\[ g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ g^* = \begin{pmatrix} -y & x+y \\ x-y & -y \end{pmatrix}, \quad g^* (x+iy) = -(x-y) + x \\ d \left( -xy + \frac{y^2}{2} \right) \]

\[ Q = 2\pi i \left( -xy + \frac{y(y-1)}{2} \right). \]

Thus

\[ y \in \mathbb{Z} \implies e^{2\pi i y} = \frac{1}{y(y-1)} \mathcal{C} (-y, x-y) \in \mathcal{C} \]

\[ Q' = \frac{1}{2\pi i} Q = -xy + \frac{y^2}{2} - \frac{x}{2} \]

\[ g^* Q' = \begin{pmatrix} x+y & -y \\ -y & -x+y \end{pmatrix} = \begin{pmatrix} -y & x+y \\ x-y & -y \end{pmatrix} \\ \begin{pmatrix} -y & x+y \\ x-y & -y \end{pmatrix} \]

\[ g^* Q' = -\begin{pmatrix} x-y & x+y \\ x-y & -y \end{pmatrix} = -\begin{pmatrix} x-y & x+y \\ x-y & -y \end{pmatrix} \]

\[ = -\begin{pmatrix} x^2 + xy - y^2 & 0 \\ 0 & x \end{pmatrix} = -\begin{pmatrix} x^2 + xy - y^2 & 0 \\ 0 & x \end{pmatrix} \]

\[ = -\begin{pmatrix} x^2 + xy - y^2 & 0 \\ 0 & x \end{pmatrix} = -\begin{pmatrix} x^2 + xy - y^2 & 0 \\ 0 & x \end{pmatrix} \]

\[ e^0 g^* = g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad Q = 2\pi i \left( -xy + \frac{y(y-1)}{2} \right) \]

\[ e^* = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathcal{C} (x+y) \rightarrow \mathcal{C} (-x-y) \]

\[ e^* Q' = e^* \left( -xy + \frac{y(y-1)}{2} \right) = -(-x)(-y) + \frac{(y)(y-1)}{2} \]

\[ = -\frac{x^2 + y^2}{2} \]

\[ e^0 g^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \mathcal{C} (x+y) \rightarrow \mathcal{C} (-x-y) \]

\[ g = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathcal{C} (x+y) \rightarrow \mathcal{C} (-x-y) \]

\[ e^{2\pi i} (-xy + \frac{x^2+y}{2}) \mathcal{C} (x-y, -x) \]

Answer: No. \]

\[ \mathcal{C} (x+y) \rightarrow e^{2\pi i} (-xy + \frac{x^2+y}{2}) \mathcal{C} (x-y, -x) \]
\[ \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \Rightarrow \tau^2 - \tau + 1 = 0 \Rightarrow \tau = \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2} \in \text{UHP} \]

Stabilizer of \( \frac{-1 + i\sqrt{3}}{2} \) is \( \{g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, -g, -g^2 \} \in \text{UHP} \)

in fact it's the cyclic group of order 6 generated by

\[ -g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

Let's go back to the problem of identifying the principal \( \mathbb{T} \)-bundle \( P \) associated to \( L \) with the Heisenberg group \( H \). What is \( H \)? It's the central group extension with cross-section

\[ \mathbb{T} \longrightarrow H \longrightarrow \mathbb{R}^2 \]

Notation \( P, L \) refer to the absence of automorphic condition associated to the bilinear cocycle \( c((x_1, y_1), (x_2, y_2)) = e^{2\pi i y_1 x_2} \).

But you should start with the operator on \( L \).

