The principal bundle for L $(= \mathbb{R}^2 \times \mathbb{C})$ is $\mathbb{T} \times \mathbb{R}^2$

You want to exploit the translation invariant of the curvature. If there existed a translation-invariant 1-form A with correct curvature then the translation group $\mathbb{R}^2$ would act preserving the connection. In nonzero curvature case you can lift translations $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} s+x \\ t+y \end{pmatrix}$ to L preserving the connection.

Consider over $\mathbb{R}^2$ "the" hermitian complex line bundle L + connection D whose curvature is $2\pi i\, dx\, dy$. (Any two such line bundles are isomorphic up to a constant scalar factor $\in \mathbb{T}$.)

Model: L = trivial line bundle, $D = d + 2\pi i x\, dy$. You want to find the group of automorphisms of this geometric object: diffeo of $\mathbb{R}^2$ + lifting to L which preserves D.

Look at a diffeo of $\mathbb{R}^2$ given by a translation $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} s+x \\ t+y \end{pmatrix}$

$$g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s+x \\ t+y \end{pmatrix} \qquad g^*(d + 2\pi i x\, dy)\psi = \underbrace{(d + 2\pi i (s+x)\, dy)}_{e^{-2\pi i s y}(d + 2\pi i x\, dy)\, e^{2\pi i s y}}(g^*\psi)$$

$$\therefore \left(\underbrace{e^{2\pi i s y}\, e^{s\partial_x}\, e^{t\partial_y}}_{e^{s(\partial_x + 2\pi i y)}\, e^{t\partial_y}}\psi\right)(x,y) = e^{2\pi i s y}\psi(s+x, t+y)$$

So it seems that you get an action of a Heisenberg group on $(\mathbb{R}^2, L)$ preserving D. It's probably a right action because $[\partial_x + 2\pi i y, \partial_y] = -2\pi i$. This would fit with the idea that there is a left action given by the components of D: $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$

You should look at the principal bundle for L

$$\begin{array}{c} L = \mathbb{C} \times \mathbb{R} \times \mathbb{R} \\ \downarrow \\ \mathbb{R} \times \mathbb{R} \end{array} \qquad \text{coords } w, x, y$$

a section is $(\psi(x,y), x, y)$

principal bundle $\mathbb{T} = \{z \mid |z| = 1\}$.

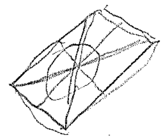
$P = \mathbb{T} \times \mathbb{R} \times \mathbb{R}$. You need to understand better the relation between the connection in the principal bundle, which in general is a Lie alg valued 1-form θ restricting to the Maurer-Cartan form of G at each point of the base, and the connection in the associated v.b. $D = d + \theta$. MC form is $g^{-1}dg$, you take the variation $g + \delta g$ and you left mult. by g^{-1} to get $1 + g^{-1}\delta g$.

$P = \mathbb{T} \times \mathbb{R} \times \mathbb{R}$ MC form on $\mathbb{T}$ should be $z^{-1}dz$ connection form should be $z^{-1}dz + 2\pi i x\,dy$. Put $z = e^{i\theta}$ so that $z^{-1}dz = +i\,d\theta$. What is the horizontal lift of the vector field ∂_x? Look for $\partial_x + f(x,y)\partial_\theta$ such that

$$\underbrace{\left(\iota(\partial_x) + f\,\iota(\partial_\theta)\right)\left(i\,d\theta + 2\pi i x\,dy\right)}_{f(x,y)\,i\,=\,0} = 0$$

$\tilde{\partial}_x = \partial_x$. Next $\tilde{\partial}_y = \partial_y + g\,\partial_\theta$

$$\left(\iota(\partial_y) + g\,\iota(\partial_\theta)\right)\left(i\theta + 2\pi i x\,dy\right)$$



$$= 2\pi i x + ig = 0$$

connection form is $\sqrt{-1}(d\theta + 2\pi x\,dy)$. The lift of ∂_x which is horizontal is ∂_x; the horizontal lift of ∂_y is $\partial_y + f\partial_\theta$ where $f + 2\pi x = 0$. ~~So~~ you seem to have a sign problem.

If the principal bundle P is to turn out to be the Heisenberg group, then the structure shouldn't depend on a sign.

Go back to the picture of the Heisenberg group as a circle bundle over $\mathbb{R}^2$. What you want is to start with H acting on itself by left + right mult.

Feb 18, 02. Let us consider the infinitesimal translation operators

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix} \; , \; \begin{matrix} \nabla_x = \partial_x + 2\pi i y \\ \nabla_y = \partial_y \end{matrix}\right] = 0$$

$$[D_x, D_y] = 2\pi i \qquad [\nabla_x, \nabla_y] = -2\pi i$$

These are diffl operators acting on $C^\infty(\mathbb{R}^2)$.

You can exponentiate these differential operators. These are differential operators acting on sections of the trivial line bundle over $\mathbb{C}$.

Your aim is get a statement about the line bundle L. Begin with the Heisenberg group. Just as the line bundle L + connection can be presented using a nice 1-form (bilinear form on $\mathbb{R}^2$ with nondeg. skew symm.), so can the Heisenberg group be presented using such a bilinear form.

So you might try to do both using $2\pi i \, x dy$ first and then passing to the general case.

Better idea. Take a suitable presentation for H then look at infinitesimal left + right mult, i.e. look at the vector field on H that you get.

Feb 19, 02. Presentation: $H = \mathbb{T} \times \mathbb{R} \times \mathbb{R}$ typical element $z e^{aX} e^{bY}$

mult.
$$z_1 e^{a_1 X} e^{b_1 Y} z_2 e^{a_2 X} e^{b_2 Y} = z_1 z_2 e^{-2\pi i b_1 a_2} e^{(a_1+a_2)X} e^{(b_1+b_2)Y}$$

Let $G \times G^{op}$ act on G $\quad (g_1, g_2)x = g_1 x g_2^{-1}$. Stabilizer of x is $\{(g_1, g_2) \mid g_1 x g_2^{-1} = x\}$

$g_1 x = x g_2$

$$g_1 x = x g_2 \qquad g_1' x = x g_2' \qquad g_1 g_1' x = g_1 x g_2' = x g_2 g_2'$$

Problem remains to determine whether you can identify the Heisenberg group H with "the" principal $\mathbb{T}$-bundle P over $\mathbb{R}^2$ having curvature $2\pi i \, dx dy$. Since you have a uniqueness result for the principal bundle you should be able to produce the desired isomorphism by constructing a connection on the principal $\mathbb{T}$-bundle

$$\mathbb{T} \rightarrowtail H \longrightarrow\!\!\!\!\rightarrow \mathbb{R}^2$$

having the desired curvature. At this point you need to describe H precisely, namely, as a set + mult. From the theory of central group extensions where the quotient groups is abelian there is an invariant, namely the commutator: take two elements of $\mathbb{R}^2$ lift them to H and take commutator. This gives a skew-symmetric $\mathbb{Z}$-bilinear pairing on $\mathbb{R}^2$ with values in $\mathbb{T}$, which lifts to $Lie(\mathbb{T}) = i\mathbb{R}$ - think of the universal covering of H.

Let's go over this again. You want to identify the Heisenberg group H with the principal $\mathbb{T}$ bundle + connection over $\mathbb{R}^2$ having curvature $2\pi i\, dx\, dy$. You have a uniqueness result for P, so it should suffice to construct a connection on the principal $\mathbb{T}$-bundle given by the group extension

$$\mathbb{T} \rightarrowtail H \longrightarrow\!\!\!\!\rightarrow \mathbb{R}^2$$

having the desired curvature.

What do you know about this group extension, in fact, what do you know about a central group extension of an abelian group? Commutator pairing

Think generally. Consider a gp extn.

$$B \overset{\iota}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} A$$

where A, B abelian and B is in the center of E. To study this you choose a section s of π, which gives a biject. $B \times A \longrightarrow E$, $(b,a) \mapsto \iota(b)s(a)$, in term of which the product in E is $\iota(b_1)s(a_1)\iota(b_2)s(a_2) = \iota(b_1 b_2)\, s(a_1)s(a_2)$ $= \iota(b_1 b_2 f(a_1,a_2))\, s(a_1 a_2)$, where $f: A \times A \longrightarrow B$ satisfies the 2 cocycle condition

$$f(a_2,a_3) - f(a_1 a_2, a_3) + f(a_1, a_2 a_3) - f(a_1,a_2) = 0$$

if product in B is written additively. Suppose we use $+$ for both A, B. Commutator pairing

$$(a_1,a_2) \longmapsto s(a_1)\, s(a_2)\, s(a_1)^{-1}\, s(a_2)^{-1}$$

$$s(a_1)\, s(a_2) = i(f(a_1,a_2))\, s(a_1+a_2)$$
$$s(a_2)\, s(a_1) = i(f(a_2,a_1))\, s(a_1+a_2)$$

$$s(a_1)\, s(a_2)\, s(a_1)^{-1} s(a_2)^{-1} = i\big(f(a_1,a_2) - f(a_2,a_1)\big)$$

So you learn that the 2-cocycle $f(a_1,a_2)$ when skew-symmetrized is $\mathbb{Z}$-bilinear. In fact this should be true for any 2-cocycle $f: A \times A \longrightarrow B$.

Make universal ~~construction~~: $Z^2(A,B) = \mathrm{Hom}(\,?\,, B)$

Have map $Z^2(A,B) \longrightarrow \mathrm{Hom}(\Lambda^2 A, B)$. You seem to recall

$$\Lambda^2 A \xrightarrow{\sim} H_2(A) \; ?$$

Universal Coeff Thm.

$$0 \to \mathrm{Ext}^1(H_1(B^{d}A), B) \hookrightarrow H^2(B^{d}A, B) \longrightarrow \mathrm{Hom}(H_2(B^{d}A), B) \to 0$$

$$\begin{array}{ccccccc} & \| & & \| & & & \\ 0 \longrightarrow & \underset{\text{abel gp extns}}{\mathrm{Ext}^1(A,B)} & \longrightarrow & \underset{\text{central gp extns}}{H^2(A,B)} & \longrightarrow & \mathrm{Hom}(\Lambda^2 A, B) & \longrightarrow 0 \end{array}$$

It appears that $Z^2(A,B) \longrightarrow \mathrm{Hom}(\Lambda^2 A, B)$

is surjective, i.e. any commutator pairing arises from a central extension. This seems to imply that there is at least one central extension $\Lambda^2 A \rightarrowtail E \longrightarrow A$ whose comm. pairing is the identity of $\Lambda^2 A$.

Look $\delta : C^1(A,B) \longrightarrow Z^2(A,B)$

$$(\delta f)(a_1,a_2) = f(a_2) - f(a_1+a_2) + f(a_1)$$

Is δf $\mathbb{Z}$-bilinear NO

$$\delta f(a_0+a_1, a_2) = f(a_2) - f(a_0+a_1+a_2) + f(a_0+a_1)$$

$$\delta f(a_0, a_2) = f(a_2) - f(a_0+a_2) + f(a_0)$$
$$\delta f(a_1, a_2) = f(a_2) - f(a_1+a_2) + f(a_1)$$

$$f(a_2,a_3) - f(a_1+a_2,a_3) + f(a_1,a_2+a_3) - f(a_1,a_2) = 0$$

Review central extensions of abelian groups

$$B \overset{i}{\rightarrowtail} E \overset{\pi}{\twoheadrightarrow} A$$

commutator pairing $\Lambda^2_{\mathbb{Z}} A \longrightarrow B$, $a_1 \wedge a_2 \overset{\lambda}{\longmapsto} \tilde{a}_1 \tilde{a}_2 \tilde{a}_1^{-1} \tilde{a}_2^{-1}$

where $\tilde{a}_i$ any elt of $\pi^{-1}\{a_i\}$. By univ. coeff thm

$$\begin{array}{ccccc} \mathrm{Ext}^1_{\mathbb{Z}}(A,B) & \rightarrowtail & H^2(A,B) & \xrightarrow{\text{commutator pairing}} & \mathrm{Hom}(\Lambda^2 A, B) \\ & & \| & & \\ & & Z^2(A,B)/\delta C^1(A,B) & & \end{array}$$

short exact sequence

and $\mathrm{Hom}(A \otimes_{\mathbb{Z}} A, B) \subset Z^2(A,B)$, i.e. any $\mathbb{Z}$-bilinear $f(a_1,a_2)$ is a 2-cocycle. Question: Does every extension arise from a bilinear cocycle? Equiv: Is the comp.

$$\mathrm{Hom}(A \otimes_{\mathbb{Z}} A, B) \subset Z^2(A,B) \longrightarrow H^2(A,B)$$

surjective? Another point is that the following square is cartesian

Quadratic maps $q : A \to B$ =

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma_2(A), B) & \longrightarrow & C^1(A,B) \\ \downarrow & & \downarrow \delta \\ \mathrm{Hom}_{\mathbb{Z}}(A \otimes A, B) & \hookrightarrow & Z^2(A,B) \end{array}$$

by definition of quadratic map $q : A \to B$ and def of $\Gamma_2(A)$.

