March 2, 02   Time to review Heisenberg group

$(\zeta_1, x_1, y_1)\cdot(\zeta_2, x_2, y_2) = (\zeta_1\zeta_2 e^{2\pi i y_1 x_2}, x_1+x_2, y_1+y_2)$

Start with where you stopped before. Objects ① L = trivial bundle (hermitian) over $\mathbb{R}^2$ equipped with a convenient connection having constant curvature. Connection is $d + A$, $dA = 2\pi i\, dx\, dy$, say $A = 2\pi i\, x\, dy$. ~~I~~ think this object is clearly defined - sections of L are identified with fns. $\psi(x,y)$ on $\mathbb{R}^2$, and the connection is described (Bott description) as a diff. operator 1st order symbol d. ② Principal $\mathbb{T}$ bundle P for L. STOP

This is a rather non intrinsic discussion. Instead you should start by describing L as a locally trivial fibre bundle over $\mathbb{R}^2$ with fibre the complex v. space $\mathbb{C}$ equipped with herm. metric $|z|^2$. Then P is the unit circle bundle inside L. X = vector field generating circular rotation $(\theta, \zeta) \longmapsto e^{i\theta}\zeta$ $\partial_\theta(e^{i\theta}\zeta)\big|_{\theta=0} = i\zeta$

$X\zeta = i\zeta$ in each fibre. (So you've given the action of $\mathbb{T}$ on P)

What is your aim? From the Poisson summ. fmla you get a line bundle over $\mathbb{T}^2$ equipped with connection. You can pull this back to $\mathbb{R}^2$ to get L, P. You are interested in the symmetries.

Somehow the result should be the identification of P with H so that on P you get commuting left + right actions of H. This is hard

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Problem: To identify H with P. How to proceed? Maybe you should show H has the desired structure for P, structure to be made precise. Exact sequence $\mathbb{T} \rightarrowtail H \twoheadrightarrow \mathbb{R}^2$ gives the principal $\mathbb{T}$ bundle, the vector field X should be part of the Lie alg. of H. Need connection form Θ on H. Θ to satisfy

$\mathcal{L}_X\Theta = 0$, $\iota_X\Theta = 1$ Curvature is $d\Theta$

IDEA Convexity naturally occurs with moment mapping. So you might look for partitions of 1 in such situations, e.g. buildings.

Two objects ① H, the Heisenberg group, central extn

$$\mathbb{T} \rightarrowtail H \xrightarrow{\pi} \mathbb{R}^2$$ of Lie groups

To pin H down you seem to need a suitable section of π and a bilinear cocycle. Be precise: H is the Lie gp central extension of $(\mathbb{R}^2, +)$ by $(\mathbb{T}, \cdot)$ such that the commutator pairing is $(x_1,y_1)\wedge(x_2,y_2) \longmapsto \exp(2\pi i(x_1 y_2 - x_2 y_1))$.

Now let $\mathbb{T} \rightarrowtail G \twoheadrightarrow \mathbb{R}^2$ be the central extn with cross section associated to the 2 cocycle $\exp(\pi i(x_1 y_2 - x_2 y_1))$. It has the same commutator pairing as H. Take the difference of these two central extensions, (Baer addition). You should get a central extension $\mathbb{T} \to E \to \mathbb{R}^2$ with 0 commutator pairing, so E is abelian and Pontryagin gives a ~~splitting~~ up to a character of $\mathbb{R}^2$.

Start again. You want to define the Heisenberg group. You are given $\mathbb{R}^2$ equipped with a symplectic form $\omega(v,v')$. You choose a bilinear form $\beta(v,v')$ such that $\omega(v,v') = \beta(v,v') - \beta(v',v)$, then construct H as $\mathbb{T} \times \mathbb{R}^2$ with product defined by the 2-cocycle $e^{2\pi i \beta(v,v')}$

Problem is to define the Heisenberg group associated to $\mathbb{R}^2 = V$ equipped with symp. form ω. Answer: it is a group extension $\mathbb{T} \rightarrowtail H \xrightarrow{\pi} V$ equipped with a section of π such that the 2-cocycle arising from this section is bilinear (probably the same as saying the 2-cocycle is $\exp 2\pi i B(v,v')$ with B a bilinear form on V), you also want the commutator pairing for this extension to be $\exp 2\pi i\, \omega(v,v')$ (should be equiv to $B(v,v') - B(v',v) = \omega(v,v')$.

The basic idea is that the Heisenberg group H is a central extension of V by $\mathbb{T}$ equipped with a section of π such that ① the associated 2 cocycle is $e^{2\pi i \cdot B(v,v')}$, where B is a bilinear form on V satisf. $B(v,v') - B(v',v) = \omega(v,v')$.

It seems now that you understand what the Heisenberg group H. It is a central extension $\mathbb{T} \rightarrowtail H \twoheadrightarrow \mathbb{R}^2$ which is equipped with a cross section such that the associated 2-cocycle is bilinear. Somehow you need to link this to structure on P. P is a principal $\mathbb{T}$ bundle over $\mathbb{R}^2$ which is equipped with a connection $\theta \in \Omega^1(P)$ (this means $\iota(X)\theta = 1$, $\mathcal{L}_X \theta = 0$) such that the curvature $d\theta = 2\pi i\, dx\, dy$.

It seems that if you are going to identify H with P you only have to produce the appropriate connection form on H. If you have two principal $\mathbb{T}$ bundles H, P with connection over $\mathbb{R}^2$, you form $H^\vee \times^{\mathbb{T}} P$ which has a connection. Assuming curvatures of H, P are the same, $H^\vee \times^{\mathbb{T}} P$ is flat so you get an isom. $H \xrightarrow{\sim} P$ which is unique up to an elt of $\mathbb{T}$.

So here's a simple problem. Take the defn.

$$(\zeta_1, x_1, y_1)(\zeta_2, x_2, y_2) = (\zeta_1 \zeta_2 e^{2\pi i y_1 x_2}, x_1 + x_2, y_1 + y_2)$$

and find "the" connection. The 2-cocycle is the bilinear form

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = (x_1 \ \ y_1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

followed by $\exp(2\pi i -)$.

$$H = \mathbb{T} \times \mathbb{R} \times \mathbb{R} = \{(e^{i\varphi}, x, y)\} \qquad \frac{d(e^{i\varphi})}{e^{i\varphi}} = \cancel{e^{i\varphi}}\, i\, d\varphi$$

$$\theta = i\, d\varphi + 2\pi i\, x\, dy \qquad d\theta = 2\pi i\, dx\, dy$$

I think you are missing the holonomy idea. Where can you ~~start~~? From Poisson stuff you get connection $d + 2\pi i\, x\, dy$. At the moment you have lots of ~~pictures~~ viewpoints.

$$\tilde f(x,y) = \sum_m e^{2\pi i m y} f(x+m)$$

$$e^{2\pi i xy} \tilde f(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m)$$

$$\partial_x \tilde f = \widetilde{\left(\frac{d}{dx} f\right)}$$

$$\partial_y \left(e^{2\pi i xy} \tilde f\right)(x,y) = e^{2\pi i xy}\, \widetilde{2\pi i x f}$$

$$\tilde f(x, y+1) = \tilde f(x,y)$$

$$\left(e^{2\pi i xy} \tilde f\right)(x+1, y) = \left(e^{2\pi i xy} \tilde f\right)(x,y)$$

$$\boxed{\begin{aligned} \partial_x \tilde f &= \widetilde{\tfrac{d}{dx} f} \\ (\partial_y + 2\pi i x) \tilde f &= \widetilde{2\pi i x f} \end{aligned}}$$

March 3, 02

I want to list the objects encountered in trying to identify H with P. A problem is that you frequently have two versions: a fibre bundle and the space of its sections.

from Poisson S.F. you get the space of sections of L over T^2 described as functions on $\mathbb{R}^2$ satisfying automorphy condition, and you get a connection on L described as two operators $D_x = \partial_x$ and $D_y = \partial_y + 2\pi i x$.

Your aim is to list the relevant objects

① From PSF you get space $\mathcal{F}$ of sections of L over T^2 with connection given by D_x, D_y.

② Actual line bundle $L \to T^2$

③ Principal $\mathbb{T}$-bundle $P = SL \subset L$

Regular rep. of H, splitting over the center $\mathbb{T}$ of H, the level. Functions on the circle.

QUESTION: Is there something interesting to be said about $\bigoplus_n C^\infty(T^2, L^{\otimes n}) = C^\infty(T^2, \bigoplus_n L^{\otimes n})$. This should be (essentially) the ring of functions on the principal $\mathbb{T}$-bundle $P = SL \to T^2$, this should be true after completing the direct sum in the C^∞ top. You can see three circles: the x, y axes of T^2 and the circle in the fibre. P here should be the quotient $H/\mathbb{Z}^2$.

IDEA Can h be in Mult(B)?

Let's continue trying to identify H with P. You have defined precisely H as a central extension of Lie groups $\mathbb{T} \rightarrowtail H \twoheadrightarrow \mathbb{R}^2$ equipped with a section, such that the 2-cocycle β assoc. to s is bilinear and skew symmetrizes to $e^{2\pi i \, vol}$. H is nice, small, set with product.

P, How does P arise? You begin with the space of sections of $\bar{L}$ over T^2 defined as functions on $\mathbb{R}^2$ modulo suitable action of $\mathbb{Z}^2$, equipped with D_x, D_y.

Forget action of $\mathbb{Z}^2$ to get functions on $\mathbb{R}^2$ equipped with D_x, D_y (left repn. of H) and commuting right Heisenberg repn. ∇_x, ∇_y

You interpret the space $C^\infty(\mathbb{R}^2)$ equipped with the ops D_x, D_y as the space of sections of the trivial line bundle equipped with the connection operator $D = d + A$.

You next want to go from (triv line bundle over $\mathbb{R}^2$, $d+A$) to the trivial principal $\mathbb{T}$ bundle P with the corresponding connection form θ (after Weil).

Then you want to relate (trivial $\mathbb{T}$-bundle P over $\mathbb{R}^2$, θ) to H.

Discuss what you know about θ. $\theta \in \Omega^1(P)$ such that $\mathcal{L}_X \theta = 0$, $\iota_X \theta = 1$ where X is the vector field $\frac{1}{i}\partial_\varphi$ on $P = (e^{i\varphi}, x, y)$. Note that $(e^{i\varphi})^{-1} d(e^{i\varphi}) = e^{-i\varphi} e^{i\varphi}\, i\, d\varphi = i\, d\varphi$. So θ should be $i\, d\varphi$ + some 1-form on base $\mathbb{R}^2$.

Now you have to link θ on P to D on sections of L. The point is that a section of L is equivalent to a function f from P to $\mathbb{C}$ transforming like $(f \circ e^{i\varphi}) = e^{i\varphi} f$, better infly $\frac{1}{i}\partial_\varphi f = f$, i.e. $Xf = f$.
When you apply d to f you get a 1-form which has vertical component, so add $\pm \theta f$ to get a basic form.

Let's repeat the transitions

1) Sections of the line bundle $\overline{L} \to T^2$ with D_x, D_y
2) Sections of line bundle $L \to \mathbb{R}^2$ with D_x, D_y $(+\nabla_x, \nabla_y)$
3) $\{\psi : \underset{\mathbb{T}\times\mathbb{R}^2}{\underset{\shortparallel}{P}} \to \mathbb{C} \mid \psi \circ e^{i\varphi} = e^{i\varphi}\psi$ or $Xf = f\}$ with θ

4) (P, θ) + symmetries.

So you have a list of transitions between pictures of L with connections. What's missing is how the connection gives a left action of H on P. You should be able to find 3 vector fields.

You still cannot get from the space of sections of $L \longrightarrow \mathbb{R}^2$ (or $\underline{L} \longrightarrow T^2$) to P (or $\underline{P}$).

Let's work in the other direction, start from H and construct the rest. H is a principal $\mathbb{T}$-bundle over $\mathbb{R}^2$, and $H/\mathbb{Z}^2$ is a principle $\mathbb{T}$-bundle over T^2. Describe sections of the assoc line bundle $(\mathbb{C} \times^{\mathbb{T}} \mathbb{T}) \times H$. These are $\psi : H \longrightarrow \mathbb{C}$, $\psi(\zeta, x, y)$ satisf $\psi(e^{i\varphi}\zeta, x, y) = e^{i\varphi}\psi(\zeta, x, y)$, maybe $e^{-i\varphi}$ instead.

You want $\theta \in \Omega^1(H) = \Omega^1(\mathbb{T} \times \mathbb{R} \times \mathbb{R})$

$$\partial_\varphi\Big|_{\varphi=0} \psi(e^{i\varphi}\zeta, x, y) = i\zeta\,\partial_\zeta \psi(\zeta, x, y), \quad \text{So } X\psi(\zeta, x, y) = i\zeta\partial_\zeta\psi(\zeta, x, y)$$

Simpler ~~maybe~~ to use $\psi(\varphi, x, y)$ where $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$

Then $\psi(\varphi' + \varphi, x, y) = e^{i\varphi'}\psi(\varphi, x, y)$

$$X\psi(\varphi, x, y) = \partial_\varphi \psi(\varphi, x, y) = i\psi(\varphi, x, y).$$

Try to understand ∥ transport with respect to the connection. In the section picture you have $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ and given $x(t), y(t)$ you have $T\left\{\exp\int_a^b dt\left(\dot{x}\,\partial_x + \dot{y}\,(\partial_y + 2\pi i x)\right)\right\}$. This is new to you and nice because it can be evaluated. So what should come next? All this is taking place on the principle bundle. So it does seem that you are constructing an action of H on P.

Return to the space of sections in order to understand the H action. You have operators $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ on sections, that is, $\psi(x,y) \in C^\infty(\mathbb{R}^2)$ satisf automorphic cond. Given a curve $(x(t), y(t)) \in \mathbb{R}^2$ $a \leq t \leq b$, you find

$$\Phi_a^t = T\left\{\exp \int_a^t dt\,(\dot{x}D_x + \dot{y}D_y)\right\}$$

$$\frac{d}{dt}\,\Phi_a^t = \Phi_a^t(\dot{x}D_x + \dot{y}D_y)$$

you want this on a specific section, so the DE should read $\partial_t \psi(t,x,y) = (\dot{x}D_x + \dot{y}D_y)\psi(t,x,y)$

There's a lot of notation here that's probably wrong, but the strategy seems clear, namely, you have a path in the Heisenberg Lie alg, which leads to two paths (left + right) in the Heisenberg group, so you get some action somewhere.

derivations satisfying the CCR

OKAY. You have $\underbrace{D_x, D_y}$ operating on $\mathcal{L}$ = space of C^∞ sections of L, so you have a Lie alg repn of H on the space $\mathcal{L}$ which is compatible with the translation action on $\mathcal{Q}$. It should be simple now to explain the action of H on L, because you have a description of points of L.

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mult in H defined $(c_1, x_1, y_1)(c_2, x_2, y_2) = (c_1 c_2\, e(y_1 x_2), x_1 + x_2, y_1 + y_2)$ necessary to straighten out signs.

$$D_x = \partial_x, \quad D_y = \partial_y + 2\pi i x \qquad [D_x, D_y] = 2\pi i$$

get inf repn of H on $\psi(x,y)$.

IDEA: Recall that G, c, h can be combined to obtain absolute units for distance time and energy. What about Avogadro's number?, Boltzman's constant k, temperature units?

Mar 4, 02. Where to start? You want to understand left and right actions of H on itself and on the space of sections of the line bundle L over $\mathbb{R}^2$. Begin with what you have done for $\Gamma(\mathbb{R}^2, L)$.

L is the trivial line bundle over $\mathbb{R}^2$; hence $\Gamma(\mathbb{R}^2, L)$ is the space of smooth $\psi(x,y) \in C^\infty(\mathbb{R}^2)$. You have two commuting families of operators

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix} \quad , \quad \begin{matrix} \nabla_x = \partial_x + 2\pi i y \\ \nabla_y = \partial_y \end{matrix}\right] = 0$$

$$[D_x, D_y] = 2\pi i \qquad [\nabla_x, \nabla_y] = -2\pi i$$

which should amount to commuting infinitesimal left and right actions of H on $C^\infty(\mathbb{R}^2)$.

$$e^{aD_x} e^{bD_y} \psi(x,y) = e^{a\partial_x} e^{b(2\pi i x)} e^{b\partial_y} \psi(x,y)$$

$$= e^{2\pi i b x} e^{2\pi i a b} e^{a\partial_x} e^{b\partial_y} \psi(x,y)$$

$$= e^{2\pi i a b} e^{2\pi i b x} \psi(x+a, y+b).$$

$$\left(c_1 e^{a_1 D_x} e^{b_1 D_y}\right)\left(c_2 e^{a_2 D_x} e^{b_2 D_y}\right) = c_1 c_2 e^{-2\pi i b_1 a_2} e^{(a_1+a_2)D_x} e^{(b_1+b_2)D_y}$$

$$e^{a_2 D_x} e^{b_2 D_y} \psi(x,y) = e^{2\pi i a_2 b_2} \underbrace{e^{2\pi i b_2 x} \psi(x+a_2, y+b_2)}_{\tilde{\psi}(x,y)}$$

$$e^{a_1 D_x} e^{b_1 D_y} \tilde{\psi}(x,y) = e^{2\pi i a_1 b_1} e^{2\pi i b_1 x} \tilde{\psi}(x+a_1, y+b_1)$$

$$= e^{2\pi i (a_1 b_1 + b_1 x + b_2(x+a_2))} \psi(x+a_2+a_1, y+b_2+b_1)$$

?

$$e^{a_2D_x}e^{b_2D_y}\psi(x,y) = e^{a_2\partial_x}e^{2\pi i b_2 x}e^{b_2\partial_y}\psi(x,y)$$

$$= e^{2\pi i a_2b_2 + 2\pi i x b_2}\psi(x+a_2, y+b_2) = \tilde\psi(x,y)$$

$$e^{a_1D_x}e^{b_1D_y}e^{a_2D_x}e^{b_2D_y}\psi(x,y) = e^{2\pi i a_1b_1 + 2\pi i x b_1}\tilde\psi(x+a_1, y+b_1)$$

$$= e^{2\pi i(a_1b_1+xb_1)}e^{2\pi i a_2b_2 + 2\pi i(x+a_1)b_2}\psi(x+a_1+a_2, y+b_1+b_2)$$

$$= e^{2\pi i(a_1b_1+a_2b_2+a_1b_2)}e^{2\pi i x(b_1+b_2)}\psi(x+a_1+a_2, y+b_1+b_2)$$

$$\overset{?}{=} e^{-2\pi i a_2b_1}e^{2\pi i(a_1b_1+a_1b_2+\boxed{a_2b_1}+a_2b_2)}e^{2\pi i x(b_1+b_2)}\psi(x+a_1+a_2, y+b_1+b_2)$$

$$T_{a_1,b_1}T_{a_2,b_2}\psi(x,y) = T_{a_1,b_1}e^{2\pi i b_2(x+a_2)}\psi(x+a_2, y+b_2)$$

$$= e^{2\pi i b_1(x+a_1)}e^{2\pi i b_2(x+a_1+a_2)}\psi(x+a_1+a_2, y+b_1+b_2)$$

$$T_{(a_1,b_1)}T_{(a_2,b)}\psi(x,y) = e^{-2\pi i b_1a_2}T_{(a_1+a_2, b_1+b_2)}\psi(x,y)$$

How to summarize this, to end up with something clean? One possibility is to replace $\mathbb{R}^2$ by V

Where are you? You are looking at the inf action of H on $\mathcal{L} = C^\infty(\mathbb{R}^2)$ with gens $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$, $[D_x, D_y] = 2\pi i$. Get operators $T_{(c,a,b)}\psi(x,y) = ce^{2\pi i b(x+a)}\psi(x+a, y+b)$

$$\boxed{T_{(c_1,a_1,b_1)}T_{(c_2,a_2,b_2)} = T_{(c_1c_2e^{-2\pi i b_1a_2}, a_1+a_2, b_1+b_2)}}$$

$$(c_1e^{a_1D_x}e^{b_1D_y})(c_2e^{a_2D_x}e^{b_2D_y}) = (c_1c_2e^{-2\pi i b_1a_2}, a_1+a_2, b_1+b_2)$$

Generalize? V real vector space of dim 2. equipped with bilinear form $B(v,v')$ such that $B(v,v') - B(v',v)$ is non degenerate

Define H to be the central extension + section corresp. to B

$$H = \mathbb{T} \times V \qquad (c_1, v_1)\cdot(c_2,v_2) = (c_1 c_2 e^{B(v_1,v_2)}, v_1+v_2)$$

You want a Lie action of H on $C^\infty(V)$. Should be made from translation with multiplication.

$$[\partial_{v_1} + \lambda_{v_1}, \partial_{v_2} + \lambda_{v_2}] = B(v_1,v_2) - B(v_2,v_1)$$

here $\lambda_v(x) = B(x,v)$

$$e^{a_1 D_x} e^{b_1 D_y} e^{a_2 D_x} e^{b_2 D_y}$$

$$= e^{-2\pi i b_1 a_2} e^{(a_1+a_2)D_x} e^{(b_1+b_2)D_y}$$

$$B\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right) = -b_1 a_2$$

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}^t \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$$

$$\partial_{\binom{1}{0}} + \underbrace{B\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}_{(x\ y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \partial_x$$

$$\partial_{\binom{0}{1}} + (x\ y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \partial_y + x$$

$$H = \mathbb{T} \times V, \quad (c_1, v_1)\cdot(c_2,v_2) = (c_1 c_2 e^{B(v_1,v_2)}, v_1+v_2)$$

$\mathcal{L} = C^\infty(V)$, have ops $\partial_v + B(v,-)$.

$$[\partial_{v_1} + B(-,v_1), \partial_{v_2} + B(-,v_2)] = B(v_1,v_2) - B(v_2,v_1)$$

Example $V = \mathbb{R}^2$ $\quad B\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = (x_1\ y_1)\, B \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$.

It's time you succeeded in understanding the principal $\mathbb{T}$-bundle P over T^2 that arises from

$$\mathcal{L} = \{ \psi \in C^\infty(\mathbb{R}^2) \mid \psi(x+m, y+n) = e^{-2\pi i m y} \psi(x,y) \}$$

$$e^{m(\partial_x + 2\pi i y)} e^{n\partial_y} \psi(x,y) = \psi(x,y)$$

What is a reasonable goal? To define a left H-action on P using the operators $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$

$P = \mathcal{S} L$, what is a point of L over a point $(a+\mathbb{Z}, b+\mathbb{Z}) \in T^2$, answer: a function $\{\psi(a+m, b+n) \mid (m,n) \in \mathbb{Z}^2\}$ satisfying automorphy conditions $\psi(a+m, b+n) = e^{-2\pi i m b}\psi(a,b)$, and a point of P is such a ψ with values in $\mathbb{T}$.

