The principal bundle for $\mathbb{L} = \mathbb{R}^2 \times \mathbb{C}$ is $\mathbb{T} \times \mathbb{R}^2$.

You want to exploit the translation invariance of the curvature. If there existed a translation-invariant 1-form $A$ with zero curvature, then the translation group $\mathbb{R}^2$ would act preserving the connection. In nonzero curvature case you can lift translations $(x, y) \rightarrow (x + s, y + t)$ to $L$ preserving the connection.

Consider over $\mathbb{R}^2$ the complex line bundle $L$ + connection $D$ whose curvature is $2\pi i \, dx \wedge dy$. (Any two such line bundles are isomorphic up to a constant scalar factor $\in \mathbb{U}$.) Note: $L = \text{twisted line bundle}$, $D = d + 2\pi i \, x \, dy$. You want to find the group of automorphisms of this geometric object: differ $\mathbb{R}^2$ + lifting to $L$ which preserves $D$.

Look at a differ $\mathbb{R}^2$ given by a translation $g^*(y) = \left( x + s \atop t + y \right)$

$$g^*(d + 2\pi i \, x \, dy) \phi = (d + 2\pi i \, (x + s) \, dy)(g^* \phi)$$

$$e^{2\pi i y}(d + 2\pi i \, x \, dy) e^{2\pi i y}$$

So it seems that you get an action of a Heisenberg group on $(\mathbb{R}^2, L)$ preserving $D$. It's probably a right action because $[2\pi i, x, y] = -2\pi i$. This would fit with the idea that there is a lift action given by the components of $D$: $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$

You should look at the principal bundle for $L$

$$L = \mathbb{C} \times \mathbb{R} \times \mathbb{R} \quad \text{coords } w, x, y$$

a section is $(\psi(x, y), x, y)$
Principal bundle \( P = \tilde{T} \times G \). You need to understand better the relation between the connection in the principal bundle, which, in general, is a Lie alg valued 1-form \( \theta \) restricting to the Maurer-Cartan form of \( G \) at each point of the base, and the connection in the associated \( \text{Vch} \).

\[ D = \partial + \theta. \]

MC form is \( g^*d\theta \), you take the variation \( g^*d\theta \) and you left mult. by \( g \). to get \( 1+g^*d\theta \).

\[ P = \tilde{T} \times G \quad \text{MC form on } \tilde{T} \text{ should be } z^{-1}dz \]

connection form should be \( z^{-1}dz + 2\pi i \times dy \). Put \( z = e^{i\theta} \) so that \( z^{-1}dz = i d\theta \). What is the horizontal lift of the vector field \( \partial_x \)? Look for \( \tilde{\partial}_x + f(x,y) \partial_\theta \), such that

\[
(\tilde{\partial}_x + f(x,y) \partial_\theta)((i\theta + 2\pi i \times dy)) = 0
\]

\[
\tilde{\partial}_x = \partial_x, \quad \tilde{\partial}_y = \partial_y + g \partial_\theta
\]

\[
\int (f(x,y)) \partial_\theta = 0
\]

\[
\text{connection form is } \sqrt{i}(d\theta + 2\pi i \times dy). \text{ The lift of } \partial_x \text{ which is horizontal is } \tilde{\partial}_x; \text{ the horizontal lift of } \partial_y \text{ is } \partial_y + f \partial_\theta \text{ where } f + 2\pi x = 0. \text{ So you seem to have a sign problem.}
\]

If the principal bundle \( P \) is to turn out to be the Heisenberg group, then the structure shouldn't depend on a sign.

So back to the picture of the Heisenberg group as a circle bundle over \( \mathbb{R} \). What you want is to stand with \( \mathbb{H} \) acting on itself by left \& right mult
Feb 18, 02

Let us consider the infinitesimal translation operators

\[
\begin{bmatrix}
D_x = \partial_x \\
D_y = \partial_y + 2\pi i x
\end{bmatrix}, \quad
\begin{bmatrix}
\nabla_x = \partial_x + 2\pi i y \\
\nabla_y = \partial_y
\end{bmatrix} = \mathbf{0}
\]

\[
[D_x, D_y] = 2\pi i, \quad [\nabla_x, \nabla_y] = -2\pi i
\]

These are differential operators acting on \( C^\infty(\mathbb{R}^2) \).

You can exponentiate these differential operators to get a simple PDE like one for \( R^2 \) with regular skew symmetrical 1-forms in \( \mathbb{R} \).

These are differential operators acting on sections of the trivial line bundle over \( \mathbb{C} \).

Your aim is to get a statement about the line bundle \( L \). Begin with the Heisenberg groups. Just as the line bundle \( L + \text{connection} \) can be presented using a skew 1-form (bilinear form on \( \mathbb{R}^2 \) with regular skew symmetrical), so can the Heisenberg groups be presented using a bilinear form.

So you might try to do both using \( 2\pi i \times dy \) first and then passing to the general case.

Better idea: Take a suitable presentation for \( H \) then look at infinitesimal left + right multi, i.e. look at the vector field on \( H \) that you get.

Feb 19, 02

Presentation: \( H = \mathbb{T} \times \mathbb{R} \times \mathbb{R} \) typical element \( \begin{bmatrix} z_1 e^{a_1 x} e^{b_1 y} \end{bmatrix} \) mult. \( z_1 e^{a_1 x} e^{b_1 y} z_2 e^{a_2 x} e^{b_2 y} \)

\( = z_1 z_2 e^{-2\pi i b_1 b_2} e^{(a_1 + a_2) x} e^{(b_1 + b_2) y} \)

let \( G \times G^0 \) act on \( G \) \( (g_1 g_2) x = g_1 x g_2^{-1} \). Stabilizer, \( g_1 x \) is \( \{ (g_1 g_2) \mid g_1 x g_2^{-1} = x \} \)

\( g_1 x = x g_2 \quad g_1 x = x g_2 \quad g_2 x = g_1 x g_2 \quad g_2 x = x g_2 g_1 \)

Problem remains to determine whether you can identify the Heisenberg group \( H \) with the principal \( \mathbb{T} \)-bundle over \( \mathbb{R}^2 \) having curvature \( 2\pi i \times dx \times dy \). Since you a uniqueness result
for the principal bundle you should be able to produce the desired isomorphism by constructing a connection on the principal $T$-bundle

$$
\begin{array}{c}
T \\
\rightarrow \\
H \\
\rightarrow \mathbb{R}^2
\end{array}
$$

having the desired curvature. At this point you need to describe $H$ precisely, namely, as a set + mult. group.

From theory of group extensions where the quotient is abelian there is an invariant, namely the commutator: take two elements of $\mathbb{R}^2$, lift them to $H$ and take commutator. This gives a skew-symmetric bilinear pairing on $\mathbb{R}^2$ with values in $T$, which lifts to $\text{Lie}(T) = i\mathbb{R}$ — think of the universal covering of $H$.

Let’s go over this again. You want to identify the Heisenberg group $H$ with the principal $T$ bundle $\mathbb{R}^2$ having curvature and thereby you have a uniqueness result for $P$, so it should suffice to construct a connection on the principal $T$-bundle given by the group extension

$$
\begin{array}{c}
T \\
\rightarrow \\
H \\
\rightarrow \mathbb{R}^2
\end{array}
$$

having the desired curvature.

What do you know about this group extension, in fact, what do you know about a central extension of an abelian group? Commutator pairing

Think generically. Consider agrp extn $\begin{array}{c}
B \\
\rightarrow \\
E \\
\rightarrow A
\end{array}$

where $A, B$ abelian and $B$ is in the center of $E$. To study this you choose a section $s$ of $\pi$, which gives a bijection $B \times A \rightarrow E$, $(b, a) \mapsto i(b)s(a)$, in terms of which the product in $E$ is $i(b_1)s(a_1)i(b_2)s(a_2) = i(b,b_2)s(a_1)s(a_2) = i(b_1b_2 f(a_1,a_2))s(a_1a_2)$ where $f: A \times A \rightarrow B$ satisfies
the 2-cocycle condition
\[ f(a_1, a_2) - f(a_1, a_3) + f(a_2, a_3) - f(a_1, a_2) = 0 \]
if product in B is written additively. Suppose we use + for both A, B. Commutator pairing

\[ (a_1, a_2) \mapsto s(a_1)s(a_2)s(a_1)^{-1}s(a_2)^{-1} \]

\[ s(a_1)s(a_2) = i(f(a_1, a_2)) s(a_1 + a_2) \]
\[ s(a_2)s(a_1) = i(f(a_2, a_1)) s(a_1 + a_2) \]

\[ s(a_1)s(a_2)s(a_1)^{-1}s(a_2)^{-1} = i(f(a_1, a_2) - f(a_2, a_1)) \]

So you learn that the 2-cocycle \( f(a_1, a_2) \) when skew-symmetric is bilinear. In fact this should be true for any 2-cocycle \( f: A \times A \to B \).

Make universal construction: \( Z^2(A, B) = \text{Hom}(\Lambda^2 A, B) \). Note we have map \( Z^2(A, B) \to \text{Hom}(\Lambda^2 A, B) \) and recall \( \Lambda^2 A \to H_2(A) \) ?

Universal Coeffs.

\[ 0 \to \text{Ext}^1(H_1(B^{\#}A), B) \to H^2(B^{\#}A, B) \to \text{Hom}(H_2(B^{\#}A), B) \to 0 \]
\[ 0 \to \text{Ext}^1(A, B) \to H^2(A, B) \to \text{Hom}(\Lambda^2 A, B) \to 0 \]

It appears that \( Z^2(A, B) \to \text{Hom}(\Lambda^2 A, B) \) is surjective, i.e. any commutator pairing arises from a central extension. This seems to imply that there is at least one central extension \( \Lambda^2 A \to E \to A \) whose commutator pairing is the identity of \( \Lambda^2 A \).
Let \( S: C'(A, B) \to Z(A, B) \)

\[(\delta f)(a_1, a_2) = f(a_2) - f(a_1 + a_2) + f(a_1)\]

\(\delta f\) \text{ is } \mathbb{Z}\text{-bilinear NC}

\(\delta f(a_0 + a_1, a_2) = f(a_2) - f(a_0 + a_1 + a_2) + f(a_0 + a_1)\)

\(\delta f(a_0, a_2) = f(a_2) - f(a_0 + a_2) + f(a_0)\)

\(\delta f(a_1, a_2) = f(a_2) - f(a_1 + a_2) + f(a_1)\)

\[f(a_1, a_2) - f(a_1 + a_2, a_2) + f(a_1, a_2 + a_3) - f(a_1, a_2) = 0\]

