Consider $\mathbb{R}^2$, coordinate functions x, y Inf. translations ∂_x, ∂_y symp structure $\omega = 2\pi i\, dx\, dy$, connection in the trivial line bundle $d + 2\pi i\, x\, dy$. ω is invariant under $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ acting on $\mathbb{R}^2$ by the vector fields

$$\partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y$$

$$\left[x\partial_y, \begin{matrix}\partial_x \\ \partial_y\end{matrix}\right] = \begin{matrix}-\partial_y \\ 0\end{matrix}$$

$$[x\partial_y, a\partial_x + b\partial_y] = -a\partial_y$$

$$[x\partial_y, \alpha x + \beta y] = \beta x \qquad \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \beta$$

$$[y\partial_x, \alpha x + \beta y] = \alpha y \qquad \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

$$[x\partial_x - y\partial_y, \alpha x + \beta y] = \alpha x - \beta y \qquad \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \longmapsto \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

So write linear function on $\mathbb{R}^2$ in the form $(x\ y)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$

$$\mathcal{L}(x\partial_y)\ (x, y)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (0\ \ x)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (x\ \ y)\begin{pmatrix}0 & 1 \\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

$$\mathcal{L}(y\partial_x)\ (x, y) = (y\ \ 0)\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = (x\ \ y)\begin{pmatrix}0 & 0 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

This is reasonable.

At this point you have ~~?~~ five nice vector fields on $\mathbb{R}^2$ preserving symplectic structure.

Now continue, persist with your viewpoint, namely that these vector fields can be adjusted by a gauge transf so as to preserve the connection

$$\mathcal{L}_{a\partial_x}(d + 2\pi i\, x\, dy) = 2\pi i\, \mathcal{L}_{a\partial_x}(x\, dy) = \cancel{d\, \iota_{a\partial_x} x\, dy} + \iota_{a\partial_x} dx\, dy \overset{2\pi i}{=} a\, dy$$

$$e^{2\pi i a y}(d + 2\pi i\, x\, dy)e^{-2\pi i a y} = d + 2\pi i\, x\, dy - 2\pi i a\, dy$$

Something is wrong. You want perhaps an infinitesimal gauge transf.

$$[2\pi i a y, d + 2\pi i\, x\, dy] = -2\pi i a\, dy$$

$$[\mathcal{L}_{\partial_x} + 2\pi i y, d + 2\pi i\, x\, dy] = 2\pi i\, dy - 2\pi i\, dy = 0$$

There should be a clearer way to proceed. Instead of operators on $\Omega(\mathbb{R}^2)$ and $\mathcal{L}_X$, why not try to work entirely with functions. Since you have a nice ~~[illegible]~~ basis $1, dx, dy, dx\,dy$ for $\Omega(\mathbb{R}^2)$ as $\Omega^0(\mathbb{R}^2)$-module things should be easier.

$$(d+A)\psi = \partial_x\psi\,dx + \partial_y\psi\,dy + 2\pi i\,x\,dy\,\psi$$

so ~~your~~ the connection amounts to $D_x = \partial_x$ and $D_y = \partial_y + 2\pi i\,x\,dy$

What does this all mean?

Go back to $\mathcal{S}(\mathbb{R}) \ni f \longmapsto \sum_m e^{2\pi i m y}\overbrace{f(x+m)}^{e^{m\partial_x}f}$

$$\sum_{m\in\mathbb{Z}} e^{m(\partial_x + 2\pi i y)} f = \tilde{f}$$

$$e^{\partial_y}\psi = \psi = e^{2\pi i y}e^{\partial_x}\psi$$

Back to $C^\infty(\mathbb{R}^2)$ $\qquad A = 2\pi i\,x\,dy$

∂_x	$\partial_x + 2\pi i y$	these two reps commute
$\partial_y + 2\pi i x$	∂_y	
gen. Heis. Lie alg	gen. Heis Lie alg.	
$d + 2\pi i x\,dy$	$d + 2\pi i y\,dx$	

Question: Are autom conditions just being fixed under e^{∂_y} and $e^{\partial_x + 2\pi i y}$? YES.