You construct an action of H on the space $\mathcal{L}$ of sections of L, namely

$$(z, a, b)\cdot\psi = z e^{aD_x} e^{bD_y} \psi(x,y)$$

$$= z e^{2\pi i ab} e^{b 2\pi i x} \psi(x+a, y+b)$$

$$\boxed{(z, a, b)\cdot\psi = z e^{2\pi i b(x+a)} \psi(x+a, y+b)}$$

This should be a well-defd action of H on $\mathcal{L}$ compat with H action on $C^\infty(T^2)$ via translation. There should be an easy way to deduce an action

$$H \times L \longrightarrow L$$

$$\mathbb{R}^2 \times T^2 \longrightarrow T^2$$

What techniques? Ignore the autom. condition, pull back via the map $\mathbb{R}^2 \twoheadrightarrow T^2$. Then you have $\mathcal{L} = \{\psi \in C^\infty(\mathbb{R}^2)\}$ with D_x, D_y

This means you forget the "right" $\mathbb{Z}^2$-action. Now you have H acting on $\psi \in C^\infty(\mathbb{R})$ as above

$$(z, a, b)\cdot\psi(x,y) = z e^{2\pi i b(x+a)} \psi(x+a, y+b)$$

Maybe all that you need is the couple $\psi(x,y), (x,y)$ that is, the graph of the section.

Describe the problem. You have a (hermitian) line bundle over $\mathbb{R}^2$ equipped with connection whose curvature is constant. Cancel the curvature — put a bilinear connection form with opposite curvature on the trivial line bundle, then tensor to get an isom. Nice uniqueness result. Possible structure – symmetries. Stick to translations in the base. You seem to be getting back to your old viewpoint.

Mar 5, 02

Problem. Consider "the" hermitian line bundle over $\mathbb{R}^2$ equipped with connection having constant (translation invariant) nonzero curvature. What is the group of symmetries of this line bundle?

Recall the uniqueness argument. Let L_1, L_2 be two hermitian line bundles over $\mathbb{R}^2$ equipped with connections D_1, D_2 having the same curvature. Then $L_1 \otimes L_2^\vee$ is a flat line bundle over $\mathbb{R}^2$ and any flat section of norm 1 will yield an isom between L_1 and L_2, unique up to a scalar factor in $\mathbb{T}$.

Let L be such a line bundle over $\mathbb{R}^2$. A symmetry of L is a pair (g, φ) where g is a diffeom of $\mathbb{R}^2$ and $\varphi : L \xrightarrow{\sim} g^*L$ is an isomorphism preserving hermitian scalar product and connection. If $\omega \in \Omega^2(\mathbb{R}^2)$ is the curvature of L, then $g^*\omega = \omega$, and conversely if this holds, then L and g^*L have the same curvature, so there is a φ unique up to a scalar factor in $\mathbb{T}$. Thus the group of symmetries of L will be a central extn by $\mathbb{T}$ of the group of diffeos. of $\mathbb{R}^2$ preserving ω. Get all symplectic diffeos of $\mathbb{R}^2$ when ω is a symplectic form, in particular $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ when $\omega = dx\,dy$.

$L = \mathbb{C} \times \mathbb{R}^2$ trivial line bundle, $D = d + 2\pi i x\, dy$.
$D^2 \overset{?}{=} 2\pi i\, dx\, dy$. I think the projective action of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ is generated by gen. translations commuting with D.

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix} , \begin{matrix} \nabla_x = \partial_x + 2\pi i y \\ \nabla_y = \partial_y \end{matrix}\right] = 0$$

You've learned various things since you looked at the $SL(2,\mathbb{R})$ action. In particular you should know that "the" hermitian line bundle L over $\mathbb{R}^2$ equipped with connection having curvature $2\pi i\, dx\, dy$ should be H.

Maybe you should go over the picture again.

$P = \underset{(e^{i\varphi}, x, y)}{\mathbb{T} \times \mathbb{R}^2}$, $\Theta \in \Omega^1(P, i\mathbb{R})$ $\quad \zeta = e^{i\varphi} \quad d\zeta = \zeta i\, d\varphi$
$\partial_\varphi = i\zeta \partial_\zeta$

$\zeta = e^{i\varphi}$ $\quad \frac{d\zeta}{d\varphi} = e^{i\varphi} i = i\zeta \quad d\varphi = \frac{1}{i\zeta} d\zeta$

2 Boyce $\quad \frac{d\varphi}{d\zeta} = \frac{1}{i\zeta} \quad \frac{d}{d\zeta} = \frac{1}{i\zeta}\frac{d}{d\varphi} \quad \Big| \quad \frac{d}{d\varphi} = i\zeta\frac{d}{d\zeta}$

do you agree? $\quad i\frac{d}{d\log\zeta} \overset{?}{=} \frac{d}{d\varphi}$?

Problem: How to clear up this confusion? Focus on "the" principal $\mathbb{T}$-bundle over $\mathbb{R}^2$ equipped with connection $\Theta \in \Omega^1(P, i\mathbb{R})$ having curvature $d\Theta = 2\pi i\, dx\, dy$. You know that any two of these are isomorphic + the isom is unique up to a scalar factor in $\mathbb{T}$. The group of autos, symmetries of P should be a central extension by $\mathbb{T}$ of the group of symplectic diffeos of $\mathbb{R}^2$. Consider the subgroup of autos of P which are affine linear on $\mathbb{R}^2$.

Let's review briefly Joachim's M_2 theory. Try to start with the best viewpoint. You want to study retracts of a free M_2-module, but instead of $M_2 \otimes V$ you want

$$M_2 e_{11} \otimes V_1 \oplus M_2 e_{22} \otimes V_2, \qquad M_2 = \begin{pmatrix} \mathbb{C}e_1 \\ \mathbb{C}e_2 \end{pmatrix} \otimes (\mathbb{C}e_1^* \ \mathbb{C}e_2^*)$$

Apply the M-equiv to get $T^* \otimes V_1 \oplus T^* \otimes V_2$. not clear. What you want is related to the fact that M_2 is a path algebra

---

Discuss retract of $M_2 \otimes V$ as left M_2-module.

Category of unital left M_2-modules is equivalent to cat of unital $\mathbb{C}$-modules, so discuss retract of $T^* \otimes V = V \oplus V$ as $\mathbb{C}$-modules. Should be same as a retract of $\begin{smallmatrix}V\\ \oplus\\ V\end{smallmatrix}$:

$$\begin{tikzcd} W & \begin{smallmatrix}V\\ \oplus\\ V\end{smallmatrix} \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \end{tikzcd} \quad , \quad \beta\alpha = 1_W \, , \; \alpha\beta = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$$

$$\in \mathrm{End}\left(\begin{smallmatrix}V\\ \oplus\\ V\end{smallmatrix}\right) = M_2(\mathrm{End}(V)) \qquad \boxed{\sum_j p_{ij} p_{jk} = p_{ik}}$$

Assertion should be that a retract of the v.s. $\begin{smallmatrix}V\\ \oplus\\ V\end{smallmatrix}$ is a family $p_{ij} \in \mathrm{End}(V)$ $i,j \in \{1,2\}$ satisfying the idemp. condition *. Now define A by gens: p_{ij} $i,j = 1,2$, + four relations. A obviously idempotent.

$$\begin{tikzcd} \begin{smallmatrix}V\\ \oplus\\ V\end{smallmatrix} & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] & \begin{smallmatrix}V\\ \oplus\\ V\end{smallmatrix} \arrow[l, two heads, "(\beta_1 \, \beta_2)"'] \end{tikzcd} \qquad p_{ij} = \alpha_i \beta_j$$

missing is the idea $\beta\alpha = 1_W$ i.e. $\beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$

$$\sum_{ij} p_{ij} V = \sum_i \alpha_i \sum_j \beta_j V = \sum_i \alpha_i W$$

Let $v \in V$ be such that $p_{ij} v = 0$ $\forall i,j \in \{1,2\}$,

$\Rightarrow \alpha_i \beta_j v = 0$ $\forall i$ $\Rightarrow \beta_j v = 0$ $j = 1,2$.

$$\bigcap_{ij} \mathrm{Ker}\, p_{ij} = (\mathrm{Ker}\,\beta_1) \cap (\mathrm{Ker}\,\beta_2)$$

Suppose V, W both $= \mathbb{C}$ so that the α_i, β_j are $\in \mathbb{C}$.
$\beta_j \neq 0$ some j
$\alpha_i \neq 0$ some i

$$\begin{tikzcd} \begin{smallmatrix}W\\ \oplus\\ W\end{smallmatrix} & V \arrow[l, tail, "\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}"'] & \begin{smallmatrix}W\\ \oplus\\ W\end{smallmatrix} \arrow[l, two heads, "(\alpha_1 \, \alpha_2)"'] & V \arrow[l, tail, "\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}"'] \end{tikzcd}$$

$$\beta_1\alpha_1 + \beta_2\alpha_2 = 1$$
$$\Downarrow$$
$$\alpha_1\beta_1 + \alpha_2\beta_2 = 1$$

$p = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2 \end{pmatrix}$ is idempotent $\qquad p = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} (\cos\theta \;\; \sin\theta)$

Mar 6, 02 Study retract of a free M_2-module $M_2 \otimes V$. By M-eq this is the same as a vector space retract

$$\begin{tikzcd} W & V \oplus V \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\binom{\alpha_1}{\alpha_2}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

This retract is equivalent to the projection

$$p = \alpha\beta = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}(\beta_1\ \ \beta_2) \qquad p_{ij} = \alpha_i\beta_j$$

which is an idempotent in $\mathrm{End}\begin{pmatrix}V\\ \oplus\\ V\end{pmatrix} = M_2 \otimes \mathrm{End}(V)$, i.e. a retract of the M_2 module $M_2 \otimes V$ is equivalent to operators $p_{ij} \in \mathrm{End}(V)$ $i,j \in \{1,2\}$ satisfying $\sum_j p_{ij}p_{jk} = p_{ik}$ 4 operators, 4 relations. Thus a retract of the free M_2 module $M_2 \otimes V$ is the same as an A-module structure on V, where A is the alg with these generators and relations. $A = A^2$

One has $\sum_{i,j} p_{ij}V = \sum_i \alpha_i W$, $\bigcap_{i,j} \mathrm{Ker}(p_{ij} \text{ on } V) = \bigcap_j \mathrm{Ker}\,\beta_j$

so V is reduced $\iff$ $V = \sum_i \alpha_i W$, $\bigcap_j \mathrm{Ker}(\beta_j : V \to W) = 0$. This means you have

$$\begin{tikzcd} W \oplus W & V \arrow[l, tail, "\binom{\beta_1}{\beta_2}"'] & W \oplus W \arrow[l, two heads, "(\alpha_1\ \alpha_2)"'] \end{tikzcd}$$

i.e. V is the image of the operator $\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}(\alpha_1\ \ \alpha_2)$ on $\begin{matrix}W\\ \oplus\\ W\end{matrix}$.

Then you have the endom. $(\alpha_1\ \ \alpha_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2$ on V which is $p_{11} + p_{22}$.

There does not seem to be much to say. You've learned that the good situation concerns a retract of $\begin{matrix}V_1\\ \oplus\\ V_2\end{matrix}$, where there is no identification between V_1 and V_2.

Let's look at the case where V is a Hilbert space, say finite dimensional to simplify, and let W be a closed subspace of the orthog dir. sum $\begin{matrix}V\\ \oplus\\ V\end{matrix}$:

$$\begin{tikzcd} W & V \oplus V \arrow[l, two heads, "(\alpha_1^*\ \alpha_2^*)"'] & W \arrow[l, tail, "\binom{\alpha_1}{\alpha_2}"'] \end{tikzcd} \qquad \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 = 1.$$

You understand part of this geometry in terms of two involutions ε, F and the dihedral group they generate. Note that you do not use the identification between the two copies of V.

IDEA It it possible that your algebra A gen by p_{ij} subject to $\sum_j p_{ij}p_{jk} = p_{ik}$ only $i,j,k \in \{1,2\}$, might be relevant for "plumbing" over the $SL(2,\mathbb{Z})$ tree?

Go over Joachim's other M.eq. Start by looking at retracts

$$\begin{tikzcd} W & V_1 \oplus \cdots \oplus V_n \arrow[l, two heads, "(\beta_1 \cdots \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}"'] \end{tikzcd}$$

$$\sum \beta_i \alpha_i = 1_W$$

$$p = \alpha\beta = (\alpha_i \beta_j)$$

A gens p_{ij} rels $p_{ij}p_{kl} = 0 \quad j \neq k$

$$\sum_j p_{ij}p_{jk} = p_{ik}$$

$\mathcal{B} = \overset{\text{unital}}{\text{alg}}$ gen by h_i $1 \le i \le n$, $\sum h_i = 1$. Thus $\mathcal{W}$ consists of v.s. W equipped with ops h_i for $1 \le i \le n$ sat $\sum h_i = 1_W$.

$\mathcal{V}$ consists of $V_1 \oplus \cdots \oplus V_n$ equipped with $\forall_{ij}$: $V_i \xleftarrow{p_{ij}} V_j$ sat $\sum_j p_{ij}p_{jk} = p_{ij}$

You want to identify $\mathcal{V}$ with cat. of red A-modules

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Set this up like for retracts of $\mathbb{C}\Gamma \otimes V$, namely, first get the Morita equivalence straight + then get the rings + Morita context.

a W in $\mathcal{W}$ is a v.s. equipped with op h_i $\ni$ $\sum h_i = 1$

Let canon. factor of h_i be $W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$

whence you have retract

$$\begin{tikzcd} W & V_1 \oplus \cdots \oplus V_n \arrow[l, two heads, "(\beta_1 \cdots \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}"'] \end{tikzcd}$$

$$\sum \beta_i \alpha_i = 1_W$$

Now forget W and put $p = \alpha\beta = (\alpha_i \beta_j) = (p_{ij})$. So it seems that an object of $\mathcal{V}$ consists of a family of v.s. $V_1, \dots, V_n$ ~~equipped~~ with op $p_{ij} : V_i \leftarrow V_j$ such that $\sum_j p_{ij}p_{jk} = p_{ik}$

You still have to handle V reduced. ~~B~~ This means that $V_i = \sum_j p_{ij} V_j$ and $\bigcap_i \operatorname{Ker}\{p_{ij} : V_i \leftarrow V_j\} = 0$

Use $p_{ij} : V_i \xleftarrow{\alpha_i} W \xleftarrow{\beta_j} V_j$, $\sum_j p_{ij} V_j = \sum_j \alpha_i \beta_j V_j = \sum \alpha_i W$

$(\forall i)\ 0 = p_{ij} v_j = \alpha_i \beta_j v_j \implies \beta_j v_j = 0.$

Get order straight. You should start with the family V_i and operators $p_{ij} : V_i \leftarrow V_j \ni \sum_j p_{ij} p_{jk} = p_{ik}$.

Assume reduced which means $V_i = \sum_j p_{ij} V_j$ and $(\forall i)\ p_{ij} v_j = 0 \implies v_j = 0.$

Start again with family V_i of $p_{ij} : V_i \leftarrow V_j$ sat $\sum_j p_{ij} p_{jk} = p_{ik}$

You have proj of $p = (p_{ij})$: $\oplus V_i \longleftarrow \oplus V_j$, + you define W to be corresp. retract yielding

$$\begin{tikzcd} W & \oplus V_i \arrow[l, two heads, "(\cdots \beta_i \cdots)"'] & W \arrow[l, tail, "\begin{pmatrix} \vdots \\ \alpha_i \\ \vdots \end{pmatrix}"'] \end{tikzcd} \qquad \sum \beta_i \alpha_i = 1_W$$

$$\begin{tikzcd} V_i & W \arrow[l, "\alpha_i"'] & \begin{pmatrix} V_1 \\ \oplus \\ \vdots \\ \oplus \\ V_n \end{pmatrix} \arrow[l, two heads, "(\beta_1 \cdots \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}"'] & V_j \arrow[l, "\beta_j"'] \end{tikzcd}$$

$\alpha_i W = \sum \alpha_i \beta_j V_j = \sum_j p_{ij} V_j$

$\forall i\ \ 0 = p_{ij} v_j = \alpha_i \beta_j v_j \implies \beta_j v_j = 0$

$\therefore \bigcap_i \operatorname{Ker}(p_{ij} : V_i \leftarrow V_j) = \operatorname{Ker}(\beta_j)$

Therefore you see that if V is reduced i.e. $(\forall i)\ V_i = \sum_j p_{ij} V_j$ and $(\forall j) \bigcap_i \operatorname{Ker} p_{ij} = 0$, this means α_i surj, β_j inj so $V_i \overset{\beta_i}{\leftarrowtail} W \overset{\alpha_i}{\twoheadleftarrow} V_i$ is the canonical fact of h_i.

You've constructed an equiv between cat $\mathcal{W}$: $W + h_i \quad \sum h_i = 1$. and $\mathcal{V}$: $V_i + p_{ij} : V_i \leftarrow V_j \quad \sum_j p_{ij} p_{jk} = p_{ik}$, $V_i = \sum_j p_{ij} V_j$,

$(\forall i)\ \ p_{ij} v_j = 0 \Longrightarrow v_j = 0.$

The next step is to identify $\mathcal{W}, \mathcal{V}$ with the cats of reduced modules over B, A. Then the Morita contexts.

Problem: You have this equivalence $\mathcal{W} \simeq \mathcal{V}$ and you want a Morita context yielding it. In particular you want an idemp. ring A whose reduced modules are the objects of $\mathcal{V}$, i.e. families V_i + maps $p_{ij} : V_i \leftarrow V_j$ satisf $\sum_j p_{ij} p_{jk} = p_{ik}$, $V_i = \sum_j p_{ij} V_j$, $(\forall i)\ p_{ij} v_j = 0 \Rightarrow v_j = 0$.

A should be the ring with gens p_{ij}, relns $p_{ij} p_{kl} = 0$ if $j \neq k$, and $\sum_j p_{ij} p_{jk} = p_{ik}$.

Now the viewpoint should be that you know the modules (reduced) so finding the ring shouldn't be hard.

You can also try to construct the Morita context via generators + relations.

$$\begin{pmatrix} \boxed{} & \begin{matrix}\alpha_1 \\ \vdots \\ \alpha_n\end{matrix} \\ \beta_1 \cdots \beta_n & B \end{pmatrix} \begin{pmatrix} V_1 \\ \vdots \\ V_n \\ W \end{pmatrix}$$

A is the ring with gens p_{ij}, rels $p_{ij} p_{kl} = 0$ $j \neq k$, $\sum_j p_{ij} p_{jk} = p_{ik}$. If V is a reduced A-module then you have to construct a splitting $V = \oplus V_i$ such that $p_{ij} V_k = 0$ for $j \neq k$ and $p_{ij} V_j \subset V_i$. Multipliers should enter.

You want to embed A as an ideal in a unital ring R containing the "units" e_{ii} objects in the groupoid M_n.

Recall the idea. A is going to be a graded alg wrt the groupoid M_n consisting of n objects and a unique arrow for each ordered pair (i,j) of objects. call this arrow e_{ij}.

The path alg of the groupoid M_n is $M_n\mathbb{C}$ with basis e_{ij} which satisfy $e_{ij} e_{kl} = 0$ $j \neq k$, $e_{ij} e_{jk} = e_{ik}$.

Note that the gens and rels for A are homogeneous for the M_n grading: p_{ij} has degree ij.

You need to review $\Gamma = M_n$ graded v.s. + algs

$$\Delta : V \longrightarrow \mathbb{C}\Gamma \otimes V \qquad \Delta(v) = \sum_{s \in \Gamma} s \otimes e_s v$$

comod map $\quad e_s e_t = \begin{cases} 0 & s \neq t \\ e_t & s = t \end{cases} \qquad (\eta \otimes 1)(\Delta v) = \sum_s e_s v = 0$

counital.

Mar 7, 02

Point: Let $\Gamma_+ = \Gamma \amalg \{0\}$ be a monoid such that the basepoint 0 is absorbing, let $\mathbb{C}\Gamma = \mathbb{C}\Gamma_+/\mathbb{C}[0]$ be the corresp reduced monoid algebra. Then a $\mathbb{C}\Gamma$ comodule

$\Delta : V \longrightarrow \mathbb{C}\Gamma \otimes V$ is the same as a v.s. graded wrt Γ_+ and V is counital iff V is graded wrt Γ

Back to $\Gamma = M_n$ and $\mathbb{C}M_n = M_n\mathbb{C}$. Apply to A, gens p_{ij} etc.

Define alg map $\Delta : A \longrightarrow M_n\mathbb{C} \otimes A$ $\qquad \Delta p_{ij} = e_{ij} \otimes p_{ij}$

$$\Delta(p_{ij})\Delta(p_{kl}) = (e_{ij} \otimes p_{ij})(e_{kl} \otimes p_{kl}) = e_{ij}e_{kl} \otimes p_{ij}p_{kl}$$
$$= 0 \qquad j \neq k.$$

$\therefore$ Δ respects $p_{ij}p_{kl} = 0 \quad j \neq k$

$$\sum_j \Delta(p_{ij})\Delta(p_{jk}) = \underbrace{e_{ij}e_{jk}}_{e_{ik}} \otimes \sum_j p_{ij}p_{jk} = e_{ik} \otimes p_{ik} = \Delta(p_{ik})$$

so Δ resp reln $\sum_j p_{ij}p_{jk} = p_{ik}$

So you have a well defined alg map Δ.

Coassociativity $\quad (\Delta \otimes 1)\Delta \overset{?}{=} (1 \otimes \Delta)\Delta$

$$\Delta p_{ij} = e_{ij} \otimes p_{ij} \qquad (\Delta \otimes 1)\Delta p_{ij} = e_{ij} \otimes e_{ij} \otimes p_{ij}$$
$$(1 \otimes \Delta)\Delta p_{ij} = e_{ij} \otimes e_{ij} \otimes p_{ij}$$

So A becomes a $\mathbb{C}M_n$-comodule, which means $\overbrace{A = \bigoplus_{ij} A_{ij}}^{A \text{ has a grading}}$

$\ni \quad \Delta \sum a_{ij} = \sum e_{ij} \otimes a_{ij}$.

