \otimes V^t$ and the image of $T$ is useful.

It would be nice to use the sesquilinear form picture of $T$ if possible.

Let's start with the case where $T = T^*$.

Maybe better to consider $V \xleftarrow{T} W$. Then there should be a standard picture for the image

$$V \hookleftarrow I \twoheadleftarrow W$$

You expect two norms on $I$, induced resp. from $V$ and $W$.

The way to handle this is to form $\begin{bmatrix} 0 & T^* \\ T & 0 \end{bmatrix}$ on $\begin{bmatrix} V \\ W \end{bmatrix}$; this should lead to characteristic values for $T$.

(419) ~~[illegible]~~ You want a Schwartz kernel for ~~take~~ operator $T$: $y^*Tx$.

Let $T = T^* \in \mathrm{End}(V)$. What can you say about the image of $T$? Use spectral decomp. The image of $T$ is the orthog complement of the kernel of $T$.

$\boxed{\text{Idea}}$: The graph of $T^*$ is a kind of orthogonal complement to the graph of $T$: $\begin{bmatrix} 1 & -T^* \\ T & 1 \end{bmatrix}$

Let's go back to the main problem, namely

$$\mathrm{End}(V) = V \otimes V^t \overset{?}{=} V \otimes \bar{V}.$$ ~~Look~~

The problem is to understand the repn

$a \longmapsto \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$ of $\mathcal{L}U(V)$ on $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$

~~Take $n=1$. $V$ is a line~~

---

rank 1 operator on $V$ has the form

$Tv = x\langle y|v\rangle$. Find $T^*$.

$\langle r|Tv\rangle =$

$$\langle r|x\langle y|v\rangle\rangle = \langle r|x\rangle \langle y|v\rangle \overset{?}{=} \langle T^*r|v\rangle$$

$$= \langle \overline{\langle r|x\rangle} y \,|\, v\rangle = \langle \langle x|r\rangle y \,|\, v\rangle = \langle y\langle x|r\rangle | v\rangle$$

$$Tv = x\langle y|v\rangle \iff T^*v = y\langle x|v\rangle$$

---

How unique are $x, y$? If you want to understand the possible $T$'s, you look at the span of $x, y$. Note $T$ is unchanged by mult $x, y$ by the same phase. You should be able to adjust the ~~weight~~ norms of $x, y$ so that $|x| = |y|$.

(420) It looks like there's ~~an~~ an angle between $x, y$ which might be an invariant of $T$.

$\boxed{\text{Idea}}$ This reminds you of the Hilbert-Schmidt inner product $\operatorname{tr}(T^*T)$, which you should have looked at much earlier.

What you want to have is a [redacted] Schwartz kernel picture of $\operatorname{End}(V)$. You want to [redacted] view $\operatorname{End}(V)$ as the tensor product of two reps of $U(V)$, namely the basic repn. $V$, and the contragredient repn: $V^t = \bar{V}$. [redacted] $\mathcal{L}U(V)$ = the vector space of skew-adjoint operators on $V$; it is a real Lie algebra under $[\ ,\ ]$ which acts on $\operatorname{End}(V)$. Better ~~to~~ say $\mathcal{L}U(V)$ is a Lie subalgebra of $\operatorname{End}(V)$, and that $\operatorname{End}(V)$ is the complexification of [redacted] $\mathcal{L}U(V)$.

Now you need to use $\operatorname{End}(V) = V \otimes V^t$.
$= V \otimes \bar{V}$

Return to rank 1 operators

$$T(v) = x\langle y|v\rangle \qquad T^*(v) = y\langle x|v\rangle.$$

$$\|T\| = \|x\|\,\|y\| = \|T^*\| \qquad \operatorname{tr}(T^*T) = \operatorname{tr}\big(\underbrace{x\langle y|}_{T}\underbrace{y\rangle\langle x|}_{T^*}\big) = \|x\|^2\|y\|^2$$

When is $T$ self-adjoint? This means $xy^* = yx^*$. $\Longrightarrow x(y^*y) = y(x^*y) \Longrightarrow x = y\,\dfrac{x^*y}{y^*y}$

$\Longrightarrow x^*x = \dfrac{(x^*y)^2}{y^*y}$. Cauchy-Schwartz tells us that

$|x^*y| \leq \|x\|\,\|y\|$ with equality $\Longleftrightarrow$ $y = \lambda x$ $\lambda \in \mathbb{R}$?

421 Recall Cauchy-Schwartz. Let $x, y$ be two nonzero vectors. Form

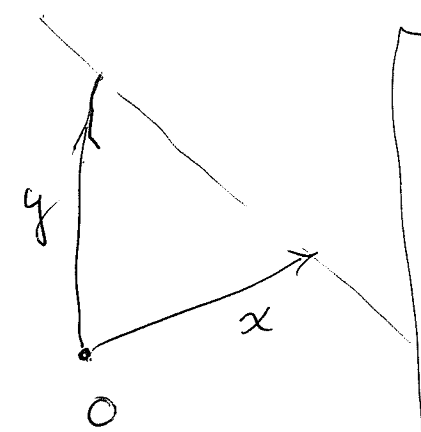
$$\|rx + sy\|^2 \quad \text{with } r, s \in \mathbb{R}$$

$$r^2 \underbrace{\|x\|^2}_{a} + rs \underbrace{(x^*y + y^*x)}_{2b} + s^2 \underbrace{\|y\|^2}_{c} \geq 0$$

$$\begin{bmatrix} x \\ y \end{bmatrix}^* \begin{bmatrix} x \\ y \end{bmatrix} = [x^* \;\; y^*] \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x^*x & y^*x \\ x^*y & y^*y \end{bmatrix}$$

$$\cancel{a + 2bs + cs^2}$$

$$ar^2 + 2br + c \qquad b^2 - ac < 0$$




$$\boxed{\begin{array}{l} T = xy^*, \quad T^* = yx^* \\ \text{Assume } x, y \neq 0 \text{ and } T = T^* \\ xy^* = yx^* \\ x\underbrace{(y^*y)}_{>0} = y\,x^*y \\ (x^*x)(y^*y) = (x^*y)^2 \quad \therefore x^*y \in \mathbb{R} \end{array}}$$

Repeat $xy^* = yx^*$

$$\Rightarrow x^*x\, y^*y = x^*y\, x^*y$$

$$\therefore x^*y \in \mathbb{R} \qquad x^*y = \pm |x||y|$$

4.22 $\quad T = xy^* = yx^*$ self-adjoint rk 1.

$\Rightarrow (x^*x)(y^*y) = (x^*y)(x^*y)$ but $x^*x, y^*y > 0$ so

$x^*y = \pm |x||y|$. $\therefore$ $x^*y \in \mathbb{R}$.

Let $u, v$ be the unit vectors s.t. $x = |x|u$, $y = |y|v$.

Then $u^*v = \frac{x^*}{|x|}\frac{y}{|y|} = \pm 1$.

---

$$0 \leq (tx+y)^*(tx+y) = |t|^2 \underbrace{x^*x}_{a} + \underset{\bar{t}b \; + t\bar{b}}{(\bar{t}x^*y + ty^*x)} + \underbrace{y^*y}_{c}$$

$$= \bar{t}ta + (\bar{t}b + \bar{b}t) + c$$

$$= \underbrace{\frac{1}{a}(\bar{t}a + \bar{b})(ta + b)}_{|ta+b|^2} + c - \frac{|b|^2}{a}$$

$|b|^2 \leq ac$ $\Uparrow$

Conclude $c - \frac{|b|^2}{a} \geq 0$

If $|b|^2 = ac$, then ~~setting~~ $t = -\frac{b}{a}$ gives 0.