$$e^{\partial_y}\psi = \psi \iff \psi(x, y+1) = \psi(x,y)$$

$$e^{\partial_x + 2\pi i y}\psi = e^{2\pi i y}e^{\partial_x}\psi = \psi \qquad e^{2\pi i y}\psi(x+1,y) = \psi(x,y)$$

predicting actions? Begin with $SL(2,\mathbb{R})\ltimes\mathbb{R}^2$ acting on $\mathbb{R}^2$ by affine symplectic transf. Stabilizer of 0 is $SL(2,\mathbb{R})$

Look at $A = 2\pi i\,x\,dy$ $\qquad D_x = \partial_x,\ D_y = \partial_y + 2\pi i x$

A is not transl. invariant $\qquad \mathcal{L}_{\partial_x}(2\pi i\,x\,dy) = 2\pi i(dy + x0) = 2\pi i\,dy$

$$\mathcal{L}_{\partial_x}\, d + 2\pi i\,x\,dy = 2\pi i\,dy$$

$$[2\pi i y,\ d + 2\pi i\,x\,dy]$$

You need to explain this invariance. Meaning?

Consider $\mathbb{R}^2$, the trivial ~~line~~ circle bdl over $\mathbb{R}^2$: ~~[illegible]~~ $\mathbb{R}^2 \times \mathbb{T}$ 9/6
with vector fields arising from inf. transl. ∂_x, ∂_y
You know what is a connection in this fibre bundle.
A connection is choice of horizontal subspaces, it allows to lift vector fields, therefore can ask whether you have a Lie homom.

Idea: Given $\mathbb{R}^2$ ∂_x, ∂_y & $dx\,dy$
Better. Given $\mathbb{R}^2$, ω you choose A, $dA = \omega$ and get a connection on $\mathbb{R}^2 \times \mathbb{C}$ or $\mathbb{R}^2 \times \mathbb{T}$. A diff choice A', $A' - A = df$,

$$d + A' = d + A + df$$
$$= e^{-f}(d+A)e^{f}$$

What's true in general for a line bundle? Think of the base $\mathbf{T}^2$. ~~The space of connections $\{d + A \mid A \in \Omega^1\}$ is contractible~~
The line bundle $L \xrightarrow{\pi} T^2$ has a principal bundle $\mathbb{T} \to P \to T^2$, a connection is a 1-form on P with values in the Lie alg, equivariant for the group action on P and $\mathfrak{g}$, restriction to fibre is ~~closed,~~ ~~[illegible]~~ an invariant diff form.

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trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ equipped with conn $d + 2\pi i x\,dy$

$$\downarrow$$
$$\mathbb{R}^2$$

x, y, θ coord

$\partial_x, \partial_y, \partial_\theta$

$$\partial_x \longmapsto \partial_x = D_x$$
$$\partial_y \longmapsto \partial_y + 2\pi i x = D_y$$

$dx\,D_x + dy\,D_y$

autom. cond. on $\psi \in C^\infty(\mathbb{R}^2)$

$$e^{\partial_y}\psi = \psi = e^{2\pi i y} e^{\partial_x}\psi$$

ψ inv. under $e^{m(\partial_x + 2\pi i y)} e^{n\partial_y}$

$$\begin{bmatrix} \partial_x & \partial_x + 2\pi i y \\ \partial_y + 2\pi i x, & \partial_y \end{bmatrix} = 0$$

~~[illegible]~~

Basic idea - you have symplectic symmetry $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$, $dx\,dy$
~~You have~~ circle action
Let's do the inf. calculations, practice.
~~[illegible]~~ IDEA = Review Hamiltonian v.f. on a symp. man.

$$0 = \mathcal{L}_X \omega = d(\iota_X \omega) + \cancel{\iota_X d\omega} \implies \iota_X \omega = d\phi \text{ locally}$$

Jan 28, 02.

begin with $\mathbb{R}^2$, ω $\mathcal{L}_X \omega = d\iota_X\omega + \cancel{\iota_X d\omega}$

The amazing thing from your viewpoint is the fine structure within.

Let's build understanding. This Ham. v.f. stuff is worth clarifying. There should be a link between the Hamiltonian v.f. stuff and connections on line bundles.