Question about Δ being counital. Confused!

Need review. Γ set, get coalg $\mathbb{C}\Gamma$, $\Delta s = s \otimes s$ coass, cocomm, counital where $\eta : \mathbb{C}\Gamma \to \mathbb{C}$ $\quad \eta(s) = 1 \quad \forall s$.

Let V be a Δ-comod: $\quad \Delta_V : V \longrightarrow \mathbb{C}\Gamma \otimes V \quad \Delta v = \sum_s s \otimes e_s v$

$$(\Delta \otimes 1)\Delta v = \sum_s s \otimes s \otimes e_s v \qquad (1 \otimes \Delta)\Delta v = \sum_s s \otimes \sum_t t \otimes e_t e_s v$$

$\therefore$ $e_t e_s = \delta_{ts} e_s$ so the e_s are "orthog" projections, so

$$V = \bigoplus_s e_s V \oplus V_*, \quad \Delta(v) = \sum_s s \otimes e_s v, \quad V_*$$

Summary. Γ set, $\mathbb{C}\Gamma$ is coalg w $\Delta s = s \otimes s$. $\mathbb{C}\Gamma$ is coassoc, cocomm, and counital with counit $\eta(s) = 1$, $\forall s$. Let V be a $\mathbb{C}\Gamma$ comodule. Then $\Delta_V : V \longrightarrow \mathbb{C}\Gamma \otimes V$ has the form $\Delta v = \sum_s s \otimes e_s v$, where e_s, $s \in \Gamma$ is a locally finite family of annihilating idempotents. Let $e_* = 1 - \sum_{s\in\Gamma} e_s$ on V. <u>The $\mathbb{C}\Gamma$-comodule structure on V is equivalent to a grading</u> $V = \bigoplus_{s\in\Gamma} e_s V \oplus e_* V$ of V <u>indexed by</u> $\Gamma_+ = \Gamma \amalg \{*\}$. V is a counital comodule over $\mathbb{C}\Gamma$ iff $e_* V = 0$. V is always counital over $\mathbb{C}\Gamma_+$, which is the counital coalgebra obtained by adjoining a counit to $\mathbb{C}\Gamma$.

Now suppose Γ_+ is a semigroup with $*$ an absorbing elt. Then $\mathbb{C}\Gamma = \mathbb{C}\Gamma_+ / \mathbb{C}\{*\}$ is naturally a bialgebra, there's a $\otimes$ operation on Γ-graded vector spaces, and a corresponding notion of Γ-graded alg.

Examine the case A gens p_{ij} rels. $p_{ij}p_{k\ell} = 0$, $p_{ik} = \sum_j p_{ij}p_{jk}$

$\mathbb{C}\Gamma = M_n\mathbb{C}$. $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. Check rels preserved.

$$0 = \Delta(p_{ij})\Delta(p_{k\ell}) = (e_{ij} \otimes p_{ij})(e_{k\ell} \otimes p_{k\ell}) = \overbrace{e_{ij}e_{k\ell}}^{0} \otimes p_{ij}p_{k\ell} = \Delta(p_{ij}p_{k\ell})$$

$$\Delta\Big(\sum_j p_{ij}p_{jk}\Big) = \sum_j (e_{ij} \otimes p_{ij})(e_{jk} \otimes p_{jk}) = e_{ik} \otimes \sum_j p_{ij}p_{jk} = e_{ik} \otimes p_{ik} \overset{\Delta(p_{ik})}{=}$$

Suppose you define A without the support condition; you think there are examples where $p_{ij}p_{k\ell} \neq 0$. Then $\Delta : A \to M_n A$ will kill $p_{ij}p_{k\ell}$

Maybe it would help to look at the general case. Start with Γ a set. TFAE:

(i) a coproduct $\Delta : V \longrightarrow \mathbb{C}\Gamma \otimes V$

(ii) a counital coproduct $\tilde{\Delta} : V \longrightarrow \mathbb{C}\Gamma_+ \otimes V$

(iii) a Γ_+-grading on V: $V = \bigoplus_{s\in\Gamma} V_s \oplus V_*$

Next look at $\otimes$, that is, you're given counital coprods. $\tilde{\Delta}_V : V \longrightarrow \mathbb{C}\Gamma_+ \otimes V$

$\tilde{\Delta}_W : W \longrightarrow \mathbb{C}\Gamma_+ \otimes W$

and you want to use the product $\overset{\mu}{}$ in $\mathbb{C}\Gamma_+$ to define a tensor product comodule:

$$\begin{tikzcd}
V\otimes W \arrow[r, "\tilde\Delta_V\otimes\tilde\Delta_W"] & \mathbb{C}\Gamma_+\otimes V\otimes \mathbb{C}\Gamma_+\otimes W \arrow[d, phantom, "\shortparallel"] \\
 & \mathbb{C}\Gamma_+\otimes\mathbb{C}\Gamma_+\otimes V\otimes W \arrow[d, "\mu\otimes 1"] \\
 & \mathbb{C}\Gamma_+\otimes V\otimes W
\end{tikzcd}$$

so you get a Γ_+-grading

$$V\otimes W = \bigoplus_{u\in\Gamma_+}\Big(\bigoplus_{\substack{st=u\\ \text{in } \Gamma_+}} V_s\otimes W_t\Big)$$

On the other hand, suppose you use noncounital coproducts:

$$\begin{tikzcd}
V\otimes W \arrow[r, "\Delta_V\otimes\Delta_W"] & \mathbb{C}\Gamma\otimes V\otimes \mathbb{C}\Gamma\otimes W \arrow[d, phantom, "\shortparallel"] \\
 & \mathbb{C}\Gamma\otimes\mathbb{C}\Gamma\otimes V\otimes W \arrow[d, "\mu\otimes 1"] \\
 & \mathbb{C}\Gamma\otimes V\otimes W,
\end{tikzcd}$$

then you get a Γ grading on $V\otimes W$.

Which way do the arrows go? $\mathbb{C}[*]$ is an ideal in $\mathbb{C}\Gamma_+$ and the quotient alg is $\mathbb{C}\Gamma$. The map

$$\mathbb{C}\Gamma_+\otimes(V\otimes W)\longrightarrow \mathbb{C}\Gamma\otimes(V\otimes W)$$

kills the $V_s\otimes W_t$ such that $st=*$, e.g. if either s or $t = *$.

Repeat. Γ set; $\mathbb{C}\Gamma$ coproduct $\Delta(s)=s\otimes s$, TFAE

(i) a coproduct $\Delta : V\longrightarrow \mathbb{C}\Gamma\otimes V$

(ii) a counital coproduct $\tilde\Delta : V\longrightarrow\mathbb{C}\Gamma_+\otimes V$

(iii) a Γ_+-grading on V: $\quad V=\bigoplus_{s\in\Gamma} V_s\oplus V_*$

note Δ is counital $\iff V_*=0$.

It seems that the good situation is a counital coproduct Δ. Γ has partially defined product

Suppose Γ equipped with semi group structure on Γ_+ such that $*$ is absorbing. Then $\mathbb{C}\Gamma$ is a bialgebra and one gets a tensor category from counital comodules for $\mathbb{C}\Gamma$, ~~equivalently~~ a tensor cat from Γ-graded v.s.

$$V = \bigoplus_{s\in\Gamma} V_s. \qquad \left(V\otimes W\right)_u = \bigoplus_{st=u} V_s \otimes W_t$$

Thus a Γ-graded alg A is an algebra with Γ-grading $A = \bigoplus_{s\in\Gamma} A_s$ satisf

$$\begin{aligned} A_s A_t &\subset A_{st} && \text{if } st\in\Gamma \\ &= 0 && \text{if } st = * \end{aligned}$$

Now go back to $\Gamma = M_n$, $\mathbb{C}\Gamma = M_n\mathbb{C}$ Recall

$$A = \mathcal{P}_\Gamma \quad \text{gens } p_s \quad s\in\Gamma, \text{ rels } \quad p_s p_t = 0 \quad \text{if } st = *$$

$$p_u = \sum_{st=u} p_s p_t$$

To show $\mathcal{P}_\Gamma$ is naturally Γ-graded. Let $\Delta : A \longrightarrow \mathbb{C}\Gamma\otimes A$ be the alg map such that $\Delta(p_s) = s\otimes p_s$. Check the rels are satisf.

$$st = * \Rightarrow \Delta(p_s)\Delta(p_t) = (s\otimes p_s)(t\otimes p_t) = \underbrace{st}_{0}\otimes p_s p_t$$

$$\Delta(p_u) \overset{?}{=} \sum_{st=u} \Delta(p_s)\Delta(p_t) = \sum_{st=u} st\otimes p_s p_t = u\otimes \sum_{st=u} p_s p_t = u\otimes p_u = \Delta(p_u)$$

So you have constructed an alg map $A \xrightarrow{\Delta} \mathbb{C}\Gamma\otimes A$ such that $\Delta p_s = s\otimes p_s$, then

$$(1\otimes\Delta)\Delta p_s = s\otimes s\otimes p_s$$
$$(\Delta\otimes 1)\Delta p_s = s\otimes s\otimes p_s$$

so Δ is a coproduct. Thus $A = \bigoplus_{s\in\Gamma} A_s \oplus A_*$. ?? Confused

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You need to be more precise. Γ a set, $\mathbb{C}\Gamma$ ~~the~~ with coalg str $\Delta s = s\otimes s$

V v.s. with $\mathbb{C}\Gamma$-comodule structure: $\Delta : V \longrightarrow \mathbb{C}\Gamma\otimes V$ satisf $(1\otimes\Delta)\Delta = (\Delta\otimes 1)\Delta$.

$$\Delta v = \sum_s s\otimes e_s v \qquad e_s e_t = \begin{cases} 0 & s\neq t \\ e_s & s = t. \end{cases}$$

$$\therefore\ V = \bigoplus_{s\in\Gamma} V_s \oplus V_\infty \qquad \text{where } V_s = e_s V,\ V_\infty = \left(1 - \sum_s e_s\right)V$$

$$\Delta v = \sum_{s\in\Gamma} s\otimes e_s v \qquad V_\infty = \operatorname{Ker}\{\Delta : V \longrightarrow \mathbb{C}\Gamma\otimes V\}$$

V counital iff $\overset{\text{TFAE}}{}$ $\sum e_s = 1$, Δ inj, $V_\infty = 0$

TFAE. (i) $\mathbb{C}\Gamma$ - comodule structure Δ on V
(ii) counital $\mathbb{C}\Gamma_+$ - comodule structure on V
(iii) Γ_+ - grading on V.

You need to work with the above V when you define the tensor product, however, the important object should be a Γ-graded vector space equivalently, a counital $\mathbb{C}\Gamma$-comodule. You want to remove the V_∞ component in analogy with ignoring nil-modules.

TFAE: (i) counital $\mathbb{C}\Gamma$ - comodule structure on V
(ii) $\mathbb{C}\Gamma$- comodule structure on V s.t. $\Delta : V \longrightarrow \mathbb{C}\Gamma\otimes V$ injective
(iii) Γ- grading on V

Now let Γ_+ be a semigroup with abs. elt ∞. You then should have a $\otimes$ operation for Γ-graded vector spaces.
Is there an interesting $\otimes$ for Morita contexts?

$$(V\otimes W)_{ik} = \bigoplus_j V_{ij}\otimes W_{jk}$$

$$\begin{pmatrix} V_{11}\otimes W_{11} \oplus V_{12}\otimes W_{21} & V_{12}\otimes W_{22} \oplus V_{11}\otimes W_{12} \\ V_{21}\otimes W_{11} \oplus V_{22}\otimes W_{21} & V_{22}\otimes W_{22} \oplus V_{21}\otimes W_{12} \end{pmatrix}$$

Aim? The basic object is a counital $\mathbb{C}\Gamma$ comodule V where Γ is a set. This is equivalent to a Γ-graded v.s. $V = \bigoplus_{s\in\Gamma} V_s$. But in the nonunital spirit you want to consider any $\mathbb{C}\Gamma$ comodule, whence the counital comodules are the analog of reduced modules.

Mar 8, 02

$\Gamma = M_n$. $\mathcal{W}$ consists of v.s. W + op h_i, $\sum_i h_i = 1_W$

$\mathcal{V}$ consists of (V_i), + op $p_{ij} : V_i \leftarrow V_j$, $p_{ik} = \sum_j p_{ij} p_{jk}$

+ red. cond. $V_i = \sum_j p_{ij} V_j$, $(\forall i)\ p_{ij} v_j = 0 \Rightarrow v_j = 0$.

$\mathcal{U}$ consists of $W, (V_i)$, $\beta_j : W \leftarrow V_j$, $\alpha_i : V_i \leftarrow W$, $\sum \beta_j \alpha_j = 1_W$

+ red. cond. β_j inj, α_i surj.

Equivalence $\mathcal{W} \simeq \mathcal{U}$. Given W, h_i canon fact.

$$h_i = W \xleftarrow{\beta_i} V_i \xleftarrow{\alpha_i} W$$

Equivalence $\mathcal{V} \simeq \mathcal{U}$ Given (V_i, p_{ij}) in $\mathcal{V}$, let p be the op on $\oplus V_i$ with component $p_{ij} : V_i \leftarrow V_j$. $p^2 = p$, let W α, β corresp. retract of $\oplus V_i$

$$\begin{tikzcd} W & \begin{pmatrix} V_1 \\ \\ V_n \end{pmatrix} \arrow[l, two heads, "(\beta_1 \ \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix} \alpha_1 \\ \alpha_n \end{pmatrix}"'] & \begin{pmatrix} V_1 \\ \\ V_n \end{pmatrix} \arrow[l, two heads, "(\beta_1 \ \beta_n)"'] \end{tikzcd}$$

α_i surj. $\quad \alpha_i W = \alpha_i \sum_j \beta_j V_j = \sum_j p_{ij} V_j = V_i$

β_j inj $\quad \beta_j v_j = 0 \Rightarrow (\forall i)\ \alpha_i \beta_j v_j = 0 \Rightarrow (\forall i)\ p_{ij} v_j = 0 \Rightarrow v_j = 0$.

At this point you've reviewed the equivalence of cats. Next you want to exhibit the idempotent rings. No problem with $\mathcal{W}$, but $\mathcal{V}$ is a category of graded vector spaces. multipliers!

Discuss the problem of recovering an idempotent ring from its reduced left module category. Better: Roos thm. You pick a generator + look at the unital ring of its endos. This is the fibre functor viewpoint.

R unital $\qquad \mathrm{Mod}(R) \longrightarrow \mathrm{Mod}(\mathbb{Z}) \qquad$ forget

$$M \longmapsto M$$

$$\{G\text{-sets}\} \xrightarrow{F} \{\text{sets}\} \qquad F(X) = X = \mathrm{Hom}_G(G, X)$$

Recall cat $\mathcal{V}$ consists of $(V_i, p_{ij} : V_i \leftarrow V_j)$
satisf $p_{ik} = \sum_j p_{ij} p_{jk}$, $V_i = \sum_j p_{ij} V_j$, $(\forall i)\, p_{ij} v_j = 0 \Rightarrow v_j = 0$.

You want to construct an idempotent ring A, whose reduced module cat is $\mathcal{V}$.

You want to replace the family (V_i) by the graded vector space $V = \bigoplus V_i$, which means that you want on V annihilating idempotents e_i with $\sum e_i = 1_V$. You don't want $e_i \in A$, otherwise A would be left unital (?).
The point is that the e_i should be in $\mathrm{Mult}(A)$

IDEA. Can you extend the canonical factorization of h, the fact that it seems to enable you to pretend h is idempotent, to handle a filtration

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, two heads] & & Y \arrow[rr, "g"] \arrow[dr, two heads] & & Z \\
& fX \arrow[ur, tail] \arrow[dr, two heads] & & gY \arrow[ur, tail] & \\
& & gfX \arrow[ur, tail] & &
\end{tikzcd}$$

You want to link this picture somehow to the Δ of mapping cones $\mathrm{Cone}(f) \longrightarrow \mathrm{Cone}(gf) \longrightarrow \mathrm{Cone}(g)$

Return to the problem of constructing an idempotent ring A whose reduced module cat is $\mathcal{V}$. You replace the family (V_i) by the direct sum $V = \bigoplus V_i$ equipped with the disjoint partition $1_V = \sum_i e_i$. Look at the operators on V that you have. Besides this ~~partition~~, you have $p \in \mathrm{End}(V)$

$p = \sum_{i,j} e_i p e_j \qquad e_i p e_j = p_{ij}$. Actually you have the ~~partition~~ $\sum e_i = 1$ and a $p = p^2$

so your are looking at the free product $\sum \mathbb{C} e_i * \mathbb{C} p$.
It remains to find A.

Repeat: cat $\mathcal{V}$ of $(V_i, p_{ij}: V_i \leftarrow V_j)$, $p_{ik} = \sum_j p_{ij}p_{jk}$

red cond. $V_i = \sum_j p_{ij}V_j$, $(\forall i)\ p_{ij}v_j = 0 \implies v_j = 0$.

Given such an object, let $V = \bigoplus_i V_i$, $e_i = \text{pr on } V_i$ so that $e_ie_j = \delta_{ij}e_j$, $\sum e_i = 1$. Define $p: V \leftarrow V$ by $e_ipe_j = p_{ij}: V_i \leftarrow V_j$, you have $p^2 = p$ since

$$p^2 = \sum_{i,j,k} e_ipe_je_jpe_k = \sum_{ik} e_i \sum_j p_{ij}p_{jk}e_k = \sum_{ik} e_ip_{ik}e_k = \sum_i e_ipe_k = p$$

Now what?? You know $R = \sum \mathbb{C}e_i * \mathbb{C}p$, and A should be the ideal generated by p.

This picture of A seems different from the alg w gens p_{ij}, rels $\overset{j \neq k \Rightarrow}{} p_{ij}p_{k\ell} = 0$, $p_{ik} = \sum_j p_{ij}p_{jk}$

Better: Given (V_i, p_{ij}) let $V = \bigoplus V_i$, $\varepsilon_i : V_i \rightarrowtail V$, $\eta_j : V \longrightarrow V_j$, so $\sum_i \varepsilon_i \eta_i = 1_V$, $\eta_j \varepsilon_i = \delta_{ji} = \begin{cases} 0 & j \neq i \\ 1_{V_i} & j = i \end{cases}$

let $p = \sum_{ij} \varepsilon_i p_{ij} \eta_j \implies p^2 = \sum_{i,j,k} \varepsilon_i p_{ij} \eta_j \varepsilon_j p_{jk} \eta_k$

$$= \sum_{i,k} \varepsilon_i \sum_j p_{ij}p_{jk}\eta_k = \sum_{i,k} \varepsilon_i p_{ik} \eta_k = p.$$

Where are you? You have gone from (V_i, p_{ij}) in $\mathcal{V}$ to graded v.s. $V = \bigoplus_i V_i$ tog. with idemp. of p.

So far no reduced conditions on (V_i, p_{ij}) have been used

$R = \bigoplus_i \mathbb{C}e_i * (\mathbb{C} \oplus \mathbb{C}p)$ what does this mean $\bigoplus \mathbb{C}e_i$ is unital

This is close to Joachim's $R(\bigoplus \mathbb{C}e_i)$

$$\begin{array}{ccc} \rho : \bigoplus \mathbb{C}e_i & \longrightarrow & R \\ \rho(e_i) & & h_i \end{array}$$

Mar 9, 02 You have just realized that this retract stuff is closely connected with Joachim's R construction. Let Λ be an algebra, say unital to begin with. Form $\Lambda * \mathbb{C}[\mathbb{Z}/2]$, i.e. you adjoin an idempotent, equivalently an involution. You get $QA \rtimes \mathbb{Z}/2$, except you want the part of this corresponding to the retract.

Check your memory. $\rho : \Lambda \longrightarrow P$ linear map $\rho(1) = 1$, $GNS(\rho : \Lambda \longrightarrow P)$ = cat of $({}_\Lambda M, {}_P N, j : M \to N, i : N \to M)$ satisfying $j \lambda i n = \rho(\lambda) n$ $(\forall \lambda \in \Lambda)(\forall n \in N)$. Take $\lambda = 1$ find $ji = 1_N$. If Λ nonunital, you presumably adjoin the condition $ji = 1_N$.

HOPE here that you find an interesting nonunital version. You want P to be nonunital

You should have the example you need sitting in front of you. Modules

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Go back to "the" principal $\mathbb{T}$-bundle P equipped with connection over $\mathbb{R}^2$ having curvature form $\omega = 2\pi i\, dx\, dy$. Treat $\mathbb{R}^2$ as an affine space, so the natural symmetry group is $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$.

(M, ω) symplectic manifold, X symplectic vector field, means $\mathcal{L}_X \omega = 0$, i.e. $d\iota_X \omega = -\iota_X d\omega = 0$, so at least locally $\exists$ function h_X unique up to add. const such that $dh_X = \iota_X \omega$

M, ω, X as above, choose $A \in \Omega^1(M)$ satisf $dA = \omega$ then $\{P = M \times \mathbb{T} \xrightarrow{pr_1} M\}$ equipped with $\Theta = z^{-1}dz + A$, is a principal $\mathbb{T}$ bundle with connection over M with curvature $dA = \omega$. A section of the assoc. line bundle $L = \{M \times \mathbb{C} \xrightarrow{pr_1} M\}$ should be equivalent to a function on P of the form $S(x,z) = \psi(x) z^{-1}$ with $\psi : M \to \mathbb{C}$. Then $(d+\Theta)(\psi(x) z^{-1})$ $(d_M + A)\psi(x) z^{-1} + \psi(x)\, dz \left(\partial_z + z^{-1}\right) z^{-1} = (d_M + A)\psi(x) z^{-1}$

$$(d+\theta)(\psi(x)z^{-1}) = \left(d_M + dz(\partial_z + z^{-1}) + A\right)(\psi(x)z^{-1})$$
$$= \left(d_M\psi(x) + A\psi(x)\right)z^{-1} + \psi(x)\,dz\underbrace{\left(\partial_z(z^{-1}) + z^{-1}z^{-1}\right)}_{0}$$

So far you have a uniqueness result for a principal $\mathbb{T}$-bundle with connection over $\mathbb{R}^2$ having the curvature $2\pi i\, dx\, dy$, i.e. translation-invariant curvature.