Recall \( D_x = \partial_x, \quad D_y = \partial_y + 2\pi i x \).

\[ e^{aD_x} e^{bD_y} \psi(x, y) = e^{a\partial_x} e^{b\partial_y} \psi(x, y) = e^{2\pi i b x} e^{a\partial_x} e^{b\partial_y} \psi(x, y) = e^{2\pi i b x} \psi(x + a, y + b) \]

\[ e^{a_1 D_x} e^{b_1 D_y} e^{a_2 D_x} e^{b_2 D_y} = e^{a_1 D_x} a_2 D_x \left[ b_1 D_y, a_2 D_x \right] e^{b_1 D_y} e^{b_2 D_y} \]

\[ = e^{2\pi i b_1 a_2 (a_1 + a_2) D_x} e^{(b_1 + b_2) D_y} \]

Let's begin with \( H \) described as \( \mathbb{T} \times \mathbb{R}^2 \) with multiple \( (e^{i\varphi_1}, a_1, b_1) \cdot (e^{i\varphi_2}, a_2, b_2) = (e^{i(\varphi_1 + \varphi_2 + 2\pi b_1 a_2)}, a_1 + a_2, b_1 + b_2) \).

You want the infinitesimal lift (left-right) translations

\[ \delta \psi(e^{i\varphi}, a, b) = \partial_x \psi(e^{i\varphi}, a, b) \psi \]

\[ \delta \psi(e^{i\varphi}, a, b) = \partial_x \psi(e^{i\varphi}, a, b) a \partial a + \partial_y \psi(e^{i\varphi}, a, b) \partial b \]

\[ \delta e^{i\varphi} = i e^{i\varphi} \delta \varphi \]

\[ \delta \varphi = \frac{\delta}{i} \psi \]
\[
\psi(x, y) = (e^{i\varphi}, x, y) \cdot (1 + i\delta\varphi, \delta x, \delta y)
\]

\[
= (e^{i\varphi} (1 + i\delta\varphi) e^{iy\delta x}, x + \delta x, y + \delta y)
\]

\[
\psi = (e^{i\varphi} (1 + i\delta\varphi + iy\delta x), x + \delta x, y + \delta y)
\]

So the variation of \(\psi\) under the infinitesimal right translation by \((e^{i\delta\varphi}, \delta x, \delta y)\) should be

\[
\delta \psi = 2\pi i \chi \psi + (\delta \psi + \chi_\varphi \delta\varphi) \delta x + (\delta \psi + \chi_y \delta y)
\]

You need to put in the 2\pi

\[
\psi(e^{2\pi i\varphi}, x, y)
\]

\[
\phi + \delta \phi = \psi(e^{2\pi i\varphi}, x, y) \cdot (e^{2\pi i\delta\varphi}, \delta x, \delta y)
\]

\[
= \psi(e^{2\pi i\delta\varphi + 2\pi iy\delta x}, x + \delta x, y + \delta y)
\]

\[
= \psi + 2\pi i \chi \psi \delta \varphi + 2\pi i \chi_\varphi \delta\varphi \delta x + 2\pi i \chi_y \delta y
\]

\[
\delta \psi = (2\pi i \chi_\varphi \delta \varphi) \delta\varphi + (2\pi i \chi_y \delta \varphi) \delta y + 2\pi i \chi \psi \delta x + 2\pi i \chi_\varphi \delta\varphi \delta x + 2\pi i \chi_y \delta y
\]

So this inf right result yields the vector fields:

\[
2\pi i \chi_\varphi \delta \varphi \chi_\varphi \delta \varphi \delta x + 2\pi i \chi_y \delta \varphi \delta y
\]

Next look at left translations

\[
\psi + \delta \psi = \psi(e^{2\pi i\delta\varphi}, \delta x, \delta y) \cdot (e^{2\pi i\varphi}, x, y)
\]

\[
= \psi(e^{2\pi i(\delta\varphi + \varphi)} e^{2\pi i(y \delta y)}, x + \delta x, y + \delta y)
\]
\[ \psi(x, y) = \psi(x + 2\pi i, y + 2\pi) = \psi(x + 2\pi, y + 2\pi) \]

\[ \Delta \psi = \psi(2\pi i, x, y) = \psi(2\pi, x, y) = \psi(x, y) \]

So inf left mult. yields the vector fields

\[ 2\pi i \partial_z, \quad D_x = \partial_x, \quad D_y = \partial_y + 2\pi i x \partial_z \]

Note that if we use \( \tau = e^{2\pi i} \), then

\[ \partial_\tau = 2\pi i \partial_\phi \]

\[ \frac{\partial_\phi}{\partial \tau} = \frac{1}{2\pi i} \frac{\partial_\tau}{\partial \phi} \]

Also \( [D_x, D_y] = D_\phi \)

Let \( \psi \in \mathcal{L} \); \( \psi(x, y) \) periodic in \( y \) but \( \psi(x + m, y) = e^{-2\pi im} \psi(x, y) \), the phase is in some sense linear in \( x \). You want perhaps to describe the situation using tools from your determinant line bundle paper. On the determinant line bundle you put a metric, you calculated the curvature, then you introduce an exponential factor to cancel the curvature, the resulting connection is flat yielding the desired determinant up to scalar factor.