Review central extensions of abelian groups

\[\begin{array}{ccc}
B & \xrightarrow{i} & E & \xrightarrow{\pi} & A \\
\text{commutator pairing} & \phi & \text{commutator pairing} & & \phi & \text{commutator pairing} \\
\text{where } \tilde{a}_1, \text{ any elt of } \pi^{-1}a_1. & & \text{By universal property} & & \text{short exact sequence} \\
\text{Ext}_Z^1(A, B) & \to & H^2(A, B) & \to & \text{Hom}(\Lambda^2 A, B) \\
\end{array}\]

\[Z^2(A, B) / SC'(A, B)\]

and \(\text{Hom}(A \otimes A, B) \subset Z^2(A, B)\), i.e. any \(\mathbb{Z}\)-bilinear \(f(a_1, a_2)\) is a \(\mathbb{Z}\)-cocycle. Question: Does every extension arise from a bilinear cocycle? Equiv. Is the comp.

\[\text{Hom}(A \otimes A, B) \subset Z^2(A, B) \to H^2(A, B)\]

surjective? Another point is that the following square is cartesian

\[\begin{array}{ccc}
\text{by definition of quadratic map } g: A \to B \text{ and } \delta f \text{ of } \Gamma_2(A) \\
\end{array}\]

Quadratic maps \(g: A \to B\)

\[\begin{array}{ccc}
\text{Hom}_Z(A \otimes A, B) & \to & Z^2(A, B) \\
\end{array}\]
There seems to be an exact sequence

\[ \Lambda^2 A \rightarrow A \otimes A \rightarrow S_2(A) \rightarrow 0 \]

and a canonical map \( S_2(A) \rightarrow \Gamma_2(A) \) whose kernel and cokernel are killed by 2. Maybe also

\[ \Gamma_2(A) \rightarrow A \otimes A \rightarrow \Lambda^2 A \rightarrow 0 \]

Combine:

\[ \begin{array}{c}
\Lambda^2 A \\
A \otimes A \\
S_2 A \\
0
\end{array} \]

\[ \begin{array}{ccc}
\Gamma_2 A & \rightarrow & \Lambda^2 A \\
\downarrow & & \downarrow 2. \\
\Lambda^2 A & \rightarrow & A \otimes A \\
\downarrow & & \downarrow \text{non-mult} \\
\Lambda^2 A & \rightarrow & 0
\end{array} \]

Somewhere introduce \( \Sigma_2 A \) and \( Z^2(A;B) = \text{Hom}_U(\Sigma_2 A, B) / \text{Hom}_U(A \otimes A, B) \)

\[ \Sigma_2 A \text{ is the abelian group of } (a_1, a_2) \in A \times A \text{ and relations making the image of } (a_1, a_2) \text{ into a 2 cycle:} \]

\[ (a_2, a_3) - (a_1 + a_2, a_3) + (a_1, a_2 + a_3) - (a_1, a_2) = 0 \]

Thus there's a canonical map \( \Sigma_2 A \rightarrow A \otimes A \) sending \( (a_1, a_2) \) to \( a_1 \otimes a_2 \). It looks like \( \Sigma_2 A \) naturally arises from the MacLane resolution of an abelian group A by free abelian groups of type \( Z[A^n] \).
back to the Heisenberg group, to identify it with "the" principal \( T \)-bundle \( P \) over \( R^2 \) with curvature \( 2 \pi i dx dy \).

What do you know about \( P \)? It is \( T \times R^2 \), consists

\[ \mathbb{Z}[A^2] \longrightarrow \mathbb{Z}_2 A \longrightarrow A \otimes A \]

\[ \mathbb{Z}[A] = \mathbb{Z}[A] \longrightarrow \Gamma_2 A \]

\[ A = A \]

This describes \( P \) as the principal \( T \)-bundle over \( R^2 \). Next you must give a connection form on \( P \),

This amounts to a 1-form \( \mathbb{Z}^1 dz + \eta \), where \( \eta \in \Omega^1(R^2) \).

The curvature is \( d\eta \) (up to sign). Not very illuminating.

To look at \( H \) which is a central extension of \( R^2 \)

by \( T \). So you have an exact sequence of groups

\[ T \longrightarrow H \longrightarrow R^2 \]

with \( T \) the center of \( H \). The commutator pairing is a skew-symmetric.
bilinear form on $\mathbb{R}^2$ with values in $\mathbb{I}$. Calculate:

Take $(a,b) \in \mathbb{R}^2$, $(a',b') \in \mathbb{R}^2$ and lift to

$(1 \alpha, \beta) (1 \alpha', \beta')$ resp. in $H = \mathbb{I} \times \mathbb{R}^2$

$$e_{a \Delta e_b \gamma} e_{a' \Delta e_{b'} \gamma} = e^{-2 \pi i b a'} e^{(a + a') \Delta e_b \gamma} e^{(b + b') \Delta e_{a'} \gamma}$$

The commutator is $e^{-2 \pi i b a' - b a}$ to get a clear link between the commutator pairing for $H$ and the curvature for $P$.

What you would like to do next is to match up presentations for $H$ with connections for $P$. When you define $H$ you pick a 2-cocycle on $\mathbb{R}^2$ with values in $\mathbb{I}$, then define $H$ to be $\mathbb{I} \times \mathbb{R}^2$ (so $H$ comes provided with a section of $\pi : H \to \mathbb{R}^2$) equipped with a defined using the cocycle.

Picking the section of $\mathbb{I}$ corresponds to trivializing the $\mathbb{I}$ bundle $P$. If we restrict 2-cocycles to those associated to bilinear forms, then our 2-cocycle corresponds to a connection, so it seems to work, but you want the details to be clean.

Crazy ideas: suggested by

$$\mathbb{I}[A^2] \to \Sigma^2 A \to \Lambda \otimes \Lambda$$

Is there anything here which might help find the good class of signals for sampling, i.e. the $f$ in $L^2(\mathbb{R})$ corresponding to continuous sections of the degree 1 line bundle over $T^2$.

Vide ratios of a finite Heisenberg group $G = \mathbb{Z}/n \mathbb{Z}$.

$$H = \mathbb{I} \times \mathbb{G} \times \mathbb{G}$$

May idea here is to understand something about the problem of linking the rep theory of $H$ with the structure of $H$. 

$$G = \mathbb{Z}/n \mathbb{Z}$$

$$G' = \mu_n$$
Feb 21, 02

Aim: Calculate left and right translation vector fields on H. Start with your favorite model for H, namely, $\mathbb{T} \times \mathbb{R}^2$ where mult is defined using the complex step $2\pi i (-b_1, a_2)$. 

$$e^{a_1 X} e^{b_1 Y} e^{a_2 X} e^{b_2 Y} = e^{-2\pi i b_1 a_2} e^{(a_1 + a_2) X} e^{(b_1 + b_2) Y}$$

$$\left(\xi_1, a_1, b_1\right), \left(\xi_2, a_2, b_2\right) = \left(\xi_1, \xi_2, e^{-2\pi i b_1 a_2}, a_1 + a_2, b_1 + b_2\right)$$

Infinitesimal left translation 

$$\left(1, \delta a, \delta b\right), \left(\xi, a, b\right) = \left(\xi, e^{-2\pi i b_1 a_2}, \delta a + a_1, \delta b + b_1\right)$$

Infinitesimal right translation

$$\left(\xi_1, y_1, z_1\right), \left(\xi_2, y_2, z_2\right) = \left(\xi_1 + y_1, \xi_2 + y_2, e^{-2\pi i b_1 a_2}, z_1 + \xi_1, z_2 + \xi_2\right)$$

Start with this description of H, calculate the infinitesimal left (+ right) translation 

$$\psi((a, b, \xi), (\xi, y, z))$$

This is the effect of left translation, now take 

$$\left(\xi, \delta a, \delta b, 1 + \delta \xi\right).$$

But first do the product

$$\psi(a + x, b + y, e^{-2\pi i b_1 a_2} z, \xi)$$

and expand to first order around $(a, 0, 1)$. 

$$= (2 \psi) \delta a + (\partial_y \psi) \delta b + (\partial_z \psi) \left(e^{-2\pi i b_1 a_2} \delta b + e^{-2\pi i b_1 a_2} \delta \xi\right)$$

$$= (2 \psi) \delta a + (\partial_y \psi) \delta b + (\partial_z \psi) \left(-2\pi i b_1 a_2 \delta b\right) + \partial_z \psi (\delta \xi)$$

$$= \left(\partial_x \psi\right) \delta a + \left(\partial_y \psi - 2\pi i b_1 \partial_z \psi\right) \delta b + \left(\partial_z \psi\right) \delta \xi$$

where $\partial_x \psi, \partial_y \psi, \partial_z \psi$ are evaluated at $x, y, z$.

So the vector fields on H we get are

$$\partial_x, \partial_y - 2\pi i b_1 \partial_z, \partial_z$$
Next find inf right translation
\[
(\psi + \delta \psi)(x, y, z) = \psi(x + \delta a, y + \delta b, z + \delta z) = \psi(x, y, z) - 2\pi i y \delta a + \delta z (1 + \delta z)
\]
\[
\psi(x, y, z) + (\partial_x \psi) \delta a + (\partial_y \psi) \delta b + (\partial_z \psi) \delta z
\]
Maybe better is
\[
\psi(x, y, z) = \psi(x + \delta x, y + \delta y, z + \delta z) = \psi(x, y, z) - 2\pi i y \delta x + \delta z (1 + \delta z)
\]
\[
= \psi(x, y, z) - 2\pi i y (\partial_z \psi) \delta z + (\partial_z \psi) (-2\pi i y \delta z + z \delta \psi)
\]
\[
= (\partial_x \psi - 2\pi i y (\partial_z \psi)) \delta x + (\partial_y \psi) \delta y + (\partial_z \psi) \delta z
\]
\[
\text{inf left mult vector fields.} \quad \partial_x, \partial_y - 2\pi i x \partial_z, \partial_z
\]
\[
\text{inf rt mult v. f.} \quad \partial_x - 2\pi i y \partial_z, \partial_y, \partial_z
\]
\[
\text{observe wrong sign} \quad [\partial_x, \partial_y - 2\pi i x \partial_z] = -2\pi i \partial_z
\]
\[
[\partial_x - 2\pi i y \partial_z, \partial_y] = +2\pi i \partial_z
\]
because of contravariance of the action on functions.

