January 16, 2002

Make some notes on Serre's theorem about the equivalence of categories between fin. gen. proj. modules over $\text{C}(X)$, $X$ compact, and (complex) vector bundles over $X$. Points:

- Abstract category staff: notions of retract $Y$ of an object $X$: $Y \rightarrow X \rightarrow Y$ (i = Y)
- Projection: $p: X \rightarrow X$, $p^2 = p$, Karoubian category, Karoubian envelope.
- Any sub $E/_{X}$ is a retract of a trivial bundle $X \times V$, $V$ fin diml vs. True locally as $E$ is locally trivial, hence $E$ finite open covering $X = U_1 \cup \cdots \cup U_n$, and retracts

$$E \leftrightsquigarrow U_\mu \times V \leftrightsquigarrow E \mu$$

$E \mu$ is id on $E \mu$.

- Partition of 1 = $\sum x^2, \text{Supp } x_\mu \subset U_\mu$.

$$E \leftrightsquigarrow X \times (V_1 \cdots V_n) \leftrightsquigarrow E$$

$$\sum x_\mu p_\mu x_\mu = \sum x^2 = 1 \text{ in } E$$

- If $p^2 = p$ is a projection on the vector bundle $E/_{X}$, then locally there exists a trivialization $E \mu \sim U \times V$ such that $p$ becomes the usual projection on $V$.

Proof: can suppose $E = X \times V$ trivial, $p$ is a continuous family $x \mapsto p_x = p_x \in \text{End}(V)$, shift from projection to involution $F_x = 2p_x - 1$, let $E = F_0$ at the point of interest, put $g_x = F_x$, so that $Eg_x^{-1} = EF_x = g_x^{-1}$.
Use \( \exp: \text{End}(V) \to \text{Aut}(V) \), local diffeomorphism near zero, to define \( g^x = \exp(\frac{1}{2} \log g_x) \).

Then

\[
\begin{align*}
\varepsilon g_x \varepsilon^{-1} &= g_x^{-1} \quad \Rightarrow \quad \varepsilon^{rac{1}{2}} g^{-\frac{1}{2}} = \varepsilon^{rac{1}{2}} \varepsilon^{-\frac{1}{2}}
\end{align*}
\]

(hint of the 1-parameter subgroup \( t \mapsto \exp(t \log g_x) \)). Finally

\[
\varepsilon^{rac{1}{2}} g^x \varepsilon^{-\frac{1}{2}} = g_x \varepsilon = F_x,
\]

which means that the bundle action \( x \mapsto g^x_x \) transforms \( F_x \) to \( \varepsilon \).

Heisenberg group + Lie alg. The Lie alg has basis \( X, Y, H \) with relations \([X, Y] = H\), \([X, H] = [Y, H] = 0\)

\[
\partial_t \left( e^{tx} Y e^{-tx} \right) = e^{tx} (XY - YX) e^{-tx} = e^{tx} He^{-tx} = H.
\]

so \( e^{tx} Y e^{-tx} = Y + tH \). Then

\[
\begin{align*}
\varepsilon x^s Y e^{-tx} e^{-tX} &= \varepsilon x^{s(x^{ty} e^{tx})} = \varepsilon x^{s(y + tH)} = \varepsilon x^{stH sY}
\end{align*}
\]

since \( H, Y \) commute.

\[
\begin{align*}
\varepsilon x^{tX} sY e^{-tx} e^{-tY} = \varepsilon x^{stH sY}
\end{align*}
\]

Also let \( U(t) = e^{-tx} e^{tx(Y+X)} \). Then

\[
\partial_t U(t) = e^{-tx} (-X)e^{tx(Y+X)} + e^{-tx} (X+Y)e^{tx(Y+X)} = e^{-tx} Y e^{tx(t^2 X + tX t(Y+X))},
\]

so

\[
\partial_t U(t) = (Y + tH) U(t) \quad \Rightarrow \quad U(t) = e^{ty - \frac{t^2}{2} H},
\]

so

\[
\varepsilon \begin{pmatrix} t(X+Y) \\ tx \\ ty \end{pmatrix} = e^{ty - \frac{t^2}{2} H},
\]

(Check: \( e^{t(X+Y)} e^{-t(X+Y)} = e^{tx ty - tx ty - t^2 H} = 1 \))
Consider the principal bundle $\pi: \mathbb{R}^2 \rightarrow \mathbb{R}^2/\mathbb{Z}^2 = T^2$
where $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translations. Let $A = C^\infty(T^2)$
equiv. $A$ is the ring of smooth functions $f(x,y)$ on $\mathbb{R}^2$
which are doubly periodic (period 1): $f(x+m, y+n) = f(x,y)$.

Let $L$ be the space of smooth functions $\psi(x,y)$ on $\mathbb{R}^2$
satisfying $\psi(x,y+1) = \psi(x,y) = e^{2\pi i y} \psi(x+1,y)$, again
$\psi(x+m, y+n) = e^{-2\pi i my} \psi(x,y)$.

Note that $(f, \psi) \mapsto f \psi$ makes $L$ into an $A$-module.

More generally, for any open set $U \subset T^2$, let $A(U) = C^\infty(U)$ = doubly periodic smooth $f(x,y)$ in $\pi^{-1}U$, and
let $L(U) = $ smooth functions on $\pi^{-1}U$ satisfying the above
automorphy condition.

Assume there is given a trivialization of the principal
bundle $\pi$ over $U$: $s: U \rightarrow \pi^{-1}U$, $\pi s = id_U$. Then
$\pi^{-1}U = sU \times \mathbb{Z}^2$, so that $A(\pi^{-1}U) = C^\infty(U)$ and
$\pi s: \pi^{-1}U \rightarrow U$, where these isos. arise by composing
with $s$. Thus one sees that locally over the 2-torus,
$L(U)$ is a free module of rank 1 over $A(U)$.

So it should be clear that $L$ is the space of smooth
sections of a smooth line bundle $L$ over the torus.

Question: Is this line bundle trivial, equiv. does there
exist $\psi \in L$ nowhere vanishing?

No, because there's a degree obstruction. Assuming $\psi \neq 0$
then $\int_{y=0}^{y=1} (\psi(x,y)^{-1} d\psi(x,y))$ is independent
of $x$ by Stokes's thm. On the other hand the automorphy
condition gives

\[ \psi(x, y)^{-1} d\psi(x, y) = 2\pi i \, dy + \psi(x+1, y) \psi(x, y)^{-1} d\psi(x+1, y) \]

and \[ \int_{y_0}^{y_1} 2\pi i \, dy = 2\pi i \], showing the degree jumps.

The next step will be to construct a connection in the v.b. L over \( T^2 \). On \( T^2 \) we have the commuting vector fields \( \partial_x, \partial_y \) which generate the tangent space at each point. A connection in \( L \) can be described as \( \partial_x, \partial_y \) which are compatible with \( \partial_x, \partial_y \) in the sense that Leibniz's rule holds:

\[ D_x (\psi) = \partial_x \psi + \psi D_x \psi, \quad \text{same for } D_y. \]

Claim: \( D_x = \partial_x + i \, 2\pi x \), \( D_y = \partial_y + i \, 2\pi y \) are operators on \( L \) compatible with multiplication by elements of \( A \) in the above sense.