Where do you start? With \( L \) and the autom. condition

\[ \psi(x + m, y + n) = e^{-2\pi im} \psi(x, y) \]

periodic in \( y \); in the vertical direction; in the \( x \) direction the phase is linear in \( m \).

Let's pay attention to the periodic lines; line means summand of \( \mathbb{Z}^2 \) of rank 1. There's the real version also. Give a line \( l \), i.e. point of \( \mathbb{P} \), what else do you need to obtain an automorphic condition?
Given a Z-line \( l \), i.e., a subgroup \( l \subset \mathbb{Z}^2 \) of rank 1 such that \( \mathbb{Z}^2/l \) is free, what is needed to obtain an automorphic condition having periodicity \( l \).

Consider the case \( l = \{(0, n) | n \in \mathbb{Z}\} \subset \mathbb{Z}^2 \); thus \( l = \mathbb{Z}(0,1) \).

Possible complement are \( \mathbb{Z}(1,k) \subset \mathbb{Z}^2 \) for any \( n \in \mathbb{Z} \).

Discuss importantly. You have a free abelian group of rank 2, the real plane \( \Gamma \otimes \mathbb{R} \), it generates, and a basis \( \gamma_1, \gamma_2 \) for \( \Gamma \). Equip \( \Gamma \) with the volume such that \( \gamma_1 \wedge \gamma_2 = 1 \). You want to understand?

Start again with the lattice \( \Gamma \) in the real plane \( \Gamma_R = \Gamma \otimes \mathbb{R} \) such that volume is 1 on \( \Gamma_R/\Gamma \). Pick a unimodular basis for \( \Gamma \), say \( \gamma_1, \gamma_2 \) so that the \( \square \) has vol = 1.

Is there a natural automorphic condition associated to this basis?

You want periodicity in the \( \gamma_1 \) direction, and also some

Try a different viewpoint, namely, consider a connection on the plane; rather, consider a line bundle \( L \) over the plane equipped with a connection whose curvature is a translation invariant purely imaginary non-degenerate 2-form. There is an associated Heisenberg group action on sections of this line bundle, because you are given a connection on \( L \). Suppose now that the plane comes from a lattice: \( \Gamma = \Gamma_R \otimes \mathbb{R} \), and the "volume" of \( \Gamma_R/\mathbb{Z} \) is integral.
Let's look at the symmetries of $L$ over $\mathbb{R}^2$.

Review the classical mechanics, Hamilton-Jacobi theory.

You have a symplectic manifold $(\mathbb{R}^2, \omega_{xy})$, this is phase space. Recall that phase space is ordinarily the cotangent bundle of configuration space. There is a canonical 1-form $\omega$ with

$$d(\omega_{xy}) = d\omega_{xy},$$

the symplectic form. Contact transformations:

$$\omega_{xy} - \omega_{pq} = dF.$$

In the case of the plane and

$$821.29 + 30,000\text{ ATM} \quad \text{Apr} 12,$$

$$-10 \text{ 1.50} \quad \text{Apr} 9,$$

$$-1 \text{ 1.50} \quad \text{May} 22,$$

$$-1 \text{ 1.95},$$

$$-107.34,$$

$$-30,000\text{ ATM} \quad \text{Mar} 22,$$

$$395 \text{ -54,75},$$

$$-18.00.$$

Your aim is to understand "geometric quantization" in the special case of 1 degree of freedom. Various things come to mind such as Feynmann's path integral picture.

**Idea:** Role of convexity

You know this appears in the Legendre transform (see book by Hörmander), convexity of the moment mapping (Atiyah, Guillemin-Sternberg, Kuran). Is there any link between convexity of this sort and partitions of unity?