Recall that you have left and right actions of the Heisenberg Lie algebra on the functions on the Heisenberg group. View the Heisenberg group as a trivial principal $\mathbb{T}$-bundle over $\mathbb{R}^2$.

\[
\mathbb{T} \rightarrow H \xrightarrow{\pi} \mathbb{R}^2 \Rightarrow H = \mathbb{R}^2 \times \mathbb{T}
\]

Function on $H: \psi(x, y, z) \in C^\infty(\mathbb{R}^2 \times \mathbb{T})$
Aims: You want to link the vector fields on $H$ to operators on sections of the line bundle $H \times T$. The first point is that a section is equivalent to a map $\psi: H \to \mathbb{C}$ satisfying

$$\psi(x, y, z) = \tau^1 \psi(x, y, z)$$

Relevant calculation:

$$H = \mathbb{R}^2 \times T \ni (x, y, z)$$

$$\psi((a, b, \xi) \cdot (x, y, z)) = \psi(a + x, b + y, e^{-2\pi i bx} \xi z)$$

$$\delta \psi(a + x, b + y, e^{-2\pi i bx} \xi z)$$

$$= (\partial_x \psi) \delta a + (\partial_y \psi) \delta b + (\partial_z \psi) \delta \xi$$

$$= (\partial_x \psi) \delta a + (\partial_y \psi - 2\pi i x \partial_z \psi) \delta b + \xi (\partial_z \psi) \delta \xi$$

three r.f. $\partial_x$, $\partial_y$, $\partial_z - 2\pi i x \partial_z$ $\partial_z \psi$

$$[\partial_x, \partial_z - 2\pi i x \partial_z] = -2\pi i \xi \partial_x$$

Summary: You have $H$ acting on itself by left and by right multiplication and some have the concept r.f.s. What's new is the vertical vector field $\xi \partial_z = \frac{1}{\partial_\theta}$ dual to $\frac{d\xi}{\xi} = i d\theta$. Next you want to deal with the principal fibre bundle $P$ and its connection.

$$P = \mathbb{R}^2 \times T$$

What is a connection in the principle $T$-bundle $P$ and is invariant under the right $T$-action.

$$A = M dx + N dy + z^{-1} d\xi$$
Let's try to identify $\mathcal{D}$ on sections of $L = \mathcal{P} \mathcal{T}$.

1. **Definition:** $G \rightarrow \mathcal{P}$ is a section of $L = \mathcal{P} \mathcal{T}$.
2. **Claim:** $\mathcal{D}$ is the fiber over $P$.

You have learned that $\mathcal{D}$ is a fiber of $L$.

Now you have seen from...

\[\phi(x,y,z) = f(y(x,y),z)\]

\[\mathcal{D} + \mathcal{N} \rightarrow \mathcal{D}(\mathcal{P}, \mathcal{I})\]

\[\phi(x, y, z) = f(x, y) + z\]

\[\phi(x, y, z) = f(x, y) + z\]
IDEA: Look at the effect of \( -1 : A \to A \) on central extensions of \( A \) (as well as \( 2: A \to A \)).

Return to your principal bundle \( P \). Problem yesterday was relating \( P \) to the associated line bundle \( L \). You need to relate a connection form on \( P \) to a differential operator \( d + A \) on sections of \( L \). You started to understand some of this, namely a section \( \phi \) is a function on \( P \) behaving according to a character of \( T \); something like \( \phi(x, y, z) = \phi(x, y, z)^3 \) but signs didn't work.

Some you need to link \( \frac{\partial \phi}{\partial z} = \frac{i}{2} \phi \) to the \( d \) in \( D = d + A \) on \( \Gamma(\mathbb{R}^2, L) \). Other ideas: canonical line bundle on projective space, better: in the case of a circle bundle, you look at the graded space of homogeneous functions of different degrees. You should also look at coadjoint orbits in the Heisenberg group, constructing the wired reps by geometric quantization.

\[
0 \to \Lambda^2 A \to A \otimes A \to S^2 A \to 0
\]

exact

The inj of \( \Lambda^2 A \) by lines led to a free gen, prod of cysc, \( \mathfrak{g} \)-s, show map is direct injection via induction on number of factors,
cross effect:

\[
\Lambda^2 (A \otimes A) = \Lambda^2 A_1 \otimes A_2 \oplus \Lambda^2 A_2 \otimes A_1,
\]

\[
(A_1 \otimes A_2)^2 = A_1^2 \otimes A_2 A_2 A_1 A_1 \oplus A_2^2.
\]

The 'brinman' part obvious direct injection.

You believe

\[
\mathbb{Z}^2(A, B) \xrightarrow{\text{comm. pairing}} \text{Hom}(\Lambda^2 A, B) \xrightarrow{\text{onto}}
\]

\[
\text{Hom}(A \otimes A, B) \xrightarrow{\text{onto?}} \text{for all } B \in A \otimes A.
\]

Heisenberg group \( T \to H \to \mathbb{R}^2 \). There should be a line bundle somewhere, look for its sections, which should yield a representation (maybe two) of \( H \). Induced reps. So (like pulling teeth) you arrive at something which should have been familiar. Too much geometry, not enough reps. theory.
Now you can get the formulas straight. The concept of induced representation, how it fits with fibre bundle stuff. If there's a subgroup H of G, V is a rep of H, then form vector bundle $G^xV$ whose sections are maps $f: G \to V$ satisfying the equivariance condition $f(gh) = h^{-1}f(g)$, because you descend $G^xV \to G^xV / H$. Every $gH$ is an $H$-orbit $(g, h^{-1}f(g))$; there is a unique representative $(g, v)$ with first component $g$. Maybe better to say a section of $G^xV / H$ is the same by descent as a section $G^xV \to V$. Let's go back to $T \to H \to \mathbb{R}^2$. You want maybe to look at a general principal bundle.

Look at a manifold $P$ with $T$ action, free, proper. $P$ is a principal $T$ bundle. There's an associated line bundle for any character $T \to T$. This is something you've forgotten about.

You probably want to look at embedding these line bundles as retracts of trivial vector bundles.

Return to $P$ with $T$ acting freely and base $B$. Assume trivial $P = B \times T$, $C^0(P) = C^0(B) \otimes C^0(T)$. Functions on $P$ are Fourier series whose Fourier coefficients are functions on $B$. Notice $\mathbb{Z}$-grading on $C^0(P)$, this is true even without $P$ being trivial.
So you have a new way to view a principal $T$-bundle $P \to B$, as a $\mathbb{Z}$-graded algebra corresponding to the irreducible characters of $T$. Thus

$$C^\infty(P) = \bigoplus_{n \in \mathbb{Z}} \Gamma(B, L^n)$$

A partition of $1$ enters you should have

$$\Gamma(B, L^n) \otimes \Gamma(B, L^m) \sim \delta_{nm}$$

so you should be able to write $1 = \sum x_i \otimes x'_i$ with $x_i \in \Gamma(L)$, $x'_i \in \Gamma(L)$. But now what? Connections. Find the link between the connection form $\omega$ on $P$ and the connection operator $D$ on $\Gamma(B, L)$.

$P$ has free action of $T$, so functions on $P$ decompose according to the characters $\chi \in \pi^\vee = \mathbb{Z}$. Let $X = z \partial_x + \frac{i}{2} \partial_y$ generate $\text{Lie}(T) = i \mathbb{R}$. Better let $\psi \in C^\infty(P)$, then $\psi(e^{i\theta} p)$ is periodic of period $2\pi$ in $\theta$, so you have

$$\psi(e^{i\theta}) = \sum_{n \in \mathbb{Z}} \psi_n e^{in\theta}$$

where $\psi_n = \int e^{-in\theta} \psi(e^{i\theta} z) \frac{d\theta}{2\pi}$?

Need better notation: $T$ acts freely on $P$, $T = \{ e^{i\theta} | \theta \in \mathbb{R} \}$

The point is that $T$ acts on $C^\infty(P)$, so $C^\infty(P)$ is a rep of $T$, and it decomposes according to the characters

$$C^\infty(P) = \bigoplus_{n \in \mathbb{Z}} \{ \psi \in C^\infty(P) \mid T \psi = e^{in\theta} \psi \}$$

It should be true that this is a $\mathbb{Z}$ grading of $C^\infty(P)$ as an alg, and the $n$-th component is $\Gamma(B, L^n)$

Put $C^\infty(P)_n = \{ \psi \in C^\infty(P) \mid T \psi = e^{in\theta} \psi \}$

$C^\infty(P)_m \cdot C^\infty(P)_n \subset C^\infty(P)_{m+n}$.
What does this look like for \( P = H \)?

Completely trivial since \( H = \mathbb{R}^2 \times T \), so that
\[
C^\infty(H) = C^\infty(\mathbb{R}^2) \otimes C^\infty(T).
\]

Go back to \( \pi: P \to B \). Aim: to understand a connection in \( P \), i.e. a \( \text{Lie}(T) \)-valued 1-form on \( P \),
call it \( \Theta \) satisfying \( \iota_X \Theta = 1 \), where \( X \) is the vector
field whose flow is the \( T \) action, also \( L_X \Theta = 0 \), \( \Theta \) is preserved by the flow,
\[
0 = L_X \Theta = d\iota_X \Theta + \iota_X d\Theta. \quad \therefore d\Theta = 0
\]
is basic.

So w

Do over the structure again. \( P \) is a manifold with
free circle \( (T) \) action, infinitesimal generator \( X \), \( \Theta \) is a
\( \text{Lie}(T) \)-valued 1-form, \( L_X \Theta = 0 \), \( \iota_X \Theta = 1 \).

It should now be true that there is some sort of
differential operator \( D \) on the vector bundle \( L \), i.e. on sections of \( L \),
it satisfy the derivation property used by Bott. What can
you do. Sections of \( L \) are functions on \( P \), so you want to
apply \( D \) to get \( df \), gives the derivation property over \( C^\infty(B) \).
Next you need to correct \( df \) in the vertical direction in
some way. \( \psi \) is not constant vertically because it has
degree \( \pm 1 \), which means that vertically \( df = \pm \psi \), i.e.
\[
\iota_X (df - \Theta \psi) = L_X \psi + \iota_X df - \psi = 0.
\]
So it seems that
the operator is \( \psi \rightarrow (d - \Theta) \psi \), where this is a 1-form on \( B \)
with values in the line bundle \( L \), because it is "basic".

OK it seems to work. Notice that you end up
working with functions and cliff forms on \( P \), you
construct things to be basic so that they descend to the
bases.

Thus, if a finite, for, abelian group \( G \), then any skew-symmetric \( \Lambda^2 A \rightarrow B \)
arrises by skew-symmetrizing bilinear form \( A \otimes A \rightarrow B \).

\[
0 \rightarrow \Lambda^2 A \rightarrow A \otimes A \rightarrow \Lambda^2 A \rightarrow 0
\]
\[
0 \rightarrow \Lambda^2 A \rightarrow \Gamma^2 A \rightarrow A \rightarrow 0
\]
$\rightarrow \text{Ext}^1(A, B) \rightarrow Z^2(A, B) \rightarrow \text{Hom}(A^2, B) \rightarrow 0$

Quadratic function $q : A \rightarrow B$ is one such that

$$\begin{align*}
\delta q(a_1, a_2) &= q(q_1 + q_2) - q(q_1) - q(q_2) \\
&\in \text{Hom}(A^2, B)
\end{align*}$$

$\text{Hom}(\Sigma^2 A, B) = Z^2(A, B) \Rightarrow \text{Hom}(A \otimes A, B)$

$\text{Hom}(\mathbb{Z}A, B) = C^1(A, B) \Rightarrow \text{Quad}(A, B) = \text{Hom}(\Gamma^2 A, B)$

Question: For $A$ a fin gen. is every central ext of $A$ given by a quadratic function?

$\mathbb{Z}[A^3] \rightarrowtail \mathbb{Z}[A^2] \twoheadrightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0$

What is the exact here?

