~~V f.d. v.s., V^t dual space consisting of all linear maps $\lambda : V \to \mathbb{C}$, canonical bilinear pairing (map) $(v, \lambda) \longmapsto \lambda(v)$, $V \times V^t \longrightarrow \mathbb{C}$.~~

Somehow you have to get back on the track. You're stuck on finding a clean picture of $H(V)$ with its three structures.

$V = \mathbb{C}^n$ (space of column vectors $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$)

equipped with pos herm. form. $x^* y = \sum \bar{x}_j y_j$.

$V^t = \mathbb{C}^n$ (space of row vectors) ?

This is not making sense. Go back to the abstract situation. V f.d. $\mathbb{C}$ v.s., $V^t = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$

canon bilinear pairing $V \times V^t \longrightarrow \mathbb{C}$

maybe you should make bilinear form a basic object.

TFAE

$V \times W \xrightarrow{B(-,-)} \mathbb{C}$ bilinear form

$V \xrightarrow{\alpha} W^t$ $\quad v \longmapsto B(v,-) \in W^t$

$W \xrightarrow{\beta} V^t$ $\quad w \longmapsto B(-,w) \in V^t$

Claim $\alpha^t = \beta$? Take $\mu \in (W^t)^t$

$\mu : W^t \longrightarrow \mathbb{C}$. You know $\mu = \mu_x$ so that

$\lambda \longmapsto \mu(\lambda)$ $\quad \mu_x(\lambda) = \lambda(x)$. $\quad V \xrightarrow{\alpha} W^t \xrightarrow{\mu_w} \mathbb{C}$

$v \mapsto B(v,-) \mapsto B(v,w)$

$\alpha^t(\mu_w) = \mu_w \alpha = B(\cancel{-}, w)$.

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TFAE

$V \times W \xrightarrow{B} \mathbb{C}$ bilinear

$V \xrightarrow{\alpha} W^t$ $\quad v \mapsto B(v, -)$

$W \xrightarrow{\beta} V^t$ $\quad w \mapsto B(-, w)$

use double dual thm.

$$V^t \xleftarrow{\alpha^t} (W^t)^t = W$$

$$(\lambda \mapsto \lambda(x)) \leftarrow\!\shortmid w$$

$\alpha^t(\lambda) = \lambda\alpha$

$\alpha^t(\lambda)(v) = \lambda\alpha(v)$

Back to $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$. You made some progress by walking. Your aim is to understand properly the notion of polarizations for $H(V)$. The simplest description should be a [redacted] Lagrangian subspace. Hence the space of polarizations of $H(V)$ is [redacted] the minimal flag manifold for $Sp(2n)$. Another idea [redacted] is that [redacted] a polarization of $H(V)$ is roughly an isomorphism of $H(V)$ with itself. More precisely ~~[illegible]~~ given a polarization of $H(V)$, i.e. a Lagrangian subspace $W \subset H(V)$, then [redacted] you should have $W^\perp$ Lagrangian, so

$$H(V) = W \oplus W^\perp = H(W) \approx H(V)$$

where the latter arises from choosing an isom $W \simeq V$. This is confused. [redacted] What point of view should you use? If you are given a polarization $W \subset H(V)$, this should be the same as an isomorphism $\mathbb{H} \otimes_{\mathbb{C}} W \xrightarrow{\sim} H(V) = \mathbb{H} \otimes_{\mathbb{C}} V$. Can you identify [redacted] the space of polarizations with $Sp(2n)/U(n)$? [redacted] This should be clear. Given W Lagrangian $\subset H(V) = \mathbb{H} \otimes_{\mathbb{C}} V$, ~~[illegible]~~ the embedding $W \hookrightarrow \mathbb{H}\otimes_{\mathbb{C}} V$ extends uniquely to an $\mathbb{H}$-module map $\mathbb{H}\otimes_{\mathbb{C}} W \xrightarrow{\sim} \mathbb{H}\otimes_{\mathbb{C}} V$ which is an isomorphism since W, V have the same dim $= n$. You are leaving out a lot of details, which have eventually to be checked.

An important point to be understand involves the inner product on $H(V)$. You believe that this inner product is a consequence of the symplectic and $\mathbb{H}$-module structures.

Today you want to clean up polarization. You want to study the space of polarizations of $H(V)$. Polarization = Lagrangian subspace should be true. ??

There are problems with the $\mathbb{H}$ structure.

Q: Is the orthogonal complement for a Lagrangian (wrt inner product) L again Lagrangian?

Let's return to the situation where you have both a symplectic structure and an inner product structure on a complex vector space V. Choose orth basis for V, so that $V = \mathbb{C}^n$ (columns) $V^t = \mathbb{C}^n$ (rows), the symp. str is a map $A : V \longrightarrow V^t$, $x \longmapsto x^t A$ $\quad A^t = -A$.

The comp. $T = *A$

$$V \xrightarrow{A} V^t \xrightarrow{*} V$$

is anti-linear $Tx = *(x^t A) = A^* \bar{x} = -\bar{A}\bar{x}$ with square $T(Tx) = T(-\bar{A}\bar{x}) = -\bar{A}\,\overline{(-\bar{A}\bar{x})} = (\bar{A}A)x$

$\bar{A}A = -(A^*A) < 0$ so polar decomp yields complex structure

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The good case is when $\bar{A}A = -(A^*A) = -1$.
This is something you didn't make explicit before.
Another formulation: Given a skew symmetric bilinear form A and a hermitian form $*$:

$$V \xrightarrow{A} V^t \xrightarrow{*} V$$

these are compatible if $A^*A = +1$. A is unitary and skew symm.

Repeat. V $\mathbb{C}$-v.s. equipped w. $V \xleftrightarrow{*} V^t$, $V \xrightarrow{A} V^t$

You want to know when $*$ and A are compatible: this means that the anti-linear ~~transf~~ operator

$T = *A : V \xrightarrow{A} V^t \xrightarrow{*} V$ has square $= -1$.

$$T(x) = (x^tA)^* = -\bar{A}\,\bar{x}$$

$$V \xrightarrow{A} V^t$$
$$x \longmapsto (Ax)^t = -x^tA$$
$$x \longmapsto x^tA$$

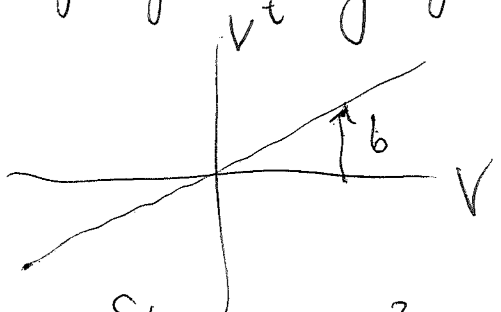
These are two possibilities differing in sign.

$$T(Tx) = -\bar{A}\,\overline{(Tx)} = -\bar{A}\,(-\bar{A}\bar{x}) = (\bar{A}A)x = (-A^*A)x$$

Therefore the compatibility condition is $A^*A = 1$ (in addition to $A^t = -A$).

Next you would like to apply the preceding to polarizations. Start naively with a Lagrangian subspace of $H(V)$.

320 Take a complex symplectic space of $\dim(2n)$, say $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ $\dim_{\mathbb{C}} V = n$. You want the dim of the space of Lagrangian subspaces of $H(V)$. You have an open set of Lag subsp given by graphs of quadratic forms. So the



$$\boxed{\dim_{\mathbb{C}} \text{ of } \{\text{Lag subspaces}\} = \frac{n(n+1)}{2}}$$

On the other hand you can consider the fibre bundle over {Lag subsp} consisting of a Lag sub together with a complete flag. This you can construct inductively by choosing a line, restricting to the symplectic quotient etc. ~~The $\dim_{\mathbb{C}}$ of {Lag subsp + complete flag up to dim p} is~~

$$\cancel{2n-1 + 2n-3 + \cdots + 2p-(2p-1) = p^2}$$

~~The $\dim_{\mathbb{C}}$ of {complete flags in $\mathbb{C}^p$} $= \frac{p(p-1)}{2}$~~

Suppose we choose $L_1 \subset H(\mathbb{C}^n)$, $\dim\{L_1\} = 2n-1$, then choose L_2 s.t. $L_1 < L_2 \subset L_1^{\perp}$, $\dim\{L_2/L_1\} = 2n-3\}$, {Lag subs with complete flag} has $\dim_{\mathbb{C}}$

$$(2n-1) + (2n-3) + \cdots + (2n-(2n-1))$$

$= n \cdot 2n - n^2 = n^2$. Each Lag subsp L_n has dim {complete flags in L_n} $= \frac{n(n-1)}{2} =$

$n-1 + n-2 + \cdots + n-(n-1)$
$(n-1)\left(\frac{n-1+n-(n-1)}{2}\right)$
$(n-1)n/2$.

$$\boxed{\dim_{\mathbb{C}}\{\text{Lag subs}\} = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}}$$

Let's study eigenvalue theory for a Hilbert space equipped with a symmetric bilinear form $S : V \longrightarrow V^t$. Associated to pos herm inner product on V is an invertible anti-linear transformation $V \xrightarrow{\sim} V^t$, $x \longmapsto x^*$. Composing

$$V \xrightarrow{S} V^t \xrightarrow{\sim} V$$
$$x \longmapsto x^t S \longmapsto (x^t S)^* = \bar{S}\bar{x}$$

yields an anti-linear transformation $T : x \longmapsto \bar{S}\bar{x}$ which should be equivalent to S.

$$\begin{tikzcd} & V^t \arrow[d, leftrightarrow, "S\ *"] \\ V \arrow[ur, "S"] \arrow[r, "T"'] & V \end{tikzcd}$$

Note that T^2 is linear:

$$T(Tx) = T(\bar{S}\bar{x}) = \bar{S}\,\overline{\bar{S}\bar{x}} = (\bar{S}S)x \qquad \text{where } \bar{S}S = S^*S \geq 0.$$

So T^2 is a (weakly) positive hermitian operator on V, so there's an eigenspace decomposition $V = \bigoplus_{\lambda \geq 0} V_\lambda$.

T commutes with $T^2 = \bar{S}S$, so T respects this eigenspace decomposition.

If $\bar{S}S\xi = \lambda\xi$, then $\lambda \geq 0$ and

$$\bar{S}ST\xi = T\bar{S}S\xi = T\lambda\xi = \lambda T\xi$$

showing that T preserves V_λ, $\forall \lambda$.

You think it should be possible to give a variational picture of this decomposition. You know already about the polar decomposition of T, the phase being a real structure: anti linear invertible transf with square $+1$. There might be a Rayleigh-Ritz theory

Rayleigh-Ritz theory for the eigenvalues, the nth eigenvalue is obtained by some variational problem involving subspaces of dim n. There are probably interesting "minimax" inequalities. I expect some similarity with Morse theory construction of eigenvalues, in which you look at critical points of a suitable function on a Grassmannian.

Today [redacted] you want the analog of the critical point construction of the spectrum of a hermitian operator in the case of a symmetric bilinear form. You start with a Hilbert space V equipped with a symmetric bilinear form $V \xrightarrow{S} V^t$. You propose to use the conjugacy theorem in the $n = \dim V$; $\mathcal{L}(Sp(2n)/U(n))$

[redacted]

Recall $\mathcal{L}(Sp(2n)/U(n))$ is the subspace of

$$\mathcal{L}(Sp(2n)) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$$

where $a = 0$, equipped with conjugation action of $U(n) \ni u \longmapsto \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}$. Thus

$$\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ -\bar{u}\bar{b}u^* & 0 \end{bmatrix}$$

$u \# b = ubu^t$ which is also

$u \# b = u b \bar{u}^{-1}$ infinitesimally $ab - b\bar{a}$.

Now you want to find [redacted] a suitable variational problems. Pick the smallest flag manifold, i.e. the space of Lagrangian subspaces, which is the orbit under $U(n)$ of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = J_0$. J_0 is the basepoint of our symmetric spaces $Sp(2n)/U(n)$.

(323) You want a function $F(J)$ for J on the symmetric space $Sp(2n)/U(n)$, which should depend on J_0, whose critical points are those J commuting with J_0.