Let's look at geometric quantization again. (M,ω) a symplectic manifold. Then symplectic vector fields correspond to functions modulo constants (at least if $H^1(M)=0$) via $dh_X = \iota(X)\omega$. Consider the operators $L_X + h_X$ for different symplectic v.fs. Then

$$\left[L_X + \tfrac{1}{2}h_X, L_Y + \tfrac{1}{2}h_Y\right] = L_{[X,Y]} + \tfrac{1}{2}\left(L_X h_Y - L_Y h_X\right)$$

$$d\left(L_X h_Y - L_Y h_X\right) = \underset{\iota_{[X,Y]}\omega}{\underset{\parallel}{L_X\iota_Y\omega}} - \underset{\iota_{[Y,X]}\omega}{\underset{\parallel}{L_Y\iota_X\omega}} = 2\iota_{[X,Y]}\omega$$

so you get a projective repn. of symplectic v.f.'s on functions. Is this related to the principal $\mathbb{T}$ bundle P with curvature ω?

Assume P trivial (e.g. if $H^2(M,\mathbb{Z})=0$), choose a trivialization $P = M\times\mathbb{T} \xrightarrow{pr_1} M$, so that the connection becomes $d+A$ on functions on M, where $dA=\omega$. You would like the operators $L_X + \frac{1}{2}h_X$ to appear, so that one has a projective rep of symp. v.f.'s on sections of L. (if $H^1=0$)

$$[L_X, d+A] = L_X A \qquad dL_XA = L_X dA = L_X\omega = 0, \text{ so get } f_X$$

mod constants with $df_X = L_X A$. Then

$$[L_X + f_X, d+A] = L_XA + [f_X, d] = L_XA - df_X = 0$$

$$[L_X + f_X, L_Y + f_Y] = L_{[X,Y]} + \underbrace{L_X f_Y - L_Y f_X}_{f_{[X,Y]}} \qquad d(L_Xf_Y - L_Yf_X) = L_XL_YA - L_YL_XA = L_{[X,Y]}A = df_{[X,Y]}$$

You have this nice "geometric quantization" situation: group of symplectic diffeoms, operating on sections of L, and L itself. The problem is to get an honest Hilbert space repn.

Now you want to explore adjoining an idempotent freely to a ring Λ, this means you form the free product $\Lambda * p = \Lambda \oplus \mathbb{C}p \oplus \Lambda p \oplus p\Lambda \oplus \cdots\cdots$ There is some relation between this free product and the Cuntz algebra $\Lambda * \Lambda = \Omega\Lambda$ with Fedosov product.

In order to understand this properly, take the simple M_2 case where the objects are retracts of a 2 fold direct sum

$$\begin{tikzcd}
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} & W \arrow[l, tail, "\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}"'] & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}"']
\end{tikzcd}$$

$\beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$

α_1, α_2 surjective

β_1, β_2 injective

What you want to do: You have $\Lambda = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ You recall that the ring $A = \mathcal{P}_{M_2}$ generated by the components p_{ij} a proj $p = (p_{ij})$ in an M_2-graded alg. should be the free product $\mathbb{C}[e_1] * \mathbb{C}[e_2]$.

Point you missed. It's clear that the alg describing whose modules are two vector spaces V_1, V_2 equipped with a projection p on $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$, is the unital alg $\mathbb{C}[e_1] * \mathbb{C}[p]$, or the non unital alg $\mathbb{C}e_1 * \mathbb{C}p$. But you want the reduced conditions to hold, you want the ideal generated by p.

Look at alg $\mathbb{C}e * \mathbb{C}p$, ideal gen by p "units"

Recall your idea of adjoining multipliers corresponding to the "objects". Go over this. Start with $\mathcal{P}_\Gamma$ $\Gamma = M_2$

alg gens: p_{ij} rels $p_{ij}p_{k\ell} = 0$ $j \neq k$, $p_{ik} = \sum_j p_{ij}p_{jk}$

March 10, 02 To show $P = P_\Gamma$ is a Γ-graded alg.,

construct alg map $\Delta : P \longrightarrow \mathbb{C}\Gamma \otimes P$ such that $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$. Check the rels satisfied, and that $(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta$. Then you know there is a unique Γ_+ grading $P = \bigoplus_{ij} P_{ij} \oplus P_\infty$ such that $(\forall x \in P_{ij})\ \Delta x = e_{ij} \otimes x$, $\Delta(P_\infty) = 0$. You now need to show that $P_\infty = 0$.

P is spanned by words $p_{s_1} \cdots p_{s_n}$ in the generators, and P_s for $s \in \Gamma_+$ is spanned by such words of degree $s_1 s_2 \cdots s_n = s$. In the case $\Gamma = M_n$, $\mathbb{C}\Gamma$ is the path algebra of a category, where $s_1 \cdots s_n = \infty \implies$ some $s_i = \infty$ or some $s_i s_{i+1} = \infty$. $(s_i = \infty \implies p_{s_i} = 0 \implies p_{s_1} \cdots p_{s_n} = 0$, $s_i s_{i+1} = \infty \implies$ the ordered pair of arrows (s_i, s_{i+1}) is not composable $\implies s_i s_{i+1} = \infty$. But the supp reln. $p_{ij} p_{kl} = 0$ $j \neq k$ says that $p_{s_i} p_{s_{i+1}} = 0 \implies p_{s_1} \cdots p_{s_n} = 0$. This concludes $P_\infty = 0$.

Recap: For $\Gamma = M_n$ you have shown that P_Γ defined by gens + rels is a Γ-graded alg. P_Γ is idempotent.

Question: What are the consequences of the Γ-grading for the reduced module category?

Your idea here was to embed P_Γ as an ideal in a unital ring, better might be to look at $Mult(P_\Gamma)$.

Adjoin 1 to P_Γ, then get $1 = \sum_i e_i$ where the e_i are identity maps for the objects of Γ.

Motivation: Arrow alg of a category, its good modules are functors. Special role played by 1_X for any object X.

Better picture $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ You want to adjoin an identity which should be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Go back to

$$\begin{tikzcd} \begin{pmatrix} V_1 \\ V_n \end{pmatrix} & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}"'] & \begin{pmatrix} V_1 \\ \\ V_n \end{pmatrix} \arrow[l, two heads, "(\beta_1 \quad \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}"'] \end{tikzcd}$$

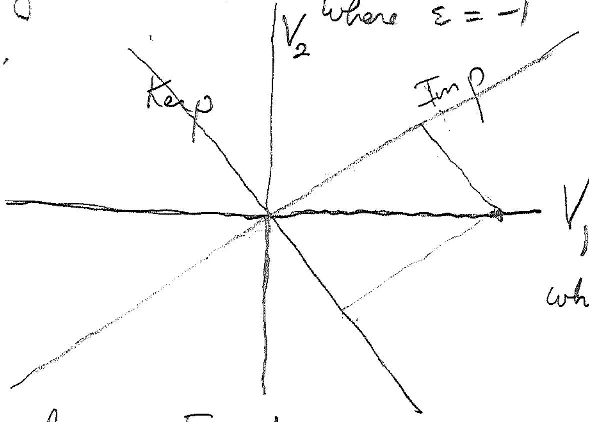
α_i surj

β_j inj.

Aim: Understand the case $n=2$ properly. This should be simple because the category $\mathcal{W}$ is the cat of unital modules over the ~~commutative~~ unital ring $\mathbb{C}\langle h_1, h_2\rangle / h_1+h_2=1$. You have the category vector spaces equipped with two projections, equivalently two involutions, hence you have the group ring of the dihedral group. There's a nuance in that you favor the image of p and want to ignore the kernel of p.

Visit again ε, F involutions on V $\quad g = \varepsilon F$

$\varepsilon g \varepsilon^{-1} = g^{-1}$.



Im p is where $F = 1$ $\qquad$ Ker p is where $F = -1$
then $g = -\varepsilon$

Review C.T. $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ $\qquad F = +1$ on $W = \operatorname{Im} p$
-1 on Ker p

$g = F\varepsilon$

$$g = \frac{1+X}{1-X} \qquad \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = 1 + X$$

$$F(1+X) = g\varepsilon(1+X) = \frac{1+X}{1-X}(1-X)\varepsilon = (1+X)\varepsilon = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\therefore \ F = +1$ on $\begin{pmatrix} 1 \\ T \end{pmatrix}$ $\qquad F = -1$ on $\begin{pmatrix} -T^* \\ 1 \end{pmatrix}$

What is the analog of α_1, α_2 surj. and β_1, β_2 inj.?
Use spectral theory for dihedral group. Decompose according to eigenvalues of $g = F\varepsilon$. $\quad \xi, \xi^{-1}$ related by $\varepsilon g \varepsilon^{-1} = g^{-1}$ leading to an irred. rep of dim 2 if $\xi \neq \xi^{-1}$. Otherwise $g = \pm 1$ so you

get 1-diml irred reps of 4 types.

V_1	V_2	W	$W^\perp$	E	F	g
$\mathbb{C}$	0	$\mathbb{C}$	0	1	1	1
0	$\mathbb{C}$	$\mathbb{C}$	0	-1	1	-1
$\mathbb{C}$	0	0	$\mathbb{C}$	1	-1	-1
0	$\mathbb{C}$	0	$\mathbb{C}$	-1	-1	1

This table not very useful.

Instead go back to

$$\begin{tikzcd} W & \begin{pmatrix} V_1 \\ V_n \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \ \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}"'] \end{tikzcd} \qquad \sum \beta_i \alpha_i = 1_W$$

and observe that W is unchanged if you shrink V_1 to $\alpha_1 W$ and afterward collapse $\alpha_1 W$ to $\alpha_1 W / \mathrm{Ker}(\beta_1 | \alpha_1 W)$. Clearly W is unchanged when we replace (β_j, V_j, α_j) by the canon. fact of h_i on W.

Discuss the situation. Study retracts

$$\begin{tikzcd} \begin{pmatrix} V_1 \\ V_n \end{pmatrix} & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_n\end{pmatrix}"'] & \begin{pmatrix} V_1 \\ V_n \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \ \beta_n)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \beta\alpha = 1_W$$

This retract is equivalent to the operator $p = \alpha\beta = (p_{ij}) = (\alpha_i \beta_j)$ on $V = \begin{pmatrix} V_1 \\ V_n \end{pmatrix}$.

Describe the problem: You have a "Morita" equivalence between $\mathcal{W}$ = unital modules over $\mathbb{C}\langle h_1, .., h_n\rangle / (\sum_i h_i = 1) = R$ and $\mathcal{V}$ = cat of graded v.s. $\begin{pmatrix} V_1 \\ \\ V_n \end{pmatrix}$ equipped with op $p_{ij} : V_i \leftarrow V_j$ satis $p_{ik} = \sum_j p_{ij} p_{jk}$, + red conditions.

You need idempotent rings whose M_r are $\mathcal{W}$, $\mathcal{V}$ and a Morita context containing these rings which yields the Mor. equiv.

Go thru the steps. Basic category has following objects:
a graded v.s. $\begin{pmatrix}V_1\\V_2\end{pmatrix}$ equipped with a retract W:

$$\begin{tikzcd} W & \begin{pmatrix}V_1\\V_2\end{pmatrix} \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}"'] \end{tikzcd}$$

$\beta\alpha = \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$

subject to $\left(\begin{matrix}\beta_j \text{ inj}\\ \alpha_i \text{ surj.}\end{matrix}\right.$

Forget V functor keeps W, $h_i = \beta_i\alpha_i$
Can recover $W \xleftarrow{\beta_i} V_i \twoheadleftarrow{\alpha_i} W$ as canon. fact of h_i

Forget W functor keeps pair $\begin{pmatrix}V_1\\V_2\end{pmatrix}$ and $p_{ij} = \alpha_i\beta_j : V_i \leftarrow V_j$
satisf reduced cond $V_i = \sum_j p_{ij}V_j$, $(\forall i)\, p_{ij}v_j = 0 \implies v_j = 0$
Can recover W as retract assoc. to operator $p = (p_{ij})$ on $\begin{pmatrix}V_1\\V_2\end{pmatrix}$

The problem now is to find the ring P behind the V-category
Method start with gens p_{ij}, rels. | supp idemp., then construct
M_2 grading $P \xrightarrow{\Delta} M_2\mathbb{C}\otimes P$ $\Delta p_{ij} = e_{ij}\otimes p_{ij}$
So $P = \begin{pmatrix}P_{11} & P_{12}\\ P_{21} & P_{22}\end{pmatrix}$ is a Morita context, + you should be able to embed P as an ideal in $\begin{pmatrix}P_{11}\oplus\mathbb{C}e_{11} & P_{12}\\ P_{21} & P_{22}\oplus\mathbb{C}e_{22}\end{pmatrix}$ which should be a unital ring. NOT VERY CLEAR.

Maybe it's possible to see that e_{11} is a multiplier
Does projection on the first row commute with right mult?
Yes. Similarly for proj. on the 2nd row, etc.

Now arises the question about whether you can construct the Morita context easily

$$\begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2 & \alpha_1\\ \alpha_2\beta_1 & \alpha_2\beta_2 & \alpha_2\\ \beta_1 & \beta_2 & R\end{pmatrix}\begin{pmatrix}V_1\\V_2\\W\end{pmatrix}$$

It looks like you might have trouble since you still need e_{11}, e_{22}. No, You should still get e_{ii} $i=1,2,3$ as multipliers.

You want a better way to do things. The idea would be to exploit the similarity between retracts and Joachim's RA construction.

IDEA ⤴ First case is where the W category is the unital modules over $\mathbb{C}\langle h_1, h_2\rangle/(h_1+h_2=1)$ = universal extension of $\mathbb{C}e_{11}\oplus\mathbb{C}e_{22}$. What ext? The GNS construction assoc. to $(\rho: A \xrightarrow[\text{connection}]{} RA)$ is something simple in terms of A. $a_1 \iota \gamma a_2 \iota \gamma a_3$

$\Gamma = GNS(\rho : A \to RA) = A \oplus A \otimes RA \otimes A$

Look at a repn of Γ on a vector space M. It consists of an A-module structure on M, a subspace retract N: $N \overset{i}{\rightarrowtail} M \overset{\gamma}{\twoheadrightarrow} N$, $\gamma i = 1_N$, a B-module structure on N, all this such that $\gamma a \iota n = \rho(a) n$.

When $\rho \overset{A \to RA}{:}$ is the canonical map?

Ask about an alg map $\Gamma \longrightarrow C$. This will be given by $A \xrightarrow{u} C$, $e \in C$ such that $e u(a) e = \rho(a)$

It seems that $\Gamma \longrightarrow C$ is the same as an alg map $A \longrightarrow C$ together with a projection $e \in C$.

INTERESTING POINT p635   where all $V_i = V$.

$$W \xleftarrow{(\beta_1 \cdots \beta_n)} \begin{pmatrix} \mathbb{C} \\ \vdots \\ \mathbb{C} \end{pmatrix} \otimes V \xleftarrow{\begin{pmatrix} \alpha_1 \\ \alpha_n \end{pmatrix}} W$$

$\sum \beta_j \alpha_j = 1_W$

$p_{ij} = \alpha_i \beta_j \in End(V)$
satisfy idemp. condition

then supp cond.!

$j \neq k \Rightarrow \underset{\overset{\|}{\alpha_i \beta_j \alpha_k \beta_l}}{p_{ij} p_{kl}} = 0$ apply $\sum \beta_i \circ$ and $\sum \circ \alpha_l$ to get $\beta_j \alpha_k = 0$ for $j \neq k$

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p646   GNS idea $\rho : A \to B$ $A = \oplus \mathbb{C} e_i$ $\rho(e_i) = h_i$ $\sum h_i = 1$.

positive hermitian-valued measures given by projections.

p651   Is any idempotent ring Mor. equi. to a ring with local units? example sequences $(f_n)_{n \geq 0}$ with limit 0; this is a C^* alg.
Mor. equiv. of comm. rings $\Rightarrow$ isom.

p662   Use embedding $\Delta : A \longrightarrow \Lambda \otimes \tilde{A}$, let $e_{ii} \otimes 1 \in \Lambda \otimes \tilde{A}$, left + right mult by e_{ii} preserves ΔA, defining a multiplier on A.

Mar 11, 02 Link of $A * p$, where $p = p^2$ with the universal extension. Work unitally.

Let A be a unital ring, e.g. $\bigoplus_{i=1}^{n} \mathbb{C} e_i$, adjoin a projection p freely to A, i.e. form $A \underset{(\mathbb{C})}{*} \mathbb{C}[p]$, $p^2 = p$. Replace p by $F = 2p - 1$, $F^2 = 1$. Then one has

$$A * \mathbb{C}[F] = (A * A) \rtimes F.$$

Recall $A * A$ is Cuntz's QA $= \Omega A$ equipped with Fedosov product.

You want the ideal generated by p. You want to ignore $\operatorname{Ker} p$.

Example: $D = \mathbb{C}[e = e^2] * \mathbb{C}[p = p^2] = \mathbb{C}[\varepsilon, \varepsilon^2 = 1] * \mathbb{C}[F, F^2 = 1]$ $=$ group alg of the dihedral group. Basis $g^n, \varepsilon g^n$ $n \in \mathbb{Z}$, where $g = F\varepsilon$. F is important.

Adjoin an involution F to a unital algebra A, say with basis $1, a$; denote this $A * F$. It has basis given by words

$$\left|\begin{array}{l} a, aF, aFa, \ldots \\ F, Fa, FaF, \ldots \end{array}\right.$$

Recall $A * F = QA \rtimes F$ where $QA = A * A$ is the subalg gen by the two alg maps $\iota a = a$, $\bar{\iota} a = FaF$. Let $pa = \frac{a + FaF}{2}$, $qa = \frac{a - FaF}{2} = \frac{1}{2} F[F, a]$. Then QA has the basis $pa (qa)^n$ $n \geq 0$. Also there's the isom

$$QA \simeq \Omega A \text{ equipped with } \omega_1 \circ \omega_2 = \omega_1 \omega_2 - (-1)^{|\omega_1|} d\omega_1 d\omega_2.$$

QA is $\mathbb{Z}/2$ graded; you've formed the cross product with $\mathbb{Z}/2$. There might be a Morita equivalence between $A * F$ and the even part of QA which is RA.

$D = \mathbb{C}[\varepsilon * F]$ group ring of dihedral group, a unital D-modules are same as a vector spaces with 2 splittings. You need

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Let's begin this over again. I think you are missing something important like the link to RA. There might be a Morita equivalence of a general sort.

Let's study the simplest case. The first structure of interest is a vector space V equipped with two projections e, p. You think of these differently, namely, e gives a grading $V = V_1 \oplus V_2$ wrt the set $\{1,2\}$ and p gives a retract (W, β, α) of V:

$V_1 = eV$
$V_2 = (1-e)V.$

$$\begin{tikzcd} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, two heads, "(\beta_1, \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] \end{tikzcd}$$

$\beta\alpha = 1_W$

$\alpha\beta = (p_{ij})$

It's clear this structure is the same as a unital module structure of $\mathbb{C}[e] * \mathbb{C}[p]$ on V. If you shift to involutions $\varepsilon = 2e - 1$, $F = 2p-1$, then you have just a unital module over the group alg of the infinite dihedral group, a representation of the dihedral group. $\mathbb{Z}/2 \ltimes \mathbb{Z}$ gen. ε, $g = F\varepsilon$

Except this is not the real structure of interest because you want to require α_i surj, β_j inj. (This means that V_i can be recovered as the image of $h_i = \beta_i \alpha_i$)

Let's examine carefully what you have. There are three module categories ~~which are equivalent~~ via forgetful functors

$$\mathcal{V} \longleftarrow \mathcal{U} \longrightarrow \mathcal{W}$$

The easiest to understand is $\mathcal{W}$, which is the cat of unital modules over $R = R(\mathbb{C}e_1 \oplus \mathbb{C}e_2)$.

You seem to struggle over the grading, yet if it's possible to do things on the level of RA, the struggle might be unnecessary.

Look carefully at $A * F = (A * A) \rtimes F$. Can you identify this with $\Gamma = GNS(A \xrightarrow{\rho} RA)$? unital modules over Γ should be the same as A-modules (unital) M equipped with a $\mathbb{C}$-linear retract $N \overset{j}{\twoheadleftarrow} M \overset{\iota}{\leftarrowtail} N$, together with an RA-mod structure (unital) on N satisfying $j a \iota n = \rho(a) n$.

Try to describe an (unital) alg map $\Gamma \longrightarrow C$

$$\Gamma = A \oplus A \otimes RA \otimes A$$

$\iota \quad ja\iota \quad j$

Mar 12, 02 Recall why $GNS(\rho: A \to RA) = A * p$

$C = A * p \simeq (A * A) \rtimes F$. You want the Morita context $\begin{pmatrix} pCp & pC \\ Cp & CpC \end{pmatrix}$ $\begin{pmatrix} RA & RA \otimes A \\ A \otimes RA & A \otimes RA \otimes A \end{pmatrix}$

The meaning of this Morita $\overset{\text{equiv.}}{\vee}$ is unclear. It looks trivial.

Back to $\mathbb{C}[\varepsilon] * \mathbb{C}[F] = \mathbb{C}[F\varepsilon] \rtimes F$, call this ring D, let $p = \frac{F+1}{2}$, consider $\begin{pmatrix} pDp & pD \\ Dp & DpD \end{pmatrix}$, look at $Dp =$

$$\left(\mathbb{C}[g, g^{-1}] \otimes (\mathbb{C} \oplus \mathbb{C}F)\right)p = \mathbb{C}[g, g^{-1}] \otimes \mathbb{C}p = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} g^n \frac{F+1}{2}$$

$$p\left(\bigoplus_n \mathbb{C} g^n \frac{F+1}{2}\right) = \sum_n \mathbb{C} \frac{F+1}{2} g^n \frac{F+1}{2} = \sum_n \mathbb{C}\left(g^n + Fg^nF + g^nF + Fg^n\right)$$

$$= \sum_n \mathbb{C}\left(g^n + g^{-n} + (g^n + g^{-n})F\right) = \sum \mathbb{C}(g^n + g^{-n})\, p$$

$\mathbb{C}[\varepsilon] * \mathbb{C}[F] = \mathbb{C}[g, g^{-1}] \rtimes \mathbb{C}[F]$ has basis g^n, g^nF better the basis g^n, g^np. Apply $p = \frac{F+1}{2}$

$$\frac{F+1}{2} g^n = \frac{1}{2}\left(g^n + g^{-n}\underbrace{F}_{2p-1}\right) = \frac{1}{2}(g^n - g^{-n}) + g^{-n}p$$

$$\frac{F+1}{2} g^n p = \frac{1}{2}\left(g^n + g^{-n}F\right)p$$

$\mathbb{C}[g, g^{-1}] \otimes \mathbb{C}[F]$ has basis g^n, g^np

$$pg^n = \frac{F+1}{2} g^n = \frac{1}{2}(g^n + g^{-n}F) = \frac{1}{2}(g^n + g^{-n}(2p-1)) = \frac{1}{2}(g^n - g^{-n}) + g^{-n}p$$

$$\frac{F+1}{2} g^n = \frac{1}{2}\left(g^n + g^{-n}F\right) = \frac{1}{2}\left(g^n + g^{-n}(2p-1)\right)$$
$$= \frac{1}{2}(g^n - g^{-n}) + g^{-n}p$$

$$\frac{F+1}{2} g^n p = \frac{1}{2}(g^n p - g^{-n}p) + g^{-n}p = \frac{1}{2}(g^n + g^{-n})p$$

You have the Morita contexts associated to a proj p in a ring D, ~~namely~~ $\begin{pmatrix} pDp & pD \\ Dp & D \end{pmatrix}$ get M.eq of pDp with DpD which is an ideal in D.

pDp subring

Question. If $D = \mathbb{C}[\varepsilon] * \mathbb{C}[F]$, what is the ideal DpD, where $p = \frac{F+1}{2}$? D/DpD is the quotient of D by the relation $F = -1$, which implies that D/DpD = $\mathbb{C}[\varepsilon] * \left(\mathbb{C}[F]/(F+1)\right) = \mathbb{C}[\varepsilon] * \mathbb{C} = \mathbb{C}[\varepsilon]$.