$(d+\iota_X)^2 = L_X$ free loop ~~space~~, supersymmetric 1 param. gp.

So let's return to $\mathbb{R}^2$, $\omega = dxdy$, symplectic mfd, trivial line bundle. You have learned that symplectic v.f. correspond to real functions mod constants, Poisson bracket $\{f,g\}$

Question Does a symplectic manifold have a canonical complex line bundle with connection? Even in the case of $\mathbb{R}^2$, $i\,dxdy$ is there something canonical to construct?

$0 = \mathcal{L}_{\partial_x}(dx\,dy) = \cancel{d}\,\iota_{\partial_x}(dx\,dy) = \cancel{d}(dy)$ Ham $= y$ mod $\mathbb{R}$

$\mathcal{L}_{\partial_y}(dxdy) = \cancel{d}\,\iota_{\partial_y} dxdy = -\cancel{d}(dx)$ $= -x$

$\mathcal{L}_{x\partial_y}(dxdy) = d(-xdx)$ $-\frac{x^2}{2}$

$\mathcal{L}_{y\partial_x}(dxdy) = d(ydy)$ $\frac{y^2}{2}$

$\mathcal{L}_{x\partial_x - y\partial_y}(dxdy) = d(xdy + ydx)$ xy

Now you propose to combine a v.f. X with its Ham to get an infinitesimal action on the line bundle

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$d+A = d + 2\pi i x dy$ $d+A$ lifts ∂_x somehow

What is the aim, motivation here? You have complex line bundle, fibre bundle. What is the naive idea? Why did you think this stuff related to Hamiltonians, because a ~~symplectic v. field~~ diffeom. preserving the curve should not affect the iso class of the line bundle.

Recall that a connection rigidifies the bundle.

So given $\mathbb{R}^2 + 2\pi i\, dx dy$. The idea somehow ~~then~~ to look all connections on the trivial line bundle over $\mathbb{R}^2$ having curvature $2\pi i\, dx\, dy$. Given two $d+A$, $d+A_1$ then $d(A_1 - A) = 0$ so $A_1 = A + df$, f unique up to an additive constant. $e^{-f}(d+A)e^{f} = d + df + A = d + A_1$

If $g : \mathbb{R}^2 \xrightarrow{\sim} \mathbb{R}^2$ is a symplectic diffeom; ~~then~~ and A is a connection form: $dA = 2\pi i\, dx dy$, then

$$g^*(dA) = d(g^*A) \quad ?$$
$$\| $$
$$g^*(2\pi i\, dx\, dy) = 2\pi i\, dx dy$$

$$d + 2\pi i x\, dy = dx \underbrace{\partial_x}_{D_x} + dx \underbrace{(\partial_y + 2\pi i x)}_{D_y} \qquad [D_x, D_y] = 2\pi i$$

$$e^{+a\partial_x}(d + 2\pi i x dy) e^{-a\partial_x} = d + 2\pi i (x+a) dy$$

$$[\partial_x, d + 2\pi i x\, dy] = 2\pi i\, dy$$

$$\left[\partial_x, \begin{matrix} D_x \\ D_y \end{matrix}\right] = \begin{matrix} 0 \\ 2\pi i \end{matrix}$$

$$\left[\partial_x + 2\pi i y, \begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix}\right] = \begin{matrix} 0 \\ 0 \end{matrix}$$

$$\left[+2\pi i y, \begin{matrix} D_x \\ D_y \end{matrix}\right] = \begin{matrix} 0 \\ -2\pi i \end{matrix}$$

$$\left[\partial_y, \begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix}\right] = \begin{matrix} 0 \\ 0 \end{matrix}$$

You may want to repeat this calculation enough times before going on.

$$(d + 2\pi i x\, dy)\psi = dx(\underbrace{\partial_x \psi}_{D_x}) + dy\underbrace{(\partial_y + 2\pi i x)}_{D_y}\psi$$

Point this connection "morally should be" invariant under the $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ action on $\mathbb{R}^2$. Why? Because the curvature $2\pi i\, dx dy$ is invariant and any two line bundles ^(equipped with connection)^ over $\mathbb{R}^2$ having this curvature are isomorphic up to a constant scalar factor.