There seems to be an exact sequence

$$\Lambda^2 A \longrightarrow A\otimes A \longrightarrow S_2(A) \longrightarrow 0$$

and a canonical map $S_2(A) \to \Gamma_2(A)$ whose kernel & cokernel are killed by 2. Maybe also

$$\Gamma_2(A) \longrightarrow A\otimes A \longrightarrow \Lambda^2 A \longrightarrow 0$$

Combine:

$$\begin{tikzcd}
& \Gamma_2 A \arrow[d] \arrow[dr, shift left] & & \\
\Lambda^2 A \arrow[r, "a_1\wedge a_2 \mapsto a_1\otimes a_2 - a_2\otimes a_1"] \arrow[dr, "2"'] & A\otimes A \arrow[r] \arrow[d, "a_1\otimes a_2 \mapsto a_1\wedge a_2"] & S_2 A \arrow[r] \arrow[ul, shift left] & 0 \\
& \Lambda^2 A \arrow[d] & & \\
& 0 & &
\end{tikzcd}$$

compositions here are mult by 2

Somewhere introduce $\Sigma_2 A$ rep $Z^2(A,B) = \mathrm{Hom}(\Sigma_2 A, B)$

$$\begin{array}{c} Z^2(A,B) \\ \cup \\ \mathrm{Hom}(A\otimes A, B) \end{array}$$

$\Sigma_2 A$ is the abelian group w gens. $(a_1, a_2) \in A\times A$ and relations making the image of (a_1,a_2) into a 2 cocycle:

$$(a_2, a_3) - (a_1+a_2, a_3) + (a_1, a_2+a_3) - (a_1, a_2) = 0$$

Then there's a canonical map $\Sigma_2 A \twoheadrightarrow A\otimes A$ sending (a_1,a_2) to $a_1\otimes a_2$. It looks like $\Sigma_2 A$ naturally arises from the MacLane resolution of an abelian group A by free abelian groups of type $\mathbb{Z}[A^n]$.

$$\begin{tikzcd}
\mathrm{Hom}(\Lambda^2 A, B) \arrow[r, equal] & \mathrm{Hom}(\Lambda^2 A, B) \arrow[r, equal] & \mathrm{Hom}(\Lambda^2 A, B) & \\
Z^2(A,B) \arrow[u] \arrow[r, equal] & \mathrm{Hom}(\Sigma_2 A, B) \arrow[u] \arrow[r, phantom, "\supset"] & \mathrm{Hom}(A\otimes A, B) \arrow[u] & \text{assoc. bil form} \\
C^1(A,B) \arrow[u, "\delta"] \arrow[r, equal] & \mathrm{Hom}(\mathbb{Z}[A], B) \arrow[u] \arrow[r, phantom, "\supset"] \arrow[ur, phantom, "\text{cart}"] & \mathrm{Hom}(\Gamma_2 A, B) \arrow[u] \arrow[r, equal] & \text{quad fns from } A \text{ to } B \arrow[u] \\
\mathrm{Hom}(A,B) \arrow[u] \arrow[r, equal] & \mathrm{Hom}(A,B) \arrow[u] \arrow[r, equal] & \mathrm{Hom}(A,B) \arrow[u] & \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
& \Lambda^2 A \arrow[r, equal] \arrow[d] & \Lambda^2 A \arrow[d] \\
\mathbb{Z}[A^2] \arrow[r, two heads] & \Sigma_2 A \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & A\otimes A \arrow[d] \\
\mathbb{Z}[A] \arrow[r, equal] & \mathbb{Z}[A] \arrow[r, two heads] \arrow[d] & \Gamma_2 A \arrow[d] \\
& A \arrow[r, equal] & A
\end{tikzcd}$$

---

back to the Heisenberg group, to identify it with "the" principal $\mathbb{T}$-bundle P over $\mathbb{R}^2$ with curvature $2\pi i\, dx\, dy$. What do you know about P? It is $\mathbb{T}\times\mathbb{R}^2$, coords z, x, y, where $\mathbb{T}$ acts trivially on $\mathbb{R}^2$, and on itself by multiplication. This describes P as the trivial principal $\mathbb{T}$ bundle over $\mathbb{R}^2$. Next you must give a connection form on P, this amounts to a 1-form $z^{-1}dz + \eta$, where $\eta \in \Omega^1(\mathbb{R}^2)$. The curvature is $d\eta$ (up to sign). Not very illuminating.

So look at H which is a central extension of $\mathbb{R}^2$ by $\mathbb{T}$. So you have an exact sequence of groups

$$\mathbb{T} \rightarrowtail H \twoheadrightarrow \mathbb{R}^2$$

with $\mathbb{T}$ = the center of H. The commutator pairing is a skew-symm.

bilinear form on $\mathbb{R}^2$ with values in $\mathbb{T}$. Calculate: take $(a,b) \in \mathbb{R}^2$, $(a',b') \in \mathbb{R}^2$ and lift to $(1, a, b)$ $(1, a', b')$ resp. in $H = \mathbb{T} \times \mathbb{R}^2$

$$\overset{\parallel}{\left(e^{a\partial_x} e^{b\partial_y}\right)} \cdot \left(e^{a'\partial_x} e^{b'\partial_y}\right) = e^{-2\pi i ba'} e^{(a+a')\partial_x} e^{(b+b')\partial_y}$$

$$\left(e^{a'\partial_x} e^{b'\partial_y}\right)\left(e^{a\partial_x} e^{b\partial_y}\right) = e^{-2\pi i b'a} e^{(a+a')\partial_x} e^{(b+b')\partial_y}$$

commutator is $e^{-2\pi i(ba'-b'a)}$. So get a clear link between the commutator pairing for H and the curvature for P.

What you would like to do next is to match up presentations for H with connections for P. When you define H you pick a 2 cocycle on $\mathbb{R}^2$ with values in $\mathbb{T}$, then define H to be $\mathbb{T} \times \mathbb{R}^2$ (so H comes provided with a section of $\pi : H \to \mathbb{R}^2$) equipped mult. defined using the cocycle.

Picking the section of π corresponds to trivializing the $\mathbb{T}$ bundle P. If we restrict 2-cocycles to those associated to bilinear forms, then our 2-cocycle corresponds to a connection. So it seems to work, but you want the details to be clean.

$$\begin{tikzcd}
 & \Lambda^2 A \arrow[r, equal] \arrow[d] & \Lambda^2 A \arrow[d] & \\
\mathbb{Z}[A^2] \arrow[r, two heads] & \Sigma_2 A \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & A \otimes A \arrow[d] \arrow[r] & S^2 A \arrow[dl] \\
\mathbb{Z}[A] \arrow[r, two heads] & \Gamma_2 A \arrow[d] & & \\
 & A & &
\end{tikzcd}$$

Crazy Idea: suggested by

Is there anything here which might help find the good class of signals for sampling, i.e. the f in $L^2(\mathbb{R})$ corresponding to continuous sections of the degree 1 line bundle over T^2.

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vired reps of a finite Heisenberg grp. $G = \mathbb{Z}/n$ $G^\vee = \mu_n$

$H = \mathbb{T} \times G^\vee \times G$ My idea here ~~too~~ was to understand something about the problem of linking the repn theory of H with the structure of H.

Feb 21, 02 Aim: Calculate left and right translation vector fields on H. Start with your favorite model for H, namely, $\mathbb{T}\times\mathbb{R}^2$ where mult is defined using the cocycle $\exp 2\pi i(-b_1 a_2)$:

$$e^{a_1X}e^{b_1Y}e^{a_2X}e^{b_2Y} = e^{-2\pi i b_1 a_2}\, e^{(a_1+a_2)X}\, e^{(b_1+b_2)Y}$$

$$(z_1, a_1, b_1)(z_2, a_2, b_2) = (z_1 z_2 e^{-2\pi i b_1 a_2}, a_1+a_2, b_1+b_2)$$

inf left translation:

$$(1, \delta a_1, \delta b_1)(z, a_2, b_2) = (z\underbrace{e^{-2\pi i\,\delta b_1 a_2}}_{1-2\pi i a_2\delta b_1}, \delta a_1 + a_1, \delta b_1 + b_2)$$

$$(x_1, y_1, \zeta_1)(x_2, y_2, \zeta_2) = (x_1+y_1, x_2+y_2, e^{-2\pi i y_1 x_2}\zeta_1\zeta_2)$$

Start with this description of H, calculate the infinitesimal left (+ right) translation

---

$\psi((a,b,\zeta)\cdot(x,y,z))$ this is the effect of left translation

now take product $(a,b,\zeta) = (\delta a, \delta b, 1+\delta\zeta)$. But first do the

$\psi(a+x, b+y, e^{-2\pi i bx}\zeta z)$ and expand to first order around $(0,0,1)$.

$$= (\partial_x\psi)\,\delta a + (\partial_y\psi)\delta b + (\partial_z\psi)\left(\cancel{e^{-2\pi i bx}}(-2\pi i(\delta b)x)\zeta z + \cancel{e^{-2\pi i bx}} z\,\delta\zeta\right)$$

$$= (\partial_x\psi)\,\delta a + (\partial_y\psi)\,\delta b + (\partial_z\psi)(-2\pi i x z\,\delta b) + \partial_z\psi(z\,\delta\zeta)$$

$$= (\partial_x\psi)\,\delta a + (\partial_y\psi - 2\pi i x(z\partial_z\psi))\,\delta b + (z\partial_z\psi)\,\delta\zeta$$

where $\partial_x\psi$, $\partial_y\psi$, $z\partial_z\psi$ are evaluated at x, y, z

So the vector fields on H we get are

$$\partial_x, \quad \partial_y - 2\pi i x(z\partial_z), \quad z\partial_z$$

Next find inf right translation

$$(\psi+\delta\psi)(x,y,z) = \psi((x,y,z)\cdot(\delta a, \delta b, 1+\delta\xi)) \qquad z(1-2\pi i y\,\delta a)(1+\delta\xi)$$

$$= \psi(x+\delta a,\ y+\delta b,\ e^{-2\pi i y\,\delta a} z(1+\delta\xi))$$

$$= \psi(x,y,z) + (\partial_x\psi)\,\delta a + (\partial_y\psi)\,\delta b + (\partial_z\psi)\, z(-2\pi i y\,\delta a + \delta\xi)$$

$$\underline{\delta\psi(x,y,z)} = (\partial_x\psi - 2\pi i y(z\partial_z\psi))\,\delta a + (\partial_y\psi)\,\delta b + (z\partial_z\psi)\,\delta\xi$$

Maybe better is

$$\delta\psi(x,y,z) = \psi((x,y,z)\cdot(\delta x, \delta y, 1+\delta z)) - \psi(x,y,z)$$

$$= \psi(x+\delta x,\ y+\delta y,\ e^{-2\pi i y\,\delta x} z(1+\delta z)) - \psi(x,y,z)$$

$$= (\partial_x\psi)\,\delta x + (\partial_y\psi)\,\delta y + (\partial_z\psi)(-2\pi i y\,\delta x\, z + z\,\delta z)$$

$$= (\partial_x\psi - 2\pi i y(z\partial_z\psi))\,\delta x + (\partial_y\psi)\,\delta y + (z\partial_z\psi)\,\delta z$$

---

inf left mult vector fields. $\quad \partial_x, \quad \partial_y - 2\pi i x\, z\partial_z, \quad z\partial_z$

inf rt mult v. f. $\quad \partial_x - 2\pi i y\, z\partial_z, \quad \partial_y, \quad z\partial_z$

observe wrong sign

$$[\partial_x, \partial_y - 2\pi i x\, z\partial_z] = -2\pi i\, z\partial_z$$

$$[\partial_x - 2\pi i y\, z\partial_z, \partial_y] = +2\pi i\, z\partial_z$$

because of contravariance of the action on functions.