At a point J you have the tangent space to the symm. sp. Note: J is the same as a polarization, so the tangent space should be canon. isom to the space of symmetric bilinear forms b, which has $\dim_{\mathbb{R}} = 2\cdot\frac{n(n+1)}{2} = n^2+n$

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digress with dim calculation

$g^t J g = J$ defines $Sp(2n,\mathbb{C})$

$$\begin{bmatrix} c & d \\ -a & -b \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d & c \\ b & -a \end{bmatrix}$$

$X^t J + J X = 0$ " $\mathcal{L}\, Sp(2n,\mathbb{C})$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = X \Rightarrow \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = X^t = JXJ = \overbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}^{=}$$

$\therefore$ $a^t = -d$ $\quad c^t = c$

$b^t = b$ $\quad d^t = -a$

$\dim_{\mathbb{C}} Sp(2n,\mathbb{C}) = \overset{2n^2+n}{\overset{\shortparallel}{\frac{2n(2n+1)}{2}}}$, $\quad \dim_{\mathbb{R}} Sp(2n) = \frac{2n(2n+1)}{2}$

Now return to the space of polarizations $Sp(2n)/U(n)$ which has $\dim_{\mathbb{R}} = (2n^2+n) - n^2 = n^2+n = 2\cdot\frac{n(n+1)}{2}$

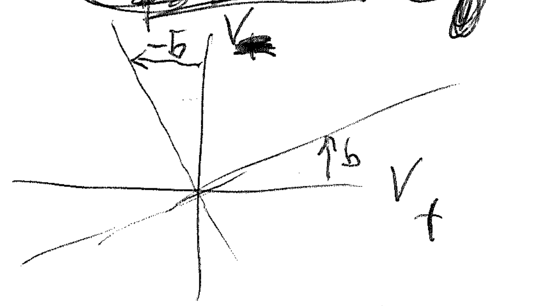
$\therefore$ space of polarizations = orbit of $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under $Sp(2n)$.

whose tangent space at J_0 is ~~the~~ space of symmetric complex matrices b, which has $\dim_{\mathbb{R}} = n^2+n$.

Repeat. The basic object is the space of polarizations of $H(\mathbb{C}^n)$, i.e. the flag manifold, homog space $Sp(2n)/U(n)$, conjugacy class of $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(324) You are studying the space of polarizations of ${}_{(-1)}H(\mathbb{C}^n)$, this space should be $Sp(2n)/U(n)$ = orbit under $Sp(2n)$ of $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and it has $\dim_{\mathbb{R}} = n^2 + n = 2 \times \dim \{$symmetric $\mathbb{C}$ bilinear forms$\}$.

In fact the tangent space to this space for a polarization J should be the $\mathbb{C}$ vector space of symmetric bilinear maps $b : V_+ \longrightarrow V_-$,

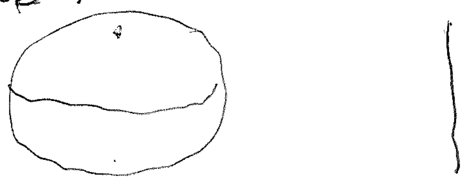


more precisely such a b yields $\begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} \in \mathfrak{p} = \mathcal{L}(Sp(2n)) / \mathcal{L}(U(n)) = \mathcal{L}(U(n))^{\perp}$

Next you want to use the basepoint polarization J_0 to construct a "Morse fn" on $P = Sp(2n)/U(n)$.

At this point you are reminded of Moment Map theory. In general the coadjoint orbits of a Lie algebra are symplectic manifolds. For a compact Lie group, coadjoint orbits = adjoint orbits. This explains why the adjoint orbits are symplectic. Note: an adjoint orbit is $G/\text{Centralizer}(X)$ for some $X \in \mathfrak{g}$.

There should also be a Duistermaat-Heckman theorem; Archimedes case: height fn.



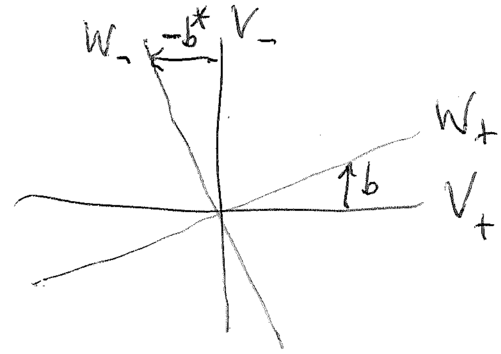
Review: Space of polarizations

$= $ the orbit of $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under $Sp(2n)$

$= Sp(2n)/U(n)$. (There may be some confusion about whether J_0 is an elt of $Sp(2n)$ or $\mathcal{L}(Sp(2n))$?)

Consider a polarization J. This means first of all that $-J = J^* = J^{-1}$ as an operator on $H(\mathbb{C}^n)$, i.e. J is unitary and its spectrum is $\{i, -i\}$. Thus $H(\mathbb{C}^n) = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$ with $J = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. In addition $V_\pm$ should be Lagrangian subspaces for the symplectic form. Your aim is to get a critical point proof that any polarization J is conjugate to J_0. Moreover you do not want to assume that the space $\mathbb{P}$ of polarizations is connected. You want to use J_0 to construct the real valued function with the desired critical points. Geometric idea is the tangent space to $\mathbb{P}$ at J.

Next look at the C.T. picture.



$H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}$, $J_0 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\varepsilon$

$J = \begin{cases} +i & \text{on } W_+ \\ -i & \text{on } W_- \end{cases}$ $\quad J \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $J(1+X) = (1+X)i\varepsilon$

$$J(1+X)(-i\varepsilon) = J(-i\varepsilon)(1-X) = 1+X \Rightarrow J J_0^{-1} = \frac{1+X}{1-X}$$

Next you want a measure of the size of the difference $J J_0^{-1}$. Remove i's: $J = iF$, $J_0 = i\varepsilon$

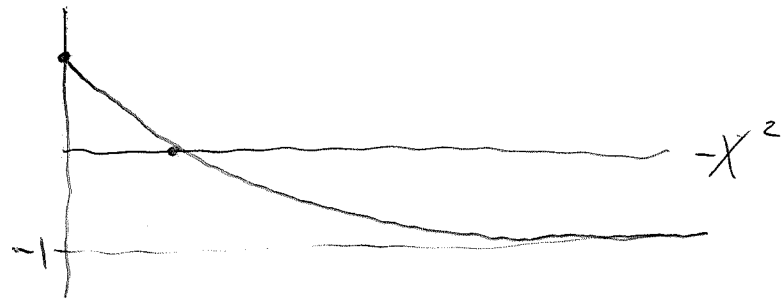
then $JJ_0^{-1} = F\varepsilon$, ~~which~~ means that you are really working in the Grassm. $U(2n)/U(n)\times U(n)$ under the embedding $Sp(2n)/U(n) \hookrightarrow$ ______.

Then $tr(F\varepsilon)$, the functional you use with F ranging over a Grassm. and ε the hermitian op, is

$$tr(F\varepsilon) = \frac{1}{2}tr(F\varepsilon + \varepsilon F) \overset{tr}{=} \frac{1}{2}(g+g^{-1}) \overset{tr}{=} \frac{1}{2}\left\{\frac{1+X}{1-X} + \frac{1-X}{1+X}\right\}$$

$$= tr\left(\frac{1+X^2}{1-X^2}\right).$$ Recall $X^2 \leqslant 0$ so that

$$\frac{1+X^2}{1-X^2}$$



$$\frac{1+X^2}{1-X^2} + 1 = \frac{2}{1-X^2}.$$ Therefore one has

$$\frac{1+X^2}{1-X^2} = \frac{2}{1-X^2} - 1$$ decreases monotonely from 1 to -1

as $-X^2$ increases from 0 to $+\infty$. Something is wrong.

Review $F \mapsto tr(FA)$, A hermitian, F ranges over a Grassm. so that $F^2 = 1$. The tangent space to the Grassm. is $\{\delta F \mid (\delta F)F + F(\delta F) = 0\}$. F is a critical point $\iff tr(\delta F)A = 0$, $\forall \delta F$ (herm and anticommuting with F. Split A into 4 components wrt eigenspaces of F: $A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$

$F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \delta F = \begin{bmatrix} 0 & b^* \\ b & 0 \end{bmatrix}$

(327) then $\operatorname{tr}(\delta F A) = \operatorname{tr}\begin{bmatrix} 0 & b^* \\ b & 0 \end{bmatrix}\begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$

$= \operatorname{tr}\begin{bmatrix} b^* A_{-+} & ? \\ ? & bA_{+-} \end{bmatrix} = \operatorname{tr}(b^* A_{-+} + bA_{+-})$ where

$A_{+-} = A^*_{-+}$, so $0 = \operatorname{tr}(b^* A_{-+} + bA^*_{-+}) \quad \forall b$

Put $b = A_{-+}$ get

$0 = \operatorname{tr}(A^*_{-+}A_{-+} + A_{-+}A^*_{-+}) \Longrightarrow A_{-+}, A_{+-} = 0$

$\Longrightarrow$ A commutes with F.

Another idea is to scale X: put in tX for X

$\frac{1}{2}(FA + AF) = F(A + FAF)\frac{1}{2} = FA_{ev}$

$\frac{1}{2}(FA - AF) = F(A - FAF)\frac{1}{2} = FA_{od}$

$\operatorname{tr}(\delta F A) = \operatorname{tr}(\delta F A_{od})$ because δF is odd

$A = \begin{bmatrix} a & b^* \\ b & d \end{bmatrix}, \quad \delta F = \begin{bmatrix} 0 & m^* \\ m & 0 \end{bmatrix}$

$\operatorname{tr}(A\delta F) = \operatorname{tr}\begin{bmatrix} a & b^* \\ b & d \end{bmatrix}\begin{bmatrix} 0 & m^* \\ m & 0 \end{bmatrix} = \operatorname{tr}(b^* m + bm^*)$

If $A_{od} \neq 0$, take $m = b$, get $\operatorname{tr}(A\delta F) = \operatorname{tr}(b^*b + bb^*) > 0$
so $\operatorname{tr}(\delta F A) = 0, \ \forall \delta F \Rightarrow A_{od} = 0.$

Next you want to adapt this to the symplectic case. Yesterday you tried using the C.T. You concluded that it didn't seem right, because the anti-linear operator was absent.

Let review the ideas again. You are studying the space of polarizations in $H(\mathbb{C}^n)$. You want to construct a suitable functional depending on the basepoint polarization, whose critical points will commute with the basepoint polarization.

Start again. You consider the space $\mathbb{P}$ of polarizations of $H(\mathbb{C}^n)$, equipped with basepoint $J_0 = i\varepsilon$. You want to construct a Morse function on $\mathbb{P}$ depending on J_0 whose critical points are polarizations centralizing J_0. In principle you should be able to do this by means of the conjugacy proof for the adjoint picture of the symmetric space $Sp(2n)/U(n)$. ~~You want to alter the~~ This means that you consider the action of the isotropy group $U(n)$ ($=$ centralizer of J_0) on the tangent space to ~~the~~ $\mathbb{P}$ at J_0, which is $\left\{\begin{bmatrix}0 & -b\\ b & 0\end{bmatrix} : b^t = b\right\}$, the space of symmetric bilinear forms.

Now you ~~need~~ want to know what happens at any polarization J. More precisely you need the tangent space to $\mathbb{P}$ at any J. This should be isom to the space of symmetric bilinear forms. ~~Think~~

You need a good picture. You have the linear space $\mathfrak{p} \simeq \mathcal{L}(Sp(2n)/U(n)) \simeq \{\text{symm. bilinear forms}\}$ inside $\mathcal{L}(Sp(2n))$ acted on by $K = U(n)$. Basic result: orbits of K on $\mathfrak{p}$ are flag manifold varieties: of the form $K/$Centralizer of a torus. (If you take an element of $\mathfrak{p}$, it generates a torus.) Maybe this result is important because it allows you to identify the flag manifolds with orbits in Lie algebra.

Aim: to use J_0 to get a Morse function on $\mathbb{P}$. This should be obvious if you knew the moment map theory well.

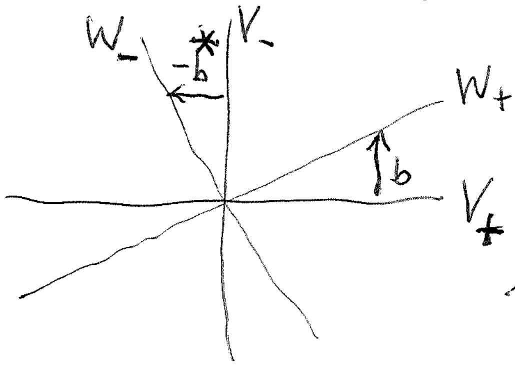
(329) You consider $\mathbb{P}$, the space of polarizations given by of $H(\mathbb{C}^n)$. An element J of [redacted] $\mathbb{P}$ should be an [redacted] an orthogonal splitting

$H = \begin{bmatrix} W_+ \\ W_- \end{bmatrix}$ (wrt the herm. inner product) such that W_+, W_- are Lagrangian subspaces of H

$J = \pm i$ on $W_\pm$ so that $-J = J^* = J^{-1}$

You have the basepoint $J_0 = i\varepsilon = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ on $\begin{bmatrix} V_+ \\ V_- \end{bmatrix}$

Let's discuss the big cell (affine open subspace) of $\mathbb{P}$ centered at J_0:



W_+ is the graph of $b : V_+ \to V_-$

$W_- = (W_+)^\perp =$ graph of $-b^* : V_- \to V_+$

You know that W_+ is Lagrangian iff $b^t = b$. $W_- = \begin{bmatrix} -b^* \\ 1 \end{bmatrix} : V_- \to V_+$

is Lagrangian iff $-b^*$ is symm. $\iff (b^*)^t = b^*$, which follows from $b^t = b$ by applying [redacted] $*$; things commute.

$$J \begin{bmatrix} 1 & -\bar{b} \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\bar{b} \\ b & 1 \end{bmatrix} i\varepsilon \qquad X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}$$

$$J(1+X)(-i\varepsilon) = (1+X) \text{ [redacted]} \qquad \therefore \boxed{J J_0^{-1} = \frac{1+X}{1-X}}$$

$$\overset{\shortparallel}{J \underbrace{(-i)\varepsilon}_{J_0^{-1}}(1-X)}$$

Note X is simultaneously skew adjoint skew symm.