$\mathbb{Z}[A]$

$\mathbb{Z}^2 A$

$\Sigma^3 A$

$C^3(A, B) \leftarrow C^2(A, B) \leftarrow C^1(A, B) \leftarrow \text{Hom}(A, B)$

$\text{U} \delta \text{U}$

$\text{U} \delta \text{U}$

$Z^2(A, B)$

$\text{Hom}(A, B)$

Question: What is the homology of

$\mathbb{Z}[A^3] \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0$?

$C^3(A, B) \leftarrow C^2(A, B) \leftarrow C^1(A, B) \leftarrow \text{Hom}(A, B) \leftarrow 0$
Feb 24, 02. Review central extensions of elementary abelian groups. \[ \beta: E \rightarrow A \] invariant is the quadratic function \( q(a) = e^2 \) \( \pi(e) = a \). Because \( H_2(E, A, Z) = \Gamma^2A \) (duel to \( H^2 = S^2A \)), one has \( H^2(A, B) = \text{Hom}(\Gamma^2A, B) \), the quadratic form is complete invariant.

\[ \Lambda^2A \rightarrow A \otimes A \rightarrow S^2A \rightarrow 0 \rightarrow S^2A \rightarrow \Gamma^2A \rightarrow A \rightarrow 0 \]

\[ S^2A \rightarrow \Gamma^2A \]

\[ a_1, a_2 \rightarrow \frac{1}{2}(a_1 + a_2) - \frac{1}{2}(a_1^2 - 2\cdot a_1 \cdot a_2) \]

\[ 0 \rightarrow A \rightarrow S^2A \rightarrow \Gamma^2A \rightarrow A \rightarrow 0 \]

So what \[ \Lambda^2A \subset \Gamma^2A \rightarrow A \]

\[ \Lambda^2A \subset A \otimes A \rightarrow S^2A \]

\[ A \otimes A \supset \Gamma^2A \supset \Lambda^2A \]

\[ \gamma(a_1, a_2) - \gamma_1 - \gamma_2 \leftrightarrow a_1, a_2 \]

\[ \Gamma^2A \supset S^2A \supset A \]

\[ A \otimes A \supset \Lambda^2A \]

\[ 0 \rightarrow A \rightarrow S^2A \rightarrow \Lambda^2A \rightarrow 0 \]
functors \( A \rightarrow A^{(3)} \), \( A \otimes A \), \( S^2A \), \( \Gamma^2A \), \( \Lambda^2A \)

\( A \otimes A \) contains \( \Lambda^2A \) \( \Gamma^2A \)

\( \Gamma^2A \rightarrow A \otimes A \)

\( \gamma(a + a_2) - \gamma(a) - \gamma(a_2) \)

\( q(a) \rightarrow a \otimes a \)

You seem to have problems linking the elementary abelian group case to the general case.

\[ 0 \rightarrow \Lambda^2A \rightarrow A \otimes A \rightarrow \Gamma^2A \rightarrow A \rightarrow 0 \]

\[ q_1, q_2 \rightarrow q_1 \otimes q_2 - q_2 \otimes q_1 \]

\[ q_1 \otimes q_2 \rightarrow \gamma(q_1 + q_2) - \gamma(q_1) - \gamma(q_2) \]

You believe that

\[ 0 \rightarrow \Lambda^2A \rightarrow A \otimes A \rightarrow S^2A \rightarrow 0 \]

\[ q_1, q_2 \rightarrow q_1 \otimes q_2 - q_2 \otimes q_1 \]

\[ q_1 \otimes q_2 \rightarrow q_1 q_2 \]

is exact. So if true you get

\[ 0 \rightarrow S^2A \rightarrow \Gamma^2A \rightarrow A \rightarrow 0 \]

exact

\[ q_1 q_2 \rightarrow \gamma(q_1 + q_2) - \gamma(q_1) - \gamma(q_2) \]

\[ q_2 \rightarrow \gamma(q_2) - 2 \gamma(q_1) \]

Look at

\[ \Gamma^2A \rightarrow A \otimes A \]

\[ \gamma(a) \rightarrow a \otimes a \]

Compose with

\[ S^2A \rightarrow \Gamma^2A \rightarrow A \otimes A \]

\[ q_1 q_2 \rightarrow \gamma(q_1 + q_2) - \gamma(q_1) - \gamma(q_2) \rightarrow q_1 \otimes q_2 + q_2 \otimes q_1 \]

\[ q_2 \rightarrow 0 \] ?

Here's where the mistake occurred. Look at

\[ 0 \rightarrow \Lambda^2A \rightarrow A \otimes A \rightarrow \Gamma^2A \rightarrow A \rightarrow 0 \]

with \( A = \mathbb{Z}/2 \). Then \( 0 \rightarrow \mathbb{Z}/2 \rightarrow \Gamma^2(\mathbb{Z}/2) \rightarrow \mathbb{Z}/2 \rightarrow 0 \) is exact so \( \Gamma^2(\mathbb{Z}/2) \) has order 4, so it won't embed in \( A \otimes A \).
It seems that quadratic functions on $\mathbb{Z}/n$ allow you to construct the extension $\mathbb{Z}/n \to \mathbb{Z}/n \to \mathbb{Z}/m$ if clear of abelian groups. General argument. Given $h: A \otimes A \to B$

- bilinear, you get a group from the set $E = \mathbb{A} \times A$ with the product $(b, a) \cdot (b', a') = (b + b' + h(a, a'), a + a')$, a central $E$-module $E$ of $A$ by $B$, whose commutator pairing is $\langle b, a \rangle = h(a, a) - h(a, a')$. If the pairing is 0, then $E$ is abelian.

- Take $\mathbb{A} = B = \mathbb{Z}$ and $h(m, n) = mn$. Then you get an abelian group $\mathbb{Z} \otimes \mathbb{Z} \to \mathbb{Z}$, which splits. Splitting means a section of $\pi$ which is additive, which means that 1-cochain $g$ with coboundary $h$, $g$ is then a quadratic function yielding $h$.

Let $g(m) = \frac{m(m-1)}{2}$, $g(m+n) - g(m) - g(n) = \frac{1}{2}[(m^2 + 2mn + n^2 - m^2 - n^2)/m^2] = mn$.

Consider $0 \to A^2 \to A \otimes A \to \Gamma^2 A \to A_0 \to 0$

where $A = \mathbb{Z}/N$ where $0 \to \mathbb{Z}/N \to \Gamma^2(\mathbb{Z}/N) \to \mathbb{Z}/N \to 0$

so that $|\Gamma^2(\mathbb{Z}/N)| = N^2$. Take $B = \mathbb{Z}/N$ and $h(m+n, n+N) = mn+N$.

Go back to $\mathbb{Z} \times \mathbb{Z}$ with the product

$$(m, n) \cdot (m', n') = (m + m', n + n')$$

and consider the maps $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$

$h \mapsto (g(h), n)$

Then $(g(n), n) \cdot (g(n'), n') = (g(n) + g(n') + nn', n + n')$

$= (g(n+n'), n+n')$

Thus one has a homomorphism splitting the extension.

Now consider $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

$\mathbb{Z}/N \to \mathbb{Z}/N \times \mathbb{Z}/N \to \mathbb{Z}/N$

not the direct product

So the order of the elt $(0, 1)$ in the bottom entry is the least $n > 0$ such that $n$ and $g(n) \equiv 0 \mod N$. 
\[ \frac{N(N-1)}{2} \quad \text{if } N \text{ odd then} \quad \frac{2N(2N-1)}{2} \quad \text{if } N \text{ even then} \]

\[ N \text{ works} \quad 2N \text{ works}. \]

**How to calculate \( \Gamma^2A \).** Probably use

\[ 0 \rightarrow S^2A \rightarrow \Gamma^2A \rightarrow \pi \rightarrow A \rightarrow 0 \]

This is an extension of abelian groups. There is a tautological section of \( \pi \) whence \( \Gamma^2A = S^2A \times A \), and a tautological 2-couple with values in \( S^2A \), yielding an addition on \( \Gamma^2A \):

\[ (b,a) \cdot (b',a') = (b + b' + aa', a + a') \]

Check that \( a \mapsto (g(a), a) \) is a hom.

\[ (g(a), a) \cdot (g(a'), a') = (\frac{g(a) + g(a') + aa'}{g(a + a')}, a + a') \]

\[ A = \mathbb{Z}/2 \quad (g(1), 1)(g(1), 1) = (g(1) + g(1) + 1, 0) \]

\[ 0 \rightarrow S^2A \rightarrow \Gamma^2A \rightarrow A \rightarrow 0 \]

\[ i(a_1, a_2) = \pi(a_1 + a_2) - \pi(a_1) - \pi(a_2) \]

It looks like you want to have a \( g : A \rightarrow S^2A \) with \( g(a_1, a_2) = a_1 a_2 \)

\[ 0 \rightarrow S^2\mathbb{Z} \rightarrow \Gamma^2\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \]

\[ (0, 1)(0, 1) = (1, 2) \quad 2 \quad (0, a)(0, a) = (a^2, 0) \]

\[ (1, 2)(0, 1) = (3, 3) \quad 3 \quad (a^2, 0)(0, 0) = (a^2, a) \]

\[ (3, 3)(0, 1) = (6, 4) \quad 3 \quad (a^2, a)(0, a) = (0, 0) \]

\[ (6, 4)(0, 1) = (10, 5) \quad (\frac{n(n-1)}{2}, n) \]
What is the length of \( \frac{n(n-1)}{2} \) and \( n \)?

\[
\begin{align*}
    p & \quad \frac{p(p-1)}{2} \\
    n \quad \frac{n(n-1)}{2} & \equiv 0 \pmod{N} \\
    N & \mid n \text{ and } \frac{n(n-1)}{2} \quad n = 2k
\end{align*}
\]

We have

\[
S^2(Z/N) \longrightarrow \Gamma^2(Z/N) \longrightarrow Z/N
\]

in here have elements

\[
\frac{n(n-1)}{2}, \quad n
\]

have

\[
S^2 \longrightarrow \Gamma^2 Z \longrightarrow Z
\]

0 \longrightarrow \Lambda^2 A \longrightarrow A \otimes A \longrightarrow \Gamma^2 A \longrightarrow A \longrightarrow 0

\[
\begin{align*}
    S^2 A & \quad (0, 1) \quad (0, 1) = (1, 0) \\
    (0, 1) & \quad (0, 1) \quad (1, 0)
\end{align*}
\]

Do the examples carefully:

\[
\begin{align*}
    \text{Abelian group extension} & \quad 28 \quad 8 \\
    (b, a)(b', a') & = (b+b'+aa', a+a')
\end{align*}
\]

also have abelian group extensions

\[
S^2 A \longrightarrow \Gamma^2 A \longrightarrow A
\]

\[
8(a+o') - 8(a) - 8(o') = i(4o')
\]

on the other hand you have the abelian group extensions.

\[
S^2 A \longrightarrow EA \longrightarrow A
\]

where

\[
EA = \{ (b, a) \mid b \in S^2 A, \quad a \in A \}
\]

\[
(b, a)(b', a') = (b+b'+aa', a+a')
\]

