Somehow you have to get back on the track. You're stuck on finding a clean picture of $H(V)$ with its three structures.

$V = \mathbb{C}^n$ (space of column vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$) equipped with Frobenius form $x^*y = \sum \bar{x}_i y_i$.

$V^t = \mathbb{C}^n$ (space of row vectors)

This is not making sense. Go back to the abstract situation. $V$ f.d. $\mathbb{C}$ v.s. $V^t = Hom_{\mathbb{C}}(V, \mathbb{C})$

Bilinear pairings $V \times V^t \rightarrow \mathbb{C}$

Maybe you should make bilinear form a basic object.

\[
\begin{array}{ccl}
V \times W & \overset{B(-,-)}{\longrightarrow} & \mathbb{C} \\
V & \overset{\alpha}{\longrightarrow} & W^t \\
W & \overset{\beta}{\longrightarrow} & V^t
\end{array}
\]

TFAE

Claim $\alpha^t = \beta$ ? Take $\mu \in (W^t)^t$ such that

$\mu_\lambda = \lambda_{\mu}(\lambda) = \lambda_{B(-,-)}(\lambda, \mu)$. 

$\alpha^t(\mu_{\mu}) = \mu_{\mu^t} = B(\mu, \mu)$. 

You know $\mu = \mu_\lambda$. 

$\alpha^t(\mu_{\mu}) = \mu_{\mu^t} = B(\mu, \mu)$. 

$V \overset{\alpha}{\rightarrow} W \overset{\mu}{\rightarrow} \mathbb{C}$

$V \overset{\alpha}{\rightarrow} B(\nu, -) \overset{\mu_{\mu}}{\rightarrow} B(\nu, \mu)$. 

$V \overset{\alpha}{\rightarrow} W \overset{\mu_{\mu}}{\rightarrow} \mathbb{C}$

$V \overset{\alpha}{\rightarrow} B(\nu, -) \overset{\mu_{\mu}}{\rightarrow} B(\nu, \mu)$.
TFAE
\[ V \times W \xrightarrow{B} C \]
\[ V \xrightarrow{\alpha} W^t \]
\[ W \xrightarrow{\beta} V^t \]

Bilinear
\[ v \mapsto B(v, \cdot) \]
\[ w \mapsto B(\cdot, w) \]

\[ V^t \xleftarrow{\alpha^t} (W^t)^t = W \]
\[ (\lambda \mapsto \lambda(x)) \leftarrow w \]
\[ \alpha^t(\lambda)(v) = \lambda \alpha(v) \]

Back to \( H(V) = [V^t] \). You made some progress by walking. Your aim is to understand properly the notion of polarizations for \( H(V) \). The simplest description should be a Lagrangian subspace. Hence the space of polarizations of \( H(V) \) is the minimal flag manifold for \( \text{Sp}(2n) \). Another idea is that a polarization of \( H(V) \) is roughly an isomorphism of \( H(V) \) with itself. More precisely, given a polarization of \( H(V) \), i.e., a Lagrangian subspace \( W \subset H(V) \), then you should have \( W \perp \text{Lagrangian} \), so \( H(V) = W \oplus W^\perp = H(W) \cong H(V) \).

Where the latter arises from choosing an isomorphism \( W \cong V \). This is confused. What point of view should you use? If you are given a polarization \( W \subset H(V) \), this should be the same as an isomorphism \( H \otimes W \cong H(V) = H \otimes V \). Can you identify the space of polarizations with \( \text{Sp}(2n)/U(n) \)? This should be clear.
embedding \( W \hookrightarrow H \otimes V \) extends uniquely to an \( H \)-module map \( H \otimes W \to H \otimes V \) which is an isomorphism since \( W, V \) have the same dim = \( n \).

You are leaving out a lot of details, which have eventually to be checked.

An important point to be understood involves the inner product on \( H(V) \). You believe that this inner product is a consequence of the symplectic and \( H \)-module structures.

Today you want to clean up polarization. You want to study the space of polarizations of \( H(V) \). Polarization = Lagrangian subspace should be true, ??

There are problems with the \( H \) structure.

Q: Is the orthogonal complement for a Lagrangian \( L \) again Lagrangian? [unclear inner product]

Let's return to the situation where you have both a symplectic structure and an inner product structure on a complex vector space \( V \). Choose a basis for \( V \) so that \( V = \mathbb{C}^n \) (columns) \( V^t = \mathbb{C}^n \) (rows), then the symplectic is a map \( A : V \to V^t, \ x \mapsto x^tA \quad A^t = -A. \)

The comp. \( T = *A \)

\[
\begin{array}{ccc}
V & \xrightarrow{A} & V^t & \xrightarrow{*} & V \\
\end{array}
\]

is anti-linear \( T x = *(x^tA) = A^t x = -A x \) with square \( T(T x) = T(-A x) = -A (-A x) = (AA)x \)

\[ \bar{A}A = -(A^tA) < 0 \]

So polar decomposition yields complex structure.
The good case is when $\bar{1}A = -(A^*A) = -1$. This is something you didn't make explicit before. Another formulation: Given a skew-symmetric form $A$ and a hermitian form $\star$:

\[ V \xrightarrow{A} V^t \xrightarrow{\star} V \]

These are compatible if $A^*A = +1$. $A$ is unitary and skew-symmetric.

Repeat. $V$ C-uss. equipped w. $\star \leftrightarrow V^t .

\[ V \xrightarrow{A} V^t \]

You want to know when $\star$ and $A$ are compatible: this means that the anti-linear operator $T = \star A : V \xrightarrow{A} V^t \xrightarrow{\star} V$ has square $= -1$.

\[ T(x) = \star(x^t A)^* = -\bar{A} \bar{x} \]

There are two possibilities differing in sign.

\[ T(Tx) = -\bar{A} (Tx) = -\bar{A} (-\bar{A} \bar{x}) = (AA)x = (-A^*A)x \]

Therefore, the compatibility condition is $A^*A = 1$ (in addition to $A^t = -A$).

Next you would like to apply the preceding to polarizations. Start naively with a Lagrangian subspace of $H(V)$.
Take a complex symplectic space of dimension $2n$, say $\dim C V = n$. You want the dimension of the space of Lagrangian subspaces of $H(V)$. You have an open set of Lag subspaces given by graphs of quadratic forms. So the dimension of these subspaces is $\frac{n(n+1)}{2}$.

On the other hand you can consider the fibre bundle over these consisting of a Lag sub together with a complete flag. This you can construct inductively by choosing a line, restricting to the symplectic quotient etc. The dimension of flag subspaces + complete flag up to dim $\ell$ is

\[ 2n - 1 + 2n - 3 + \cdots + 2n - (2n - \ell) = \ell^2 \]

The dimension of complete flags in $\mathbb{C}^n$ is $\frac{n(n+1)}{2}$.

Suppose we choose $L_1 \subset H(C)$, $\dim \{L_1\} = 2n-1$, then choose $L_2$ s.t. $L_1 < L_2 \subset \frac{1}{n} \ell$, $\dim \{L_2/L_1\} = 2n-3$. A Lag subspace with complete flag has dimension

\[ (2n-1) + (2n-3) + \cdots + (2n - \ell) = n \cdot 2n - n^2 = n^2. \]

Each Lag subspace has dimension $\binom{n}{\ell}$ of complete flags in $L_n = \frac{n(n-1)}{2} = n-1 + n-2 + \cdots + \frac{n-1}{2}$

\[ \binom{n}{\ell} = \frac{n(n-1)}{2} \left( \frac{h-1}{n-1} + n - (n-1) \right) \]

\[ \binom{n}{\ell} = \frac{(n-1)(h-1 + n - (n-1))}{(n-1)(n-1)} \]

\[ \binom{n}{\ell} = \frac{n(n-1)}{2(n-1)} \]
Let's study eigenvalue theory for a Hilbert space equipped with a symmetric bilinear form \( S : V \rightarrow V^* \). Associated to positive inner product on \( V \) is an anti-linear transformation \( \tilde{V} \), \( x \mapsto x^* \).

\[
V \xrightarrow{S} V^* \xrightarrow{\tilde{V}} V
\]

\[
x \mapsto x^S \mapsto (x^S)^* = \tilde{x}
\]
yields an anti-linear transformation \( \tilde{T} : x \mapsto \tilde{x} \) which is equivalent to \( S \).

Note that \( \tilde{T}^2 \) is linear:

\[
\tilde{T}(T x) = \tilde{T}(\tilde{x}) = \tilde{\tilde{x}} = (\tilde{S}S)x = (\tilde{S}S)x
\]

where \( \tilde{S}S = S^*S \geq 0 \).

\( \tilde{T} \) is a (weakly) positive hermitian operator on \( V \), so there's an eigenspace decomposition \( V = \bigoplus V_\lambda \), \( \lambda \geq 0 \).

\( \tilde{T} \) commutes with \( \tilde{T}^2 = \tilde{S}S \), so \( T \) respects this eigenspace decomposition.

If \( \tilde{S}S \xi = \lambda \xi \), then \( \lambda \geq 0 \) and

\[
\tilde{S}S T \xi = T \tilde{S}S \xi = T \lambda \xi = \lambda T \xi
\]

showing that \( T \) preserves \( V_\lambda \), \( \forall \lambda \).

You think it should be possible to give a variational picture of this decomposition. You know already about the polar decomposition of \( T \), the phase being a real structure; anti-linear invertible transform with square +1. There might be a Rayleigh-Ritz theory.
Rayleigh-Ritz theory for the eigenvalues, the $n$th eigenvalue is obtained by some variational problem involving subspaces of dimension $n$. There are probably interesting "minimax" inequalities, I expect some similarity with Morse theory construction of eigenvalues, in which you look at critical points of a suitable function on a Grassmannian.

Today you want the analogous construction of the spectrum of a hermitian operator in the case of a symmetric bilinear form. You start with a Hilbert space $V$ equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. You propose to use the conjugation theorem in the subspace $L(\text{Sp}(2n)/U(n))$.

Recall $L(\text{Sp}(2n)/U(n))$ is the subspace of $L(\text{Sp}(2n)) = \{ [a, b] : \begin{pmatrix} a & -b^* \\ -b & a^* \end{pmatrix} \}$ where $a = O$, equipped with conjugation action of $U(n)$: $\exists \, u \mapsto \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}$. Thus

\[
\begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} = \begin{pmatrix} u & bu^* \\ -b^*u & u^* \end{pmatrix} \]

This is also

\begin{align*}
&u \# b = ubu^* \\
&u \# b = u b \bar{u}^{-1} \quad \text{infinitesimally } \quad ab - b\bar{u},
\end{align*}

Now you want to find a suitable variational problem. Pick the flag manifold, i.e. the space of Lagrangian subspaces, which is the orbit under $U(n)$ of $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = J_0$. $J_0$ is the fixed point of our symmetric spaces $\text{Sp}(2n)/U(n)$.
You want a function $F(J)$ for $J$ on the symmetric space $\text{Sp}(2n)/U(n)$, which should depend on $J_0$, whose critical points are those $J$ commuting with $I_0$.