Easy for \( D_x \):

\[ \psi(x, y+1) = \psi(x, y) = e^{2\pi i y} \psi(x, y+1) \]

\[ \Rightarrow (\partial_x \psi)(x, y+1) = (\partial_x \psi)(x, y) = e^{2\pi i y} (\partial_x \psi)(x, y+1) \]

For \( D_y \) use alternate form of the automorphy condition:

\[ e^{2\pi i x} \psi(x, y+1) = e^{2\pi i x} \psi(x, y) = e^{2\pi i (x+1)} \psi(x+1, y) \]

We apply \( e^{-2\pi i x} \partial_y \) to these three terms to get

\[ (2\pi i x + \partial_y) \psi(x, y+1) = (2\pi i x + \partial_y) \psi(x, y) \]

\[ = e^{2\pi i y} (2\pi i (x+1) \psi(x+1, y) + e^{2\pi i y} (\partial_y \psi)(x+1, y) \]

\[ = e^{2\pi i y} (2\pi i \psi(x+1, y) + e^{2\pi i y} (\partial_y \psi)(x+1, y)) \]

Showing \( (\partial_y + 2\pi i x) \psi \) also satisfies the automorphy condition.
Simpler method uses the isomorphism
\[ f(x) \mapsto \hat{f}(x, y) = \sum_m e^{2\pi i m y} f(x+m) \]
between \( S(\mathbb{R}) \) and \( \mathbb{L} \).

\[ \partial_x \hat{f}(x, y) = \sum_m e^{2\pi i m y} \frac{d}{dx} f(x+m) = \frac{df}{dx} \]

\[ (\partial_y + 2\pi i x) \hat{f}(x, y) = \sum_m e^{2\pi i m y} \left( \frac{2\pi i m + 2\pi i x}{2\pi i (x+m)} \right) f(x+m) \]

\[ \approx (2\pi i x f) \]

Thus the operators \( D_x, D_y \) when viewed in the \( S(\mathbb{R}) \) picture are

\[ D_x f = \frac{d}{dx} f \]

\[ D_y f = 2\pi i x f \]

Note that \([D_x, D_y] = 2\pi i\) so that we have an action of the Heisenberg Lie algebra on \( \mathbb{L} \) compatible with the action of \( IR^2 \) on \( A \).
January 24, 2002 (David is 38)

Recall \( L = \{ \psi(x,y) \in C^\infty(R^2) \mid \psi(x,y) \text{ period 1 in } y \} \)

Claim \( L \) closed under the operators \( D_x = \partial_x, D_y = \partial_y + 2\pi i x \partial_y \)

\( \partial_x \psi(x,y) \) preserves translations \((x,y) \mapsto (x+a, y+b)\). So \( \partial_x \psi(x,y) \) has period 1 in \( y \)

\[
\partial_x (e^{2\pi i x \psi(x,y)}) = 2\pi i y e^{2\pi i x \psi(x,y)} + e^{2\pi i x \psi(x,y)} \partial_x \psi(x,y)
\]

period 1 in \( x \) and \( e^{2\pi i x \psi(x,y)} \partial_x \psi(x,y) \) has period 1 in \( x \).

Next, \( D_y \psi(x,y) = \partial_y \psi(x,y) + 2\pi i x \psi(x,y) \) period 1 in \( y \).

\[
\partial_y (e^{2\pi i x \psi(x,y)}) = e^{2\pi i x \psi(x,y)} (\partial_y \psi(x,y) + 2\pi i x \psi(x,y))
\]

period 1 in \( x \) and \( e^{2\pi i x \psi(x,y)} \partial_y \psi(x,y) \) has period 1 in \( x \).

so \( D_y \psi \in L \).

Recall that \( L \) is a module over the ring \( A = C^\infty(T^2) \) of \( f(x,y) \) having period 1 in both \( x,y \).

\( D_x \) has the derivative (Leibniz) property

\[
D_x(f \psi) = \partial_x f \psi + f \partial_x \psi
\]

and similar for \( D_y \). This implies upon exponentiating that

\[
e^{aD_x} f(x,y) \psi(x,y)
= (e^{a\partial_x} f(x,y)) (e^{aD_x} \psi(x,y)) = f(x+a, y) e^{aD_x} \psi(x,y)
\]

and similarly for \( bD_y \). This means that we obtain operators \( e^{aD_x} e^{bD_y} \) on \( L \) compatible with the translation operators \( e^{a\partial_x} e^{b\partial_y} f(x,y) = f(x+a, y+b) \) on \( A \).

Let's calculate \( e^{aD_x} e^{bD_y} \) on \( L \). Since
\[ D_y = \partial_y + 2\pi i x \quad \text{where} \quad \partial_y, 2\pi i x \quad \text{commute} \]

we have \[ e^{bD_y} \psi(x, y) = e^{b2\pi i x} e^{bD_y} \psi(x, y) \]
\[ = e^{2\pi ibx} \psi(x, y + b). \]

Then \[ e^{aD_x} e^{bD_y} \psi(x, y) = e^{aD_x} e^{2\pi ibx} \psi(x, y + b) \]
\[ = e^{2\pi ib(x+a)} \psi(x+a, y+b). \]

Also \[ e^{aD_x} e^{bD_y} \psi(x, y) = e^{2\pi ibx} e^{bD_y} e^{aD_x} \psi(x, y) \]
\[ = e^{2\pi ibx} \psi(x+a, y+b). \]

Thus \[ e^{aD_x} e^{bD_y} = e^{2\pi ib} e^{bD_y} e^{aD_x} \quad \text{as expected} \]
in the Heisenberg group since \[ [aD_x, bD_y] = 2\pi i ab. \]

Let's check: that \[ e^{2\pi ibx} \psi(x+a, y+b) \in L. \]

It's clearly periodic if period 1 in y. Consider \[ e^{2\pi i(xy+xb)} \psi(x+a, y+b) \]
and \[ e^{2\pi i(xy+xb+ay+ab)} \psi(x+a, y+b). \]

We want the former to have period 1 in x. The latter has period 1 in x since \[ e^{2\pi i(xy)} \psi(x,y) \] does. The two expressions differ by the factor \[ e^{2\pi i(ay+ab)} \] which is constant in x. So it's clear.
Let $L = \{ \psi \in C^\infty(\mathbb{R}^2) \text{ satisfying } \psi(x+m,y+n) = e^{-2\pi i m y} \psi(x,y) \}$.

This is the space of smooth sections of the line bundle of degree $1$ over $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$. Let's propose to construct a projective action of $SL(2,\mathbb{Z})$ on $L$ corresponding to the natural action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2 \times \mathbb{R}^2 \times T^2$.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$, let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be the corresponding matrix multiplication

$$(c_1 \, c_2) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} ac_1 + bc_2 \\ cc_1 + dc_2 \end{pmatrix}$$


and let $x : \mathbb{R}^2 \to \mathbb{R}$, $y : \mathbb{R}^2 \to \mathbb{R}$ be the coordinate functions: $x = pr_1$, $y = pr_2$. Use $g$ to pull back functions, differential forms on $\mathbb{R}^2$. Thus

$$g^* x(c_1, c_2) = x(ac_1 + bc_2, cc_1 + dc_2) = ac_1 + bc_2,$$

so that

$$g^* x = ax + by$$

$$g^* y = cx + dy$$

$$(g^* y)(x, y) = y(ax + by, cx + dy).$$

Ex. 1. $g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$(g^* y)(x, y) = y(y, -x)$$

Suppose $\psi_0 \in L$, and 

$$\psi_1(x, y) = (g^* \psi_0)(x, y) = \psi_0(y, -x).$$

Corresponding to the automorphic condition for $\psi_0 \in L$
we have
\[ \psi_1(x+m, y+n) = \psi_0(y+n, x-m) = e^{-2\pi in(-x)} \psi_0(y, x) \]
\[ \implies \psi_1(x+m, y+n) = e^{2\pi i nx} \psi_1(x, y) \]

i.e.
\[ \psi_1(x+m, y+n) = e^{-2\pi i xy} \psi_1(x, y) \]

Now put \( \psi_2(x, y) = e^{-2\pi i xy} \psi_1(x, y) \). Then
\[ \psi_2(x+m, y+n) = e^{-2\pi i (x+m)(y+n)} \psi_1(x+m, y+n) \]
\[ = e^{-2\pi i (xy+my+nx+mn)} e^{2\pi i nx} \psi_1(x, y) \]
\[ \implies \psi_2(x+m, y+n) = e^{-2\pi i my} \psi_2(x, y) \]

\[ \psi(x, y) \mapsto e^{-2\pi i xy} \psi(y, x) \]
maps \( L \) into \( L \)

Ex.2. \( g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \)
\( (g^* \psi)(x, y) = \psi(x+y, y) \)