Question: Are these two group extensions isomorphic? Yes, let

\[
s: A \longrightarrow EA \quad \text{be the section} \quad s(a) = (0, a)
\]
\[ q(a) = \frac{a(a-1)}{2} + 6a \quad N=3 \]

\[ (g(0) mod 3) = \left( \frac{a(a-1)}{2} + 6a \right) \mod 3. \]

Want 3 times \((b, 1)\) to be non-zero.

\[ N=5 \quad \frac{3+3}{10+6} = \frac{3}{5} \]

\[ p(\frac{p-1}{2}) + bp_0, p \]

\[ \frac{N(N-1)}{2} + Nb, N \quad N \text{ odd no good.} \]

\[ N=4 \quad 4x(0, 1) = \left( \frac{4(3)}{2}, 4 \right) = (2, 4) \equiv (2, 0) \mod 4 \]

\[ 8x(0, 1) = (28, 8) \equiv (0, 0) \mod 4 \]

\[ N=8 \quad 8x(0, 1) = (28, 8) \equiv (4, 0) \mod 8 \]

\[ 16x(0, 1) = (8+16, 16) = (0, 0) \mod 16 \]

\[ H^2(A, B) = \text{Ham} \left( \Gamma_{\mathbb{Z}/2}^2 A, B \right) \]

\[ \Gamma_{\mathbb{Z}/2}^2 A = \Gamma^2 A \otimes \mathbb{Z}/2 \]

\[ 0 \rightarrow S^2 A \rightarrow \Gamma^2 A \rightarrow A \rightarrow 0 \]

\[ 0 \rightarrow A \rightarrow S^2 A \rightarrow \Gamma_{\mathbb{Z}/2}^2 A \rightarrow A \rightarrow 0 \]

\[ 0 \rightarrow \Lambda^2 A \rightarrow \Gamma^2 A \rightarrow A \rightarrow 0 \]

\[ \Gamma^2 \quad E \quad A \]
Feb 25, 02  

Something is unclear in the elementary abelian case. Consider begin with

\[ 0 \to \Lambda^0 A \to A \otimes A \to \Gamma^2 A \to A \to 0 \quad \text{in general} \]

Then \[ 0 \to s^2 A \to \frac{1}{2} A \to A \to 0. \]

Apply \( \text{Hom}(\_ , B) \) to get

\[ 0 \to \text{Hom}(A , B) \to \text{Hom}(\frac{1}{2} A , B) \to \text{Hom}(s^2 A , B) \to \text{Ext}^1(A , B) \]

(B \to E \xrightarrow{\pi} A)

Assume \( 2A = 0 \quad g(a) = x^2 \quad \text{if} \quad \pi x = a \)

Given \( x, y \in E \)

\[
(\pi x) y x y (\pi y)^{-1} = x^{-1} y x y^{-1}
\]

\[
\pi x^{-1} = \pi x^{-1}
\]

\[
\pi y^{-1} = \pi y^{-1}
\]

\[
x^{-1} = x^2
\]

\[
x = x^2
\]

\[
x^{-1} y x^{-1} = x^2 y x^{-1} y^{-1} = x^2 y^{-1}
\]

So if \( A \) is elementary 2-algebra, then what? You have a map

\[ H^2(A , B) \to \text{Hom}(\frac{1}{2} A , B) \quad \text{for} \quad 2A = 0 \]

\[ H^2(A , B) \to \text{Hom}(s^2 A , B) \]

Basic question: Is \( \Gamma^2(A \otimes A') = \Gamma^2(A) \oplus (A \otimes A') \oplus \Gamma^2(A') ? \)

It seems that for \( 2A = 0 \) you

\[ H^2(A , B) \to \text{Hom}(\frac{1}{2} A , B) \]

\[ \text{Ext}^1(A , B) \to H^2(A , B) \to \text{Hom}(s^2 A , B) \]

As over again carefully. \( B \xrightarrow{\pi} E \xrightarrow{\pi} A \)

\[
g(\pi x) = x^2 \quad g(\pi x + \pi y) - g(\pi x) - g(\pi y) =
\]

\[
x y x (x x)^{-1} (y y)^{-1} = x^2 x^{-1} y x y^{-1} = x^{-1} y x y^{-1}
\]

But \( x^{-1} = x^2 \quad z \in B \)

\[
x y x (x x)^{-1} (y y)^{-1} = x y x y^{-1} x^{-1} = x y x y^{-1}
\]
$S^2A \xrightarrow{i} \Gamma^2A \xrightarrow{\gamma} A$

$(b, \gamma a)(b', \gamma a') = (b + b' + aa', \gamma(\alpha + \beta))$

$(0, \gamma a)(0, \gamma a) = (0 + aa', 0)$

use $S^2A \times A \xrightarrow{\sim} \Gamma^2A$  

$\alpha \rightarrow \gamma(\alpha)$

group law

$(b, \gamma a)(b', \gamma a') = (b + b' + aa', \gamma(\alpha + \beta))$

The quadratic map $A \rightarrow S^2A$ is

$(0, \gamma a)(0, \gamma a) = (aa', 0)$

The quadratic map associ. to the extn.

$S^2A \xrightarrow{\sim} \Gamma^2A \xrightarrow{\sim} A$

is $\alpha \rightarrow \alpha^2$.

$\Lambda^2A \xrightarrow{\sim} \Gamma^2A/2 \xrightarrow{\sim} A$

\[\begin{array}{c}
A \\
\downarrow \\
S^2A \\
\downarrow \\
\Lambda^2A
\end{array}\quad \begin{array}{c}
\rightarrow \\
\Gamma^2A \\
\rightarrow \\
\Gamma^2A/2 \\
\rightarrow \\
A
\end{array}\]

$\text{Ext}'(A,B) \rightarrow H^2(A,B) \rightarrow \text{Ham}(\Lambda^2A,B)$ possibly OK because $\text{Ext}'(A,B)$ and $\text{Ham}(A,B)$ related by the Bockstein

$0 \rightarrow A \rightarrow S^2A \rightarrow \Gamma^2A/2 \rightarrow A \rightarrow 0$
Let \( hBh^{-1} \) and \( hB \) be group case.

Review everything today.

Begin with \( C : \sum h_s h_t \neq 0 \Rightarrow \sum h_s h_t = h_t = \sum h_t h_s \),

\( C \) has local unit and right unit idempotent \( M = C \sum h_s M \), \( C = \sum h_s C \).

\[ B = \Gamma \times C_{\Gamma, \Xi} \]

\[ A = \mathbb{Z}_{\Gamma, \Xi} \]

\[ p(s) \sum_{t} p(st^{-1})p(t) = p(s) \]

\[ p(s) \neq 0 \Rightarrow s \Xi \Xi \]

\[ \Gamma, \Xi \text{ given } C = \text{ any defined via } \gamma \text{ and } \pi \text{ as above.} \]

If \( M \) a \( C \)-module, then \( M = CM \Leftrightarrow \forall m \in M \Rightarrow \sum h_s m \).

in which case \( M \) is reduced + finite \( C \otimes CM = M \).

\( \Gamma \) acts on \( C \) as \( \gamma \text{ form } \Gamma \text{ product } \Gamma \times C = B \),

\( \Gamma \times C^+ \text{ semi direct product } \Gamma \times B \), \( \Gamma \rightarrow \text{Mult}(B) \).

Question: \( \Gamma = \mathbb{Z}^2 \), \( \mathbb{P}_\gamma \) should classify retracts of

trivial vector bundles over \( \mathbb{T}^2 \text{ map } \mathbb{T}^2 \rightarrow \mathbb{B}. \) Are

there interesting examples, with \( \Xi \) finite.

Today you want to go over the details of \( (\Gamma, \Xi) \) in

a systematic way.

Begin with \( B = \Gamma \times C \) def

1. Red \( B \)-module = \( \Gamma \)-module \( W \) with \( h \in \text{End}(W) \)

\[ h \neq 0 \Rightarrow \sum_s h_s w = w \quad \forall w. \]

\[ \Rightarrow W = \sum_s W \]
Given \( W \) as above put \( V = h W \), \( W \leftarrow V \leftarrow h W \).

2. Claim \( V \) is a \( \Lambda \)-module. 

A module, \( A = \frac{\mathbb{Z}}{\langle s \rangle} \), \( p(s) = js \in \mathbb{Z}(V) \)

\[ p(s) f = l g s i f = h s h \]

\[ p(s) \neq 0 \left( + x \ rel, y \ rel \right) \Rightarrow h s h \neq 0 \Rightarrow s e \in \Lambda . \]

\[ \sum_{t} p(st^{-1}) p(t) = \sum_{t} p(st^{-1} t) = js \sum t^{-1} h t c = js c = p(s) \text{.} \]

\( \therefore \) \( V \) an \( \Lambda \)-module. Next \( W = \sum_{s} s \cdot V \Rightarrow \)

\[ V = s W = \sum_{s} js i V = \sum_{s} p(s) V \quad \therefore \quad V = AV. \]

Let \( \nu \in V \) satisfy \( p(s) \nu = 0 \quad \forall s \).

Then \( \nu = \sum s \ h s^{-1} \nu = \sum s \ j g s^{-1} \nu = 0 \quad \Rightarrow \nu = 0. \)

Define maps \( \sum_{s} \sum_{s} p(s) t f(t) \leftarrow \sum_{s} t \circ f(t) \)

\[ \begin{array}{ccc} \sum_{s} \sum_{s} p(s) t f(t) \leftarrow & \sum_{s} t \circ f(t) \\
W \leftarrow & \Lambda \otimes V \leftarrow & \Lambda \otimes V \\
w = \sum s \ j g s^{-1} w \leftarrow & \sum_{s} s \circ g s^{-1} w \leftarrow w \\
\sum_{s} t \circ f(t) \leftarrow & \sum_{s} t \circ f(t) \\
\alpha \text{ well defined because} \quad W = \sum t h i W \\
\text{and} \quad j g s^{-1} t = p(s^{-1} t) \end{array} \]

Define \( \beta, \alpha \) show \( \beta \alpha = 1_{W} \)

and \( \alpha \beta \left( \sum_{s} t \circ f(t) \right) = \sum_{s} \sum_{s} p(s) t f(t) \)

3. Twine \( V \) any \( \Lambda \)-module define \( p \) on \( \Lambda \otimes V \).

\( W = \left\{ \sum \alpha t \circ f(t) \mid \sum_{s} p(s^{-1} t) f(t) = f(s) \right\} \).