$-\frac{b}{a}x + y = 0$ $\therefore$ $x, y$ are proportional

NO signs are wrong $\quad \frac{1}{a}|ta+b|^2 + c - \frac{|b|^2}{a} \geq 0$

---

Good viewpoint: You have a complex Hilbert space generated by two vectors $x, y$. What does this mean? Possibilities: 1) A complex v.s. spanned by elts $x, y$ and equipped with a positive hermitian form. 2) Take two vectors $x, y$ in a complex Hilbert space and form what they generate. Clearly this is the same as 1)

$$(sx + ty, sx + ty) = \bar{s}s(x,x) + \bar{t}s(y,x) + \bar{s}t(x,y) + \bar{t}t(y,y)$$

Let's review Cauchy-Schwartz.

It concerns two vectors in a complex Hilbert space $V$, call them $x$ and $y$. wlog we may suppose $V = \mathbb{C}x + \mathbb{C}y$ with the pos. herm. form on $\mathbb{C}x + \mathbb{C}y$ induced by that form on $V$.

$\forall s, t \in \mathbb{C}$ one has $\|sx + ty\|^2 = (sx + ty, sx + ty) =$

$$\bar{s}s\underbrace{\|x\|^2}_{a} + \bar{t}s\underbrace{(y,x)}_{b} + \bar{s}t\underbrace{(x,y)}_{\bar{b}} + \bar{t}t\underbrace{\|y\|^2}_{c} \geq 0$$

Try $t = 1$: $\underbrace{\bar{s}sa + sb + \bar{s}\bar{b} + c}_{F} \geq 0$

$\forall s \in \mathbb{C}$. Look for critical point.

$$\left.\begin{aligned} \partial_{\bar{s}} F &= sa + \bar{b} = 0 \Rightarrow s = -\frac{\bar{b}}{a} \\ \partial_s F &= \bar{s}a + b = 0 \Rightarrow \bar{s} = \frac{-b}{a} \end{aligned}\right\} \text{consistant}$$

$$\underbrace{\left(-\frac{b}{a}\right)\left(-\frac{\bar{b}}{a}\right)a + \left(-\frac{\bar{b}}{a}\right)b + \left(-\frac{b}{a}\right)\bar{b}}_{0} + c \geq 0 \quad \therefore \; c \geq \frac{|b|^2}{a}$$

---

How did you get involved with this? You are trying to ~~understand~~ produce a simple picture, an intrinsic description, of self-adjoint operators on $V$. Look at rank 1: $T = xy^*$. The adjoint is $T^* = yx^*$. If $T$ is self adjoint, then

$x^*x\, y^*y = x^*y\, x^*y$ which means

$(x^*y)^2 = \|x\|^2\|y\|^2$ so $x^*y = \pm\|x\|\,\|y\|$.

Because Cauchy Schwartz

is an equality, it should be true that $x$ and $y$ are proportional. Look at this differently. $xy^* = yx^*$. Recall
$$(x^*y)^2 = x^*x\, y^*y > 0$$
so that $x^*y = \pm \|x\|\|y\|$. Then $u = \frac{x}{\|x\|}$, $v = \frac{y}{\|y\|}$ are unit vectors satisfying $u^*v = \frac{x^*y}{\|x\|\|y\|} = \pm 1$, also $v^*u = \pm 1$  7.99 $138  1150

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Let's go over the program: Given Hilbert space $V$ (fin. diml), you want to understand the infinitesimal repns of $U(V)$ on $V$ and $V^t$.

An element of $\mathcal{L}U(V)$ is a skew adjoint operator $T$.

Let's reconstruct the reasoning. First the obstruction: Given $T \in \mathrm{End}(V) = V \otimes V^t = V \otimes \bar{V}$; here you have used the canon isom. $\bar{V} \xrightarrow{\sim} V^t$, $x \mapsto x^*$, where $x^*y$ is the hermitian form $h(x,y)$. But you need to understand the tensor product $V \otimes_{\mathbb{C}} \bar{V}$. This should be the v.s. with generates $x \otimes y^*$ and suitable bilinearity relations. If $\lambda \in \mathbb{C}$, then $\lambda x \otimes y^* = x\lambda \otimes y^* = x \otimes \lambda y^* = x \otimes (\bar{\lambda} y)^*$? This is confused - let's try to understand the rank 1 case.

If $x \in V$, recall that $x^*$ is the element of $V^t$ defined by $x^*(y) = h(x,y)$, simpler: $x^*y = h(x,y)$.

Let's repeat what you learned about a rank 1 operator $T = xy^*$. Its adjoint is $T^* = yx^*$. $T$ is self-adjoint when $T = T^*$: $xy^* = yx^*$. This implies
$$\|x\|^2\|y\|^2 = (x^*x)(y^*y) = (x^*y)(x^*y) \Rightarrow x^*y = \pm\|x\|\|y\|$$

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Assume $x, y \neq 0$. The unit vectors $u = \frac{x^*}{\|x\|}$ and $v = \frac{y^*}{\|y\|}$ satisfy $u^*v = \pm 1$, $v^*u = \pm 1$.

Look at the matrix of inner products

$$\begin{bmatrix} u^*u & u^*v \\ v^*u & v^*v \end{bmatrix} = \begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}$$

which is positive of rank 1. Two cases $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ yielding $u = v$, $u = -v$ resp.

Conclude that

$$xy^* = yx^* = \lambda uv^* = \lambda vu^*, \quad \lambda = \|x\| \cdot \|y\|$$

Now you need to understand the operator $uv^*$ where $u, v$ are unit vectors with either $u = v$ or $u = -v$. If $v = u$, then you have $uu^*$ which is the orthogonal projection onto the line $\mathbb{C}u$, and if $v = -u$ you have $-$ this projection.

---

Next project is to analyze a self adjoint operator of the form $xy^* + yx^*$. What you are doing is to start with the rank 1 operator $xy^*$. What comes to mind is a kind of $*$alg with generators

$$\begin{bmatrix} xx^* & xy^* \\ yx^* & yy^* \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

426

Review the problem: To find an intrinsic picture of the action of $\mathcal{L}\,U(V)$ on $V$ and $V^t$. You haven't made much progress so far, so maybe you should try a different approach.

Begin with $V$ a finite dim'l complex Hilb. space. Let $\mathrm{End}(V)$ be the $^*$algebra of linear transformations on $V$. Define $U(V) = \{ g \in \mathrm{End}(V) : g^* g = 1 \}$; this is naturally both a group and a smooth manifold; these two structures are compatible making $U(V)$ a compact Lie group. The associated Lie algebra is $\mathcal{L}\,U(V) = \{ X \in \mathrm{End}(V) : X^* + X = 0 \}$. It consists of all skew adjoint operators under Lie bracket. There's an obvious action of $\mathcal{L}\,U(V)$, skew adjoint operators, on $V$. What about on $V^*$?

Idea: You understand how to represent elts of $\mathcal{L}\,U(V)$ by matrices once you choose an orthonormal basis for $V$. Not clear.

You want an intrinsic picture of $\mathcal{L}\,Sp(HV)$ acting on $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$ etc. You know how to handle this via matrices once you choose $V \simeq \mathbb{C}^n$, an orthonormal basis. You might get useful information by describing what happens to the matrix picture when the orthonormal basis is changed.

$$\mathcal{L}\,Sp(2n) = \left\{ X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{array}{l} a^* = -a \\ b^t = b \end{array} \right\}$$

Still the real problem is present about how $\mathcal{L}\,U(V) = \{\text{skew adj matrices}\}$ acts on $V^t = \bar{V}$.

