January 16, 2002

Make some notes on Serre's thm about the equivalence of categories between fin. gen. proj modules over $C(X)$, X compact, and (complex) vector bundles over X. Points:

- abstract category stuff. notions of [redacted] retract Y of an object X: $Y \xrightarrow{i} X \xrightarrow{j} Y$ $\quad ji = 1_Y$, projection: $p: X \to X$ $\quad p^2 = p$, Karoubian category, Karoubian envelope.
- Any v.b. E/X is a retract of a trivial bundle $X \times V$, V fin diml v.s. True locally as E is locally trivial, hence $\exists$ finite open covering $X = U_1 \cup \cdots \cup U_n$ and retracts

$$E_{U_\mu} \xleftarrow{j_\mu} U_\mu \times V_\mu \xleftarrow{i_\mu} E_{U_\mu} \qquad j_\mu i_\mu = id \text{ on } E_\mu.$$

$\exists$ partition of $1 = \sum \chi_\mu^2$, $\operatorname{Supp} \chi_\mu \subset U_\mu$.

$$E \xleftarrow{(\cdots \chi_\mu j_\mu \cdots)} X \times \begin{pmatrix} V_1 \\ \vdots \\ \vdots \\ V_n \end{pmatrix} \xleftarrow{\begin{pmatrix} \vdots \\ \chi_\mu i_\mu \\ \vdots \end{pmatrix}} E \qquad \begin{aligned} &\sum \chi_\mu j_\mu \chi_\mu i_\mu \\ &= \sum \chi_\mu^2 = 1 \text{ on } E \end{aligned}$$

- If $p^2 = p$ is a projection on the vector bundle E/X, then locally there exists a [redacted] trivialization $E_U \simeq U \times V$ such that p [redacted] becomes $1_U \times$ a fixed projection on V.

Proof: can suppose $E = X \times V$ trivial, p is a continuous family $x \mapsto p_x = p_x^2 \in \operatorname{End}(V)$. Shift from projections to involutions $F_x = 2p_x - 1$. Let $\varepsilon = F_0$ at the point 0 of interest, put $g_x = F_x \varepsilon$, so that $\varepsilon g_x \varepsilon^{-1} = \varepsilon F_x = g_x^{-1}$

Use $\exp: \mathrm{End}(V) \longrightarrow \mathrm{Aut}(V)$, local diffeom near zero, to define $g_x^{1/2} = \exp(\frac{1}{2}\log g_x)$ a continuous family of autos of V. Then $\varepsilon g_x \varepsilon^{-1} = g_x^{-1} \implies \varepsilon g_x^{1/2} \varepsilon^{-1} = g_x^{-1/2}$ (think of the 1-parameter subgroup $t \mapsto \exp(t \log g_x)$. Finally $g_x^{1/2}\, \varepsilon\, g_x^{-1/2} = g_x \varepsilon = F_x$, [redacted] which means that the bundle autom $x \mapsto g_x^{1/2}$ transforms F_x to ε.

Heisenberg group + Lie alg. The Lie alg has basis X, Y, H with relations $[X,Y] = H$, $[X,H] = [Y,H] = 0$

$$\partial_t \left(e^{tX} Y e^{-tX}\right) = e^{tX}(XY - YX)e^{-tX} = e^{tX} H e^{-tX} = H.$$

so $e^{tX} Y e^{-tX} = Y + tH$. Then [redacted]

$$e^{tX} e^{sY} e^{-tX} = e^{s(e^{tX} Y e^{-tX})} = e^{s(Y+tH)} = e^{stH} e^{sY}$$

since H, Y commute. $\therefore$ $\boxed{e^{tX} e^{sY} e^{-tX} e^{-sY} = e^{stH}}$

Also let $u(t) = e^{-tX} e^{t(X+Y)}$. Then $\partial_t u(t) =$

$$e^{-tX}(-X)e^{t(X+Y)} + e^{-tX}(X+Y)e^{t(X+Y)} = e^{-tX} Y e^{tX}\left(e^{-tX} e^{t(X+Y)}\right),$$

so $\partial_t u(t) = (Y - tH)\, u(t) \implies u(t) = e^{tY - \frac{t^2}{2}H}$, so

$$\boxed{e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}H}}$$

(Check: $e^{t(X+Y)} e^{-t(X+Y)} = e^{tX} e^{tY} e^{-tX} e^{-tY} e^{-t^2 H} = 1$)

January 22, 2002

Consider the principal bundle $\pi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2 = T^2$ where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translations. Let $\mathcal{A} = C^\infty(T^2)$, equiv. $\mathcal{A}$ is the ring of smooth functions $f(x,y)$ on $\mathbb{R}^2$ which are doubly periodic (period 1): $f(x+m, y+n) = f(x,y)$. Let $\mathcal{L}$ be the space of smooth functions [redacted] $\psi(x,y)$ on $\mathbb{R}^2$ satisfing $\psi(x,y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)$; equiv.

$$\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y).$$

Note that $(f,\psi) \longmapsto f\psi$ makes $\mathcal{L}$ into an $\mathcal{A}$-module.

More generally for any open set $U \subset T^2$, let $\mathcal{A}(U) = C^\infty(U)$ = doubly periodic smooth $f(x,y)$ on $\pi^{-1}U$, and let $\mathcal{L}(U)$ = smooth functions on $\pi^{-1}U$ satisfying the above automorphy conditions.

Assume there is given a trivialization of the principal bundle π over U: $s : U \to \pi^{-1}U$, $\pi s = id_U$. Then $\pi^{-1}U \simeq sU \times \mathbb{Z}^2$, so that $\mathcal{A}(\pi^{-1}U) \simeq C^\infty(U)$ and $\mathcal{L}(\pi^{-1}U) \simeq C^\infty(U)$, where these isos. arise by composing with s. Thus one sees that locally over the 2-torus, $\mathcal{L}(U)$ is a free module of rank 1 over $\mathcal{A}(U)$.

So it should be clear that $\mathcal{L}$ is the space of smooth sections of a smooth line bundle L over the torus.

Question: Is this line bundle trivial, equiv. does there exist $\psi \in \mathcal{L}$ nowhere vanishing?

No, because there's a degree obstruction. Assuming [redacted] $\psi \neq 0$ [redacted] then [redacted] $\displaystyle\int_{y=0}^{y=1} \psi(x,y)^{-1}\, d\psi(x,y)$ is independent of x by Stokes's thm. On the other hand the automorphy condition gives

$$\psi(x,y)^{-1}d\psi(x,y) = 2\pi i\, dy + \psi(x+1,y)^{-1}d\psi(x+1,y)$$

and $\displaystyle\int_{y=0}^{y=1} 2\pi i\, dy = 2\pi i$, showing the degree jumps.

The next step will be to construct a connection on the v.b. L over T^2. On T^2 we have the commuting vector fields ∂_x, ∂_y which generate the tangent space at each point. A connection on L can be described as operators D_x, D_y on the space of sections $\mathcal{L}$ which are compatible with ∂_x, ∂_y in the sense that Leibniz's rule holds:

$$D_x(f\psi) = \partial_x f\,\psi + f\,D_x\psi \quad , \quad \text{sim. for } y.$$

Claim $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ are operators on $\mathcal{L}$ compatible with multiplication by elements of $\mathcal{A}$ in the above Leibniz sense.

Easy for D_x:

$$\psi(x,y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)$$

$$\Longrightarrow (\partial_x\psi)(x,y+1) = (\partial_x\psi)(x,y) = e^{2\pi i y}(\partial_x\psi)(x+1,y)$$

For D_y use the alternate form of the automorphy condition:

$$e^{2\pi i xy}\psi(x,y+1) = e^{2\pi i xy}\psi(x,y) = e^{2\pi i (x+1)y}\psi(x+1,y)$$

We apply $e^{-2\pi i xy}\partial_y$ to these three terms to get

$$(2\pi i x + \partial_y)\psi(x,y+1) = (2\pi i x + \partial_y)\psi(x,y)$$

$$= e^{2\pi i y}\, 2\pi i(x+1)\,\psi(x+1,y) + e^{2\pi i y}(\partial_y\psi)(x+1,y)$$

$$= e^{2\pi i y}\Big(\big(2\pi i\, x\psi + \partial_y\psi\big)(x+1,y)\Big)$$

showing $(\partial_y + 2\pi i x)\psi$ also satisfies the automorphy condition.