Recall what you knew. Given $\rho : A \to B$, $\rho(1) = 1$. Define product μ on $A \otimes B \otimes A$ by

$$(a_1' \otimes b_1 \otimes a_1'')(a_2' \otimes b_2 \otimes a_2'') = a_1' \otimes b_1 \rho(a_1'' a_2') b_2 \otimes a_2'$$

Let $e = 1 \otimes 1 \otimes 1$.

$$e(a' \otimes b \otimes a'') = 1 \otimes \rho(a') b \otimes a''$$
$$(a' \otimes b \otimes a'')e = a' \otimes b\rho(a'') \otimes 1$$

So $e^2 = e$ and

$$e(A \otimes B \otimes A) = 1 \otimes B \otimes A$$
$$(A \otimes B \otimes A)e = A \otimes B \otimes 1$$
$$e(A \otimes B \otimes A)e = 1 \otimes B \otimes 1.$$

Note that

$$(A \otimes B \otimes A)e(A \otimes B \otimes A) = (A \otimes B \otimes 1)(1 \otimes B \otimes A)$$
$$= A \otimes B \otimes A.$$

so you have the kind of Morita equivalence between B and $A \otimes B \otimes A$, which arises from an idempotent.

Next form semi-direct product $C = A \oplus A \otimes B \otimes A$. You are in the unital setting. An alg. map (unital) $C \longrightarrow D$ is specified by a unital alg map $A \xrightarrow{u} D$, an alg map $v : B \longrightarrow D$ not necessarily respecting unit (thus $v(1_B) = e$ is idempotent in D and $v : B \longrightarrow eDe$ is unital alg map. Impose condition $eu(a)e = v(\rho(a))$)

$$\boxed{\begin{array}{l} C \longrightarrow D \text{ equivalent to } A \xrightarrow{u} D \text{ unital}, \ e = e^2 \in D, \\ v : B \longrightarrow eDe \text{ unital} \quad eu(a)e = v(\rho(a)) \end{array}}$$

Assume $B = RA$. v is equivalent to linear map $A \longrightarrow eDe$ preserving 1, so v determ. by e, u.

It seems that the Morita equivalence between $A \otimes B \otimes A$ and B associated to a linear map $\rho : A \longrightarrow B$, $\rho(1) = 1$ is not going to be useful. So let's return to the special case under discussion, where $\Lambda = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ handles graded vector spaces wrt $\{1,2\}$, these are the "objects" of M_2, and where $p = (p_{ij})$ is a graded projection wrt the arrows of M_2.

In this situation $R = R\Lambda = \mathbb{C}\langle h_1, h_2\rangle / (h_1 + h_2 = 1)$ describes W, and you need an M_2-graded ring for V

$$\begin{array}{ccc} R & \alpha_1 & \alpha_2 \\ \beta_1 & & \\ \beta_2 & & \end{array}$$

$$\begin{tikzcd} W & \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\binom{\alpha_1}{\alpha_2}"'] \end{tikzcd}$$

And you need an M_3 graded ring for the Morita context.

$$\Delta : C \longrightarrow M_3\mathbb{C} \otimes C$$

$$\Delta(\alpha_i) = e_{i0} \otimes \alpha_i \qquad i = 1, 2$$

$$\Delta(\beta_j) = e_{0j} \otimes \beta_j \qquad j = 1, 2$$

$$\Delta(\beta_j)\Delta(\alpha_i) = e_{0j}e_{i0} \otimes \beta_j\alpha_i = \begin{cases} 0 & j \neq i \\ e_{00} \otimes \beta_i\alpha_i & j = i \end{cases}$$

$$\sum_i \Delta(\beta_i)\Delta(\alpha_i) = e_{00} \otimes \sum \beta_i\alpha_i = e_{00} \otimes 1$$

$$e_{ii} \otimes 1 \in M_3\mathbb{C} \otimes \tilde{C}$$

$$M_3 C \oplus M_3\mathbb{C}$$

$$\Delta c_{ij} = e_{ij} \otimes c_{ij}$$

March 13, 02

Consider a Morita context, i.e. a Γ-graded alg A where $\Gamma = M_2$ = arrows in the groupoid with objects $\{1, 2\}$ an a unique map from one obj to another. This means one has

$$A = \bigoplus_{ij} A_{ij} \quad \text{such that} \quad \Delta : A \longrightarrow M_2\mathbb{C} \otimes A \quad \text{is an alg. homom.}$$

$$\Delta(a_{ij}) = e_{ij} \otimes a_{ij}$$

You want to prove that you can enlarge A to a unital Morita context $A \oplus \mathbb{C}e_{11}$

$$\begin{pmatrix} \mathbb{C}e_{11} & 0 \\ 0 & \mathbb{C}e_{22} \end{pmatrix} \oplus \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}^+ & A_{12} \\ A_{21} & A_{22}^+ \end{pmatrix}$$

So the idea was simply to look at

$$\Delta : A \longrightarrow M_2\mathbb{C} \otimes A \subset M_2\mathbb{C} \otimes A^+ = M_2\mathbb{C} \ltimes (M_2\mathbb{C} \otimes A)$$

You want to adjoin

$$\begin{tikzcd}
A \arrow[d, "\Delta"] & e_{ii} \otimes 1 \arrow[d, phantom, sloped, "\in"] & \mathbb{C}e_{ii} \\
M_2\mathbb{C} \otimes A \arrow[r, tail] & M_2\mathbb{C} \otimes A^+ \arrow[r, two heads] & M_2\mathbb{C}
\end{tikzcd}$$

let $a \in A_{ij}$ $\quad \Delta a = e_{ij} \otimes a$

$$(e_{kk} \otimes 1)\Delta a = e_{kk}e_{ij} \otimes a = \begin{cases} 0 & k \neq i \\ e_{ij} \otimes a & k = i \end{cases} = \begin{cases} 0 & k \neq i \\ \Delta a & k = i. \end{cases}$$

You need to put this clearly.

You have $A = \begin{pmatrix} e_{11} \otimes A_{11} & e_{12} \otimes A_{12} \\ e_{21} \otimes A_{21} & e_{22} \otimes A_{22} \end{pmatrix}$

$$\begin{tikzcd}
A = \bigoplus_{ij} A_{ij} \arrow[r, "\Delta"] & M_2\mathbb{C} \otimes A^+ \arrow[d, phantom, "\cup"] \\
A_{ij} \arrow[r, "\Delta", "\sim"'] & e_{ij} \otimes A_{ij}
\end{tikzcd}$$

Maybe it's just a matter of checking that the subalg $\bigoplus e_{ij} \otimes A_{ij}$ of $M_2 \otimes A^+$ is closed under left + rt mult by $e_{kk} \otimes 1$

When is a Morita context $\underbrace{\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}}_{A=}$ unital.

Look at left mult, which preserves the two columns. So you get right mult. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$\mathrm{Im}\left\{\begin{tikzcd} A\otimes_A V \arrow[rr] \arrow[dr] & & \mathrm{Hom}_A(A,V) \\ & V \arrow[ur] & \end{tikzcd}\right\}$$

$$\begin{tikzcd} a\otimes v \arrow[rr, mapsto] \arrow[dr] & & (a'\mapsto a'av) \\ & av \arrow[ur, mapsto] & \end{tikzcd}$$

Let $\mu = (\mu_\ell, \mu_r)$

$$\begin{tikzcd} a\otimes v & (a'\mapsto a'av) \\ A\otimes_A V \arrow[r] \arrow[d, "\mu_\ell\otimes 1"'] & \mathrm{Hom}_A(A,V) \arrow[d, "\mu_r"] \\ A\otimes_A V \arrow[r] & \mathrm{Hom}_A(A,V) \\ & (a'\mapsto(a'\mu_r)av) \\ \mu_\ell a\otimes v \arrow[r, mapsto] & (a'\mapsto(a'\mu_\ell a)v) \end{tikzcd}$$

Let's return to examples.

$$\begin{tikzcd} \Lambda\otimes V \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1"] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1"] \\ V \arrow[u, shift left, "\varepsilon_1"] & & V \arrow[u, shift left, "\varepsilon_1"] \arrow[ul, "\beta_1"] & \end{tikzcd}$$

$$\beta\alpha w = \sum_s s\underbrace{\beta_1\alpha_1}_{h=h_1}s^{-1}w$$

$$\beta\sum_t t\otimes f(t) = \sum_t t\beta_1 f(t)$$

$$\alpha w = \sum_s s_1\otimes \alpha_1 s^{-1}w$$

$$\alpha\beta\sum_t t\otimes f(t) = \sum_s s\otimes\sum_t \underbrace{\alpha_1 s^{-1}t\beta_1}_{p(s^{-1}t)} f(t)$$

$$\mathrm{Ker}(\beta_1) = \mathrm{Ker}(\alpha\beta_1) = \bigcap_s \mathrm{Ker}(\alpha_1 s^{-1}\beta_1) = \bigcap_s \mathrm{Ker}\, p(s^{-1}) \text{ on } V$$

$$\alpha_1 W = \alpha_1\beta(\Lambda\otimes V) = \left\{\sum_t \alpha_1 t\beta_1 f(t)\right\} = \left\{\sum_t p(t) f(t)\right\} = \sum_t p(t) V$$

Need to set up the Morita context.

Aim: You have two approaches to the Morita context, one based on pretending h is idempotent, the other on grading wrt Γ, this means $\alpha_s = \alpha_1 s^{-1}$, $\beta_t = t\beta_1$. You need to get at least one of these understood completely.

Note that there is no $\overset{\text{obvious}}{\text{grading}}$ around in the Morita context $\begin{pmatrix} hBh & hB \\ Bh & BhB \end{pmatrix}$. But Bh "contains" $\beta_t = t\beta_1$ and hB "contains" $\alpha_1 s^{-1} = \alpha_s$. What exactly would you like to do?

Compare examples

$$\begin{pmatrix} R & \beta_1 & \beta_2 \\ \alpha_1 & & \\ \alpha_2 & & \end{pmatrix}\begin{pmatrix} W \\ V_1 \\ V_2 \end{pmatrix} \qquad \begin{pmatrix} B & \cdots t\beta_1 \cdots \\ \vdots & \\ \alpha s^{-1} & ▨ \\ \vdots & \end{pmatrix}\begin{pmatrix} W \\ V \end{pmatrix}$$

Program: Construct the M. context with the appropriate grading

Assign degree – what is Γ? A groupoid with two objects? B is a Γ algebra, A is a Γ-graded alg.

You forgot that W is a Γ-module, V is a vector space. So your Γ will have two objects, one W having Γ for its autos, the other V having only the identity, and then arrows $t\beta_1 : V \to W$, $\alpha_1 s^{-1} : W \to V$. Not clear

Two objects W, V $\quad Aut(W) = \Gamma \quad Aut(V) = pt.$

You have arrow $V \xleftarrow{\alpha_1} W \xleftarrow{\beta_1} V$. Questions: What structure do you have? a groupoid, a category, quiver?

$$\begin{tikzcd} W \arrow[loop above, "s"] \arrow[d, bend right, "\alpha_1"'] \\ V \arrow[u, bend right, "\beta_1"'] \end{tikzcd} \qquad \begin{pmatrix} \Gamma & \Gamma\beta_1 \\ \alpha_1\Gamma & \alpha_1\Gamma\beta_1 \end{pmatrix}\begin{pmatrix} W \\ V \end{pmatrix}$$

What do you have? You do not have a groupoid, but you might have a category with two objects. This picture is close to GNS, where W is Γ-module

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Start again. Aim is to find an alg yielding the red modules over the Morita context $\mathbb{C}$. A reduced C-mod should yield a Γ-module W and a vector space V together with $W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W \xleftarrow{\beta_1} V$

You are puzzled by the fact that W is a Γ- module and V has some dual $\Gamma^\vee$- structure. Example: If Γ abelian and W is a Hilbert representation of Γ with generating subspace $V \underset{i}{\overset{j}{\rightleftarrows}} W$, then $\rho(s) = jsi$, then $\rho(s) \in \mathcal{L}(V)$ is a positive measure on the dual.

Start again. Recall GNS. You have Γ acting on W and maps $V \xleftarrow{j} W \xleftarrow{i} V$, whence a function $\rho(s) = jsi : \Gamma \longrightarrow \mathcal{L}(V)$. Assuming Hilbert space context ρ is completely positive function on Γ. If Γ abelian, then you should get a positive matrix measure on $\Gamma^\vee$.

So if you take $\Gamma = \mathbb{Z}$

It's possible that the grading you seek is not present. Maybe that's the lesson arising from your proof that the Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ works.

$B = \mathcal{E} \rtimes \Gamma$ is naturally a Γ-graded algebra

$hB = h\mathcal{E} \times \Gamma$

GNS case $\rho(s) = i^* s\, i \in \mathcal{L}(V)$ completely positive operator valued functions - means you get a positive hermitian form $\sum_{s,t} \langle f(s) | \rho(s^{-1}t) | f(t) \rangle$ on $\Gamma \otimes V$

You restrict to idempotent hermitian case. If $\Gamma = \mathbb{Z}$ then you get a family of projections on the trivial bundle with fibre V over the circle.

You are studying Γ grading for the Morita context. The two rings $\mathcal{E} \rtimes \Gamma$ and P are Γ graded. So the Morita context should be $M_2 \times \Gamma$ graded.

Mar 14, 02   Construct the Morita context C as an $M_2 \times \Gamma$ graded algebra. Generators α_s, β_t

$$\begin{pmatrix} & \alpha_s \\ \beta_t & \end{pmatrix}\begin{pmatrix} V \\ W \end{pmatrix}$$

$$\Delta\alpha_s = e_{12} \otimes s^{-1} \otimes \alpha_s \in M_2\Lambda \otimes C$$

$$\Delta\beta_t = e_{21} \otimes t \otimes \beta_t$$

Relations $\sum_s \beta_s\alpha_s = 1$

$$\sum_s \Delta(\beta_s)\Delta(\alpha_s) = \sum_s e_{22} \otimes 1 \otimes \beta_s\alpha_s = e_{22} \otimes 1 \otimes 1$$

$$\Delta(\alpha_s\beta_t) = \Delta(\alpha_s)\Delta(\beta_t) = e_{11} \otimes s^{-1}t \otimes \alpha_s\beta_t$$

You need to impose a relation for each pair of non-composable arrows.

Start again. To construct a graded alg with resp. $M_2 \times \Gamma$. Ultimately you want a unital alg which describes your module category with an idempotent ideal inside. Actually you construct an idempotent alg and enlarge it to a unital ring using multipliers.

Generators are α_s, β_t where $s, t \in \Gamma$

Relations: support $\alpha_s\beta_t \neq 0 \implies s^{-1}t \in \Phi$

completeness $\sum_s \beta_s\alpha_s\beta_t = \beta_t$, $\sum_s \alpha_t\beta_s\alpha_s = \alpha_t$

Let C be the ring defined by these generators and relations.

The program is to multipliers of C that will yield on any reduced C-module a splitting into a Γ-module W and a v.s. V together with etc.

Actually all you want is $\Gamma \longrightarrow \mathrm{Mult}(C) \longleftarrow \mathbb{C}e_{11}$?

Want $\begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C}\Gamma \end{pmatrix} \longrightarrow \mathrm{Mult}(C)$

Go over again the two versions for the Morita context.

$h=\beta_1\alpha_1$ $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ $\begin{pmatrix} p & (P\otimes\Lambda)p \\ p(\Lambda\otimes P) & B=\mathcal{E}\rtimes\Gamma \end{pmatrix}$

You have the functors $V \longmapsto p(\Lambda\otimes V) = W$
$W \longmapsto hW = V$

why they are inverse

$$\begin{tikzcd}
\Lambda\otimes V & W \arrow[l, tail, "\alpha"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1"] \\
 & & V \arrow[ul, "\beta_1"] \arrow[u, shift left, "\varepsilon_1"] &
\end{tikzcd}$$

Given W (Γ-module $+ h$, support cond., $\sum_s = 1$) $\exists!$
($W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$, $h=\beta_1\alpha_1$, β_1 surj, α_1 inj)

$\mathcal{W}$ an obj $=$ Γ-mod W $+$ $h \in \mathrm{Hom}_{\mathbb{C}}(W,W)$ $\ni hsh \neq 0 \Rightarrow s \in \Phi$
$w = \sum shs^{-1}w$ (finite sum)

$\mathcal{V}$ an obj $=$ v.s. V $+$ $p(s) \in \mathrm{Hom}_{\mathbb{C}}(V,V)$ supp + idemp. conditions, 2 red. conds.

$\mathcal{U}$ an obj $=$ $(V, W, \alpha_1, \beta_1)$ V v.s., W Γ-mod
$\alpha_1 : V \twoheadleftarrow W$, $\beta_1 : W \leftarrowtail V$ sat. $\alpha_1 s \beta_1 \neq 0 \Rightarrow s \in \Phi$
$\sum_s s\beta_1\alpha_1 s^{-1} w = w$

Given W, let $h = \beta_1\alpha_1 : W \leftarrowtail V \twoheadleftarrow W$ be canon fact of h
then $\alpha_1 s \beta_1 \neq 0 \Rightarrow hsh \neq 0 \Rightarrow s \in \Phi$. Also completeness.
Given V, $p(s)$ define $p(\sum_t t\otimes f(t)) = \sum_s s \otimes \sum_t p(s^{-1}t)f(t)$ on $\Lambda\otimes V$
$p^2 = p$, $pu = up$. Get (W, α, β)

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1"] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \\
 & V \arrow[ul, "\beta_1"] & & V \arrow[ul, "\beta_1"] \arrow[u, "\varepsilon_1"']
\end{tikzcd}$$

You are looking at $\mathcal{W} \doteq \mathcal{U} \simeq \mathcal{V}$.

Given W, h let $W \xleftarrow{\beta_1} V \xtwoheadleftarrow{\alpha_1} W$ be an an fact $h = \beta_1\alpha_1$. Then $0 \neq \underset{\overset{\shortparallel}{p(t)}}{\alpha_1 t \beta_1} \Longrightarrow \beta_1\alpha_1 t \beta_1 \alpha_1 \neq 0 \Longrightarrow t \in \Phi$.

$\boxed{\text{let } p(t) = \alpha_1 t \beta_1 \in \mathcal{L}(V)}$ $\quad \sum_t p(s^{-1}t)p(t^{-1}u)v =$

$\sum_t \alpha_1 s^{-1} t \beta_1 \alpha_1 t^{-1} u \beta_1 v = \alpha_1 s^{-1} u \beta_1 v$. Given $v \in V = \alpha_1 W$, write $v = \alpha_1 w = \sum_s \alpha_1 s \beta_1 \alpha_1 s^{-1} w \in \sum_s p(s) V$. Also

$(\forall s)\ p(s^{-1})v = 0 \Longrightarrow 0 = \sum s \beta_1 \alpha_1 s^{-1} \beta_1 v = \beta_1 v \Longrightarrow v = 0$ as β_1 inj.

Next given $V, p(s)$ define p on $\Lambda \otimes V$ $\quad up = pu$, $p^2 = p$

let $W = p(\Lambda \otimes V)$, let $\alpha_1 = \eta_1 \alpha \quad \beta_1 = \beta\varepsilon_1$

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \end{tikzcd}$$

β, α comm. with u

Four things to prove: (i) $0 \neq \alpha_1 s \beta_1 = \eta_1 p \varepsilon_1 = p(s) \Longrightarrow s \in \Phi$ ✓

(ii) $w = \beta\alpha w = \sum_{s\,t} \beta s \varepsilon_1 \eta_1 s^{-1} \alpha w = \sum_s s \beta_1 \alpha_1 s^{-1} w$

(iii) α_1 surj? $\alpha_1 W = \eta_1 \alpha\beta(\Lambda\otimes V) = \sum \eta_1 s \otimes \sum_t p(s^{-1}t) V = \sum p(s^{-1}) V$

(iv) β_1 inj? $\beta_1 v = 0 \Longrightarrow 0 = \alpha\beta_1 v = \alpha\beta\varepsilon_1 v = {}^t\sum s \otimes p(s^{-1}) v$

$\Longrightarrow \forall s\ p(s^{-1}) v = 0 \Longrightarrow v = 0$.

So now you have the basic Morita equivalence under control and you can hunt for the Morita context. So begin with ~~tl~~ typical $\mathcal{U}$-module $U = \{W_\Gamma \overset{\beta_1}{\underset{\alpha_1}{\rightleftarrows}} V\}$. Your aim is to find an idempotent ring C whose red. module cat is $\mathcal{U}$. The important structure comes from $\mathrm{Mult}(C)$.

Is it possible to get at the interesting structure in a better way? Let's go back to the simplest case namely $\mathcal{U}$, an object is a triple of v.s. $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \overset{\beta}{\underset{\alpha}{\rightleftarrows}} W$ and 4 maps.

$$\begin{tikzcd} W & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, two heads, "{(\beta_1, \beta_2)}"'] & W \arrow[l, tail, "{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}}"'] & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, two heads, "{(\beta_1, \beta_2)}"'] \end{tikzcd}$$

such that $\beta_1\alpha_1 + \beta_2\alpha_2 = 1$, β_1, β_2 inj α_1, α_2 surj.