(427) Let $V$ be a f. dim. Hilbert space, $\mathcal{U}(V)$ = the group of unitary operators on $V$, $\mathcal{L}\mathcal{U}(V)$ = the Lie algebra of skew adjoint operators on $V$. The problem is to give an intrinsic description of the natural action of $\mathcal{L}\mathcal{U}(V)$ on $V^t$. One idea is to choose an orthonormal basis for $V$, take the dual basis for $V^t$, and then calculate the induced action of $\mathcal{U}(V)$ and $\mathcal{L}\mathcal{U}(V)$ on $V^t$ using the dual basis. On the matrix level you should have $g \in \mathcal{U}(V) \simeq \mathcal{U}(n)$ acting on $V^t = (\mathbb{C}^n)^t$ = row vectors. You should get a matrix picture which might be invariant under change of the orthonormal basis on $V$. Some calculations:

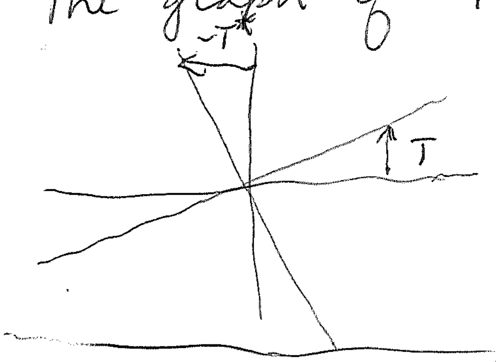
The action of $\mathcal{U}(V)$ on $V^t$ is restriction of the $GL(V)$ action to $\mathcal{U}(V)$, which preserves the fundamental pairing $(y^t, x) \longmapsto y^t x$, $V^t \times V \longrightarrow \mathbb{C}$. So you want $(gy)^t(gx) = y^t x$ i.e. $g^t g = 1$, which should mean that the action on $V^t$ induced by $g$ over $V$ should be $g \longmapsto (g^t)^{-1}$. Infinitesimal version: $T \in \mathrm{End}(V)$ preserves $y^t x$ means

$$(Ty)^t x + y^t T x = 0 \qquad \text{i.e. } T^t + T = 0$$

or also $(Ty)^t = -y^t T$

Second idea: Replace linear transformations by their graphs. You probably want explicit complements for these graphs which might arise from the pos. herm. inner product on $V$.

So the first thing to look at is the graph of $T : V_+ \longrightarrow V_-$



$$X = \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix} \in \operatorname{End} \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$$

Maybe you ~~[illegible]~~ should try for a splitting of $\begin{bmatrix} V \\ V^t \end{bmatrix}$

Still struggling with making $\mathcal{L}U(V)$ acts on $V^t$. ~~[illegible]~~ $\mathcal{L}U(V)$ is a Lie subalgebra of $\mathcal{L}GL(V) = \operatorname{End}(V)$ equipped with Lie bracket. ~~[illegible]~~

~~[illegible]~~

Working with algs seems easier than with Lie algebras. Let $A = \operatorname{End}(V)$, then $V$ is a left $A$-module and $V^t = \operatorname{Hom}(V, \mathbb{C})$ is a right $A$-module. Have compatibility with pairing

$$V^t \times A \times V \rightrightarrows V^t \times V \longrightarrow \mathbb{C}$$

$$\begin{aligned} (y^t, T, x) &\longmapsto (y^t T, x) \\ &\longmapsto (y^t, Tx) \end{aligned} \Big\} \longrightarrow y^t T x$$

what happens in the case of a Lie homom. $\mathfrak{g} \xrightarrow{\rho} A$ $\rho[X,Y] = [\rho X, \rho Y]$? Same as an alg hom $U\mathfrak{g} \longrightarrow A$.

$$V^t \times A \times A \times V \qquad V^t \times A \times V \qquad V^t \times V$$

$$\left.\begin{matrix} y^t, \rho X, \rho Y, x \\ -y^t, \rho Y, \rho X, x \end{matrix}\right\} \longrightarrow \begin{matrix} y^t, [\rho X, \rho Y], x \\ y^t, \rho[X,Y], x \end{matrix}$$

(429) Try matrices again. One has $\mathcal{L}\,GL(V) = \mathrm{End}(V)$, so any $T \in \mathrm{End}(V)$ yields an infinitesimal symmetry $\delta_T$ of $V$, given by $\delta_T x = Tx$. What is the inf. symm. on $V^t$ corresp to $T$. Intrinsic, or canonical, pairing $y^t x$: $V^t \times V \longrightarrow \mathbb{C}$, $(y^t, x) \longmapsto y^t x$; this is to be preserved by $\delta_T$ so $0 = \delta_T(y^t)x + y^t \delta_T(x)$. Now $\delta_T(y^t) = (\delta_T y)^t = (Ty)^t = y^t T^t$. $\therefore 0 = y^t T^t x + y^t T x$ $\Rightarrow T^t = -T$ and $\boxed{\delta_T(y^t) = -y^t T}$.

Next you want to check that $T \longmapsto \delta_T$ respects the Lie bracket. Given $T_1, T_2 \in \mathrm{End}\,V$

$$\delta_{T_1}(\delta_{T_2}(y^t)) = \delta_{T_1}(-y^t T_2) = y^t T_2 T_1$$

$$\delta_{T_2}(\delta_{T_1}(y^t)) = \qquad = y^t T_1 T_2$$

$$[\delta_{T_1}, \delta_{T_2}](y^t) = y^t[T_2, T_1] = -y^t[T_1, T_2] = \delta_{[T_1,T_2]}(y^t)$$

---

Review: $\mathrm{End}(V)$ equipped with Lie bracket is the space of infinitesimal symmetries of the vector space $V/\mathbb{C}$. $\mathcal{L}\,GL(V) = (\mathrm{End}(V), [\ ,\ ])$. One has an induced action on $V^t = \mathrm{Hom}(V, \mathbb{C})$ by $GL(V)$ given by $(g^t)^{-1}\lambda = (g^{-1})^t \lambda = \lambda g^{-1}$, better: given by $(g, \lambda) \mapsto \lambda g^{-1}$ ~~from~~ $GL(V) \times V^t \longrightarrow V^t$. The infinitesimal action $\delta_T$ on $V^t$ is $\delta_T \lambda = -\lambda T$.

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$$HV = \begin{bmatrix} V \\ V^t \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 - \xi_1^t x_2 \approx z_1^t J z_2$$

$$Sp(HV) = \{ g \in GL(HV) \mid (gz_1)^t J (gz_2) = z_1^t J z_2 \text{ [redacted] } \therefore g^t J g = J \}$$

$$\mathcal{L}\, Sp(HV) = \{ X \in End(HV) : X^t J + JX = 0 \}.$$

Let's restrict attention to homogeneous symmetries, that is, [redacted] the subgroup of $\begin{bmatrix} GL(V) & 0 \\ 0 & GL(V^t) \end{bmatrix}$ respecting the duality: $\left\{ \begin{bmatrix} g & 0 \\ 0 & (g^t)^{-1} \end{bmatrix} : g \in GL(V) \right\}$

---

[redacted] Consider $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$ equipped with $z_1^t J z_2$ $\quad z_i = \begin{bmatrix} x_i \\ \xi_i \end{bmatrix}$, $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$Sp(HV) = \{ g \in GL(HV) : (gz_1)^t J (gz_2) =$ [redacted] $z_1^t J z_2$ equiv: $g^t J g = J \}$

You want to understand the subgroup of $g \in Sp(HV)$ which respect the grading, that is $g = \begin{bmatrix} g_+ & 0 \\ 0 & g_- \end{bmatrix}$ such that $g^t J g = J$, which translates to

$$\begin{bmatrix} g_+^t & 0 \\ 0 & g_-^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} g_+^{-1} & 0 \\ 0 & g_-^{-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \text{[redacted]} \begin{bmatrix} 0 & g_-^{-1} \\ -g_+^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} g_-^{-1} & 0 \\ 0 & g_+^{-1} \end{bmatrix}$$

$\therefore\ g_- = (g_+^t)^{-1}$ is the contragredient repn. of $g_+$ on $V$.

inf version $X = \begin{bmatrix} X_+ & 0 \\ 0 & X_- \end{bmatrix}$ should satisfy $X_- = -X_+^t$. In prior notation: $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ one has $d = -a^t$

Next comes imposing the condition that $g$ respect a given positive hermitian form. [redacted] This form is equiv. to an antilinear isomorphism $V \longrightarrow V^t$, [redacted] satisfying a positive hermitian symmetry. Notation [redacted] $g^* x$ for the pos hermitian form, Then ??