Go back to phase space for 1 degree of freedom: the $(\mathbb{R}, \mathbb{P})$-plane equipped with the 1-form $\omega_{xy}$ giving the action leading to Hamilton's principle saying the dynamics are paths of stationary phase for the action $\int dt^2$.

What are the symmetries of phase space? Because you've given the 1-form $\omega_{xy}$, you should have a canonical line bundle with connection having constant curvature, non-degenerate purely imaginary. The line bundle is...
April 20, 02. Consider phase space for 1 degree of freedom, in other words, the cotangent bundle of an affine line \( T^*X \).

Choose a coord \( x: X \rightarrow \mathbb{R} \); a point of \( T^*X \) over \( x \) is linear functional \( ydx \) on the tangent space \( (T^*X)_x \), and \( (x,y): T^*X \rightarrow \mathbb{R}^2 \). There's a canonical 1-form, the contact form \( ydx \) on \( T^*X \), whose diff \( dy dx = -dx dy \) is a symplectic form on \( T^*X \).

To study symplectic diff of phase space arising from affine linear transfs, which must be in \( SL(2, \mathbb{R}) \times \mathbb{R}^2 \) to preserve the sym. form. Your aim is to understand the automorphic condition under these diff, respecting the behavior of integral structure.

The first thing to understand is whether and if so how there is a link between the 1-form and the automorphic condition.

Take \( dx + y dy = dx(2x + 2\pi i y) + dy(3y) \)

\[
\begin{align*}
d + 2\pi i y dx &= dx(2x) + dy(3y + 2\pi i x) \\
&= dx(2x) + dy(3y) + 2\pi i x dy
\end{align*}
\]

Recall usual form

\[
e^m(2x + 2\pi i y) e^n(3y) \psi(x,y) = e^{2\pi i y} \psi(x + m, y + n) = \psi(x,y)
\]

What does this mean? \( \psi \) is \( \psi \) under \( x \rightarrow y+n \).

What is special about the 1-forms that occur? They have the same curvature as \( ydx \), hence differ from \( ydx \) by an exact 1-form, the differential of a form of degree \( 2 \).
You ought to review symplectic stuff, especially the link between Lagrangian subspaces and quadratic forms. Let $V$ be a $\mathbb{R}$ vector space, $W = V \oplus V^*$ equipped with the anti-symmetric form $A(v \otimes \omega, v' \otimes \omega') = \Lambda(v^\top (\omega') - (\omega^\top v')).$ Let $\Gamma_T = \{ v \otimes T v \}$ be isotropic i.e. 0 = $A(v \otimes T v, v' \otimes T v') = (T v^\top (\omega')) - (\omega^\top (T v')).$ Equivalently $\Gamma_T \subset V^*$ is defined by $(T^\top v)^\top (\omega') = (T v^\top (\omega)).$ Thus $\Gamma_T$ is a Lagrangian subspace $\subset T^\top = T^\perp.$

How is this related to something like $ydx$? You can interpret $ydx$ as a bilinear form in 2 variables

$$(x, y) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

which skew-symmetrizes to

$$(x, y) \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = dydx - dxdy.$$

**Answer:** Consider two symplectic vector spaces $W_1, W_2$ and a symplectic isomorphism $U: W_1 \longrightarrow W_2.$ Form a.s.i.

$W_1 \oplus W_2$ with symplectic form $\omega_1$ on $W_1$ and $-\omega_2$ on $W_2.$ The graph of $U$ should be lagrangian which should translate into a quadratic form.

But the situation might be more interesting when you introduce translation.

**Idea:** Novikov in his talk about metals mentioned quasi-momenta, points in a 3 true, which contains the Fermi surface, on which there is some kind of dynamics.