Simpler method uses the isomorphism

$$f(x) \longmapsto \tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m)$$

between $\mathcal{S}(\mathbb{R})$ and $\mathcal{L}$.

$$\partial_x \tilde{f}(x,y) = \sum_m e^{2\pi i m y} \frac{d}{dx} f(x+m) = \widetilde{\frac{df}{dx}}$$

$$(\partial_y + 2\pi i x)\tilde{f}(x,y) = \sum_m e^{2\pi i m y} \underbrace{(2\pi i m + 2\pi i x)}_{2\pi i (x+m)} f(x+m)$$

$$= \widetilde{(2\pi i x f)}$$

Thus the operators D_x, D_y when viewed in the $\mathcal{S}(\mathbb{R})$ picture are

$$D_x f = \frac{d}{dx} f$$

$$D_y f = 2\pi i x f$$

Note that $[D_x, D_y] = 2\pi i$ so that we have an action of the Heisenberg Lie algebra on $\mathcal{L}$ compatible with the action of $\mathbb{R}\partial_x + \mathbb{R}\partial_y$ on $\mathcal{A}$.

January 24, 2002 (David is 38)

Recall $\mathcal{L} = \left\{ \psi(x,y) \in C^\infty(\mathbb{R}^2) \;\middle|\; \begin{array}{l} \psi(x,y) \text{ periodic } 1 \text{ in } y \\ e^{2\pi i x y}\psi(x,y) \text{ period } 1 \text{ in } x \end{array} \right\}$

Claim $\mathcal{L}$ closed under the operators $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$:

∂_x, ∂_y preserves translations $(x,y) \mapsto (x+a, y+b)$. So $\partial_x \psi(x,y)$ has period 1 in y, and

$$\underbrace{\partial_x\left(e^{2\pi i x y}\psi(x,y)\right)}_{\text{per } 1 \text{ in } x} = \underbrace{2\pi i y\, e^{2\pi i x y}\psi(x,y)}_{\text{per } 1 \text{ in } x} + e^{2\pi i x y}\partial_x\psi(x,y) \quad \text{so } \partial_x\psi \in \mathcal{L}.$$

Next, $D_y\psi(x,y) = \partial_y\psi(x,y) + 2\pi i x\,\psi(x,y)$ period 1 in y.

$$\underbrace{\partial_y\left(e^{2\pi i x y}\psi(x,y)\right)}_{\text{per } 1 \text{ in } x} = e^{2\pi i x y}\left(\partial_y\psi(x,y) + 2\pi i x\,\psi(x,y)\right)$$
$$= e^{2\pi i x y} D_y\psi(x,y) \quad \text{has period } 1 \text{ in } x$$

so $D_y\psi \in \mathcal{L}$.

Recall that $\mathcal{L}$ is a module over the ring $\mathcal{A} = C^\infty(T^2)$ of $f(x,y)$ having period 1 in both x,y.
D_x has the derivation (Leibniz) property

$$D_x(f\psi) = \partial_x f\,\psi + f D_x\psi$$

and similarly for D_y. This implies upon exponentiating that

$$e^{aD_x} f(x,y)\psi(x,y)$$
$$= \left(e^{a\partial_x} f(x,y)\right)\left(e^{aD_x}\psi(x,y)\right) = f(x+a,y)\, e^{aD_x}\psi(x,y)$$

and similarly for bD_y. This means that we obtain operators $e^{aD_x}e^{bD_y}$ on $\mathcal{L}$ compatible with the translation operators $e^{a\partial_x}e^{b\partial_y} f(x,y) = f(x+a, y+b)$ on $\mathcal{A}$.

Let's calculate $e^{aD_x}e^{bD_y}$ on $\mathcal{L}$. Since

$$D_y = \partial_y + 2\pi i x \quad \text{where } \partial_y,\ 2\pi i x \text{ commute}$$

we have $e^{bD_y}\psi(x,y) = e^{b2\pi i x} e^{b\partial_y}\psi(x,y)$

$= e^{2\pi i b x}\psi(x, y+b)$. Then

$$e^{aD_x} e^{bD_y}\psi(x,y) = e^{a\partial_x} e^{2\pi i b x}\psi(x, y+b)$$
$$= e^{2\pi i b(x+a)}\psi(x+a, y+b).$$

Also $e^{bD_y} e^{aD_x}\psi(x,y) = e^{2\pi i b x} e^{b\partial_y} e^{a\partial_x}\psi(x,y)$

$$= e^{2\pi i b x}\psi(x+a, y+b).$$

Thus $e^{aD_x} e^{bD_y} = e^{2\pi i a b} e^{bD_y} e^{aD_x}$ as expected in the Heisenberg group since $[aD_x, bD_y] = 2\pi i a b$.

Let's check that $e^{2\pi i b x}\psi(x+a, y+b) \in \mathcal{L}$.

It's clearly periodic of period 1 in y. Consider

$$e^{2\pi i(xy + xb)}\psi(x+a, y+b) \quad \text{and}$$

$$e^{2\pi i(xy + xb + ay + ab)}\psi(x+a, y+b).$$

We want the former to have period 1 in x. The latter has period 1 in x since $e^{2\pi i xy}\psi(x,y)$ does. The two expressions differ by the factor $e^{2\pi i(ay+ab)}$ which is constant in x. So it's clear.

Feb. 7, 2002 Let $\mathcal{L} = \{\psi \in C^\infty(\mathbb{R}^2)$ satisfying the automorphic condition $\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)\}$.

This is the space of smooth sections of the line bundle of degree 1 over $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We propose to construct a projective action of $SL(2,\mathbb{Z})$ on $\mathcal{L}$ corresponding to the natural action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2, \mathbb{R}^2, T^2$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the corresponding matrix multiplication

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} ac_1 + bc_2 \\ cc_1 + dc_2 \end{pmatrix}$$

and let $x : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $y : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the coordinate functions: $x = pr_1$, $y = pr_2$. Use g to pull back functions, differential forms on $\mathbb{R}^2$. Thus

$$g^*x\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = x\begin{pmatrix} ac_1 + bc_2 \\ cc_1 + dc_2 \end{pmatrix} = ac_1 + bc_2, \text{ etc.}$$

so that

$$g^*x = ax + by$$
$$g^*y = cx + dy$$
$$(g^*\psi)(x,y) = \psi(ax+by, cx+dy).$$

Ex. 1. $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad (g^*\psi)(x,y) = \psi(y,-x)$

Suppose $\psi_0 \in \mathcal{L}$, and put

$$\psi_1(x,y) = (g^*\psi_0)(x,y) = \psi_0(y,-x).$$

Corresponding to the automorphic condition for $\psi_0 \in \mathcal{L}$

we have

$$\psi_1(x+m, y+n) = \psi_0(y+n, -x-m) = e^{-2\pi i n(-x)}\psi_0(y,-x)$$

i.e. $$\boxed{\psi_1(x+m, y+n) = e^{2\pi i n x}\psi_1(x,y)}$$

Now put $\psi_2(x,y) = e^{-2\pi i xy}\psi_1(x,y)$. Then

$$\psi_2(x+m, y+n) = e^{-2\pi i (x+m)(y+n)}\psi_1(x+m,y+n)$$

$$= e^{-2\pi i (xy+my+\cancel{nx}+\cancel{mn})} e^{2\pi i n x}\psi_1(x,y)$$

i.e. $\psi_2(x+m, y+n) = e^{-2\pi i m y}\psi_2(x,y)$. Thus $\psi_0 \mapsto \psi_2$

~~[illegible]~~ i.e.

$$\boxed{\psi(x,y) \longmapsto e^{-2\pi i xy}\psi(y,-x) \text{ maps } \mathcal{L} \text{ into } \mathcal{L}}$$

Ex. 2. $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\quad (g^*\psi)(x,y) = \psi(x+y, y)$

Let $\psi_0 \in \mathcal{L}$, put $\psi_1(x,y) = g^*\psi_0 = \psi_0(x+y,y)$. Then

$$\psi_1(x+m, y+n) = \psi_0(x+y+m+n, y+n) = e^{-2\pi i (m+n) y}\psi_0(x+y, y)$$

Put $\psi_2(x,y) = e^{h(y)}\psi_1(x,y)$ where $h(y)$ is to be determined. Then

$$\psi_2(x+m, y+n) = e^{h(y+n)}\psi_1(x+m, y+n)$$

$$= e^{h(y+m)} e^{-2\pi i (m+n) y} e^{-h(y)} e^{h(y)}\psi_1(x,y)$$

$$\psi_2(x+m, y+n) = e^{-2\pi i m y}\psi_2(x,y)\, e^{h(y+n)-h(y)-2\pi i n y}$$

Put $h(y) = 2\pi i \frac{y(y-1)}{2}$. $\quad \frac{(y+n)^2 - y - n - y^2 + y}{2} = ny + \frac{n^2-n}{2}$