(431) herm form $h(y,x)$ is sesquilinear: antilinear in $y$, linear in $x$; $\overline{h(x,y)} = h(y,x)$ is the hermitian condition. [redacted] Note $y \mapsto h(y,x)$ is an anti linear map from $V$ to $V^t$. Little idea: What is the dual of $V$? You have been defining it as the space $\mathrm{Hom}(V,\mathbb{C})$ of linear functionals on $V$. But it seems better to define the dual space up to canonical isomorphism. (The functorial approach). Thus [redacted] the dual space [redacted] is any vector space $W$ equipped with a bilinear pairing $W \times V \longrightarrow \mathbb{C}$ which is nondegerate in both variables. Use $\varphi(w,v)$ for the bilinear form. [redacted] For $\varphi$ to be nondegenerate in the variable $v$, this means that the linear map from $V$ to $W^t$ given by $v \longmapsto \varphi(-,v)$ is an isom.

---

You want to understand better the notation required for the dual space. Recall what's involved: a bilinear pairing [redacted] $Y \times X \longrightarrow \mathbb{C}$, $(y,x) \longmapsto y^t x$ which is nondegenerate: this means

$$Y \longrightarrow X^t, \quad y \longmapsto (x \longmapsto y^t x) \quad \text{is an isom.}$$

$$X \longrightarrow Y^t, \quad x \longmapsto (y \longmapsto y^t x) \quad \text{———}$$

Here $X^t = \mathrm{Hom}(X, \mathbb{C})$. [redacted] So far you have two vector spaces, better: an ordered pair $(Y,X)$ of vector spaces, and a nondegenerate bilinear map $Y \times X \longrightarrow \mathbb{C}$, $(y,x) \longmapsto y^t x$, between them.

Next you [redacted] want to examine the notation $y^t \in X^t$, $x^t \in Y$ in the case of a ~~non~~ dual pair (ordered pair $(X,Y)$ w. non ~~dg~~ deg pairing)

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Change notations:

$$Y \times X \longrightarrow \mathbb{C}, \quad (y,x) \longmapsto b(x,y)$$

$$Y \xrightarrow{\sim} X^t, \quad y \longmapsto (x \longmapsto b(x,y))$$

$$X \xrightarrow{\sim} Y^t, \quad x \longmapsto (y \longmapsto b(x,y))$$

There are still problems with the notation $x^t$, $y^t$. Again try a functorial approach, where functorial means wrt automorphisms eventually. Start with the canonical isomorphism $X \simeq (X^t)^t$. Thus you consider an ordered pair $(X,Y)$ equipped with a bilinear form $b : X \times Y \longrightarrow \mathbb{C}$, $(x,y) \xrightarrow{b(x)} b(x,y)$.

---

Assuming $b$ nondegenerate in the variable $x$, which means $\forall x \neq 0 \ \exists y$ s.t. $b(x,y) \neq 0$, you get a linear transf. $X \longrightarrow Y^t$, $x \longmapsto b(x)$ which $(y \longmapsto b(x,y))$ is injective, so $\dim(X) \leq \dim(Y)$. Similarly assuming $b$ nondegenerate in the variable $y$, you get a linear transf $b^y : X \longrightarrow Y^t$, $x \longmapsto b^y(x) = b(x,y)$

---

Consider $(X,Y)$ equipped with bilinear $b : X \times Y \to \mathbb{C}$,
let $b_x \in Y^t$ be defined by $b_x(y) = b(x,y)$, and
let $b^y \in X^t$ $\qquad b^y(x) = b(x,y)$.

$b$ nondeg in $x$ means $X \to Y^t$, $x \mapsto b_x$ injective
$b$ —— $y$ —— $Y \to X^t$, $y \mapsto b^y$ "

In fin dim $\Rightarrow \dim(X) \leq \dim(Y) \leq \dim(X)$

You are studying a complex vector space $V$ equipped with a bilinear form and a sesquilinear form, where both forms have a symmetry property. A bilinear form $b(v_1, v_2)$ is equivalent to a linear map $V \longrightarrow V^t$, $v \mapsto (v' \mapsto b(v,v'))$, also equivalent to $V \longrightarrow V^t$, $v' \mapsto (v \mapsto b(v,v'))$, and when $b$ is symmetric (resp. skew-symm.) these two maps agree with the appropriate sign.

A sesquilinear form $h(v_1, v_2)$ is linear in $v_1$, and anti-linear in $v_2$, so $h$ is equivalent to $V \xrightarrow{\text{linear}} V^*$, $v \longmapsto (v' \mapsto h(v,v'))$, also to $V \xrightarrow{\text{anti lin}} V^t$, $v' \longmapsto (v \longmapsto h(v,v'))$. Hermitian symmetry: $\overline{h(v,v')} = h(v',v)$ relates these two maps.

You want to look at the <u>images</u> of these maps. Motivation: A linear map $T: V \longrightarrow W$ factors canonically into

$$V \twoheadrightarrow V/K \xrightarrow{\sim} I \rightarrowtail W$$

This should be related to expressing

$$T \in \mathrm{Hom}(V,W) = W \otimes V^t$$

as $\sum_{i=1}^{n} w_i \otimes \lambda_i$ in a minimal way. $\{w_i\}$ is a basis for the image $I$; $\{\lambda_i\}$ should be a basis for $(V/K)^t$, the space of linear functionals on $V$ which vanish on $K$. Also these two bases are dual wrt the isom $V/K = I$.

Repeat: Given $T : V \longrightarrow W$ linear, you have the canonical factorization

$$V \twoheadrightarrow V/K \simeq I \rightarrowtail W.$$

This is related to expressing

$$T \in \mathrm{Hom}(V, W) = W \otimes V^t$$

in the form $\quad T = \sum_{i=1}^{r} w_i \otimes \lambda_i \quad$ where $r$ is minimal

$\{w_i\}$ is a basis for $I$ and $\{\lambda_i\}$ is a basis for $(V/K)^t$, the space of linear fnls on $V$ which vanish on $K$. You also should know that these two bases are dual relative to the isom. $V/K = I$. ??

---

Go back to $\xi \in X \otimes Y$. Choose $x_i \in X$, $y_i \in Y$ $1 \le i \le r$ s.t. $\xi = \sum_{i=1}^{r} x_i \otimes y_i$ and $r$ minimal.