Phase space is just $\mathbb{R}^2$ equipped with the 1-form $ydx$. Here $\mathbb{R}^2$ is viewed as affine space. A better way to put this might be to use the affine plane $z = 1$ in $\mathbb{R}^3.$ When does $g \in GL(n+1, \mathbb{R})$ preserve the last coordinate? $(0 \ldots 0)$? 

\[ \begin{pmatrix} x \\ z \end{pmatrix} \]
1 degree freedom, configuration space = affine line, a tensor $L$ for the Lie group $(\mathbb{R}, +)$. $\mathbb{R}^\times = \text{GL}(1, \mathbb{R})$ is the group of symmetries of the Lie group. Then $\mathbb{R}^\times$ and $(\mathbb{R}, +)$ combined suitably (semi-direct product $\mathbb{R}^\times \ltimes (\mathbb{R}, +)$) yield the symmetries of the affine line. Next comes phase space, the cotangent bundle $T^*\mathbb{R}$.

Problem: Intrinsic definition of an affine line $L$ as a nonzero coset of a 1-dim subspace $l$ of a 2-dim space $V$. Any coset of $l$ is a tensor for $L$. Can you reconstruct $V$ from $L$ and $l$?

The idea here is that a standard way to represent an affine space is via a linear Map $f: V \to \mathbb{R}$ as $L = f^{-1}(1)$.

At some point you need to understand Suslin’s theorem on excision for K-theory of h-unital rings.

Do back to affine line $\mathbb{R}$. Do affine spaces form a category like vector spaces?

April 22, 2022: Discuss the category of affine spaces. Hopefully these are related to partitions of $L$. The first step should be to identify an affine space $L$ of dimension $n$ with a pair $(V, f)$, where $V$ is $\mathbb{R}^n$ of dim $n+1$ and $f: V \to \mathbb{R}$ is a nonzero linear map. Then $L = f^{-1}(1)$. These $f: V \to \mathbb{R}$ form a category.
Another definition of an affine space is a tensor \( L \) for a vector space \( L_0 \), i.e. an action \( L \times L_0 \rightarrow L \) such that \( L \times L_0 \overset{\text{affine}}{=} L \times L \).

Can you construct \( f: V \rightarrow \mathbb{R} \) with \( f(1) = L \)?

Given a tensor \( L \) for the vector space \( W \), you want to construct \( V \overset{f}{\rightarrow} \mathbb{R} \) such that \( L = f^{-1}(1) \), \( W = f^{-1}(0) \).

It seems that \( V \) should be described by generators and relations, something like \( aL + w \). Note that every element of \( V \) not in \( W \) is uniquely \( v = aL \) with \( a \in L \) and \( a \in \mathbb{R}^* \). You should only need to get elements of \( W \). So you consider \( aL + w \) with \( a \in \mathbb{R} \) and \( w \in W \).

These form a vector space \( V = \mathbb{R} \times W \).

Exact sequence \( 0 \rightarrow W \rightarrow V \rightarrow \mathbb{R} \rightarrow 0 \)

Legendre transform in 1-dim = poor man's F.T.

\[
\int e^{-x^2} e^{F(x)} \, dx \sim e^{-(x^2 - F)} \quad \text{where} \quad \frac{d}{dx}(x^2 - F) = 0,
\]

i.e. \( x = \frac{dF}{dx} \).

Use \( x = \frac{dF}{dx} \) and Implicit Func. Thm. to make \( G = x^2 - F \) to regard \( G \) as a function of \( x \). Then

\[
\frac{dG}{dx} = x + \frac{dx}{\frac{dF}{dx}} \frac{dF}{dx} \frac{dx}{dx} = x.
\]
Apr 23, 02

\[ \text{[Affine space]} \quad \Leftrightarrow \quad \text{[V \rightarrow \mathbb{R}^3]} \]

\[ f^{-1}(1) \quad \Leftrightarrow \quad (V \rightarrow \mathbb{R}) \]

Here define affine space as a tensor under the group of a vector space, \( W \). Thus there is an action \( L \times W \rightarrow L \times L \) such that \( L \times W \xrightarrow{(m, \mu)} L \times L \) is an isom. But \( W \) also has scalar multiplication by scalars \( \alpha \in \mathbb{R} \), yielding

\[ \lambda \cdot (\lambda_0, \lambda_1) = (\lambda_0, \lambda_1 - \lambda_0) \]

**Problem:** Define for this operation. Barycentric linear combination.

**IDEA:** Convexity \( \Leftrightarrow \) positive partitions of \( 1 \). Can you weaken positivity to some kind of semi-boundedness? From Poisson Sum Fula work you get an affine line with \( \mathbb{Z} \)-translations, an \( x \)-circle (1-torus) of quasi-momenta.