Recall that you have computed the ~~left~~ left and right inf translations actions of the Heisenberg Lie algebra on the functions on the Heisenberg group. View the Heisenberg group as a trivial principal $\mathbb{T}$-bundle over $\mathbb{R}^2$.

$$\mathbb{T} \rightarrowtail H \xrightarrow{\pi} \mathbb{R}^2 \implies H = \mathbb{R}^2\times\mathbb{T}$$

(given section of π)

Function on H: $\quad \psi(x,y,z) \in C^\infty(\mathbb{R}^2\times\mathbb{T})$

Aim: You want to link the vector fields on H to operators on sections of the line bundle

$$\begin{array}{c} H \times^{\mathbb{T}} \mathbb{C} \\ \downarrow \\ \mathbb{R}^2 \end{array}$$

The first point is that a section is equivalent to a map $\psi : H \to \mathbb{C}$ satisfying

$$\psi(x, y, z\zeta) = \zeta^{-1}\psi(x, y, z)$$

Repeat calculation $\quad H = \mathbb{R}^2 \times \mathbb{T} \ni (x, y, z)$

$$\psi((a, b, \zeta)\cdot(x, y, z)) = \psi(a+x,\ b+y,\ e^{-2\pi i bx}\zeta z)$$

$$\delta\psi(a+x,\ b+y,\ e^{-2\pi i(bx)}\zeta z)$$

$$= (\partial_x\psi)\,\delta a + (\partial_y\psi)(\delta b) + (\partial_z\psi)\ \underbrace{\delta\left(e^{-2\pi i(bx)}\zeta z\right)\Big|_{(a,b,\zeta)=(0,0,1)}}$$

$$e^{-2\pi i bx}\left[(-2\pi i\,\delta b\, x)\zeta z + \delta\zeta\, z\right]$$

set $b=0$, $\zeta = 1$ $\quad\left[-2\pi i x\,\delta b\, z + \delta\zeta\, z\right]$

$$= (\partial_x\psi)\,\delta a + (\partial_y\psi)\,\delta b$$

$$(z\partial_z\psi)(-2\pi i x)\,\delta b + (z\partial_z\psi)\,\delta\zeta$$

$$= (\partial_x\psi)\,\delta a + (\partial_y\psi - 2\pi i x\, z\partial_z\psi)\,\delta b + z\partial_z\psi\,\delta\zeta$$

three v.f. $\quad \partial_x,\quad \partial_y - 2\pi i x\, z\partial_z,\quad z\partial_z\psi$

$$[\partial_x,\ \partial_y - 2\pi i x\, z\partial_z] = -2\pi i\, z\partial_z$$

Summary: You have H acting on itself by left and by right mult and you have the corresp. v.fs. What's new is the vertical vector field $z\partial_z = \frac{1}{i}\partial_\theta$ dual to $\frac{dz}{z} = i\,d\theta$. Next you want to deal with the principal $\mathbb{T}$-bundle P and its connection.

$$\begin{array}{c} P = \mathbb{R}^2 \times \mathbb{T} \\ \downarrow \\ \mathbb{R}^2 \end{array}$$

What is a connection in the principal $\mathbb{T}$ bundle P

Answer: a 1-form with values in $\mathrm{Lie}(\mathbb{T}) = i\mathbb{R}$ whose restriction to the fibres is $\frac{dz}{z} = i\,d\theta$ and is invariant under the right $\mathbb{T}$ action.

$$\boxed{\underbrace{M\,dx + N\,dy}_{A} + z^{-1}dz}$$

Let's try to identify D on sections of $L = P \times^{\pi} \mathbb{C}$

Section $= \psi(x,y,z) : P \longrightarrow \mathbb{C}$ such that

$\boxed{\psi(x,y,z\zeta) = \zeta^{-1}\psi(x,y,z).}$ Now you have conn. form

$$M\,dx + N\,dy + z^{-1}dz \in \Omega^1(P, i\mathbb{R})$$

You want $D\psi \in \Omega^1(\mathbb{R}^2, i\mathbb{R})$. Guess is to mult.

$$(M\,dx + N\,dy + z^{-1}dz)\,\psi(x,y,z)$$

Somehow you want $\psi(x,y,z\zeta) = \zeta^{-1}\psi(x,y,z)$

$$\therefore \psi(x,y,\zeta) = \zeta^{-1}\psi(x,y,1)$$

You've learned that ψ is homogeneous of degree ± 1 (?) on the fibre

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$$\begin{array}{c} G \to P \\ \downarrow \\ B \end{array}$$

You have to decide whether G acts on the left or the right. Weil algebra, equivariant DR cohomology.

Feb 22, 02

Digression: Central extensions for elementary abelian 2 groups.

$$0 \longrightarrow B \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 0$$

Instead of the commutator pairing you have a finer invariant, namely, a quadratic map $q : A \to B$ defined by $i\,q(\pi x) = x^2$. Well-def'd. $(xb)^2 = xbxb = x^2b^2 = x^2$, quadratic means $q(xy)\,q(x)^{-1}q(y)^{-1} = q(xy)\,q(x)\,q(y)$ is $\mathbb{Z}$-bilinear

$$xyxyxxyy = xyx^3y^3 = xyx^{-1}y^{-1}$$

which is the commutator pairing.

$$H^2(A,B) = \mathrm{Hom}(\Gamma_2 A, B)$$

universal central extn

$$\Gamma_2 A \rightarrowtail EA \twoheadrightarrow A$$

$$\begin{tikzcd}
\Lambda^2 A \arrow[r, equal] \arrow[d] & \Lambda^2 A \arrow[d, "{a_1\wedge a_2 \,\mapsto\, a_1\otimes a_2 - a_2\otimes a_1}"] & \\
\Gamma_2 A \arrow[r, tail, "\gamma_2(a)\mapsto a\otimes a"] \arrow[d] & A\otimes A \arrow[r, two heads, "a_1\otimes a_2 \mapsto a_1\wedge a_2"] \arrow[d, "a_1\otimes a_2 \mapsto a_1a_2"] & \Lambda^2 A \arrow[d, equal] \\
A^{(2)} \arrow[r, tail] & S_2 A \arrow[r] & \Lambda^2 A
\end{tikzcd}$$

Notice that $\Gamma_2 A$ is a summand of $A\otimes A$, (vector spaces over $\mathbb{F}_2$), so that bilinear cocycles yield all extensions.

IDEA: Look at the effect of $-1 : A \to A$ on central extensions of A, (as well as $2 : A \to A$?)

Return to your principal $\mathbb{T}$ bundle P. Problem yesterday with relating P to the associated line bundle L. You need to relate a connection form on P to a differential operator $d + A$ on sections of L. You started to understand some of this, namely a section ψ is a function on P behaving according to a character of $\mathbb{T}$, something like $\psi(x,y,z\zeta) = \psi(x,y,z)\zeta$, but signs didn't work.

Some you need to link $z\partial_z = \frac{1}{i}\partial_\theta$ to the d in $D = d + A$ on $\Gamma(\mathbb{R}^2, L)$. Other ideas: canonical line bundle on projective space, better: in the case of a circle bundle you look at the graded space of homogeneous functions of different degrees. You should also look at coadjoint orbits in the Heisenberg group, constructing the irred reps by geometric quantization.

$$0 \to \Lambda^2 A \longrightarrow A \otimes A \longrightarrow S^2 A \longrightarrow 0 \quad \text{exact}$$

the inj of by lins led to A fin. gen, prod of cyc. gps; show map direct injection via induction no. of factors, cross effect: $\Lambda^2(A_1 \oplus A_2) = \Lambda^2 A_1 \oplus A_1 \otimes A_2 \oplus \Lambda^2 A_2$

$$(A_1 \oplus A_2)^{\otimes 2} = A_1^{\otimes 2} \oplus \overbrace{A_1 \otimes A_2 \oplus A_2 \otimes A_1} \oplus A_2^{\otimes 2}$$

the 'bilinear' part obvious direct injection

You believe $Z^2(A,B) \xrightarrow[\text{pairing}]{\text{comm.}} \operatorname{Hom}(\Lambda^2 A, B)$ onto

inj since $\mathbb{Z}[A^2] \twoheadrightarrow A \otimes A$

$\operatorname{Hom}(A \otimes A, B)$ onto? for all $B \iff \Lambda^2 A \to A \otimes A$ direct injection

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Heisenberg group $\mathbb{T} \longrightarrow H \longrightarrow \mathbb{R}^2$. There should be a line bundle L, somewhere, look for its sections, which should yield a representation (maybe two) of H. Induced repn. So (like pulling teeth) you arrive at something which should have been familiar. Too much geometry, not enough repn. theory.

Now you can get the formulas straight. The concept of induced representation, how it fits with fibre bundle stuff. H subgroup of G, V repn of H, then form vector bundle $\begin{matrix} G\times^H V \\ \downarrow \\ \boxed{G/H} \end{matrix}$ whose sections are maps $f: G \longrightarrow V$ satisfying the equivariance condition

$$f(gh) = h^{-1}f(g),$$

because you descend

$$\begin{tikzcd} G\times V \arrow[r] \arrow[d] & G\times^H V \arrow[d] \\ G \arrow[r] & G/H \end{tikzcd}$$

a point of $G\times^H V$ over a coset gH is an H-orbit $\{(gh, h^{-1}v) \mid h\in H\}$, there is a unique representative (g,v) with first component g. Maybe better to say a section of

$$\begin{tikzcd} G\times^H V \arrow[d] \\ G/H \arrow[u, bend left, "s"] \end{tikzcd}$$

is the same by descent as a section

$$\begin{tikzcd} G\times V \arrow[d] \\ G \arrow[u, bend left] \end{tikzcd}$$

which is H-equivariant

$$\begin{matrix} (g, f(g)) & \{(gh, h^{-1}f(g))\} \\ \uparrow & \\ g & \{gh\} \end{matrix}$$

Let's go back to $\mathbb{T} \rightarrowtail H \twoheadrightarrow \mathbb{R}^2$. You want maybe to look at a general principal bundle

Look at a manifold P with $\mathbb{T}$ action, free, slices So P is a principal $\mathbb{T}$ bundle. There's an assoc. line bundle for any character $\mathbb{T} \longrightarrow \mathbb{T}$. This is something you'd forgotten about.

You probably want to look at embedding these line bundles as retracts of trivial vector bundles.

Return to P with $\mathbb{T}$ acting freely and base B. Assume trivial $P = B\times\mathbb{T}$. $C^\infty(P) = C^\infty(B)\otimes C^\infty(\mathbb{T})$. Functions on P are Fourier series whose Fourier coefficients are functions on B. So there's a $\mathbb{Z}$-grading on $C^\infty(P)$. This is true even without P being trivial

So you have a new way to view a principal $\mathbb{T}$-bundle $\mathbb{T} \to P \to B$, as a $\mathbb{Z}$-graded algebra corresponding to the irreducible characters of $\mathbb{T}$. Thus

$$C^\infty(P) = \bigoplus_{n\in\mathbb{Z}} \Gamma(B, L^{\otimes n})$$

A partition of 1 enters you should have

$$\Gamma(B, L^\vee) \otimes_{\mathcal{O}_B} \Gamma(B, L) \xrightarrow{\sim} \mathcal{O}_B$$

so you should be able to write $1 = \sum_i x_i \otimes x_i'$ with $x_i \in \Gamma(L)$, $x_i' \in \Gamma(L^\vee)$.

But now what? Connections. Find the link between the connection form A on P and the connection operator D on $\Gamma(B, L)$.

P has free action of $\mathbb{T}$, so functions on P decompose according to the characters $\chi \in \mathbb{T}^\vee = \mathbb{Z}$. Let $X = z\partial_z = \frac{1}{i}\partial_\theta$ generate $\mathrm{Lie}(\mathbb{T}) = i\mathbb{R}$. Better let $\psi \in C^\infty(P)$, then $\psi(e^{i\theta}p)$ is periodic of period 2π in θ, so you have

$$\psi(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \psi_n e^{in\theta}$$

where $\psi_n = \int e^{-in\theta}\psi(e^{i\theta}z)\frac{d\theta}{2\pi}$?

Need better notation: $\mathbb{T}$ acts freely on P, $\mathbb{T} = \{e^{i\theta} | \theta = \mathbb{R}/2\pi\}$

The point is that $\mathbb{T}$ acts on $C^\infty(P)$, so $C^\infty(P)$ is a repn. of $\mathbb{T}$, and it decomposes according to the characters.

$$C^\infty(P) = \bigoplus_{n\in\mathbb{Z}} \{\psi \in C^\infty(P) \mid T_\theta\psi = e^{in\theta}\psi\}$$

It should be true that this is a $\mathbb{Z}$ grading of $C^\infty(P)$ as an alg, and the n-th component is $\Gamma(B, L^{\otimes n})$

Put $C^\infty(P)_n = \{\psi \in C^\infty(P) \mid T_\theta\psi = e^{in\theta}\psi\}$

$$C^\infty(P)_m \cdot C^\infty(P)_n \subset C^\infty(P)_{m+n}.$$

What does this look like for $P = H$? It's completely trivial since $H = \mathbb{R}^2 \times \mathbb{T}$, so that

$$C^\infty(H) = C^\infty(\mathbb{R}^2) \otimes C^\infty(\mathbb{T}).$$

Go back to $\mathbb{T} \longrightarrow P \xrightarrow{\pi} B$. Aim: to understand a connection in P, i.e. a $\mathrm{Lie}(\mathbb{T}) = i\mathbb{R}$ valued 1-form on P, call it Θ satisfying $\iota_X \Theta = 1$, where X is the vector field whose flow is the $\mathbb{T}$ action, also $\mathcal{L}_X \Theta = 0$, Θ is preserved by the flow. $0 = \mathcal{L}_X \Theta = \cancel{d\iota_X\Theta} + \iota_X d\Theta$. $\therefore d\Theta$ is basic. So w [redacted]

Go over the structure again. P is a manifold with free circle ($\mathbb{T}$) action, infinitesimal generator X, Θ is a $\mathrm{Lie}(\mathbb{T})$ valued 1-form, $\mathcal{L}_X \Theta = 0$, $\iota_X \Theta = 1$.

It should now be true that there is some sort of differential operator D on the vector bundle L, i.e. on sections of L, it satisfy the derivation property used by Bott. What can you do. Sections of L are certain functions ψ on P, so you want to apply d to get $d\psi$ which gives the derivation property over $C^\infty(B)$. Next you need to correct $d\psi$ in the vertical direction in some way. ψ is not constant vertically because it has degree ± 1, which means that vertically $d\psi = \pm \Theta \psi$, i.e.