Begin with $A = 2\pi i x\, dy$, $(d+A)\psi = D_x\psi\, dx + D_y \psi\, dy$

$$D_x = \partial_x \qquad D_y = \partial_y + 2\pi i x$$

resolved $d+A$ into D_x, D_y. Suppose given a vector field on $\mathbb{R}^2$ preserving the curvature form $2\pi i\, dx\, dy$. Breaking the connection D into D_x, D_y might not be reasonable when the x, y directions are not preserved.

Use $\mathcal{L}_X = d\,\iota_X + \iota_X d$ on $\Omega(\mathbb{R}^2)$.

$$\mathcal{L}(x\partial_y)\big(d + 2\pi i x\, dy\,(\omega)\big)$$

$$= \mathcal{L}(x\partial_y)(d\omega + 2\pi i x\, dy\, \omega) = d\,\mathcal{L}(x\partial_y)\omega + 2\pi i x\, dx\, \omega$$
$$+ 2\pi i x\, dy\, \mathcal{L}(x\partial_y)\omega$$

$$= (d + 2\pi i\, dy)\, \mathcal{L}(x\partial_y)\, \omega$$

$$[\mathcal{L}(x\partial_y), d + 2\pi i x\, dy]\,\omega = 2\pi i x\, dx\, \omega$$

$$[\pi i x^2, d + 2\pi i x\, dy]\,\omega = -2\pi i x\, dx\, \omega$$

$$[\mathcal{L}(x\partial_y) + \pi i x^2,\ d + 2\pi i x\, dy] = 0$$

$$[\underset{+\pi i y^2}{\mathcal{L}(y\partial_x)},\ d + 2\pi i x\, dy] = \underset{-2\pi i y\, dy}{2\pi i y\, dy} = 0$$

$$[\mathcal{L}(x\partial_x - y\partial_y),\ d + 2\pi i x\, dy] = 2\pi i x\, dy + 2\pi i x\, d(-y) = 0$$

$$[\mathcal{L}(x\partial_y) + \pi i x^2,\ \mathcal{L}(y\partial_x) + \pi i y^2]$$
$$= \mathcal{L}(x\partial_x - y\partial_y) + \cancel{2\pi i x y - \pi i y 2x}$$

five operators preserving $d + 2\pi i x dy$

$\mathcal{L}(\partial_x)$ [redacted] $+ 2\pi i y$

$\mathcal{L}(\partial_y)$

$\mathcal{L}(x\partial_y)$ [redacted] $+ \pi i x^2$

$\mathcal{L}(y\partial_x)$ [redacted] $+ \pi i y^2$

$\mathcal{L}(x\partial_x - y\partial_y)$

preserving means $[\quad , d + 2\pi i x dy] = 0$

Now suppose you replace $d + 2\pi i x dy$ by its components [redacted] Recall

$D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$

$$\begin{bmatrix} \partial_x + 2\pi i y & \partial_x \\ \partial_y & \partial_y + 2\pi i x \end{bmatrix} = 0$$

You can interprete the other mixed ops (vect flds + functions) as follows

$$\left[x\partial_y + \pi i x^2, \begin{pmatrix} \partial_x \\ \partial_y + 2\pi i x \end{pmatrix} \right] = \begin{pmatrix} -(\partial_y + 2\pi i x) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y + 2\pi i x \end{pmatrix}$$

$$\left[y\partial_x + \pi i y^2, \begin{pmatrix} \partial_x \\ \partial_y + 2\pi i x \end{pmatrix} \right] = \begin{pmatrix} 0 \\ -\partial_x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y + 2\pi i x \end{pmatrix}$$

$$[x\partial_y + \pi i x^2, y\partial_x + \pi i y^2] = x\partial_x - y\partial_y$$

$$\left[x\partial_x - y\partial_y \quad \begin{matrix} \partial_x \\ \partial_y + 2\pi i x \end{matrix} \right] = \begin{pmatrix} -\partial_x \\ \partial_y + 2\pi i x \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y + 2\pi i x \end{pmatrix}$$