$\mathcal{W}$ consists of unital $\mathbb{C}\langle\langle h_1, h_2\rangle\rangle / 1 = h_1 + h_2$ modules

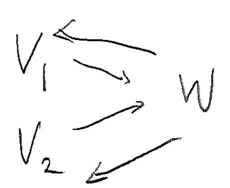
A Module in $\mathcal{U}$ is a triple (V_1, V_2, W) + maps

$$\begin{tikzcd} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] & \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \arrow[l, two heads, "(\beta_1 \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] \end{tikzcd}$$

satisf. $\beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$, α_i surj, β_i inj. You want a unital ring whose unital modules are triples with indicated maps. Look at the ~~rings~~ ring of operators on $\begin{matrix} V_1 \\ \oplus \\ V_2 \\ \oplus \\ W \end{matrix}$ that you get

$$\begin{pmatrix} e_1 & & \alpha_1 \\ & e_2 & \alpha_2 \\ \beta_1 & \beta_2 & e_W \end{pmatrix}$$

Your data consists of vector spaces V_1, V_2, W and 4 maps $\alpha_i : W \longrightarrow V_i$, $\beta_j : V_j \longrightarrow W$ subject to 1 relation $\beta_1\alpha_1 + \beta_2\alpha_2 = e_W$. So you have a quiver



Let's try to organizing the confusion. Look at Γ case. A module in $\mathcal{U}$ is $(W\ \Gamma \text{ module}, V\ \mathbb{C}\text{-mod}, W \xleftarrow{\beta_1} V \twoheadleftarrow{\alpha_1} W)$

Example: $\Gamma = \mathbb{Z}/2$. The module cat $\mathcal{U}$ consists of $(W, V, \alpha_1, \beta_1)$ where W is a $\mathbb{Z}/2$ module, V a vector space $W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$, $1_W = \beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon^{-1}$.

Start again. You want to compare two situations.

(1) retract of a graded vector space $\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$ wrt $\{1, 2\}$.

(2) retract of a free $\mathbb{Z}/2$ - module

Describe (1).

$$\begin{tikzcd} W & \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1$$

three spaces V_1, V_2, W four maps β_i, α_j, one relation

Describe (2).

$$\begin{tikzcd} W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \end{tikzcd}$$

β, α are $\mathbb{Z}/2$-mod maps

$\beta\alpha = 1_W$

$\mathbb{Z}/2$ module structure same as ± 1 eigenspace splitting for ε

So you get

$$\begin{tikzcd} W_+ & V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \end{tikzcd} \qquad \beta_+\alpha_+ = 1$$

$$\begin{tikzcd} W_- & V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"'] \end{tikzcd} \qquad \beta_-\alpha_- = 1.$$

It seems you end up with two projections on V. Summarize: You consider a $\mathbb{Z}/2$-module retract W of the free $\mathbb{Z}/2$ module $\Lambda\otimes V$, equivalently, a projection operator on the $\mathbb{Z}/2$ module $\Lambda\otimes V$. Now use ±1 eigenspace decomposition of $\mathbb{Z}/2$ modules $\Lambda = \frac{1+\varepsilon}{2}\Lambda + \frac{1-\varepsilon}{2}\Lambda$

$$\Lambda\otimes V = \frac{1+\varepsilon}{2}\otimes V \oplus \frac{1-\varepsilon}{2}\otimes V \simeq \begin{pmatrix} V \\ V \end{pmatrix} \quad \text{with } \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So you have learned that a $\mathbb{Z}/2$ module retract of $\Lambda\otimes V$ is equivalent to retracts W_+, W_- of V, i.e. two projections on V, which are independent.

So V is naturally a module over $\mathbb{C}p_+ * \mathbb{C}p_-$

How does a projection in a $\mathbb{Z}/2$ graded alg look?

$p = p_+ * p_-$ — you get $Q(\mathbb{C}e)$.

Repeat: Consider a $\mathbb{Z}/2$-module retract W of $\Lambda\otimes V$, split into ±1 eigenspaces for ε = the generator of $\mathbb{Z}/2$, then W is equiv. to retracts $W_\pm$ of $\frac{1\pm\varepsilon}{2}\otimes V \simeq V$, which in turn are equivalent to projections $p_\pm$ on V.

Conclude: a Λ-retract of $\Lambda\otimes V$ is equivalent to a module structure on V for the free product $\mathbb{C}e_+ * \mathbb{C}e_- = \mathcal{P}$

V is reduced when $V = e_+V + e_-V$, $\quad e_+v = e_-v = 0 \implies v = 0$.

$$\begin{tikzcd} W_+ \oplus W_- & V \arrow[l, tail, "\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}"'] & W_+ \oplus W_- \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] \end{tikzcd}$$

Repeat. $\mathbb{Z}/2$ module retract W of $\Lambda\otimes V$ splits into ±1 eigenspaces for the gen. ε. Get

$$\begin{tikzcd} W_+ & \frac{1+\varepsilon}{2}\otimes V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \\ W_- & \frac{1-\varepsilon}{2}\otimes V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"'] \end{tikzcd}$$

So you end up with two projections $p_+ = \alpha_+\beta_+$, $p_- = \alpha_-\beta_-$.

So V becomes a module over the free product $\mathbb{C}p_+ * \mathbb{C}p_-$.

Don't lose sight of the fact that you should be able to handle this case, since it concerns $\Gamma \ltimes \mathcal{E}_\Gamma$, $\mathcal{P}_\Gamma$.

$\mathcal{P}_\Gamma$ gens p_1, p_ε ; better p_0, p_1 $\qquad p_n = \sum p_i p_{n-i}$ $\qquad$ $p_0 = p_0^2 + p_1^2$
$p_1 = p_0p_1 + p_1p_0$

which means that $p_0 \pm p_1$ idempotent. Go back to

$$\begin{tikzcd} W_+ & V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \end{tikzcd} \qquad \beta_+ \alpha_+ = 1_{W_+}$$

$$\begin{tikzcd} W_- & V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"'] \end{tikzcd} \qquad \beta_- \alpha_- = 1_{W_-}$$

can suppose $V = \alpha_+ W_+ + \alpha_- W_-$ without changing W. Similarly can suppose $\mathrm{Ker}(\beta_+, \beta_-) = 0$, and then you get

$$\begin{tikzcd} V & W_+ \oplus W_- \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] & V \arrow[l, tail, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"'] & W_+ \oplus W_- \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] \end{tikzcd}$$

$$\begin{pmatrix} 1 & \beta_+ \alpha_- \\ \beta_- \alpha_+ & 1 \end{pmatrix}$$

Somehow this picture so far has to have an explanation in terms of a vector space V equipped with two involutions, projections.

Focus on $\mathbb{C}[\varepsilon] * \mathbb{C}[F] = \mathbb{C}[\varepsilon] \ltimes \mathbb{C}[\mathbb{Z}]$. What does this say about a unital module? So any module over the dihedral group has got to exhibit a kind of $\mathbb{Z}$ symmetry. What you have is

$$\begin{tikzcd} V & W_+ \oplus W_- \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] & V \arrow[l, tail, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"'] & W_+ \oplus W_- \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] & V \arrow[l, "\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}"'] \end{tikzcd}$$

The obvious question is whether $\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} (\alpha_+ \ \alpha_-) = \begin{pmatrix} 1 & \beta_+ \alpha_- \\ \beta_- \alpha_+ & 1 \end{pmatrix}$ is invertible? What is the simplest V you can write down? Modules over a P.I.D. Are there any interesting indecomposables?

Mar 16, 02. Discuss representations of the dihedral group $\mathbb{Z}/2 \ltimes \mathbb{Z}$; these should be very close to representations of $\mathbb{Z}$, that is, modules over the ring $L = \mathbb{C}[g,g^{-1}]$ of Laurent polys. Look at some examples. L being a PID, its finitely gen. modules split into primary cyclic modules L/m_z^{n+1} where $m_z \subset L$ is the maximal ideal of L vanishing at $z \in \mathbb{C}^\times$. Also there is L itself.

Take a f.g. module M over $\mathbb{C}[\varepsilon] \ltimes \mathbb{C}[g,g^{-1}]$, restrict it to $\mathbb{C}[g,g^{-1}]$, split into cyclic modules. The auto ε of order 2 transforms L/m_z^{n+1} into $L/m_{z^{-1}}^{n+1}$.

Instead of struggling with descent, you should try to understand the diagrams from yesterday for the indecomposable representations of the dihedral group. Repeat this: Consider

$$\begin{tikzcd} W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \end{tikzcd} \qquad \beta\alpha = 1_W$$

a Λ-module retract of a free Λ-module, split into ± 1 eigenspaces for ε to get

$$\begin{tikzcd} W_\pm & \frac{1\pm\varepsilon}{2} \otimes V \arrow[l, two heads, "\beta_\pm"'] \arrow[d, phantom, "\wr"] & W_\pm \arrow[l, tail, "\alpha_\pm"'] \\ & V & \end{tikzcd} \qquad \begin{matrix} \beta_\pm\alpha_\pm = 1 \\ \text{on } W_\pm \end{matrix}$$

two independent retracts of the v.s. V, given by projection ops $p_\pm = \alpha_\pm\beta_\pm$, same as a representation of $D = \mathbb{Z}/2 \ltimes \mathbb{Z} = \langle \sigma, g \rangle$, $\sigma^2 = 1$, $\sigma g \sigma^{-1} = g^{-1}$! Be careful about ε.

Find some examples. Take $V = W_+ = W_- = \mathbb{C}$

$\alpha_\pm = \beta_\pm = 1$.

$$\begin{tikzcd} V & W \arrow[l, "(1\ 1)"'] & V \arrow[l, tail, "\begin{pmatrix}1\\1\end{pmatrix}"'] \arrow[ll, bend left, "2"] & W \arrow[l, "(1\ 1)"'] & V \arrow[l, tail, "\begin{pmatrix}1\\1\end{pmatrix}"'] \arrow[ll, bend left, "2"] \end{tikzcd}$$

Consider $V = P$ as a left module over itself and try to get a good picture of the rest. Actually you should work on the Morita context in this case, which should tie in $\mathbb{Z}/2 \ltimes \mathcal{E}$ where $\mathcal{E} = R(\mathbb{C}[\varepsilon])$, maybe you want $\mathbb{C}[\varepsilon] = \mathbb{C}e_+ \oplus \mathbb{C}e_-$ $\quad e_\pm^2 = e_\pm \quad e_+ + e_- = 1$

Let's check this $\Lambda \otimes V$. $\Lambda = \mathbb{C}[\mathbb{Z}/2]$

Let W be a Λ-module retract of the free module $\Lambda \otimes V$

$$\begin{tikzcd} W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \end{tikzcd} \qquad \beta\alpha = 1.$$

$$\begin{tikzcd} W_+ & \frac{1+\varepsilon}{2} \otimes V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \\ \oplus & & \oplus \\ W_- & \frac{1-\varepsilon}{2} \otimes V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"'] \end{tikzcd} \qquad \begin{matrix} \beta_+\alpha_+ = 1_{W_+} \\ \beta_-\alpha_- = 1_{W_-} \end{matrix}$$

Compare this with a retract of $M_2\mathbb{C} \otimes V$

$$\begin{tikzcd} V \oplus V & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}"'] & V \oplus V \arrow[l, two heads, "(\beta_1 \ \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

What structure do you get on V? $\rho_{ij} = \alpha_i\beta_j \in \mathcal{L}(V)$ satisfying $\rho_{ik} = \sum_j \rho_{ij}\rho_{jk}$. V is reduced means $V = \alpha_1 W + \alpha_2 W$, $\operatorname{Ker}(\beta_1, \beta_2) = 0$. $\therefore$ you get

$$\begin{tikzcd} W \oplus W & V \arrow[l, tail, "\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}"'] & W \oplus W \arrow[l, two heads, "(\alpha_1 \ \alpha_2)"'] \end{tikzcd} \qquad \begin{pmatrix} \beta_1\alpha_1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & \beta_2\alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & \beta_1\alpha_2 \\ \beta_2\alpha_1 & 1 \end{pmatrix}$$

So it seems true that the $\rho_{\mathbb{Z}/2}$ algebra describes retracts of a free $M_2\mathbb{C}$ algebra.

IDEA. Improving interpolation from $\mathbb{Z}$ to $\mathbb{R}$ by dyadic subdivision, could this explain the Haar basis for $L^2(\mathbb{R})$?

Back to retracts. Repeat. Let W be a retract of the free Λ-module $\Lambda \otimes V$, where $\Lambda = \mathbb{C}[\mathbb{Z}/2] = \mathbb{C} \oplus \mathbb{C}\varepsilon$. W is equivalent to v.s. retracts $W_\pm$ of $\Lambda_\pm \otimes V$ where $\Lambda_\pm = \frac{1\pm\varepsilon}{2}\Lambda$ are the ± 1 eigenspaces of ε. If we choose an isom. $\frac{1+\varepsilon}{2}\Lambda \simeq \frac{1-\varepsilon}{2}\Lambda$, then W is equivalent to a pair of projections on V. It would be better to choose isos $\mathbb{C} \xrightarrow{\sim} \frac{1+\varepsilon}{2}\Lambda$, $\mathbb{C} \xrightarrow{\sim} \frac{1-\varepsilon}{2}\Lambda$ and transport the projections $p_\pm = \alpha_\pm \beta_\pm$ on $\frac{1\pm\varepsilon}{2}\Lambda \otimes V$ to projections on V. Changing the isos. amounts to multiplication by a nonzero scalar on V, which commutes with $p_\pm$.

So you find that retracts of the Λ-module $\Lambda \otimes V$ are given by ordered pairs (p_+, p_-) of projections on V.

Next consider retracts of the $M_2\mathbb{C}$-module $M_2\mathbb{C} \otimes V$. By Morita equivalence these should be the same as vector space retracts:

$$\begin{tikzcd} W & \begin{matrix} V \\ \oplus \\ V \end{matrix} \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 1_W$$

which are the same as projections $p \in \mathrm{End}\begin{pmatrix}V \\ V\end{pmatrix} = M_2\mathbb{C} \otimes \mathrm{End}(V)$

i.e. $p_{ij} \in \mathrm{End}(V)$ $i, j \in \{1, 2\}$ such that $p_{ik} = \sum_j p_{ij} p_{jk}$

Next consider vector space retracts

$$\begin{tikzcd} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}"'] & \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \arrow[l, two heads, "(\beta_1\ \beta_2)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_2\end{pmatrix}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \beta_2\alpha_2 = 0.$$

which are equivalent to projections

$$\begin{tikzcd} \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} & \begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix} \arrow[l, "\begin{pmatrix}p_{11} & p_{12} \\ p_{21} & p_{22}\end{pmatrix}"'] \end{tikzcd}$$

in $\mathrm{End}\begin{pmatrix}V_1 \\ \oplus \\ V_2\end{pmatrix}$, that is

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Start again and try to keep your cats straight.

① projections on the $\mathbb{C}[\mathbb{Z}/2]$-module $\mathbb{C}[\mathbb{Z}/2] \otimes V$

② projections on the $M_2\mathbb{C}$-module $M_2\mathbb{C} \otimes V$

③ projections ——— $\mathbb{C}$-module $\begin{matrix} V_1 \\ \oplus \\ V_2 \end{matrix}$

First case: $\Gamma = \mathbb{C}[\mathbb{Z}/2]$, $\Lambda = \mathbb{C}[\Gamma]$, consider a vector space V equipped with a Λ-module retract of $\Lambda\otimes V$:

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \end{tikzcd} \qquad \beta\alpha = 1_W$$

equiv description: consider a vector space V equipped with a Λ-module projection on $\Lambda\otimes V$. Use alg isom $\mathbb{C}[\mathbb{Z}/2] = \mathbb{C}e_+ \oplus \mathbb{C}e_-$, $e_\pm = \frac{1\pm\varepsilon}{2}$. W amounts to retracts

$$\begin{tikzcd} W_+ & e_+\Lambda\otimes V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \end{tikzcd} \qquad \beta_+\alpha_+ = 1_{W_+}$$

$$\begin{tikzcd} W_- & e_-\Lambda\otimes V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"'] \end{tikzcd} \qquad \beta_-\alpha_- = 1_{W_-}$$

now $e_+\Lambda \simeq \mathbb{C} \simeq e_-\Lambda$, so choosing such isos. you get two retracts of V. The choices are nonzero scalars so get same projections.

Conclude: A projection π on the $\Lambda = \mathbb{C}[\mathbb{Z}/2]$-module $\Lambda\otimes V$ is equivalent to a pair (π_+, π_-) of projections on V.

Other picture which might be useful is that

$$p\Big(\sum_t t\otimes f(t)\Big) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t), \quad \text{where } \underset{\substack{\parallel\\ p_+}}{p(1)}, \underset{\substack{\parallel\\ p_-}}{p(\varepsilon)} \in \mathcal{L}(V)$$

satisfy $\displaystyle\sum_{u=st} p(s)p(t) = p(u)$.

$$p_+ = p_+p_+ + p_-p_-, \quad p_- = p_+p_- + p_-p_+$$

$$\therefore \quad (p_+ \pm p_-)^2 = p_+ \pm p_- .$$

not to be confused with ± 1 eigenspaces (?) (?)

Look at W, where is the partition of 1_W

$$1 = e_+ + e_- \quad \text{on } \Lambda\otimes V$$

$$1_W = \beta e_+\alpha + \beta e_-\alpha$$

NOTATION terribly confused.

$$\begin{tikzcd} W_+ & e_+\Lambda\otimes V \arrow[l, "\beta"'] \arrow[d, "s"'] & W_+ \arrow[l, "\alpha"'] \arrow[dl, "\alpha_+"] \\ & V \arrow[ul, bend left, "\beta_+"] & \end{tikzcd}$$

$\Lambda = \mathbb{C}[\mathbb{Z}/2]$. Let W be retract of the free Λ-mod $\underbrace{}_{\Lambda\otimes V}$

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1"] \\
& V \arrow[u, shift left, "\varepsilon_1"] \arrow[ul, bend left, "\beta_1"] &
\end{tikzcd}$$

$\beta\alpha = 1$, $\alpha\beta = p$

$$p\sum_t t\otimes f(t) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t)$$

$$w = \sum_s \beta s \varepsilon_1 \eta_1 s^{-1}\alpha w = \sum_s s\overset{h_1}{\boxed{\alpha_1\beta_1}}s^{-1}w \qquad p(s^{-1}t) = \alpha_1 s^{-1}t\beta_1$$

$$\sum_{u=st} p(s)p(t) = p(u)$$

$\mathbb{Z}/2$ cons. of $+1, -1$ under mult.

$$p_+ = p_+^2 + p_-^2, \qquad p_- = p_-p_+ + p_+p_-$$

So the proj p on $\Lambda\otimes V$ should be equivalent to 2 projections on V. Idea use F.T. to replace convolution $f(t) \mapsto \sum_t p(s^{-1}t)f(t)$ by multiplication by a function on the group.

$$\frac{1+\varepsilon}{2}\sum t\otimes f(t) = \frac{1+\varepsilon}{2}\left(1\otimes f(1) + \varepsilon\otimes f(\varepsilon)\right) = \frac{1+\varepsilon}{2}\left(f(1)+f(\varepsilon)\right)$$

$$\frac{1-\varepsilon}{2}\sum t\otimes f(t) = \frac{1-\varepsilon}{2}\left(1\otimes f(1) + \varepsilon\otimes f(\varepsilon)\right) = \frac{1-\varepsilon}{2}\left(f(1)-f(\varepsilon)\right)$$

Similarly

$$\frac{1+\varepsilon}{2}\sum_s s\otimes\sum_t p(s^{-1}t)f(t) = \frac{1+\varepsilon}{2}\sum_t\left(p(t)+p(\varepsilon t)\right)f(t)$$
$$= \frac{1+\varepsilon}{2}\left(p(1)+p(\varepsilon)\right)\left(f(1)+f(\varepsilon)\right)$$

$$\frac{1-\varepsilon}{2}\sum_s s\otimes\sum_t p(s^{-1}t)f(t) = \frac{1-\varepsilon}{2}\sum_t\left(p(t)f(t) - p(\varepsilon t)f(t)\right)$$
$$= \frac{1-\varepsilon}{2}\left(p(1)f(1) + p(\varepsilon)f(\varepsilon) - p(\varepsilon)f(1) - p(1)f(\varepsilon)\right)$$
$$\left(p(1)-p(\varepsilon)\right)\left(f(1)-f(\varepsilon)\right)$$

$$p\sum_t t\otimes f(t) = \sum_u\sum_t \underbrace{tu^{-1}}_{s}\otimes p(u)f(t) = \sum_s\sum_t s\otimes p(s^{-1}t)f(t)$$

$$p = \sum s\otimes p(s) \in \Lambda\otimes\mathcal{L}(V)$$

March 17, 02

$\Gamma = \mathbb{Z}/2$, $\Lambda = \mathbb{C}\Gamma$, a $\overbrace{\text{retract}}^{\Lambda\text{-module}}$ W of $\Lambda \otimes V$

$$\begin{tikzcd} W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \end{tikzcd}$$

$\beta\alpha = 1$
$\alpha\beta = p$

is the same as an idempotent operator p on the Λ-module $\Lambda \otimes V$. Describe p

$$\boxed{\begin{aligned} p &= \sum_u u \otimes p(u) \quad \in \Lambda \otimes \mathcal{L}(V) \\ p \sum_t t \otimes f(t) &= \sum_t \sum_u tu^{-1} \otimes p(u) f(t) \\ &= \sum_s s \otimes \sum_t p(s^{-1}t) f(t) \\ p(u) &= \sum_{u = st} p(s)\, p(t) \end{aligned}}$$

$tu^{-1} = s$
$u = s^{-1}t$

Now use $\Lambda \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$ $\quad$ $1 \longmapsto (1,1)$
$\varepsilon \longmapsto (1,-1), (1,-1)$

$$p = \underset{1}{[0]} \otimes p_0 + \underset{\varepsilon}{[1]} \otimes p_1 \longmapsto p_0 \pm p_1$$

In this way you see that a projection on $\Lambda \otimes V$ is the same as two projections on V.