$\iota_X(d\psi - \Theta\psi) = L_X\psi + \cancel{\ldots} - \psi = 0$. So it seems that the operator is $\psi \longmapsto (d - \Theta)\psi$, where this is a <u>1-form on B</u> with values in the line bundle L. "because it is "basic"

OK it seems to work. Notice that you end up working with functions and diffl forms on P, you construct things to be <u>basic</u> so that they descend to the base.

Thm? If A fin. gen. abelian group, then any commut. pairing $\Lambda^2 A \longrightarrow B$ arises by skew-symmetrizing some bilinear form $A \otimes A \longrightarrow B$.

$$0 \longrightarrow \Lambda^2 A \longrightarrow A \otimes A \longrightarrow S^2 A \longrightarrow 0$$

$$0 \longrightarrow S^2 A \longrightarrow \Gamma^2 A \longrightarrow A \longrightarrow 0$$

Feb 23, 02

$$0 \to \mathrm{Ext}^1(A,B) \longrightarrow Z^2(A,B) \longrightarrow \mathrm{Hom}(\Lambda^2 A, B) \to 0$$

Quadratic function $q : A \to B$ is one such that

$$(\delta q)(a_1, a_2) = \underbrace{q(a_1+a_2) - q(a_1) - q(a_2)}_{\mathrm{Hom}(\Lambda^2 A, B)} \in \mathrm{Hom}(A \otimes A, B)$$

$$\begin{tikzcd}
\mathrm{Hom}(\Sigma^2 A, B) \arrow[r, phantom, "="] & Z^2(A,B) \arrow[r, phantom, "\supset"] \arrow[dr, phantom, "\text{cart}"] & \mathrm{Hom}(A \otimes A, B) \\
\mathrm{Hom}(\mathbb{Z}[A], B) \arrow[r, phantom, "="] & C^1(A,B) \arrow[u, "\delta"] \arrow[r, phantom, "\supset"] & \mathrm{Quad}(A,B) \arrow[u, "\delta"] \arrow[r, phantom, "="] & \mathrm{Hom}(\Gamma^2 A, B)
\end{tikzcd}$$

Question For A fin. gen. is every central extn of A given by a quadratic function.

$$\underbrace{\mathbb{Z}[A^3] \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; \mathbb{Z}[A^2]}_{\Sigma_2 A} \;\substack{\xrightarrow{t_g}\\\longrightarrow\\\longrightarrow}\; \underset{\text{exact here}}{\mathbb{Z}[A]} \longrightarrow A \longrightarrow 0$$

what is the

$$\begin{tikzcd}
C^3(A,B) & C^2(A,B) \arrow[l, "\delta"'] & C^1(A,B) \arrow[l, "\delta"'] \arrow[dl, "\delta"'] \\
& Z^2(A,B) \arrow[u, phantom, "\cup"] & \mathrm{Hom}(A,B) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

Question: What is the homology of

$$\mathbb{Z}[A^3] \longrightarrow \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}[A] \longrightarrow A \longrightarrow 0 \quad ?$$

$$C^3(A,B) \longleftarrow C^2(A,B) \longleftarrow C^1(A,B) \longleftarrow \mathrm{Hom}(A,B) \longleftarrow 0$$

Feb 24, 02. Review central extensions of elementary abelian groups. $B \xrightarrow{\lambda} E \xrightarrow{\pi} A$, invariant is the quadratic function $i\,q(a) = e^2$ $\quad \pi(e) = a$. Because $H_2(B_c A, \mathbb{Z}/2) = \Gamma^2 A$ (dual to $H^2 = S^2 A$), one has $H^2(A,B) = \mathrm{Hom}(\Gamma^2 A, B)$, the quadratic fn is complete invariant

$$\Lambda^2 A \longrightarrow A \otimes A \longrightarrow S^2 A \qquad\qquad 0 \to S^2A \longrightarrow \Gamma^2 A \longrightarrow A \to 0$$

$$\Gamma^2 A \longrightarrow A \otimes A \longrightarrow \Lambda^2 A$$

$$\begin{tikzcd}
 & \Lambda^2 A \arrow[r, equal] \arrow[d] & \Lambda^2 A \arrow[d] & & \\
0 \arrow[r] & \Gamma^2 A \arrow[r] \arrow[d] & A \otimes A \arrow[r] \arrow[d] & \Lambda^2 A \arrow[r] \arrow[d, equal] & 0 \quad ? \\
0 \arrow[r] & A \arrow[r] & S^2 A \arrow[r] & \Lambda^2 A \arrow[r] & 0
\end{tikzcd}$$

$$S^2 A \xrightarrow{?} \Gamma^2 A \qquad\qquad S^2 A \longrightarrow A \otimes A$$

$$a_1 a_2 \longmapsto \gamma^2(a_1 + a_2) - \gamma^2(a_1) - \gamma^2(a_2) \qquad\qquad a_1 a_2 \quad a_1 \otimes a_2 + a_2 \otimes a_1$$

is zero if $a_1 = a_2$

$$0 \longrightarrow A \longrightarrow S^2 A \longrightarrow \Gamma^2 A \longrightarrow A \longrightarrow 0$$

$$\begin{tikzcd}
0 \arrow[r] & \Lambda^2 A \arrow[r] \arrow[d] & A \otimes A \arrow[r] \arrow[d] & \Gamma^2 A \arrow[r] & A \arrow[r] & 0 \\
 & \Gamma^2 A \arrow[ur] & S^2 A \arrow[ur, leftrightarrow] & & &
\end{tikzcd}$$

So what

$$\begin{tikzcd}
\Lambda^2 A \arrow[r, phantom, "\subset"] \arrow[d, equal] & \Gamma^2 A \arrow[r] \arrow[d, phantom, "\cap"] & A \arrow[d, phantom, "\cap"] \\
\Lambda^2 A \arrow[r, phantom, "\subset"] & A \otimes A \arrow[r, two heads] \arrow[d] & S^2 A \arrow[d] \\
 & \Lambda^2 A \arrow[r, equal] & \Lambda^2 A
\end{tikzcd}$$

$$A \otimes A \supset \Gamma^2 A \supset \Lambda^2 A$$

$$\gamma(a_1 + a_2) - \gamma a_1 - \gamma a_2 \longleftarrow a_1 a_2$$

$$\begin{tikzcd}
\Gamma^2 A \arrow[r, phantom, "\supset"] \arrow[d, phantom, "\cap"] & S^2 A \arrow[r, phantom, "\supset"] \arrow[d] & A \\
A \otimes A & \Lambda^2 A &
\end{tikzcd}$$

$$0 \longrightarrow A \longrightarrow S^2 A \longrightarrow \Lambda^2 A \longrightarrow 0$$

functors $A \mapsto A^{(2)}, A\otimes A, S^2A, \Gamma^2 A, \Lambda^2 A$

$A\otimes A$ contains $\Lambda^2 A$ $\Gamma^2 A$

$$\Gamma^2 A \longrightarrow A\otimes A \qquad\qquad \gamma(a_1+a_2)-\gamma(a_1)-\gamma(a_2)$$
$$\gamma(a) \longmapsto a\otimes a \qquad\qquad = a_1\otimes a_2 + a_2\otimes a_1$$

You seem to have problems linking the elementary abelian group case to general case

$$0 \longrightarrow \Lambda^2 A \longrightarrow A\otimes A \xrightarrow{d} \Gamma^2 A \longrightarrow A \longrightarrow 0$$
$$a_1\wedge a_2 \longmapsto a_1\otimes a_2 - a_2\otimes a_1$$
$$a_1\otimes a_2 \longmapsto \gamma(a_1+a_2)-\gamma(a_1)-\gamma(a_2)$$

You believe that

$$0 \longrightarrow \Lambda^2 A \longrightarrow A\otimes A \longrightarrow S^2 A \longrightarrow 0$$
$$a_1\wedge a_2 \longmapsto a_1\otimes a_2 - a_2\otimes a_1$$
$$a_1\otimes a_2 \longmapsto a_1 a_2$$

is exact. So if true you get

$$0 \longrightarrow S^2 A \longrightarrow \Gamma^2 A \longrightarrow A \longrightarrow 0$$

exact
$$a_1a_2 \longmapsto \gamma(a_1+a_2)-\gamma(a_1)-\gamma(a_2)$$
$$a^2 \longmapsto \gamma(2a)-2\gamma(a)$$

Look at $\quad \Gamma^2 A \longrightarrow A\otimes A$

$$\gamma(a) \longmapsto a\otimes a$$

Compose with $\quad S^2 A \longrightarrow \Gamma^2 A \longrightarrow A\otimes A$

$$a_1a_2 \longmapsto \gamma(a_1+a_2)-\gamma(a_1)-\gamma(a_2) \longmapsto a_1\otimes a_2 + a_2\otimes a_1$$

$$a^2 \longmapsto \qquad\qquad 0 \quad ?$$

Here's where the mistake occurred. Look at

$$0 \longrightarrow \Lambda^2 A \longrightarrow A\otimes A \xrightarrow{d} \Gamma^2 A \longrightarrow A \longrightarrow 0$$

with $A = \mathbb{Z}/2$. Then $0 \to \mathbb{Z}/2 \longrightarrow \Gamma^2(\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2 \to 0$ is exact so $\Gamma^2(\mathbb{Z}/2)$ has order 4, so it won't embed in $A\otimes A$.

Not clear

It seems that quadratic functions on $\mathbb{Z}/n$ allow you to construct the extension $\mathbb{Z}/n \rightarrowtail \mathbb{Z}/n^2 \twoheadrightarrow \mathbb{Z}/n$ of abelian groups. General argument. Given $h: A\otimes A \to B$ $\mathbb{Z}$-bilinear, you get a group from the set $B\times A$ with the product $(b,a)\cdot(b',a') = (b+b'+h(a,a'), a+a')$, a central extn E of A by B, whose commutator pairing is $\Lambda^2 A \to B$, $a_1\wedge a_2 \mapsto h(a_1,a_2) - h(a_2,a_1)$. If the comm pairing is 0, then E is abelian.

Take $A = B = \mathbb{Z}$ and $h(m,n) = mn$: Then you get an abelian group extn $\mathbb{Z} \rightarrowtail E \xrightarrow{\pi} \mathbb{Z}$, which splits. Splitting means a section of π which is additive, which means that 1-cochain g with coboundary h, g is then a quadratic function yielding h.

Let $g(m) = \frac{m(m-1)}{2}$, $\quad g(m+n) - g(m) - g(n) = \frac{1}{2}\left[\cancel{m^2} + 2mn + \cancel{n^2} - \cancel{m} - \cancel{n} - \cancel{m^2} + \cancel{m} - \cancel{n^2} + \cancel{n}\right] = mn$

Consider $\quad 0 \to \Lambda^2 A \to A\otimes A \to \Gamma^2 A \to A \to 0$

where $A = \mathbb{Z}/N$ whence $\quad 0 \to \mathbb{Z}/N \to \Gamma^2(\mathbb{Z}/N) \to \mathbb{Z}/N \to 0$

so that $|\Gamma^2(\mathbb{Z}/N)| = N^2$. Take $B = \mathbb{Z}/N$ and $h(m+N\mathbb{Z}, n+N\mathbb{Z}) = mn+\mathbb{Z}$

Go back to $\mathbb{Z}\times\mathbb{Z}$ with the product

$$(m,n)\cdot(m',n') = (m+m'+nn', n+n')$$

and consider the maps
$$\mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}$$
$$n \longmapsto (g(n), n)$$

Then $\quad (g(n),n)\cdot(g(n'),n') = (g(n)+g(n')+nn', n+n')$
$$= (g(n+n'), n+n')$$

Thus one has a homomorphism splitting the extension.

Now consider
$$\begin{tikzcd}
\mathbb{Z} \arrow[r, tail] \arrow[d] & \mathbb{Z}\times\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \arrow[l, bend right, "(g(n){,}n)\ \longleftarrow\ n"'] \\
\mathbb{Z}/N \arrow[r, tail] & \mathbb{Z}/N\times\mathbb{Z}/N \arrow[r, two heads] & \mathbb{Z}/N
\end{tikzcd}$$

not the direct product

So the order of the elt $(0,1)$ in the bottom extn. is the least $n>0$ such that n and $g(n) \equiv 0 \mod N$.

$\dfrac{N(N-1)}{2}$, $\dfrac{2N(2N-1)}{2}$

If N odd then N works

If N even then $2N$ works.