$$[x\partial_y + \pi i x^2, (x \quad y)] = (0 \quad x) = (x \quad y) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[y\partial_x + \pi i y^2, (x \quad y)] = (y \quad 0) = (x \quad y) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$[x\partial_x - y\partial_y, (x \quad y)] = (x \quad -y) = (x \quad y) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Explore a simpler, more transparent, viewpoint. The idea is that on the trivial line bundle over $\mathbb{R}^2$ there is only one line bundle + connection having the curvature $2\pi i\, dx\, dy$. More precisely there is a unique up to scalar factor isomorphism between two such objects: Given $d+A$, $d+A_1$ w $dA = dA_1$, then $A_1 - A = df$ and $e^{-f}(d+A)e^{f} = d+df+A = d+A_1$.

Repeat. $[L(x\partial_y), d+2\pi i x\, dy] = 2\pi i x\, dx$

$$\left[L(x\partial_y) + \pi i x^2,\ d+2\pi i x\, dy\right] = 0$$

$$\underbrace{dx\,\partial_x + dy(\partial_y + 2\pi i x)}$$

$x\partial_y + \pi i x^2$ op. on functions

$$\left[\mathcal{L}(x\partial_y) + \pi i x^2, (x\ \ y)\right] = (0\ \ x) = (x\ \ y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\mathcal{L}(y\partial_x) + \pi i y^2, (x\ \ y)\right] = (y\ \ 0) = (x\ \ y)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left[\mathcal{L}(x\partial_x - y\partial_y), (x\ \ y)\right] = (x\ \ {-y}) = (x\ \ y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

above holds with $x\ y \mapsto dx\ dy$

$$\left[\mathcal{L}(x\partial_y) + \pi i x^2,\ d+2\pi i x\, dy\right] = 0.$$

$$\left[\mathcal{L}(x\partial_y) + \pi i x^2,\ dx\,\partial_x + dy(\partial_y + 2\pi i x)\right] = 0$$

$$\mathcal{L}(x\partial_y)(dx\,\partial_x) = d0\,\partial_x + dx\,[x\partial_y, \partial_x] = -dx\,\partial_y$$

$$\mathcal{L}(x\partial_y)(dy(\partial_y + 2\pi i x)) = dx(\partial_y)$$

$$[L(x\partial_y),\ d + 2\pi i x\, dy] = 2\pi i x\, dx$$

$$[L(x\partial_y) + \pi i x^2,\ d + 2\pi i\, dy] = 0$$

Given ψ, what is $e^{\cancel{tx}\partial_y}\psi$

$$e^{tx\partial_y}(x \quad y) = (x \quad y) + t(0 \quad x) + \frac{t^2}{2}(0\,0) +$$
$$= (x,\ y + tx)$$

$$e^{sx\partial_y}\psi = \sum \frac{(sx)^n}{n!}\partial_y^n \psi$$

$$e^{a\partial_x + b\partial_y} x^3 y^2 = (x+a)^3 (y+b)^2$$

$$e^{sx\partial_y}\psi\, e^{-sx\partial_y}$$

---

patience. $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\quad \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$

$a\delta - bc = 1.$

$$dx\, dy = (a\, du + b\, dv) \wedge (c\, du + \delta\, dv)$$
$$= (a\delta - bc)\, du\, dv$$

$$(du \ dv)\begin{pmatrix} a & c \\ b & \delta \end{pmatrix} = (dx \ dy)$$

$$\partial_u f = \partial_x f \frac{\partial x}{\partial u} + \partial_y f \frac{\partial y}{\partial u} = a\partial_x f + c\partial_y f$$