Let \( \psi_0 \in L \), put \( \psi_1(x, y) = g^* \psi_0 = \psi_0(x+y, y) \). Then
\[ \psi_1(x+m, y+n) = \psi_0(x+ym, y+n) = e^{-2\pi i (m+n)y} \psi_0(x, y) \]

Put \( \psi_2(x, y) = e^{h(y)} \psi_1(x, y) \) where \( h(y) \) is to be determined. Then
\[ \psi_2(x+m, y+n) = e^{h(y+n)} \psi_1(x+m, y+n) \]
\[ = e^{h(y+m)} e^{-2\pi i (m+n)y} e^{-h(y)} e^{h(y)} e^{h(y)} \psi_1(x, y) \]
\[ \psi_2(x+m, y+n) = e^{-2\pi i my} \psi_2(x, y) e^{h(y+n)-h(y)-2\pi i y} \]

Put \( h(y) = 2\pi i \frac{y(y-1)}{2}, \quad \frac{(y+n)^2 - y - m}{2} - \frac{y^2 + y}{2} = ny + n^2 x/2 \)
Conclude \( \psi(x, y) \in L \Rightarrow e^{2\pi i \frac{y^2}{2} - xy} \psi(y, y-x) \in L \)

Ex. 3 \( \mathbf{g} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \), \( \psi_1(x, y) = \psi_0(x, x+y) \).

Then for \( \psi_0 \in L \) : \( \psi_1(x+m, y+n) = \psi_0(x+m, x+y+m+n) = e^{-2\pi im(x+y)/x} \psi_0(x, y) \)

or \( \psi_1(x+m, y+n) = e^{-2\pi im(x+y)} \psi_1(x, y) \).

Put \( \psi_2(x, y) = e^{h(x) \psi_1(x, y)} \)

\( \psi_2(x+m, y+n) = e^{h(x+m) - 2\pi im(x+y)} \psi_1(x, y) \)

\( = e^{-2\pi imy} \psi_2(x, y) e^{h(x+m) - h(x) - 2\pi imx} \)

The last exponential is 1 if \( h(x) = \frac{2\pi i x(1-x)}{\lambda} \), so

Conclude \( \psi(x, y) \in L \Rightarrow e^{2\pi i \frac{y(x-1)}{\lambda}} \psi(x, x+y) \in L \)

Ex. 4. \( \mathbf{g} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \). The corresponding fractional linear transform satisfies \( g(0) = (\frac{1}{1}) = 1 \), \( g(1) = (\frac{1}{0}) = \infty \), \( g(\infty) = (0) = 0 \).

So \( \mathbf{g} \) denotes the non-Euclidean \( \Delta \) with vertices 0, 1, \( \infty \) in \( \mathbb{P} \mathbb{R} \) by 120°. If the center of the \( \Delta \) is fixed, then \( (0, 1) \in \mathbb{C} \Rightarrow \frac{1}{-z+1} = z \), \( z^2 - z + 1 = 0 \).

\( z = \frac{1 \pm \sqrt{-3}}{2} \), so the fixed in the UHP is \( \frac{1 \pm i\sqrt{3}}{2} \), a primitive 6th root of \( 1 \). Actually \( \mathbf{g} \) has order 6 and \( \mathbf{g}^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) generates the center of \( \text{SL}(2, \mathbb{Z}) \).

One calculates that

\( \psi_1(x, y) \in L \Rightarrow e^{2\pi i \frac{y^2}{2} - xy} \psi(y, y-x) \in L \)
Discuss central extensions of an abelian group $A$.

$1 \rightarrow B \xrightarrow{i} E \xrightarrow{\pi} A \rightarrow 1.$

A basic invariant of such an extension is the commutator pairing $h(a_1, a_2) \in \text{Hom}(\Lambda^2 A, B)$, which is defined by $\pi(h(a_1, a_2)) = e_1 e_2 e_1^{-1} e_2^{-1}$ where $\pi(e_i) = a_i$.

One has a short exact sequence

(1) $0 \rightarrow \text{Ext}_2^1(A, B) \rightarrow H^2(A, B) \rightarrow \text{Hom}(\Lambda^2 A, B) \rightarrow 0$

obtained from the universal coefficient theorem for the classifying space $B_\mu A$. One needs to know that $H_1(B_\mu A) = A$, $H_2(B_\mu A) = \Lambda^2 A$. The first is Hurewicz, the second is checked for cyclic groups, then holds for products by

$\Lambda^2(A_1 \times A_2) \cong \Lambda^2 A_1 \oplus A_1 \otimes A_2 \oplus \Lambda^2 A_2$

$H_2(B_\mu A_1 \times B_\mu A_2) \cong H_2(B_\mu A_1) \oplus H_1(B_\mu A_1) \otimes H_1(B_\mu A_2) \oplus H_2(B_\mu A_2)$

hence it holds for all $A$ by taking colimits.

Note that in (1) the left exactness is obvious: central extensions with zero commutator pairing are abelian group extensions. You would like to understand why every skew-symmetric bilinear map $\Lambda^2 A \rightarrow B$ comes from a central extension.

Let $Z^2(A, B)$ be the space of group 2-cocycles:

$f(a_1, a_2) : Z[A \times A] \rightarrow B$ satisfying

$(df)(a_1, a_2, a_3) = f(a_2, a_3) - f(a_1 + a_2, a_3) + f(a_1, a_2 + a_3) - f(a_2, a_2) = 0$
Note that a bilinear map \( f: A \otimes A \to B \) is a 2-cocycle:
\[ (b_1, a_1)(b_2, a_2) = (b_1 + b_2, f(a_1, a_2), a_1 + a_2) \]

One has
\[ (b_2, a_2)(b_1, a_1) = (b_1 + b_2 + f(a_2, a_1), a_1 + a_2) \]

so the commutator pairing is \( f(a_1, a_2) - f(a_2, a_1) \).

So you find that process of skew-symmetrization converts any 2-cocycle into a bilinear skew-symmetric (means \( f(a, a) = 0 \)) 2-cocycle.

Question: Is every map \( \Lambda^2 A \to B \) obtained as the commutator pairing for the extension corresponding to a map \( A \otimes A \to B \)? Taking \( B = \Lambda^2 A \) one sees this implies that \( a_1 \otimes a_2 \mapsto a_1 \otimes a_2 - a_2 \otimes a_1 \) from \( \Lambda^2 A \) to \( A \otimes A \) is a direct injection. Using
\[ \Lambda^2(A_1 \oplus A_2) = \Lambda^2 A_1 \oplus A_1 \otimes A_2 \oplus \Lambda^2 A_2 \]

one checks this property persists under taking direct sums, since it's true for \( A \) cyclic, one sees it's true for \( A \) fin. gen. Then one gets injectivity of \( \Lambda^2 A \to A \otimes A \) by colims, and the exact sequence
\[ 0 \to \Lambda^2 A \to A \otimes A \to \text{Im} \to 0 \]
in general.
Def. A quadratic function \( g : \Lambda^2 \rightarrow C \) is one such that 
\[- (g_1, g_2) = g(q_1 + q_2) - g(q_1) - g(q_2) \]
is \( C \)-bilinear. Thus we have a cartesian square

\[
\begin{array}{ccc}
\text{Hom}(\Lambda^2, C) & \cong & \text{Quad}(A, B) \\
\downarrow & & \downarrow S^6 \\
\text{Hom}(A \otimes A, B) & \cong & \text{Quad}(A, B) = \text{Hom}(\Sigma_2 A, B)
\end{array}
\]

of representable functors in \( B \). Here \( \Sigma_2 A \) is the universal abelian group generated by a \( 2 \)-cocycle; its the cokernel of a map \( \mathbb{Z}[A^3] \rightarrow \mathbb{Z}[A^2] \). Corresponding to the above cartesian square is a cocartesian square

\[
\begin{array}{ccc}
\Lambda^2 A & \xrightarrow{\text{comm}} & \Sigma_2^1 A \\
\downarrow \text{pairing} & & \downarrow (S^8) \\
\Sigma_2^2 A & \rightarrow & \mathbb{Z}[A] \\
\downarrow \text{cocart} & & \downarrow \\
0 & \rightarrow & \Lambda^2 A \\
\end{array}
\]

Bottom row is exact. Think of the bottom row as best you can do with \( 2 \)-nd degree tensors.