How to calculate $\Gamma^2 A$. probably use

$$0 \longrightarrow S^2 A \longrightarrow \Gamma^2 A \xrightarrow{\pi} A \longrightarrow 0$$

This is an extension of abelian groups. There is a tautological section of π whence $\Gamma^2 A = S^2 A \times A$, and a tautological 2cocycle with values in $S^2 A$, yielding an addition on $\Gamma^2 A$.

$$(b,a)\cdot(b',a') = (b+b'+aa', a+a')$$

check that $a \longmapsto (g(a), a)$ is a hom.

$$(g(a),a)\cdot(g(a'),a') = (\underbrace{g(a)+g(a')+aa'}_{g(a+a')}, a+a')$$

$A = \mathbb{Z}/2$ $\quad (g(1),1)(g(1),1) = (\cancel{g(1)+g(1)}+1, 0)$

$$0 \longrightarrow S^2 A \xrightarrow{i} \Gamma^2 A \underset{\pi}{\overset{\gamma}{\rightleftarrows}} A \longrightarrow 0$$

$$i(a_1 a_2) = \gamma(a_1+a_2) - \gamma(a_1) - \gamma(a_2)$$

It looks like you want to have a $g: A \to S^2 A$ with $(\delta g)(a_1,a_2) = a_1 a_2$

$$0 \to S^2\mathbb{Z} \longrightarrow \Gamma^2\mathbb{Z} \xrightarrow[(0,1)]{} \mathbb{Z} \longrightarrow 0$$

$(0,1)(0,1) = (1,2)$ $\quad 2 \quad (0,a)(0,a) = (a^2, 0)$

$(1,2)(0,1) = (3,3)$ $\quad 3 \quad (a^2,0)(0,a) = (a^2, a)$

$(3,3)(0,1) = (6,4)$ $\quad 2 \quad (a^2,a)(0,a) = (0,0)$

$(6,4)(0,1) = (10,5)$

$$\left(\frac{n(n-1)}{2}, n\right)$$

What is the lcm of $\frac{n(n-1)}{2}$ and n

$p \quad \frac{p(p-1)}{2}$

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want $\quad n, \ \frac{n(n-1)}{2} \equiv 0 \quad (N)$

$N \mid n$ and $\frac{n(n-1)}{2}$ $\qquad n = 2k$

$$\begin{tikzcd} S^2(\mathbb{Z}/N) \arrow[r] \arrow[d, equal] & \Gamma^2(\mathbb{Z}/N) \arrow[r] & \mathbb{Z}/N \\ \mathbb{Z}/N & & \end{tikzcd}$$

in here have elements $\frac{n(n-1)}{2}, n$

have $\quad S^2\mathbb{Z} \longrightarrow \Gamma^2\mathbb{Z} \longrightarrow \mathbb{Z}$

$$\begin{tikzcd} 0 \arrow[r] & \Lambda^2 A \arrow[r] & A \otimes A \arrow[rr] \arrow[dr] & & \Gamma^2 A \arrow[r] & A \arrow[r] & 0 \\ & & & S^2 A \arrow[ur] & & & \end{tikzcd}$$

$(0,1)\cdot(0,1) = (1,0)$

$\frac{n(n-1)}{2}$	n
(0	1)
(1	2)
(3	3)
(6	4)
28	8

$$\begin{tikzcd} 0 \arrow[r] & S^2\mathbb{Z} \arrow[r] \arrow[d] & \Gamma^2\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & S^2(\mathbb{Z}/2) \arrow[r] & \Gamma^2(\mathbb{Z}/2) \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

Do the examples carefully: $\quad$ abelian group extension

$$S^2A \rightarrowtail S^2A \times A \twoheadrightarrow A \qquad (b,a)(b',a') = (b+b'+aa', a+a')$$

also have abelian group extension

$$S^2A \overset{i}{\rightarrowtail} \Gamma^2 A \underset{\gamma}{\overset{}{\twoheadrightarrow}} A \qquad \gamma(a+a') - \gamma(a) - \gamma(a') = i(aa')$$

on the other hand you have the abelian group extensions.

$$S^2A \rightarrowtail EA \twoheadrightarrow A$$

where $EA = \{(b,a) \mid b \in S^2A,\ a \in A\}$

$(b,a)(b',a') = (b+b'+a\cdot a', a+a')$

Question: Are these two group extensions isomorphic? Yes. let $s: A \to EA$ be the section $s(a) = (0,a)$

$$g(a) = \frac{a(a-1)}{2} + ba \qquad N=3.$$

$$(g(a), a) = \Big(\underbrace{\frac{a(a-1)}{2} + ba, a}_{a\cdot\text{times the element } (b,1)}\Big) \qquad \text{modulo } 3.$$

want $3\cdot$times $(b,1)$ to be non-zero

$$3 + b3, 3$$

$N=5$ $\quad 10 + b5, 5 \qquad \frac{p(p-1)}{2} + bp, p$

$$\frac{N(N-1)}{2} + Nb, N \qquad N \text{ odd no good.}$$

N even $\quad \Big(\Big(\frac{N}{2}\Big)(N-1) + Nb, N\Big) \equiv -\frac{N}{2} \mod N.$

$$\left.\begin{aligned} N=4.\ 4\times(0,1) &= \Big(\frac{4(3)}{2}, 4\Big) = (6,4) \equiv (2,0) \\ 8\times(0,1) &= (28, 8) \equiv (0,0) \mod 4 \end{aligned}\right\} \begin{aligned} &\text{mod } 4. \\ &\text{exponent } 8 \end{aligned}$$

$$\left.\begin{aligned} N=8 \qquad 8\times(0,1) &= (28, 8) \equiv (4,0) \mod 8 \\ 16\times(0,1) &= (8\times 15, 16) \equiv (0,0) \mod 16 \end{aligned}\right\} \text{exponent } 16$$

$$H^2(A,B) = \operatorname{Hom}(\Gamma_e^2 A, B) \qquad \Gamma_e^2 A = \Gamma^2 A \otimes \mathbb{Z}/2$$

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & S^2A & \longrightarrow & \Gamma^2 A & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \| & & \downarrow & & & & \\ 0 \to A & \longrightarrow & & & S^2A & \longrightarrow & \Gamma_e^2 A & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

$$0 \longrightarrow \Lambda^2 A \longrightarrow \Gamma_e^2 A \longrightarrow A \longrightarrow 0$$

$$\Gamma^2 \qquad E \qquad A$$

Feb 25, 02 Something is unclear in the elementary abelian case. ~~Consider~~ Begin with

$$0 \to \Lambda^2 A \to A \otimes A \to \Gamma^2 A \to A \to 0 \quad \text{in general}$$

Then $0 \to S^2 A \to \Gamma^2 A \to A \to 0$.

Apply $\mathrm{Hom}(-,B)$ to get

$$0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(\Gamma^2 A, B) \to \mathrm{Hom}(S^2 A, B) \to \mathrm{Ext}^1(A,B)$$

$B \rightarrowtail E \xrightarrow{\pi} A$ Assume $2A = 0$ $q(a) = xx$ if $\pi x = a$

Given $x, y \in E$ $(xx)^{-1} xy\, xy (yy)^{-1} = x^{-1} y x y^{-1}$ ~~$y^{-1} z y z$ z center~~

~~$x^{-1} z^{-1} y^{-1} x y z = x^{-1} y^{-1} x y$~~ ~~$x^{-1} = xz$~~

~~$= x^{-1} z^{-1} x y x y^{-1}$~~

$x^{-1} = xz$
$x^{-1} z^{-1} = x$ $\quad x^{-1} y x y^{-1} = xzyx^{-1}z^{-1}y^{-1} = xyx^{-1}y^{-1}$

So if A is elementary 2-abelian, then what? You have a map

$$H^2(A,B) \to \mathrm{Hom}(\Gamma^2 A, B) \qquad \text{for } 2A = 0$$
$$\downarrow$$
$$H^2(A,B) \to \mathrm{Hom}(\Lambda^2 A, B)$$

Basic question: Is $\Gamma^2(A \oplus A') = \Gamma^2(A) \oplus (A \otimes A') \oplus \Gamma^2(A')$?

It seems that for $2A = 0$ you

$$\begin{tikzcd} & H^2(A,B) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(\Gamma^2 A, B) \arrow[d] \\ \mathrm{Ext}^1(A,B) \arrow[r, tail] & H^2(A,B) \arrow[r] & \mathrm{Hom}(\Lambda^2 A, B) \end{tikzcd}$$

Go over again carefully. $B \overset{i}{\rightarrowtail} E \xrightarrow{\pi} A$ central extn.

$i\, q(\pi x) = x^2$ $\qquad q(\pi x + \pi y) - q(\pi x) - q(\pi y) =$

$= xy\, xy (\cancel{x}x)^{-1} (yy)^{-1} = x^{-1} x^{-1} xy\, xy\, y^{-1} y^{-1} = x^{-1} y x y^{-1}$

But $x^{-1} = xz$ $\quad z \in iB$ $\qquad = xzyx^{-1}z^{-1}y^{-1} = xyx^{-1}y^{-1}$

$xy\, xy (xx)^{-1} (yy)^{-1} = xy\, x\cancel{y}\cancel{y}^{-1} y^{-1} x^{-1} x^{-1} = xyxy^{-1}x^{-1}x^{-1} = xy\cancel{xx^{-1}}x^{-1}\cancel{y}^{-1}$

$$\begin{tikzcd} S^2A \arrow[r, tail, "i"] & \Gamma^2 A \arrow[r, two heads] & A \arrow[l, bend right, "\gamma"'] \end{tikzcd}$$

$$\cancel{(b, \gamma a)(b', \gamma a') = (b+b'+aa', \gamma(a+a'))}$$

$$\cancel{(0, \gamma a)(0, \gamma a) = (0 + aa, 0)}$$

use $\quad S^2A \times A \xrightarrow{\sim} \Gamma^2 A \qquad (b,a) \longmapsto i(b) + \gamma(a)$

group law $\quad (b,a)(b',a') = (b+b'+aa', a+a')$

the quadratic map $A \to S^2A$ is $\quad (0,a)(0,a) = (aa, 0)$

The quadratic map assoc. to the extn.

$$S^2A \rightarrowtail \Gamma^2 A \twoheadrightarrow A$$

is $\quad a \longmapsto a^2$.

$$\Lambda^2 A \rightarrowtail \Gamma^2 A/2 \twoheadrightarrow A$$

$$\begin{tikzcd}
A \arrow[r, equal] \arrow[d] & A \arrow[d, "\gamma"] & \\
S^2A \arrow[r, tail] \arrow[d] & \Gamma^2 A \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\
\Lambda^2 A \arrow[r, tail] & \Gamma^2 A/2 \arrow[r, two heads] & A
\end{tikzcd}$$

$\mathrm{Ext}^1(A,B) \rightarrowtail H^2(A,B) \twoheadrightarrow \mathrm{Hom}(\Lambda^2 A, B)$ possibly OK because $\mathrm{Ext}^1(A,B)$ and $\mathrm{Hom}(A,B)$ related by the Bockstein

$$0 \to A \to S^2A \to \Gamma^2 A/2 \to A \to 0$$

Feb 26, 02

$$\begin{pmatrix} hBh & hB \\ & \\ Bh & B \end{pmatrix}$$

go over group case

Review everything today

Begin with C: gen h_s $s \in \Gamma$

$h_s h_t \neq 0 \Rightarrow s^{-1}t \in \Phi$, $\sum_s h_s h_t = h_t = \sum_s h_t h_s$

C has local left unit and right unit idempotent

$M = CM = \sum h_s M$ $\qquad$ $C = \sum_t h_t C$

$B = \Gamma \ltimes C_{\Gamma,\Phi}$

$A = \mathcal{P}_{\Gamma,\Phi}$ $\qquad$ $p(s)$ $s \in \Gamma$
$\sum_t p(st^{-1})p(t) = p(s)$
$p(s) \neq 0 \Rightarrow s \in \Phi$

idempotent defined by gen. + relns.

Γ, Φ given $\quad C =$ alg defined via gens + rels as above.

If M a C-module, then $M = CM \iff \forall m \quad m = \sum_{s \in \Gamma} h_s m$

in which case M is reduced + firm $C \otimes_C M \xrightarrow{\sim} M$.

Γ acts on C so can form cross product $\Gamma \ltimes C = B$.

$\Gamma \ltimes C^+$ = semi direct product $\mathbb{C}\Gamma \oplus B$ $\qquad$ $\mathbb{C}\Gamma \longrightarrow Mult(B)$

Question: $\Gamma = \mathbb{Z}^2$, $\mathcal{P}_\Gamma$ should classify retracts of trivial vector bundles over T^2, map $T^2 \longrightarrow BU$. Are there interesting examples with Φ finite.

Today you want to go over the details of (Γ, Φ) in a systematic way.

Begin with $B = \Gamma \ltimes C$ def.

① Red B-module = Γ-module W with $h \in End(W)$

sat $hsh \neq 0 \Rightarrow s \in \Phi$, $\sum shs^{-1}w = w$ $\quad \forall w$.

$\Rightarrow W = \sum_s shW$

Given W as above put $V = hW$, $W \overset{\text{inc.}}{\underset{\iota}{\longleftarrow}} V \xleftarrow{j = h} W$

(2) Claim V is red. A-module, $A = P_{\Gamma, \Phi}$, $p(s) = jsi \in \mathcal{L}(V)$

$$\iota p(s) j = \iota j s \iota j = hsh$$

$p(s) \neq 0$ (+ ι inj, j surj) $\Rightarrow hsh \neq 0 \Rightarrow s \in \Phi$.

$$\sum_t p(st^{-1})p(t) = \sum_t j s t^{-1} \iota j t \iota = j s \sum t^{-1} h t \iota = \underset{p(s)}{\underset{\shortparallel}{j s \iota}}.$$

$\therefore$ V an A-module. Next $W = \sum_s s \iota V \Rightarrow$

$$V = jW = \sum_s j s \iota V = \sum_s p(s) V \qquad \therefore V = AV.$$

Let $v \in V$ satisfy $p(s)v = 0 \quad \forall s$

Then $\iota v = \sum s h s^{-1} \iota v = \sum s \iota \underbrace{(j s^{-1} \iota)}_{p(s^{-1})} v = 0 \Rightarrow v = 0$.