At a point $J$ you have the tangent space to the sym. sp. Note: $J$ is the same as a polarization, so the tangent space should be canon. ian to the space of symmetric bilinear forms $b$, which has $\text{dim}_R = 2 \cdot \frac{n(n+1)}{2} = n^2 + n$.

Definition of the tangent space:

$$X^t J + JX = 0 \quad \Leftrightarrow \quad [a \ b] = X \Rightarrow \begin{bmatrix} a^t & c^t \\ b^t & d^t \end{bmatrix} = X^t = JXJ = \begin{bmatrix} 0 & a \\ -1 & 0 \end{bmatrix} \begin{bmatrix} b & 0 \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Now return to the space of polarizations $\text{Sp}(2n)/U(n)$ which has $\text{dim}_R = (2n^2 + n) - n^2 = n^2 + n = 2 \cdot \frac{n(n+1)}{2}$.

Space of polarizations = orbit of $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ under $\text{Sp}(2n)$.

The basic object is the space of polarizations of $\text{H}(\mathbb{C}^n)$, i.e. the flag manifold, homogeneous space $\text{Sp}(2n)/U(n)$, conjugacy class of $J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.
You are studying the space of polarizations of \( \mathcal{H}(\mathbb{C}^n) \), this space should be \( \text{Sp}(2n)/U(n) \), orbit under \( \text{Sp}(2n) \) of \( J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

and it has dimension \( n^2 + n = 2 \times \text{dim} \{ \text{symmetric bilinear forms} \} \).

In fact the tangent space to this space for a polarization \( J \) should be the \( \mathbb{C} \) vector space of symmetric bilinear maps \( b: V_+ \rightarrow V_- \), more precisely such a \( b \) yields \( \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \) \( \in \mathfrak{b} = \mathcal{L}(\text{Sp}(2n))/\mathcal{L}(U(n)) = \mathcal{L}(U(n)) \).

Next you to use the basepoint polarization \( J_0 \) to construct a "Morse fn" on \( P = \text{Sp}(2n)/U(n) \).

At this point you are reminded of Moment Maps theory. In general the co-adjoint orbits of a Lie algebra are symplectic manifolds. For a compact Lie group, co-adjoint orbits = adjoint orbits. This explains why the \( J_0 \) adjoint orbits are symplectic.

Note: an adjoint orbit is \( G/\text{centralizer}(X) \) for some \( X \in \mathfrak{g} \).

There should also be a Duistermaat-Heckman theorem; Archimedean cases height fn.
Review: Space of polarizations \( \mathcal{P} \) is the orbit of \( J_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) under \( \text{Sp}(2n) \) \( = \text{Sp}(2n)/U(n) \). (There may be some confusion about whether \( J_0 \) is an elt of \( \text{Sp}(2n) \) or \( \text{I}(\text{Sp}(2n)) \).

Consider a polarization \( J \). This means \( J = J^* = J^{-1} \) as an operator on \( H(\mathbb{C}^n) \), i.e. \( J \) is unitary and its spectrum is \( \{-i, i\} \). Thus \( H(\mathbb{C}^n) = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \) with \( J = \begin{bmatrix} \text{I} & 0 \\ 0 & -i \end{bmatrix} \). In addition \( V_+ \) should be Lagrangian subspaces for the symplectic form. Your aim is to get a critical point proof that any polarization \( J \) is conjugate to \( J_0 \). Moreover you do not want to assume that the space of polarizations is connected. You want to use \( J_0 \) to construct the real valued function with the desired critical points. Geometric idea is the tangent space to \( \mathcal{P} \) at \( J_0 \).

Next look at the C.T. picture.

\[
H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix}, \quad J_0 = \begin{bmatrix} \text{I} & 0 \\ 0 & -i \end{bmatrix} = \mathcal{C} \mathcal{T}
\]

\[
J = \begin{bmatrix} \text{I} & \text{I} \\ -\text{I} & \text{I} \end{bmatrix} \quad J = \begin{bmatrix} 1 & \text{I} \\ \text{I} & 1 \end{bmatrix} \quad J(1+X)(-i\mathcal{T}) = J(-i)(1-X) = 1+X \Rightarrow J J_0^{-1} = \begin{bmatrix} 1+X \\ 1-X \end{bmatrix}
\]

Next you want a measure of the size of the difference \( J J_0^{-1} \). Remove \( i \)'s: \( J = \mathcal{C} \), \( J_0 = \mathcal{T} \)
\[ \text{then } \mathcal{D}^+_0 = F \varepsilon \text{ which means that you are really working in the Grassm. } U(2n)/U(n) \times U(n) \text{ under the embedding } S^1(2n)/U(n) \rightarrow S^1 \text{.} \]

Then \( \text{tr} (F \varepsilon) \), the functional you use with \( F \) ranging over a Grassm. and \( \varepsilon \) the hermitian of \( F \), is

\[
\text{tr} (F \varepsilon) = \frac{1}{2} \text{tr} (F \varepsilon + \varepsilon F) = \frac{1}{2} (g + g^{-1}) = \frac{\text{tr} (1 + X^2)}{2} \left[ \frac{1}{1 - X^2} + \frac{1}{1 + X^2} \right] = \text{tr} \left( \frac{1 + X^2}{1 - X^2} \right).
\]

Recall \( X^2 \leq 0 \) so that

\[
\frac{1 + X^2}{1 - X^2} \quad -X^2
\]

\[
\frac{1 + X^2}{1 - X^2} + 1 = \frac{2}{1 - X^2}. \quad \text{Therefore one has}
\]

\[
\frac{1 + X^2}{1 - X^2} = \frac{2}{1 - X^2} - 1 \quad \text{decreases monotonically from } 1 \text{ to } -1
\]

as \( -X^2 \) increases from 0 to \( +\infty \). Something is wrong.

Review \( F \mapsto \text{tr}(FA) \), A hermitian, \( F \) ranges over a Grassm. so that \( F^2 = 1 \). The tangent space to the Grassm. is \( \{ \delta F \mid (\delta F) F + F (\delta F) = 0 \} \). \( F \) is a critical point \( \iff \text{tr}(\delta F) A = 0 \), \( \forall \delta F \) (herm. and anticommuting with \( F \)).

\[ F = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}, \quad \delta F = \begin{bmatrix} b^* & 0 \\ 0 & -b \end{bmatrix} \text{.} \]
\[ t_n(\delta F A) = t_n \begin{bmatrix} 0 & b^* \\ b & 0 \end{bmatrix} \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix} \]
\[ = t_n \begin{bmatrix} b^* A_{++} & A_{+-} \\ 0 & b A_{-+} \end{bmatrix} = t_n(b^* A_{++} + b A_{-+}) \]

where \( A_{++} = A_{--} \), so \( 0 = t_n(b^* A_{++} + b A_{-+}) \) \( \forall b \)

Put \( b = A_{-+} \) get

\[ 0 = t_n(A_{++} A_{-+} + A_{-+} A_{-+}^*) \Rightarrow A_{++} A_{-+} = 0 \]

\[ \Rightarrow A \text{ commutes with } F. \]

Another idea is to scale \( X \); put in \( tX \) for \( X \)

\[ \frac{1}{2}(FA + AF) = F(A + FAF) \]
\[ \frac{1}{2}(FA - AF) = F(A - FAF) \]

\[ t_n(\delta F A) = t_n(\delta F A_{od}) \quad \text{because } \delta F \text{ is odd} \]

\[ A = \begin{bmatrix} a & b^* \\ b & d \end{bmatrix}, \quad \delta F = \begin{bmatrix} 0 & m^* \\ m & 0 \end{bmatrix} \]

\[ t_n(A \delta F) = t_n \begin{bmatrix} a & b^* \\ b & d \end{bmatrix} \begin{bmatrix} 0 & m^* \\ m & 0 \end{bmatrix} = t_n(b^* m + b m^*) \]

If \( A_{od} \neq 0 \), take \( m = b \), get \( t_n(A \delta F) = t_n(b^* b + b b^*) \) \( > 0 \)

so \( t_2(\delta F A) = 0, \forall \delta F \Rightarrow A_{od} = 0. \)

Next you want to adapt this to the symplectic case. Yesterday you tried using the C.T. You concluded that it didn’t seem right, because the anti-linear operator was absent.

Let review the ideas again. You are studying the space of polarizations in \( H(\mathbb{C}^n) \). You want to construct a suitable functional depending on the basepoint polarization, whose critical points
will commute with the basepoint polarization.

Start again. You consider the space $\mathbb{P}$ of polarizations of $H(\mathbb{C})$, equipped with basepoint $J_0 = \text{id}$. You want to construct a Morse function on $\mathbb{P}$ depending on $J_0$ whose critical points are polarizations centralizing $J_0$. In principle you should be able to do this by means of the conjugacy proof for the adjoint picture of the symmetric space $\text{Sp}(2n)/U(n)$. You need to write a little more.

This means that you consider the action of the isotropy group $U(n)$ (centralizer of $J_0$) on the tangent space to $\mathbb{P}$ at $J_0$, which is $\{ [O,b]; b^t = b \}$, the space of symmetric bilinear forms.

Now you want to know what happens at any polarization $J$. More precisely, you need the tangent space to $\mathbb{P}$ at any $J$. This should be isomorphic to the space of symmetric bilinear forms.

You need a good picture. You have the linear space $\mathfrak{p} = L(\text{Sp}(2n)/U(n)) \approx \{\text{symm. bilinear forms}\}$ inside $L(\text{Sp}(2n))$ acted on by $K \subset U(n)$. Basic result: orbits of $K$ on $\mathfrak{p}$ are flag manifold varieties: of the form $K/\text{Centralizer of a torus}$. (If you take an element of $\mathfrak{p}$, it generates a torus.) Maybe this result is important because it allows you to identify the flag manifolds with orbits in Lie algebra.

Aim: to use $J_0$ to get a Morse function on $\mathbb{P}$. This should be obvious if you knew the moment map theory well.
You consider \( P \), the space of polarizations of \( \mathbb{H}(\mathbb{C}^2) \). An element \( J \) of \( P \) should be an orthogonal splitting \( H = \begin{bmatrix} W_+ \\ W_- \end{bmatrix} \) (not the herm inner product) such that \( W_+ \cap W_- = \{ 0 \} \), \( W_+ \) are Lagrangian subspaces of \( H \).

\( J = \pm \iota \) on \( W_+ \) so that \( -J = J^* = J^{-1} \).

You have the basepoint \( J_0 = i\mathbb{E} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \) on \( \begin{bmatrix} V_+ \\ V_- \end{bmatrix} \).

Let's discuss the big cell (affine open subspace) of \( P \) centered at \( J_0 \):

- \( W_+ \) is the graph of \( b: V_+ \to V_- \)
- \( W_- = (W_+)^\perp = \) graph of \(-b^*: V_- \to V_+\).