Novikov's metal talk suggests looking at a surface with \( \mathbb{Z} \)-translational symmetry, 2-torus of quasi-momenta, Fermi curves. Is this Fermi curve related to Novikov's version of Morse theory where the Morse function is \( 
\end{eqnarray*}

In your situation you have a complex line bundle over \( T^2 \)

\[ L \times W \rightarrow L \times L \]

\[ (\lambda, \omega) \rightarrow (\lambda, \lambda + \omega) \]

\[ L \quad \sum \quad \text{somehow you should be able to construct V from this.} \]

\[ W \quad 0 \quad (\lambda, \omega, \alpha) \rightarrow \]

\[ \text{Maybe V is a quotient of } L \times W \times \mathbb{R} \text{ by something in n dimens.} \]
Go back to \( W \xrightarrow{\pi} V \xrightarrow{f} R \), \( L = f^{-1}(1) \), \( W = f^{-1}(0) \).

You want to obtain \( V \) as a quotient of \( W \) and \( L \) in some way, but \( L \) is not a vector space; moreover, you need products \( \Delta \) for \( \alpha \in R, \lambda \in L \). Note that \( R \times L \) is also not a vector space; you need \( R[L] \). There is a canonical map \( R[L] \rightarrow V \) sending \( \sum \alpha_j \lambda \) to \( \sum \alpha_j \lambda \) in \( V \). Is this map onto? You need to show that \( W \) is in the image. \( \Rightarrow \) \( f(\sum \alpha_j \lambda) = \sum \alpha_j = 0 \), so it should be clear, namely given \( w \) chooses \( \lambda_0 - \lambda_1 = w \) and then take \( \alpha_0 = 1 \), \( \alpha_1 = -1 \).

So \( V \) is the vector space generated by the elements \([\lambda]\) for \( \lambda \in L \) subject to the relations: \([\lambda + w] = [\lambda] + i w \) for all \( \lambda \in L \) and \( w \in W \). If \( \lambda_0 \in L \) is chosen, then one has \( \forall \lambda \): \([\lambda] = [\lambda - \lambda_0 + \lambda_0] = [\lambda_0] + i (\lambda - \lambda_0) \), so that \( V = R[\lambda_0] + iW \), etc.

Now you should be able to write up an account of affine spaces. These form a category equivalent to the category of v.s. \( V \) equipped with a map \( f: V \rightarrow R \). There is a final object \( R \rightarrow R \), but no initial object.

It should be true that free a" map of affine spaces is a set map respecting barycentric linear combinations. So the join of two affine spaces should be the direct sum.

Given \( W \xrightarrow{\pi} V \xrightarrow{f} R \), \( W' \xrightarrow{\pi'} V' \xrightarrow{f'} R \) get

\[ W \times W' \xrightarrow{1 \times 1} V \times V' \xrightarrow{f \times f'} R \times R \]
\[ U \xrightarrow{\Delta} \]
\[ W \times W' \xrightarrow{1 \times 1} V \times V' \xrightarrow{f \times f'} R \]

the product affine space \( L \times L' \).
Let's now look at the cotangent bundle of the affine line \( \mathbb{A}^1 \). The tangent space of an affine space \( L \) at any point is the vector space \( W \), so the tangent bundle is \( L \times W \), and the cotangent bundle should be \( L \times W^* \).

What's happening? You have the affine group operating on the affine space \( L \) and also on its cotangent bundle \( L \times W^* \). Somewhere here should be Hamiltonian symmetry, maybe?

You are looking at symmetries of the affine line first, this is configuration space, and then symmetries of phase space. You know the affine symmetries of the line \( \mathbb{A}^1 \) are \( x \mapsto ax + b \), \( a \neq 0 \). What are the natural symmetries of phase space?