Define maps

$$\sum_s s \otimes \sum_t p(s^{-1}t) f(t) \longleftarrow \sum t \otimes f(t)$$

$$\begin{array}{ccccccc}
W & \xleftarrow{\beta} & \Lambda \otimes V & \xleftarrow{\alpha} & W & \xleftarrow{\beta} & \Lambda \otimes V \\
 & & & & \iota v & & 1 \otimes v \\
w = \sum s \iota j s^{-1} w & \longleftarrow & \sum s \otimes j s^{-1} w & \longleftarrow & w & & \\
\sum t \iota f(t) & \longleftarrow & \sum t \otimes f(t) & & & &
\end{array}$$

α well defined because $W = \sum t h' W$ and $j s^{-1} t i = p(s^{-1}t)$

Define β, α show $\beta\alpha = 1_W$ and $\alpha\beta\left(\sum_s t \otimes f(t)\right) = \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$

---

(3) Given V any A-module define p on $\Lambda \otimes V$.

$W = \left\{ \sum t \otimes f(t) \;\middle|\; \sum_t p(s^{-1}t) f(t) = f(s) \right\}$. Define $j = \eta_1 \alpha$, $\iota = \beta \varepsilon_1$

$h = \iota j = \beta(\varepsilon_1 \eta_1)\alpha$

$$\begin{tikzcd}
W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, "\alpha"'] \arrow[dl, "j"] & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left] \\
 & V \arrow[ul, "i"] \arrow[u, shift left, "\varepsilon_1"] & & V \arrow[ul, "i"] \arrow[u, shift left]
\end{tikzcd}$$

Move on to the Morita context $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$

$$F(V) = \mathrm{Im}\{Bh \otimes_A V \longrightarrow \mathrm{Hom}_A(hB, V)\}$$
$$G(W) = \mathrm{Im}\{hB \otimes_B W \longrightarrow \mathrm{Hom}_B(Bh, W)\}$$

Point here is, that $AhB = hBhB = hB$

so $A\left(hB \otimes_B W\right) = hB \otimes_B W$ same for $G(W)$

Let $\lambda \in \mathrm{Hom}_B(Bh, W)$, $0 = \lambda(BhA) = \lambda(BhBh) = \lambda(Bh)$
$\therefore \lambda = 0$.

So you've checked that $G(W)$ is A-reduced.

$$hB \otimes_B W \twoheadrightarrow hW \longrightarrow \mathrm{Hom}_B(Bh, W)$$
$$hw \longmapsto \underset{\substack{\| \\ 0}}{(bh \longmapsto bhw)} \text{ means } Bhw = 0$$

$hw = 0$
$\Uparrow$
but ${}_BW = 0$
$\in W$

Look at $F(V)$. This should be $p(\Lambda \otimes V) = p(\Lambda \otimes A) \otimes_A V$

You've shown that $hW = G(W)$ is A reduced for any reduced B-module W, in particular hB and Bh are reduced for both A, B hence also $A = hBh$ is A-reduced on both sides. (B reduced is easy by the partition of 1)

General case of $\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix} \ni BhB = B$ If W is B-red see p477
then $\mathrm{Im}\{hB \otimes_B W \to \mathrm{Hom}_B(Bh, W)\} = hW$ is hBh-reduced

Problem: Identify $F(V)$ above with $p(\Lambda \otimes V)$ (group case)

Concerning $\begin{pmatrix} \overset{A}{\overset{\shortparallel}{hBh}} & hB \\ Bh & B \end{pmatrix}$ where $BhB = B$

maybe the first point is this context is completely idempotent so you then have the actual Morita equivalence on the level of red. modules

$$F(V) = \mathrm{Im}\{ Bh \otimes_A V \longrightarrow \mathrm{Hom}_A(hB, V)\}$$

$$G(W) = \mathrm{Im}\{ hB \otimes_B W \longrightarrow \mathrm{Hom}_B(Bh, W)\}$$

These functors respect reduced modules. Claim $G(W) = hW$

$$hb_1 \otimes w \longmapsto \big(b_2 h \longmapsto (b_2 h * hb_1) \otimes w = b_2 h b_1 w \big)$$

$$\begin{tikzcd} hB \otimes_B W \arrow[dr] & & \mathrm{Hom}_B(Bh, W) \\ & hW \arrow[ur, hook] & \end{tikzcd}$$

$$\begin{tikzcd} hb_1 \otimes w \arrow[dr, mapsto] & & (b_2 h \mapsto b_2 h w) \\ & hb_1 w \quad hw \arrow[ur, mapsto] & \end{tikzcd}$$

$\alpha(w) = 0$ means $Bhw = 0 \Rightarrow hw = 0$ assuming W reduced.

W	B-red	$\Longrightarrow$	hW	A-red.	leave for a while.
B	B-red	$\Longrightarrow$	hB	A-red	
B	B^{op}-red	$\Longrightarrow$	Bh	A^{op}-red.	

Want to show $F(V) = p(\Lambda \otimes V)$ $\qquad$ V has ops. $p(s)$

$$\begin{tikzcd} \cancel{\xleftarrow{\alpha}} W & \Lambda \otimes V \arrow[l, "\beta"'] \arrow[d, shift left, "\pi"] & W \arrow[l, "\alpha"'] \arrow[dl, "j"] \\ & V \arrow[ul, "i"] \arrow[u, shift left, "\iota"] & \end{tikzcd}$$

$$\alpha w = \sum s \otimes g s^{-1} w$$

$$\beta \sum t \otimes f(t) = \sum t \, f(t)$$

$$\alpha\beta \sum t \otimes f(t) = \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$$

notice that you see that

$$jW = \sum_t p(t) V$$

$$\mathrm{Ker}\, i = \bigcap_s p(s) V$$

How close are you to $F(V)$. You want the image of the map

$$\mathrm{Hom}_A(hB, V) \longleftarrow Bh \otimes_A V$$

arises from $hb_1 \cdot b_2 h \cdot v$

compare to

$$\Lambda \otimes V \xleftarrow{\ \alpha\beta\ } \Lambda \otimes V$$

$$\begin{tikzcd}
\mathrm{Hom}_A(hB, V) & W \arrow[l, tail] \arrow[dl, "j"'] \arrow[dr, "i"] & Bh \otimes_A V \arrow[l, two heads] \\
V & V & V
\end{tikzcd}$$

$$\begin{tikzcd}
W & \Lambda \otimes V \arrow[l, two heads] & W \arrow[l, tail] \arrow[dl, two heads, "j"'] & \Lambda \otimes V \arrow[l, two heads] \\
& V \arrow[ul] & & V \arrow[ul, "i"']
\end{tikzcd}$$

Aim: Construct diagram

canonical map based on the pairing $hB \otimes_B Bh \longrightarrow hBh = A$

$$\begin{tikzcd}
(hb_1 \mapsto hb_1 b_2 h v) & b_2 h \otimes v \arrow[l, mapsto] \\
\mathrm{Hom}_A(hB, V) \arrow[d, "?"] & Bh \otimes_A V \arrow[l] \arrow[dl] \\
\Lambda \otimes V & W \arrow[l, tail, "\alpha"'] \arrow[ul] & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \arrow[ul, "?"]
\end{tikzcd}$$

$$\sum_s s \otimes \sum_t p(s^{-1}t) f(t) \longleftarrow \sum_t t\, c f(t) \qquad \sum t \otimes f(t)$$

$$hj \otimes v \longleftarrow v$$

$h = ij$

maybe the point is that $h \in Bh$

$$\mathrm{Hom}_B(W, \mathrm{Hom}_A(hB, V)) = \mathrm{Hom}_A(hB \otimes_B W, V)$$

$$\mathrm{Hom}_B(Bh \otimes_A V, W) = \mathrm{Hom}_A(V, \mathrm{Hom}_B(Bh, W))$$

Now there should be a map

$$\begin{tikzcd} hB\otimes_B W \arrow[rr] \arrow[dr, two heads] & & \mathrm{Hom}_B(Bh, W) \\ & hW \arrow[ur, tail] & \end{tikzcd}$$

$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

So you get

$$\begin{tikzcd} \mathrm{Hom}_B(Bh\otimes_A V, W) = \mathrm{Hom}_A(V, \mathrm{Hom}_B(Bh, W)) \\ \mathrm{Hom}_A(V, hW) \arrow[u] \end{tikzcd}$$

So there seem to be canonical maps in the right direction.

Feb 27, 02 Check ideas of last night. The point is to identify $W \mapsto hW$, $V \mapsto p(\Lambda\otimes V)$ with

$$G(W) = \mathrm{Im}\{hB\otimes_B W \xrightarrow{1} \mathrm{Hom}_B(Bh, W)\}$$

$$F(V) = \mathrm{Im}\{Bh\otimes_A V \xrightarrow{2} \mathrm{Hom}_A(hB, V)\}$$ resp.

Note that 1 factors $\quad hw \longmapsto (bh \mapsto bh * hw = bhw)$

$$\begin{array}{ccccc} hB\otimes_B W & \twoheadrightarrow & hW & \rightarrowtail & \mathrm{Hom}_B(Bh, W) \\ hb\otimes w & & hbw & & \end{array}$$

If $\uparrow = 0 \ \forall bh$ i.e. all b, then $hw = 0$ since W red.

$$\begin{tikzcd} \mathrm{Hom}_A(hB, V) & & Bh\otimes_A V \arrow[ll, "2"'] \arrow[dl, "3"] \\ \Lambda\otimes V & W \arrow[l, tail, "\beta"'] \arrow[ul, "4"] & \Lambda\otimes V \arrow[l, two heads, "\alpha"'] \end{tikzcd}$$

3 is $bh\otimes hw \mapsto bhw$ and is surj since $BhW = BhBW = BW = W$

4 is $w \longmapsto (hb \mapsto hbw)$

If $hbw = 0$ for all b then $hBw = 0 \Longrightarrow Bw = BhBw = 0$

$\therefore w = 0 \quad \therefore$ 4 inj.

compose 3 then 4 $\quad b_1h\otimes hw \xmapsto{3} b_1hw \xmapsto{4} (hb_2 \mapsto hb_2b_1hw)$

to write a version (permanent)

category $\mathcal{W}$ objects are Γ-mod W with $h \in \operatorname{End}_{\mathbb{C}}(W)$

sat $hsh \neq 0 \Rightarrow s \in \Phi$

$\forall w$ $\sum_{t \in \Gamma} tht^{-1} w = w$ (this means the sum is finite)

category $\mathcal{V}$ objects are v.s. V with $p(s) \in \operatorname{End}_{\mathbb{C}}(V)$

sat $p(s) \neq 0 \Rightarrow s \in \Phi$

$$\sum_t p(st^{-1}) p(t) = p(s).$$

$$\sum_s p(s) V = V, \quad \bigcap_s \operatorname{Ker} p(s) \text{ on } V = 0.$$

Construct: $\mathcal{W} \longrightarrow \mathcal{V}$ Given W, let $V = hW$,

let $j : W \xrightarrow{h} V$, let $i : V \xrightarrow{\text{inc.}} W$; $h = ij$ is canonical factor. into surj followed by inj.)

let $p(s) = jsi \in \operatorname{End}(V)$

i inj
j surj

$$p(s) \neq 0 \Rightarrow \underset{\underset{0}{\shortparallel}}{i\,p(s)\,j} = ijsij = hsh \Rightarrow s \in \Phi$$

$$\sum_t p(st^{-1}) p(t) v = \sum_t js\boxed{t^{-1} ij t} iv = jsiv = p(s)v.$$

$$V = jW = \sum_s j\overset{ij}{s}hW = \sum_s p(s) V$$

$\forall s$ $p(s)v = 0$ $\qquad \sum shs^{-1} iv = iv$

$$\shortparallel$$

$$0 = \sum si\underbrace{js^{-1}i}_{p(s^{-1})}v \Rightarrow iv = 0 \Rightarrow v = 0$$

Next show how to recover W from V. Let $\Lambda = \mathbb{C}\Gamma$, $\Lambda \otimes V$ is the free Γ module gen by V.

$$\Lambda \otimes V = \left\{ \sum_t t \otimes f(t) \,\middle|\, f : \Gamma \to V \text{ finite supp} \right\}$$

$$\Lambda\otimes V \xleftarrow{\ \alpha\ } W \xleftarrow{\ \beta\ } \Lambda\otimes V$$

$$\sum_t t\iota f(t) \longleftarrow\!\shortmid \sum t\otimes f(t)$$

$$\sum_s s\otimes js^{-1}w \longleftarrow\!\shortmid w$$

β is the unique Γ-mod map extending i

Show α well-defined, i.e. $s\mapsto js^{-1}w$ has finite support

use $W = \sum_t t i W = \sum_A t\iota V$, can suppose $w = t\iota v$

then $js^{-1}t\iota v = p(s^{-1}t)v \neq 0 \implies s^{-1}t \in \Phi$
or $s^{-1} \in \Phi t^{-1}$

Note α is the unique Γ-mod map coextending j

i.e. ∃ $\eta_1\alpha = j$ $\qquad t\sum_s s\otimes js^{-1}w = \sum_s ts\otimes js^{-1}w$

$= \sum_s s\otimes j(t^{-1}s)^{-1}w = \sum_s s\otimes js^{-1}(tw)$. Also

$$\beta\alpha(w) = \sum_s s\iota js^{-1}w = w.$$

$$\alpha\beta\Big(\sum_t t\otimes f(t)\Big) = \sum_s s\otimes\sum_t \underbrace{js^{-1}t i}_{p(s^{-1}t)} f(t)$$

Therefore you can recover W as the image of $\overbrace{p=\alpha\beta}^{\text{the projection}}$.
on $\Lambda\otimes V$.