You know that \( W_+ \) is Lagrangian iff \( b^t = b \). \( W_- = \begin{bmatrix} -b \\ 1 \end{bmatrix}: V_- \to V_+ \) is Lagrangian iff \( -b^* \) is symm. \( \iff (b^*)^t = b^* \), which follows from \( b^t = b \) by applying \( \star \); things commute.

\[
J \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix} \iota \mathbb{E}.
\]

\[
J (1 + X) (-i \mathbb{E}) = (1 + X) \mathbb{E}.
\]

\[
J (-i \mathbb{E})(1 - X) \]

\( J J_0^{-1} = \begin{bmatrix} 1 + X \\ 1 - X \end{bmatrix} \)

Note \( X \) is simultaneously skew adjoint skew symm.

\( \mathbb{N} \):

\[
X \mathbb{E} = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = \begin{bmatrix} 0 & b^t \\ -b^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \neq -X
\]

The problem should be that the symplectic form \( \Omega \) on \( H \) is defined \( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \not= \iota \mathbb{E} \).
You have to clarify the notion of polarization of $H(C^n)$ complex.

First consider the symplectic structure on $H$.

\[
H = \begin{bmatrix} V \\ V^t \end{bmatrix}, \begin{bmatrix} x_1^t \\ y_1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} x_2^t \\ y_2 \\ 0 \\ 1 \end{bmatrix} = x_1^t y_2 - y_1^t x_2
\]

A polarization in this setting is an ordered pair of complementary Lagrangian subspaces. It should be clear that $Sp(2n, C)$ acts transitively on these polarizations, and the stabilizer of a basepoint polar is $GL(n, C)$.

Look at $n=1$, where $Sp(2, C) = SL(2, C)$ is given by \[
\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad - bc = 1 \right\},
\]
and $GL(1, C) = C^*$ is embedded as \[
\varepsilon : \mathbb{R} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
A polar is an ordered pair of distinct lines.

\[
\begin{bmatrix} v^t \\ w \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{This ought to show the two lines} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix}
\]

\[
\varepsilon, \begin{bmatrix} a \\ b \end{bmatrix} \in \ v, \ w \subseteq \mathbb{R}^2 \quad \text{which are close to} \quad \varepsilon_v, \varepsilon_w \quad \text{resp.}
\]

This reminds you of the Bruhat cell.

Note that an operator $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $\begin{bmatrix} V \\ V^t \end{bmatrix}$ satisfies

\[
X^t J + J X = 0 \iff b = b^t, c = c^t, d = -a^t.
\]

So a tangent vector to the space of polarizations at the basepoint has the form $X = \begin{bmatrix} 0 & c \\ b & 0 \end{bmatrix}$ with $b, c$ symmetric. There should be a similar picture for the tangent space at any polar.

Next you bring in the inner product on $H$. 
Idea: The space of polarizations is a flag manifold associated to $\text{Sp}(2n)$, and therefore it is an adjoint orbit: $\text{Sp}(2n)/\text{centralizer of some torus } T$, where $T$ should be a circle group. This idea should allow one to identify the space $P$ of polarizations with a conjugacy class in $\mathbb{L}/\text{Sp}(2n)$.

Let's go back to $\text{Sp}(2n, \mathbb{C})$,

$$
\begin{bmatrix}
  x_1 \\
  y_1 \\
-1 & 0 \\
  y_2 \\
  x_2
\end{bmatrix} =
\begin{bmatrix}
  x_1 \\
  y_2 \\
  -x_2
\end{bmatrix} = x^t y_2 - y^t_1 x_2.
$$

Let $X \in \text{M}(2n, \mathbb{C})$ preserve this symplectic form: $X^t J + J X = 0$, say $X = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $X = J X^t J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$,

$$
\begin{bmatrix}
  c^t & d^t \\
  -a^t & -b^t
\end{bmatrix}
\begin{bmatrix}
  0 & 1 \\
  -1 & 0
\end{bmatrix}
\begin{bmatrix}
  a & c \\
  b & d
\end{bmatrix}
$$

Yesterday's idea of $\text{Sp}(2n, \mathbb{C})$: This group should act transitively on polarizations (these are defined as ordered pairs of complementary Lagrangian subspaces), so that the manifold of polarizations is $\text{Sp}(2n, \mathbb{C})/\text{GL}(n, \mathbb{C})$.

Graph picture of Lagrangian subspace $V_+$

$$
W_+ = \begin{bmatrix} 1 \\ b \end{bmatrix} V_+ \text{ is Lagrangian iff } V_+^t \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} V_+ = 0
$$

$$
V_+^t \begin{bmatrix} 1 & b \\ -1 & 1 \end{bmatrix} V_+ = 0 \iff b - b^t = 0
$$

So one way to get a polarization is from an ordered pair of symmetric forms $b : V_+ \to V_-$ and $b' : V_\to V_+$ such that the graphs are transversal, which should mean that $\begin{bmatrix} 1 & b' \\ b & 1 \end{bmatrix}$ is invertible.
So what's the problem? Consider \( \text{Sp}(2n, \mathbb{C}), \ n = 1 \), i.e. \( \text{SL}(2, \mathbb{C}) \). Then \( \text{SL}(2, \mathbb{C})/\mathbb{C} \) is the space of ordered pairs of lines in \( \mathbb{C}^2 \) which are independent. Here \( \mathbb{C}^2 \cong \mathbb{C} \times \mathbb{C} \). To study this, let \( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \text{SL}(2, \mathbb{C}) \). How to study this? Let \( \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \text{SL}(2, \mathbb{C}) \) be too hard.

Instead consider the inner product \( [x_1]^{*} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^* x_2 + y_1^* y_2 \). Restrict attention to ordered pairs of lines which are \( \perp \) for the inner product. You can assume \( |a|^2 + |b|^2 = 1 \) and \( |c|^2 + |d|^2 = 1 \), also that \( a \bar{d} \neq 0 \). Then orthogonality \( a \bar{c} + \bar{b} d = 0 \), \( \text{ad} - \text{bc} = 1 \). You want to understand what \( \begin{bmatrix} a \\ b \end{bmatrix} \) means. \( a \bar{c} + \bar{b} d = 0 \) ?

You have two vectors \( v, w \in \mathbb{C}^2 \). And you want to show that \( v \perp w \iff |v \wedge w| = 1 \). This should be obvious because it reduces to \( \mathbb{R}^2 \).

Review: Any two polarizations \( H(v) = \begin{bmatrix} v \\ v^t \end{bmatrix} \) w.r.t. symp. form \( \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix} \). Def. polarization as ordered pair of complementary Lag subspaces. Any two polarizations \( \text{Sp}(2n, \mathbb{C}) \) stable; for \( \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \).

Any polarization near to \( \begin{bmatrix} v \end{bmatrix} \) has the form \( \begin{bmatrix} w_1 \ w_2 \end{bmatrix} = \begin{bmatrix} b & b' \\ -b' & b' \end{bmatrix} \begin{bmatrix} v \end{bmatrix} \) where \( b, b' \) Again, \( n = 1 \) if \( b \neq bb' \).
Next consider the 3 minor products of $\text{SU}(2)$ with itself $\text{SU}(2) \times \text{SU}(2)$. 

The space of polarizations is $\text{SU}(2)/\{[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}], a \in \mathbb{U}(1)\}$. 

A polarization is an ordered pair of orthogonal lines.

Idea: Recall Dominic Joyce's H4-theory where you have the antipodal map on the Riemann sphere $z \mapsto -\frac{1}{z}$, so in this special case you see the antilinear map giving rise to $\jmath$.

Is it true that the subgroup $U(1) = \{[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}], |a| = 1\} \subset \text{SU}(2)$ is the centralizer of an element of $X(\text{SU}(2)) = \{[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}]: a+b = 0\}$?

Yes $[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}]$ when $0 \neq x \in \mathbb{R}$.

What are you trying to find out?

What questions to ask? Consider $\text{SU}(2)$ acting on $\mathbb{C}^2$ preserving inner product and symplectic form.

Notion of polarization: an ordered pair of lines in $\mathbb{C}^2$.

Basepoint polar con $[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$, stabilizer is max torus $\{[\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}]: a \in \mathbb{U}(1)\}$. 

Take polarization close to basepoint $[W_+, W_-] = \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix} [V_+, V_-]$. 

What should question be? Point you missed: X is skew adjoint, so $1 \pm X$ invertible.
So now you can try to correct past problem. If \( b^t = b \), then \(-b^* = -b^t = -b\),
so \( X = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}, \quad X^t = \begin{bmatrix} 0 & b^t \\ -b^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \).

so \( X + X^t = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \iff X^* = -X. \)

Now you hoped before that \( X + X^* = 0 \) and \( X \) of the form \( \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \implies X + X^t = 0. \) This is true when the symplectic form is
\[
\begin{bmatrix} x_1^t & 0 & 1 & x_2 \\ y_1 & 0 & -1 & y_2 \end{bmatrix}
\]
This may be wrong because of the shift orth \( \leftrightarrow \text{symp} \) upon dividing rank by 2.

Another approach might be to look at the three simplest symplectic forms:
\[
\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]
These form a nice basis for \( \mathfrak{l}(SU(2)) \).

\[ H(\mathbb{C}^n) = \begin{bmatrix} \mathbb{C}^n \\ \mathbb{C}^n \end{bmatrix} \] inner product
\[ \text{symp. form} \]
\[ \left[ \begin{bmatrix} x_1^t & 0 & 1 & x_2 \\ y_1 & 0 & -1 & y_2 \end{bmatrix} \right] = x_1^* x_2 + y_1^* y_2 \]
Let \( X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in \text{End } H \)

satisfy
(i) \( X^* + X = 0 \)
(ii) \( X^t J + JX = 0 \) \( \implies \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -d & b \\ -a & -c \end{bmatrix} \] because \( X = -X^t \)

(iii) \( JX = XJ \)

\( \therefore b^t = b, \quad c^t = c, \quad d = -a^t. \)
Also \( X^* + X = 0 \) \( \Rightarrow \begin{align*} &a^* = -a \quad c^* = -b \\ &b^* = -c \quad d^* = -d \end{align*} \)

\( -c^* = b^* = b \), also \(-b = c\), \(d = -a^* = -a\).

Define a polarization to be an ordered pair of Lagrangian subspaces which are orthogonal in the inner product. Consider the case close to the basepoint:

\[
\begin{bmatrix} W_+ & W_- \end{bmatrix} = \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} \begin{bmatrix} V_+ \\ V_- \end{bmatrix}
\]

Because \( W_+ \text{ Lagrangian} \) we have \( b^* = b \)

\( (-b^*) \text{ symm.} \Rightarrow -b^* = -b \)

so \( X = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \in \mathcal{L}(J_0(2n)) \Rightarrow X^t J + J X = 0 \)

Check it:

\[
\begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}
\]

At this point you understand polarizations close to the basepoint, recall \( X = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \) satisfies \( X^t J + J X = 0 \)

\( \Rightarrow \begin{bmatrix} a^t & b^t \\ c^t & d^t \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \). Notice that

\[
J \begin{bmatrix} a & c \\ b & d \end{bmatrix} J^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & d \\ -a & -c \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

which is like the 2x2 rule: interchange diagonal elts, NO NO, change sign of off-diagonal elts, NO NO.
Conclude that \( X = \begin{bmatrix} 0 & b' \\ b & 0 \end{bmatrix} \) satisfies
\[ X^t J + J X = 0 \iff b, b' \text{ symm.} \quad \text{Also} \quad X \text{ preserves both symplectic form and inner product} \iff X = \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix}. \]

The preceding gives a complete picture of polarizations close to the basepoint. They correspond to symmetric \( b : V_+ \to V_- \) i.e. to Lagrangian subspaces transversal to \( V_- \).