No. $$X^t = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}^t = \begin{bmatrix} 0 & b^t \\ -\bar{b}^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \neq -X$$

The problem should be that the symplectic form ~~on~~ on H is defined $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ not $i\varepsilon$.

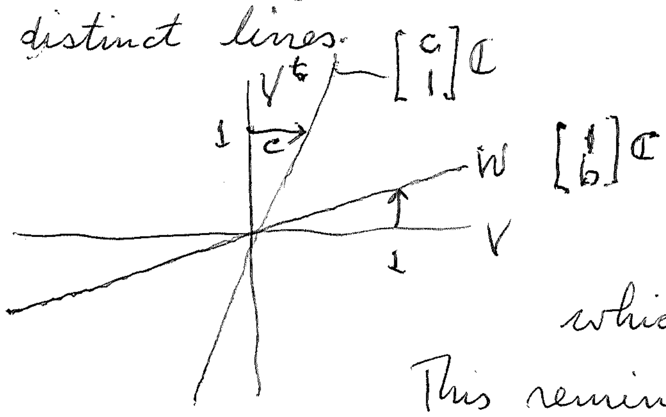
You still have to clarify the notion of polarization, polarization of $H(\mathbb{C}^n)$,

First consider the complex symplectic structure on H.

$$H = \begin{bmatrix} V \\ V^t \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$$

A polarization in this setting is an ordered pair of complementary Lagrangian subspaces. It should be clear that $Sp(2n,\mathbb{C})$ acts transitively on these polarizations, and the stabilizer of the basepoint polarn. is $GL(n,\mathbb{C})$.

Look at $n=1$, where $Sp(2,\mathbb{C}) = SL(2,\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad-bc=1 \right\}$, and $GL(1,\mathbb{C}) = \mathbb{C}^\times$ is embedded as $z \longmapsto \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$. A polar. is an ordered pair of distinct lines.



This ought to show the two lines $\begin{bmatrix} 1 \\ b \end{bmatrix}\mathbb{C}$, $\begin{bmatrix} c \\ 1 \end{bmatrix}\mathbb{C}$ which are close to V, V^t resp.

This reminds you of the big Bruhat cell.

Note that an operator $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ on $\begin{bmatrix} V \\ V^t \end{bmatrix}$ satisfies

$X^t J + JX = 0 \iff b = b^t,\ c = c^t,\ d = -a^t$. So a tangent vector to the space of polarizations at the basepoint has the form $X = \begin{bmatrix} 0 & c \\ b & 0 \end{bmatrix}$ with b, c symmetric. There should be a similar picture for the tangent space at any polarn.

Next you bring in the inner product on H.

(331) <u>Idea</u>: The space of polarns. is a flag manifold assoc. to $Sp(2n)$, and therefore it is an adjoint orbit: $Sp(2n)/$centralizer of some torus T, where T should be a circle group. This idea should allow one to identify the space $\mathbb{P}$ of polarizations with a conjugacy class in $\mathcal{L}(Sp(2n))$.

Let's go back to $Sp(2n, \mathbb{C})$

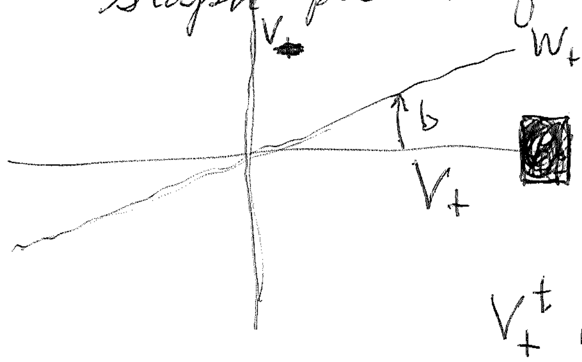
$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [x_1 \ \ y_1] \begin{bmatrix} y_2 \\ -x_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2.$$

Let $X \in M(2n, \mathbb{C})$ preserve this symplectic form: $X^t J + JX = 0$, say $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $X = J X^t J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} c^t & d^t \\ -a^t & -b^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d^t & c^t \\ b^t & -a^t \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \iff \begin{array}{l} b^t = b \\ c^t = c \\ d = -a^t. \end{array}$$

Yesterday's idea of $Sp(2n, \mathbb{C})$: This group should act transitively on polarizations (these are defined as ordered pairs of complementary Lagrangian subspaces), so that the manifold of polarizations is $Sp(2n, \mathbb{C}) / GL(n, \mathbb{C})$.

Graph picture of Lagrangian subspace



$W_+ = \begin{bmatrix} 1 \\ b \end{bmatrix} V_+$ is Lagrangian iff

$$\left(\begin{bmatrix} 1 \\ b \end{bmatrix} V_+ \right)^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} V_+ = 0$$

$$\parallel$$

$$V_+^t [1 \ \ b^t] \begin{bmatrix} b \\ -1 \end{bmatrix} V_+ = 0 \iff b - b^t = 0$$

So one way to get a polarization is from an ordered pair of symmetric forms $b : V_+ \longrightarrow V_-$ and $b' : V_- \longrightarrow V_+$ such that the graphs are transversal, which should mean that $\begin{bmatrix} 1 & b' \\ b & 1 \end{bmatrix}$ is invertible.

(332) So what's the problem? Consider $Sp(2n, \mathbb{C})$, $n = 1$, i.e $SL(2,\mathbb{C})$. Then $SL(2,\mathbb{C})/\mathbb{C}^*$ is the space of ordered pairs of lines in $\mathbb{C}^2$ which are independent. Here $\mathbb{C}^\times \xrightarrow{z \mapsto} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \subset SL(2,\mathbb{C})$

How to study this? Let $\begin{bmatrix} a & c \\ b & d \end{bmatrix} \in SL(2,\mathbb{C})$? too hard.

Instead consider the inner product $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \cdot \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^* x_2 + y_1^* y_2$.

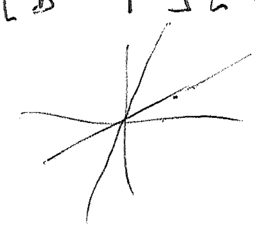
Restrict ~~attention~~ to ordered pairs of lines which are $\perp$ for the inner product. You can assume $|a|^2 + |b|^2 = 1$ and $|c|^2 + |d|^2 = 1$ also that $a, d \geq 0$. Then orthogonality $a\bar{c} + \bar{b}d = 0$, $ad - bc = 1$? You want to understand what $\begin{bmatrix} a \\ b \end{bmatrix} \perp \begin{bmatrix} c \\ d \end{bmatrix}$ means. $a\bar{c} + d\bar{b} = 0$?

You have two unit vectors $v, w \in \mathbb{C}^2$ and you want to show that $v \perp w \iff |v \wedge w| = 1$.

This should be obvious because it reduces to $\mathbb{R}^2$.

Review: ~~Any two polarizations~~ $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ eg w. symp. form $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$. Def polarization as ordered pair of complementary Lag subspaces. Any two poln conj. by $Sp(2n, \mathbb{C})$, stab. is $GL(n, \mathbb{C})$

Any polarization $\begin{bmatrix} W \\ W^t \end{bmatrix}$ near to $\begin{bmatrix} V \\ V^t \end{bmatrix}$ has the form $[W \ \ W^t] = \begin{bmatrix} 1 & b' \\ b & 1 \end{bmatrix} \begin{bmatrix} V \\ V^t \end{bmatrix}$ where b, b' symm and $\begin{bmatrix} 1 & b' \\ b & 1 \end{bmatrix}^{-1}$ ∃. ~~so what?~~



$n = 1$ $\quad 1 \neq bb'$

333 Next consider the [redacted] inner product. [redacted] You have 3 structures on $H(V)$. Def poln: ordered pair of orthogonal Lagrangian subspaces. Trans. of $Sp(2n)$ with stab $U(n)$ is clear. Take $n=1$, where $Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : |a|^2+|b|^2=1 \right\}$.

[redacted] The space of polarizations is $SU(2) / \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, a \in U(1) \right\}$. [redacted]

[redacted] A polarization is ~~an~~ ordered pair of orthogonal lines.

Idea: Recall Dominic Joyce's $\mathbb{H}$-theory where you have the antipodal map on the Riemann sphere: $z \mapsto -\bar{z}^{-1}$. So [redacted] in this special case you see the antilinear map giving rise to j.

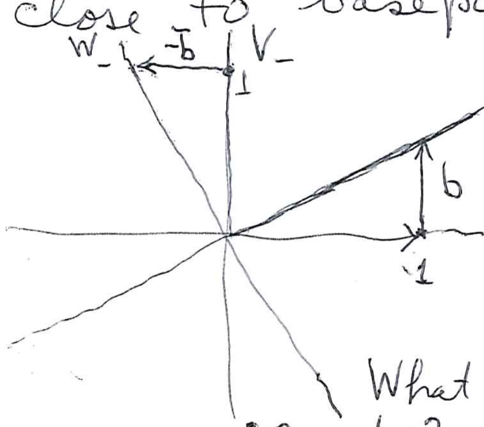
[redacted] Is it true that the subgroup $U(1) \simeq \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}, |a|=1 \right\} \subset SU(2)$ is the centralizer of an element of $\mathcal{L}(SU(2)) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : a+\bar{a}=0 \right\}$?

Yes $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \alpha$ where $0 \neq \alpha \in \mathbb{R}$.

What are you trying to find out?

What questions to ask? Consider $SU(2)$ acting on $\mathbb{C}^2$ preserving inner product and symplectic form. Notion of polarization: an ordered pair of $\perp$ lines in $\mathbb{C}^2$. Basepoint polarzn $\begin{bmatrix} \mathbb{C} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbb{C} \end{bmatrix}$, [redacted] stabilizer is max torus $\left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : a \in U(1) \right\}$. [redacted] Take polarization close to basepoint

$$[W_+ \ W_-] = \begin{bmatrix} 1 & -\bar{b} \\ b & 1 \end{bmatrix} \begin{bmatrix} V_+ \\ V_- \end{bmatrix}.$$



[redacted]

$\dfrac{b}{1} \longrightarrow \dfrac{1}{-\bar{b}}$ goes from line [redacted] to $\perp$ line.

What should question be? Point you missed: $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ is skew adjoint, so $1 \pm X$ invertible.

334 So now you can try to correct past problem. If $b^t = b$, then $-b^* = -\bar{b}^t = -\bar{b}$,

so $X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}$, $X^t = \begin{bmatrix} 0 & b^t \\ -\bar{b}^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$

so $X + \overline{X^t} = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{b} \\ -b & 0 \end{bmatrix} = 0 \quad \therefore X^* = -X.$

Now you hoped before that $X + X^* = 0$ and X of the form $\begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} \Longrightarrow X + X^t = 0$. This is true when the symplectic form is $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ (?)

This may be wrong because of the shift orth $\longleftrightarrow$ symp upon dividing rank by 2.

Another approach might be to look at the three simplest symplectic forms: NO, only one is skew-symm. The other two are symm.

$$\underset{\hat{i}}{\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}}, \quad \underset{\hat{j}}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}, \quad \underset{\hat{k}}{\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}}$$

These form a nice basis for (over $\mathbb{R}$) $\mathcal{L}(SU(2))$.

$H(\mathbb{C}^n) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ inner production $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^* x_2 + y_1^* y_2$

symp. form $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{J} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$

Let $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \operatorname{End} H$

satisfy (i) $X^* + X = 0$

(ii) $X^t J + JX = 0 \Longrightarrow \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(iii) $JX = \bar{X} J$ $\quad = \begin{bmatrix} b & d \\ -a & -c \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$

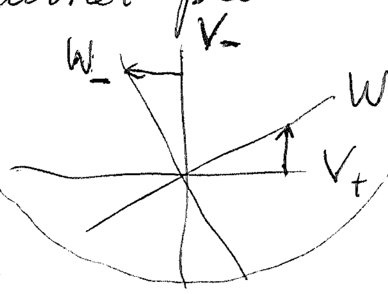
(because $\bar{X} = -X^t$) $\quad \therefore b^t = b,\ c^t = c,\ d = -a^t.$

Also $X^* + X = 0 \Rightarrow$ $a^* = -a$, $c^* = -b$, $b^* = -c$, $d^* = -d$

$-c = b^* = \overline{b^t} = \bar{b}$, also $-b = \bar{c}$, $\bar{d} = -\overline{a^t} = -a^* = a$. ∴

$$\boxed{X = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : \begin{matrix} a^* + a = 0 \\ b^t = b \end{matrix} \right\}}$$

Go over what you did earlier. Define a polarization to be an ordered pair of Lagrangian subspaces which are orthogonal wrt the inner product. Consider the case close to the basepoint:



Then

$$[W_+ \; W_-] = \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} [V_+ \; V_-]$$

You know that $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ skew adjoint $\Rightarrow (\pm X)^{-1} \exists$

Because W_+, W_- Lagrangian one has $b^t = b$

$(-b^*)$ symm. $\Rightarrow -b^* = -\bar{b}$

so $X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} \in \mathcal{L}(Sp(2n)) \Longrightarrow X^t J + JX = 0$

Check it: $\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} =$

$= \begin{bmatrix} b & 0 \\ 0 & \bar{b} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ Yes.