Conclude $\boxed{\psi \in \mathcal{L} \implies e^{2\pi i \frac{y(y-1)}{2}} \psi(x+y, y) \in \mathcal{L}}$

Ex. 3 $\quad g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \psi_1(x,y) = \psi_0(x, x+y).$

Then for $\psi_0 \in \mathcal{L}$: $\quad \psi_1(x+m, y+n) = \psi_0(x+m, x+y+m+n) = e^{-2\pi i m(x+y)} \psi_0(x, x+y)$

or $\quad \psi_1(x+m, y+n) = e^{-2\pi i m(x+y)} \psi_1(x,y)$. Put $\psi_2(x,y) = e^{h(x)} \psi_1(x,y)$

$$\psi_2(x+m, y+n) = e^{h(x+m)} e^{-2\pi i m(x+y)} \psi_1(x,y)$$
$$= e^{-2\pi i m y} \psi_2(x,y)\, e^{h(x+m)-h(x)-2\pi i m x}$$

The last exponential is 1 if $h(x) = 2\pi i \frac{x(x-1)}{2}$ so

Conclude $\boxed{\psi \in \mathcal{L} \implies e^{2\pi i \frac{x(x-1)}{2}} \psi(x, x+y) \in \mathcal{L}}$

Ex 4. $\quad g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$. The corresponding fractional linear transf satisfies $\quad g\left(\frac{0}{1}\right) = \left(\frac{1}{1}\right) = 1, \quad g\left(\frac{1}{1}\right) = \left(\frac{1}{0}\right) = \infty, \quad g\left(\frac{1}{0}\right) = \left(\frac{0}{-1}\right) = 0$

so g rotates the non Euclidean Δ with vertices $0, 1, \infty \in P_1\mathbb{R}$ by $120°$. If the center z of the Δ is fixed, then $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}(z) = \frac{1}{-z+1} = z, \quad z^2 - z + 1 = 0,$

$z = \frac{1 \pm \sqrt{-3}}{2}$, so the fix pt in the UHP is $\frac{1}{2} + \frac{i\sqrt{3}}{2}$, a primitive 6th root of 1. Actually g has order 6 and $g^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ generates the center of $SL(2,\mathbb{Z})$.

One calculates that

$$\boxed{\psi(x,y) \in \mathcal{L} \implies \overbrace{e^{2\pi i \left(\frac{y^2-y}{2} - xy\right)} \psi(y, y-x)}^{g^*\psi(x,y)} \in \mathcal{L}}$$

February 23, 2002

Discuss central extensions of an abelian group A.

$$1 \longrightarrow B \xrightarrow{i} E \xrightarrow{\pi} A \longrightarrow 1.$$

A basic invariant of such an extension is the commutator pairing $h(a_1, a_2) \in \mathrm{Hom}(\Lambda^2 A, B)$, which is defined by $i(h(a_1,a_2)) = e_1 e_2 e_1^{-1} e_2^{-1}$ where $\pi(e_i) = a_i$.

One has a short exact sequence

$$(1) \qquad 0 \longrightarrow \mathrm{Ext}^1_{\mathbb{Z}}(A,B) \longrightarrow H^2(A,B) \longrightarrow \mathrm{Hom}(\Lambda^2 A, B) \to 0$$

obtained from the universal coefficient thm for the classifying space $B_c A$. One needs to know that $H_1(B_c A) = A$, $H_2(B_c A) = \Lambda^2 A$. The first is Hurewicz, the second is checked for cyclic groups, then holds for products by

$$\Lambda^2(A_1 \times A_2) = \Lambda^2 A_1 \oplus A_1 \otimes A_2 \oplus \Lambda^2 A_2$$

$$H_2(B_c A_1 \times B_c A_2) = H_2(B_c A_1) \oplus H_1(B_c A_1) \otimes H_1(B_c A_2) \oplus H_2(B_c A_2)$$

hence it holds for A f.g. abelian, hence for all A by taking colimits.

Note that in (1) the left exactness is obvious: central extensions with zero commutator pairing are simply abelian group extensions. You would like to understand why every skew-symmetric bilinear map $\Lambda^2 A \to B$ comes from a central extension.

Let $Z^2(A,B)$ be the space of group 2-cocycles: $f(a_1,a_2) : \mathbb{Z}[A \times A] \longrightarrow B$ satisfying

$$(\delta f)(a_1,a_2,a_3) = f(a_2,a_3) - f(a_1+a_2,a_3) + f(a_1,a_2+a_3) - f(a_1,a_2) = 0$$

Note that a bilinear map $f : A \otimes A \longrightarrow B$ is a 2-cocycle.

A 2cocycle f yields a central extension where $E = B \times A$ as a set, and the product is

$$(b_1, a_1)(b_2, a_2) = (b_1 + b_2 + f(a_1, a_2),\ a_1 + a_2)$$

One has

$$(b_2, a_2)(b_1, a_1) = (b_1 + b_2 + f(a_2, a_1),\ a_1 + a_2)$$

so the commutator pairing is $f(a_1, a_2) - f(a_2, a_1)$. So you find that process of skew-symmetrization converts any 2-cocycle into a bilinear skew-symmetric (means $f(a,a) = 0$) 2-cocycle.

Question: Is every map $\Lambda^2 A \longrightarrow B$ obtained as the commutator pairing for the extension corresponding to a map $A \otimes A \longrightarrow B$? Taking $B = \Lambda^2 A$ one sees this implies that $a_1 \wedge a_2 \longmapsto a_1 \otimes a_2 - a_2 \otimes a_1$ from $\Lambda^2 A$ to $A \otimes A$ is a direct injection. Using

$$\Lambda^2(A_1 \oplus A_2) = \Lambda^2 A_1 \oplus A_1 \otimes A_2 \oplus \Lambda^2 A_2$$

$$(A_1 \oplus A_2) \otimes (A_1 \oplus A_2) = A_1^{\otimes 2} \oplus (A_1 \otimes A_2 \oplus A_2 \otimes A_1) \otimes A_2^{\otimes 2}$$

one checks this property persists under taking direct sums. Since it's true for A cyclic, one sees it's true for A fin. gen. Then one gets injectivity of $\Lambda^2 A \longrightarrow A \otimes A$ by colims, and the exact sequence

$$0 \longrightarrow \Lambda^2 A \longrightarrow A \otimes A \longrightarrow S^2 A \longrightarrow 0$$

in general.

Def. A *quadratic function* $q : A \to B$ is one such that $-(\delta q)(a_1, a_2) = q(a_1+a_2) - q(a_1) - q(a_2)$ is $\mathbb{Z}$-bilinear. Thus one has a cartesian square

$$\begin{tikzcd}
\mathrm{Hom}(\Gamma^2 A, B) = \mathrm{Quad}(A,B) \arrow[d] \arrow[r, phantom, "\subset"] \arrow[dr, phantom, "\text{cart}"] & C^1(A,B) \arrow[d, "\delta"] \arrow[r, phantom, "="] & \mathrm{Hom}(\mathbb{Z}[A], B) \\
\mathrm{Hom}(A \otimes A, B) \arrow[r, phantom, "\subset"] & Z^2(A,B) \arrow[r, phantom, "="] & \mathrm{Hom}(\Sigma_2 A, B)
\end{tikzcd}$$

of representable functors in B. Here $\Sigma_2 A$ is the universal abelian group generated by a 2-cocycle; it's the cokernel of a map $\mathbb{Z}[A^3] \longrightarrow \mathbb{Z}[A^2]$. Corresponding to the above cartesian square is a cocartesian square

$$\begin{tikzcd}
 & \Lambda^2 A \arrow[r, "\text{comm}", "\text{pairing}"'] \arrow[d, equal] & \Sigma_2 A \arrow[r, "(\delta^t)"] \arrow[d] \arrow[dr, phantom, "\text{cocart}"] & \mathbb{Z}[A] \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda^2 A \arrow[r] & A \otimes A \arrow[r] \arrow[d] & \Gamma^2 A \arrow[r] & A \arrow[r] & 0 \\
 & & S^2 A \arrow[ur] & & &
\end{tikzcd}$$

Bottom row is exact. Think of the bottom row as the best you can do with 2nd degree tensors.