Start with contact transformations: 4 variables

\[ P, Q, p, q; \quad \text{such that} \quad P \frac{dQ}{dt} - Q \frac{dp}{dt} = dS. \]

Example: Hamilton's principle

\[ S = \int_a^b L(q, \dot{q}, t) \, dt \]

\[ \int_a^b L(q, \dot{q}, t) \, dt = \int_a^b \left( \frac{\partial L}{\partial q} \dot{q}^2 + \frac{\partial L}{\partial \dot{q}} \ddot{q} \right) \, dt \]

\[ = \int_a^b \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) \dot{q} - \frac{\partial L}{\partial q} \right] \, dt \]

\[ = \left[ \frac{\partial L}{\partial \dot{q}} \right]_b^a - \int_a^b \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right) \dot{q} \, dt \]

This vanishes for \( \dot{q} \) vanishing at \( a, b \). \[ \Rightarrow \quad \int \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} \right) \dot{q} \, dt = 0 \]

Solve eqn of motion to get \( q(t) \), then integrate
Solve Lagrange again to get a family of paths $g(t)$ for $a \leq t \leq b$ which relate the initial conditions at $a$ to the ones at $b$. Integrating $L$ over these paths yields the action $S$ satisfying

$$SS = \left[ p\dot{q} \right]_a^b$$

where $p = \frac{\partial L}{\partial \dot{q}}$

whence a contact transformation.

Suppose $p, q, p\dot{q}$ are variables such that $p\dot{q} = p\dot{q}$, for example $p = a^{-1}p$, $q = a q$. Another example: $p = p$ and $q = q + b$. Combining these two types yields

$$q = a q + b$$

$$p = a^{-1} p \quad a^{-1} p \, d(a q + b) = p \, dq$$

This should be the symmetry of phase space induced by the affine symmetry $q \mapsto a q + b$ of $\mathbb{R}$. Converse: do any diffeomorphism of $\mathbb{R}^2$ preserving $p\dot{q}$ necessarily affine? No because any diffeo of the line induces a diffeom of its phase space preserving the canonical 1-form. Note that $p = 0$ is where the 1-form $p\dot{q}$ vanishes, and that vertical lines $q = \text{constant}$ are the null curves for $p\dot{q}$.

A diffeomorphism of phase space preserving $p\dot{q}$ has to induce a diffeomorphism of the line $p = 0$. It seems likely that latter induces the former, for consider the "difference" between $g^*$ on phase space and $g^*$ the induced diffeom on phase space. This should be the identity on the line $p = 0$, and one has

$$p\dot{q} = g^*(p\dot{q}) = g^*(p) \, dq^*(q) = g^*(p) \, dq$$

so $g^*(p) = p$.

Idea. It may be important to use scaling transformations because of wavelet theory, although there only $a = 2$ appears.
Consider phase space $\mathbb{R}^2$ with the canonical 1-form $\eta = pdx$. Define a contact transformation to be a pair $(f, S_f)$ consisting of a diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a function $S_f$ on $\mathbb{R}^2$ satisfying $f^*\eta - \eta = dS_f$.

Consider two contact transformations $(f, S_f)$ and $(g, S_g)$. Then $f^*\eta - \eta = dS_f$, $g^*f^*\eta - g^*\eta = d(g^*S_f)$, $g^*\eta - \eta = dS_g$. Thus, $(fg)^*\eta - \eta = d(g^*S_f + S_g)$, so composition can be defined for contact transformations by:

$$(f, S_f)(g, S_g) = (fg, g^*S_f + S_g)$$

This seems to be the semi direct product of the symplectic diffeomorphism groups operating the additive group of functions on phase space.

Now $f^*\eta - \eta = dS_f \Rightarrow f^*(d\eta) = d\eta$, i.e., $f$ is symplectic. Conversely, if $f$ is a symplectic diffeomorphism, then $f^*\eta - \eta$ is closed, hence exact $= dS_f$ because phase space is contractible. $S_f$ is determined up to an additive constant. So it seems that the group of contact transformations is a central extension by $\mathbb{R}$ of the group of symplectic diffeomorphisms.