At this point you should understand polarizations close to the basepoint. Recall $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ satisfies $X^t J + JX = 0$

$\Longleftrightarrow \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$. Notice that

$$J \begin{bmatrix} a & c \\ b & d \end{bmatrix} J^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} b & d \\ -a & -c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ +1 & 0 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

which is like the 2×2 rule: interchange diagonal elts change sign of off-diagonal elts.

NO NO NO NO

Conclude that $X = \begin{bmatrix} 0 & b' \\ b & 0 \end{bmatrix}$ satisfies
$X^t J + J X = 0 \iff b, b'$ symm. Also
X preserves both symplectic form and inner
product $\iff X = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$.

The preceding gives a complete picture of polarizations close to the basepoint. They correspond to symmetric $b: V_+ \longrightarrow V_-$ i.e. to Lagrangian subspaces transversal to V_-.

You want to identify the space of polarizations with an orbit in $\mathcal{L}(Sp(2n))$.

You are still trying to find a clear picture of the space of polarizations. A polarization is like a point

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You're still missing something about polarizations. Today's idea is to understand the symmetric space structure on the space of polarizations $Sp(2n)/U(n)$. A point of a symmetric space determines a reflection through that point and all these reflections should generate the symmetry group $Sp(2n)$. The obstruction to your understanding is probably the fact that the reflections are anti-linear. $\therefore$ a polarization should be an anti-linear transformation whose square is in the center.

I think you want to look for an automorphism of $Sp(2n)$ of order 2 with the fixed group $U(n)$. Conjugation by an anti-linear transformation should be what's needed. Note that elements of $Sp(2n)$ are linear transformations on H

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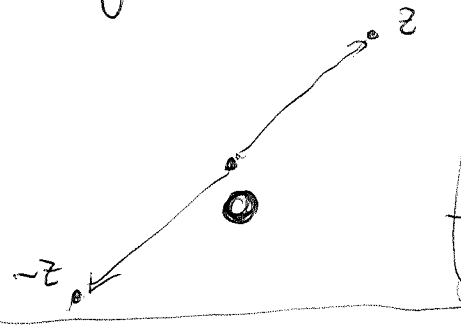
$n=1$, $Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\}$

acts on $\mathbb{P}^1\mathbb{C} = S^2$ = Riemann sphere. S^2 is the flag manifold of polarizations, where the polarn corresp. to $L \subset \mathbb{C}^2$ is $L, L^\perp$. So if $L = \begin{bmatrix} a \\ 1 \end{bmatrix}\mathbb{C}$ then $L^\perp = \begin{bmatrix} -\bar{a}^{-1} \\ 1 \end{bmatrix}\mathbb{C} = \begin{bmatrix} 1 \\ -\bar{a} \end{bmatrix}\mathbb{C}$.

Now $S^2 = SU(2)/U(1)$ is a symmetric space, and so there should be at each point a a reflection through that point. Reflection means that you join a variable point z to a by a geodesic (in the sense of spherical geometry, so this ought to amount to using a circle or straight line) and then you continue the geodesic through a an equal amount to the opposite side of a. If $a = 0$ or ∞ the reflection is $z \mapsto -z$



For a general a use an elt of $SU(2)$ to move a to 0.

You need notation change

$n=1$, $Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\}$.

$SU(2)$ acts on $\mathbb{C}^2 = H(\mathbb{C})$ in the obvious. A polarization is equivalent to a line L in $\mathbb{C}^2$, so the space of polarizations is the Riemann sphere $\mathbb{P}^1\mathbb{C} = S^2 = \mathbb{C} \cup \{\infty\}$. If the line L is $\begin{bmatrix} z \\ 1 \end{bmatrix}\mathbb{C}$, then the orthogonal line is $L^\perp = \begin{bmatrix} 1 \\ -\bar{z} \end{bmatrix}$; of course these lines are 1-diml $\Rightarrow$ Lagrangian. Action of $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ on $\begin{bmatrix} z \\ 1 \end{bmatrix}\mathbb{C}$ is $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}\begin{bmatrix} z \\ 1 \end{bmatrix} = \begin{bmatrix} az+b \\ \bar{a}-\bar{b}z \end{bmatrix} = \ ?$

338 Still struggling with the space of polarizations. Let's return to spectral theory for symmetric bilinear forms on a complex V with pos. herm. inner product.

$$V \xrightarrow[\sim]{*} V^t \xrightarrow[\sim]{*} V$$

$$\begin{array}{ccccc} V & \xrightarrow{b} & V^t & \xrightarrow{*} & V \\ x & \longmapsto & x^t b & \longmapsto & *(x^t b) = b^* \bar{x} = \bar{b} \bar{x} \end{array}$$

$$*b(*b(x)) = *b(\bar{b}\bar{x}) = \bar{b}\,\overline{\bar{b}\bar{x}} = (\bar{b} b)x$$

$$(*b)^2 = \bar{b} b = b^* b > 0.$$

It seems that what you need is the theory of an antilinear hermitian operator ??

You need only basic harmonic oscillator stuff. You are given two forms, one hermitian bilinear the other symmetric bilinear. The "difference" is an antilinear operator whose square is hermitian ≥ 0.

$$\begin{array}{ccccccc} V & \xrightarrow{b} & V^t & \xrightarrow{*} & V & \xrightarrow{b} & V^t \\ x & & x^t b & & \bar{b}\bar{x} & & (\bar{b}\bar{x})^t b = x^*(b^* b) \\ & & \overset{\parallel}{(bx)^t} & & & & \end{array}$$

Ultimately you want these antilinear ops to be the analog of hermitian operator.

Let's go over spectral theory for a symm. bilinear form on a complex Hilbert space V. So $V = \mathbb{C}^n$ with x^*y pos. herm. form and bilinear form $x^t b y$, b a symmetric matrix. Then get transformations (maps) assoc. to these two forms.

$$V \xrightarrow{t \cdot b} V^t \xrightarrow{*} V \longrightarrow V^t \longrightarrow V$$

$$x \longmapsto x^t b$$

$$\begin{array}{l} \quad\; \| \\ (bx)^t \longmapsto \overline{bx} \longmapsto \bar{b}\,\overline{\overline{bx}} \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\quad (\bar{b}b)x \end{array}$$

So $T_b(x) = \bar{b}\bar{x}$ $\quad T_b : V \longrightarrow V$ anti-linear

$$T_b T_b(x) = \bar{b}\,\overline{T_b(x)} = \bar{b}\,\overline{\bar{b}\bar{x}} = (\bar{b}b)x$$

where $\bar{b}b = b^*b \geq 0$.

So T_b and b^*b commute. ~~You can extract~~ Since b^*b hermitian, V splits into eigenspaces ~~for b*b~~ $V = \bigoplus_{\lambda \geq 0} V_\lambda$, where $V_\lambda = \operatorname{Ker} (b^*b) - \lambda^2$.

What's important is to restrict attention to V_λ.

Then you have the same situation V, b etc. but with b^* ? $\quad [T_b, b^*b] = 0$.

There might be a problem working with matrices - the eigenspace splitting V_λ won't be compatible with the basis for V chosen.

340 Assume OK. Ultimately you need a clear intrinsic formulation which you seem to have:

$$V \xrightarrow{b} V^t \xrightarrow[\sim]{*} V$$

These maps $b, *$ are intrinsically defined, so $T_b : V \longrightarrow V$ is intrinsically defined.

Focus on the pure case $T_b^2 = b^*b = \lambda^2$

Then $\lambda^{-1}T_b$ is anti-linear on V with square $+1$.

$\lambda^{-1}T_b$ is anti-linear unitary?

If $b^*b = 1$, then T_b is a real structure on V. Then $\underbrace{b^*b = \bar{b}b = 1}$, so

$b^*b = 1$ means b unitary $\qquad b^* = \bar{b} = b^{-1}$

g orthogonal means $\underline{g^tg} = 1$? $\qquad g^*\bar{g} = 1$

It seems that $T_b(x) = *(x^tb) = \bar{b}\bar{x}$

$T_b^2(x) = (\bar{b}b)x$ with $\underbrace{\bar{b}b = 1}$ is important.

What's interesting is T_b with $b^*b = 1$

$$\begin{array}{ccccc} V & \xrightarrow{b} & V^t & \xrightarrow{*} & V \\ x & \longmapsto & x^tb & \longmapsto & b^*\bar{x} \end{array}$$

$T_b(T_b(x)) = b^*\overline{T_b(x)}$

$= b^*\overline{b^*\bar{x}} = (b^*b^t)x$

(341) Aim: To link polarization to anti-operator T_b with $T_b^2 = 1$. This is the real puzzle.

Can you fit the preceding with orth case?

$$j \qquad \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$$

$$Sp(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

$$\text{Lie } SU(2) = \left\{ \begin{bmatrix} & \\ & \end{bmatrix} : a + \bar{a} = 0 \right\}.$$

$$= \left\{ x \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}$$

You want to study $S^2 = SU(2) / \left\{ \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix} : |a| = 1 \right\}$. This is a symmetric space, so you think that the isotropy group is the fixpt of an involution on the group $SU(2)$.

Conjugation by $i\varepsilon = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ has fixpoint subgp. $\varphi U(1) = \begin{bmatrix} a & 0 \\ 0 & \bar{a} \end{bmatrix}$. ~~...~~

So you can identify S^2 with an ~~...~~ orbit of $SU(2)$ on $\mathcal{L}SU(2)$, or on $SU(2)$.

~~...~~ Program: Study the symm. space $Sp(2)/\varphi\mathbb{T}$ where $\varphi(\zeta) = \begin{bmatrix} \zeta & 0 \\ 0 & \bar{\zeta} \end{bmatrix}$. ~~Inf~~ version consists of $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b \in \mathbb{C} \right\}$ with $\varphi(\zeta)$ acting by conjugation

$$\begin{bmatrix} \zeta & \\ & \bar{\zeta} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} \begin{bmatrix} \bar{\zeta} & \\ & \zeta \end{bmatrix} = \begin{bmatrix} 0 & \zeta^2 b \\ -\bar{\zeta}^2 \bar{b} & 0 \end{bmatrix}$$

one orbit for each $|b|$.

Now look at $Sp(2n)/\varphi U(n)$. Inf. picture is $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix} : b^t = b \right\}$, conjugation action of $\underset{K}{\varphi U(n)}$: $\begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}\begin{bmatrix} u^* & 0 \\ 0 & u^t \end{bmatrix} = \begin{bmatrix} 0 & ubu^t \\ -\bar{u}\bar{b}u^* & 0 \end{bmatrix}$

and $(ubu^t)^t = ub^tu^t = ubu^t$. You want the orbit structure of K on $\mathfrak{p}$, equivalently, the orbit structure of $U(n)$ acting on symmetric matrices via $u \# b = ubu^t$. Orbit structure means eigenvalue theory?

Let's try to link σ_b stuff (unitary equivalence for symmetric forms) to the infinitesimal symmetric space $\mathcal{L}(Sp(2n)/\varphi U(n))$. Can you construct the spectral decomposition for σ_b within $\mathfrak{p}$?

Repeat: $b: V \longrightarrow V^t$ symm. $\mathbb{C}$-bilinear
$x \longmapsto x^t b$

alternative $x \longmapsto bx$ column vector; transpose to get row $(bx)^t = x^t b$.

What's clear is that the spectral theory of ~~complex~~ symmetric ~~matrices~~ bilinear forms on an n-dim complex V with pos. herm. inner prod ~~as the~~ is simply the $K = \varphi U(n)$ action on $\mathfrak{p} = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$ $b^t = b$.