A puzzle is how the splitting $W = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ into ± 1 eigenspaces for ε relates to the two retracts

$$\begin{tikzcd} W_+ & e_+ \otimes V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \\ W_- & e_- \otimes V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"'] \end{tikzcd}$$

You know that W can be described as a Γ-module equipped with $h \in \mathcal{L}(V)$ such that $h + \varepsilon h \varepsilon = 1$. You have to compare

$$1_{\Lambda \otimes V} = \varepsilon_1 \eta_1 + \varepsilon \varepsilon_1 \eta_1 \varepsilon^{-1}$$

$$1_\Lambda = e_+ + e_- \qquad e_\pm = \frac{1 \pm \varepsilon}{2}$$

You have two ways of viewing $\Lambda \otimes V$: $\begin{pmatrix} e_+ \Lambda \otimes V \\ e_- \Lambda \otimes V \end{pmatrix}$ and $\begin{pmatrix} 1 \otimes V \\ \varepsilon \otimes V \end{pmatrix}$

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \end{tikzcd}$$

Try $p \in \Lambda\otimes\mathcal{L}(V)$, $\quad p = 1\otimes p_1 + \varepsilon\otimes p_\varepsilon$

$p = e_+\otimes p_+ + e_-\otimes p_-$

$$p = \frac{1+\varepsilon}{2}\otimes p_+ + \frac{1-\varepsilon}{2}\otimes p_-$$

$$= 1\otimes\left(\frac{p_+ + p_-}{2}\right) + \varepsilon\otimes\left(\frac{p_+ - p_-}{2}\right)$$

$$p_1 = \alpha_1\beta_1 = \frac{p_+ + p_-}{2}$$

$$p_\varepsilon = \alpha_1\varepsilon\beta_1 = \frac{p_+ - p_-}{2}$$

Start again. $\Gamma = \mathbb{Z}/2 = \{1, \varepsilon\}$, $\varepsilon^2 = 1$, $\Lambda = \mathbb{C}[\Gamma]$,

Λ has two bases of interest $\{1, \varepsilon\}$, $\{e_\pm = \frac{1\pm\varepsilon}{2}\}$.

If $p =$ a projection on $\Lambda\otimes V$ as Λ-module, then

$$p = 1\otimes p_1 + \varepsilon\otimes p_\varepsilon = e_+\otimes p_+ + e_-\otimes p_-$$

where p_+ and p_- are projections on V, because

$\Lambda \xrightarrow{\sim} e_+\Lambda \times e_-\Lambda = \mathbb{C}e_+ \times \mathbb{C}e_-$. Let's check this.

$$p = \frac{1+\varepsilon}{2}\otimes p_+ + \frac{1-\varepsilon}{2}\otimes p_- = 1\otimes\underbrace{\frac{p_+ + p_-}{2}}_{p_1} + \varepsilon\otimes\underbrace{\frac{p_+ - p_-}{2}}_{p_\varepsilon}$$

where $\underbrace{\begin{array}{l} p_1 = p_1^2 + p_\varepsilon^2 \\ p_\varepsilon = p_1p_\varepsilon + p_\varepsilon p_1 \end{array}}$ $\Bigg\}$ you get this from $p^2 = p$ where $p = 1\otimes p_1 + \varepsilon\otimes p_\varepsilon$

these yield $\quad p_1 \pm p_\varepsilon = (p_1 \pm p_\varepsilon)^2 \qquad \begin{array}{l} p_+ = p_1 + p_\varepsilon \\ p_- = p_1 - p_\varepsilon \end{array}$

You want to find $h_1 = \beta_1\alpha_1$. So how are $\alpha_\pm$ linked to $\alpha_1, \alpha_\varepsilon$?

$$\begin{tikzcd} W \arrow[d, equal] & \begin{pmatrix} V \\ V \end{pmatrix} \arrow[l, two heads, "(\beta_+\ \beta_-)"'] \arrow[d, "S"] & W \arrow[l, tail, "\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}"'] \arrow[d, equal] \\ W & \begin{pmatrix} V \\ V \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \beta_\varepsilon)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_\varepsilon\end{pmatrix}"'] \end{tikzcd}$$

guess

$$\begin{pmatrix}\alpha_1 \\ \alpha_\varepsilon\end{pmatrix} = \begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}$$

$$(\beta_1\ \beta_\varepsilon) = (\beta_+\ \beta_-)\,\tfrac{1}{2}\begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}$$

Clearly

$$(\beta_1\ \beta_\varepsilon)\begin{pmatrix}\alpha_1 \\ \alpha_\varepsilon\end{pmatrix} = (\beta_+\ \beta_-)\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix} = 1_W$$

Start again

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"']
\end{tikzcd}$$

$\beta\alpha = 1_W$, $p = \alpha\beta$ is a proj on Λ-mod $\Lambda\otimes V$.

Use the fact that Λ modules are the same as $\mathbb{C}e_+ \oplus \mathbb{C}e_-$ modules. So a retract of the Λ module $\Lambda\otimes V$ is the same as two retracts of V.

$$\begin{tikzcd}
W_+ & V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \\
W_- & V \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"']
\end{tikzcd}
\qquad
\begin{aligned}
\beta_+\alpha_+ &= 1_{W_+} \\
\beta_-\alpha_- &= 1_{W_-}
\end{aligned}$$

What sort of structure is on W? You know W has an action of Γ and an h. The Γ-action is handled by the splitting $W = W_+ + W_-$. So h must arise from $\beta_-\alpha_+$ and $\beta_+\alpha_-$.

The $\Gamma = \mathbb{Z}/2$ action on W is given by $\varepsilon = \pm 1$ on $W_\pm$. You seek an $h \in \mathcal{L}(W)$ such that $h + \varepsilon h \varepsilon = 1_W$.

Can you find β_1, α_1?

$$\begin{tikzcd}
W & \begin{pmatrix} 1V \\ \varepsilon V \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \ \varepsilon\beta_1)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_1\varepsilon\end{pmatrix}"']
\end{tikzcd}$$

$$\begin{tikzcd}
H & \begin{pmatrix} e_+V \\ e_-V \end{pmatrix} \arrow[l, two heads, "(\beta_+\ \ \beta_-)"'] & W \arrow[l, "\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}"']
\end{tikzcd}
\qquad
\begin{aligned}
\alpha_1 &= \frac{\alpha_+ + \alpha_-}{2} \\
\alpha_1\varepsilon &= \frac{\alpha_+ - \alpha_-}{2}
\end{aligned}$$

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, equal] & W \arrow[l, tail, "\alpha"'] \\
W & \begin{pmatrix} \iota_1 V \\ \iota_\varepsilon V \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \ \beta_\varepsilon)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_\varepsilon\end{pmatrix}"']
\end{tikzcd}$$

$$\beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon = 1_W$$

$$\begin{tikzcd}
W & \begin{pmatrix} \iota_+ V \\ \iota_- V \end{pmatrix} \arrow[l, two heads, "(\beta_+\ \ \beta_-)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}"']
\end{tikzcd}$$

$\iota_1 V = 1\otimes V$

$\iota_\varepsilon V = \varepsilon\otimes V$

$\iota_+ V = e_+\otimes V$ $\qquad e_\pm = \frac{1\pm\varepsilon}{2}$

$\iota_\pm = \frac{1}{2}(\iota_1 \pm \iota_\varepsilon)$

$$\alpha_{\pm} w = e_{\pm}\alpha w = \frac{1\pm\varepsilon}{2}\alpha w$$

---

work out details for the basis $\iota_1\,\varepsilon$ of Λ.

$$p = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\varepsilon\beta_1 \\ \alpha_1\varepsilon\beta_1 & \alpha_1\beta_1 \end{pmatrix}$$

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \end{tikzcd}$$

$$\begin{tikzcd} W & \begin{pmatrix} V \\ V \end{pmatrix} \arrow[l, two heads, "(\beta_1\ \ \varepsilon\beta_1)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_1\varepsilon\end{pmatrix}"'] \end{tikzcd}$$

where $\beta_1 v = \beta\,\iota_1 v$
$\alpha_1 w = j_1 \alpha w$

In particular you have the partition $\beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon^{-1} = 1_W$

Now you want to pass to the $\varepsilon = \pm 1$ eigenspaces.

Not yet clear because you don't know the action of ε on $\begin{pmatrix} V \\ V\end{pmatrix}$ which contains $\begin{pmatrix} \iota_1 v_1 \\ \iota_\varepsilon v_\varepsilon \end{pmatrix} = \begin{pmatrix} \iota_1 v_1 \\ \varepsilon\iota_1 v_\varepsilon \end{pmatrix}$. Thus

$\varepsilon\begin{pmatrix} v_1 \\ v_\varepsilon \end{pmatrix} = \begin{pmatrix} v_\varepsilon \\ v_1 \end{pmatrix}$. Let's check this. Let's use $\begin{pmatrix} v' \\ v'' \end{pmatrix}$ for a typical elt of $\begin{pmatrix} V \\ V\end{pmatrix}$ which is our model of $\Lambda\otimes V$.

check α commutes with ε

$$\begin{pmatrix} \alpha_1 \\ \alpha_1\varepsilon \end{pmatrix}(\varepsilon w) = \begin{pmatrix} \alpha_1\varepsilon w \\ \alpha_1 w \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_1 w \\ \alpha_1\varepsilon w \end{pmatrix}$$

$$\varepsilon(\beta_1\ \ \varepsilon\beta_1)\begin{pmatrix} v' \\ v'' \end{pmatrix} = \varepsilon(\beta_1 v' + \varepsilon\beta_1 v'') =$$

check β commutes with ε

$$(\beta_1\ \ \varepsilon\beta_1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v' \\ v'' \end{pmatrix} = (\beta_1\ \ \varepsilon\beta_1)\begin{pmatrix} v'' \\ v' \end{pmatrix} = \beta_1 v'' + \varepsilon\beta_1 v'$$

Now you want to pass to the ± 1 eigenspaces for ε

$$\begin{pmatrix} V \\ V \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} V + \begin{pmatrix} 1 \\ -1 \end{pmatrix} V$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1\ \ 1)\begin{pmatrix} \alpha_1 \\ \alpha_1\varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\alpha_1\overbrace{\left(\frac{1+\varepsilon}{2}\right)}^{e_+}$$

$$\frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}(1\ \ -1)\begin{pmatrix} \alpha_1 \\ \alpha_1\varepsilon \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\alpha_1\left(\frac{1-\varepsilon}{2}\right)$$

$$(\beta_1 \quad \varepsilon\beta_1)\,\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1 \quad 1) = \frac{1+\varepsilon}{2}\beta_1(1 \quad 1)$$

$$(\beta_1 \quad \varepsilon\beta_1)\,\frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1 \quad -1) = \frac{1-\varepsilon}{2}\beta_1(1 \quad -1)$$

Take the image of e_+

$$e_+W \longleftarrow \begin{pmatrix}1\\1\end{pmatrix}V$$

Return to

$$\begin{tikzcd} W & \begin{pmatrix}V\\V\end{pmatrix} \arrow[l, two heads, "(\beta_1 \quad \varepsilon\beta_1)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\\alpha_1\varepsilon\end{pmatrix}"'] \end{tikzcd} \qquad \beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon = 1_W$$

these β, α maps commute with ε acting by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ on $\begin{pmatrix}V\\V\end{pmatrix}$

Next you want the $+1$ eigenspace for ε from the above

$$\begin{tikzcd} e_+W & \begin{pmatrix}1\\1\end{pmatrix}V \arrow[l, "\frac{1+\varepsilon}{2}\beta_1(1 \quad 1)"'] & e_+W \arrow[l, tail, "\begin{pmatrix}1\\1\end{pmatrix}\alpha_1\left(\frac{1+\varepsilon}{2}\right)"'] \end{tikzcd}$$

$2e_+\beta_1\alpha_1e_+ = e_1$

$$\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1 \quad 1)\begin{pmatrix}\alpha_1\\\alpha_1\varepsilon\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\alpha_1e_+$$

$$(\beta_1 \quad \varepsilon\beta_1)\,\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1 \quad 1) = e_+\beta_1(1 \quad 1)$$

$$\frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1 \quad 1) + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1 \quad -1) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$$

$$\frac{1}{2}(\beta_1+\varepsilon\beta_1)(\alpha_1+\alpha_1\varepsilon) + \frac{1}{2}(\beta_1-\varepsilon\beta_1)(\alpha_1-\alpha_1\varepsilon) = \beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon$$

$$= 2\left(e_+\beta_1\alpha_1e_+ + e_-\beta_1\alpha_1e_-\right).$$

Go over this again Basic partition of 1 is $\beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon = 1_W$

$$\begin{tikzcd} W & \begin{pmatrix}V\\V\end{pmatrix} \arrow[l, two heads, "(\beta_1 \quad \varepsilon\beta_1)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1\\\alpha_1\varepsilon\end{pmatrix}"'] \end{tikzcd}$$

$$1_{\begin{pmatrix}V\\V\end{pmatrix}} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}(1 \quad 1) + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}(1 \quad -1)$$

$$1_W = 2e_+\beta_1\alpha_1e_+ + 2e_-\beta_1\alpha_1e_-$$

Start with

⊛
$$\begin{tikzcd} W & \begin{pmatrix} V \\ V \end{pmatrix} \arrow[l, two heads, "(\beta_1 \ \ \varepsilon\beta_1)"'] & W \arrow[l, tail, "\begin{pmatrix}\alpha_1 \\ \alpha_1\varepsilon\end{pmatrix}"'] \end{tikzcd}$$

$$\beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon = 1_W$$

$\varepsilon = \pm 1$ eigenspace decomposition of $\begin{pmatrix} V \\ V \end{pmatrix}$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix}$$

compress using
$\beta\alpha = 1$.

$$1_W = \frac{1}{2}(1+\varepsilon)\beta_1\alpha_1(1+\varepsilon) + \frac{1}{2}(1-\varepsilon)\beta_1\alpha_1(1-\varepsilon) = \beta_1\alpha_1 + \varepsilon\beta_1\alpha_1\varepsilon$$

as cross terms cancel (degree 1 in ε)

$$1_W = 2(e_+\beta_1\alpha_1 e_+ + e_-\beta_1\alpha_1 e_-)$$

You need to understand the operator $h = \beta_1\alpha_1$ on W. $\beta_1\alpha_1$ ~~is~~ is not compatible with the Γ action. Go back to ⊛ apply $e_\pm$

$$\begin{tikzcd} W_+ & V \arrow[l, "(1+\varepsilon)\beta_1"'] & W_+ \arrow[l, "\alpha_1(1+\varepsilon)"'] \\ W_- & V \arrow[l, "(1-\varepsilon)\beta_1"'] & W_- \arrow[l, "\alpha_1(1-\varepsilon)"'] \end{tikzcd}$$

factor of $\frac{1}{2}$ is needed . .

You are interested in $h = \beta_1\alpha_1$ on W

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, tail, "\alpha"'] \\ W & \begin{pmatrix} \iota_1 V \\ \varepsilon\,\iota_1 V \end{pmatrix} \arrow[l, "(\beta_1 j_1 \ \ \varepsilon\beta_1 j_1\varepsilon)"'] & W \arrow[l, "\begin{pmatrix}\iota_1\alpha_1 \\ \varepsilon\iota_1\alpha_1\varepsilon\end{pmatrix}"'] \end{tikzcd}$$

$\iota_1 v = 1\otimes v$
$j_1(1\otimes v) = v$
$j_1(\varepsilon\otimes v) = 0$
$\alpha w = \sum_s s\,\iota_1 j_1 s^{-1}\alpha w$
$= \sum_s s\,\iota_1\alpha_1 s^{-1} w$

$$\begin{pmatrix} \beta_1 j_1 & \varepsilon\beta_1 j_1\varepsilon \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} \iota_1\alpha_1 \\ \varepsilon\iota_1\alpha_1\varepsilon \end{pmatrix} =$$

$$\frac{1}{2}\left(\beta_1 j_1 + \varepsilon\beta_1 j_1\varepsilon\right)\left(\iota_1\alpha_1 + \varepsilon\iota_1\alpha_1\varepsilon\right)$$

To simplify notation define $\alpha_1 w = 1\otimes j_1\alpha w = e_1\alpha w$
$\beta_1 = \beta e_1$

March 18, 02

$$W \xleftarrow{\beta} \Lambda \otimes V \xleftarrow{\alpha} W$$

$$1_{\Lambda\otimes V} = \sum_s{}' e_s = e_1 + \varepsilon e_1 \varepsilon$$

$$1_{\Lambda\otimes V} = e_+ + e_- = \frac{1+\varepsilon}{2} + \frac{1-\varepsilon}{2}$$

$$1_W = \beta(e_1 + \varepsilon e_1 \varepsilon)\alpha = \beta e_1 \alpha + \varepsilon(\beta e_1 \alpha)\varepsilon$$

$$1_W = \frac{1+\varepsilon}{2} + \frac{1-\varepsilon}{2}$$

Having found the mistake what next? Recall the program. $\Gamma = \mathbb{Z}/2 = \{1, \varepsilon\}$, $\Lambda = \mathbb{C}\Gamma$, $W =$ a Λ-module retract of $\Lambda \otimes V$. Then $W_\pm = e_\pm W$ is a retract of $e_\pm \Lambda \otimes V = \mathbb{C}e_\pm \otimes V$, so you get two projections $\alpha_\pm \beta_\pm$ on V.

You have the other splitting $1_{\Lambda\otimes V} = e_1 + \varepsilon e_1 \varepsilon$ which yields $1_W = \beta e_1 \alpha + \varepsilon(\beta e_1 \alpha)\varepsilon$; this is the equivariant partition of 1. What to do? You have operators on W, two partitions of 1, which have to be related.

You have $W = W_+ \oplus W_-$ and $h = \beta e_1 \alpha (= \beta_1 \alpha_1)$ so h has 4 components

$$h = \left(\begin{array}{c|c} \frac{1}{2} & y \\ \hline x & \frac{1}{2} \end{array}\right) \qquad \varepsilon h \varepsilon = \left(\begin{array}{c|c} \frac{1}{2} & -y \\ \hline -x & \frac{1}{2} \end{array}\right)$$

Let's revise. You want a proper understand of the case $\Gamma = \mathbb{Z}/2$. Consider a retract W of the free Λ-module $\Lambda \otimes V$: $W \xleftarrow{\beta} \Lambda \otimes V \xleftarrow{\alpha} W$, the same as retracts $W_\pm \xleftarrow{\beta_\pm} e_\pm \Lambda \otimes V \xleftarrow{\alpha_\pm} W_\pm$, two retracts of V. So V is a module over $\mathbb{C}p_+ * \mathbb{C}p_-$. W is a Γ-module equipped with $h = \beta_1 \alpha_1 = \beta e_1 \alpha_1$ v.s

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There's still too much you need to understand. $\Gamma = \mathbb{Z}/2 = \{1, \varepsilon\}$, $p = p^2 \in \Lambda \otimes \mathcal{L}(V)$ has form $1 \otimes p_1 + \varepsilon \otimes p_\varepsilon$ where $p_1 = p_1^2 + p_\varepsilon^2$
$p_\varepsilon = p_1 p_\varepsilon + p_\varepsilon p_1$

$$p = \frac{1+\varepsilon}{2} \otimes \underbrace{(p_1 + p_\varepsilon)}_{p_+} + \frac{1-\varepsilon}{2} \otimes \underbrace{(p_1 - p_\varepsilon)}_{p_-}$$

There seems to be a nuance that you don't understand. Describe Λ-module retracts of a free Λ module $\Lambda \otimes V$. These are the same as projections $P = (p_+, p_-)$ in $(\mathbb{C}e_+ + \mathbb{C}e_-) \otimes \mathcal{L}(V)$, that is, two projections on V.

What can you say about the corresp. W?

$$\begin{tikzcd}
W_+ & V \arrow[l, two heads, "\beta_+"'] & W_+ \arrow[l, tail, "\alpha_+"'] \\
W_- & V \arrow[u, equal] \arrow[l, two heads, "\beta_-"'] & W_- \arrow[l, tail, "\alpha_-"']
\end{tikzcd}$$

W has the ± grading since it's a Γ-module.

You have 4 ops

$$\begin{tikzcd}
V & \begin{pmatrix} W_+ \\ \\ W_- \end{pmatrix} \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"'] & V \arrow[l, tail, "\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}"'] & \begin{pmatrix} W_+ \\ \\ W_- \end{pmatrix} \arrow[l, two heads, "(\alpha_+ \ \alpha_-)"']
\end{tikzcd}$$

$$\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}(\alpha_+ \ \alpha_-) = \begin{pmatrix} 1 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 1 \end{pmatrix}$$

When V reduced you know $\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}$ inj, $(\alpha_+ \ \alpha_-)$ surjective

Somewhere here there has to be an operator h on W such that $h + \varepsilon h \varepsilon = 1_W$

$$h = \left(\begin{array}{c|c} a & b \\ \hline c & d \end{array}\right) \qquad \varepsilon h \varepsilon = \left(\begin{array}{c|c} a & -b \\ \hline -c & d \end{array}\right) \qquad h + \varepsilon h \varepsilon = \left(\begin{array}{c|c} 2a & 0 \\ \hline 0 & 2d \end{array}\right)$$

It seems that there is some geometry related to two projections that you need.

Focus upon what structure is to be found on W: grading + odd operator. So consider the category consisting of a $\mathbb{Z}/2$ graded v.s. $W = W_+ \oplus W_-$ equipped with an odd operator.

Digress $g = F\varepsilon$ $\varepsilon g \varepsilon^{-1} = \varepsilon F = g^{-1}$. Assume $g-1$ small so that $g^{1/2}, g^{-1/2}$ can be constructed via binomial series.

Also need $\varepsilon g^{1/2} \varepsilon^{-1} = (\varepsilon g \varepsilon^{-1})^{1/2} = (g^{-1})^{1/2} = g^{-1/2}$. Then

$$F = g\varepsilon = g^{1/2} g^{1/2} \varepsilon = g^{1/2} \varepsilon g^{-1/2}$$

F ε $F\varepsilon = g$

$$F = g\varepsilon = g^{1/2} g^{1/2} \varepsilon = g^{1/2} \varepsilon g^{-1/2}$$

Better to use exponential map.

Look carefully at what you have. Two projections on V leading to two retracts $W_\pm$ of V.

Do the Morita equivalence in the $\Gamma = \mathbb{Z}/2$ case, better do in general for a finite gp Γ.