In the top row, \( \Lambda^2 A \) is a direct summand of \( \Sigma_2^1 A \) in some mysterious way, since the identity map of \( \Lambda^2 A \) is the commutator pairing of some \( 2 \)-cocycle with values in \( \Lambda^2 A \).

It turns out that the top row is exact at \( \Sigma_2^1 A \) and then the mystery disappears.
Start with the complex

\[ \mathbb{Z}[A^3] \xrightarrow{\partial} \mathbb{Z}[A^2] \xrightarrow{\partial} \mathbb{Z}[A] \to \mathbb{Z} \]

of chains in the classifying space \( B\Sigma A \), which is the realization of the simplicial group \( A \).

\[ A^3 \quad \Rightarrow \quad A^2 \quad \Rightarrow \quad A \Rightarrow \mathbb{Z} \]

The homology of this complex is \( H_0 = \mathbb{Z}, \ H_1 = A, \ H_2 = \Lambda^2 A \).

Look at the truncated complex of length 1 having the homology groups \( \Lambda^2 A \) and \( A \):

\[ \Lambda^2 A \xrightarrow{d} \sum_2 A \xrightarrow{d} \mathbb{Z}[A] \xrightarrow{d} A \]

where \( \sum_2 A = \text{Coker} \{ \mathbb{Z}[A^3] \xrightarrow{\partial} \mathbb{Z}[A^2]\} \) is the universal abelian group for a 2-cocycle on \( A \). Note that because \( \mathbb{Z}[A] \) is a free abelian group, so is the image of \( d \), so there exists a lifting of this image into \( \sum_2 A \).

Thus you see that \( \Lambda^2 A \) splits off \( \sum_2 A \).

So the complex \( \sum_2 A \xrightarrow{d} \mathbb{Z}[A] \) is quasi-

\[ \Lambda^2 A[1] \oplus A[0], \] which leads to the short exact sequence

\[ 0 \to \text{Ext}_2^1(A, B) \to H^2(A, B) \to \text{Hom}(\Lambda^2 A, B) \to 0 \]
as well as the fact that this sequence splits. (You need to use

\[ \text{Hom} \left( \sum_2 A \to \mathbb{Z}[A], B \right) = \left[ C^1(A, B) \xrightarrow{\partial} \mathbb{Z}^2(A, B) \right] \]
Now compare this complex with the complex using 2nd degree tensors:

\[
\begin{align*}
\Lambda^2 A & \longrightarrow \Sigma_2 A \longrightarrow \mathbb{Z}[A] \longrightarrow A \\
\Lambda^2 A & \longrightarrow A \otimes A \longrightarrow \Gamma^2 A \longrightarrow A
\end{align*}
\]

So the conclusion is that one has a surjective map:

\[
\Sigma_2 A \longrightarrow \mathbb{Z}[A] \longrightarrow A \otimes A \longrightarrow \Gamma^2 A
\]

of complexes, which means for some purposes, such as for central extensions with \(B\) divisible, that the two complexes yield the same information.

Feb 24, '02

Consider the short exact sequence of abelian groups:

\[
S^2 A \longrightarrow \Gamma^2 A \longrightarrow A
\]

Notice that the projection \(\pi\) has a tautological section \(\xi\) satisfying \(\xi (a+a') = \xi (a) + \xi (a') = \) the tautological symmetric bilinear form \(a_1 \otimes a_2 \mapsto a_1 a_2 \in S^2 A\). Thus identifying \(S^2 A \times A \cong \Gamma^2 A\) one has an abelian group law on the set \(\Gamma^2 A\) given by \((b, a) \cdot (b', a') = (b + b' + aa', a + a')\). The map \(\xi : A \rightarrow S^2 A \times A, a \mapsto (\xi (a), a)\) satisfies \((\xi (a), a) \cdot (\xi (a'), a') = (\xi (a + a'), a + a')\).
Start again: A symmetric bilinear form
\[ s^2 A \to B \] yields an abelian group extension
\[ B \to E \to A \]
with a section giving back the 2-cocycle \( h \). On the other hand, if \( h \) is the coboundary of a map \( g: A \to B \) (\( g \) is necessarily quadratic), then the extension \( E \) splits.

There are two abelian group extensions of \( A \) by \( s^2 A \).
First is the extension corresponding to 2-cocycle \( (a_1, a_2) \to a_1 a_2 \in s^2 A \).

\[
\begin{align*}
S^2 A &\to S^2 A \times A \\ &\to A,
\end{align*}
\]

Choose \( EA \)

\[
\{ (b, a) | b \in s^2 A, a \in A \}
\]

Next is the exact sequence
\[ S^2 A \to \Gamma^2 A \to A \] Note that the universal quadratic map \( \gamma \) is a section of \( \pi \), and the corresponding 2-cocycle is \( \gamma(a+a')-\gamma(a)-\gamma(a') = 2(aa') \).

Therefore one has an obvious canonical isomorphism between \( EA \) and \( \Gamma^2 A \).

A splitting of the extension \( EA \) is given by a section \( a \to (g(a), a) \) of \( \pi \): \( EA \to A \) which is a group homomorphism, i.e.

\[ (g(a), a) \cdot (g(a'), a') = (g(a)+g(a')+aa', a+a') \]
is equal to \( (g(a+a'), a+a') \), in other words \( g: A \to S^2 A \) is a quadratic map with associated symmetric bilinear form \( aa' \). (Note also that \( g \) induces \( \Gamma^2 A \to S^2 A \) which splits the \( \Gamma^2 A \) extension)

Example: Let \( A = \mathbb{Z} \), whence \( S^2 \mathbb{Z} = \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z} \), hence the symmetric bilinear form is \( (m, n) \to mn \). So our abelian group \( EZ \) is \( \mathbb{Z} \times \mathbb{Z} \) with \( (b, a)(b', a') = (b+b'+aa', a+a') \), where now \( a, b, c \in \mathbb{Z} \). The extension split by lifting \( a=1 \) to \( (b, 1) \) for
any $b \in \mathbb{Z}$ and then adding successively ($a \in \mathbb{Z}$)

to get the section homomorphism $(q(a), a)$.

Note that $q(a) = \frac{a(a-1)}{2}$ is a quadratic map from

$\mathbb{Z}$ to $\mathbb{Z}$ with the associated symm. form $aa'$:

$$q(a+a') - q(a) - q(a') = \frac{1}{2} \left( a^2 + 2aa' + a'^2 - a^2 - a'^2 \right) = a a'$$

Then $q(a) = \frac{a(a-1)}{2} + ba$ is a quadratic map

with $(q(1), 1) = (b, 1)$ as desired.

Examples: $A = \mathbb{Z}/2$. Here $\mathbb{S}^2(\mathbb{Z}/2) \cong \mathbb{Z}/2$ with symmetric

pairing $aa'$. Then

$$n(0, 1) = \left( \frac{n(n-1)}{2}, n \right) \quad \text{modulo } 2.$$ 

Take $n=2$ so that $2^2(0, 1) = (1, 0) \equiv 1 \pmod{2}$, hence the element

$(0, 1)$ of $E(\mathbb{Z}/2)$ has order 4.

$$\mathbb{S}^2(\mathbb{Z}/2) = \mathbb{Z}/4$$

$A = \mathbb{Z}/N$. The question is whether $\mathbb{S}^2(A)$ is killed by $N$.