So it should be possible to directly construct the decomp. of b. Put $X = \begin{bmatrix} 0 & b \\ -\bar{b} & 0 \end{bmatrix}$, form

$$-X^2 = \begin{bmatrix} b\bar{b} & 0 \\ 0 & \bar{b}b \end{bmatrix} \geq 0.$$

[redacted] Properties of $X=\begin{bmatrix}0 & b\\ -\bar{b} & 0\end{bmatrix}$ w. $b^t=b$ [redacted]

$$X^*+X=0,\quad \underbrace{X^tJ+JX}=0,\quad JX=\bar{X}J$$

$$-X^t=+JXJ^{-1}$$ [redacted]

$$J\begin{bmatrix}a & b\\ c & d\end{bmatrix}J^{-1}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}c & d\\ -a & -b\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}d & -c\\ -b & a\end{bmatrix}$$

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check $\begin{bmatrix}-a^t & -c^t\\ -b^t & -d^t\end{bmatrix}=\begin{bmatrix}d & -c\\ -b & a\end{bmatrix}$ $\qquad b^t=b,\ c^t=c,\ d=-a^t$

So [redacted] where next? How do you link

[redacted] $V\longrightarrow V^t\xrightarrow{*}V$

$x\longmapsto x^tb\longmapsto\overline{bx}$

$x^tb = (bx)^t$

$\sigma_b \cancel{x}=\overline{bx}$

$\sigma_b(\sigma_b x)=\overline{b\,\overline{bx}}=(\bar{b}b)x$

~~Can you work this into~~

~~$g^tJg=J \qquad g^t=Jg^{-1}J^{-1} \qquad g^{-1}=\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$~~

~~$g=\begin{bmatrix}a & b\\ c & d\end{bmatrix} \qquad \begin{bmatrix}a & c\\ b & d\end{bmatrix}=\begin{bmatrix}a & b\\ b & d\end{bmatrix}$~~

$X=\begin{bmatrix}0 & b\\ -\bar{b} & 0\end{bmatrix}$ where $b^t=b$. Form $-X^2=\begin{bmatrix}b\bar{b} & 0\\ 0 & \bar{b}b\end{bmatrix}$

There's still no link to anti-linear maps.

Canonical commutation relations.

$$[a_i, a_j] = 0, \quad [a_i^*, a_j^*] = 0, \quad [a_i, a_j^*] = \delta_{ij}$$

basic structure is $*$ together with $[\ ,\ ]$

Considering a symplectic form + pos herm. form.

Maybe it would be good to list all the ideas you want to organize:

- polarization, maximal abelian subspace
- compatibility of $*$ with a symplectic form
- symmetric bilinear forms in dimension 2 where you have factorization into 2 linear forms.
- Given a symmetric form, you can restrict to each line in V, where you have a well-defined $\lambda \geq 0$. Maximize this over $\mathbb{P}V$.

Given $b = b^t$, let x be a unit vector, form $x^t b x$. The absolute value $|x^t b x|$ is independent of the phase of x, so you get a well-defined function on $\mathbb{P}V$. Properties: smooth function on V when $x^t b x \neq 0$. Look at $\dim V = 2$, where $x^t b x$ is a quadratic form in x_1, x_2. Look at the product of 2 linear factors: $x_1 x_2$. $|x_1 x_2|$ is not a smooth function of x_2 for $x_1 \neq 0$.

So it seems that you want to square the absolute value: $x^t b x \, \overline{x^t b x}$, no problem with smooth since $|z|^2$ is smooth fn of z and $x^t b x$ is a smooth fn of x.

So consider $|x^t b x|^2$ on $\mathbb{P}V$,

$$|x^t b x|^2 = x^t b x \, \overline{x^t} \, \overline{b} \, \overline{x} = \mathrm{tr}(x^t b x \, x^* \overline{b} \, \overline{x})$$

$$= \mathrm{tr}(b \underbrace{x x^*}_{p} \overline{b} \underbrace{\overline{x} x^t}_{\overline{p}}) = \mathrm{tr}(b p \overline{b} \overline{p})$$

So you now have a nice functional defined on $\mathbb{P}V$. What are its critical points?

$$\delta \, \mathrm{tr}(b p \overline{b} \overline{p}) = \mathrm{tr}(b \delta p \overline{b} \overline{p} + b p \overline{b} \delta \overline{p})$$

$$= \mathrm{tr}(\delta p \, \overline{b} \overline{p} b + b p \overline{b} \, \delta \overline{p})$$

$p = x x^*$ $\qquad$ $\cancel{\delta p = \delta x \, x^* + x \delta x^*}$

$p = p^2$ $\qquad$ $\delta p = p \delta p + \delta p \, p$

$p^\perp \delta p = \delta p \, p$ $\qquad$ $\delta p = p \delta p + p^\perp \delta p$

$$\mathrm{tr}(\delta p \, \overline{b} \overline{p} b) = \mathrm{tr}(\delta p \, (p + p^\perp) \overline{b} \overline{p} b)$$

$$\delta p = p \delta p + p^\perp \delta p = p p \delta p + p^\perp p^\perp \delta p$$

$$= p \delta p p^\perp + p^\perp \delta p p$$

$$\mathrm{tr}\left((p \delta p p^\perp + p^\perp \delta p p)(T)\right) \qquad T = \overline{b} \overline{p} b$$

$$= \mathrm{tr}\left(\delta p (p^\perp T p) + \delta p (p T p^\perp)\right)$$

$\therefore$ $p T p^\perp$ and $p^\perp T p$ $= 0$

(346) From an invariant viewpoint you need to explain $\bar{p}$, $\bar{b}$ etc. Let's go over what you learned yesterday. You consider a symmetric matrix b of rank n and use it to construct a function on $\mathbb{P}(\mathbb{C}^n)$. Given a unit vector x you form $|x^t b x|$ which is ind of the phase of x. Unfortunately not smooth in general, so you take $|x^t b x|^2 = x^t b x\, x^* \bar{b} \bar{x}$ which you can rewrite as $tr(b\, xx^* \bar{b}\, \overline{xx^*})$, note that xx^* = orthogonal projection operator on $\mathbb{C}^n$ whose image is line $x\mathbb{C}$. Try replacing xx^* by

$$2xx^* - 1 = F \qquad xx^* = \frac{1+F}{2} \qquad tr\left(b\,\frac{1+F}{2}\,\bar{b}\,\frac{1+\bar{F}}{2}\right)$$

$$\frac{1}{4} tr\left(b\bar{b} + bF\bar{b} + b\bar{b}\bar{F} + bF\bar{b}\bar{F}\right).$$ ~~This doesn't m~~

You still have the same problem: How to handle $\bar{p}$, $\bar{b}$ intrinsically, which probably means that you want a semi direct product with j.

Nice smooth function $tr(bp\bar{b}\bar{p})$ for ~~where p = orthogonal projection on the~~ $p \in \mathbb{P}V$, that is, p is any orthogonal projections of rank 1.

$$\delta\, tr(bp\bar{b}\bar{p}) = tr\left(\delta p\, \bar{b}\bar{p}b + bp\bar{b}\,\delta\bar{p}\right)$$

What conclusion to draw is probably $p^\perp \bar{b}\bar{p}b = 0$

You find that $\bar{b}\bar{p}bV \subset pV$

$$x^* A x = tr(pA) \overset{\delta}{\longmapsto} tr(\delta p A) = 0 \quad \text{all } \delta p \Rightarrow A^{odd} = 0$$

$$p^\perp A p = pAp^\perp = 0$$

$$ApV \subset pV.$$

Hope $\bar{b}\bar{p}bpV \subset pV$

(347) $\delta p = p\,\delta p\, p^{\perp} + p^{\perp}\delta p\, p$

$$tr(\delta p\ \bar b \bar p b) = tr(p\,\delta p\, p^{\perp} \bar b \bar p b) + tr(p^{\perp}\delta p\, p\, \bar b \bar p b)$$

$$= tr\left(\delta p\,(p^{\perp} \bar b \bar p b p + p \bar b \bar p b p^{\perp})\right)$$

So this is all very reasonable. If the function is stationary wrt $\delta p \iff p^{\perp} \bar b \bar p b p = 0$

$\delta \bar p \iff \bar p^{\perp} b p \bar b \bar p = 0$

But you don't know what this means.

So it becomes important to understand the meaning of replacing b by $\begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}$.

Idea: p is a projection with image a line $L \subset V$, p should induce a projection on $H(V)$, which should be $\pi = \begin{bmatrix} p & 0 \\ 0 & \bar p \end{bmatrix}$, $\beta = \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}$. What's the function

$tr(bp\bar b \bar p)$?

$$\begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & \bar p \end{bmatrix}\begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}\begin{bmatrix} p & 0 \\ 0 & \bar p \end{bmatrix}$$

$$\begin{bmatrix} 0 & b\bar p \\ -\bar b p & 0 \end{bmatrix}\begin{bmatrix} 0 & b\bar p \\ -\bar b p & 0 \end{bmatrix} = \begin{bmatrix} -b\bar p \bar b p & 0 \\ 0 & -\bar b p b \bar p \end{bmatrix}$$

It might help to replace: $\begin{bmatrix} p & 0 \\ 0 & \bar p \end{bmatrix} = \begin{bmatrix} xx^* & 0 \\ 0 & \bar x x^t \end{bmatrix}$

$= \begin{bmatrix} x \\ \bar x \end{bmatrix} \begin{bmatrix} x \\ \bar x \end{bmatrix}^*$?

$= \begin{bmatrix} x \\ \bar x \end{bmatrix}\begin{bmatrix} x^* & x^t \end{bmatrix} = \begin{bmatrix} xx^* & xx^t \\ \bar x x^* & \bar x x^t \end{bmatrix}$?

(348) Look at the case $n=1$. $b \in \mathbb{C}$, trivial since $\mathbb{P}V$ is a point.

The critical value is $|b|^2$; here you're using the function $x^t b x\, \overline{x^t b x}$ for x any unit vector in the given line.

Next generalize (from $\mathbb{P}V$) to a Grassmannian $\{p : p^* = p = p^2,\ \mathrm{rank}(p) = d\}$. x becomes a $d \times n$ matrix of orthogonal unit vectors, so that $x^* x = 1$, $x x^*$ is the projection on the span of these unit vectors. The function on the Grassmannian is $\mathrm{tr}(b p \bar{b} \bar{p})$.

You still don't understand $\bar{b}$ and $\bar{p}$. b and p have straightforward meanings: b symmetric bilinear form, p is retract / subspace on V.

Viewpoint: Consider V pos. herm space + $b^t = b$.

$$V \xrightarrow{b} V^t \xrightarrow{*} V \xrightarrow{b} V^t \xrightarrow{*} V$$

$$x \mapsto (bx)^t = x^t b \longmapsto \overline{bx} = \bar{b}\bar{x} \mapsto (b\bar{b}\bar{x})^t = x^* b^* b \mapsto (\bar{b} b) x$$

$$x \longmapsto \bar{b}\bar{x}$$

$$y^t \overset{*}{\longmapsto} \bar{y} \overset{b}{\longmapsto} (b\bar{y})^t = y^* b$$

So you get $y^t \longmapsto y^* b \longmapsto (y^* b)^* |$?

$$V \xrightarrow{b} V^t \xrightarrow{*} V \xrightarrow{b} V^t \xrightarrow{*} V \xrightarrow{b} V^t$$

$$y^t \longmapsto \bar{y} \longmapsto (b\bar{y})^t = y^* b \longmapsto \bar{b} y \mapsto (b\bar{b} y)^t = y^t(\bar{b} b)$$

(349) Here's a good picture of the anti-linear map associated to a symmetric b.

$$V \xrightarrow{\;b\;} V^t \xrightarrow{\;*\;} V \xrightarrow{\;b\;} V^t \xrightarrow{\;*\;} V$$

$$x \overset{b\cdot}{\longmapsto} bx \overset{t}{\longmapsto} x^t b \overset{*}{\longmapsto} \bar b \bar x \mapsto b\bar b \bar x \mapsto x^* \bar b b \longmapsto (\bar b b) x$$

So the antilinear map is $T_b x = \bar b \bar x$ and $T_b^2 x = (\bar b b) x$

You also want the antilinear map on V^t.

$$V^t \xrightarrow{\;*\;} V \xrightarrow{\;b\;} V^t \xrightarrow{\;*\;} V \xrightarrow{\;b\;} V^t$$

$$y^t \longmapsto \bar y \overset{b\cdot}{\mapsto} b\bar y \overset{t}{\mapsto} y^* b \longmapsto \bar b y \overset{b\cdot}{\mapsto} b\bar b y \overset{t}{\mapsto} y^t(\bar b b)$$

You want to compare the preceding with the picture [redacted] from the symmetric space: $\mathfrak{p} = \left\{ \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix} : b^t = b \right\}$ with conjugation action by $K = \left\{ \begin{bmatrix} u & 0 \\ 0 & \bar u \end{bmatrix} : u \in U(n) \right\}$. [redacted] You want a spectral decomposition of $X = \begin{bmatrix} 0 & b \\ -\bar b & 0 \end{bmatrix}$. This is a skew adjoint operator, [redacted] so it has a spectral [redacted] decomposition arising from $-X^2 = \begin{bmatrix} b\bar b & 0 \\ 0 & \bar b b \end{bmatrix} = \begin{bmatrix} bb^* & 0 \\ 0 & b^*b \end{bmatrix}$ which is [redacted] self adjoint ≥ 0. Note that $\mathfrak{p}$, K [redacted] operate on [redacted] $\begin{bmatrix} V \\ V^t \end{bmatrix}$. Now use the spectral decomposition of $-X^2$ to split [redacted] the skew adjoint op X on $\mathfrak{p}$ canonically into "blocks" where $-X^2$ is constant ≥ 0. There should be a similar decomposition for your antilinear transformation pictures. Assume b nonsingular, rescale the [redacted] blocks via polar decomp of X so that $X^2 = -1$

(350)   Review:

$$V \xrightarrow{b} V^t \xrightarrow[\sim]{*} V \xrightarrow{b} V^t \xrightarrow[\sim]{*} V$$

$$x \mapsto bx \overset{t}{\mapsto} x^t b \overset{*}{\longmapsto} \bar{b}\bar{x} \mapsto b\bar{b}\bar{x} \overset{t}{\mapsto} x^* b^* b \longmapsto (\bar{b}b)x$$

(Alternative maybe: replace V^t by $\bar{V}$ using $* : V^t \to \bar{V}$, $(y^t)^* = \bar{y}$

$$\begin{array}{ccc} V & \longrightarrow & V^t \\ x & \longmapsto & x^t b \\ & & \downarrow * \\ & & \bar{b}\bar{x} \end{array} \qquad \begin{array}{ccc} V^t & & \\ \downarrow * & & \\ V^{(-)} & \longrightarrow & V^* \\ & & \downarrow * \\ & & V \end{array}$$

not clear

$T_b(x) = \bar{b}\bar{x}$, $T_b^2(x) = (\bar{b}b)x$

Assume now that $T_b^2 =$ $\bar{b}b = b^*b = 1$. T_b is then a real structure on V, i.e. T_b is antilinear of square 1, and its fixpts $x = T_b(x) = \bar{b}\bar{x}$ form a real subspace whose complexification is V.