You want to identify the space of polarizations with an orbit in \( \mathcal{L}(\text{Sp}(2n)) \).

You are still trying to find a clear picture of the space of polarizations. A polarization is like a point.

You're still missing something about polarizations. Today's idea is to understand the symmetric space structure on the space of polarizations \( \text{Sp}(2n)/\text{U}(n) \). A point of a symmetric space determines a reflection through that point and all these reflections should generate the symmetry group \( \text{Sp}(2n) \). The obstruction to understanding is probably the fact that the reflections are anti-linear.

A polarization should be an anti-linear transformation whose square is in the center.

I think you want to look for an automorphism of \( \text{Sp}(2n) \) of order 2 with the fixed group \( \text{U}(n) \).

\textbf{Conjugation} by an anti-linear transformation should be what's needed. Note that elements of \( \text{Sp}(2n) \) are linear transformations on \( \mathbb{H} \).
\[ n = 1, \quad \text{Sp}(2) = \text{SU}(2) = \left\{ \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\} \]

acts on \( \mathbb{P}^1 \mathbb{C} = S^2 \) = Riemann sphere. \( S^2 \) is the flag manifold of polarizations, where the polar cone to \( L \subset \mathbb{C}^2 \) is \( L \cup L^\perp \). So if \( L = \begin{bmatrix} a \\ \overline{c} \end{bmatrix} \mathbb{C} \) then \( L^\perp = \begin{bmatrix} -\overline{a} \\ 1 \end{bmatrix} \mathbb{C} = \begin{bmatrix} 1 \\ -\overline{a} \end{bmatrix} \mathbb{C} \).

Now \( S^2 = SU(2)/U(1) \) is a symmetric space, and so there should be at each point a a reflection through that point. Reflection means that you join a variable point \( z \) to a by a geodesic (in the sense of spherical geometry, so this ought to amount to using a circle or straight line) and then you continue the geodesic through \( a \) an equal amount to the opposite side of \( a \). If \( a = e^{i\theta} \), the reflection is \( z \mapsto -\overline{z} \).

For a general \( a \) use an orbit of \( SU(2) \) to move \( a \) to 0.

You need notation change.

\[ n = 1, \quad \text{Sp}(2) = \text{SU}(2) = \left\{ \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\} \]

\( SU(2) \) acts on \( \mathbb{C}^2 = H(\mathbb{C}) \) in the obvious. A polarization is equivalent to a line \( L \) in \( \mathbb{C}^2 \), so the space of polarizations is the Riemann sphere \( \mathbb{P}^1 \mathbb{C} = S^2 = \mathbb{C} \cup \mathbb{\infty} \).

\( L = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mathbb{C} \), then the orthogonal line is \( L^\perp = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \mathbb{C} \); of course these lines are 1-dimensional \( \Rightarrow \) Lagrangian. Action of

\[ \begin{bmatrix} \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \end{bmatrix} \]

\( \begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \end{bmatrix} \) is

\[ \begin{bmatrix} a & -\overline{b} \\ b & \overline{a} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a^2 + b^2 \\ \overline{a} - \overline{b} \end{bmatrix} \]
Still struggling with the space of polarizations. Let’s return to spectral theory for symmetric bilinear forms on a complex $V$ with pos. herm. inner product.

$$V \xrightarrow{\ast} V^t \xrightarrow{\ast} V$$

$$V \xrightarrow{b} V^t \xrightarrow{\ast} V$$

$$\begin{align*}
\star b & \quad \star b(x) = b^* \overline{x} = \overline{b^* x} \\
\star b(\star b(x)) & = \star b(b \overline{x}) = b^* \overline{b^* x} = (b^* b)x \\
(\star b)^2 & = b^* b = b^* b > 0.
\end{align*}$$

It seems that what you need is the theory of an antilinear hermitian operator $\star$.

You need only basic harmonic oscillator stuff. You are given two forms, one hermitian bilinear. The other symmetric bilinear. The “difference” is an antilinear operator whose square is hermitian $> 0$.

$$V \xrightarrow{b} V^t \xrightarrow{\ast} V \xrightarrow{b} V^t$$

$$\begin{align*}
x & \quad x^t b \\
\overline{b^* x} & = (b^* x)^t
\end{align*}$$

$$(b^* x)^t b = x^*(b^* b)$$

Ultimately you want these antilinear ops to be the analog of hermitian operators.
Let's go over spectral theory for a symmetric bilinear form on a complex Hilbert space $V$. So $V = \mathbb{C}^n$ with hermitian form and bilinear form $x^t b y$, $b$ a symmetric matrix. Then get transformations (maps) assoc. to these two forms. 

$$V \xrightarrow{t \cdot b} V^t \xrightarrow{x} V \xrightarrow{V^t} V$$

$$x \mapsto x^t b \\
\Rightarrow (bx)^t \mapsto b x \mapsto \overline{b x} \mapsto \overline{b b x} \mapsto (\overline{b b} x)$$

So 

$$T_b(x) = \overline{b x}$$

$$T_b : V \rightarrow V$$

anti-linear

$$T_b T_b(x) = \overline{b \overline{b x}} = \overline{\overline{b b x}} = (\overline{b b}) x$$

where $\overline{b b} = b^* b \geq 0$.

So $T_b$ and $b^* b$ commute. Since $b^* b$ hermitian, $V$ splits into eigenspaces $V = \bigoplus \nu \lambda$, where

$$\nu = \ker (b^* b) - \lambda^2$$

What's important is to restrict attention to $V_\lambda$.

Then you have the same situation $V$, $b$ etc. but with $b^*$? $[T_b, b^* b] = 0$.

There might be a problem working with matrices - the eigenspace $V_\lambda$ won't be compatible with the basis for $V$ chosen.
Assume OK. Ultimately you need a clear intrinsic formulation which you seem to have:

$$V \xrightarrow{b} V^t \xrightarrow{\times} V$$

These maps $b$, $\times$ are intrinsically defined, so $T_b : V \to V$ is intrinsically defined.

Focus on the pure case $T_b^2 = b^*b = \lambda^2$.

Then $\lambda^{-1}T_b$ is anti-linear in $V$ with square $+1$.

$\lambda^{-1}T_b$ is anti-linear unitary?

If $b^*b = 1$, then $T_b$ is a real structure on $V$. Then $b^*b = bb = 1$ so $b^* = \bar{b} = b^{-1}$.

$g$ orthogonal means $g^*g = 1 \Rightarrow g^*\bar{g} = 1$.

It seems that $T_b(x) = \times (\delta x^tb) = \bar{b}x$

$T_b^2(x) = (bb)x$ with $\bar{b}b = 1$ is important.

What's interesting is $T_b$ with $b^*b = 1$:

$$V \xrightarrow{b} V^t \xrightarrow{\times} V$$

$$\times \xrightarrow{x^tb} b^*x$$

$$T_b(T_b(x)) = b^*\overline{T_b(x)}$$

$$= b^* b^*x = (b^*b^t)x$$
Aim: To link polarization to operator $T_b$ with $T_b^2 = 1$. This is the real puzzle.

Can you fit the preceding with other case?

$$\begin{bmatrix} 0 & b \\ -\overline{b} & 0 \end{bmatrix}$$

$$\text{Sp}(2) = SU(2) = \left\{ \begin{bmatrix} a & b \\ -\overline{b} & \overline{a} \end{bmatrix} : |a|^2 + |b|^2 = 1 \right\}$$

Lie $SU(2) = \left\{ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} : \alpha + \overline{\alpha} = 0 \right\}$

$$= \{ x \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \}$$

You want to study $S^2 = SU(2)/\left\{ \begin{bmatrix} a & 0 \\ 0 & \overline{a} \end{bmatrix} : |a| = 1 \right\}$. This is a symmetric space, so you think that the isometry group is the fixed point of an involution in the group $SU(2)$.

Conjugation by $x = i \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has fixed point subgroup

$\Phi(\mathbb{U}(1)) = \begin{bmatrix} 0 & \overline{\alpha} \\ \alpha & 0 \end{bmatrix}$. So you can identify $S^2$ with an orbit of $SU(2)$ on $\mathbb{Z}$ or $\mathbb{SU}(2)$.

Program: Study the symmetric space $\text{Sp}(2)/\Phi(\mathbb{U}(1))$ where $\Phi(\mathbb{U}(1)) = \begin{bmatrix} 0 & \overline{\alpha} \\ \alpha & 0 \end{bmatrix}$. Invariant consists of $\Phi(\mathbb{U}(1)) = \left\{ \begin{bmatrix} 0 & b \\ -\overline{b} & 0 \end{bmatrix} : b \in \mathbb{C} \right\}$ with $\Phi(\mathbb{U}(1))$ acting by conjugation

$$\begin{bmatrix} 0 & b \\ -\overline{b} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ b & \overline{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \overline{\beta} & \overline{\alpha} \end{bmatrix} \begin{bmatrix} 0 & b \\ -\overline{b} & 0 \end{bmatrix}$$

One orbit for each $|b|$. 
Now look at \( \text{Sp}(2n)/\text{U}(n) \). In

the picture is \( \mathfrak{sp} = \{ [\begin{array}{cc} 0 & b \\ -b & 0 \end{array}] : b^T = b \} \),

the conjugation action of \( \text{U}(n) \):

\[
K \cdot \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix} \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \begin{bmatrix} u^* & 0 \\ 0 & u^* \end{bmatrix} = \begin{bmatrix} 0 & ub^t \\ -b^u & 0 \end{bmatrix}
\]

and

\[
(ub^t)^T = ub^T u^* = ub^t.
\]

You want the orbit structure of \( K \) on \( \mathfrak{g} \), equivalently, the orbit structure of \( \text{U}(n) \) acting on symmetric matrices via \( u = b = ub^t \).

Orbit structure means eigenvalue theory?

Let's try to link \( \sigma_b \) stuff (unitary equivalence for symmetric forms) to the infinitesimal symmetric space \( \mathfrak{g}(\text{Sp}(2n)/\text{U}(n)) \). Can you construct the spectral decomposition for \( \sigma_b \) within \( \mathfrak{g} \)?

Repeat: \( b : \mathcal{V} \to \mathcal{V}^T \) symm. bilinear

\( x \mapsto x^T b \)

Alternative: \( x \mapsto bx \) column vector transpose to get row

\( (bx)^T = x^T b \).

What's clear is that the spectral theory of symmetric bilinear forms on \( \mathbb{C}^n \) is simply the \( K = \text{U}(n) \)

action on \( \mathfrak{g} = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \) \( b^T = b \).