$\mathcal{W}$ = cat of Γ-modules W equipped with $h \in \mathcal{L}(W)$ sat $\sum_s shs^{-1} = 1_W$

$\mathcal{V}$ = cat of v.s. V equipped with $\overbrace{\ \ p(s)}^{\forall s \in \Gamma} \in \mathcal{L}(V)$ satif $\sum_{u=st} p(s)p(t) = p(u)$. + red. conditions
$V = \sum p(s) V$
$0 = \bigcap \operatorname{Ker} p(s)$

$\mathcal{U}$ = cat of $W \xleftarrow{\beta_1} V \xleftarrow{\alpha_1} W$ where W is a Γ module, V a v.s., β_1, α_1 linear $\ni \sum_s s\beta_1\alpha_1 s^{-1} = 1_W$
red conds: β_1 inj, α_1 surj.

Given V in $\mathcal{V}$, let $\Lambda = \mathbb{C}[\Gamma]$, = the group ring of Γ, form the "free" Γ-module gen by V: $\Lambda \otimes V = \{ \sum_t t \otimes f(t) \mid f : \Gamma \to V \}$

Let p be the operator on $\Lambda \otimes V$ defined

$$p\Big(\sum_t t \otimes f(t)\Big) = \sum_t \sum_u tu^{-1} \otimes p(u) f(t) \qquad \begin{matrix} s = tu^{-1} \\ u = s^{-1}t \end{matrix}$$

$$= \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

$p^2 = p$

$$\sum_t p(s^{-1}t)\, p(t^{-1}u) = \sum_{s^{-1}u = t'u'} p(t')p(u') = p(s^{-1}u)$$

Define W = the retract of the Λ-mod $\Lambda \otimes V$ corresp to p

$$\begin{tikzcd}
\Lambda\otimes V & W \arrow[l, tail, "\alpha"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, two heads, "j_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "\alpha_1 = j_1\alpha"] & \Lambda\otimes V \arrow[l] \\
 & & V \arrow[ul, "\beta_1 = \beta i_1"] \arrow[u, shift left, "i_1"] & &
\end{tikzcd}$$

$\beta\alpha = 1_W$
$p = \alpha\beta$

Set $h = \beta_1\alpha_1 = \beta i_1 j_1 \alpha$.

$$1_W = \alpha \cdot 1_{\Lambda\otimes V} \beta = \sum_s \beta s\, i_1 j_1 s^{-1} \alpha = \sum_s shs^{-1}$$

$\mathcal{W}$ = cat of left (unital) modules over $\mathcal{E}_\Gamma \rtimes \Gamma$

$$\mathcal{E}_\Gamma = \mathbb{C}\langle h_s, s\in\Gamma\rangle \Big/ \Big(1 = \sum_s h_s\Big) \qquad t h_s = h_{ts} t$$

$\mathcal{E}_\Gamma$ is always $R\Big(\bigoplus_s \mathbb{C}e_s\Big)$ $\qquad$ e_s annihilating idempotents

so the cross product seems different.

Now construct the Morita context. Recall that there is a natural grading around.

$$\begin{bmatrix} & \alpha_s \\ \beta_t & B \end{bmatrix} \begin{bmatrix} V \\ W \end{bmatrix}$$

$$\deg(\beta_t\alpha_s] = e_{22}\otimes ts^{-1}$$
$$\deg(\alpha_s) = e_{12}\otimes s^{-1}$$
$$\deg(\beta_t) = e_{21}\otimes t$$
$$\deg(\alpha_s\beta_t) = e_{11}\otimes s^{-1}t$$

So what appears to be happening is that V, W together amount to a grading wrt Γ ?? ? ?

Is it possible for the Morita context you seek to have graded modules of the form $\begin{pmatrix} \Lambda\otimes V \\ W \end{pmatrix}$. This probably won't work.

Try to finish the proof. $\qquad B = \mathcal{E}_\Gamma \rtimes \Gamma \qquad h = h_1$

Morita context. $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$ $\qquad$ where you pretend $h = h^2$

$$\langle b_1 h | h b_2\rangle = (b_1 h) b_2$$

Let W be a reduced B-module. To establish an isom. between hW and $\operatorname{Im}\{hB\otimes_B W \longrightarrow \operatorname{Hom}_B(Bh, W)\}$

$$\begin{tikzcd} hB\otimes_B W \arrow[rr] \arrow[dr, two heads] & & \operatorname{Hom}_B(Bh, W) \\ & hW \arrow[ur] & \end{tikzcd}$$

$$hb_1\otimes w \longmapsto (b_2h \longmapsto (b_2hb_1)w)$$

$$hb_1\otimes w \longmapsto (b_2h \mapsto b_2hb_1w)$$

$$hw \mapsto (b_2h \mapsto b_2hw)$$

$$\operatorname{Im}\{hB\otimes_B W \longrightarrow \operatorname{Hom}_B(Bh, W)\}$$

$$\begin{tikzcd} hB\otimes_B W \arrow[r] & hW \arrow[r, tail] & \operatorname{Hom}_B \end{tikzcd}$$

$$hb_1\otimes w \mapsto hb_1w \qquad hw' \mapsto (b_2h \mapsto b_2hw') \qquad w' = b_1w$$

$$b_1h\otimes v \longmapsto (hb_2b_1h)v$$

$$\begin{tikzcd} Bh\otimes_A V \arrow[r] & \operatorname{Hom}_A(hB, V) \arrow[d] \\ \Lambda\otimes V \arrow[u] \arrow[r, two heads, "\beta"] & W \arrow[r, tail, "\alpha"] & \Lambda\otimes V \end{tikzcd}$$

March 19, 02. $e_{\pm} = \frac{1 \pm \varepsilon}{2}$

$\Gamma = \mathbb{Z}/2, \quad \Lambda = \mathbb{C}\Gamma = \mathbb{C}1 \oplus \mathbb{C}\varepsilon = \mathbb{C}e_+ \oplus \mathbb{C}e_-$

Consider a ret of the free Λ module $\Lambda \otimes V$

$$\begin{tikzcd}
\Lambda\otimes V & W \arrow[l, hook', "\alpha"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & W \arrow[l, "\alpha"']
\end{tikzcd}$$

$\beta\alpha = 1_W \quad p = \alpha\beta = 1 \otimes p(1) + \varepsilon \otimes p(\varepsilon) = e_+ \otimes p_+ + e_- \otimes p_-$

where $p(1) = \frac{p_+ + p_-}{2} \quad p(\varepsilon) = \frac{p_+ - p_-}{2} \quad p_{\pm} = p(1) \pm p(\varepsilon)$

$$p^2 = 1 \otimes \underbrace{\left(p(1)^2 + p(\varepsilon)^2\right)}_{p(1)} + \varepsilon \otimes \underbrace{\left(p(1)p(\varepsilon) + p(\varepsilon)p(1)\right)}_{p(\varepsilon)}, \quad p_{\pm}^2 = p_{\pm}$$

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, "\beta"'] \arrow[d, shift left, "j_1"] & W \arrow[l, "\alpha"'] \arrow[dl, "\alpha_1"] & \Lambda\otimes V \arrow[l, "\beta"'] \arrow[ll, bend left, "p"] \\
& V \arrow[ul, "\beta_1"] \arrow[u, shift left, "\iota_1"] & &
\end{tikzcd}$$

$1_{\Lambda\otimes V} = \iota_1 j_1 + \varepsilon \iota_1 j_1 \varepsilon = e_1 + \varepsilon e_1 \varepsilon$

$1_W = \beta e_1 \alpha + \varepsilon(\beta e_1 \alpha)\varepsilon \qquad h = \beta e_1 \alpha$

$$\begin{tikzcd}
e_+W & e_+\Lambda\otimes V \arrow[l, "\beta_+"'] & e_+W \arrow[l, "\alpha_+"'] & e_+\Lambda\otimes V \arrow[l, "\beta_+"']
\end{tikzcd}$$

$$\begin{tikzcd}
e_-W & e_-\Lambda\otimes V \arrow[l, "\beta_-"'] & e_-W \arrow[l, "\alpha_-"'] & e_-\Lambda\otimes V \arrow[l, "\beta_-"']
\end{tikzcd}$$

Aim? To understand all operators naturally arising, appearing on W. You believe that these are given by the action of $B = \mathcal{E}_\Gamma \rtimes \Gamma$, $\mathcal{E}_\Gamma = \mathbb{C}[h]$ where $\varepsilon h \varepsilon = 1 - h$. Can you work around the point $\frac{1}{2}$.

To replace h by $h - \frac{1}{2}$ so that $\varepsilon\left(h - \frac{1}{2}\right)\varepsilon = 1 - h - \frac{1}{2}$
$= \frac{1}{2} - h$
$= -\left(h - \frac{1}{2}\right)$

$h - \frac{1}{2} = \beta\left(e_1 - \frac{1}{2}\right)\alpha \qquad e_1 - \frac{1}{2} = e_1 - \frac{1}{2}(e_1 + e_\varepsilon) = \frac{1}{2}(e_1 - e_\varepsilon)$

Repeat: Aim to understand all operators on W arising from its being a retract of the Λ-module $\Lambda \otimes V$. W has this Γ-action

You know $\overset{W}{\wedge}$ is a unital $\mathcal{E}_\Gamma \rtimes \Gamma$ module, that is W has operator ε, $\varepsilon^2 = 1$, and an operator h such that $h + \varepsilon h \varepsilon = 1_W$. Subtract $\frac{1}{2} + \frac{1}{2} = 1$ from this to get $(h - \frac{1}{2}) + \varepsilon(h - \frac{1}{2})\varepsilon = 0$, in other words, $h - \frac{1}{2}$ is an odd operator wrt $W = W_+ \oplus W_-$.

Prop. Let $\varepsilon^2 = 1$ on W a v.s. Then $\exists$ 1-1 corresp. between operator $h \in \mathcal{L}(W)$ satisf $h + \varepsilon h \varepsilon = 1$ and operators $h - \frac{1}{2} \in \mathcal{L}(W)$ satisfying $\varepsilon(h - \frac{1}{2})\varepsilon + (h - \frac{1}{2}) = 0$.

Is it possible now to see the Morita equivalence in this picture. Given p_+, p_- on V you put $W_\pm = p_\pm V$ and on $\begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ you put the odd op $\begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$

whence $h = \begin{pmatrix} \frac{1}{2} & \beta_+\alpha_- \\ \beta_-\alpha_+ & \frac{1}{2} \end{pmatrix}$ satisfies $h + \varepsilon h \varepsilon = 1$

can you see a factorization $h = \beta_1 \alpha_1$? Candidate:

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

See if you can exploit the fact that you really have two retracts

$$W_+ \xleftarrow{\beta_+} \mathbb{C}e_+ \otimes V \xleftarrow{\alpha_+} W_+$$

$$W_- \xleftarrow{\beta_-} \mathbb{C}e_- \otimes V \xleftarrow{\alpha_-} W_-$$

and you have to choose generators for $\mathbb{C}e_\pm$ before $(\alpha_+ \ \alpha_-)$ and $\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}$ are defined.

$$\alpha_1 = j_1 \alpha : V \leftarrow W$$

$$\beta_1 = \beta i_1 : W \leftarrow V$$

$$(1 \ \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\alpha_1 = j_1 \alpha \left(\frac{1+\varepsilon}{2} \quad \frac{1-\varepsilon}{2} \right)$$

$$\beta_1 = \begin{pmatrix} \frac{1+\varepsilon}{2} \\ \frac{1-\varepsilon}{2} \end{pmatrix} \beta i_1$$

$$\beta_1 \alpha_1 = \beta \begin{pmatrix} \frac{1+\varepsilon}{2} \\ \frac{1-\varepsilon}{2} \end{pmatrix} j_1 i_1 \left(\frac{1+\varepsilon}{2} \quad \frac{1-\varepsilon}{2} \right) \alpha$$

$$\begin{pmatrix} \frac{1+\varepsilon}{2} \\ \frac{1-\varepsilon}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \end{pmatrix}$$

$$\begin{bmatrix} \frac{1+\varepsilon}{2} & 0 \\ \frac{1-\varepsilon}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1+\varepsilon}{2} & 0 \\ 0 & \frac{1-\varepsilon}{2} \end{bmatrix}$$

no good

Begin again with

$$\begin{tikzcd} W & \Lambda \otimes V \arrow[l, "\beta"'] & W \arrow[l, "\alpha"'] \end{tikzcd}$$

$$\begin{tikzcd} W_+ & \Lambda_+ \otimes V \arrow[l, "e_+\beta = \beta e_+"'] & W_+ \arrow[l, "e_+\alpha"'] \end{tikzcd}$$

$$\begin{tikzcd} W_- & \Lambda_- \otimes V \arrow[l, "e_-\beta"'] & W_- \arrow[l, "e_-\alpha"'] \end{tikzcd}$$

$$\Lambda = \mathbb{C}1 \oplus \mathbb{C}\varepsilon = \mathbb{C}e_+ \oplus \mathbb{C}e_- \qquad e_\pm = \frac{1 \pm \varepsilon}{2}$$

corresp to each splitting you have $1_{\Lambda\otimes V} = \iota_1 j_1 + \iota_\varepsilon j_\varepsilon$
$= \iota_+ j_+ + \iota_- j_-$

$$\begin{tikzcd} & V \arrow[dl, "\beta\iota_-"'] \arrow[d, shift right, "\iota_-"'] & \\ W & \Lambda \otimes V \arrow[l, "\beta"'] \arrow[u, shift right, "j_-"'] \arrow[d, shift left, "j_+"] & W \arrow[l, "\alpha"'] \arrow[ul, "j_-\alpha"'] \arrow[dl, "j_+\alpha"] \arrow[d, "e_+"] \\ W_+ \arrow[u] & V \arrow[l, "\beta\iota_+"'] \arrow[ul] \arrow[u, shift left, "\iota_+"] & W_+ \arrow[l, "j_+\alpha"] \end{tikzcd}$$

I think you want to express ι_1 in terms of $\iota_\pm$
j_1 in terms of $j_\pm$

$\iota_1 v = 1 \otimes v$
$\iota_+ v = \frac{1+\varepsilon}{2} \otimes v$
$\iota_- v = \frac{1-\varepsilon}{2} \otimes v$

$$\iota_1 = \iota_+ j_+ \iota_1 + \iota_- j_- \iota_1$$

$$\iota_\varepsilon v = \varepsilon \otimes v = \frac{1+\varepsilon}{2} \otimes v - \frac{1-\varepsilon}{2} \otimes v$$

$$\boxed{\iota_\varepsilon = \iota_+ - \iota_-} \qquad \boxed{\iota_1 = \iota_+ + \iota_-}$$

$j_+\left(\frac{1+\varepsilon}{2}\otimes v\right) = v$ $\qquad j_+ = j_1 + j_\varepsilon$

$j_+\left(\frac{1-\varepsilon}{2}\otimes v\right) = 0$ $\qquad j_1 = \frac{j_+ + j_-}{2}$, $j_\varepsilon = \frac{j_+ - j_-}{2}$

$j_-\left(\frac{1+\varepsilon}{2}\otimes v\right) = 0$ $\qquad j_- = j_1 - j_\varepsilon$

$j_-\left(\frac{1-\varepsilon}{2}\otimes v\right) = -v$

Check:

$$i_1 j_1 + i_\varepsilon j_\varepsilon = (i_+ + i_-)\frac{j_+ + j_-}{2} + (i_+ - i_-)\frac{j_+ - j_-}{2}$$
$$= \frac{i_+ j_+ + i_- j_-}{2} + \frac{i_+ j_+ + i_- j_-}{2} = 1$$

So now take

$$i_1 j_1 = \frac{1}{2}(i_+ + i_-)(j_+ + j_-)$$
$$\beta i_1 j_1 \alpha = \frac{1}{2}\left(\beta i_+ + \beta i_-\right)\left(j_+\alpha + j_-\alpha\right)$$
$$= \frac{1}{2}\left(\underbrace{\beta i_+ j_+ \alpha}_{1_{W_+}} + \underbrace{\beta i_- j_- \alpha}_{1_{W_-}} + \beta i_+ j_- \alpha + \beta i_- j_+ \alpha\right)$$

Remark: It is scandalous that you took 4 days + 20 pgs to straighten this out. The moral is perhaps that using direct sums like $\begin{pmatrix}V_1\\V_2\end{pmatrix}$ is sometimes awkward, maybe even unreliable. When the situation becomes confusing you should use "orthogonality + completeness" relations.

$\varepsilon \quad F \qquad g = F\varepsilon \qquad$ assume $\|g-1\|$ small

$$g = \frac{1+X}{1-X} = \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} X \qquad X = \begin{pmatrix}+1 & -1\\ 1 & +1\end{pmatrix} g = \frac{g-1}{g+1}$$

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} \qquad g = -1 + \frac{2}{1-X}$$

$\Gamma = \mathbb{Z}/2$, $\Lambda = \mathbb{C}\Gamma$
$= \{1, \varepsilon\}$, $\varepsilon^2 = 1$.

$$\begin{tikzcd}
 & V \arrow[d, shift right, "\iota_+"'] & \\
W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \arrow[u, shift right, "j_+"'] \arrow[d, shift left, "j_-"] & W \arrow[l, tail, "\alpha"'] \\
 & V \arrow[u, shift left, "\iota_-"] &
\end{tikzcd}$$

object retract W of $\Lambda \otimes V$
$p = p^2$ in $\Lambda \otimes \mathcal{L}(V)$
$p = \sum_s s \otimes p(s)$

$\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_-$ $\qquad e_\pm = \frac{1 \pm \varepsilon}{2}$

$\Lambda \otimes V = e_+ \otimes V \oplus e_- \otimes V$

$\iota_\pm v = e_\pm \otimes v$ $\qquad j_\pm(e_+ \otimes v + e_- \otimes v')$
$= v$ if $+$
$ v'$ if $-$

$$1 = \iota_+ j_+ + \iota_- j_-$$

$$\begin{pmatrix} j_+ \\ j_- \end{pmatrix} \begin{pmatrix} \iota_+ & \iota_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

What is the goal? You want to look at Λ as $\mathbb{C}e_+ \oplus \mathbb{C}e_-$ rather than $\mathbb{C}1 \oplus \mathbb{C}s$

So you consider a Λ-module retract W of $\Lambda \otimes V$. It is exactly the same as two retracts W_+ W_- of V, provided you use the appropriate partition $1_{\Lambda \otimes V} = \iota_+ j_+ + \iota_- j_-$

$$\begin{tikzcd}
W_+ \arrow[d, shift right] & V \arrow[l, "\beta_+"'] \arrow[d, shift right, "\iota_+"'] & W_+ \arrow[l, "\alpha_+"'] \arrow[d, shift right] \\
W \arrow[u, shift right] \arrow[d, shift left] & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \arrow[u, shift right, "j_+"'] \arrow[d, shift left, "j_-"] & W \arrow[l, tail, "\alpha"'] \arrow[u, shift right] \arrow[d, shift left] \arrow[ul, "j_+\alpha"'] \arrow[dl, "j_-\alpha"] \\
W_- \arrow[u, shift left] & V \arrow[l, two heads, "\beta_-"'] \arrow[u, shift left, "\iota_-"] & W_- \arrow[l, tail, "\alpha_-"'] \arrow[u, shift left]
\end{tikzcd}$$

$$\begin{tikzcd}
W_+ & V \arrow[l, "\beta\iota_+"'] \arrow[d, equal] & W_+ \arrow[l, "j_+\alpha"'] \\
W_- & V \arrow[l, "\beta\iota_-"'] & W_- \arrow[l, "j_-\alpha"']
\end{tikzcd}$$

Go from W ret of $\Lambda \otimes V$

to

$$\begin{tikzcd}
W_+ & V \arrow[l, "\beta\iota_+"'] \arrow[d, equal] & W_+ \arrow[l, "j_+\alpha"'] \\
W_- & V \arrow[l, "\beta\iota_-"'] & W_- \arrow[l, "j_-\alpha"']
\end{tikzcd}$$

get $j_\pm \alpha \beta \iota_\pm = p_\pm$ on V

and on W you get

$$\begin{pmatrix} 1 & \beta\iota_+ j_- \alpha \\ \beta\iota_- j_+ \alpha & 1 \end{pmatrix}$$

V

V

Is there an elementary reason why two projections which are close are conjugate in a canonical way?

$g = F\varepsilon = \frac{1+X}{1-X}$ $\qquad \frac{g-1}{g+1}$ $\qquad g = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ $\qquad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} g = \frac{g-1}{g+1}$

$$\varepsilon X \varepsilon^{-1} = \varepsilon \frac{g-1}{g+1} \varepsilon^{-1} = \frac{g^{-1}-1}{g^{-1}+1} = \frac{1-g}{1+g} = -\frac{g-1}{g+1} = -X$$

$g = F\varepsilon$ assume $g - 1 = \frac{2X}{1-X}$ small $\therefore X$

$$g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}} \qquad g = \frac{(1+X)^2}{1-X^2} = \frac{1+X}{1-X}$$

$$\begin{tikzcd}
W_+ \oplus W_- \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"'] & V \arrow[l, tail, "\begin{pmatrix} \beta\iota_+ \\ \beta\iota_- \end{pmatrix}"'] \arrow[d, "\wr"] & W_+ \oplus W_- \arrow[l, two heads, "(j_+\alpha \ j_-\alpha)"'] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}"] \\
W & V \arrow[l, tail, "\begin{pmatrix} \beta\iota_+ \\ -\beta\iota_- \end{pmatrix}"'] & W \arrow[l, two heads, "(j_+\alpha \ -j_-\alpha)"']
\end{tikzcd}$$

V is reduced iff V is the image of $1+X$ in which case it's also the image of $1-X$. It seems clear that you get an autom. of V, which in the invertible case is $\frac{1+X}{1-X}$

The best case is when $1+X$ is invertible on W

$$\begin{tikzcd}
W \arrow[d, "\varepsilon"'] & (1-X)W \arrow[l, phantom, "\supset"] & W \arrow[l] \arrow[d, "\varepsilon"] \\
W & (1+X)W \arrow[l, phantom, "\supset"] & W \arrow[l]
\end{tikzcd}$$

$$\varepsilon(1-X)W = (1+X)\varepsilon W$$

A retract of $\Lambda \otimes V$ is the same as 2 retracts of V, i.e. two involutions on V whose "difference" is an autom. of V. Recall three types

① retract of the free module $\Lambda \otimes V$ where $\Lambda = M_2\mathbb{C}$
same as a retract of $\begin{pmatrix} V \\ V \end{pmatrix}$

② retract of free module $\Lambda \otimes V$ where $\Lambda = \mathbb{C}e_+ \oplus \mathbb{C}e_- = \mathbb{C}[1, \varepsilon]$
same as 2 projections on V.

③ retract of $\begin{pmatrix} V_1 \\ \oplus \\ V_2 \end{pmatrix}$