One has $N$ times $(b, 1) = (Nb + \frac{N(N-1)}{2}, N)$ will be $0 \pmod{N}$

provided $\frac{N(N-1)}{2} \equiv 0 \pmod{N}$. True if $N$ odd. If $N$ is

even then $N$ times $(0, 1) = \frac{N(N-1)}{2}$ one less power of 2,

so it seems that the $2^e$ exponent of $\mathbb{S}^2(\mathbb{Z}/N)$ is one more

than for $N$. For example if $N = 2^k$, then

$$2^k \cdot (0, 1) = (2^k(2^k-1), 2^k) \not\equiv 0 \pmod{2^k}$$

$$2^k \cdot (0, 1) = (2^k(2^k-1), 2^{k+1}) \equiv 0 \pmod{2^k}$$

Warning: When $B$ is not divisible, you have to be careful trying
to use the sequence $0 \rightarrow A^2A \rightarrow A \otimes A \rightarrow \mathbb{S}^2A \rightarrow A \rightarrow 0$. In

the case of central extensions of elementary abelian 2-groups, one

has $H^2(A, B) = \text{Hom}(\mathbb{Z}/4, A)$.

The invariant of the extension is the quadratic map $A \rightarrow B$ obtained by lifting $a \in A$ to $E$

and squaring. Exact sequence $0 \rightarrow \mathbb{S}^2A \rightarrow \mathbb{S}^2A/2 \rightarrow A \rightarrow 0$ yields

$0 \rightarrow \text{Hom}(A, B) \rightarrow \text{Hom}(\mathbb{S}^2A, B) \rightarrow \text{Hom}(\mathbb{S}^2A, B) \rightarrow 0$ ??
Feb 25,02. Here's what happens for $A_{\text{elementary}}$
abelian 2 groups. Given a central extension $B \rightarrow E \rightarrow A$
you get a quadratic map $q : A \rightarrow B$ by lifting $a \in A$ to $E$ and squaring; the associated bilinear form
is the commutator pairing $\Lambda^2 A \rightarrow B$. Here $\Lambda$ can be any abelian group. (Check: Let $x, y \in E$. Then

$$xyxy(xx)^{-1}(yy)^{-1} = xyx(xx)^{-1}y(yy)^{-1} = xyx^{-1}y^{-1}$$

since $xx$ is in the center.)

For example consider the canonical extension

$$\begin{array}{ccc}
S^2A & \rightarrow & \Gamma^2A & \rightarrow & A
\end{array}$$

where $\Gamma^2A = \{(b,a) \in S^2A \times A\}$ with group law

$$(b,a)(b',a') = (b+b'+aa', a+a')$$

one has $(0,a)(0,a) = (aa,0)$, whence $(0,a)$ has order 2 for $a \neq 0$. The quadratic map

$q : A \rightarrow S^2A$ is $q(a) = a^2$, which is linear.

You know that $H^2(B_{\text{el}}, A, \mathbb{Z}/2) = \Gamma^2A/2$ and hence

$$H^2(A, B) = \text{Hom}(\Gamma^2A/2, B)$$

for $B$ elementary abelian. Diagram:

$$\begin{array}{ccc}
A & \rightarrow & A \\
\downarrow & & \downarrow \\
S^2A & \rightarrow & \Gamma^2A & \rightarrow & A \\
\downarrow & & \downarrow & & \downarrow \\
\Lambda^2A & \rightarrow & \Gamma^2A/2 & \rightarrow & A
\end{array}$$

The last row yields

$$\text{Hom}(A, B) \rightarrow H^2(A, B) \rightarrow \text{Hom}(\Gamma^2A, B)$$

by Bockstein.

$$\text{Ext}^1(A, B) \quad \text{(by Bockstein)}.$$
Feb 28, 02. Background: You've seen that the algebra \( A(P) \) of functions (smooth) on a principal \( \mathbb{T} \)-bundle \( P \) is \( \mathbb{Z} \)-graded naturally, because you have a representation of the circle group on \( A(P) \) which then splits according to \( \mathbb{T}^\mathbb{Z} = \mathbb{Z} \). So

\[
A(P) = \bigoplus_{n \in \mathbb{Z}} A(P_n)
\]

where \( \psi \in A(P_n) \) iff \( e^{2\pi i n t} \psi = \psi \). Thus \( A(P_0) = A(B) \) where \( B \) is the base. By the usual, description of sections of an associated fibre bundle to a principal bundle, one should know that \( A(P_n) \) is the space of sections of a line bundle over \( B \) which is the \( n \)-th tensor power of the line bundle \( L \) corresponding to \( A(P_1) \). Thus

\[
A(P) = \bigoplus_n \Gamma(B, \mathcal{L}^n).
\]

This situation reminds you of the Toeplitz algebra, which is the universal algebra \( R \) with generators \( x, y \) subject to the relation \( yx = 1 \). \( R \) acts on \( \mathbb{C}[z] \) with \( xz^n = z^{n+1}, \quad yz^n = z^{n-1} \) if \( n > 1 \), \( y1 = 0 \). \( R \) is clearly spanned by the words \( x^m y^n, \quad m, n \geq 0 \).

Let \( I \) be the ideal in \( R \) generated by \( 1 - xy \). Then \( I = R(1 - xy)R \) is spanned by \( x^m(1 - xy)y^n, \quad m, n \geq 0 \). One has

\[
x^m(1 - xy)y^n z^k = \begin{cases} 
0 & \text{if } k \neq n \\
z^m & \text{if } k = n
\end{cases}
\]

because \( 1 - xy \) acting on \( \mathbb{C}[z] \) is projection onto \( z^0 = 1 \) killing \( z^k \) for \( k > 0 \). Thus the matrix of the operator \( x^m(1 - xy)y^n \) relative to the basis \( z^k, \quad k \geq 0 \) has entry 1 in the \( m, n \) position and 0 elsewhere.
Conclude that the $x^m(1-xy)y^n$ form a basis for $I$ and that $I$ can be identified with the matrices of finite support.

The quotient ring $R/I$ is clearly the ring of Laurent polynomials $\mathbb{C}[z, z^{-1}]$, where $x, y \rightarrow z, z^{-1}$ respectively.

Let's show that $x^m y^n$ for $m, n > 0$ is a basis for $R$. You recall doing this by defining an action of $R$ on $\mathbb{C}[x, y]$ respecting the multi-map $\mathbb{C}[x, y] \rightarrow R$. Then acting on $1 \in \mathbb{C}[x, y]$ gives a $R$-module map $\mathbb{C}[x, y] \rightarrow R$ sending $1$ to $1$, etc.

Here's a direct method:

$$
x^2 \quad (1-xy) \quad y^2
$$

$$
x^3 \quad x(1-xy) \quad (1-xy)y \quad y^3
$$

These elements are independent because they give bases for both $I$ and $R/I$.

Remark that the extension $I \rightarrow R \rightarrow \mathbb{C}[z, z^{-1}]$ you can't lift $z$ to an invertible element of $R$.

Futz's generalization used by Pimsner: $R$ gen'red by $x_i, y_i \quad i=1, \ldots, n$ acting on $T(V), \quad V = \mathbb{C}x_1 + \ldots + \mathbb{C}x_n$, with $x_i$ multiplying on $T(V)$ and $y_j x_i = \delta_{ji} w$. Then you get the $C_\infty$ algebra

$$
y_j x_i = \delta_{ji} \sum_{i=1}^n x_i y_i = 1$$
Let $\Gamma$ be a group, $I$ a finite subset of $\Gamma$.
Define two categories of modules $W$ and $V$.

An object $W$ of $\Gamma$-modules $W$ (scalars $= 0$) equipped with a linear operator $h \in \text{End}(W)$ satisfying

1. $hsh = 0 \Rightarrow seI$

2. $\forall w \in W, \left( \exists s \in \Gamma \mid shs^{-1}w = 0 \right)$ is finite and
   $$\sum_{s} shs^{-1}w = w$$

An object of $V$ is a vector space $V$ together with operators $p(s) \in \text{End}(V)$ for $s \in \Gamma$ satisfying

1. $p(s) \neq 0 \Rightarrow seI$

2. $\sum_{t} p(st^{-1})p(t) = p(s)$

3. $\sum_{s} p(s)V = V$

4. $(\forall s) p(s)v = 0 \Rightarrow v = 0$.

Morphisms in $W$ and $V$ are maps preserving the structure.

Claim: The categories $W$ and $V$ are naturally equivalent.

Define a functor $W \rightarrow V$ as follows.