Let $V\in\mathcal{V}$ define $\Lambda\otimes V \xleftarrow{\ p\ } \Lambda\otimes V$

by $p\big(\sum_t t\otimes f(t)\big) = \sum_s s\otimes\sum_t p(s^{-1}t)f(t)$. Properties

$p\cdot u = u\cdot p$, $p^2 = p$.

$$u\sum_t t\otimes f(t) = \sum t\otimes f(u^{-1}t)$$

$$pu\sum_t t\otimes f(t) = \sum_s s\otimes\sum_t p(s^{-1}t)f(u^{-1}t)$$

$$= \sum_s s\otimes\sum_t p(s^{-1}ut)f(t) = \sum_s us\otimes\sum_t p($$

$up\sum_t t\otimes f(t)$

$= u\sum_s s\otimes\sum_t p(s^{-1}t)f(t)$

$$\Lambda\otimes V \xleftarrow{\;p\;} \Lambda\otimes V$$

$$p\left(\sum_t t\otimes f(t)\right) = \sum_s s\otimes \sum_t p(s^{-1}t)\, f(t)$$

$$p\left(u\sum_t t\otimes f(t)\right) = p\left(\sum_t ut\otimes f(t)\right) = p\left(\sum_t \cancel{uu^{-1}}t \otimes f(u^{-1}t)\right)$$

$$= \sum_s s\otimes \sum_t p(s^{-1}t)\, f(u^{-1}t) = \sum_s s\otimes \sum_t p(s^{-1}ut)\, f(\cancel{u^{-1}}ut)$$

$$= \sum_s us\otimes \sum_t p(\underbrace{(us)^{-1}ut}_{s^{-1}t})\, f(t) = u\, p\left(\sum_t t\otimes f(t)\right).$$

$$pp\left(\sum_u u\otimes f(u)\right) = p\left[\sum_t t\otimes \sum_u p(t^{-1}u) f(u)\right]$$

$$= \sum_s s\otimes \sum_t p(s^{-1}t) \sum_u p(t^{-1}u)\, f(u)$$

$$= \sum_s s\otimes \sum_u \underbrace{\left(\sum_t p(s^{-1}t)\, p(t^{-1}u)\right)}_{p(s^{-1}u)} f(u)$$

$W = p(\Lambda\otimes V)$. To find i, j $\quad W \xleftarrow{\;i\;} V \xleftarrow{\;j\;} W$

$$\begin{tikzcd}
\Lambda\otimes V \arrow[d, "\eta_1"] & W \arrow[l, "\alpha"'] \arrow[dl] & \Lambda\otimes V \arrow[l, "\beta"'] \arrow[ll, bend right, "p"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, "\alpha"'] \\
V & & V \arrow[u, shift left, "\varepsilon_1"] \arrow[ul, "i"] &
\end{tikzcd}$$

Think of α as the inclusion β as the proj.

$$\overline{\sum t\otimes \delta_1(t) v}$$

$$\iota v = p\,\overline{\varepsilon_1 v} = \sum_s s\otimes p(s^{-1}) v$$

any elt of W

$$\sum_s s\otimes \sum_t p(s^{-1}t) f(t) = p\left(\sum t\otimes f(t)\right)$$

$$j\left(p\left(\sum t\otimes f(t)\right)\right) = \eta_1 \sum_s s\otimes \sum_t p(s^{-1}t) f(t) = \sum_t p(t) f(t).$$

$$h\left(p\sum t\otimes f(t)\right) = i\sum p(t) f(t) = \sum_s s\otimes p(s^{-1}) \sum_t p(t) f(t)$$

Seems too hard.

Somehow you should be able to start with V and construct W. Given V with the operators $p(s)$ define p on $\Lambda\otimes V$ by $p\left(\sum_t t\otimes f(t)\right) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t)$.

Then p is the linear op on column vectors indexed by $t\in\Gamma$ given by matrix $p(s^{-1}t)$. Then because this kernel is invariant under $(s,t)\mapsto(us,ut)$ (under left translation) $\boxed{pu = up}$, one has $p^2=p$ as $\sum_t p(s^{-1}t)p(t^{-1}u) = p(s^{-1}u)$

$\sum_{u=st} p(s)p(t) = p(u)$. Then define $W = p(\Lambda\otimes V)$ so you get W retract of $\Lambda\otimes V$

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift right, "\eta_1"] & W \arrow[l, "\alpha"'] \arrow[dl, "j"] \\ & V \arrow[u, shift right, "\varepsilon_1"] \arrow[ul, "i"] \arrow[ur, bend right=60, "h"'] & \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, shift left] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "j"] \\ & V \arrow[ul, "i"] \arrow[u, shift left, "\varepsilon_1"] \end{tikzcd} \qquad p = \alpha\beta$$

$$1_{\Lambda\otimes V} = \sum s\varepsilon_1\eta_1 s^{-1}$$

$$1 = \beta\alpha = \sum \beta s\varepsilon_1\eta_1 s^{-1}\alpha = \sum s h s^{-1}$$

Want matrix elements $\quad \eta_1 s^{-1}\alpha\beta t\varepsilon_1 = j s^{-1} t i = p(s^{-1}t)$

Review: Given V with $p(s)$, get $p(s^{-1}t)$ inv. under left mult $s,t\longmapsto us,ut$ and idemp. ~~Get~~ proj p on $\Lambda\otimes V$ $pu=up$, $p^2=p$, know $\eta_1 s^{-1}pt\varepsilon_1 = p(s^{-1}t)$

$$\eta_1 s^{-1}\alpha\beta t\varepsilon_1 = \eta_1\alpha s^{-1}t\beta\varepsilon_1 = j(s^{-1}t)i$$

explain α,β

Given V, $\Lambda = \mathbb{C}\Gamma$, $\Lambda\otimes V = \{f:\Gamma\to V \mid \text{fin supp}\}$

$(uf)(t) = f(u^{-1}t)$, define $(pf)(s) = \sum_t p(s^{-1}t)f(t)$. Then

$(puf)(s) = \sum_t p(s^{-1}t)f(u^{-1}t) = \sum_t p(s^{-1}ut)f(t)$, $(u(pf))(s) = (pf)(u^{-1}s) = \sum_t p(s^{-1}ut)f(t)$

$pu=up$, $p^2=p$. Let $W = p(\Lambda\otimes V)$, $\beta = p:\Lambda\otimes V\to W$, $\alpha = \text{inc}\ W\to\Lambda\otimes V$

W is a Γ-module retract of $\Lambda\otimes V$

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift right, "\eta_1"] & W \arrow[l, "\alpha"'] \arrow[dl, "j=\eta_1\alpha"] \\ & V \arrow[u, shift right, "\varepsilon_1"] \arrow[ul, "i=\beta\varepsilon_1"] & \end{tikzcd} \qquad \begin{aligned} \sum s\varepsilon_1\eta_1 s^{-1} &= 1_{\Lambda\otimes V} \\ \sum s\,\underbrace{ij}_{h}\,s^{-1} &= 1_W \end{aligned}$$

Start with V with $p(s)$ $\quad p(s) \neq 0 \Rightarrow s \in \overline{\Phi}$

$$\sum_{u=st} p(s)p(t) = p(u)$$

free Γ module $\Lambda \otimes V = \{ f : \Gamma \to V \mid \text{fin. supp.} \}$ $\quad (uf)(t) = f(u^{-1}t)$

define $(pf)(s) = \sum_t p(s^{-1}t) f(t)$, $\quad pu = up$, $p^2 = p$

let $W = p(\Lambda \otimes V)$, $\quad \beta = p : \Lambda \otimes V \twoheadrightarrow W$, $\quad \alpha = \text{inc.} : W \to \Lambda \otimes V$

then α, β are Γ-maps

$\ni \beta\alpha = 1_W$, $\alpha\beta = p$. $\quad$ Let $\underbrace{\iota = \beta\varepsilon_1}_{: V \to W}$, $j = \eta_1\alpha : W \to V$, $\quad h = ij$
$= \beta\varepsilon_1\eta_1\alpha$

Then $1_{\Lambda \otimes V} = \sum s\varepsilon_1\eta_1 s^{-1} \quad \Rightarrow \quad \sum s h s^{-1} = 1$

Digression: Intermediate cat $\mathcal{U}$ of $u = (V, W, i, j)$ where W is a Γ-module, V a vector space, and $i : V \to W$, $j : W \to V$ are $\mathbb{C}$-linear maps satisfying

$$jsi \neq 0 \Rightarrow s \in \overline{\Phi}, \quad \sum_s sijs^{-1} w = w \quad \forall w \in W$$

i inj, j surj.

functor $\mathcal{U} \to \mathcal{V}$ $\quad (V, W, i, j) \longmapsto V$ with $p(s) = jsi$

$$\sum_s sijs^{-1}w = w \Rightarrow \sum_s p(s)\, js^{-1}w = jw \Rightarrow \sum p(s) V = V$$

Ass $\forall s$ $p(s)v = 0$, i.e. $\forall s$ $jsiv = 0$, $\Rightarrow 0 = \sum_s sijs^{-1}iv = iv$, $\therefore v = 0$.

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, "\eta_1"] & W \arrow[l, "\alpha"'] \arrow[dl, "j"] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \\
 & V \arrow[ul, "i"] & & V \arrow[ul, "i"'] \arrow[u, "\varepsilon_1"']
\end{tikzcd}$$

Confused!

$\mathcal{U}$ = cat of $V, W, \iota : V \to W$, $j : W \to V$ where W is Γ module, V v.s., ι inj, j surj $\ni$

$$jsi \neq 0 \Rightarrow s \in \overline{\Phi} \quad (\text{maybe } js^{-1}ti \neq 0 \Rightarrow s^{-1}t \in \overline{\Phi})$$

$$\forall w \quad \sum_s sijs^{-1}w = w$$

functor $\mathcal{U} \to \mathcal{V}$ $\quad (V, W, \iota, j) \longmapsto V$ with $p(s) = jsi$

$p(s) \neq 0 \Rightarrow s \in \Phi$ ✓ $\qquad \sum_{s=tu} p(t)p(u) = \sum_{s=tu} jtijui$

$\sum_t jtijt^{-1}si = jsi = p(s)$

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, "\beta"'] \\ & V \arrow[u, "\xi_1"'] \arrow[ul, "j"] \end{tikzcd}$$

$$W = \sum_s siV \Rightarrow V = jW = \sum_s p(s)V$$

$$\begin{tikzcd} \Lambda\otimes V \arrow[d] & W \arrow[l, "\alpha"'] \\ V & V \arrow[u, "i"'] \end{tikzcd}$$

$$0 = p(s)v = jsiv, \ \forall s \Rightarrow 0 = \sum_s s^{-1}ijsiv = iv \Rightarrow v = 0.$$

have just showed $\forall (V, W, i, j) \in \mathcal{U}$ that $(V, p(s) = jsi) \in \mathcal{V}$

Digress to review the Toeplitz algebra, simplest case. It the unital algebra R gen. x, y subject to reln $yx = 1$. Natural R-module $\mathbb{C}[z]$ with $xz^n = z^{n+1}$ and $yz^n = z^{n-1}$ for $n \geq 1$, and $y1 = 0$. R is spanned by words in x, y which can be replaced by words of smaller length where a y is followed by ~~any~~ x. R spanned by $x^m y^n$ with $m, n \geq 0$.