Next consider the infinitesimal situation. An infinitesimal contact transformation on phase space should be $(X, \eta_X)$, where $X$ is a vector field and $\eta_X$ a 1-form satisfying

$$L_X\eta = d\eta_X \quad (\Rightarrow i_X(d\eta) = 0 \text{ so } X \text{ Hamiltonian})$$

Conversely, if $0 = L_X(d\eta) = d i_X d\eta$, then $i_X d\eta = d\eta_X$, where $\eta_X$ is a function determined up to an additive constant.
Let's make infinitesimal transform into a Lie algebra.

Let \( L_X \psi = d\phi x \) and \( L_Y \psi = d\phi y \). Then

\[
[L_{[x,y]} \psi] = [L_x, L_y] \psi = L_x (d\phi y) - L_y (d\phi x) = d(X\phi_y - Y\phi_x)
\]

so \( [x, \phi_x], [y, \phi_y] \) should be \( (X, Y) \), \( X\phi_y - Y\phi_x \). One can probably check easily the Jacobi identity, but instead let's exhibit a representation of the infinitesimal transform on functions.

Try the connection \( \phi \mapsto (d + \eta) \phi \) on the trivial line bundle.

\[
L_x (d\psi + \eta \psi) = d(X\psi) + (d\phi x) \psi + \eta X\psi
\]

\[
L_x (d + \eta) \psi = (d + \eta)X\psi + d\phi x \psi
\]

\[
(d + \eta) X\psi = (d\phi x) \psi + \phi_x d\psi + \phi_x \eta \psi
\]

\[
\begin{aligned}
(d + \eta) \phi_x &= d\phi_x + \phi_x (d + \eta) \\
(d + \eta) L_x &= -d\phi_x + L_x (d + \eta)
\end{aligned}
\]

\[
\Rightarrow [L_x + \phi_x, d + \eta] = 0
\]

Check: \( [L_x + \phi_x, L_y + \phi_y] = L_{[x,y]} + X\phi_y - Y\phi_x \). This should make clear the Jacobi identity for the bracket on line 4 above, provided you can recover from the operator \( L_x + \phi_x \) the vector field \( X \) and function \( \phi_x \). Clear because \( (L_x + \phi_x)_T = \phi_x \) and because \( X \) is the symbol of the first order operator \( L_x + \phi_x \).

So what to do about quantization. You want to quantize the real affine plane equipped with a translation invariant symplectic structure (i.e. volume form + orientation). Notion of polarization - two intersecting lines...
April 28, 02

Viewpoint: configuration space = the affine line $R$, that is, $R$ equipped with translational symmetries, in other words a torso for the Lie group $(R_+, +)$.

phase space = cotangent bundle of configuration space. The important point is that phase space has more symmetries than configuration space, and these are relevant for physics. Look at the cotangent bundle of the affine line $R$.

Let $q$ be position coord, then the tangent bundle is $R^2$ with coords $(q, \dot{q})$, the cotangent bundle is $R^2$ with coords $(q, p)$, the pairing between tangent and cotangent spaces is $p\dot{q} \mapsto p\dot{q}$, so that the contact form on phase space is $\eta = p dq$. Phase space is an affine plane, however, its translation group does not preserve the contact form $\eta$, although it does preserve the symplectic form $dpq$.

\[
\begin{align*}
L_{\dot{q}} (pdq) &= 0, & L_{\dot{p}} (pdq) &= dq \\
\varphi_{\dot{q}} &= 0 & \varphi_{\dot{p}} &= \dot{q}
\end{align*}
\]

\[
\begin{bmatrix}
(\alpha \dot{p} + \dot{q}) (d + \eta) \\
[ L_{\dot{q}}, d + \eta ]
\end{bmatrix} = L_{\dot{p}} (\eta) + [ \dot{q}, d ] = dq - [d, \dot{q}] = 0
\]
The Poisson summation formula is a basic result linking Fourier series and Fourier integrals, which has many important applications. This formula leads naturally to a complex line bundle over the 2-dimensional torus, whose geometric structure will be discussed. Sections of this line bundle provide a realization of the basic representation of the Heisenberg commutation relations different from the familiar ones from quantum mechanics. I will also discuss the connection with "aliasing" and "sampling" in electrical engineering.