So it should be possible to directly construct the desc. of \( b \). Put \( X = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \). Form

\[
-X^2 = \begin{bmatrix} bb & 0 \\ 0 & bb \end{bmatrix} \geq 0.
\]
Properties of $X = \begin{bmatrix} 0 & b \\ \frac{a}{b} & 0 \end{bmatrix}$, $b^t = b$.  

\[ X^* + X = 0, \quad X^* J + JX = 0, \quad JX = \overline{XJ} \]

\[-X^* = -JXJ^{-1} \]
canonical commutation relations:
\[ [a_i, a_j^*] = \delta_{ij}, \quad [a_i, a_j] = 0, \quad [a_i^*, a_j^*] = 0 \]

basic structure is \( x \) together with \([\ , \ ]\)

Considering a symplectic form \( \psi \) on a Herm. form.

Maybe it would be good to list all the ideas you want to organize:

- polarization, maximal abelian subspace
- compatibility of \( \psi \) with a symplectic form
- symmetric bilinear forms in dimension 2 where you have factorization into 2 linear forms.

Given a \( \psi \) symmetric form, you can restrict to each line in \( V \), where you have a well-defined \( \lambda > 0 \). Maximize this over \( PV \).

Given \( b = b^t \), let \( x \) be a unit vector, form \( x^t b x \). The absolute value \( |x^t b x| \) is independent of the phase of \( x \), so you get a well-defined function on \( PV \). Properties: smooth function on \( V \) except when \( x^t b x \neq 0 \). Look at \( \dim V = \mathbb{R} 2 \), where \( x^t b x \) is a quadratic form in \( x_1 x_2 \). Look at the product of 2 linear factors: \( x_1 x_2 \). \( |x_1 x_2| \) is not a smooth function of \( x_2 \) for \( x_1 \neq 0 \).

So it seems that you want to square the absolute value: \( x^t b x \frac{1}{2} x^t b x \), no problem with smooth since \( |z|^2 \) is smooth, \( x^t b x \) is a smooth function of \( z \).
So consider $\| x^T b x \|^2$ on $\mathbb{P}^V$.

\[ |x^T b x|^2 = x^T b x x^T b x = \text{tr} (x^T b x x^T b x) \]
\[ = \text{tr} (b x x^T b x^T) = \text{tr} (bp \bar{b}) \]

So you now have a nice functional defined on $\mathbb{P}^V$. What are its critical points?

\[ \delta \text{tr} (bp \bar{b}) = \text{tr} (b \delta p \bar{b} \bar{p} + bp \bar{b} \delta \bar{p}) \]
\[ = \text{tr} (\delta p \bar{b} \bar{p} b + bp \bar{b} \delta \bar{p}) \]

\[ p = xx^T \]
\[ \delta p = p \delta p + p \delta p = \delta p \]

\[ \delta p = p \delta p + p^4 \delta p = \delta p \]
\[ \delta p = p \delta p + p^4 \delta p \]

\[ \text{tr} (\delta p \bar{b} \bar{p} b) = \text{tr} (\delta p (p + p^4) \bar{b} \bar{p} b) \]

\[ \delta p = p \delta p + p^4 \delta p = p \delta p + p_4 \delta p \]
\[ = p \delta p + p^4 \delta p \]

\[ \text{tr} \left( \delta p \left( p \delta p^+ \right) + \delta p \left( p^4 \delta p \right) \right) \]

\[ T = \bar{b} \bar{p} b \]

\[ \text{tr} \left( \delta p \left( p^+ T \right) + \delta p \left( p T p^+ \right) \right) \]

\[ : p T p^+ \text{ and } p^4 T p = 0 \]
From an invariant viewpoint you need to explain $\bar{p}$, $b$ etc. Let’s go over what you learned yesterday. Consider a symmetric matrix $b$ of rank $n$ and use it to construct a function $p(\mathbb{C}^n)$. Given a unit vector $x$ you form $|x^*b x|$ which is independent of the phase of $x$. Unfortunately not smooth in general, so you take $|x^*b x|^2 = x^* b x x^* b x$ which you can rewrite as $tr(b x x^* b x x^*)$. Note that $x x^*$ = orthogonal projection operator on $\mathbb{C}^n$ whose image is line $x \mathbb{C}$. Try replacing $x x^*$ by $2 x x^* - I = F$.

$$2 x x^* - I = F \quad x x^* = \frac{1 + F}{2} \quad tr(b \frac{1 + F}{2} \frac{1 + F}{2})$$

$$\frac{1}{4} tr(b \bar{b} + b F \bar{b} + b \bar{b} F + b F \bar{F} F).$$

You still have the same problem: How to handle $\bar{p}$ intrinsically, which probably means that you want a semi-direct product with $p \in \mathbb{P} V$, that is, $p$ is any orthogonal Projection of rank 1.

$$8 tr(b \bar{p} b \bar{p}) = tr(8p \bar{p} b + b \bar{p} b \bar{p})$$

What conclusion to draw is probably $\rho b b' = 0$ in $x^* A x = tr(\rho A) \rightarrow tr(8\rho A) = 0 \quad \forall \rho = A^{odd} = 0$.

Hence $\bar{p} b b' V \subset p V$ and $\bar{p} \rho b V \subset \rho V$. 

$$\rho \bar{\rho} = \rho A \bar{p} = \rho A^d \bar{p} = 0 \quad \rho V \subset p V.$$
\[ \delta p = \rho \delta p + \rho^* \delta p \rho \]

\[ t_2(\delta p \bar{b} \bar{b}) = t_1(\rho \delta p \rho + \bar{b} \bar{b}) + t_1(\rho^* \delta p \rho \bar{b} \bar{b}) \]

\[ = \text{tr}(\delta p (\rho \rho^* \bar{b} b + \rho^* \bar{b} \bar{b} \rho \rho^*)) \]

This is all very reasonable. If the function is stationary and \( \delta p \Rightarrow \rho \rho^* \bar{b} b = 0 \) but you don't know what this means.

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It becomes important to understand the meaning of replacing \( b \) by \( \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \).

Idea: \( \rho \) is a projection with range a line \( \overline{bc} \in V \), \( \rho \) should induce a projection in \( H(V) \), which should be

\[ \pi = \begin{bmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}. \]

What's the function

\[ \text{tr}(\beta \rho \bar{b} \bar{b})? \]

\[ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \begin{bmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{bmatrix} = \begin{bmatrix} \rho \bar{b} \bar{b} & 0 \\ 0 & \bar{\rho} \bar{b} \bar{b} \end{bmatrix} \]

It might help to replace:

\[ \begin{bmatrix} \rho & 0 \\ 0 & \bar{\rho} \end{bmatrix} = \begin{bmatrix} xx^* & 0 \\ 0 & \bar{x} \bar{x}^* \end{bmatrix} \]

\[ \begin{bmatrix} x \\ \bar{x} \end{bmatrix} \begin{bmatrix} x^* & \bar{x}^t \end{bmatrix} = \begin{bmatrix} xx^* & \bar{x} \bar{x}^t \\ \bar{x} x^* & \bar{x} \bar{x}^t \end{bmatrix} \]
Look at the case $n=1$, $b \in \mathbb{C}$, trivial since PV is a point.

The critical value is $|b|^2$; here you're using the function $x^t b x$ for any unit vector $x$.

Next generalize to a Grassmannian $\{p : p^* = p = p^2, \text{rank}(p) = d \}$. $x$ becomes a $d \times n$ matrix of orthogonal unit vectors, so that $x^x = 1$, $x^* x$ is the projection on the span of these unit vectors. The function on the Grassmannian is $\text{tr}(bp \overline{p})$.

You still don't understand $b$ and $\overline{p}$. $b$ and $p$ have straightforward meanings: $b$ symmetric bilinear form; $p$ is subspace on $V$.

**Viewpoint:** Consider $V$ inner-product space + $b^t = b$.

\[
\begin{array}{cccccc}
V & b & \rightarrow & V^t & \times & V \\
\times & x \mapsto (bx)^t = x^t b & \times & b & \rightarrow & V \\
\rightarrow & \overline{b} x & \mapsto & b & \rightarrow & (b \overline{b} x)^t = x^* b^* b & \mapsto (bb)^x
\end{array}
\]

\[
\begin{array}{cccccc}
y^t & \times & \bar{y} & \mapsto & b \\
\rightarrow & \overline{b} \bar{y} & \mapsto & (b \bar{y})^t = y^* b
\end{array}
\]

So you get: $y^t \mapsto y^* b \mapsto (y^* b)^t$.

\[
\begin{array}{cccccc}
V & b & \rightarrow & V^t & \times & V \\
\times & y^t \mapsto & (by)^t \mapsto y^* b & \mapsto & V \\
\rightarrow & \overline{b} y & \mapsto & (b \overline{b} y)^t = y^t (bb)
\end{array}
\]
Here's a good picture of the anti-linear map associated to a symmetric $b$.

$$
\begin{align*}
V \overset{b}{\longrightarrow} V^t \overset{\ast}{\longrightarrow} V \overset{b}{\longrightarrow} V^t \overset{\ast}{\longrightarrow} V \\
\otimes \overset{b^*}{\longrightarrow} (b^* b) \overset{(b^* b)^*}{\longrightarrow} (b^* b) \overset{b}{\longrightarrow} V
\end{align*}
$$

So the antilinear map is $T_b x = b^* x$ and $T_b^2 x = (b^* b) x$.

You also want the antilinear map on $V^t$.

$$
\begin{align*}
V^t \overset{\ast}{\longrightarrow} V \overset{b}{\longrightarrow} V^t \overset{\ast}{\longrightarrow} V \overset{b}{\longrightarrow} V^t \\
\otimes \overset{b}{\longrightarrow} (b^* b) \overset{(b^* b)^*}{\longrightarrow} (b^* b) \overset{b}{\longrightarrow} V
\end{align*}
$$

You want to compare the preceding with the picture from the symmetric space $\mathbb{P} = \{ [b^* b] : b^* b = b \}$ with conjugation action by $K = \{ [t^* t] : t \in \mathbb{U}(n) \}$.

You want a spectral decomposition of $\begin{bmatrix} b^* & 0 \\ 0 & b \end{bmatrix}$. This is a skew-adjoint operator, so it has a spectral decomposition arising from $-X^2 = \begin{bmatrix} b^* b & 0 \\ 0 & b b^* \end{bmatrix} = \begin{bmatrix} b b^* & 0 \\ 0 & b^* b \end{bmatrix}$, which is self-adjoint $\geq 0$. Note that $\mathbb{P}, K$ operate on $\begin{bmatrix} V \otimes V^t \\ V^t \otimes V \end{bmatrix}$.

Now use the spectral decomposition of $-X^2$ to split the skew-adjoint of $X$ on $\mathbb{P}$ canonically into "blocks" where $-X^2$ is constant $\geq 0$. There should be a similar decomposition for your antilinear transformation picture. Assume $b$ nonsingular, rescale the blocks via polar decomp of $X$ so that $X^2 = -1$. 

...
Fix a subspace \( W \) from the quaternionic space \( H(V) \).

Assume that \( b \in \mathbb{R} \), then a real structure on \( V \) is defined by \( T_b(x) = b \cdot x \).