Define \( \eta = \eta \), \( \iota = \beta, e_{1} \)

\[ \begin{array}{ccc} W & \leftarrow & \Lambda \otimes V \\
\alpha \downarrow & \leftarrow & \alpha \downarrow \\
W & \leftarrow & W \\
\beta \downarrow & \leftarrow & \beta \downarrow \\
\iota \downarrow & \leftarrow & \iota \downarrow \\
V & \leftarrow & \Lambda \otimes V \end{array} \]

\[ h = \iota \alpha = \beta_{2} e_{1} \]
More on to the Moita context \((hBh, hB)\) \(F(V) = \text{Im} \{ Bh \otimes_A V \rightarrow \text{Hom}_A(hB, V) \}\)

\(G(W) = \text{Im} \{ hB \otimes_B W \rightarrow \text{Hom}_B(hB, W) \}\)

Point here is that \(A(hB) = hBhB = hB\)

so \(A(hB \otimes_B W) = hB \otimes_B W\) same for \(G(W)\)

Let \(\lambda \in \text{Hom}_B(hB, W), 0 = \lambda(hB A) = \lambda(hBhB) = \lambda(hB)\)

\(\lambda = 0\).

So you've checked that \(G(W)\) is \(A\)-reduced.

\[hB \otimes_B W \rightarrow hW \rightarrow \text{Hom}_B(hB, W)\]

\[\begin{array}{c}
\text{hw} \\
\downarrow
\end{array}\]

\[\begin{array}{c}
(bh \mapsto bhw) \\
\downarrow
\end{array}\]

\(0\) means \(Bhw = 0\)

Look at \(F(V)\). This should be \(p(\Lambda \otimes V) = p(\Lambda \otimes A) \otimes V\)

You've shown that \(hW = G(W)\) is \(A\)-reduced for any reduced \(B\)-module \(W\), in particular \(hB\) and \(Bh\) are reduced for both \(A, B\), hence also \(A = hBh\) is \(A\)-reduced on both sides. \((B\) reduced is easy by the partition of \(1\))

General case of \((hBh, hB) \nRightarrow hBh = B\) if \(W\) is \(B\)-red

then \(\text{Im} \{ hB \otimes_B W \rightarrow \text{Hom}_B(hB, W) \} = hW\) is \(hBh\)-reduced

Problem: Identify \(F(V)\) above with \(p(\Lambda \otimes V)\) (group case)
Concerning \( (\hat{A}Bh_B \quad hB) \) where \( B h_B = B \)

maybe the first point in this context is completely idempotent. Do you then have the actual Morita equivalence on the level of red. modules

\[
\begin{align*}
F(V) &= \text{Im} \{ B h_B \otimes_A V \to \text{Hom}_A (B h_B, V) \} \\
G(W) &= \text{Im} \{ h B \otimes_B W \to \text{Hom}_B (B h, W) \}
\end{align*}
\]

These functors respect reduced modules. Claim \( G(W) = h W \)

\[
\begin{align*}
h b_1 \otimes w &\mapsto (b_2 h \mapsto (b_2 h * h b_1) \otimes w = b_2 (h b_1 w) \\

&\downarrow \\
\text{Hom}_B (B h, W) &\mapsto (b_2 h \mapsto b_2 h w)
\end{align*}
\]

\[\alpha(w) = 0 \quad \text{means} \quad B h w = 0 \Rightarrow h w = 0\]

assuming \( W \) reduced.

\[
\begin{align*}
W &\text{ B-red } \Rightarrow h W & A\text{-red.} \\
B &\text{ B-red } \Rightarrow h B & A\text{-red.} \\
B &\text{ B-red } \Rightarrow B h & A^0\text{-red.}
\end{align*}
\]

\[
\begin{align*}
V \text{ has qfs. } p(t) \\
\text{Want to show } F(V) = p(\Lambda \otimes V)
\end{align*}
\]

\[
\begin{align*}
&\leftarrow \leftarrow W \leftarrow B \leftarrow \Lambda \otimes V \leftarrow W \leftarrow B \\
&\updownarrow \downarrow \updownarrow \downarrow \updownarrow \downarrow \downarrow \downarrow \downarrow \\
&\leftarrow \leftarrow V \leftarrow \leftarrow \\
&\text{notice that you see that} \\
&\sum_t p(t) V \\
&\text{Ker } i = \bigcup_s p(s) V
\end{align*}
\]

\[
\begin{align*}
\alpha w &= \sum s \otimes s w \\
(\sum t \otimes f(t)) &= \sum t f(t) \\
\alpha (\sum t \otimes f(t)) &= \sum s \otimes \sum_t p(s) f(t) g(t)
\end{align*}
\]
How close are you to $F(V)$. You want the image of the map

$$\text{Hom}_A(hB, V) \leftarrow B \otimes_A V \quad \text{arises from} \quad h b_1 \cdot b_h \cdot v$$

Compare to

$$\Lambda \otimes V \leftarrow \Lambda \otimes V$$

$$\text{Hom}_A(hB, V) \leftarrow W \leftarrow B \otimes_A V$$

$$\Lambda \otimes V \leftarrow \Lambda \otimes V$$

$$\Lambda \otimes V \leftarrow W \leftarrow \Lambda \otimes V$$

Aim: Construct diagram

$$(h b_1 \mapsto h b b_1 b_2 b_1) \leftarrow \text{canonical map based on the pairing}$$

$$b_1 \otimes b_2 \otimes v$$

$$\text{Hom}_A(hB, V) \leftarrow B \otimes_A V \quad b \otimes v$$

$$\Lambda \otimes V \leftarrow \Lambda \otimes V$$

$$\sum_s \sum_t \rho(s^{-1} t) f(y) \leftrightarrow \sum t c f(t) \quad \sum t^2 c f(t)$$

Maybe the point is that $h \in B h$

$$\text{Hom}_B(h \otimes_A V, W) = \text{Hom}_B(V, \text{Hom}_B(hB, W))$$

$$\text{Hom}_B(h \otimes_A V, W) = \text{Hom}_A(V, \text{Hom}_B(hB, W))$$
Now there should be a map

\[ hB \otimes B \rightarrow \text{Hom}_B (hB, W) \]

So you get

\[ \text{Hom}_B (hB \otimes V, W) = \text{Hom}_B (V, \text{Hom}_B (hB, W)) \]

\[ \text{Hom}_B (V, hW) \]

So there seem to be canonical maps in the right direction.

Feb 27, 02: Check ideas of last night. The point is to identify

\[ W \rightarrow hW \quad V \rightarrow p(\Lambda \otimes V) \quad \text{with} \]

\[ G(W) = \text{Im} \left\{ hB \otimes B \rightarrow \text{Hom}_B (hB, W) \right\} \]

\[ F(V) = \text{Im} \left\{ hB \otimes V \rightarrow \text{Hom}_B (hB, V) \right\} \quad \text{resp.} \]

Note that 1. factors \[ hW \rightarrow (hB \rightarrow hB \otimes hW = hB hW) \]

\[ hB \otimes B \rightarrow hW \rightarrow \text{Hom}_B (hB, W) \]

\[ hB \otimes W \rightarrow hB \otimes hW \]

\[ \text{Hom}_B (hB, V) \leftrightarrow \text{Hom}_B (hB, W) \]

\[ \Lambda \otimes V \leftarrow W \leftarrow \Lambda \otimes V \]

1. is \[ \Lambda \otimes W \rightarrow (hB \rightarrow hB \otimes hW) \]

- If \[ hB \otimes hW = 0 \] for all \( B \)

then \[ hB \otimes W = 0 \]

hence \[ \Lambda \otimes W = 0 \]

3. is \[ hB \otimes hW \rightarrow hB hW \]

and is surj since \[ B hW = B hB W = B W \]

\[ \Lambda \otimes V \leftarrow \Lambda \otimes V \]

3. is \[ B hW = B hB W = B W \]

compose 3 then 4

\[ b, h \otimes hW \rightarrow b hW \rightarrow (hB \rightarrow hB \otimes hW) \]
to write a version (permanent)

Category $W$ objects are $\Gamma$-mod $W$ with $h \in \text{End}_\Gamma(W)$

$s 
\forall w \sum_{t \in \mathbb{T}} wh^{-1} w = w$ (This means the sum is finite)

Category $V$ objects are $\text{u.s. } V$ with $p(s) \in \text{End}_\Gamma(W)$

Set $p(s) \neq 0 \Rightarrow s \in \mathbb{T}$

$\sum_{t} p(st^{-1}) p(t) = p(s)$:

$\sum_{s} p(s) V = V, \bigcap_{s} \ker p(s) \cap V = 0$.


canonical factor into our $\tilde{V}$ followed by any $j$

let $p(s) = \tilde{j} i s i \in \text{End}_\Gamma(V)$

$p(s) \neq 0 \Rightarrow \sum_{t} p(st^{-1}) p(t) u = \sum_{t} j s t i g t i v = j s u = p(s) v$.

$V = \sum_{s} j s h W = \sum_{s} p(s) V$

$\forall s p(s) \nu = 0 \Rightarrow \sum s h s^{-1} \nu = \nu$

$0 = \sum s i \tilde{j} s^{-1} i v \Rightarrow \nu = 0 \Rightarrow \tilde{v} = 0$

Next show how to recover $W$ from $V$. Let $\Lambda = \text{Gr}, \Lambda \otimes V$ is the free $\Gamma$ module gen by $V$.

$\Lambda \otimes V = \{ \sum_{t \otimes f(t)} | f: \Gamma \rightarrow V \text{ finite supp} \}$
Let \( \Lambda \otimes V \) define \( \Lambda \otimes V \leftarrow^p \Lambda \otimes V \)
by
\[ p(\sum_t t \otimes f(t)) = \sum_s s \otimes \sum_t p(s^{-1} t) f(t). \]

Properties:
\[ p \cdot u = u \cdot p, \quad p^2 = p. \]

- \[ u \sum_t t \otimes f(t) = \sum_t t \otimes f(u^{-1} t) = u \sum_t s \otimes \sum_t p(s^{-1} t) f(t) \]
- \[ p u \sum_t t \otimes f(t) = p \sum_t t \otimes f(t) = \sum_s s \otimes \sum_t p(s^{-1} t) f(t) \]

Therefore you can recover \( W \) as the image of \( p = \alpha \beta \).

\[ \beta \] is the unique \( \Gamma \)-mod map extending \( i \)

Note \( \alpha \) is the unique \( \Gamma \)-mod map extending \( j \).

\[ \sum_s s \otimes g s^{-1} \omega = \sum_s s \otimes \sum_t g s^{-1} t \omega. \]
\[ \wedge \otimes V \leq p ( \Lambda \otimes V ) \]

\[ p \left( \sum_t t \otimes f(t) \right) = \sum_s \otimes \sum_t p(s^{-1} t) f(t) \]

\[ p \left( u \sum_t t \otimes f(t) \right) = p \left( \sum_t u t \otimes f(t) \right) = p \left( \sum_t u s^{-1} t \otimes f(u^{-1} t) \right) \]

\[ = \sum_s \otimes \sum_t p(s^{-1} t) f(u^{-1} t) = \sum_s \otimes \sum_t p(s^{-1} u t) f(u^{-1} t) \]

\[ = \sum_s u s \otimes \sum_t p ((us)^{-1} u t) f(t) = u p \left( \sum_t t \otimes f(t) \right) \]

\[ \prod p \left( \sum_u u \otimes f(u) \right) = \left[ \sum_t \prod \sum_u p(t^{-1} u) f(u) \right] \]

\[ = \sum_s \otimes \sum_t p(s^{-1} t) \sum_u p(t^{-1} u) f(u) \]

\[ = \sum_s \otimes \sum_u \left( \sum_t p(s^{-1} t) p(t^{-1} u) \right) f(u) \]

\[ = \sum_s \otimes \sum_u p(s^{-1} t) \sum_t p(t^{-1} u) f(u) \]

\[ W = p ( \Lambda \otimes V ) \]

To find \( W \leftarrow V \leftarrow W \)

Think of \( \alpha \) as the inclusion, \( \beta \) as the proj.