Claim $\{x_i\}$ is lin. ind. If false some $x_i$ (may assume $i=1$) is such that $x_1 = \sum_{j=2}^{r} c_j x_j$. Then

$$\xi = \sum_{j=2}^{r} c_j (x_j \otimes y_1) + \sum_{i=2}^{r} x_i \otimes y_i$$

$$= \left(\sum_{j=2}^{r} c_j x_j\right) \otimes y_1 + \sum_{i=2}^{r} x_i \otimes y_i$$

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$$\xi = \sum_{i=1}^{r} x_i \otimes y_i \qquad r \text{ least, assume this}$$

Claim [redacted] $x_1, \dots, x_r$ are linearly independent. If not, wlog may ass. $\exists$ $x_1 = \sum_{j=2}^{r} c_j x_j$

then
$$x_1 \otimes y_1 = \sum_{j=2}^{r} x_j \otimes c_j y_1$$

$$\xi = \sum_{i=1}^{r} x_i \otimes y_i = \sum_{j=2}^{r} x_j \otimes c_j y_1 + \sum_{j=2}^{r} x_j \otimes y_j$$

$$= \sum_{j=2}^{r} x_j \otimes (c_j y_1 + y_j)$$

contradicts choice of $r$.

---

Lets review "images" retracts.

$$W \xleftarrow{[\beta_+ \ \beta_-]} \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \xleftarrow{\begin{bmatrix}\alpha_+ \\ \alpha_-\end{bmatrix}} W \xleftarrow{[\beta_+ \ \beta_-]} \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$$

assume $\beta_+\alpha_+ + \beta_-\alpha_- = 1_W$

$$p = \alpha\beta = \begin{bmatrix}\alpha_+ \\ \alpha_-\end{bmatrix} [\beta_+ \ \beta_-] = \begin{bmatrix} \alpha_+\beta_+ & \alpha_+\beta_- \\ \alpha_-\beta_+ & \alpha_-\beta_- \end{bmatrix} \text{ on } V = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$$

You remember a different notation: $V = \bigoplus_{i \in I} V_i$

~~what's the notation~~ You look for a retract

$$W \xleftarrow{\beta = [\dots \beta_i \dots]} \bigoplus V_i \xleftarrow{\alpha = \begin{bmatrix} \vdots \\ \alpha_i \\ \vdots \end{bmatrix}} \qquad \sum_i \beta_i \alpha_i = 1$$

$$W \xleftarrow{\beta} \bigoplus_i V_i \xleftarrow{\alpha} W \qquad \text{retract}$$

$h_i = \beta_i \alpha_i$, $\sum h_i = 1$. | Go back to $\pm$ case

$$W \xleftarrow{[\beta_+ \ \beta_-]} \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \xleftarrow{\begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix}} W \qquad \beta_\pm \alpha_\pm = h_\pm, \quad h_+ + h_- = 1$$

Steps. From the retract $W$ of $V$ you get a projection $p = \alpha\beta = \begin{bmatrix} \alpha_+ \\ \alpha_- \end{bmatrix} [\beta_+ \ \beta_-] = \begin{bmatrix} \alpha_+\beta_+ & \alpha_+\beta_- \\ \alpha_-\beta_+ & \alpha_-\beta_- \end{bmatrix} \to$ [redacted] a graded projection wrt the groupoid $M_2$. Defer this – instead explain the minimal choice for $V$ as an image. [redacted] You start with $h_\pm$ on $W$, you [redacted] let $V_\pm = h_\pm W$ and let $W \xleftarrow{\beta_\pm} V_\pm \xleftarrow{\alpha_\pm} W$

$$\begin{tikzcd} & V_+ \arrow[dl, "\beta_+"'] & \\ W & & W \arrow[ul, "\beta_-"'] \arrow[dl, "\alpha_-"] \\ & V_- \arrow[ul, "\beta_-"] & \end{tikzcd}$$

---

[redacted] What next? Go back to $T: V \to W$

$T \in \mathrm{Hom}(V, W) = W \otimes V^t$. Maybe $V, W$ have pos. herm. structures?

Let's try to understand the image. One has

$$W \leftarrow I \simeq V/K \twoheadleftarrow V$$

~~What do~~ [redacted] You want splittings at the ends, a complement to $I$ in $W$, and a complement to $K$ in $V$. a projection $W \twoheadrightarrow I$, and a lifting $V/K \rightarrowtail V$.

OK. you don't [redacted] see what you want, but what is the good viewpoint? [redacted]

[redacted] <u>nuclearity</u>. Recall that for K-theory purposes you need to take a colimit over the category of direct embeddings.

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Start with $T \in \mathrm{Hom}(V, W) = W \otimes V^t$.

You want to understand the nuclear picture of a linear transformation $T : V \longrightarrow W$.

Fact: Any $\xi \in X \otimes Y$ where $X, Y$ are v.s. has a repn. $\xi = \sum_{i=1}^{r} x_i \otimes y_i$ with smallest $r$. Better consider ~~expressions sequences~~ ways of splitting $\xi$ into decomposable tensors $x_i \otimes y_i$) and choose a repn. $\xi = \sum_{i=1}^{r} x_i \otimes y_i$ with $r$ least.

~~Claim the sequence $x_i$ is lin. indep. because~~

~~if $x_j = \sum_{i \neq j} c_{ji} x_i$, then $\xi = \sum_{j \neq i} x_j \otimes y_j + x_i \otimes y_i$~~

~~$\sum_{i \neq j} x_i \otimes c_{ji} y_j$~~

~~who~~

~~$\xi = \sum_{j \neq i}$~~

Claim $x_1, \dots, x_k$ are lin. ind. because if not $\exists i$ s.t. $x_i = \sum_{j \neq i} c_{ij} x_j$. Then

$$\xi = \sum_{j \neq i} x_j \otimes y_j + x_i \otimes y_i$$

$$x_i \otimes y_i = \sum_{k \neq i} c_{ij} x_j \otimes y_i = x_j \otimes \sum_{j \neq i} c_{ij} y_i$$

---

$$x_0 = \sum_{j \neq 0} c_j x_j, \quad \xi = \sum_j x_j \otimes y_j = \sum_{j \neq 0} x_j \otimes y_j + x_0 \otimes y_0$$

$$x_0 \otimes y_0 = \Big(\sum_{j \neq 0} c_j x_j\Big) \otimes y_0 = \sum_{j \neq 0} x_j \otimes c_j y_0$$

(438) $\xi = \sum_{i=1}^{r} x_i \otimes y_i \in X \otimes Y$ $r$ least.

Claim $x_1, \dots, x_r$ are linearly indep. If not $\exists i$, $c_j$ for $j \neq i$ s.t. $x_i = \sum_{j \neq i} c_j x_j$. Then

$$\xi = \sum_{j \neq i} x_j \otimes y_j + x_i \otimes y_i$$

$$x_i \otimes y_i = \sum_{j \neq i} c_j x_j \otimes y_i = \sum_{j \neq i} x_j \otimes c_j y_i$$

$$\xi = \sum_{j \neq i} x_j \otimes (y_j + c_j y_i)$$

---

simpler version

$$\xi = \sum_{i=1}^{r} x_i \otimes y_i \qquad x_0 = \sum_{i \neq 0} c_i x_i$$

$$\xi = \sum_{i \neq 0} x_i \otimes y_i + \underbrace{x_0 \otimes y_0}_{\sum_{i \neq 0} c_i x_i \otimes y_0 \; = \; \sum_{i \neq 0} x_i \otimes c_i y_0}$$

$$\boxed{\xi = \sum_{i \neq 0} x_i \otimes (y_i + c_i y_0)}$$

contradicting the choice of $r$.

---

What next? A similar argument shows that $y_1, \dots, y_r$ are linear independent.