**Alternate version:** replace \( V \) by \( V_t \) ending with a real structure on \( V_t \).

Assume now that \( T_b(x) = b \cdot x \) is a real structure on \( V_t \).

Then a real structure on \( V_t \) is given by \( T_b(x) = b \cdot x \) and its lifts to \( V_t \) do not change.
Review things. You want to understand the space of polarizations of \( H(V) \). A polarization is an ordered pair of orthogonal Lagrangian subspaces. It should be determined by the 1st subspace. This is clear from the CCR viewpoint.

\[
[a_i, a_j] = 0, \quad [a_i^*, a_j^*] = 0, \quad [a_i, a_j^*] = \delta_{ij}
\]

Let \( H(V) \) be the complex v.s. having basis \( a_i, a_i^* \). You have a symplectic form on \( H(V) \) with \( \omega(a_i, a_j) = [a_i, a_j] \) etc. Conjugation of \( \star \) on \( H(V) \): \( \star \) is anti-linear square = 1. Conventions are that

\[
[a_i, a_j^*]^* = -[a_i^*, a_j]
\]

How do you express the CCR structure?

\[
H(V) = V \oplus V^* = V^* \oplus V
\]

Complex vector space with basis \( a_1, \ldots, a_n, a_1^*, \ldots, a_n^* \) with skew-symmetric form:

\[
\omega = [a_i, a_j^*] = [a_i^*, a_j^*], \quad [a_i, a_k^*] = \delta_{ik}
\]

You should have \( [a_i, a_i^*]^* = -[a_i^*, a_i^*] \)

\[
\delta_{jk} = [a_j, a_k^*]^* = [a_k, a_j^*] = \delta_{kj} = \delta_{jk}
\]

Do we should have what? A 2n-dim v.s., a symplectic form \( \omega(x, y) \), also anti-linear inv. \( \star \).

\[
\omega(x, y) = \omega(y^*, x^*)
\]

Once you get the structure correct? A 2n-dim complex symplectic space \( H \) + \( \star \) anti-linear involution \( \phi \) on \( H \) s.t. \( \omega(\phi x, y) = \omega(x, \phi y) \)

Then use unitary equivalence theory for skew-symmetric forms. Possible problem here is how to ensure that the real structure given by \( \star \) is compatible. What's going on is that you're given the
The symplectic form and a conjugation $\ast$. You should first understand possible conjugations on a complex $V$.

Look carefully at $[a_j, q_k] = 0 = [a^*_j, a^*_k]$.

$[a_j, a^*_k] = \delta_{jk} = [a^*_k, a^*_j]$. You have a space $V \oplus V^*$ with basis $a_j, a^*_k$, $1 \leq j, k \leq n$, and a skew-symmetric form.

You haven't made precise the aim. It seems that you have a $2n$-dim $\mathbb{C}$ v.s. which is hyperbolic i.e. $[V, V^*]$

You want a good formulation of the structure.

$$W = \left\{ x \otimes a_j + y \otimes a^*_k \right\} \ni [x]^t [a]$$

$$(x \otimes a_j + y \otimes a^*_k)^* = \overline{x} \otimes a^*_j + \overline{y} \otimes a^*_k = [\overline{x}]^t [a^*]$$

when is $\sum x \otimes a_j + y \otimes a^*_k$ real?

$$\overline{\sum x \otimes a_j + y \otimes a^*_k}$$

when $y \otimes \bar{a}^*_j = \bar{x} \otimes a^*_j$ or $y = \bar{x}$ when $y \otimes \bar{a}^*_j = \bar{x} \otimes a^*_j$

Next

$$\left[ x \otimes a_j + y \otimes a^*_j, x \otimes a \right]$$
You are trying to understand the symmetries of the CCR. You have a vector space $H/C$ with basis $a_j, a_j^*$ for $1 \leq j \leq n$, and equipped with two structures:

1) antilinear involution $*$ defined by

\[(a_j)^* = a_j^* , \quad (a_j^*)^* = a_j .\]

2) skew symmetric bilinear form $[\xi, \eta]$, defined by

\[ [a_j, a_k] = [a_j^*, a_k^*] = 0, \quad [a_j, a_k^*] = \delta_{jk} = -[a_k^*, a_j] .\]

So you this $v.s. H$ with basis $a_j, a_k^*$ of $2n$ elts. You have real structure given by first subspace of $\mathbb{R}$.

Obvious question whether $[\xi, \eta]^* \in \mathbb{R}$ where $\xi^* = \xi$ and $\eta^* = \eta$. From the operator interp. you have the following link between $*$ and $[\xi, \eta]$:

\[[\xi, \eta]^* = (\xi^* \eta - \eta^* \xi)^* = \eta^* \xi^* - \xi^* \eta^* = -[\xi^*, \eta^*] \]

or \[[\xi, \eta]^* = [\eta^*, \xi^*] \]

\[[a_j, a_k^*]^* = [a_k^*, a_j^*] \]

If $\xi, \eta$ are real $\xi = \xi^*, \eta = \eta^*$, then $[\xi, \eta]^* = [\eta, \xi] = -[\xi, \eta]$ so $[\xi, \eta] \in i\mathbb{R}$ interesting.

Let $n=1$,

\[ H = \{ xa + y a^* \mid x, y \in \mathbb{C} \} . \]

\[(xa + ya^*)^* = xa^* + ya \]

antilinear sq $= 1$:

\[ [y] = \begin{bmatrix} y_1 \\ -1 \\ y_2 \end{bmatrix} \]

\[ [x] = \begin{bmatrix} x_1 \\ 0 \\ x_2 \end{bmatrix} \]

\[ [x] [y] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} [x^*] = \begin{bmatrix} x_1 y_2 - y_1 x_2 \\ y_1 \end{bmatrix} \]
So you get something unexpected, namely that the \(*\) operator on the space of creation + annihilation operators seems different? Go over this with a review of $H(V)$ and its structure. $H(V) = \left[ \begin{array}{c} \mathbf{V} \\ \mathbf{t} \end{array} \right]$ with symplectic form.

$$\left[ \begin{array}{c} x_1' \\ y_1' \end{array} \right] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \left[ \begin{array}{c} x_2 \\ y_2 \end{array} \right] = \left[ \begin{array}{c} x_1 \\ y_1 \\ y_2 \\ -x_2 \end{array} \right] = x_1y_2 - y_1x_2$$

$g \in \text{Sp}(2n, \mathbb{C}) : g^T J g = J. \quad \text{inf} \quad x^T J + J x = 0 \quad \Rightarrow \quad -x^T J = J x J^{-1}$

$X = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right]$  $x^T = \begin{bmatrix} -a^* & -c^* \\ b^* & -d^* \end{bmatrix} = J X J^{-1} = \left[ \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right] \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right]$

$= \begin{bmatrix} c d & 0 & -1 \\ -a - b & 1 & 0 \end{bmatrix} = \begin{bmatrix} d & -c \\ -a & b \end{bmatrix}$  $b = b^* , c = c^* , d = -a^*$

$X^* = X$  $a^* = -a, \ a = -a^*$

$X^* + X = 0$  $a^* = -a, \ a = -a^*$

$X = \left[ \begin{array}{cc} a & b \\ -b^* & -a^* \end{array} \right]$  $b^* = b$  $a = b^* b$.

$X = \left[ \begin{array}{cc} a & b \\ -b & a \end{array} \right]$  3 cases. $X^* + X = 0, \ x^T J + J x = 0, \ J X = X J$

Start again with $H$, its symplectic form, and the antilinear involution:

$$\left[ \begin{array}{c} x \\ y \end{array} \right]^* = \begin{bmatrix} y \\ x \end{bmatrix}.$$  Now combine the symplectic form and $\ast$:  $H \xrightarrow{J} H^t \xrightarrow{\ast} H$?

To get a hermitian form.
Calculate: \[ (x_1 a + y_1 a^*)^* x_2 a + y_2 a^* ] = x y_1 - y x_1

This is the symplectic form on \( H = \{ x a + y a^* \} \).

Next \((x a + y a^*)^* \) is \( x a^* + y a \). Try

\[
\left[ \begin{array}{c}
 x_1 a + y_1 a^* \\
 x_2 a + y_2 a^*
\end{array} \right] = \left[ \begin{array}{c}
 x_1 x^*_2 - y_1 y^*_2
\end{array} \right]
\]

You probably want the antilinear part on the left.

\[
\left[ \begin{array}{c}
 x_1 a + y_1 a^* \\
 x_2 a + y_2 a^*
\end{array} \right] = \left[ \begin{array}{c}
 y y^*_2 - x x^*_2
\end{array} \right]
\]

Start again \([x a + y a^*, x a + y, a^*] = x y_1 - y x_1 \)

\((x a + y a^*)^* = \bar{y} a + \bar{x} a^* \)

\[
\left[ \begin{array}{c}
 x_1 a + y_1 a^* \\
 x_2 a + y_2 a^*
\end{array} \right] = \left[ \begin{array}{c}
 \bar{y} y^*_2 - \bar{x} x^*_2
\end{array} \right]
\]

What you've done is to compare the antilinear of \((x a + y a^*)^* \) with the symplectic form, to get a hermitian form.

What's the symplectic form restricted to real ends.
and you have the anti-involution

\[ \star : x a + y a^* \mapsto (x a + y a^*)^* = \overline{y} a + \overline{x} a^* \]

You can combine this operator with the symplectic form to get a sesquilinear form:

\[ \overline{y} g_2 - \overline{x} g_1 \]

\[ \left[ (x a + y a^*), x_2 a + y_2 a^* \right] = \left[ \overline{y} g_2 + \overline{x} a^*, x_2 a + y_2 a^* \right] \]

which is hermitian symmetric.