In the top row $\Lambda^2 A$ is a direct summand of $\Sigma_2 A$ in some mysterious way, since the identity map of $\Lambda^2 A$ is the commutator pairing of some 2-cocycle with values in $\Lambda^2 A$.

It turns out that the top row is exact at $\Sigma_2 A$ and then the mystery disappears.

Start with the bar complex

$$\longrightarrow \mathbb{Z}[A^3] \xrightarrow{\partial} \mathbb{Z}[A^2] \xrightarrow{\partial} \mathbb{Z}[A] \longrightarrow \mathbb{Z}$$

of chains on the classifying space $B_{cl}A$, which is the realization of the simplicial group.

$$A^3 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow}\; A^2 \;\substack{\longrightarrow\\\longrightarrow\\\longrightarrow}\; A \rightrightarrows pt$$

The homology of this complex is $H_0 = \mathbb{Z}$, $H_1 = A$, $H_2 = \Lambda^2 A$. Look at the truncated complex of length 1 having the homology groups $\Lambda^2 A$ and A:

$$\Lambda^2 A \rightarrowtail \Sigma_2 A \xrightarrow{d} \mathbb{Z}[A] \twoheadrightarrow A$$

where $\Sigma_2 A = \mathrm{Coker}\{\mathbb{Z}[A^3] \xrightarrow{\partial} \mathbb{Z}[A^2]\}$ is the universal abelian group for a 2-cocycle on A. Note that because $\mathbb{Z}[A]$ is a free abelian group, so is the image of d, so there exists a lifting (with respect to d) of this image into $\Sigma_2 A$. Thus you see that $\Lambda^2 A$ splits off $\Sigma_2 A$.

So the complex $\Sigma_2 A \xrightarrow{d} \mathbb{Z}[A]$ is quis $\Lambda^2 A[1] \oplus A[0]$, which should lead to the short exact sequence

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(A, B) \to H^2(A, B) \longrightarrow \mathrm{Hom}(\Lambda^2 A, B) \to 0$$

as well as the fact that this sequence splits. (You need to use

$$\mathrm{Hom}^\bullet([\Sigma_2 A \to \mathbb{Z}[A]], B) = [C^1(A,B) \xrightarrow{\delta} Z^2(A,B)]\Big).$$

Now compare this d complex with the complex using 2nd degree tensors.

$$\begin{tikzcd}
\Lambda^2 A \arrow[r, tail] \arrow[d, equal] & \Sigma_2 A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{bicart}"] & \mathbb{Z}[A] \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\
\Lambda^2 A \arrow[r, tail] & A \otimes A \arrow[r] \arrow[dr, two heads] & \Gamma^2 A \arrow[r, two heads] & A \\
& & S^2 A \arrow[u, tail] &
\end{tikzcd}$$

So the conclusion is that one has a surjective quis

$$\begin{tikzcd}
\Sigma_2 A \arrow[r] & \mathbb{Z}[A] \arrow[d] \\
A \otimes A \arrow[r] & \Gamma^2 A
\end{tikzcd}$$

of complexes, which means for some [redacted] purposes, such as for central extensions with B divisible, that the two [redacted] complexes yield the same information.

Feb 24, 02

Consider the short exact sequence of abelian groups

$$S^2 A \xrightarrow{\iota} \Gamma^2 A \underset{\pi}{\overset{q}{\rightleftarrows}} A$$

Notice that the projection π has a tautological section q satisfying $q(a+a') - q(a) - q(a') =$ the [redacted] tautological symmetric bilinear form $a_1 \otimes a_2 \longmapsto a_1 a_2 \in S^2 A$. Thus identifying [redacted] $S^2 A \times A \xrightarrow[\sim]{(\iota, q)} \Gamma^2 A$ one has an abelian group law on the set $\Gamma^2 A$ given by $(b,a)\cdot(b',a') = (b+b'+aa', a+a')$. The map [redacted] $A \to S^2 A \times A$, $a \longmapsto (q(a), a)$ satisfies

$$(q(a),a)\cdot(q(a'),a') = (q(a)+q(a')+aa', a+a') = (q(a+a'), a+a').$$

Start again: A symmetric bilinear form $S^2A \xrightarrow{h} B$ yields an abelian group extension

$$\begin{tikzcd} B \arrow[r, tail] & E \arrow[r, two heads] & A \arrow[l, bend right, dashed] \end{tikzcd}$$

with a section giving back the 2-cocycle h. On the other hand if h is the coboundary of a map $q : A \to B$ (q is necessarily quadratic) then the extension E splits.

There are two abelian group extensions of A by S^2A. First is the extension corresponding to 2-cocycle $(a_1, a_2) \mapsto a_1 a_2 \in S^2A$.

Call this EA $\left\{ S^2A \rightarrowtail S^2A \times A \twoheadrightarrow A, \quad (b,a)(b',a') = (b+b'+aa', a+a') \right.$

$$S^2A \times A = \{(b,a) \mid b \in S^2A, a \in A\}$$

Next is the abelian group exact sequence $\begin{tikzcd} S^2A \arrow[r, tail, "\iota"] & \Gamma^2A \arrow[r, two heads, "\pi"] & A \arrow[l, bend right, "\gamma"'] \end{tikzcd}$. Note that the universal quadratic map γ is a section of π, and the corresponding 2-cocycle is $\gamma(a+a') - \gamma(a) - \gamma(a') = \iota(aa')$. Therefore one has an obvious canonical isomorphism between EA and Γ^2A.

A splitting of the extension EA is given by a section $a \mapsto (q(a), a)$ of $\pi : EA \to A$ which is a group homomorphism, i.e.

$$(q(a), a) \cdot (q(a'), a') = (q(a) + q(a') + aa', a + a')$$

is equal to $(q(a+a'), a+a')$, in other words $q : A \to S^2A$ is a quadratic map with associated symmetric bilinear form aa'. (Note also that q induces $\Gamma^2A \to S^2A$ which splits the Γ^2A extension.)

<u>Example</u>: Let $A = \mathbb{Z}$, whence $S^2\mathbb{Z} = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}$, hence the symmetric bilinear form is $(m,n) \mapsto mn$. So our abelian group $E\mathbb{Z}$ is the set $\mathbb{Z} \times \mathbb{Z}$ with $(b,a)(b',a') = (b+b'+aa', a+a')$, where now $a, b \in \mathbb{Z}$. The extension can be split by lifting $a=1$ to $(b, 1)$ for

any $b \in \mathbb{Z}$ and then adding successively (in $E\mathbb{Z}$) to get the section homomorphism $(q(a), a)$.

Note that $q(a) = \frac{a(a-1)}{2}$ is a quadratic map from $\mathbb{Z}$ to $\mathbb{Z}$ with the associated symm. form aa':

$$q(a+a') - q(a) - q(a') = \frac{1}{2}\left[\begin{matrix} \cancel{a^2} + 2aa' + \cancel{a'^2} - \cancel{a^2} - \cancel{a'^2} \\ -\cancel{a} - \cancel{a'} \quad + \cancel{a} + \cancel{a'} \end{matrix}\right] = aa'$$

Then $q(a) = \frac{a(a-1)}{2} + ba$ is a quadratic map with $(q(1),1) = (b,1)$ as desired.