Try something different, drop $b^t = b$ condition

$$x \mapsto bx \overset{t}{\longmapsto} x^t b^t \overset{*}{\longmapsto} \bar{b}\bar{x} = T_b(x) \qquad (\bar{b}b)x$$

$$T_b(\bar{b}\bar{x}) = \bar{b}\,\overline{\bar{b}\bar{x}} = (\bar{b}b)x \quad \text{better} \quad T_b(\bar{b}\bar{x}) = \bar{b}\,\overline{\bar{b}\bar{x}}$$

Assume that $\bar{b}b = 1$. It seems that you get a real structure - this must be the cocycle condition for descent. Ex $\underline{b = e^{i\theta}}$. $T_b(z) = e^{-i\theta}\bar{z} \overset{?}{=} z$

$$e^{-i\frac{\theta}{2}}\bar{z} = e^{i\frac{\theta}{2}}z \quad \text{or} \quad e^{i\frac{\theta}{2}}z = e^{-i\frac{\theta}{2}}\bar{z} \quad \therefore$$

Fixpt subspace is $\{z \in e^{i\frac{\theta}{2}}\mathbb{R}\}$.

Slight puzzle is that you get a real structure on V from the quaternionic space $H(V)$.

Review things. You want to understand the space of polarizations of $H(\mathbb{C}^n)$. A polarization is an ordered pair of orthogonal Lagrangian subspaces. It should be determined by the 1st " subspace. This is clear from the CCR viewpt.

$$[a_i, a_j] = 0 \;,\; [a_i^*, a_j^*] = 0 \;,\; [a_i, a_j^*] = \delta_{ij}$$

Let $H(V)$ be the complex v.s. having basis a_i, a_i^*. You have a symplectic form on $H(V)$ with $f(a_i, a_j) = [a_i, a_j]$ etc. Conjugation of $*$ on $H(V)$: $*$ is antilinear square $=1$. Conventions are that [redacted] $[v_1, v_2]^* = -[v_1^*, v_2^*]$

How do you [redacted] express the CCR structure?

$$H(V) = V \oplus V^* = V \oplus V \qquad \begin{matrix} a(v) \in V \\ a^*(v) \in V^* \end{matrix}$$

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complex vector space with basis $a_1, \dots, a_n, a_1^*, \dots, a_n^*$ with skew symmetric form: $0 = [a_j, a_k] = [a_j^*, a_k^*]$, $[a_j, a_k^*] = \delta_{jk}$.

You should have $[a, a']^* = -[a'^*, a^*]$

$$\delta_{jk} = [a_j, a_k^*]^* = [a_k, a_j^*] = \delta_{kj} = \delta_{jk}$$

So we should have what? A $2n$ diml v.s, a symplectic form $\omega(x, y)$, also anti-linear inv. $*$.

$\overline{\omega(x, y)} = \omega(y^*, x^*)$. Once you get the structure correct? A $2n$-diml complex symplectic space H + $*$ antilinear involution on H s.t. $\overline{\omega(x,y)} = \omega(y^*, x^*)$

Then use unitary equivalence theory for skew-symmetric forms. Possible problem here is how to ensure that the real structure given by $*$ is compatible. What's going on is that you're given the

352 [redacted] The symplectic form and a conjugation $*$. [redacted] You should first understand possible conjugations on a complex V.

Look carefully at $[a_j, a_k] = 0 = [a_j^*, a_k^*]$,

$[a_j, a_k^*] = \delta_{jk} = [a_k, a_j^*]$. You have a space $V \oplus V^*$ with basis a_j, a_k^* $1 \le j, k \le n$. and a skew symm. form

You haven't made precise the aim. [redacted] It seems that you have a $2n$ dim'l $\mathbb{C}$ v.s. which is hyperbolic i.e. $\begin{bmatrix} V \\ V^* \end{bmatrix}$

You want good formulation of the structure.

$W = \{ x^j a_j + y^k a_k^* \} \ni \begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} a \\ a^* \end{bmatrix}$

$$(x^j a_j + y^k a_k^*)^* = \bar{x}^j a_j^* + \bar{y}^k a_k = \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^t \begin{bmatrix} a \\ a^* \end{bmatrix}$$

when is $\sum x^j a_j + y^j a_j^*$ real?

$$\Big\downarrow *$$

$\sum \bar{x}^j a_j^* + \bar{y}^j a_j$

when $\boxed{y^j = \bar{x}^j}$ or $\boxed{\begin{matrix} y = \bar{x} \\ \bar{y} = x \end{matrix}}$

Next $[x^j a_j + y^j a_j^*, \; x^k a$

(353) You are trying to understand the symmetries of the CCR. You have a [redacted] vector space $H/\mathbb{C}$ with basis a_j, a_j^* for $1 \le j \le n$, and equipped with two structures:

1) antilinear involution $*$ defined by

$(a_j)^* = a_j^*$, $(a_j^*)^* = a_j$.

2) skewsymm bilinear form [redacted] $[\xi, \eta]$ $/\mathbb{C}$ defined by

$$[a_j, a_k] = [a_j^*, a_k^*] = 0, \quad [a_j, a_k^*] = \delta_{jk} = -[a_k^*, a_j]$$

So you this v.s. H with basis a_j, a_k^* of $2n$ elts. You have real structure given by fixpt subspace of $*$.

Obvious question whether $[\xi,\eta] \in$ [redacted] $\mathbb{R}$ when $\xi^* = \xi$ and $\eta^* = \eta$. From the operator interp. you have the following link between $*$ and $[\xi,\eta]$:

$$[\xi,\eta]^* = (\xi\eta - \eta\xi)^* = \eta^*\xi^* - \xi^*\eta^* = -[\xi^*,\eta^*]$$

or $\boxed{[\xi,\eta]^* = [\eta^*, \xi^*]}$ $\quad [a_j, a_k^*]^* = [a_k, a_j^*]$

$$\begin{array}{ccc} [a_j, a_k^*]^* & = & [a_k, a_j^*] \\ \| & & \| \\ \delta_{jk} & & \delta_{kj} \end{array}$$

If ξ, η are real $\xi = \xi^*, \eta = \eta^*$ then $[\xi,\eta]^* = [\eta,\xi] = -[\xi,\eta]$

so $[\xi,\eta] \in i\mathbb{R}$. interesting

[redacted] Let $n = 1$,

$H = \{ xa + ya^* \mid x, y \in \mathbb{C} \}$. Then $[x_1 a + y_1 a^*, x_2 a + y_2 a^*]$

$$= x_1 y_2 - y_1 x_2$$

$$= \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$(xa + ya^*)^* = \bar{x} a^* + \bar{y} a$

antilinear sqr $= 1$: $\begin{bmatrix} x \\ y \end{bmatrix}^* = \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix}$

So you get something unexpected, namely that the $*$ operator on the space of creation + annihilation operators seems different? Go over this with a review of $H(V)$ and its structures: $H(V) = \begin{bmatrix} V \\ V^t \end{bmatrix}$ with symp form.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{J} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1^t & y_1^t \end{bmatrix} \begin{bmatrix} y_2 \\ -x_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$$

$g \in Sp(2n, \mathbb{C})$: $g^t J g = J$ inf: $X^t J + J X = 0$

$-X^t = J X J^{-1}$

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \; -X^t = \begin{bmatrix} -a^t & -c^t \\ -b^t & -d^t \end{bmatrix} = J X J^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} c & d \\ -a & -b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \qquad b = b^t, \; c = c^t, \; d = -a^t$$

~~$g \in U(2n)$ $g^* g = 1$ inf $X^* + X = 0$, $X = \begin{bmatrix} a & b \\ -b^* & -a^t \end{bmatrix}$~~

~~$\therefore X = \begin{bmatrix} a & b \\ -b & d \end{bmatrix}$: $a^* = -a$, $d^* = -d$ $-b$ $d = -a^t$~~

$$X^* + X = 0 \qquad X = \begin{bmatrix} a & b \\ -b^* & -a^t \end{bmatrix} \qquad a^* = -a, \; -a^t = +\bar{a}, \quad b^* = \bar{b} \text{ as } b^t = b.$$

$$\therefore X = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \qquad \text{3 cond. } X^* + X = 0, \; X^t J + J X = 0, \; J X = \bar{X} J$$

Start again with H, its symplectic form, and $*$ the antilinear involution:

$$\begin{bmatrix} x \\ y \end{bmatrix}^* = \begin{bmatrix} \bar{y} \\ \bar{x} \end{bmatrix}.$$

Now combine the symplectic form and $*$. $H \xrightarrow{J} H^t \xrightarrow{*} H$? to get a hermitian form.

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Calculate: $[x_1 a + y_1 a^*, x_2 a + y_2 a^*] = x_1 y_2 - y_1 x_2$

This is the symplectic form on $H = \{xa + ya^*\}$.

Next $(xa + ya^*)^* = \bar{x} a^* + \bar{y} a$. Try

$$[x_1 a + y_1 a^*, \bar{x}_2 a^* + \bar{y}_2 a] = x_1^t \bar{x}_2 - y_1^t \bar{y}_2$$

You probably want the antilinear part on the left.

$$[(xa + ya^*)^*, x_2 a + y_2 a^*]$$

$$= [\bar{x} a^* + \bar{y} a, x_2 a + y_2 a^*] = \bar{y}^t y_2 - \bar{x}^t x_2$$

Start again $[xa + ya^*, x_1 a + y_1 a^*] = x y_1 - y x_1$

$(xa + ya^*)^* = \bar{y} a + \bar{x} a^*$

$[(xa + ya^*)^*, (x_1 a + y_1 a^*)] = [\bar{x} a^* + \bar{y} a, x_1 a + y_1 a^*] = \bar{y} y_1 - \bar{x} x_1$

or $[xa + ya^*, \bar{x}_1 a^* + \bar{y}_1 a] = x \bar{x}_1 - y \bar{y}_1$

What you've done is to compare the antilinear op $(xa + ya^*) \xmapsto{*} \bar{y} a + \bar{x} a^*$ with the symplectic form to get a hermitian form (non pos.)

What's the symplectic form restricted to real elts.

and you have the anti-involution

$$* : \quad xa + ya^* \longmapsto (xa+ya^*)^* = \bar y a + \bar x a^*$$

You can combine this antilinear operator with the symplectic form to get a sesquilinear form.

$$[(x_1a+y_1a^*)^*,\ x_2a+y_2a^*] = \overbrace{[\bar y_1 a + \bar x_1 a^*,\ x_2a+y_2a^*]}^{\bar y_1 y_2 - \bar x_1 x_2}$$

which is hermitian symmetric.

Another point is $\overline{[z_1, z_2]} = [z_2^*, z_1^*]$; this comes comes from the operator interpretation of the bracket and should be checked: $[z_1^*, z_2^*] = -\overline{[z_1,z_2]}$?

$$[z_1^*, z_2^*] = [\bar y_1 a + \bar x_1 a^*,\ \bar y_2 a + \bar x_2 a^*] = \bar y_1 \bar x_2 - \bar x_1 \bar y_2$$

$$[z_1,z_2] = x_1y_2 - y_1x_2, \quad \overline{[z_1,z_2]} = \bar x_1 \bar y_2 - \bar y_1 \bar x_2 . \quad \therefore \text{OK}$$

So now you want to leave $Sp(2n)$ and $SO(2n)$ and move on to periodicity.