Another point is \( \left[ z_1, z_2 \right] = \left[ z_1^*, z_2^* \right] \); this comes from the operator interpretation of the bracket and should be checked:

\[ \left[ z_1^*, z_2^* \right] = - \overline{\left[ z_1, z_2 \right]} \]

\[ \left[ z_1^*, z_2^* \right] = \left[ \overline{y}_1 x + \overline{x}_1 a^*, \overline{y}_2 a + \overline{x}_2 a^* \right] = \overline{y}_1 x_2 - \overline{x}_1 y_2 \]

\[ \left[ z_1, z_2 \right] = y_1 y_2 - y_1 x_2, \quad \left[ z_1, z_2 \right] = \overline{\left[ z_1, z_2 \right]} \]

So now you want to leave \( \text{Sp}(2n) \) and \( \text{SO}(2n) \) and move on to periodicity.

\[ \text{Idea: Symmetries of the CCR.} \quad \text{You know that these should give the real symplectic group. Let's check.} \]

You want \( g \in \text{GL}_2(\mathbb{C}) \) respecting symplectic form - which means \( g \in \text{SL}(2,\mathbb{C}) \) - but also \( g \) should respect the hermitian form \( \overline{y}_1 x_2 - \overline{y}_1 x_2 = \left[ x_1^*, x_2^* \right] \)

\[ \left[ x_1^*, x_2^* \right] = \left[ 0 1 \right] \left[ x_2 \right] = x_1 y_2 - y_1 x_2. \quad \text{So if } g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right], \text{ then} \]

\[ g^t J g = J \Rightarrow (\det g)^2 = 1 \]

\[ g^t = \left[ \begin{array}{cc} a & c \\ d & b \end{array} \right], \quad g^t = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \left[ \begin{array}{cc} d & b \\ -c & a \end{array} \right], \quad (\det g)^2 = 1 \]

\[ \left[ a c \right] \left[ \begin{array}{cc} a & c \\ d & b \end{array} \right] = \left[ a e \right] \left[ \begin{array}{cc} a & c \\ d & b \end{array} \right] \text{ if } \det g = 1, \text{ this only condition is } a \neq e. \]
\[ \mathbb{C}^2 \]

Next, suppose you have \( g \in \text{SL}(2, \mathbb{C}) \) and \( g^* \varepsilon g = \varepsilon \) \( \Rightarrow g^* = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} = \varepsilon [d-b \varepsilon] = [d+b] \varepsilon [c-a] = [c-a] \varepsilon [d-b] \varepsilon \]

so \( a = \bar{a}, \quad b = \bar{b}, \quad c = \bar{c}, \quad \alpha = \bar{\alpha} \) \( \Rightarrow g = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} : \quad |a|^2 - |b|^2 = 1 \)

Check again \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}(2, \mathbb{C}) \), assume \( g^* \varepsilon g = \varepsilon \)

i.e. \( g^* = \varepsilon g^{-1} \varepsilon = \varepsilon \begin{bmatrix} d-b \varepsilon & d-b \varepsilon \\ -c-a \varepsilon & -c-a \varepsilon \end{bmatrix} = \begin{bmatrix} d-b \varepsilon & d-b \varepsilon \\ -c-a \varepsilon & -c-a \varepsilon \end{bmatrix} = g^* \)

Go back to real periodicity thru. via Morse theory, where you have problems with real Clifford algebras. Passing from \( \text{Cliff}_n \) to \( \text{Cliff}_{n+1} \)

\[ \mathbb{Z} \times \mathbb{BO} \]

Try to describe the spaces which occur - these are symmetric spaces - a homogeneous space of a Lie group by the centralizer of an involution.

0th space is \( \mathbb{Z} \times \mathbb{BO} \) infinite real Grass. \( \pi_0 = \mathbb{Z} \)

1st " " \( \Omega \mathbb{BO} = 0 \)

2nd " " \( \Omega \mathbb{BO} \) going from the basepoint to some "antipodal" point \( \Omega \mathbb{BO} \)

What is the rough idea? You consider the loop space of the symmetric space and find a family of geodesics going from the basepoint to some "antipodal" point of the geodesics. The symmetry groups (of the symmetric space) should act transitively on these family of geodesics. This isn't clear. Example needed.
\[ H = \{ [x] \in \mathbb{C}^n \mid \left[ \begin{array}{cccc} x_1^t \otimes I \mid x_2 \end{array} \right] = x_1^t g_2 - g_1^t x_2 \} \]

\[ O(2n, \mathbb{C}) = \{ g \in GL(2n, \mathbb{C}) \mid g S g^t = S \} \quad n = 1 \quad g = \left[ \begin{array}{cc} a & b \\ c & d \end{array} \right] \]

\[ g^t = \left[ \begin{array}{cc} a & c \\ b & d \end{array} \right] = S g^{-1} S = \frac{1}{\det(g)} \left[ \begin{array}{cc} d & -b \\ -c & a \end{array} \right] S = \frac{1}{\det(g)} \left[ \begin{array}{cc} a & -c \\ -b & d \end{array} \right] \]

If \( \det(g) = 1 \): \( b = c = 0 \) \( \Rightarrow g = \left[ \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right] \)

If \( \det(g) = -1 \): \( a = d = 0 \) \( \Rightarrow g = \left[ \begin{array}{cc} 0 & b \\ b & 0 \end{array} \right] \) dihedral picture.

\[ L O(2n, \mathbb{C}) = \{ X \in M_{2n}(\mathbb{C}) \mid X^t S + S X = 0 \} \]

\[ X^t = \left[ \begin{array}{cc} -a^t & -c^t \\ -b^t & -d^t \end{array} \right] = S X S = \left[ \begin{array}{cc} d & c \\ b & a \end{array} \right] \]

\[ b^t = -b \quad c^t = -c \quad a = -a^t \quad d = d^t \]

\[ L U(2n) = \{ X \in M_{2n}(\mathbb{C}) \mid X^* X = 0 \} \]

\[ -X^* X = X \Rightarrow -X^t = X \]

\[ \begin{array}{c}
    \left[ \begin{array}{cc}
    a & b \\
    c & d
    \end{array} \right] = \left[ \begin{array}{cc}
    \frac{a}{b} & \frac{c}{d}
    \end{array} \right] \quad X = \left[ \begin{array}{cc}
    a & b \\
    c & d
    \end{array} \right]
\end{array} \]

Def. \( O(2n) = U(2n) \cap O(2n, \mathbb{C}) = \{ g \in U(2n) \mid g^t S g = S \} \)

Next

You want \( \text{The family of minimal geodesics} \), you want \( \text{the maximal torus} \) and \( \text{Cartan subalgebras} \). Try

\[ e^{i\theta} = \left[ \begin{array}{cc}
    e^{i\theta} & 0 \\
    0 & e^{-i\theta}
    \end{array} \right] \quad 0 \leq \theta \leq \pi \]

space of these geodesics is \( O(2n)/\text{centralizer of } \left[ \begin{array}{cc}
    0 & 1 \\
    1 & 0
    \end{array} \right] = i\varepsilon \)

which should be \( \{ [u \bar{u}] : u \in U(n) \} \).
Next you want to understand Clifford algebras. But first you might try the next best case. So far you have a hag with increasing range $\Omega(SO(2n), 1, -1) \rightarrow O(2n)/U(n)$ two components. Two components.

So let's try to understand the space $O(2n)/U(n)$ which should be the h-fibre of $BU(n) \rightarrow BO(2n)$

Idea: These symmetric spaces you encounter such as $O(2n)/U(n)$, $Sp(2n)/U(n)$, $U(n)/O(n)$, and $U(2n)/Sp(2n)$ - do they describe polarizations of some sort? This should fit with Clifford algebras easily.

Let's try to clarify Clifford algebras. Def.

Cliff($\mathbb{R}^n$). Consider C as a 2/2 graded algebra over $\mathbb{R}$ where $i: 1$ is even and $i: i$ is odd.

Basic odd-even grading by complex conjugation.

$\text{Cliff}_n = C \otimes C \otimes \ldots \otimes C$ tensor product of superalgs.

Do this $\otimes$ ind: Let $A = A^+ \oplus A^-$ be a superalg then $A \otimes C = \begin{bmatrix} A^+ \otimes 1 & A^- \otimes 1 \\ A^+ \otimes i & A^- \otimes i \end{bmatrix}$?

$\mathbb{R} C C \otimes C$

usual Euclidean product. $\text{Cliff}_n$ takes defn Clifford alg of $\mathbb{R}^n$ basis vector $e_1 \ldots e_n$
Clifford alg defined like exterior alg

For linear \( s: \mathbb{R}^n \to \text{Cliff}(\mathbb{R}^n) \) s.t. \( s(x)^2 = -(x,x) \). Then
\[
(s(x) + s(y))^2 = -(x,x) + s(x)s(y) + s(y)s(x) + \overline{s(y)}s(y)
\]
\[-s(x+y) = -(x,x) \]

\( \mathbf{e}_i \) th i-th basis
\[
s(e_i)^2 = -(e_i,e_i) = -1
\]

\[ s: \mathbb{R}^n \to A \text{ linear } s(x)^2 = -(x,x) \]
\[
s(x+y)^2 = (s(x) + s(y))^2 = s(x)^2 + s(x)s(y) + s(y)s(x) + s(y)^2
\]
\[-s(x+y) = -(x,x) \]
\[
\{s(x), s(y)\} = -(x,y) - (y,x)
\]
\( s(x), s(y) \) anti-commute when \( (x,y) = 0 \).

Also if \( (x,x) = 1 \), then \( s(x)^2 = -1 \). \( \Rightarrow \) an orthogonal basis yielding anticommuting operators of square \( -1 \).

\( V, W \) real v.s. equipped with nondeg quadratic form
\[
C(V) \quad \text{Alg} \text{ A generated by } x(v), v \in V \text{ subject to the relations } s \text{ linear/R, } s(v)^2 = (v,v)
\]
\( \Rightarrow s(v), s(v') + s(v')s(v) = 2(v,v') \)
\( \Rightarrow s(v), s(v') \text{ anti-commute } \iff V \perp V' \).

Do you want
\[ V \text{ quadratic space over } \mathbb{R}, \quad \text{Def } C(V) = \text{alg A gen by } s : V \rightarrow A \text{ sat } \]
\[ s \text{ linear } \mathbb{R}, \quad s(v)^2 = (v, v), \]
\[ s(v + v')^2 = (s(v) + s(v'))^2 = s(v)^2 + s(v)s(v) + s(v')s(v') + s(v)s(v') + s(v)s(v') \]
\[ (v + v', v + v') = (v, v) + (v, v') + (v', v) + (v', v') \]
\[ \Rightarrow s(v)s(v') + s(v')s(v) = 2(v, v') \]

Prop. For orth \( \oplus \)
\[ C(V \oplus w) = C(V) \otimes C(w) \]

where \( \otimes \) means superalgebra. \( \otimes \)

Sylvester says any quadratic space over \( \mathbb{R} \)

is determined up to isom by \( \dim \) and signature
\[ \sum_{i=1}^p x_i^2 - \sum_{j=1}^q y_j^2 \quad \text{signature} = p-g \]
\[ \dim = p+q \]

\( C_{10} = \mathbb{R}(\mathbb{R}, (x, x) = x^2) = \mathbb{R}[s, s^2 = 1] = \mathbb{R}[x] \).

\( C_{10} = C_2 \otimes \cdots \otimes C_2 \otimes [s] = \bigwedge (\mathbb{R}^p) \text{ exterior alg.} \)

Point of Clifford algebras is to construct \( \Lambda V = \Lambda C^n \)

classes. In the complex case you have \( \Lambda V = \Lambda C^n \)

equipped with \( \Lambda e(v) + i(v^*) \) \( \Lambda \) operations satisfying
\[ (e(v) + i(v^*))^2 = e(v) + i(v^*) + i(v^*)e(v) = v^*v \]
\[ i(v^*)e(v) = e(v^*)v \]
\[ i(v^*)i(v) = (v^*v) - v^*i(v^*) \]

In this example the Clifford module is \( \Lambda V \) with \( \mathbb{Z}/2 \) grading and the self adjoint operators \( e(v) + i(v^*) \).