Given $W$, let $W \xrightarrow{i} V \xrightarrow{\epsilon} W$ be the canonical factorization of $h$ into a surjection followed by an injection. (Thus $V = hW$, $\epsilon = h : W \rightarrow V$ and $i$ is the inclusion $hW \subset W$.)
Define $p(s) = jsi \in \text{End}(V)$. Check the four conditions to see that $V$ is in $W$

(1) If $p(s) = jsi$ is $\neq 0$, then $j$ is injective and $f$ surjective one has $u(jsi)f = hsh \neq 0$ hence $s \in I$.

(2) $\sum_t p(st^{-1}) p(t) v = \sum_t jst^{-1} y t u v = \mathbb{0} \Rightarrow jst^{-1} y t u v = \mathbb{0}$

$\mathbb{0} \sum_t t^{-1} h t v = jsi v = p(s) v$.

(3) $\omega = \sum_s sh s^{-1} \omega \Rightarrow j \omega = \sum_s p(s) y s^{-1} \omega \in \sum_s p(s) V$

so $V = \sum_s p(s) V$ as $f$ is surjective.

(4) If $(y) p(s) v = 0$, then $\omega = \sum_s s i y s^{-1} \omega = 0$, so $\omega = 0$ as $i$ is injective.

Next we construct a function $V \rightarrow W$. First:

Let $\Lambda = \mathbb{C} \Gamma$ be the group algebra of $\Gamma$, and let $\Lambda \otimes V$ be the free $\Gamma$-module generated by the vector space $V$. An element $\sum_t t \otimes f(t) \in \Lambda \otimes V$ is described by a function $f : \Gamma \rightarrow V$ of finite support. The $\Gamma$-action is given by the left regular representation on functions:

$u \sum_t t \otimes f(t) = \sum_t ut \otimes f(t) = \sum_t t \otimes f(u^{-1} t)$. 
The vector space $\Lambda \otimes V = \bigoplus_{s \in \Gamma} s \otimes V \cong \bigoplus_{s \in \Gamma} V$

is naturally graded with respect to $\Gamma$. Define maps

\[
\begin{align*}
\Lambda \otimes V & \quad \varepsilon_1 \otimes V = 1 \otimes V \\
\eta_1 & \quad \sum_{t} t \otimes f(t) = f(1)
\end{align*}
\]

so that $\varepsilon_1, \eta_1$ describe the summand corresponding
to the identity $1 \in \Gamma$. Clearly $s \varepsilon_1, \eta_1 s^{-1}$ describe
the summand corresponding to $s$. One has the
"orthogonality " and "completeness" relations:

\[
\eta_1 s^{-1} t \varepsilon_1 = \delta_{st} \text{ in } V, \quad \sum_{s} s \varepsilon_1 \eta_1 s^{-1} = 1_{\Lambda \otimes V}.
\]

Now let $V$ be an object of $\mathcal{U}$, i.e. equipped with
operator $p(s)$ satisfying 4 condition on p43. Define the
operator $p$ on $\Lambda \otimes V$ by

\[
p\left(\sum_{t} t \otimes f(t)\right) = \sum_{s} s \otimes \sum_{t} p(s^{-1} t) f(t).
\]

In other words, viewing $f$ as a column vector, $p$ is
multiplication by the matrix with entries $p(s^{-1} t)$.

Properties of $p$: (1) $\forall s \in \Gamma, \, u \mapsto pu$. (2) $p^2 = p$.

(1) $p$ is well-defined since for each $t, \, s \mapsto p(s^{-1} t)$ has
finite support.

(1) is clear since $p(s^{-1} t)$ is invariant under left
multiplication $(s, t) \mapsto (hs, at)$.

(2) follows from $\sum_{t} p(s^{-1} t) p(t^{'} u) = p(s^{-1} u)$ which is
clear if the idempotence condition
is written $p(s) = \sum_{tu=s} p(t) p(u)$.
Let $W$ be the summand of the $\Gamma$-module $\Lambda \otimes V$ corresponding to the idempotent operator $p$. There are canonical $\Gamma$-module maps $\alpha, \beta$

$$W \xrightarrow{\beta} \Lambda \otimes V \xrightarrow{\alpha} W \xleftarrow{\beta} \Lambda \otimes V$$

such that $\beta \alpha = 1_W$, $\alpha \beta = p$. Componentwise, the condition $\alpha \beta = p$ means

$$\eta_i s^{-1} p t \epsilon_1 = p(s^{-1}t).$$

Let $l = \beta \epsilon_1$, $\gamma = \eta_i \alpha$, and $h = i j : W \rightarrow W$.

One has $p(s^{-1}t) = \eta_i s^{-1} \alpha \beta t \epsilon_1 = s^{-1}t \epsilon_1$.

Next we show $W$ equipped with $h$ is in $W$.

Check two conditions: (1) $0 \neq hsh = (j s i) s = (p s) s$.

$\Rightarrow p(s) \neq 0 \Rightarrow s e i$. (2) $1_W = \beta \Lambda \otimes V = \sum_s \beta s e_i \eta_i s^{-1}$

$= \sum_s s \beta e_i \eta_i s^{-1} = \sum_s shs^{-1}$.

At this point we have constructed functors $W \rightarrow hW$ from $W$ to $V$ and $V \rightarrow p(\Lambda \otimes V)$ from $V$ to $W$. In the case of the second functor we have used the 3rd and 4th conditions: $\sum_s p(s) V = V$, $\forall p(s) v = 0 \Rightarrow v = 0$. Return to (x) to find $jW$ and $\text{Ker}(i)$. $jW = \eta_i s \beta(\Lambda \otimes V)$ since $\beta$ surjective. $jW$ consists of all $\eta_i p \sum s \otimes f(t) = \eta_i s \sum s \otimes p(s) t f(t)$

$= \sum_s (p(s)) f(t)$, i.e. $jW = \sum_s p(t) V$. 
Since \( \alpha \) is injective, one has \( \text{Ker}(i) = \text{Ker}(\alpha i) = \text{Ker}(\rho \phi) \)
\[ = \{ v \in V \mid \rho(p \otimes v) = \sum_s s \otimes p(s^{-1})v = 0 \} \text{.} \]
Hence
\[ \text{Ker}(i) = \bigcap_s \text{Ker}(p(s)) \text{.} \]
Thus \( V \) satisfies 3rd + 4th conditions \( \iff \) \( \phi \) surjective, which means that there is a canonical isomorphism of \( V \) with the image \( hW \). Thus the composite functor \( V \rightarrow W \rightarrow V \) is isomorphic to \( hW \).

Next, given \( W \) in \( W \) let
\[ W \leftarrow \mathcal{V} \leftarrow W \] be the canonical factorization of the operator \( h \). Define maps \( \alpha, \beta \)
\[ W \leftarrow \mathcal{V} \leftarrow W \text{ by } \]
\[ \alpha \omega = \sum_s s \otimes \eta^\top_j w \]
\[ \beta \sum_t \sigma f(t) = \sum_t \sum_j t \otimes f(t) \]

Proof (1) \( \beta \) is the unique \( \Gamma \)-module map extending \( i \) in the sense that \( \beta \epsilon_i = i \).
(2) \( \alpha \) is the unique \( \Gamma \)-module map coextending \( f \) in the sense that \( \eta \alpha = f \).

(1) is clear since \( \beta(t \omega) = \beta(t \epsilon \omega) = t \omega \).
(2) uniqueness. Let \( \alpha \omega = \sum_t t \otimes g(t) \). Then
\[ js^{-1} \omega = \eta_1 s^{-1} \alpha \omega = \eta_1 \sum_t s^{-1} t \otimes g(t) = \sum_t g(s) \]
if a map \( \alpha \) with the stated properties exists it is given by the above formula. We must show that \( s \mapsto js^{-1} \omega \) has finite support in order that \( \omega \) be well-defined. Since \( \sum_s shs^{-1} \omega = \omega \), any element of \( W \) is a finite sum of elements of the form \( \epsilon \mathcal{V} \) so we can assume \( \omega = t \omega \), whence \( js^{-1} \omega = js^{-1} t \omega = p(s^{-1}) \omega \).
has finite support as a function of $\omega$. Then
$$\omega \star \omega = \sum_{s} \sum_{s} \omega \star f(s^{-1} \omega) \star \omega = \sum_{s} f(s^{-1} \omega) \star \omega = \omega \star \omega,$$
and
$$\eta \star \omega = \eta \sum_{s} \omega \star f(s^{-1} \omega) = f\omega$$
finishing the proof.