Example: $A = \mathbb{Z}/2$. Here $S^2(\mathbb{Z}/2) \simeq \mathbb{Z}/2$ with symmetric pairing aa'. Then [redacted]

$\underset{\text{means adding } n \text{ times}}{n \cdot (0,1)} = \left(\frac{n(n-1)}{2}, n\right)$ modulo 2. Take $n=2$

[redacted] so that $2 \cdot (0,1) = (1,0)$ mod 2, hence the element $(0,1)$ of $E(\mathbb{Z}/2)$ has order 4. $\boxed{\Gamma^2(\mathbb{Z}/2) \simeq \mathbb{Z}/4}$

$A = \mathbb{Z}/N$. The question is whether $\Gamma^2(A)$ is killed by N. One has N times $(b,1) = \left(Nb + \frac{N(N-1)}{2}, N\right)$ will be $0 \pmod N$ provided $\frac{N(N-1)}{2} \equiv 0 \pmod N$. True if N odd. If N is even then N times $(0,1) = \frac{N}{2}\underset{\text{odd}}{(N-1)}$ one less power of 2, so it seems that the 2-exponent of $\Gamma^2(\mathbb{Z}/N)$ is one more than for N. For example if $N = 2^k$, then

$$2^k [redacted] \times (0,1) = \left(2^{k-1}(2^k-1), 2^k\right) \not\equiv 0 \pmod{2^k}$$

$$2^{k+1} \times (0,1) = \left(2^k(2^{k+1}-1), 2^{k+1}\right) \equiv 0 \pmod{2^k}$$

Warning: When B is not divisible, you have be careful trying to use the exact sequence $0 \to \Lambda^2 A \to A \otimes A \to \Gamma^2 A \to A \to 0$. In the case of central extensions of elementary abelian 2-groups, one has $H^2(A,B) = \mathrm{Hom}(\Gamma^2 A/2, B)$. The invariant of the extension $B \rightarrowtail E \twoheadrightarrow A$ is the quadratic map $A \to B$ obtained by lifting $a \in A$ to E and squaring. Exact sequence $0 \to \Lambda^2 A \to \Gamma^2 A/2 \to A \to 0$ yields $0 \to \mathrm{Hom}(A,B) \to \mathrm{Hom}(\Gamma^2 A, B) \to \mathrm{Hom}(\Lambda^2 A, B) \to 0$??

Feb 25, 02. Here's what happens for A an elementary abelian 2 group. Given a central extension $B \rightarrowtail E \twoheadrightarrow A$ you get a quadratic map $q: A \to B$ by lifting $a \in A$ to E and squaring; the associated bilinear form is the commutator pairing $\Lambda^2 A \to B$. Here B can be any abelian group. (Check: Let $x, y \in E$. Then

$$xyxy(xx)^{-1}(yy)^{-1} = xyx(xx)^{-1}y(yy)^{-1} = xyx^{-1}y^{-1}$$

since xx is in the center.)

For example consider the canonical extension

$$S^2A \rightarrowtail \Gamma^2 A \twoheadrightarrow A$$

where $\Gamma^2 A = \{(b,a) \in S^2A \times A\}$ with group law $(b,a)(b',a') = (b+b'+aa', a+a')$, one has $(0,a)(0,a) = (aa, 0)$, whence $(0,a)$ has order 4 for $a \neq 0$. The quadratic map $q: A \to S^2A$ is $q(a) = a^2$, which is linear.

You know that $H_2(B_{el}A, \mathbb{Z}/2) = \Gamma^2 A/2$ and hence

$$H^2(A, B) = \mathrm{Hom}(\Gamma^2 A/2, B)$$

for B elementary abelian. Diagram:

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, tail] & A \arrow[d] & \\ S^2A \arrow[r, tail] \arrow[d] & \Gamma^2 A \arrow[r, two heads] \arrow[d] & A \arrow[d, equal] \\ \Lambda^2 A \arrow[r, tail] & \Gamma^2A/2 \arrow[r, two heads] & A \end{tikzcd}$$

The last row yields

$$\begin{array}{c} \mathrm{Hom}(A,B) \rightarrowtail H^2(A,B) \twoheadrightarrow \mathrm{Hom}(\Lambda^2 A, B) \\ \| \\ \mathrm{Ext}^1(A,B) \end{array} \quad \text{(by Bockstein).}$$

Feb 28, 02. Background: You've seen that the algebra $A(P)$ of functions (smooth) on a principal $\mathbb{T}$-bundle P is $\mathbb{Z}$-graded naturally, because you have a representation of the circle group on $A(P)$ which then splits according to $\mathbb{T}^\vee = \mathbb{Z}$. So

$$A(P) = \bigoplus_{n\in\mathbb{Z}}^{(top)} A(P)_n$$

where $\psi \in A(P)_n$ iff $e^{tX}\psi = e^{2\pi i n t}\psi$. Thus $A(P_0) = A(B)$ where B is the base. By the usual description of sections of an associated fibre bundle to a principal bundle, one should know that $A(P)_n$ is the space of sections of a line bundle over B which is the n-th tensor power of the line bundle L corresponding to $A(P)_1$. Thus

$$A(P) = \bigoplus_n \Gamma(B, L^{\otimes n}).$$

This situation reminds you of the Toeplitz alg, which is the unital algebra R with generators x, y subject to the relation $yx = 1$. R acts on $\mathbb{C}[z]$ with $xz^n = z^{n+1}$, $yz^n = z^{n-1}$ if $n \geq 1$, $y1 = 0$. R is clearly spanned by the words $x^m y^n$, $m, n \geq 0$.

Let I be the ideal in R generated by $1-xy$. Then $I = R(1-xy)R$ is spanned by $x^m(1-xy)y^n$, $m,n\geq 0$. One has

$$x^m(1-xy)y^n z^k = \begin{cases} 0 & k \neq n \\ z^m & k = n \end{cases}$$

because $1-xy$ acting on $\mathbb{C}[z]$ is projection onto $z^0 = 1$ killing z^k for $k>0$. Thus the matrix of the operator $x^m(1-xy)y^n$ relative to the basis $z^k, k\geq 0$ has entry 1 in the m,n position and 0 elsewhere.

Conclude that the $x^m(1-xy)y^n$ form a basis for I and that I can be identified with the matrices of finite support.

The quotient ring R/I is clearly the ring of Laurent polynomials $\mathbb{C}[z,z^{-1}]$, where $x, y \rightsquigarrow z, z^{-1}$ respectively.

Let's show that $x^m y^n$ for $m,n \geq 0$ is a basis for R. You recall doing this by defining an action of R on $\mathbb{C}[x,y]$ respecting the mult. map $\mathbb{C}[x,y] \to R$. Then acting on $1 \in \mathbb{C}[x,y]$ gives R-module maps $R \twoheadrightarrow \mathbb{C}[x,y] \longrightarrow R$ sending 1 to 1, etc.

Here's a direct method

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & (1-xy) & & y^2 & \\ x^3 & & x(1-xy) & & (1-xy)y & & y^3 \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & xy^2 & & y^3 \end{array}$$

these elements are independent because they give bases for both I and R/I.

Since the increasing row filtration is the same for both arrows, it follows that monomials here are independent.

Remark that in the extension $I \rightarrowtail R \twoheadrightarrow \mathbb{C}[z,z^{-1}]$ you can't lift z to an invertible element of R.

Cuntz's generalization used by Pimsner: R gens x_i, y_i $i = 1, \dots, n$ acting on $T(V)$, $V = \mathbb{C}x_1 + \dots + \mathbb{C}x_n$, with x_i left multiplying on $T(V)$ and $y_j x_i w = \delta_{ji} w$. Then you get the $\mathcal{O}_n$ algebra

$$y_j x_i = \delta_{ji}, \quad \sum_{i=1}^{n} x_i y_i = 1.$$

Let Γ be a group, Φ a finite subset of Γ. Define two categories of modules $\mathcal{W}$ and $\mathcal{V}$.

[redacted] An object of $\mathcal{W}$ is a Γ-module W (scalars $= \mathbb{C}$) equipped with a linear operator $h \in \mathrm{End}(W)$ satisfying

(1) $hsh \neq 0 \implies s \in \Phi$

(2) $\forall w \in W$, $\left(\{s \in \Gamma \mid shs^{-1}w \neq 0\}\text{ is finite}\right)$ and

$$\sum_s shs^{-1}w = w$$

An object of $\mathcal{V}$ is a vector space V together with operators $p(s) \in \mathrm{End}(V)$ for $s \in \Gamma$ satisfying

(1) $p(s) \neq 0 \implies s \in \Phi$

(2) $\sum_t p(st^{-1})p(t) = p(s)$

(3) $\sum_s p(s)V = V$

(4) $(\forall s)\ p(s)v = 0 \implies v = 0$.

Morphisms in $\mathcal{W}$ and $\mathcal{V}$ are maps preserving the structures.

Claim: The categories $\mathcal{W}$ and $\mathcal{V}$ are naturally equivalent.

Define a functor $\mathcal{W} \longrightarrow \mathcal{V}$ as follows.

Given W let $W \xrightarrow{j} V \xrightarrow{i} W$ be the canonical factorization $h = ij$ of h into a surjection followed by an injection. (Thus $V = hW$, $j = h : W \to V$ and i is the inclusion $hW \subset W$.)

Define $\rho(s) = j s i \in \mathrm{End}(V)$. Check the four conditions to see that V is in $\mathcal{V}$.

(1) If $\rho(s) = jsi$ is $\neq 0$, then as i injective and j surjective one has $i(jsi)j = hsh \neq 0$ hence $s \in \Phi$.