You can describe the above "calculation" as saying that a relation ~~between the x~~

$$x_0 = \sum_{i \neq 0} c_i x_i \implies \sum_i x_i \otimes y_i = \sum_{i \neq 0} x_i \otimes (y_i + c_i y_0)$$

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The obvious question is whether $\xi = \sum_{i=1}^{r} x_i \otimes y_i \in X \otimes Y$ with $r$ minimum implies a duality relation between ~~X and Y~~ [redacted] $\sum_{i=1}^{r} \mathbb{C} x_i$ and $\sum_{i=1}^{r} \mathbb{C} y_i$. You can suppose these subspaces [redacted] $= X, Y$ resp. Why should this duality [redacted] be true? because

$1_X \in \mathrm{Hom}(X,X) = X \otimes X^t$ has the expansion

$$1 = \sum_{i=1}^{r} x_i \otimes \xi_i$$ where $\xi_1, \dots, \xi_n$ is the dual basis to $x_1, \dots, x_n$

Canonical map. Given $\xi \in X \otimes Y$ you get maps $X^t \longrightarrow Y$ and $Y^t \longrightarrow X$, sort of the opposite of a bilinear form.

[redacted] Some ideas: [redacted] One has

$$\mathrm{Hom}(X, Y) = Y \otimes X^t$$

$$\mathrm{Hom}(Y, X) = X \otimes Y^t$$

"clearly" compatible with transpose.

---

~~Go back to the puzzle~~ Finish Image stuff

$T \in \mathrm{Hom}(X,Y) = Y \otimes X^t$, $T = \sum_{i=1}^{r} y_i \otimes \lambda_i$

assertion should be that $T$ has a unique repn. up to $GL(n, \mathbb{C})$.

~~[scribbled out]~~ $T : X \to Y$ factors canonically into surj followed by injection

$$Y \leftarrowtail I \simeq X/K \twoheadleftarrow X$$

~~Suppose T invertible~~

$T \in \mathrm{Hom}(X,Y)$ factors canonically into a surj followed by an injection

$$Y \leftarrowtail \underset{\mathbb{C}^r}{\overset{}{I}} \twoheadleftarrow X \qquad I \simeq \mathbb{C}^r$$

Choosing basis for $I \simeq \mathbb{C}^r$ yields $y_i \in Y$, $\xi_i \in X^t$ $1 \leq i \leq r$ such that $T = \sum_i^r y_i \otimes \xi_i$

---

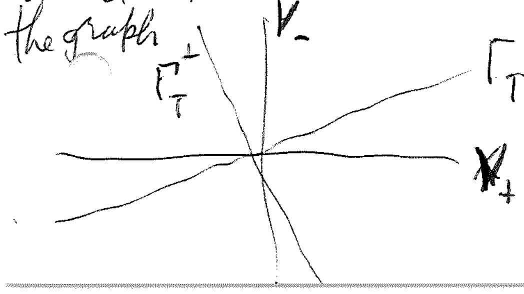
variants: You want to introduce pos. hermitian forms somehow. Suppose $X, Y$ equipped with pos herm forms, let $T : X \to Y$ be linear, and let $T^* : Y \to X$ be its adjoint. One has $K = \mathrm{Ker}(T)$, $I = \mathrm{Img}(T)$, canon factorn

$$Y \leftarrowtail I \xleftarrow{\sim} X/K \twoheadleftarrow X$$

$I$ inherits an inner product from $Y$, and $X/K$ inherits an inner product from $K^\perp$, so there are two natural inner products on $I = X/K$. These lead to an eigenspace decomposition of $I$.

---

Case of $T : X \to Y$ where $X, Y$ have inner prods. Again you have canon fact $Y \leftarrowtail I \twoheadleftarrow X$, but you get two inner products on $I$, leading to a spectral decomp. You want to link this to char values. So introduce the graph




$$X = \begin{bmatrix} 0 & -T^* \\ T & 0 \end{bmatrix}, \quad X^2 = \begin{bmatrix} -T^*T & 0 \\ 0 & -TT^* \end{bmatrix}$$

(44 I) Recall problem: Given $V$ with inner product you want an intrinsic description of the natural action of $\mathcal{L}U(V)$ on $V^t$. $\mathcal{L}U(V)$ is the real Lie algebra of skew-adjoint operators on $V$. Matrix case: Suppose $V = \mathbb{C}^n$ (column vectors). Then $End(V) = gl(n, \mathbb{C}) = M_n \mathbb{C}$ = the alg of matrices/$\mathbb{C}$. $\mathcal{L}U(V) = \{$skew-adj matrices$\} = \{a \in M_n\mathbb{C} : a^* = -a\}$. $V$ = space of column vectors $x \in \mathbb{C}^n$, $V^t$ = space of row vectors $\xi \in \mathbb{C}^n$. The action of $a$ on $V$ is left mult $a : x \mapsto ax$ and the action of $a$ on $\xi \in V^t$ should be right mult. $a : \xi \longmapsto \xi a$ up to sign.

Both Column + row vectors is confusing. ~~Instead~~ Let $V^t = \{\xi^t : \xi \in \mathbb{C}^n \text{ (column)}\}$. Then the action of $a$ on $\xi^t \in V^t$ is $a : \xi^t \longmapsto \xi^t a = (a^t \xi)^t$. So you find that the action of $a \in \mathcal{L}U(n)$ on $HV = \begin{bmatrix} V \\ V^t \end{bmatrix} = \left\{ \begin{bmatrix} x \\ \xi^t \end{bmatrix} \right\}$ should be $a \longmapsto \begin{bmatrix} a & 0 \\ 0 & -a^t \end{bmatrix}$. NOT CLEAR

---

You should find the action of $gl(V)$ on $HV = \begin{bmatrix} V \\ V^t \end{bmatrix}$. You've learned something new about the dual, namely that there's a canonical element $\delta_V \in Hom(V, V) = V \otimes V^t$ 'governing' the duality.

Let $g \in GL(V)$. One has $g \delta g^{-1} = \delta$ where $\delta = \sum_i x_i \otimes \xi_i = 1 \in End(V)$. Hence

$$\sum_{i=1}^{n} g x_i \otimes \xi_i g^{-1} = \sum_{i=1}^{n} x_i \otimes \xi_i$$

which should tell you that the action of $g$ on $\xi \in V^t$ is $\xi g^{-1}$

(442) inf form of $g\delta g^{-1}=\delta$ is $a\delta = \delta a$ or

$$\sum_i a x_i \otimes \xi_i = \sum_i x_i \otimes \xi_i a \qquad \sum_i \xi_j a x_i \otimes \xi_i = \xi_j a$$

$$\sum_i a x_i \otimes \underbrace{\xi_i x_j}_{\delta_{ij}} = \sum_i x_i \otimes \xi_i a x_j \qquad \therefore\ a x_j = \sum x_i (a_{ij})$$

$$\sum_i a_{ji}\, \xi_i = \xi_j a$$

---

$$HV = \begin{bmatrix} V \\ V^t \end{bmatrix} \qquad End(HV) = \begin{bmatrix} V\otimes V^t & V\otimes V \\ V^t \otimes V^t & V^t \otimes V \end{bmatrix} \qquad \begin{bmatrix} V \\ V^t \end{bmatrix}$$

Now you want to get the Lie subalgebra of End(HV) which respects the duality, i.e. the canonical diagonal elements $\delta_V \in V\otimes V^t$ and $\delta_{V^t} \in V^t \otimes V$. How to clarify?

Go back to matrix picture:

$$\begin{bmatrix} x_1 \\ \xi_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ \xi_2 \end{bmatrix} = x_1^t \xi_2 + \xi_1^t x_2$$

The duality between $V$ and $V^t$ is given by the pairing $(x,\xi) \longmapsto x^t\xi = \xi^t x$. ∴ You have interpreted ???