Finally one has
$$\beta \omega = \beta \sum_{s} \omega \star f(s^{-1} \omega) = \sum_{s} \omega \star f(s^{-1} \omega) = \omega$$
and
$$\alpha \beta \sum_{t} \omega \star f(t) = \alpha \sum_{t} i f(t) = \sum_{s} \sum_{t} \omega \star f(s^{-1} t) \frac{1}{p(s^{-1} t)}$$
which identifies $W$ with the summand of $\Lambda \otimes V$ corresponding to the projection $p$. Thus the composite functor $W \to V \to W$ is isomorphic to the identity $W$. 

March 19, 02.

Consider $\Gamma = \mathbb{Z}/2 = \{1, \varepsilon\}$ where $\varepsilon^2 = 1$, $\Lambda = \mathbb{C}[\Gamma]$, and a retract $W$ of the free $\Lambda$ module $\Lambda \otimes V$:

$$\beta \alpha = 1_W$$

and its compression to a partition of $1$ for $W$:

$$1_W = (\beta_1 g_1) + \varepsilon(\beta_1 g_1)\varepsilon$$

Let $h = \beta_1 g_1$ so that $W$ becomes a unital module over $E_\Gamma = \mathbb{C}[h,h_\varepsilon]/(h+h_\varepsilon=1)$.

The retract $W$ is equivalent to the projection $p = x\beta$ on the $\Lambda$-module $\Lambda \otimes V$ which has the form $p \sum t \otimes f(t) = \sum_s s \otimes \sum_t p(s^{-1}t)f(t)$, where $p(s) \in \mathbb{L}(V)$ for each $s \in \Gamma$. This formula is a consequence of

$$\sum_u u \otimes p(u) \sum_t t \otimes f(t) = \sum_{u,t} tu^{-1} \otimes p(u) f(t)$$

In the $\Gamma = \mathbb{Z}/2$ case $p = 1 \otimes p(1) + \varepsilon \otimes p(\varepsilon)$ and so $p^2 = 1 \otimes (p(1)^2 + p(\varepsilon)^2) + \varepsilon \otimes (p(1)p(\varepsilon) + p(\varepsilon)p(1))$, so $p^2 = p$ means $p(1) = p(1)^2 + p(\varepsilon)^2$, $p(\varepsilon) = p(1)p(\varepsilon) + p(\varepsilon)p(1)$. This may be rewritten as $p(1) = p(\varepsilon)$ being idempotent.
Thus one finds that a retract of $\Lambda \otimes V$ as a module is the same as a pair of projections $p_+, p_-$ on $V$.

So far you have used the basis $1, z$ for $\Lambda$ and the corresponding partition of 1:

$$1 \otimes V = z_{1} z_{1} + z_{2} z_{2}$$

But $\Lambda$ has a nicer basis given by the projections $\frac{1 \pm \epsilon}{2} = e_\pm$ with corresponding partition

$$1 \otimes V = z_{1} z_{1} + z_{2} z_{2}$$

where $z_+ = \frac{1 + \epsilon}{2} \otimes V$  
$z_- = \frac{1 - \epsilon}{2} \otimes V$

$$j_+ \left( \frac{1 + \epsilon}{2} \otimes V \right) = V$$
$$j_+ \left( \frac{1 - \epsilon}{2} \otimes V \right) = 0$$

$$j_- \left( \frac{1 + \epsilon}{2} \otimes V \right) = 0$$
$$j_- \left( \frac{1 - \epsilon}{2} \otimes V \right) = V$$

$$j = \frac{j_+ + j_-}{2}$$

$$j_\epsilon = \frac{j_+ - j_-}{2}$$

Check

$$z_1 z_1 + z_2 z_2 = \left( z_+ z_- + z_- z_+ \right) \left( z_+ z_- + z_- z_+ \right) = 1$$

Now take

$$z_1 z_1 = \frac{1}{2} \left( z_+ z_- + z_- z_+ \right)$$

$$z_1 z_1 = \frac{1}{2} \left( e_+ e_- + e_- e_+ + e_+ e_- + e_- e_+ \right)$$
Try to summarize the important points.

You are looking at a $\Lambda$-retract of $\Lambda \otimes V$ from the viewpoint of the $\pm 1$ eigenspaces for $e$.

We identify the decomposition $\Lambda \otimes V = \Lambda_+ \otimes V \oplus \Lambda_- \otimes V$ with $V \xrightarrow{\lambda^+} \Lambda_+ \otimes V \xleftarrow{\lambda^-} V$. Then one has

retracts associated to each $\pm 1$ eigenspace:

\[
\begin{align*}
W_+ & \xrightarrow{\beta^+} V \xrightarrow{\lambda^+} W_+ \\
W_- & \xleftarrow{\beta^-} V \xleftarrow{\lambda^-} W_-
\end{align*}
\]

So again you see that there are two projections $p = \beta \lambda \alpha$ on $V$.

On $W$ besides the $\pm$ grading (which is the same as the $\Gamma$-action) you have the operator $h = \beta \lambda \alpha \beta^\dagger \lambda^\dagger \alpha$

\[
\begin{pmatrix}
W_+ \\
W_-
\end{pmatrix} \xleftarrow{\frac{1}{2}} \begin{pmatrix} 1 & \beta \lambda \alpha \\
\beta^\dagger \lambda^\dagger \alpha & 1
\end{pmatrix}
\begin{pmatrix}
W_+ \\
W_-
\end{pmatrix}
\]

which has the form $h = \frac{1}{2} (1 + X)$ where $X$ is odd: $eXe = -X$.

Note that $h + ehe = \frac{1}{2} (1 + X) + \frac{1}{2} (1 - X) = 1_W$.

**Question:** Is the odd operator $X$ related by Cayley transform to the 2 projections on $V$?
March 25, 2002

Let \( \Lambda = \mathbb{C}[Z/2] = \mathbb{C}e_+ \oplus \mathbb{C}e_- \), \( e_+ = \frac{1 + z}{2} \).

A (unital) \( \Lambda \)-module \( M \) is the same as a vector space with splitting \( M = (M^+) \oplus (M^-) \) where \( e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) and \( e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

Let \( \Lambda \otimes V \) be the "free" \( \Lambda \)-module generated by the v.s. \( V \). One has

\[ \Lambda \otimes V = e_+ \otimes V \oplus e_- \otimes V \]

Using the basis \( e_+, e_- \) for \( \Lambda \) we make the identification

\[ \Lambda \otimes V = \begin{pmatrix} V \\ V \end{pmatrix} \quad e_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad e_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

Similarly, we can identify the endo ring of the \( \Lambda \)-module \( \Lambda \otimes V \):

\[ \text{End}_\Lambda (\Lambda \otimes V) = \left\{ \begin{pmatrix} x_+ & 0 \\ 0 & x_- \end{pmatrix} \mid x_+, x_- \in \mathcal{L}(V) \right\} \]

In particular, a projection \( p \) on \( \Lambda \otimes V \) has the form

\[ \begin{pmatrix} p_+ & 0 \\ 0 & p_- \end{pmatrix} \]

where \( p_+, p_- \) are projections in \( \mathcal{L}(V) \).