Idea: Symmetries of the CCR. You know that these should give the real symplectic group. Let's check. You want $g \in GL(2,\mathbb{C})$ respecting the symplectic form — which means $g \in SL(2,\mathbb{C})$ — but also g should resp the hermitian form $\bar x_1 x_2 - \bar y_1 y_2 = \begin{bmatrix}x_1\\y_1\end{bmatrix}^*\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix}$

$\begin{bmatrix}x_1\\y_1\end{bmatrix}^t\begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}x_2\\y_2\end{bmatrix} = x_1y_2 - y_1x_2$. So if $g = \begin{bmatrix}a&b\\c&d\end{bmatrix}$, then

$\underline{g^tJg = J \Rightarrow (\det g)^2 = 1}$ $\quad g^t = \begin{bmatrix}a&c\\b&d\end{bmatrix} = \begin{bmatrix}0&1\\-1&0\end{bmatrix}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix}$

$= \begin{bmatrix}-c&a\\-d&b\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}a&c\\b&d\end{bmatrix}$ if $\det g = 1$, this only condition; $\det g = -1$ none

 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$g^t J g = J \iff g \in SL(2,\mathbb{C})$$

Next, suppose you have $g^* \varepsilon g = \varepsilon \Rightarrow g^* = \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = \varepsilon \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \varepsilon = \begin{bmatrix} d & +b \\ c & a \end{bmatrix}$

so $d = \bar a$, $+b = \bar c$, $+c = \bar b$, $a = \bar d$ $\therefore g = \begin{bmatrix} a & b \\ \bar b & \bar a \end{bmatrix}$; $|a|^2 - |b|^2 = 1$.

Check again $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{C})$, assume $g^* \varepsilon g = \varepsilon$

i.e. $g^* = \varepsilon g^{-1} \varepsilon = \varepsilon \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \varepsilon = \begin{bmatrix} d & b \\ c & a \end{bmatrix} \overset{?}{=} \begin{bmatrix} \bar a & \bar c \\ \bar b & \bar d \end{bmatrix} = g^*$

Go back to real periodicity thm. via Morse theory, where you have problems with real Clifford algebras. Passing from Cliff$_n$ to Cliff$_{n+1}$

$\mathbb{Z} \times BO$. Try to describe the spaces which occur - these are symmetric spaces - a homogeneous space of a Lie group by the centralizer of an involution.

0th space is	$\mathbb{Z} \times BO$	infinite real Grass.	$\pi_0 = \mathbb{Z}$
1th " "	$\Omega BO = O$		$\pi_1 = \mathbb{Z}/2$
2nd " "	ΩSO	spinor gp.	$\pi_2 = \mathbb{Z}/2$

What is the rough idea? You consider the loop space of the symmetric space and find a nice family of geodesics going from the basepoint to some "antipodal" point. The symmetry group (of the symm. sp.) should act transitively on this family of geodesics. This isn't clear. Example needed.

Go back to $O(2n,\mathbb{C})$

$$H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix} \right\}, \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{S} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 - y_1^t x_2$$

$$O(2n,\mathbb{C}) = \{ g \in GL(2n,\mathbb{C}) \mid g^t S g = S \}. \quad n=1 \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$g^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = S g^{-1} S = \frac{1}{\det(g)} S \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} S = \frac{1}{\det g} \begin{bmatrix} a & -c \\ -b & d \end{bmatrix}$$

If $\det(g) = 1 \implies b = c = 0 \implies g = \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$

If $\det(g) = -1 \implies a = d = 0 \implies g = \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix}$ dihedral picture.

$$\mathcal{L}\, O(2n,\mathbb{C}) = \{ X \in M_{2n}(\mathbb{C}) : X^t S + S X = 0 \}$$

$$-X^t = \begin{bmatrix} -a^t & -c^t \\ -b^t & -d^t \end{bmatrix} = SXS = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \qquad b^t = -b,\ c^t = -c \quad d = -a^t$$

$$\mathcal{L}\, U(2n) = \{ X \in M_{2n}(\mathbb{C}) : \underbrace{X^* + X = 0}_{-X^* = X \implies -X^t = \bar{X}} \}, \text{ both conds}$$

$$\implies SX = -X^t S = \bar{X} S$$

$$\implies X = S\bar{X}S \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \bar{d} & \bar{c} \\ \bar{b} & \bar{a} \end{bmatrix} \quad \therefore X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \text{ with } \boxed{b^t = -b} \quad \boxed{a^* = -a}$$

$\bar{a} = -a^t$

Def. $O(2n) = U(2n) \cap O(2n,\mathbb{C}) = \{ g \in U(2n) \mid g^t S g = S \}$

$$\mathcal{L}\, O(2n) = \left\{ X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = -b \end{matrix} \right\}. \qquad \text{Next}$$

You want ~~geodesics~~ in $O(2n)$ going from 1 to -1

The family of minimal geodesics. You want the maximal torus and Cartan subalg. Try

$$e^{i\theta\varepsilon} = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \qquad 0 \le \theta \le \pi$$

space of these geodesics is $O(2n)/$ centralizer of $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\varepsilon$ which should be $\left\{ \begin{bmatrix} u & 0 \\ 0 & \bar{u} \end{bmatrix} : u \in U(n) \right\}$.

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Next you want to understand real Clifford algebras. But first you might try the next Bott case. So far you have a heq with increasing range

$$\Omega(SO(2n), 1, -1) \longleftrightarrow O(2n)/U(n)$$

two components. two components

So let's try to understand the space $O(2n)/U(n)$ which should be the h-fibre of $BU(n) \longrightarrow BO(2n)$

Idea: These symmetric spaces you encounter such as $O(2n)/U(n)$, $Sp(2n)/U(n)$, $U(n)/O(n)$, and $U(2n)/Sp(2n)$ - do they describe polarizations of some sort? This should fit with Clifford algebras easily.

Let's try to clarify Clifford algebras. Def.

$\mathrm{Cliff}(\mathbb{R}^n)$. Consider $\mathbb{C}$ as a $\mathbb{Z}/2$ graded algebra over $\mathbb{R}$ where $\mathbb{R}\cdot 1$ is even and $\mathbb{R}i$ is odd.

$\therefore$ Basic odd-even grading given by complex conjugation.

$$\mathrm{Cliff}_n = \underbrace{\mathbb{C}\otimes\mathbb{C}\otimes\cdots\otimes\mathbb{C}}_{n}$$ tensor product of superalgs.

Do this ind: Let $A = A^+ \oplus A^-$ be a superalg then $A\otimes\mathbb{C} = \begin{bmatrix} A^+\otimes 1 & A^-\otimes 1 \\ A^+\otimes i & A^-\otimes i \end{bmatrix}$?

$\mathbb{R}$ $\quad\mathbb{C}\quad$ $\mathbb{C}\otimes\mathbb{C}$

$1\otimes 1$
$i\otimes 1$
$1\otimes i$
$i\otimes i$

take defn Cliff_n Clifford alg of $\mathbb{R}^n$ usual Euclidean product. Corresp to basis vector $e_1, \ldots, e_n$

(361) Clifford alg defined like exterior alg

~~generators $s_1, \dots, s_n$ relations s_i^2~~ $\forall x \in \mathbb{R}^n$ you have $s(x) \in \mathrm{Cliff}(\mathbb{R}^n)$ linear in x satif $s(x)^2 = -(x,x)$. Then

$$s(x+y)^2 = (s(x)+s(y))^2 = -(x,x) + s(x)s(y) + s(y)s(x) - (y,y)$$

$\shortparallel$ $-(x+y,x+y)$ $\therefore$ get $s(x)s(y)+s(y)s(x) = -2(x,y)$

Enough to know $s_i = s(e_i)$ e_i the i-th basis elt.

$s(e_i)^2 = -(e_i,e_i) = -1$

$s : \mathbb{R}^n \longrightarrow A$ linear $s(x)^2 = -(x,x)$

$$s(x+y)^2 = (s(x)+s(y))^2 = \underbrace{s(x)^2}_{-(x,x)} + s(x)s(y)+s(y)s(x) + \underbrace{s(y)^2}_{-(y,y)}$$

$\shortparallel$ $-(x+y,x+y)$

$\therefore$ $\{s(x), s(y)\} = -(x,y)-(y,x)$

So $s(x), s(y)$ anti-commute when $(x,y)=0$.

Also if $(x,x)=1$, then $s(x)^2 = -1$. $\therefore$ an orthogonal basis yielding anticommuting operators of square $= -1$.

V, W real v.s. equipped with nondeg quadratic form

$C(V)$ $\mathbb{R}$-alg A generated by $s(v)$, $v \in V$ subject to the relations s linear/$\mathbb{R}$, $s(v)^2 = (v,v)$

$\Longrightarrow$ $s(v)s(v') + s(v')s(v) = 2(v,v')$

$\Longrightarrow$ $s(v), s(v')$ anti commute $\Longleftrightarrow$ $v \perp v'$.

So you want

(362) V quadratic space over $\mathbb{R}$, ~~non deg~~

Def $C(V)$ = alg A gen by $s: V \longrightarrow A$ sat s linear$/\mathbb{R}$, $s(v)^2 = (v,v)$.

$$s(v+v')^2 = \big(s(v)+s(v')\big)^2 = s(v)^2 + s(v)s(v') + s(v')s(v) + s(v')^2$$
$$\Vert$$
$$(v+v', v+v') = \cancel{(v,v)} + (v,v') + (v',v) + \cancel{(v',v')}$$

$$\Longrightarrow \quad s(v)s(v') + s(v')s(v) = 2(v,v')$$

Prop. For orth $\oplus$ $\quad C(V\oplus W) = C(V)\hat{\otimes} C(W)$ where $\hat{\otimes}$ means superalgebra $\otimes$. Sylvester says any quadratic space over $\mathbb{R}$ is determined up to isom by dim and signature

$$\sum_{i=1}^{p} x_i^2 - \sum_{j=1}^{q} y_j^2 \qquad \begin{aligned} \text{signature} &= p-q \\ \text{dim} &= p+q. \end{aligned}$$

$$C^{1,0} = Cl(\mathbb{R}, (x,x)=x^2) \cong \mathbb{R}[s, s^2=1] = \mathbb{R}[\varepsilon].$$

$$C^{p,0} = \mathbb{C}[s_1]\hat{\otimes}\cdots\hat{\otimes}\mathbb{C}[s_p] \cong \Lambda_{\mathbb{R}}(\mathbb{R}^p) \quad \text{exterior alg.}$$
NO

Point of Clifford ~~algebras~~ modules is to construct Thom classes. In the complex case you have $\Lambda V = \Lambda\mathbb{C}^n$ equipped with $e(v) + i(v^*)$ operators, satisfying

$$\big(e(v)+i(v^*)\big)^2 = e(v)i(v^*) + i(v^*)e(v) = v^*v$$

$$i(v^*)e(v)w = i(v^*)(v\wedge w) = (v^*v)w - \underbrace{v\wedge(i(v^*)w)}_{e(v)i(v^*)w}$$

In this example the Clifford module is ΛV with $\mathbb{Z}/2$-grading and the self adjoint operators $e(v)+i(v^*)$. If $n=1$ $\quad v = z,\ v^* = \bar{z}$

$$\Lambda^0\mathbb{C} \underset{z}{\overset{\bar{z}}{\rightleftarrows}} \Lambda^1\mathbb{C}$$

363 Clifford modules, a tool for understanding real Bott periodicity. [redacted] Start with a real vector space V equipped with a non-degenerate quadratic form. Define $\mathrm{Cliff}(V)$ to the alg [redacted] over $\mathbb{R}$ generated by an $\mathbb{R}$-linear map $s : V \longrightarrow \mathrm{Cliff}(V)$ subject to the relation $s(v)^2 = (v, v) \quad \forall v \in V$. Then [redacted] by polarization you have

$$s(v)s(v') + s(v')s(v) = 2(v, v')$$

whence $s(v)$ and $s(v')$ anticommute $\Longleftrightarrow (v, v') = 0$. From this you deduce: [redacted]

If V is the orthogonal direct sum of subspaces V' and V'' then one [redacted] has a canon isom of $\mathbb{R}$ algs

$$C(V') \hat{\otimes} C(V'') \xrightarrow{\sim} C(V)$$

Describe: If $V = V' \overset{\perp}{\oplus} V''$, then the quadratic form on V restricts to non-degenerate quad forms on V' and V''. Why: $V = V' \overset{\perp}{\oplus} V''$ means that

[redacted] $V = V' \oplus V''$ and $(v', v'') = 0 \quad \forall v' [redacted] \in V', v'' \in V''$

[redacted]

What does it mean for $(\; , \;)_V$ to be nondegnerate? [redacted] Ans. the map

$$V \longrightarrow V^t$$

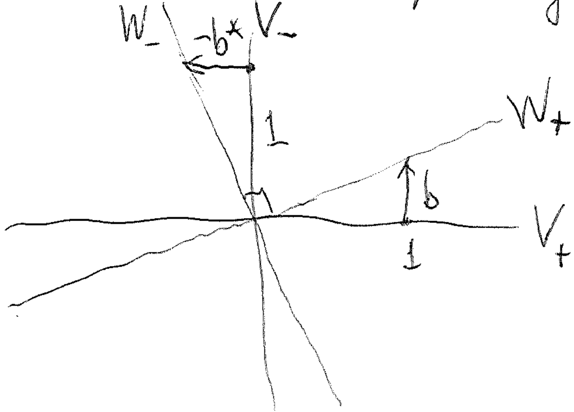
$$v \longmapsto \{v_1 \longmapsto (v, v_1)\}$$

is bijective. By f.d. enough to show that $v \neq 0 \Longrightarrow \exists\, v_1$ s.t. $(v, v_1) \neq 0$. You can factor this map

$$V' \oplus V'' \xrightarrow[\sim]{} V \longrightarrow V^t \xrightarrow[\sim]{} (V')^t \oplus (V'')^t$$

into a 2×2 matrix of maps $\begin{bmatrix}(V')^t\\(V'')^t\end{bmatrix} \xleftarrow{[\ \ ]} \begin{bmatrix}V'\\V''\end{bmatrix}$ and the point is that the off-diagonal maps are 0 because $V'\perp V''$ (and $V''\perp V'$). Since the matrix is invertible so must be the maps $(V')^t \leftarrow V'$, $(V'')^t \leftarrow V''$, and so $(\ ,\ )_V$ ~~when~~ restricted to both V' and V'' is non degenerate.