Start with V, produce W,

$$\Lambda\otimes V = \{\sum_t t\otimes f(t) \mid f : \Gamma \to V \text{ fin supp}\}$$

$$u\sum_t t\otimes f(t) = \sum_t t\otimes f(u^{-1}t)$$

define
$$p\Big(\sum_t t\otimes f(t)\Big) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t) \qquad \boxed{\begin{matrix} pu = up \\ p^2 = p \end{matrix}}$$

$$\begin{aligned} pu\Big(\sum t\otimes f(t)\Big) &= p\sum_t t\otimes f(u^{-1}t) \\ &= \sum_s s\otimes \sum_t p(s^{-1}t)f(u^{-1}t) \\ &= \sum_s s\otimes \sum_t p(s^{-1}ut)f(t) \\ &= \sum_s us\otimes \sum_t p(s^{-1}\cancel{u^{-1}u}t)f(t) = up\Big(\sum t\otimes f(t)\Big) \end{aligned}$$

Define $W = p(\Lambda\otimes V)$. $\quad W$ Γ-module retract of $\Lambda\otimes V$

means $\exists$ Γ-maps

$$\begin{tikzcd} \Lambda\otimes V & W \arrow[l, tail, "\alpha"'] \arrow[dl, "j"] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \\ V & & V \arrow[ul, "i"] \arrow[u, shift left, "\xi_1"] & \end{tikzcd}$$

Now you've defined $W, \iota, j, h = \iota j$

$hsh \neq 0 \implies \overset{p(s)}{js i} \neq 0 \implies s \in \Phi.$

$\sum shs^{-1}w = \sum s\beta\varepsilon_1\eta_1\alpha s^{-1}w = \sum \beta s\varepsilon_1\eta_1 s^{-1}\alpha w = \beta\alpha \underset{=w}{w}$

So W has the desired properties. So from V you have constructed W with all its properties, although you haven't used V reduced. $jW = j(\beta \Lambda\otimes V) = \eta_1\alpha\beta(\Lambda\otimes V)$

$$= \left\{ \sum_t p(t)f(t) \,\middle|\, f : \Gamma \underset{\text{fin. supp}}{\to} V \right\} = \sum p(s)V$$

So if $p(s)v = 0$ $\forall s$ then $\alpha\iota v = 0$

$$\alpha\iota v = p\varepsilon_1 v = \sum_s s\otimes p(s^{-1})v$$

$$\iota v = 0 \iff (\forall s)\ p(s^{-1})v = 0$$

$\therefore$ V reduced $\iff$ ι inj + j surj

W Γ-mod w h $\qquad hsh \neq 0 \implies s \in \Phi$

$$\sum_s shs^{-1}w = w$$

$V = hW \qquad h = \iota j : W \hookleftarrow V \twoheadleftarrow W$

$p(s) = jsi \in \mathrm{End}(V) \qquad p(s) \neq 0 \implies hsh \neq 0 \implies s \in \Phi$

$\iota j s\iota j \neq 0 \implies p(s) \neq 0$

ι inj

j surj

Crazy idea: nil modules could they be states with zero energy? Is there something interesting arising from $A = P_{\Gamma,\Phi}$-modules which are not reduced?

So given V explain $\Lambda\otimes V$ $\quad u\left(\sum_t t\otimes f(t)\right) = \sum_t t\otimes f(u^{-1}t)$

$$p\left(\sum_t t\otimes f(t)\right) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t)$$

$hsh = \iota j s \iota j \qquad$ need $jsi = p(s)$

better $js^{-1}ti = p(s^{-1}t)$

You failed to discuss $\Lambda\otimes V$ adequately. $\qquad p = \alpha\beta$

orth. $\eta_1 s^{-1}t\varepsilon_1 = \delta_{st}$ $\qquad$ $\varepsilon_1 \uparrow \downarrow \eta_1$ V $\qquad \eta_1 s^{-1}pt\varepsilon_1 = \eta_1\alpha s^{-1}t\beta\varepsilon_1$

$\sum_s s\varepsilon_1\eta_1 s^{-1} = 1_{\Lambda\otimes V}. \qquad = j s^{-1}t\iota = p(s^{-1}t).$

What do you want for $\Lambda\otimes V$? $\Lambda\otimes V = \{ f:\Gamma\to V \text{ fin supp}\}$

Γ actions. Want splitting $\Lambda\otimes V = \bigoplus_s V$ need ε_1, η_1

and $\eta_1 s^{-1} t \varepsilon_1 = \delta_{st}$ on V, $\sum s\varepsilon_1\eta_1 s^{-1} = 1_{\Lambda\otimes V}$

$\varepsilon_1 v = 1\otimes v$

$\eta_1 \sum_t t\otimes f(t) = f(1)$ $\Lambda\otimes V$ is the d. sum of subspaces $s\otimes V$

IDEA. orthogonality & completeness relations, the latter can be compressed to a summand. So far you've been looking at discrete cases, but the holomorphic representation of the CCR gives completeness but not orthogonality in a continuous setting

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There are canon. maps

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "j"] \\ & V \arrow[u, shift left, "\varepsilon_1"] \arrow[ul, "\iota"] & \end{tikzcd}$$

$\beta\alpha = 1_W$, $\alpha\beta = p$

Let $j = \eta_1\alpha$, $\iota = \beta\varepsilon_1$, $h = \iota j$

Claim W with $h \in \mathcal{W}$.

$hsh = \iota j s \iota j$ need $\eta_1 \alpha s \beta \varepsilon_1 = p(s)$.

$\qquad\qquad\qquad\qquad\qquad \eta_1 s p \varepsilon_1 \qquad \eta_1 s p \varepsilon_1 = p$?

confused again. $p(s^{-1}t) = \eta_1 s^{-1} p t \varepsilon_1$

$1_W = \beta\alpha = \beta \sum s\varepsilon_1\eta_1 s^{-1}\alpha$

Next composition. $W \longmapsto V = hW$ and then you must construct $W = p(\Lambda\otimes V)$. What you have is just ι, j, Γ on W. Other side is probably easy, namely

$V \mapsto p(\Lambda\otimes V) \longmapsto hp(\Lambda\otimes V)$ why?

$$\begin{tikzcd} \Lambda\otimes V & \overbrace{p(\Lambda\otimes V)}^{W} \arrow[l, tail, "\alpha"'] \arrow[dl, "j"'] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] & p(\Lambda\otimes V) \arrow[l, tail, "\alpha"'] \\ V & & V \arrow[u, "\varepsilon_1"] \arrow[ul, "\iota"] & \end{tikzcd}$$

$\alpha\iota v = p\varepsilon_1 v$
$= \sum s\otimes p(s^{-1})v$

$jW = \eta_1 p(\Lambda\otimes V)$ $\qquad \eta_1 \sum s\otimes \sum_t p(s^{-1}t)f(t) = \sum_t p(t)f(t)$

Next you to go from W to $V = hW$ to $p(\Lambda \otimes V)$. Given W you form: can fact.

$$\begin{tikzcd} W & \Lambda\otimes V \arrow[l, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, "\alpha"'] \arrow[dl, "j"] \\ & V \arrow[ul, "\iota"] \arrow[u, shift left, "\varepsilon_1"] & \end{tikzcd}$$

$$\beta \sum_t t \otimes f(t) = \sum_t t\iota f(t)$$

$$\alpha w = \sum_s{}' s_1 \otimes j s^{-1} w$$

Claim β = unique Γ-module map ext. ι in the sense that $\beta\varepsilon_1 = \iota$

α unique Γ-module map coext. j in the sense that $\eta_1 \alpha = j$.

Ch $\quad u\alpha w = \sum_s us \otimes j s^{-1} w = \sum s \otimes j s^{-1} u w$

$\alpha w \qquad j s^{-1} w \qquad (u\alpha w)(s) = \alpha w(u^{-1}s) = j(u^{-1}s)^{-1} w = j s^{-1}(uw)$

What's left is to calculate that $\quad \beta\alpha = \beta \sum s \otimes j s^{-1} w$
$= \sum s \iota j s^{-1} w = w$

$$\alpha\beta \sum t \otimes f(t) = \alpha \sum_t t\iota f(t)$$
$$= \sum_s s \otimes \sum_t j s^{-1} t \iota f(t)$$

Next you want to identify $\mathcal{V}$ and $\mathcal{W}$ with $\mathcal{M}_r$ for idempotent rings

IDEA: Recall you GNS algebra associated to linear map $\rho : A \longrightarrow B$ between unital rings satisfying $\rho 1 = 1$. Module category consists of A-module M, B-module N, and maps $N \xrightarrow{\iota} M \xrightarrow{j} N$ satisfying $\rho(a)n = j a \iota n$.

Question: Is there some nonunital version of this? GNS-alg is $\Gamma(\rho : A \to B) = A \oplus A \otimes B \otimes A$
$a_1 \otimes b \otimes a_2 \longmapsto a_1 \iota b j a_2$.

Dilation of factorization B-module N; you want to construct M, can use any

$$A \otimes N \xrightarrow[\text{extn}]{\tilde{\iota}} M \xrightarrow[\text{coext}]{\tilde{j}} \mathrm{Hom}(A, N)$$
$$a \otimes n \longmapsto a\iota n \longmapsto (a' \longmapsto (j a' a \iota) n) \qquad \rho(a'a)$$
$$m \longmapsto (a' \longmapsto j a' m)$$

minimal M is the image of this $\tilde{j}\tilde{\iota}$ map.

So concentrate on idemp. rings. Clearly $\mathcal{V}$ is the cat. of $M_r(A)$

$A = \mathcal{P}_{\Gamma,\Phi} \qquad B = \Gamma \ltimes \mathcal{C}_{\Gamma,\Phi}$. So what can

$$\begin{pmatrix} hBh & hB \\ Bh & B \end{pmatrix}$$

First examine the idea of completeness without orthogonality. You have free case

$$\Lambda \otimes V = \bigoplus_s s \otimes V = \bigoplus_s V$$

$$\eta_s\Big(\sum_t t \otimes f(t)\Big) = f(s) \qquad \Lambda \otimes V \underset{\varepsilon_t}{\overset{\eta_t}{\rightleftarrows}} V \qquad \eta_s = \eta_1 s^{-1}$$

$$\varepsilon_t v = t \otimes v \qquad \varepsilon_t = t\varepsilon_1$$

orth $\eta_s \varepsilon_t = \eta_1 s^{-1}t\, \varepsilon_1 = \delta_{st}$

$$\sum_s s\, \varepsilon_1 \eta_1 s^{-1} = 1_{\Lambda \otimes V}$$

I think you encountered this before when you tried to construct a Morita context by gens + rels. The generators are off diagonal & they yield the primary maps between V & W. Morita context

$$W \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \Lambda \otimes V \underset{t\varepsilon_1}{\overset{\eta_1 s^{-1}}{\rightleftarrows}} V \qquad W \xrightarrow{\;js^{-1}\;} V \qquad V \xrightarrow{\;ti\;} W$$

$$\begin{pmatrix} A & y_s = j s^{-1} \\ t i = x_t & B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

It seems that you defined D to be the M_2-graded algebra with generators x_t of degree $(2,1)$ and y_s of degree $(1,2)$. Here $s, t \in \Gamma$. The relations are

$\sum_s x_s y_s = 1$ in Cuntz's sense i.e. $\sum_s x_s y_s x_t = x_t$

and $\sum_t y_s x_t y_t = y_s$ need $y_s x_t \neq 0 \Rightarrow s^{-1}t \in \Phi$

Recall you need also the left Γ-invariance: $y_s x_t$ depends only on $s^{-1}t$.

A generated by $y_s x_t = j s^{-1} t i$ $\quad \Gamma$

B generated by $x_t y_s = t i j s^{-1}$ $\quad \Gamma \times \Gamma$

$= t s^{-1} h_s = h_t t s^{-1}$

<u>All this is fascinating</u>, but

dual pair over B with $\quad Y = B$

$X = B \quad B$

pairing
$(b_1, b_2) \mapsto b_1 h b_2$
$(x, y) = x h y$

then $A = Y \otimes_B X$

dual pair over B consisting of $X = B$ left mult, $Y = B$ right mult

$\langle x | y \rangle = xhy$ assume $BhB = B$

$A = Y \otimes_B X = B$ if B firm

$(yx)y = yxhy$ $ay = ahy$

$x(yx) = xhyx$ $xa = xha$

So you get a Morita context $\begin{pmatrix} A = B & Y = B \\ X = B & B \end{pmatrix}$

$$\begin{matrix} YX & Y \\ X & B \end{matrix}$$

$$\begin{pmatrix} y_1 x_1 & y_1 \\ x_1 & b_1 \end{pmatrix} \begin{pmatrix} y_2 x_2 & y_2 \\ x_2 & b_2 \end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix} y_1 x_1 h y_2 x_2 + y_1 x_2 & y_1 x_1 h y_2 + y_1 b_2 \\ x_1 h y_2 x_2 + b_1 x_2 & x_1 h y_2 + b_1 b_2 \end{pmatrix}$$

$$\begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

right module

$$\begin{pmatrix} M & N \end{pmatrix} \begin{pmatrix} jBi & jB \\ Bi & B \end{pmatrix}$$

$M \otimes_A jB \to N$

$N \otimes_B Bi \to M$

$$M \xrightarrow{\cdot j} N \xrightarrow{\cdot i} M$$

$$\begin{tikzcd} N \arrow[r, "\cdot i"] \arrow[rr, bend right, "\cdot ij"'] & M \arrow[r, "\cdot j"] & N \end{tikzcd}$$

the only problem is that canonically factoring

$\cdot h = \cdot i \cdot j$ $\cdot i$ is surjective

$\cdot j$ - injective

IDEA: This pretending an operator h is idempotent, could it generalize to a chain of operators $\to \to \overset{?}{\to}$, and perhaps be useful for higher K-theory purposes?