686.22

$$\partial_v f = \partial_x f \frac{\partial x}{\partial v} + \partial_y f \frac{\partial y}{\partial v} = b\,\partial_x f + \delta\,\partial_y f$$

$$(du \ dv)\begin{pmatrix} \partial_u f \\ \partial_v f \end{pmatrix} = (du \ dv)\begin{pmatrix} a & c \\ b & \delta \end{pmatrix}\begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix} = (dx \ dy)\begin{pmatrix} \partial_x f \\ \partial_y f \end{pmatrix}$$

$$x\, dy = (au + bv)\, d(cu + \delta v) = (a \ b)\begin{pmatrix} u \\ v \end{pmatrix}(du \ dv)\begin{pmatrix} c \\ \delta \end{pmatrix}$$
$$= (u \ v)\begin{pmatrix} a \\ b \end{pmatrix}(c \ \delta)\begin{pmatrix} du \\ dv \end{pmatrix}$$

bilinear form

$ac \quad a\delta$

$bc \quad b\delta$ skew sym. $a\delta - bc$

pulling back $d + 2\pi i x\, dy$ yields

$$d + 2\pi i \underbrace{(au+bv)(c\,du+\delta\,dv)}$$
$$(a\,du+b\,dv)\wedge(c\,du+\delta\,dv) = (a\delta - bc)\,du\,dv$$

$$A_1 = (au+bv)(c\,du+\delta\,dv)$$
$$A = u\,dv$$

$$A_1 - A = (u \;\; v)\underbrace{\begin{pmatrix} ac & a\delta-1 \\ bc & b\delta \end{pmatrix}}\begin{pmatrix} du \\ dv \end{pmatrix}$$

symmetric since $a\delta - bc = 1 \Rightarrow a\delta - 1 = bc$.

$$\frac{1}{2}(u \;\; v)\begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$abc\delta - b^2c^2$$
$$= bc(1+bc) - b^2c^2 = bc$$

Example $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\frac{1}{2}(u\,v)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -uv$

$$\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

seems okay

$$A_1 = (u+v)\,dv \qquad A_1 - A = v\,dv = d\left(\tfrac{1}{2}v^2\right).$$

Other example: $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$$A_1 = u(du+dv) \qquad A_1 - A = u\,du = d\left(\tfrac{1}{2}u^2\right).$$

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Start again: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $f(x,y)$

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$x = au + bv$
$y = cu + \delta v$

$$\partial_u f = \partial_x f\, \partial_u x + \partial_y f\, \partial_u y = (a\partial_x + c\partial_y) f$$

$$\partial_v f = \partial_x f\, \partial_v x + \partial_y f\, \partial_v y = (b\partial_x + \delta\partial_y) f$$

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$$\begin{pmatrix} \partial_u \\ \partial_v \end{pmatrix} = \begin{pmatrix} a & c \\ b & \delta \end{pmatrix}\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}$$

$$\begin{pmatrix} \partial_u f & \partial_v f \end{pmatrix} = \begin{pmatrix} \partial_x f & \partial_y f \end{pmatrix}\begin{pmatrix} a & b \\ c & \delta \end{pmatrix}$$

$$\begin{pmatrix} \partial_u f & \partial_v f \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} = \begin{pmatrix} \partial_x f & \partial_y f \end{pmatrix}\underbrace{\begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}}_{\begin{pmatrix} dx \\ dy \end{pmatrix}}$$

$$A = x\,dy \mapsto A_1 = (au + bv)(c\,du + \delta\,dv)$$

$$dA = dx\,dy \mapsto dA_1 = (a\,du + b\,dv)(c\,du + \delta\,dv)$$
$$= (a\delta - bc)\,du\,dv$$

to compare A_1 with $A = u\,dv$

$$A_1 - A = (u \;\; v)\begin{pmatrix} a \\ b \end{pmatrix}(c \;\; \delta)\begin{pmatrix} du \\ dv \end{pmatrix} - (u \;\; v)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$$= (u \;\; v)\begin{pmatrix} ac & \overset{bc}{a\delta - 1} \\ bc & b\delta \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} = d\left\{\frac{1}{2}(u \;\; v)\begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right\}$$

Examples: $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} = \begin{pmatrix} 0 & +1 \\ +1 & 0 \end{pmatrix}$

$$A_1 - A = (u \;\; v)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$
$$= -u\,dv - v\,du = -d(uv)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \qquad \tfrac{1}{2}(u\ v)\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{v^2}{2}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \tfrac{1}{2}(u\ v)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{u^2}{2}$$