If \( n = 1 \)

\[ v = z, v^* = \bar{z} \]

\[ \Lambda^0 \mathbb{C} \xrightarrow{\mathbb{Z}/2} \Lambda \mathbb{C} \]
Clifford modules, a tool for understanding real Bott periodicity. Start with a real vector space $V$ equipped with a non-degenerate quadratic form. Define $\text{Cliff}(V)$ to the alg over $\mathbb{R}$ generated by an $\mathbb{R}$-linear map $s: V \to \text{Cliff}(V)$ subject to the relation $s(v)^2 = (v, v)$ $\forall v \in V$. Then by polarization you have

$$s(v)s(v') + s(v')s(v) = 2(v, v')$$

whence $s(v)$ and $s(v')$ anticommute $\Rightarrow (v, v') = 0$. From this you deduce:

If $V$ is the orthogonal direct sum of subspaces $V'$ and $V''$ then one has a canonical isomorphism of $\mathbb{R}$-algebras

$$C(V') \otimes C(V'') \simto C(V)$$

Describe: If $V = V' \oplus V''$, then the quadratic form on $V$ restricts to non-degenerate quadratic forms on $V'$ and $V''$. Why: $V = V' \oplus V''$ means that $V = V' \oplus V''$ and $(v', v'') = 0 \forall v' \in V', v'' \in V''$

What does it mean for $(\cdot, \cdot)_V$ to be non-degenerate? Ans: the map

$$V \longrightarrow V^*$$

$$v \longmapsto \{v_i \longmapsto (v, v_i)^2\}$$

is bijective. By f.d. enough to show that $v \neq 0$ $\Rightarrow \exists v_i$ s.t. $(v, v_i) \neq 0$. You can factor the map

$$V' \oplus V'' \simto V \longrightarrow V^* \longrightarrow (V^*)^\tau \oplus (V'')^\tau$$
of maps $\begin{pmatrix} V' \\ V'' \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ V' \end{pmatrix}$ and the point is that the off-diagonal maps are 0 because $V' \perp V''$ (and $V'' \perp V'$). Since the matrix is invertible so must be the maps $(V')^* \rightarrow V', (V'')^* \rightarrow V''$, and so $(\cdot, \cdot)_V$ then restricted to both $V'$ and $V''$ is non-degenerate.

Degression: Consider $H = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ symplectic equipped with another polarization close to the given one.

$W_+ \rightarrow V_+$

$W_+ \cap V_-$

$1 \perp V_+$

$1 \perp V_-$

$F = \pm 1 \text{ on } W_+$

$X = \begin{pmatrix} 0 & -b^* \\ b & 0 \end{pmatrix}$

$F(e^{1+X}) = (1+X)e^{1+X}$

$F(e^{1-X}) = (1+X)e^{1+X}$, $g = F_X = \frac{1+X}{1-X}$

and also $g^{1/2} = \frac{1+X}{(-X^2)^{1/2}}$. Since $W_+$ Lagrangian you know that $b = b^*$ and conversely. Note then that $W_-$ is Lagrangian because $-b^* = -b$ is symmetric.

(This shows clearly that $W_+$ Lagrangian $\iff (W_+)^\perp$ Lagrangian)

The remaining point you would like to check is that $g^{1/2} \in Sp(2n)$ i.e. $u^T J u = J$ where $u = g^{1/2}$.

First look at $g = \frac{1+X}{1-X}$.

Let's work out the details. Can you set up a DE?

$g_{1t} = \frac{1+tx}{1-tX} \\
\dot{g} = \frac{(1-tx)(x) + (1+tx)(+x)}{(1-tx)^2} = \frac{2x}{(1-tx)^2}$

$g^{-1} \dot{g} = \frac{1-tx}{1+tx} \frac{2x}{(1-tx)^2} = \frac{2x}{1-tx^2}$
\[ g = F \frac{1+X}{1-X}, \text{ make path } g_t = \frac{1+tx}{1-tX}, \text{ then } \]
\[ g = \frac{(1-tx)(x) - (1+tx)x}{(1-tx)^2} = \frac{2x-tx^2+tx^2}{(1-tx)^2} = \frac{2X}{(1-tx)^2} \]
\[ g^{-1} = \frac{1-tx}{1+tx} \frac{2X}{(1-tx)^2} = \frac{2X}{1-x^2X^2}, \text{ What's interesting here is that } g \text{ and } g^{-1} \text{ are functions of } X, \text{ so you should be able to study what happens by decomposing } X \text{ into its eigenspaces. Suppose you restrict to the eigenvalue } \lambda. \text{ Look at a simple case. } X = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \]
\[ g^{-1/2} = \begin{bmatrix} 1 & -b \\ b & 1 \end{bmatrix} (1+|b|^2)^{-1/2} \]

Repeat. You considered a polarizer close to the basepoint polarization
\[ W_+ = \begin{bmatrix} 1 \\ b \end{bmatrix} V_+, \quad W_- = \begin{bmatrix} -b^* \\ 1 \end{bmatrix} V_- \]
\[ F = \pm 1 \text{ in } W_+, \text{ then } F \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} = \begin{bmatrix} 1 & -b^* \\ b & 1 \end{bmatrix} \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^* \end{bmatrix} \]
\[ F(1+X)\varepsilon = (1+X)\varepsilon \quad \varepsilon(1-X) \]
\[ F \frac{1+X}{1-X}, \quad g^{-1/2} = \frac{1+X}{1-x^2X^2} \]

Question: So far you've looked the unitary (or Hilbert space) picture. Next consider the symplectic structure. Then \( W_+ \) is Lagrangian, so you know \( b^t = b \). This implies also that \( -b^* = -b \) is symmetric, so \( W_- \) is Lagrangian. You now want to show that \( g^{1/2} \in \text{Sp}(2n) \). Let \( u = g^{1/2} \), you know \( u \in U(2n) \) and need to show \( u^*J_u = J \).
Review: You noticed that the CCR yield a symplectic form and a real structure (anti-involution) which is not the same as what you get from the basic representation of $Sp(2n)$. Recall in case $n=1$. The basic rep is $\mathbb{C}^2$ equipped with the symplectic form:

$$\begin{bmatrix} x_1^+ & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_2 - y_1 x_2 \\ \end{bmatrix}$$

and the positive herm. form $\begin{bmatrix} x_1^* & x_2 \\ y_1 & y_2 \end{bmatrix} = x_1^* x_2 + \bar{y}_1 \bar{y}_2$. The "quotient" of these forms is an anti-linear operator on $H$:

$$H \rightarrow H^t \xrightarrow{\ast} H$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = z \mapsto z^t J \mapsto -J \bar{z} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} -\bar{y} \\ x \end{bmatrix}$$

having square $= -1$.

In the case of the CCR you have the $\mathbb{C}$-vector space of elts. $x a + y a^*$, $x, y \in \mathbb{C}$ with symp. form $\begin{bmatrix} x_1 a + y_1 a^* & x_2 a + y_2 a^* \end{bmatrix} = x_1 y_2 - y_1 x_2$.
The basic problem seems to be how invariant the situation is. You start with

\[ X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} \]

which is skew-adjoint. Note that \( b^* = b \Rightarrow -b^* = -b \), so

\[ X = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} \]

Recall \( \text{Sp}(2n) = \{ \begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix} : \ a^* = -a, \ b^* = b \} \). Therefore \( X^T J + JX = 0 \), so \( X \) is an inf. symmetry of the symplectic structure. You want to go from this fact to show \( g = \frac{1 + X}{1 - X} \) and \( u \bar{u} = \frac{1 + X}{(1 - X)^{1/2}} \)

are global symmetries of the symplectic structure.

\[ u^T J u = J \iff J u = \bar{u} J \]

Therefore \( JX = (-X^T)J = -XJ \) because

\[ J(1 + X) = (1 + X)J, \quad J(1 - X) = (1 - X)J \]

\[ J(1 - X)^T = (1 - X)^T J \]

\[ JgJ^{-1} = J(1 + X)(1 - X)^{-1}J^{-1} = (1 + X)(1 - X)^{-1} = \bar{g} \]

\[ J(1 - X)(1 + X)J^{-1} = (1 - X)(1 + X), \quad J(1 - X^2)J^{-1} = (1 - X^2) \]

There should be a clearer way to proceed, probably by working in the algebra of \( X \in \mathfrak{sp}(2n, \mathbb{C}) \) such that \( JXJ^{-1} = X \).

\[ u^*u = 1 \quad u^2 \bar{u} = 1. \]
Review. $H = [V_+] = \left[ \begin{array}{c} C^n \\ C^n \end{array} \right]$ equipped with $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^*x_2 + y_1^*y_2$, $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1^*y_2 - y_1^*x_2$.

Positive Herm form.

Symplectic form

$H \xrightarrow{J} H^t \xrightarrow{*} H$

$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x^t \\ y^t \end{bmatrix}$

$\begin{bmatrix} x^t & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} x^t \\ y^t \end{bmatrix}$

$\begin{bmatrix} -y^t & x^t \\ x^t & y^t \end{bmatrix}$

operator amounting to an $H$-structure on $H$.

Look at operators on $H$. Linear symmetries $X^* + X = 0$

$J \in \text{sym} \left[ \begin{array}{c} a \\ b \end{array} \right]$ $X^tJ + JX = 0$

$L(\text{Sp}(2n)) = \{ X : a^* = -a, b^* = b \}$, $JX = XJ$

$X = \left[ \begin{array}{cc} 0 & -b \\ b & 0 \end{array} \right]$, $b^* = b$

You want to show that the C.T. of $X$ lies in $\text{Sp}(2n)$. Better: you know that $u = \frac{1 + X}{(1 - X^2)^{1/2}} \in U(2n)$. You want $\overline{u}^tJu = J$ or $\overline{u}u = J$

So you want to show that $\overline{u}uJ^{-1} = \overline{u}$. Important point is that $\{ X : JXJ^{-1} = X \}$ is an algebra, a f.d. algebra; probably $M_n H$. It should be true that this alg is $\{ \begin{bmatrix} a & b \\ \overline{a} & \overline{b} \end{bmatrix} : a, b \in M_n C \}$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ \overline{a} & \overline{b} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \overline{a} & b \\ b & \overline{a} \end{bmatrix}$
Let's see if we now understand polarizations. Let's do the orthogonal case. \( H = \begin{bmatrix} V_+ \\ V_- \end{bmatrix} = \begin{bmatrix} e^{i\theta} \\ e^{-i\theta} \end{bmatrix} \). Two forms:

\[
\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \begin{bmatrix} y_2 \\ x_2 \end{bmatrix} = x_1 y_2 + y_1 x_2, \quad \begin{bmatrix} x_1^* \\ y_1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = x_1 y_2 + y_1^* x_2.
\]

\( X \in M_{2n} \mathbb{C} \) operators on \( H \). \( X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2n} \mathbb{C} \)

(i) \( X^* + X = 0 \)

(ii) \( X^* S + S X = 0 \)

(iii) \( S X S = X \)

Next consider a polarization close to the basepoint polarizations. Again in the unitary picture, you have \( g = F E = \frac{1 + X}{1 - X} \), \( u = \frac{1 + X}{(1 - X)^{1/2}} \).

Now impose the condition \( W_+ \) is Lagrangian, then \( X = \begin{bmatrix} 0 & -b^* \\ b & 0 \end{bmatrix} \) must belong to alg sat \( S X S = X \). Therefore \( u \) should lie in this alg.