Corresponding to such a \( p \) one has a \( \Lambda \)-module retract

\[ \overset{\beta}{\xrightarrow{\alpha}} \begin{pmatrix} V \\ V \end{pmatrix} \overset{\alpha}{\xleftarrow{\beta}} W \]

\[ \beta \alpha = 1 \quad \alpha \beta = p \]

which splits into two retracts of \( V \):

\[ \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \overset{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}}{\xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}}} \begin{pmatrix} V \\ V \end{pmatrix} \overset{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}}{\xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \]

\[ \beta_+ \alpha_+ = 1 \quad \alpha_+ \beta_+ = 1 \]

\[ \beta_- \alpha_- = 1 \quad \alpha_- \beta_- = 1 \]

In addition, \( W \) is equipped with:

\[ \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta_+ \alpha_+ & 0 \end{pmatrix} \]

In addition, \( W \) is equipped with:

\[ \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \beta_+ \alpha_+ & 0 \end{pmatrix} \]
Conversely given a $\Lambda$-module $W$ together with an odd operator $X$, let
\[
\begin{pmatrix}
W_+ \\
W_-
\end{pmatrix} \leftarrow \begin{pmatrix}
(\beta_+^t) \\
(\beta_-^t)
\end{pmatrix} \vee \begin{pmatrix}
(\alpha_+^t) \\
(\alpha_-^t)
\end{pmatrix} \begin{pmatrix}
W_+ \\
W_-
\end{pmatrix}
\]
be an arbitrary factorization of $1_W + X$:
\[
\begin{pmatrix}
\beta_+ \alpha_+ \\
\beta_- \alpha_-
\end{pmatrix} = \begin{pmatrix}
1_W & T' \\
T & 1_W
\end{pmatrix}, \quad X = \begin{pmatrix} 0 & T' \\
T & 0 \end{pmatrix}.
\]
Thus $W_+$ is the retract of $V$ corresponding to the projection $p_\pm = \alpha_\pm \beta_\pm$, and $X$ is the associated odd operator.

**Question:** Assume that $1 + X$ is invertible and that $V$ is in the image, so that $(\beta_+^t)$ and $(\alpha_+^t)$ are invertible. Then on $V$ one has two splittings. Are these related to $p_\pm$?

First case: Assume $V$ equals the $W$ on the right, i.e.
\[
V = \begin{pmatrix}
W_+ \\
W_-
\end{pmatrix}, \quad W_+ \xrightarrow{\alpha_+ = (0)} W_+, \quad W_- \xrightarrow{\alpha_- = (0)} W_-
\]
\[
\beta_+ = \begin{pmatrix} 1 & 0 \\
T & 1 \end{pmatrix}, \quad \beta_- = \begin{pmatrix} 0 & 1 \\
T & 1 \end{pmatrix}.
\]
Then
\[
p_+ = \alpha_+ \beta_+ = \begin{pmatrix} 1 & 0 \\
0 & T \end{pmatrix}, \quad p_- = \alpha_- \beta_- = \begin{pmatrix} 0 & 1 \\
T & 1 \end{pmatrix}
\]
\[
\beta_+ p_+ = \begin{pmatrix} 1 & 0 \\
0 & T' \end{pmatrix}, \quad \beta_- p_- = \begin{pmatrix} 0 & 1 \\
T & 1 \end{pmatrix}
\]
\[
\alpha_+ p_+ = \begin{pmatrix} 1 & 0 \\
0 & T \end{pmatrix}, \quad \alpha_- p_- = \begin{pmatrix} 0 & 1 \\
T & 1 \end{pmatrix}
\]
\[
\beta_+ p_- = \begin{pmatrix} 0 & 1 \\
T & 1 \end{pmatrix}, \quad \beta_- p_+ = \begin{pmatrix} 1 & 0 \\
0 & T' \end{pmatrix}
\]
2nd case: $V = W$ on the left

\[
\begin{pmatrix}
\frac{1}{T}
\end{pmatrix}
\begin{pmatrix}
(1, 0)
\end{pmatrix}
\begin{pmatrix}
\beta_+
\end{pmatrix} = \begin{pmatrix}
\alpha_+
\end{pmatrix}
\begin{pmatrix}
\beta_-
\end{pmatrix} = \begin{pmatrix}
\alpha_-
\end{pmatrix}
\begin{pmatrix}
\frac{1}{T'}
\end{pmatrix}
\begin{pmatrix}
W_+
\end{pmatrix}
\begin{pmatrix}
W_-
\end{pmatrix}
\begin{pmatrix}
W_+
\end{pmatrix}
\begin{pmatrix}
W_-
\end{pmatrix}
\]

Then

\[
p_+ = \alpha_+ \beta_+ = \begin{pmatrix}
\frac{1}{T}
\end{pmatrix}
\begin{pmatrix}
1
0
\end{pmatrix} = \begin{pmatrix}
\frac{1}{T'}
0
\end{pmatrix}
\]

\[
p_- = \alpha_- \beta_- = \begin{pmatrix}
T'
1
\end{pmatrix}
\begin{pmatrix}
0
1
\end{pmatrix} = \begin{pmatrix}
0
T'
\end{pmatrix}
\]

\[
\alpha_+ \beta_- = \begin{pmatrix}
\frac{1}{T}
\end{pmatrix}
\begin{pmatrix}
0
1
\end{pmatrix} = \begin{pmatrix}
0
\frac{1}{T'}
\end{pmatrix}
\]

\[
\alpha_- \beta_+ = \begin{pmatrix}
T'
1
\end{pmatrix}
\begin{pmatrix}
1
0
\end{pmatrix} = \begin{pmatrix}
T'
0
\end{pmatrix}
\]

Let $F = (1 + X) \varepsilon (1 + X)^T = \frac{1 + X}{1 - X} \varepsilon$

so that

\[
F = +1 \text{ on } \mathbb{R} \begin{pmatrix}
1
\frac{1}{T}
\end{pmatrix} W_+
\]

\[
F = -1 \text{ on } \mathbb{R} \begin{pmatrix}
T'
1
\end{pmatrix} W_-
\]

You conclude that the Cayley transform $F$, which yields the projection $\frac{F + 1}{2}$ onto $\begin{pmatrix}
1 \\
\frac{1}{T}
\end{pmatrix} W_+$ with kernel $\begin{pmatrix}
T'
1
\end{pmatrix} W_-$ is unrelated to $p_+$ and $p_-$. 
Let $u : R \rightarrow S$ be a map of commutative rings such that the restriction of scalars functor $u^* : \text{Mod}(S) \rightarrow \text{Mod}(R)$ is an equivalence of categories. Then $u$ is an isomorphism.

Proof. Recall that $u^*$ admits a left adjoint $u_! : \text{Mod}(R) \rightarrow \text{Mod}(S)$, with adjunction maps $u_! u^* N = S \otimes_R N \overset{\alpha}{\longrightarrow} N$, $s \otimes n \mapsto sn$.

$M \overset{\beta}{\longrightarrow} u^* u_! M = S \otimes_R M$, $m \mapsto 1_S \otimes m$.

Actually it's useful to factor $\beta : M = R \otimes_R M \overset{u \otimes 1}{\longrightarrow} S \otimes_R M$.

From category theory we know that when one adjoint functor is an equivalence then both adjunction maps are isomorphisms, and the two functors are quasi-inverse. So $\beta$ is an isomorphism for all $M$; taking $M = R$ yields that $u : R \rightarrow S$ is bijective.

Here's another proof which uses the adjoint pair $\text{Mod}(R) \overset{u^*}{\underset{u_!}{\rightleftarrows}} \text{Mod}(S)$, $u_! M = \text{Hom}_R(S, M)$.

with adjunction maps $\beta : N \rightarrow u_! u^* N = \text{Hom}_R(S, N)$, $n \mapsto (s \mapsto sn)$, $\alpha : u^* u_! M \rightarrow M$, $\text{Hom}_R(S, M) \rightarrow M$.

It's useful to factor $\alpha$ into $\text{Hom}_R(S, M) \rightarrow \text{Hom}_R(R, M) = M$, $f \mapsto f(1_S)$, $f \mapsto f u(1_R) = f(1_S)$. 

March 29, 2002
When $u^*$ is an equivalence this map $\alpha$ is an isomorphism for all $R$-modules $M$, which implies $\alpha : R \rightarrow S$ is an isom. of $R$-modules, hence of rings.

\textbf{INTERVAL}

\begin{align*}
\text{I} & \rightarrow S \\
\text{I} & \rightarrow S
\end{align*}