Diagram:

\[ \Lambda \otimes V \xrightarrow{\alpha} W \xrightarrow{\beta} \Lambda \otimes V \xrightarrow{\alpha} W \]

\[ \text{an elt of } W, \sum_s \otimes \sum_t p(s^{-1} t) f(t) = p \left( \sum t \otimes f(t) \right) \]

\[ j \left( p \left( \sum_t t \otimes f(t) \right) \right) = \sum_t \otimes \sum_s \sum_t p(s^{-1} t) f(t) = \sum_s \otimes \sum_t p(s^{-1}) \sum_t p(t) f(t) \]

Seems too hard.
somehow you should be able to start with $V$ and construct $W$. Given $V$ with the operators $p(s)$ define $p$ on $\bigwedge V$ by
\[ p\left(\sum t \otimes f(t)\right) = \sum s \otimes \sum p(s^{-1})f(t) \]
Thus $p$ is the linear span of vectors indexed by $t \in T$ given by matrix $p(s^{-1}t)$. Then because this kernel is invariant under $(s, t) \mapsto (us, ut)$ (under left translation) $\sum p(s) p(t) = p(u)$. Then define $W = p(\bigwedge V)$ so you get $W$ retract of $\bigwedge V$.

\[ W \xleftarrow{\beta} \bigwedge V \xleftarrow{\alpha} W \xrightarrow{p} \bigwedge V \xrightarrow{\beta} W \]

\[ 1 \bigwedge V = \sum s \otimes \gamma s^{-1} \]

\[ 1 = p \alpha = \sum \beta s \otimes \gamma s^{-1} = \sum s h s^{-1} \]

Want matrix elements
\[ \eta, s^{-1} \alpha \beta \epsilon_1 = \eta s^{-1} \epsilon_1 = p(s^{-1}) \]

Review: Given $V$ with $p(s)$ get $p(s^{-1})$ via: under left multipl $s t \mapsto us, ut$ and identity. Let proj $p$ on $\bigwedge V$ $pu = up, p^2 = p$, know $\eta, s^{-1} p \epsilon_1 = p(s^{-1})$

Explain $\alpha, \beta$

\[ \eta s^{-1} \alpha \beta \epsilon_1 = \eta s^{-1} \beta \epsilon_1 = \eta s^{-1} i \epsilon_1 = p(s^{-1})i \]

Given $V$, $\Lambda = o T$, $\bigwedge V = \{f : T \to V \mid \text{fin supp } f\}$

$(uf)(t) = f(u^{-1}t)$, define $(pf)(s) = \sum p(s^{-1}u) f(t)$. Then

$(puf)(s) = \sum p(s^{-1}u) f(u^{-1}t) = \sum p(s^{-1}ut) f(t)$, $(u(pf))(s) = (pf)(u^{-1}s) = \sum p(s^{-1}ut)f(t)$

$p u = u p$, $p^2 = p$. Let $W = p(\bigwedge V)$, $\beta = p : \bigwedge V \to W$, $\beta^T = \text{mic}$ $W \to \bigwedge V$.

\[ W \xleftarrow{\beta} \bigwedge V \xleftarrow{\alpha} W \xrightarrow{\beta} \bigwedge V \]

\[ \sum s \otimes \eta, s^{-1} = 1 \bigwedge V \]

\[ \sum s \otimes \eta, s^{-1} = 1 W \]
\[ \text{Consider with } V \text{ with } p(s) \neq 0 \Rightarrow s \in \mathbb{F} \]

\[ \sum_{s \in \mathbb{F}} p(s)p(s) = p(0) \quad \text{free } \Gamma \text{ module } \bigwedge V = \{ f: \Gamma \to V | \text{fin. supp } \} \quad (ut)(t) = f(u \cdot t) \]

\[ (pf)(s) = \sum_{t \in \Gamma} p(s \cdot t) f(t), \quad pu = up, \quad p^2 = p \]

\[ W = p(\bigwedge V), \quad \beta = p: \bigwedge V \to W \]

\[ \alpha = \text{inj} \quad \xi = \text{surj} \]

\[ \text{Then } \bigwedge V = \sum_{s \in \mathbb{F}} \eta_s s^{-1} \Rightarrow \sum s h s^{-1} = 1 \]

**Disjcn:** Intermediate cut of \( U = (V, W, i, f) \)

where \( W \) is a \( \Gamma \)-module, \( V \) a vector space, and

\[ i: V \to W, \quad f: W \to V \]

an \( \mathbb{C} \)-linear maps satisfying

\[ j s i \neq 0 \Rightarrow s \in \mathbb{F}, \quad \sum s h s^{-1} = s, \quad \text{w} = w \quad \forall w \in W \]

i inj, j surj.

\[ \text{function } U \to \bigwedge V \quad \gamma \quad V \text{ with } p(s) = j s i \]

\[ \sum_{s \in \mathbb{F}} s h s^{-1} = w \quad \Rightarrow \sum_{s \in \mathbb{F}} p(s) j s^{-1} w = j w \quad \Rightarrow \sum_{s \in \mathbb{F}} p(s) W = V \]

Assume \( p(s) v = 0 \), i.e., \( j s i v = 0 \), \( \Rightarrow q = \sum_{s \in \mathbb{F}} s h s^{-1} v = 0 \)

\[ U = \text{cut of } V, W, i: V \to W, \quad j: W \to V \]

where

\[ W \text{ is } \Gamma \text{-module, } V \text{ n.s., } i \text{ inj, } j \text{ surj } \]

\[ j s i \neq 0 \Rightarrow s \in \mathbb{F} \]

\[ \forall w \sum_{s \in \mathbb{F}} s h s^{-1} w = w \]

**Disjcn:** Intermediate cut of \( U = (V, W, i, f) \)

\[ \text{function } U \to \bigwedge V \quad \gamma \quad V \text{ with } p(s) = j s i \]
\[ p(s) \neq 0 \implies s \in \Xi \quad \implies \quad \sum_i p(t) p(w) = \sum_{s \in \Xi} g_{t s} g_{s i} = \sum_{s \in \Xi} g_{t s} g_{s i} \]

\[ W = \sum_s s \cdot \Xi \implies V = j W = \sum_s p(s) V \]

\[ \forall \sigma \in W \quad 0 = p(\sigma) \sigma = g_{s \sigma} \sigma, \quad \forall \sigma \implies 0 = \sum_s g_{s \sigma} s \cdot \Xi \sigma = 0 \implies v = 0. \]

V have just showed \( \forall (V, W, v, g) \in \mathcal{U} \)

that \( (V, p(s) = g s \sigma) \in \mathcal{U} \)

Digress to review the Toopology algebra, simplest case. It the limited algebra \( R \) gen. \( \times, y \) subject to reln \( y x = 1 \).

Natural \( R \)-module \( C’[\Xi] \) with \( x \Xi^n = \Xi^{n+1} \) and 
\( y \Xi^n = \Xi^{n-1} \) for \( n > 1 \), and \( y \Xi^1 = 0 \). \( R \) is spanned by words \( n, x \) which can be replaced by words of smaller length

where \( y \) is followed by \( x \).

\( R \) spanned by \( x^n y^m \) with \( m, n \geq 0 \).

Start with \( V \), produce \( W \),
\[ \Lambda \otimes V = \{ \sum_t t \otimes f(t) \mid f : \Gamma \to V \text{ fin supp} \} \]

\[ u \sum_t t \otimes f(t) = \sum_t t \otimes f(u \cdot t) \]

Define
\[ p(\sum_t t \otimes f(t)) = \sum_s s \otimes \sum_t p(s \cdot t) f(t) \]

\[ p u (\sum_t t \otimes f(t)) = p \sum_t t \otimes f(u \cdot t) \]

\[ = \sum_s s \otimes \sum_t p(s \cdot t) f(u \cdot t) \]

\[ = \sum_s s \otimes \sum_t p(s \cdot t) f(t) \]

\[ = \sum_s s \otimes \sum_t p(s \cdot t) f(t) = u p(\sum_t t \otimes f(t)) \]

Define \( W = p(\Lambda \otimes V) \). \( W \) \( \Gamma \)-module retract of \( \Lambda \otimes V \)

means \( \exists \Gamma \text{-maps} \)
\[ \Lambda \otimes V \xrightarrow{\alpha} W \xleftarrow{\beta} \Lambda \otimes V \]

\[ V \xrightarrow{\delta} W \]

\[ \Lambda \otimes V \leftrightarrow W \leftrightarrow \Lambda \otimes V \]

\[ W \leftrightarrow W \]
Now you've defined $W$, $s$, $h = y$

$h \neq 0 \Rightarrow \forall i \neq \emptyset \Rightarrow \text{see } \Xi$

$\sum s \cdot h \cdot s^{-1} w = \sum s \beta s \cdot s^{-1} w = \beta \chi s \cdot s^{-1} w$

So $W$ has the desired properties. So from $V$ you have constructed $W$ with all its properties, although you haven't used $V$ reduced. $W = \{h \beta \land \Theta V \} = \eta_{\alpha} \beta \land \Theta V$

$\sum p(t) f(t) | f : \Gamma \to V \} = \sum p(t) V$

So if $p(s) \neq 0$ and $\alpha \neq 0$

$V \neq 0 \Rightarrow \forall i \neq \emptyset \Rightarrow \text{see } \Xi$

$W = \{h \beta \land \Theta V \}

\begin{align*}
W & \Gamma \text{-mod } w \ h \ h \neq 0 \Rightarrow s \in \Xi \\
\sum s \cdot h \cdot s^{-1} w & = w \\
V & = h W \ h = y : W \leftarrow V \leftarrow h W \\
p(s) & = \{s \in \text{End}(V) \} \\
p(s) \neq 0 & \Rightarrow h \neq 0 \Rightarrow s \in \Xi
\end{align*}$

Crazy idea: nil modules could they be states with zero energy? Is there something interesting arising from $A = \Phi_{\Xi}^\Gamma$ modules which are not reduced?