Digression: Consider $H = \begin{bmatrix}V_+\\V_-\end{bmatrix}$ symplectic equipped with another polarization close to the given one.



$$[W_+\ \ W_-] = \begin{bmatrix}1 & -b^*\\ b & 1\end{bmatrix}\begin{bmatrix}V_+\\V_-\end{bmatrix}$$

$F = \pm 1$ on $W_\pm$ $\qquad X = \begin{bmatrix}0 & -b^*\\ b & 0\end{bmatrix}$

$$F(1+X) = (1+X)\varepsilon$$

$$F\varepsilon(1-X) = (1+X), \quad g = F\varepsilon = \frac{1+X}{1-X}$$

and also $g^{1/2} = \dfrac{1+X}{(1-X^2)^{1/2}}$. Since W_+ is Lagrangian you know that $b = b^t$ and conversely. Note than that W_- is Lagrangian because $-b^* = -\bar{b}$ is symmetric. (This shows clearly that W_+ Lagrangian $\Leftrightarrow (W_+)^\perp$ Lagrangian)

The remaining point you would like to check is that $g^{1/2} \in Sp(2n)$ i.e. $u^t J u = J$ where $u = g^{1/2}$. First look at $g = \dfrac{1+X}{1-X}$.

Let's work out the details. Can you set up a D.E?

$$g_t = \frac{1+tX}{1-tX} \qquad \dot{g} = \frac{(1-tX)(X)+(1+tX)(+X)}{(1-tX)^2} = \frac{2X}{(1-tX)^2}$$

$$g^{-1}\dot{g} = \frac{\cancel{1-tX}}{1+tX}\,\frac{2X}{(1-tX)^{\cancel{2}}} = \frac{2X}{1-t^2X^2}$$

$u_t = \frac{1+tX}{(1-t^2X^2)^{1/2}}$. Begin again.

$g = F\varepsilon = \frac{1+X}{1-X}$, make path $g_t = \frac{1+tX}{1-tX}$, then

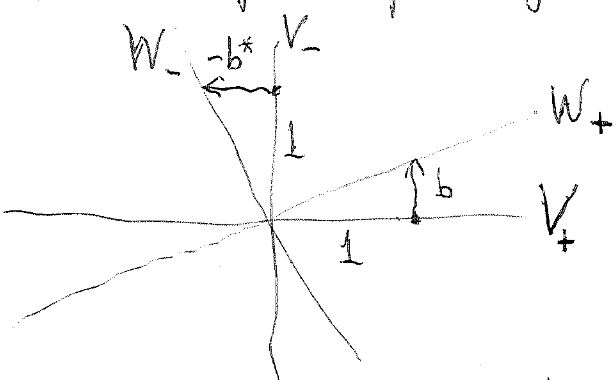
$$\dot g = \frac{(1-tX)(X) - (1+tX)(-X)}{(1-tX)^2} = \frac{2X - tX^2 + tX^2}{(1-tX)^2} = \frac{2X}{(1-tX)^2}$$

$$g^{-1}\dot g = \frac{\cancel{1-tX}}{1+tX}\,\frac{2X}{(1-tX)^{\cancel{2}}} = \frac{2X}{1-t^2X^2}.$$

What's interesting here is that g and $\dot g$ are functions of X, so you should be able to study what happens by decomposing X by means of its eigenspaces. Suppose you restrict to the eigenvalue ia. Look at a simple case. $X = \begin{bmatrix}0 & -\bar b\\ b & 0\end{bmatrix}$

$$g^{1/2} = \begin{bmatrix}1 & -\bar b\\ b & 1\end{bmatrix}(1+|b|^2)^{-1/2}$$

Repeat. You considered a polarzn ~~...~~ close to the basepoint polarization



$W_+ = \begin{bmatrix}1\\ b\end{bmatrix}V_+$ $\quad W_- = \begin{bmatrix}-b^*\\ 1\end{bmatrix}V_-$

$F = \pm 1$ on $W_\pm$, then

$$F\begin{bmatrix}1 & -b^*\\ b & 1\end{bmatrix} = \begin{bmatrix}1 & -b^*\\ b & 1\end{bmatrix}\varepsilon$$

$$F\underset{\varepsilon(1-X)}{(1+X)\varepsilon} = (1+X)\overset{1}{\overbrace{\varepsilon\varepsilon}} \qquad F\varepsilon = g = \frac{1+X}{1-X}, \quad g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

Question: So far you've looked the unitary (or Hilbert space) picture. Next consider the symplectic structure. Then W_+ is Lagrangian, so you know $b^t = b$. This implies also that $-b^* = -\bar b$ is symmetric, so W_- is Lagrangian. You now want to show that $g^{1/2} \in Sp(2n)$. Let $u = g^{1/2}$; you know $u \in U(2n)$ and need to show $u^t J u = J$.

$$[x_1 a + y_1 a^*, x_2 a + y_2 a^*] = x_1 y_2 - y_1 x_2$$

$$(xa + ya^*)^* = \bar{y} a + \bar{x} a^* \qquad xa + ya^* \text{ s. adj} \iff \bar{x} = y.$$

$$[x_1 a + \bar{x}_1 a^*, x_2 a + \bar{x}_2 a^*] = x_1 \bar{x}_2 - \bar{x}_1 x_2$$

Review You noticed that the CCR yield a symplectic form and a real structure (anti-involution $*$) which is not the same as what you get from the basic representation of $Sp(2n)$.

Recall in case $n=1$. The basic repn is $\mathbb{C}^2$ equipped with ~~pos. herm. form~~ symplectic form

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{J} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 y_2 - y_1 x_2$$

and the positive herm. form $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \bar{x}_1 x_2 + \bar{y}_1 y_2$

The "quotient" of these 2 forms is an antilinear operator on H.

$$H \longrightarrow H^t \xrightarrow{\ *\ } H$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = z \longmapsto z^t J \longmapsto -J\bar{z} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} -\bar{y} \\ \bar{x} \end{bmatrix}$$

having square $= -1$.

In the case of the CCR you have the $\mathbb{C}$-vector space of elts: $xa + ya^*$, $x, y \in \mathbb{C}$ with symp. form $[x_1 a + y_1 a^*, x_2 a + y_2 a^*] = x_1 y_2 - y_1 x_2$

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The basic problem seems to be how invariant the situation is. ? You start with $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ which is skew-adjoint. Note that $b^t = b \Rightarrow -b^* = -\bar{b}$, so $X = \begin{bmatrix} 0 & -\bar{b} \\ b & 0 \end{bmatrix}$. Recall $\mathcal{L}\,Sp(2n) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} : \begin{matrix} a^* = -a \\ b^t = b \end{matrix} \right\}$.

Therefore $X^t J + JX = 0$, so X is an inf. symmetry of the symplectic structure. You want to go from this fact to show $g = \dfrac{1+X}{1-X}$ and $u = \dfrac{1+X}{(1-X^2)^{1/2}}$ are global symmetries of the symplectic structure.

$u^t J u = J \iff Ju = \bar{u} J$

You know that $JX = (-X^t) J = \bar{X} J$ because $\underbrace{X + X^* = 0 \iff \bar{X} + X^t = 0}$

Therefore $J(1+X) = (1+\bar{X}) J$, $J(1-X) = (1-\bar{X}) J$

$J(1-X)^{-1} = (1-\bar{X})^{-1} J$.

$$J g J^{-1} = J(1+X)(1-X)^{-1} J^{-1} = (1+\bar{X})(1-\bar{X})^{-1} = \bar{g}$$

$$J(1-X)(1+X) J^{-1} = (1-\bar{X})(1+\bar{X}), \quad J(1-X^2) J^{-1} = (1-\bar{X}^2)$$

There should be a clearer way to proceed, probably by working in the algebra of $X \in gl(2n, \mathbb{C})$ such that $J X J^{-1} = \bar{X}$.

$u^* u = 1 \qquad u^t \bar{u} = 1.$

Review. $H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$ equipped with

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^* x_2 + y_1^* y_2, \qquad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{matrix} x_1^t y_2 \\ -y_1^t x_2 \end{matrix}$$

pos herm form. symplectic form

difference ratio

$$H \xrightarrow{J} H^t \xrightarrow{*} H$$

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} x^t & y^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -y^t & x^t \end{bmatrix}$$

$$\begin{bmatrix} -y^t & x^t \end{bmatrix} \overset{*}{\longmapsto} \begin{bmatrix} -\bar y \\ \bar x \end{bmatrix}$$ operator amounting to an $\mathbb{H}$-structure on H.

Look at operators on H ~~preserving~~ $X^* + X = 0$

inf symmetries $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad X^t J + JX = 0$

$\mathcal{L}(Sp(2n)) = \{ \; : \; a^* = -a, \; b^t = b \}$, $\quad JX = \bar X J$

You want to show that the C.T. of $X = \begin{bmatrix} 0 & -\bar b \\ b & 0 \end{bmatrix}$, $b^t = b$ lies in $Sp(2n)$. Better: you know that

$$u = \frac{1+X}{(1-X^2)^{1/2}} \in U(2n).$$ You want $u^t J u = J$ or $J u = \bar u J$

So you want to show that $J u J^{-1} = \bar u$. Important point is that $\{X : JXJ^{-1} = \bar X\}$ is an algebra, a f.d. algebra; probably $M_n \mathbb{H}$. It should be true that this alg is $\left\{ \begin{bmatrix} a & -\bar b \\ b & \bar a \end{bmatrix} : a, b \in M_n \mathbb{C} \right\}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & -\bar b \\ b & \bar a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & \bar a \\ -a & \bar b \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \bar a & -b \\ \bar b & a \end{bmatrix}$$

(369) Let's see if we now understand polarizations. Let's do the orthogonal case. $H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix}$. Two forms.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^* \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^* x_2 + y_1^* y_2 , \quad \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^t y_2 + y_1^t x_2$$

$X \in M_{2n}\mathbb{C} = \overset{\mathbb{C}\text{-linear}}{\text{operators}}$ on H. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a, b, c, d \in M_n\mathbb{C}$

(i) $X^* + X = 0$

(ii) $X^t S + SX = 0$

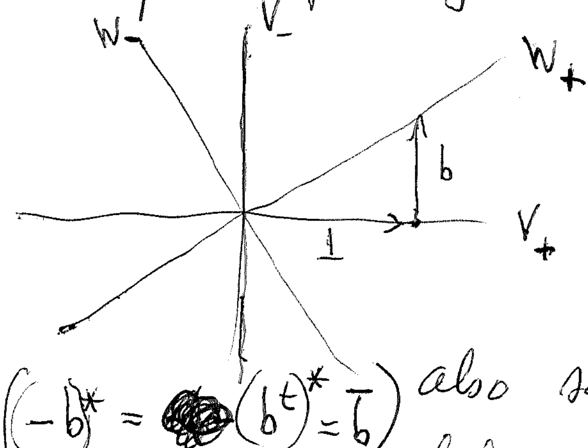
(iii) $SXS = \bar{X}$

(iii) becomes $\begin{bmatrix} d & c \\ b & a \end{bmatrix} = \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{bmatrix}$, $X = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}$

(i) becomes $a^* = -a$, $(\bar{a})^* = -\bar{a}$, $c = -b^*$

but from (iii) $c = \bar{b} \Rightarrow b = -b^t$

Next consider a polarization close to the basepoint polarization: Again in the unitary picture you have



$$g = F\varepsilon = \frac{1+X}{1-X}, \quad u = \frac{1+X}{(1-X^2)^{1/2}}$$

Now impose the condition W_+ is Lagrangian, then $X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix}$ must also satisfy $b^t = -b$ so $X = \begin{bmatrix} 0 & \bar{b} \\ b & 0 \end{bmatrix}$ belongs to alg sat $SXS = \bar{X}$. Therefore u should lie in this alg.

$\left(-b^* = (b^t)^* = \bar{b}\right)$