Let $X$, $Y$ be vector spaces of the same dimension. When can you say that $X$ and $Y$ are dual (or in duality)? There seem to be two structures specifying this: (i) bilinear pairing $X\otimes Y \longrightarrow \mathbb{C}$ which is non degenerate in each variable (ii) a 'diagonal' elt $\delta \in X\otimes Y$ with rank $= \dim(X) = \dim(Y)$.

(i) means ~~the~~ maps $X \longrightarrow Y^t$, $Y \longrightarrow X^t$ are 1-1 *

(ii) ______ $X^t \longrightarrow Y$, $Y^t \longrightarrow X$ are onto *

* and therefore bijective since $\dim(X) = \dim(Y)$.

(443) Let $X, Y$ be in duality, let $x_i$, $1 \le i \le r$ be a basis for $X$, and let $y_i$, $1 \le i \le r$ be the dual basis for $Y$. Use the definition of "in duality" via a bilinear pairing $x^t y$. Then one has $x_i^t y_j = \delta_{ij}$. Next use the defn by means of a $\delta \in X \otimes Y$ of rank $= r$. One has $\delta = \sum_{i=1}^{r} x_i \otimes \xi_i$ where $\xi_1, \dots, \xi_r$ is a basis for $Y$. Then

$$y_j^t \delta = \sum_i y_j^t x_i \otimes \xi_i = \sum_i \delta_{ji} \xi_i = \xi_j .$$

This means that the map $Y \longrightarrow Y$, $y \longmapsto y^t \delta = \sum_i y^t x_i \xi_i$ is $id_Y$.

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Repeat: $X, Y$ same $\dim^r$ are in duality when you give a nondegenerate pairing $X \otimes Y \longrightarrow \mathbb{C}$, and also when you give ~~[illegible]~~ a diagonal elt $\delta \in X \otimes Y$ of rank $r$.

These two structures: | nondegenerate pairing $X \otimes Y \to \mathbb{C}$
| diagonal elt. $\delta \in X \otimes Y$ of rank $r$

should be related. You expect that the pairing and diagonal elt. are related by the duality between $X \otimes Y$ and $Y \otimes X$: $(X \otimes Y)^t \simeq X^t \otimes Y^t = Y \otimes X$.

Alternate:

$$X \otimes Y = X \otimes X^t = \mathrm{End}(X)$$
$$Y \otimes X = X^t \otimes X = \mathrm{End}(X^t)$$

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$X, Y$ both dim $r$, $b: X \otimes Y \longrightarrow \mathbb{C}$ bilinear,

$b$ same as $Y \longrightarrow X^t$, $y \longmapsto (x \mapsto b(x,y))$ assume 1-1,

then equal dim $\Rightarrow$ $Y \xrightarrow{\sim} X^t \Rightarrow X \otimes Y \simeq X \otimes X^t = \mathrm{End}(X)$

try coords. Assume $x_i$ basis for $X$, $y_j$ dual basis for $Y$,

$Y \xrightarrow{\sim} X^t$ $\qquad$ $X \otimes Y = \bigoplus x_i \otimes Y$

$y \longmapsto (x \mapsto b(x,y))$

(444) Suppose given $Y \xrightarrow{\sim} X^t$

$$y \longmapsto (x \mapsto b(x,y))$$

Better: Start with $b : X \times Y \longrightarrow \mathbb{C}$ nondeg.
choose $x_1, \dots, x_r$ basis for $X$, get ~~dual basis~~ elements
~~$y_1, \dots, y_r$ for $Y$ such that~~ $b(x_i, y) \in$ ?
Given $b(x,y)$, this is the same as $b(x_i,y) \in Y^t$ ?

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Confused. You want to start with a bilinear form $b(x,y)$, that is, a pairing $X \otimes Y \xrightarrow{b} \mathbb{C}$, assumed nondegenerate in the variable $y$, which means the assoc. map $Y \xrightarrow{\sim} X^t$ is 1-1, and $\therefore$ an isom.
What do you mean by Given a basis $x_1, \dots, x_r$ of $X$ the dual basis $y_1, \dots, y_r$ for $Y$? Ans. $b(x_i, y_j) = \delta_{ij}$

~~So you conclude~~

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Let $X \otimes Y \xrightarrow{b} \mathbb{C}$ be a bilinear form ~~that identifies~~
~~between vector spaces of the same dim. Better thought~~
which is nondegenerate in $Y$, meaning the map

$$Y \longrightarrow X^t, \quad y \longmapsto (x \mapsto b(x,y))$$

is 1-1, that is, $y \neq 0 \Rightarrow \exists x$ s.t. $b(x,y) \neq 0$. Then $\dim(Y) \leq \dim(X^t) = \dim X$. Assume dims $=$ then $Y \xrightarrow{\sim} X^t$. Moreover

$$X \otimes Y \xrightarrow{\sim} X \otimes X^t = \mathrm{End}(X)$$

so $b$ yields a linear functional on $\mathrm{End}(X)$

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$b : X \otimes Y \longrightarrow \mathbb{C}$ bilinear form which is nondegenerate on $Y$, meaning: $\forall y \neq 0,\ \exists x$ s.t. $b(x,y) \neq 0$.

One has injection $Y \rightarrowtail X^t$, $y \longmapsto (x \mapsto b(x,y))$

so $\dim(Y) \leq \dim(X^t) = \dim(X)$. Assume [redacted] $\dim(\cancel{Y}) = \dim(X)$; one has $Y \xrightarrow{\sim} X^t$. Moreover

$$X \otimes Y \xrightarrow{\sim} X \otimes X^t = End(X)$$

so $b$ yields a linear functional on $End(X)$; in particular one has a number $b(id_X)$; a kind of trace?

[redacted]

You need to [redacted] understand this better.
What can you do with an isomorphism $Y \xrightarrow{\sim} X^t$?
It seems that you get an algebra structure on $X \otimes Y$. ~~[illegible]~~

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Object to study is ~~[illegible]~~ a perfect dual pair of vector spaces $(X, Y)$ which means that ~~[illegible]~~ one has $Y \otimes X \longrightarrow \mathbb{C}$

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~~$X, Y$ two v.s. dim $r$, are in duality~~

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$X, Y$ two vector spaces of same dimension $r$. ~~[illegible]~~

~~[illegible]~~

(i) $b : X \otimes Y \longrightarrow \mathbb{C}$ nondeg bilinear form

(ii) $\delta : \mathbb{C} \longrightarrow X \otimes Y$ nondeg "diagonal"

Choose basis $x_i$ for $X$. Then $(y \mapsto b(x_i, y))$ is a basis for $Y^t$.

Why? nondeg $\Rightarrow X \xrightarrow{\sim} Y^t$, $x \longmapsto (y \mapsto b(x,y))$, and then you get a basis [illegible] $y_i$ for $Y$ s.t. $b(x_i, y_j) = \delta_{ij}$

$$\delta = \sum x_i \otimes \tilde{y}_i$$

446 Repeat: $X$ and $Y$ are v.s. of same dim

You have two different ways to construct a duality isomorphism between $X$ and $Y$.

(i) $b : X \otimes Y \longrightarrow \mathbb{C}$, a non degenerate bilinear form

(ii) $\delta : \mathbb{C} \longrightarrow X \otimes Y$, a ———— diagonal elt.