(2) $\sum_t \rho(st^{-1})\rho(t)v = \sum_t j s t^{-1} i j t i v =$

$js \sum_t t^{-1} h t i v = j s i v = \rho(s) v$.

(3) $w = \sum_s s h s^{-1} w \Rightarrow j w = \sum_s \rho(s) j s^{-1} w \in \sum_s \rho(s) V$

so $V = \sum_s \rho(s) V$ as j is surjective.

(4) If $(\forall s)\, \rho(s) v = 0$, then $i v = \sum_s s i \overbrace{j s^{-1} i v}^{\rho(s^{-1})v} = 0$,

so $v = 0$ as i is injective.

Next we construct a functor $\mathcal{V} \longrightarrow \mathcal{W}$. First:

Let $\Lambda = \mathbb{C}\Gamma$ be the group algebra of Γ, and let $\Lambda \otimes V$ be the free Γ-module generated by the vector space V. An element $\sum_t t \otimes f(t) \in \Lambda \otimes V$ is described by a function $f : \Gamma \longrightarrow V$ of finite support. The Γ action is given by the left regular representation on functions:

$$u \sum_t t \otimes f(t) = \sum_t ut \otimes f(t) = \sum_t t \otimes f(u^{-1}t).$$

~~For $V \in \mathcal{V}$ we define an operator ρ on $\Lambda \otimes V$ by~~

~~$\rho\left(\sum_t t \otimes f(t)\right) = \sum_s s \otimes \sum_t \rho(s^{-1}t) f(t)$~~

~~In other words, if f is viewed as a column vector indexed by the elements of Γ, then ρ is multiplication by the matrix with entries $\rho(s^{-1}t)$.~~

~~[illegible]~~

The vector space $\Lambda\otimes V = \bigoplus_s s\otimes V \simeq \bigoplus_s V$

is naturally graded with respect to Γ. Define maps

$$\begin{tikzcd} \Lambda\otimes V \arrow[d, shift left, "\eta_1"] \\ V \arrow[u, shift left, "\varepsilon_1"] \end{tikzcd} \qquad \begin{aligned} \varepsilon_1 v &= 1\otimes v \\ \eta_1 \sum_t t\otimes f(t) &= f(1). \end{aligned}$$

so that ε_1, η_1 describe the summand corresponding to the identity $1\in\Gamma$. Clearly $s\varepsilon_1, \eta_1 s^{-1}$ describe the summand corresponding to s. One has the "orthogonality" and "completeness" relations:

$$\eta_1 s^{-1} t\varepsilon_1 = \delta_{st} \text{ on } V, \qquad \sum_s s\varepsilon_1\eta_1 s^{-1} = 1_{\Lambda\otimes V}.$$

Now let V be an object of $\mathcal{V}$, i.e. equipped with operator $p(s)$ satisfying 4 condition on p43. Define the operator p on $\Lambda\otimes V$ by

$$p\Big(\sum_t t\otimes f(t)\Big) = \sum_s s\otimes \sum_t p(s^{-1}t)f(t).$$

In other words, viewing f as a column vector, p is multiplication by the matrix with entries $p(s^{-1}t)$.

Properties of p: (1) $\forall u\in\Gamma$, $up = pu$. (2) $p^2=p$.

(0) p is well-defined since for each t, $s\mapsto p(s^{-1}t)$ has finite support.

(1) is clear since $p(s^{-1}t)$ is invariant under left multiplication $(s,t)\longmapsto(us,ut)$.

(2) follows from $\sum_t p(s^{-1}t)p(t^{-1}u) = p(s^{-1}u)$ which is clear if the idempotence condition is written $p(s) = \sum_{tu=s} p(t)p(u)$.

Let W be the summand of the Γ-module $\Lambda\otimes V$ corresponding to the idempotent operator p. There are canonical Γ-module maps α, β

(*)

$$\begin{tikzcd}
W & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "j"] & \Lambda\otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "\eta_1"] \\
& V \arrow[ul, "\iota"] \arrow[u, shift left, "\varepsilon_1"] & & V \arrow[ul, "\iota"] \arrow[u, shift left, "\varepsilon_1"]
\end{tikzcd}$$

such that $\beta\alpha = 1_W$, $\alpha\beta = p$. Componentwise the condition $\alpha\beta = p$ means

$$\eta_1 s^{-1} p t \varepsilon_1 = p(s^{-1}t).$$

Let $\iota = \beta\varepsilon_1$, $j = \eta_1\alpha$, and $h = ij : W \to W$.

One has $p(s^{-1}t) = \eta_1 s^{-1}\alpha\beta t\varepsilon_1 = j s^{-1} t i$.

Next we show W equipped with h is in $\mathcal{W}$. Check two conditions: (1) $0 \neq hsh = i(jsi)j = i\,p(s)\,j \Rightarrow p(s) \neq 0 \Rightarrow s \in \Phi$. (2) $1_W = \beta 1_{\Lambda\otimes V}\alpha = \sum_s \beta s\varepsilon_1\eta_1 s^{-1}\alpha = \sum_s s\beta\varepsilon_1\eta_1\alpha s^{-1} = \sum_s s h s^{-1}$.

At this point we have constructed functors $W \mapsto hW$ from $\mathcal{W}$ to $\mathcal{V}$ and $V \mapsto p(\Lambda\otimes V)$ from $\mathcal{V}$ to $\mathcal{W}$. In the case of the second functor we have not yet used the 3rd + 4th conditions: $\sum_s p(s)V = V$, $\forall s\ p(s)v = 0 \Rightarrow v = 0$. Return to (*) to find jW and $\mathrm{Ker}(i)$. $jW = \eta_1 \overset{p}{\alpha\beta}(\Lambda\otimes V)$ since β surjective. jW consists of all $\eta_1 p \sum_t t\otimes f(t) = \eta_1 \sum_s s\otimes \sum_t p(s^{-1}t)f(t) = \sum_t p(t)f(t)$, i.e. $\boxed{jW = \sum_t p(t)V}$.

Since α is injective one has $\mathrm{Ker}(i) = \mathrm{Ker}(\alpha i) = \mathrm{Ker}(p\varepsilon_1)$
$= \{v \in V \mid p(1\otimes v) = \sum_s s \otimes p(s^{-1})v = 0\}$. Hence

$$\boxed{\mathrm{Ker}(i) = \bigcap_s \mathrm{Ker}\,(p(s))}.$$

Thus V satisfies 3rd + 4th conditions $\Longleftrightarrow$ j surjective & i injective.

which means that one has a canonical isomorphism of V with the image hW. Thus the composite functor $\mathcal{V} \longrightarrow \mathcal{W} \longrightarrow \mathcal{V}$ is isomorphic to $1_{\mathcal{V}}$.

Next given W in $\mathcal{W}$ let

$W \xleftarrow{i} V \twoheadleftarrow{j} W$ be the canonical factorization of the operator h. Define maps α, β

$$W \overset{\beta}{\twoheadleftarrow} \Lambda \otimes V \xleftarrow{\alpha} W \quad \text{by} \quad \begin{aligned} \alpha w &= \sum_s s \otimes j s^{-1} w \\ \beta \sum_t t \otimes f(t) &= \sum_t t\, i\, f(t) \end{aligned}$$

Prop. (1) β is the unique Γ-module map extending i in the sense that $\beta\varepsilon_1 = i$.

(2) α is the unique Γ-module map coextending j in the sense that $\eta_1 \alpha = j$.

(1) is clear since $\beta(t\otimes v) = \beta(t\varepsilon_1 v) = t\, i\, v$

(2) uniqueness. Let $\alpha w = \sum_t t \otimes g(t)$. Then

$$j s^{-1} w = \eta_1 s^{-1} \alpha w = \eta_1 \sum_t s^{-1} t \otimes g(t) = g(s),$$

if a map α with the stated properties exists it is given by the above formula. ¶ We must show that $s \longmapsto j s^{-1} w$ has finite support in order that αw be well-defined. Since $\sum_s s h s^{-1} w = w$ any element of W is a finite sum of elements of the form $t\, i\, v$, so we can assume $w = t\, i\, v$, whence $j s^{-1} w = j s^{-1} t\, i\, v = p(s^{-1}t)v$

has finite support as a function of s. Then

$$u\alpha w = \sum_s us \otimes j s^{-1} w = \sum_s s \otimes j(u^{-1}s)^{-1} w = \sum_s j s^{-1} u w = \alpha u w,$$

and $\eta_1 \alpha w = \eta_1 \sum s \otimes j s^{-1} w = j w$ finishing the proof.