So given $V$ explain $\Lambda \otimes V = \{s \in f(t) \} = \sum s \otimes f(t) \cdot$$p(s) \neq 0 \Rightarrow h \neq 0 \Rightarrow s \in \Xi$

$h \neq 0 \Rightarrow \forall i \neq \emptyset$

\begin{align*}
\sum s \cdot h \cdot s^{-1} w & = w \\
V & = h W \ h = y : W \leftarrow V \leftarrow h W \\
p(s) & = \{s \in \text{End}(V) \} \\
p(s) \neq 0 & \Rightarrow h \neq 0 \Rightarrow s \in \Xi
\end{align*}$

hsh = y $s \in \emptyset$

\begin{align*}
\text{need } \forall i \neq \emptyset \\
\text{better } s^{-1} i \in = \text{p}(s^{-1} i)
\end{align*}$

You failed to discuss $\Lambda \otimes V$ adequately. $p = \alpha \beta$

With $\eta_{\alpha} s^{-1} t e_i = \delta_{\alpha i} = \delta_{\alpha i} \beta e_i$.

$\sum s \cdot \eta_{\alpha} s^{-1} = \lambda \otimes V$. $\Lambda \otimes V$

\begin{align*}
\sum s \cdot \eta_{\alpha} s^{-1} & = \lambda \otimes V, \\
\eta_{\alpha} s^{-1} p \neq \epsilon & = \eta_{\alpha} x s^{-1} t \beta e_i \\
\eta_{\alpha} s^{-1} t l & = y^{-1} s^{-1} t l = p(s^{-1} t)
\end{align*}$
What do you want for $\Lambda \otimes V$? $\Lambda \otimes V = \{ f : I \to V : \text{supp}_I f \leq 0 \}$

For splitting $\Lambda \otimes V = \oplus V$ need $\varepsilon_1 \eta_1$

and $\eta_1 s^{-1} \varepsilon_1 = \delta_s \otimes V$, $\sum s \varepsilon_1 \eta_1 s^{-1} = \Lambda \otimes V$

$\varepsilon_1 V = 1 \otimes V$

$\eta_1 \sum t \varepsilon(t) = f(1)$ $\Lambda \otimes V$ is the direct sum of subspaces $s \otimes V$

**IDEA** Orthogonal, completeness relations, the latter can be expressed as a summand. So far you've been looking at discrete cases, but the holomorphic representation of the CR gives completeness but not orthogonality in a continuous setting.

There are some maps

$$W \hookrightarrow \Lambda \otimes V \hookleftarrow W$$

$$\varepsilon_1 \downarrow \eta_1 \downarrow$$

$$\beta_0 \mapsto f \mapsto h$$

Let $g = \eta_1 \alpha$, $c = \beta \varepsilon_1$, $h = y$

Claim $W$ with $h \in W$.

$h s h = (g s c) h$ need $\eta_1 \varepsilon_1 = p(s)$

$\eta_1 s p \varepsilon_1$, $\eta_1 s p \varepsilon_1 = p$?

Confused again. $p(s^{-t}) = \eta_1 s^{-1} p t \varepsilon_1$

$\beta_0 = \alpha = \beta \sum s \varepsilon_1 \eta_1 s^{-1} \varepsilon_1$

Next composition. $W \hookrightarrow V = h W$ and then you must construct $W = p(\Lambda \otimes V)$. What you have is just $\varepsilon_1$, $\Gamma$ on $W$. Other side is probably easy, namely $V \mapsto p(\Lambda \otimes V) \mapsto h p(\Lambda \otimes V)$ why?

$$\Lambda \otimes V \hookrightarrow p(\Lambda \otimes V) \hookleftarrow p(\Lambda \otimes V) \hookleftarrow \Lambda \otimes V$$

$$\varepsilon_1 \downarrow \downarrow \downarrow$$

$W = \eta_1 p(\Lambda \otimes V)$

$\eta_1 \sum s \otimes \sum t \varepsilon(t) f(t) = \sum t p(t) f(t)$
Next you to go from \( W \) to \( V = hW \) to

\[
p(x \forall V), \quad \text{given } W \text{ you form } \cdot \text{ can fact.}
\]

\[
\begin{array}{c}
W
\downarrow^B
\Lambda \circ V
\downarrow^t
\end{array}
\begin{array}{c}
\uparrow^t
\ni
\uparrow^J
\end{array}
\]

Claim \( \beta \) \( \beta \) is unique \( \Gamma \)-module map ext. \( \gamma \) in the sense that \( \beta \gamma = \iota \nabla \gamma = t \).

\[
\begin{aligned}
\kappa \cdot \varsigma &= \sum_{s} \sum_{\nu} s \circ \varsigma \nu \\
\kappa \cdot \varsigma &= \sum_{s} \sum_{\nu} s \circ \varsigma \nu \cdot \nu
\end{aligned}
\]

\[
\begin{aligned}
\kappa \cdot \varsigma &= \sum_{s} \sum_{\nu} s \circ \varsigma \nu \\
\kappa \cdot \varsigma &= \sum_{s} \sum_{\nu} s \circ \varsigma \nu \cdot \nu
\end{aligned}
\]

\[
\begin{aligned}
\kappa(\varsigma) &= \sum_{s} \sum_{\nu} s \circ \varsigma \nu \\
\kappa(\varsigma) &= \sum_{s} \sum_{\nu} s \circ \varsigma \nu \cdot \nu
\end{aligned}
\]

What's left \( n \) to calculate that \( \beta \gamma = \sum_{s} \sum_{\nu} s \circ \varsigma \nu \cdot \nu = \sum_{s} \sum_{\nu} s \circ \varsigma \nu \cdot \nu = \sum_{s} \sum_{\nu} s \circ \varsigma \nu \cdot \nu
\]

Next you want to identify \( V \) and \( W \) with \( M_\nu \) for idempotent rings.

IDEA: Recall you \( GNS \) algebra associated to linear map \( f : A \rightarrow B \) between \( \nu \)-noid rings satisfying \( \rho 1 = 1 \).

Module category consists of \( A \)-module \( M \), \( B \)-module \( N \), and maps \( N \rightarrow M \rightarrow N \) satisfying \( \rho(a)n = \iota(a)n \).

Question: Is there some \( \nu \)-motional version of this? \( GNS \) alg

\[
\begin{array}{c}
\Gamma(\varsigma) = A \circ A \circ B \circ A
\end{array}
\]

\[
\begin{array}{c}
\alpha \circ \beta \circ \gamma \circ \delta
\end{array}
\]

Dilatation of \( B \)-module \( N \); you want to construct \( M \), can use any factorization

\[
\begin{array}{c}
A \circ N \rightarrow M \rightarrow \text{Hom}(A, N)
\end{array}
\]

\[
\begin{array}{c}
\alpha \circ n \rightarrow \rho(a)n \rightarrow (a' \rightarrow (a \circ a'n)n)
\end{array}
\]

\[
\begin{array}{c}
\alpha \circ m \rightarrow \rho(a'm)m
\end{array}
\]

\[
\begin{array}{c}
\text{minimal } M \text{ is the image of this } \rho \text{ map.}
\end{array}
\]

So concentrate on idemp. rings. Clearly \( V \) is the cat. of \( M_\nu(A) \)

\[
\begin{array}{c}
A = \rho_1
\end{array}
\]

\[
\begin{array}{c}
B = \rho_2
\end{array}
\]

So what can
First examine the idea of amplitudes without orthogonality. You have the free case

\[ \Lambda \otimes V = \bigoplus_s s \otimes V = \bigoplus_s V \]

\[ \eta_s (\sum_t t \otimes f(t)) = f(s) \]

\[ \varepsilon_t V = t \varepsilon_t V \quad \text{with} \quad \eta_s \varepsilon_t = \eta_t s^{-1} \varepsilon_s = \delta_{st} \]

\[ \sum_s \varepsilon_s \eta_s s^{-1} = 1 \quad \Lambda \otimes V \]

I think you encountered this before when you tried to construct a Morita context by gauss + h Felix. The generators die off diagonal \( \mathfrak{s} \) and \( \mathfrak{t} \) yield the primary maps between \( V \) and \( W \). Morita context

\[ W \xrightarrow{\varepsilon_t} \Lambda \otimes V \xrightarrow{\eta_s s^{-1}} V \]

\begin{align*}
W & \xrightarrow{\varepsilon_t} \Lambda \otimes V \\
& \xrightarrow{\eta_s s^{-1}} V
\end{align*}

\[ \left( \begin{array}{cc}
A & y_t s^{-1} \\
\varepsilon_t = x_t & B
\end{array} \right) \]

It seems that you defined \( D \) to be the \( M_2 \)-graded algebra with generators \( x_t \) of degree \( \{3,1\} \) and \( y_t \) of degree \( \{1,2\} \). Here \( s, t \in \Gamma \). The relations are

\[ \sum_s x_s y_s = 1 \quad \text{in Cuntz's sense i.e.} \quad \sum_s x_s y_s x_t = x_t \]

\[ \text{and} \quad \sum_t y_t x_t y_t = y_s \]

Recall you need also the left \( \Gamma \)-invariance: \( y_t x_t \) depends only on \( s^{-1} t \). A generated by \( y_t x_t = y_t s^{-1} x_t \)

\[ \Gamma \]

\[ B \] generated by \( x_t y_s = t i s^{-1} \)

\[ \Gamma \times \Gamma \]

\[ x_t y_s = t i s^{-1} h_s = h_t h_s \]

All this is fascinating, but let's be careful over \( B \) with

\[ Y = B \]

\[ X = B \]

\[ \langle b_1, b_2 \rangle \rightarrow b_1 h b_2 \]

\[ (x, y) = x h y \]
dual pair over $B$ consisting of $X = B$ left multi
$
Y = B$ right multi,
\[
\langle x, y \rangle = x y
\]
assume $x B B = B$.

$A = \frac{Y \otimes X}{B} = B$ if $B$ finit

\[
(y x) y = y x h y \quad a y = a h y \quad xa = x h a
\]

So you get a Morita context \((A = B, Y = B)\)

\[
\begin{array}{cccc}
Y & X & Y & X \\
\hline
Y & x_1 & y_1 & y_2 \\
& x_1 & b_1 & b_2 \\
X & y_1 x_2 & y_2 & x_1 h y_2 + y_1 b_2 \\
& x_1 h y_2 + b_1 b_2 & x_h y_2 + b_1 b_2 \\
\end{array}
\]

\[
\left( \begin{array}{cc}
Y & X \\
\end{array} \right) \left( \begin{array}{c}
V \\
W \\
\end{array} \right) \quad \text{right module}
\]

right module \((M, N) \left( \begin{array}{cc} j B_i & j B \\
B_i & B \\
\end{array} \right) \)

\[
M \otimes_{A_1} B \rightarrow N
\]

$N \otimes_{B_i} B_i \rightarrow M$

\[
\begin{array}{c}
M \rightarrow N \rightarrow M \\
N \rightarrow M \rightarrow N \\
\end{array}
\]

the only problem is that factoring
\[h \cdot j \rightarrow \text{cancellation} \]
\[j \cdot l \rightarrow \text{surjective} \]
\[j \rightarrow \text{injective} \]

IDEA: This pretending an operator $h$ is idempotent, could it
generalize to a chain of operators $\cdots \rightarrow \cdots$, and perhaps
be useful for higher K-theory purposes.