Either of these will associate to each basis $\{x_i\}$ of $X$ a basis $\{y_j\}$ of $Y$. In the first case because $b$ is non degenerate you have an isom.

$$X \xrightarrow{\sim} Y^t, \quad x \longmapsto (y \longmapsto b(x,y))$$

~~so~~ a basis $\{x_i\}$ for $X$ yields a basis for $Y^t$ consisting of the linear functionals $y \longmapsto b(x_i, y)$, which in turn yields a basis $\{y_j\}$ for $Y$ defined by $b(x_i, y_j) = \delta_{ij}$

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Claim: If $b : X \otimes Y \longrightarrow \mathbb{C}$ is a non deg bilinear form, and $\{x_i\}$ is a basis for $X$, then there is a unique basis $\{y_j\}$ for $Y$ such that $b(x_i, y_j) = \delta_{ij}$.

Proof: nondeg $\Longrightarrow$ isom $X \xrightarrow{\sim} Y^t$, $x \mapsto (y \mapsto b(x,y))$ so $(y \mapsto b(x_i, y))$ is a basis for $Y^t$, giving isom

$$Y \xrightarrow[\sim]{} \mathbb{C}^n \qquad y \longmapsto \begin{bmatrix} \vdots \\ b(x_i, y) \\ \vdots \end{bmatrix}$$

whence a unique basis $\{y_j\}$ s.t. $b(x_i, y_j) = \delta_{ij}$

There's probably a better way to introduce the matrix picture: A non degenerate bilinear form on $X \otimes Y$ is an invertible $n \times n$ matrix $b_{ij} = b(x_i, y_j)$ where $\{x_i\}$, $\{y_j\}$ are bases for $X, Y$ resp.

$x^t b y$

Now look at $\delta \in X \otimes Y$. If you pick bases $\{x_i\}$, $\{y_j\}$ for $X$ and $Y$ resp. then you get a matrix $\delta_{ij}$ of components.

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$$\mathrm{Hom}(X \otimes Y, \mathbb{C}) \simeq \mathrm{Hom}(X, \mathrm{Hom}(Y, \mathbb{C})) = \mathrm{Hom}(Y, \mathrm{Hom}(X, \mathbb{C}))$$

$$(X \otimes Y)^t \simeq \mathrm{Hom}(X, Y^t) = Y^t \otimes X^t$$

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non deg $\delta \in X \otimes Y$ given, choose basis $\hat{x}_i$ for $X$, then $X \otimes Y = \bigoplus_i \hat{x}_i \otimes Y$, so $\delta = \sum_i \hat{x}_i \otimes \hat{y}_i$, and you know $\hat{y}_i$ is a basis for $Y$

You can view $\delta \in X \otimes Y$ as an isom $X \xleftarrow{\sim} Y^t$ or as an isom $X^t \longrightarrow Y$. What do you need to make clear the idea that $\delta$ maps bases in $X$ to bases in $Y$ in a precise way.

Two basic examples.

(i) $Y = X^t$. A basis in $Y$ yields a basis in $X^t$ which in turns yields a dual basis in $X^{tt} = X$.

(ii) $Y = X$. A basis in $Y$ yields a basis in $X$

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Review. Given nondeg $\phi \in X \otimes Y$, one has isos. $X^t \longrightarrow Y$, $Y^t \longrightarrow X$ ~~which are isos.~~ mutually transpose whose inverses hopefully come from a nondeg. bilinear form $b : X \otimes Y \to \mathbb{C}$

Calculate: Choose basis $x_i$ for $X$, ~~write~~ then $X \otimes Y = \bigoplus_i x_i \otimes Y$, so one has ! $y_i$ such that $\phi = \sum x_i \otimes y_i$, nondeg $\Rightarrow$ $y_i$ basis for $Y$.

448 Calculate $X^t \to Y$. need dual basis $\xi_i$ in $X^t$ to $x_i$

$\xi_j x_i = \delta_{ji}$ Then $\xi_j \mapsto \sum_i \xi_j x_i \otimes y_i = y_j$

Similarly let $\eta_i \in Y^t$ be dual basis to $y_i$: $\eta_j y_i = \delta_{ji}$

so $Y^t \to X$ is $\eta_j \mapsto \sum_i x_i \otimes \eta_j y_i = x_j$

Summ. $X^t \xrightarrow{\sim} Y$ is $\xi_i \mapsto y_i$

$Y^t \xrightarrow{\sim} X$ is $\eta_i \mapsto x_i$

to calc transpose of $X^t \to Y$, take $\xi_i \mapsto y_i$ followed by $\eta_j : Y \to \mathbb{C}$, getting $\xi_i \to y_i \to \eta_j y_i = \delta_{ji}$, so you have the lin. fnl on $X^t$ given by $\xi_i \mapsto \delta_{ji}$ which is exactly $x_j$. $\therefore$ $X^t \xrightarrow{f} Y$ is $\xi_i \mapsto y_i$

$f^t : Y^t \to X$ is $\eta_j \mapsto x_j$

Inverting $f$ gives $f^{-1} : Y \to X^t$, $y_i \mapsto \xi_i$

$$b(x_j, y_i) = \xi_i(x_j) = \delta_{ij}$$

Inverting $f^t$ gives $(f^t)^{-1} : X \to Y^t$, $x_i \mapsto \eta_i$

$$b(x_i, y_j) = \eta_i(y_j) = \delta_{ij}$$

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$X, Y$ v.s. same dim, $\phi \in X \otimes Y$ nondeg

If $\{x_i\}$ basis for $X$, there is unique $\{y_i\}$ sequence s.t. $\phi = \sum x_i \otimes y_i$, moreover $\{y_i\}$ is a basis for $Y$.

If $\{x_i\}$ basis for $X$, $\{y_i\}$ basis for $Y$, then $\phi = \sum x_i \otimes y_i$ is a nondeg elt of $X \otimes Y$.

Q: When is $\sum x_i \otimes y_i = \sum x_i' \otimes y_i'$?

$$x_i' = g_{i1}x_1 + g_{i2}x_2 + \cdots + g_{in}x^n$$

$$x_i' = \sum_{j=1}^{n} g_{ij} x_j$$

$$\phi = \sum_i x_i' \otimes y_i' = \sum_{i,j} g_{ij} x_j \otimes y_i' = \sum_j x_j \otimes \sum_i y_i' g_{ij}$$

$$= \sum_j x_j \otimes y_j \qquad \therefore \quad y_j = \sum_i y_i' g_{ij}$$

Statement $\quad \sum x_j \otimes y_j = \sum x_i' \otimes y_i'$

$\iff \exists\, (g_{ij}) \in GL(n)$ s.t. $\underset{\sim}{x}' = g\underset{\sim}{x}$ and $\underset{\sim}{y}'g = \underset{\sim}{y}$

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Repeat. ~~X, Y same dim Let $\phi \in X \otimes Y$ be nondeg~~ ~~which means~~

$X, Y$ f.d. v.s., $\phi \in X \otimes Y$, $\phi$ is a sum of decomposable tensor $x \otimes y$, choose a repn $\phi = \sum_{i=1}^{r} x_i \otimes y_i$ with $r$ least. You know that the sequences $\{x_i\}$ in $X$, $\{y_i\}$ in $Y$ are lin ind get subspaces $\sum \mathbb{C}x_i < X$, $\sum \mathbb{C}y_i < Y$ rank $r$ and an isomorphism $x_i \longmapsto y_i$. not natural because

$X^t \ni \xi \longmapsto \xi \lrcorner \phi = \sum_i \xi(x_i)\, y_i$, $X^t \longrightarrow Y$

sends dual basis $\xi_j(x_i) = \delta_{ji}$ to $\xi_j \lrcorner \phi = y_j$

450 $X, Y$ vector spaces, fin dim'l, $\varphi \in X \otimes Y$.
You know that $\varphi$ is a sum of decomposable tensors:
$\varphi = \sum_{i=1}^{r} x_i \otimes y_i$. Choose such a sum with $r$ least.