Finally one has

$$\beta\alpha w = \beta \sum_s s \otimes j s^{-1} w = \sum_s s \overbrace{ij}^{h} s^{-1} w = w$$

$$\alpha\beta \sum_t t \otimes f(t) = \alpha \sum_t t i f(t) = \sum_s s \otimes \sum_t \underbrace{j s^{-1} t i}_{p(s^{-1}t)} f(t)$$

which identifies W with the summand of $\Lambda \otimes V$ corresponding to the projection p.

Thus the composite functor $\mathcal{W} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}$ is isomorphic to the identity $1_{\mathcal{W}}$.

March 19, 02.

Consider $\Gamma = \mathbb{Z}/2 = \{1, \varepsilon\}$ where $\varepsilon^2 = 1$, $\Lambda = \mathbb{C}\Gamma$, and a retract W of the free Λ module $\Lambda \otimes V$:

$$\beta\alpha = 1_W \qquad \begin{tikzcd} W & \Lambda \otimes V \arrow[l, two heads, "\beta"'] \arrow[d, shift left, "j_1"] & W \arrow[l, tail, "\alpha"'] \arrow[dl, "j_1\alpha"] \\ & V \arrow[u, shift left, "\iota_1"] \arrow[ul, "\beta\iota_1"] & \end{tikzcd} \qquad \iota_1 v = 1 \otimes v, \quad j_1\begin{pmatrix} 1\otimes v \\ \varepsilon \otimes v\end{pmatrix} = \begin{pmatrix} v \\ 0 \end{pmatrix}.$$

We are using the basis $1, \varepsilon$ for Λ, which yields the "orthogonal" partition of unity

$$1_{\Lambda\otimes V} = \iota_1 j_1 + \varepsilon \iota_1 j_1 \varepsilon$$

and its compression to a partition of 1 for W:

$$1_W = (\beta \iota_1 j_1 \alpha) + \varepsilon(\beta \iota_1 j_1 \alpha)\varepsilon$$

Let $h = \beta\iota_1 j_1 \alpha$ so that W becomes a unital module over $\mathcal{E}_\Gamma \rtimes \Gamma$ where $\mathcal{E}_\Gamma = \mathbb{C}[h_1, h_\varepsilon]/(h_1 + h_\varepsilon = 1)$.

The retract W is equivalent to the projection $p = \alpha\beta$ on the Λ-module $\Lambda \otimes V$ which has the form $p \sum_t t \otimes f(t) = \sum_s s \otimes \sum_t p(s^{-1}t) f(t)$, where $p(s) \in \mathcal{L}(V)$ for each $s \in \Gamma$. This formula is a consequence of

$$\Big(\sum_u u \otimes p(u)\Big) \sum_t t \otimes f(t) = \sum_{u,t} tu^{-1} \otimes p(u) f(t) \qquad \begin{matrix} s = tu^{-1} \\ u = s^{-1}t \end{matrix}$$

$$\in \Lambda \otimes \mathcal{L}(V) \qquad = \sum_s s \otimes \sum_t p(s^{-1}t) f(t).$$

In the $\Gamma = \mathbb{Z}/2$ case $p = 1 \otimes p(1) + \varepsilon \otimes p(\varepsilon)$ and so $p^2 = 1 \otimes (p(1)^2 + p(\varepsilon)^2) + \varepsilon \otimes (p(1)p(\varepsilon) + p(\varepsilon)p(1))$, so $p^2 = p$ means $p(1) = p(1)^2 + p(\varepsilon)^2$, $p(\varepsilon) = p(1)p(\varepsilon) + p(\varepsilon)p(1)$. This may be rewritten as $p(1) \pm p(\varepsilon)$ being idempotent.

Thus one finds that a retract of $\Lambda\otimes V$ as Λ module is the same a pair of projections p_+, p_- on V.

So far you have used the basis $1, \varepsilon$ for Λ and the corresponding partition of 1:

$$1_{\Lambda\otimes V} = i_1 j_1 + \varepsilon i_1 j_1 \varepsilon$$

But Λ has a nicer basis given by the projections $\frac{1\pm\varepsilon}{2} = e_\pm$ with corresponding partition

$$1_{\Lambda\otimes V} = i_+ j_+ + i_- j_-$$

where

$$i_+ v = \frac{1+\varepsilon}{2}\otimes v \qquad\qquad i_1 = i_+ + i_-$$

$$i_- v = \frac{1-\varepsilon}{2}\otimes v \qquad\qquad i_\varepsilon = i_+ - i_-$$

$$j_+\left(\frac{1+\varepsilon}{2}\otimes v\right) = v \qquad\qquad j_+(1\otimes v) = v$$

$$j_+\left(\frac{1-\varepsilon}{2}\otimes v\right) = 0 \qquad\qquad j_+(\varepsilon\otimes v) = v$$

$$j_-\left(\frac{1+\varepsilon}{2}\otimes v\right) = 0 \qquad\qquad j_-(1\otimes v) = v$$

$$j_-\left(\frac{1-\varepsilon}{2}\otimes v\right) = v \qquad\qquad j_-(\varepsilon\otimes v) = -v$$

$$j_1 = \frac{j_+ + j_-}{2} \qquad\qquad j_\varepsilon = \frac{j_+ - j_-}{2}$$

Check

$$i_1 j_1 + i_\varepsilon j_\varepsilon = \left[(i_+ + i_-)(j_+ + j_-) + (i_+ - i_-)(j_+ - j_-)\right]/2$$

$$= i_+ j_+ + i_- j_- = 1$$

Now take

$$i_1 j_1 = \frac{1}{2}(i_+ + i_-)(j_+ + j_-)$$

$$\beta i_1 j_1 \alpha = \frac{1}{2}\left(\beta i_+ j_+ \alpha + \beta i_- j_- \alpha + \beta i_+ j_- \alpha + \beta i_- j_+ \alpha\right)$$

Try to summarize the important points.
You are looking at a Λ-retract of $\Lambda\otimes V$ from the viewpoint of the ± 1 eigenspaces for ε.
We identify the decomposition $\Lambda\otimes V = \Lambda_+\otimes V \oplus \Lambda_-\otimes V$ with

$$V \underset{\iota_+}{\overset{j_+}{\leftrightarrows}} \Lambda\otimes V \underset{\iota_-}{\overset{j_-}{\rightleftarrows}} V.$$

Then one has retracts associated to each ± 1 eigenspace:

$$\begin{tikzcd}
W_+ & V \arrow[l, two heads, "\beta\iota_+"'] \arrow[d, equal] & W_+ \arrow[l, tail, "j_+\alpha"'] \\
W_- & V \arrow[l, two heads, "\beta\iota_-"'] & W_- \arrow[l, tail, "j_-\alpha"']
\end{tikzcd}$$

So again you see that there are two projections $p_\pm = \beta\iota_\pm j_\pm\alpha$ on V.

On W besides the $\pm$ grading (which is the same as the Γ action) you have the operator $h = \beta\iota_1 j_1\alpha$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\frac{1}{2}\begin{pmatrix} 1_{W_+} & \beta\iota_- j_+\alpha \\ \beta\iota_+ j_-\alpha & 1_{W_-} \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

which has the form
$h = \frac{1}{2}(1+X)$ where X is odd: $\varepsilon X\varepsilon = -X$.

Note that $h + \varepsilon h\varepsilon = \frac{1}{2}(1+X) + \frac{1}{2}(1-X) = 1_W$.

Question: Is the odd operator X related by Cayley transform to the 2 projections on V?

March 25, 2002

Let $\Lambda = \mathbb{C}[\mathbb{Z}/2] = \mathbb{C}e_+ \oplus \mathbb{C}e_-$, $\quad e_\pm = \frac{1\pm\varepsilon}{2}$.

A (unital) Λ-module M is the same as a vector space with splitting $M = \begin{pmatrix} M_+ \\ M_- \end{pmatrix}$ where $e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. ~~We identify the~~

~~the Λ-module $\Lambda\otimes V$ generated by the v.s. V [illegible]~~

Let $\Lambda \otimes V$ ~~$e_+\otimes V \oplus e_-\otimes V$~~ be the "free" Λ-module generated by the v.s. V. One has

$$\Lambda\otimes V = e_+\otimes V \oplus e_-\otimes V$$

Using the basis e_+, e_- for Λ we make the identification

$$\Lambda\otimes V = \begin{pmatrix} V \\ V \end{pmatrix} \qquad e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly we can identify the endo ring of the Λ-module $\Lambda\otimes V$:

$$\mathrm{End}_\Lambda(\Lambda\otimes V) = \left\{ \begin{pmatrix} x_+ & 0 \\ 0 & x_- \end{pmatrix} \middle| \; x_+, x_- \in \mathcal{L}(V) \right\}$$

In particular a projection p on $\Lambda\otimes V$ has the form $\begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix}$ where p_+, p_- are projections in $\mathcal{L}(V)$.

Corresponding to such a p one has a Λ-module retract

$$W \xleftarrow{\beta} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\alpha} W \qquad \begin{matrix} \beta\alpha = 1 \\ \alpha\beta = p \end{matrix}$$

which splits into two retracts of V:

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad \begin{matrix} \beta_\pm\alpha_\pm = 1_{W_\pm} \\ \alpha_\pm\beta_\pm = p_\pm \end{matrix}$$

In addition W is equipped with an odd operator $X = \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$.

Conversely given a Λ-module W together with an odd operator X let

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}} V \xleftarrow{(\alpha_+ \;\; \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

be an arbitrary factorization of $1_W + X$:

$$\begin{pmatrix} \beta_+\alpha_+ & \beta_+\alpha_- \\ \beta_-\alpha_+ & \beta_-\alpha_- \end{pmatrix} = \begin{pmatrix} 1_{W_+} & T' \\ T & 1_{W_-} \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & T' \\ T & 0 \end{pmatrix}.$$

Thus $W_\pm$ is the retract of V corresponding to the projection $p_\pm = \alpha_\pm\beta_\pm$, and X is the associated odd operator.

Question: Assume that $1+X$ is invertible and that V is isomorphic to the image, so that $\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix}$ and $(\alpha_+ \;\alpha_-)$ are invertible. Then on V one has two splittings. Are these related to $p_\pm$?

First case: Assume V equals the W on the right, i.e. $V = \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$, $W_+ \xleftarrow{\alpha_+ = \begin{pmatrix}1\\0\end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$, $W_- \xleftarrow{\alpha_- = \begin{pmatrix}0\\1\end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$

$$\beta_+ = (1 \;\; 0)\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix} = (1 \;\; T'), \quad \beta_- = (0 \;\; 1)\begin{pmatrix} 1 & T' \\ T & 1 \end{pmatrix} = (T \;\; 1).$$

Then

$$p_+ = \alpha_+\beta_+ = \begin{pmatrix}1\\0\end{pmatrix}(1 \;\; T') = \begin{pmatrix} 1 & T' \\ 0 & 0 \end{pmatrix}$$

$$p_- = \alpha_-\beta_- = \begin{pmatrix}0\\1\end{pmatrix}(T \;\; 1) = \begin{pmatrix} 0 & 0 \\ T & 1 \end{pmatrix}$$

$$\alpha_+\beta_- = \begin{pmatrix}1\\0\end{pmatrix}(T \;\; 1) = \begin{pmatrix} T & 1 \\ 0 & 0 \end{pmatrix}$$

$$\alpha_-\beta_+ = \begin{pmatrix}0\\1\end{pmatrix}(1 \;\; T') = \begin{pmatrix} 0 & 0 \\ 1 & T' \end{pmatrix}$$

2nd case: $V = W$ on the left

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ \\ \beta_-\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{(\alpha_+ \ \alpha_-) = \begin{pmatrix} 1 & T' \\ T & 1\end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

Then

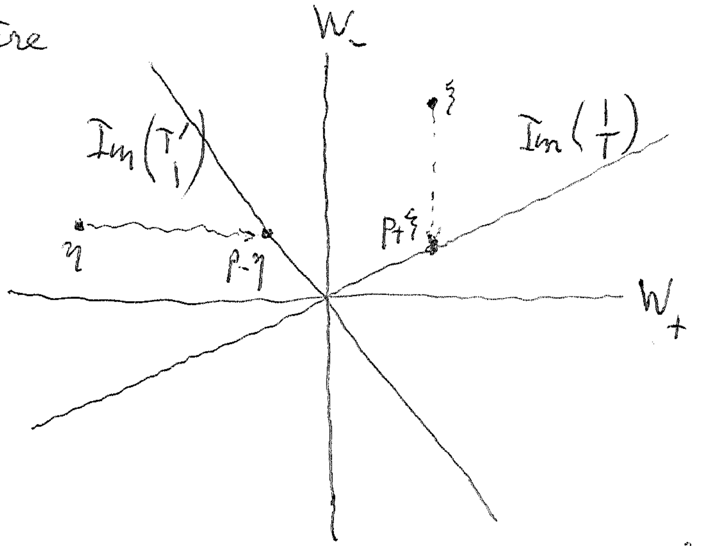
$$p_+ = \alpha_+ \beta_+ = \begin{pmatrix} 1 \\ T \end{pmatrix} (1 \ 0) = \begin{pmatrix} 1 & 0 \\ T & 0 \end{pmatrix}$$

$$p_- = \alpha_- \beta_- = \begin{pmatrix} T' \\ 1 \end{pmatrix} (0 \ 1) = \begin{pmatrix} 0 & T' \\ 0 & 1 \end{pmatrix}$$

$$\alpha_+ \beta_- = \begin{pmatrix} 1 \\ T \end{pmatrix} (0 \ 1) = \begin{pmatrix} 0 & 1 \\ 0 & T \end{pmatrix}$$

$$\alpha_- \beta_+ = \begin{pmatrix} T' \\ 1 \end{pmatrix} (1 \ 0) = \begin{pmatrix} T' & 0 \\ 1 & 0 \end{pmatrix}$$

Picture



Let $F = (1+X)\varepsilon(1+X)^{-1}$

$$= \frac{1+X}{1-X}\varepsilon$$

so that

$F = +1$ on $\begin{pmatrix} 1 \\ T \end{pmatrix} W_+$

$F = -1$ on $\begin{pmatrix} T' \\ 1 \end{pmatrix} W_-$

You conclude that the Cayley transform F, which yields the projection $\frac{F+1}{2}$ onto $\begin{pmatrix} 1 \\ T \end{pmatrix} W_+$ with kernel $\begin{pmatrix} T' \\ 1 \end{pmatrix} W_-$ is unrelated to p_+ and p_-.

March 29, 2002

Prop. Let $u : R \longrightarrow S$ be a map of unital rings such that the restriction of scalars functor
$$u^* : \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(R)$$
is an equivalence of categories. Then u is an isomorphism.

Proof. Recall that u^* admits a left adjoint $u_! M = S \otimes_R M$, with adjunction maps
$$u_! u^* N = S \otimes_R N \xrightarrow{\alpha} N \ , \quad s \otimes n \longmapsto sn$$
$$M \xrightarrow{\beta} u^* u_! M = S \otimes_R M \ , \quad m \longmapsto 1_S \otimes m$$
Actually it's useful to factor $\beta : M = R \otimes_R M \xrightarrow{u \otimes 1} S \otimes_R M$.

From category theory one knows that when one adjoint functor is an equivalence then both adjunction maps are isomorphisms, and the two functors are quasi-inverse. So β is an isomorphism for all M; taking $M = R$ yields that $u : R \to S$ is bijective.

Here's another proof which uses the adjoint pair
$$\mathrm{Mod}(R) \underset{u_*}{\overset{u^*}{\rightleftarrows}} \mathrm{Mod}(S) \qquad u_* M = \mathrm{Hom}_R(S, M)$$
with adjunction maps
$$\beta : N \longrightarrow u_* u^* N = \mathrm{Hom}_R(S, N) \ , \qquad n \longmapsto (s \longmapsto sn)$$
$$\alpha : u^* u_* M \longrightarrow M \ , \qquad \begin{array}{c} \mathrm{Hom}_A(S, M) \longrightarrow M \\ f \longmapsto f(1_S) \end{array}$$
It's useful to factor α into
$$\begin{array}{ccccc} \mathrm{Hom}_R(S, M) & \longrightarrow & \mathrm{Hom}_R(R, M) & = & M \\ f & \longmapsto & fu & \longmapsto & fu(1_R) = f(1_S). \end{array}$$

When u^* is an equivalence this map α is an isomorphism for all R-modules M, which implies $u: R \to S$ is an isom. of R-modules, hence of rings.

INTERVAL

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