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now can you finish? You have $\psi(x,y) \in C^\infty(\mathbb{R}^2)$

$$\psi(x,y) \xmapsto{\text{composition}} \psi(u+v, v) \xmapsto{d} \partial_x \psi$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$x\,dy = (au+bv)(c\,du + \delta\,dv)$$

$$A_1 = (u\ v)\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}(c\ \ \delta)}_{\begin{pmatrix} ac & a\delta \\ bc & b\delta \end{pmatrix}}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$$A = u\,dv = (u\ v)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$$

$$A_1 - A = (u\ v)\begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} = d\,\tfrac{1}{2}\begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \qquad \tfrac{1}{2} t v^2 \qquad \begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} \qquad \tfrac{1}{2} t u^2$$

$$e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} -\cos t \sin t & -\sin^2 t \\ -\sin^2 t & \sin t \cos t \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

---

what would you like to do? Want operators on functions

$$\underbrace{\left(e^{t\,x\partial_y}\psi\right)}_{\psi(t,x,y)}|x,y\rangle = \psi|x, y+tx\rangle$$

$$\partial_t \psi(t,x,y) = x\,\partial_y \psi(t,x,y)$$

$$\partial_t \psi(x, y+tx) = (\partial_y \psi)(x, y+tx)\, x$$

You have to go over this many times.

5 vector fields $\partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y$ on R^2

$(x\partial_y \psi)$ $\qquad x\partial_y (x \;\; y) = (0 \;\; x) = (x \;\; y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$L(x\partial_y)\psi$ $\qquad y\partial_x (x \;\; y) = (y \;\; 0) = (x \;\; y)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$x\partial_x - y\partial_y \; (x \;\; y) = (x \;\; -y) = (x \;\; y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[x\partial_y, y\partial_x] = x\partial_x - y\partial_y$

$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Question is whether $(L(x\partial_y)\psi)(x \;\; y) \overset{\text{NO}}{=} \psi \, x\partial_y (x \;\; y)$

$= \psi(0 \;\; x) = \psi(x \;\; y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$L(x\partial_y)(\psi) = (x\partial_y\psi - \psi x\partial_y)(1) = x(\partial_y\psi)$

$R^2 \longrightarrow R^2, \; x\,dy$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$ $\qquad a\delta - bc = 1$

$(au+bv)(c\,du + \delta\,dv) \longleftarrow x\,dy$

$\underbrace{(u \;\; v)\begin{pmatrix} a \\ b \end{pmatrix}(c \;\; \delta)\begin{pmatrix} du \\ dv \end{pmatrix}}_{A_1} = (u \;\; v)\begin{pmatrix} ac & a\delta \\ bc & b\delta \end{pmatrix}\begin{pmatrix} du \\ dv \end{pmatrix} \in V^* \otimes dV^*$

to compare with $A_0 = u\,dv$

$A_1 - A_0 = (u \;\; v)\begin{bmatrix} ac & bc \\ bc & b\delta \end{bmatrix}\begin{pmatrix} du \\ dv \end{pmatrix}$

$= d\left\{\frac{1}{2}\begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\right\}$

What does this mean? Given $g : M' \longrightarrow M$ smooth map get $g^* : \Omega(M) \longrightarrow \Omega(M')$, hence given $A \in \Omega^1(M)$, get $g^*A \in \Omega^1(M')$. So $g^*(d+A)\psi = d\,g^*\psi + g^*A\, g^*\psi$ need invert.

$g^*(d+A)(g^*)^{-1}\phi = (d + g^*A)\phi$

Repeat again Jan 30, 02

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}} & \mathbb{R}^2 \\ \cup & & \\ \begin{pmatrix} u \\ v \end{pmatrix} & \longmapsto & \begin{pmatrix} x \\ y \end{pmatrix} \end{array}$$

still confused. Instead look at a smooth map $g: M' \longrightarrow M$, you get $g^*: C^\infty(M) \longrightarrow C^\infty(M')$ $g^* f = fg$, so

if $M' = \mathbb{R}^2$ with coords u, v and $g^* x = au + bv$
$M = \mathbb{R}^2$ ——— x, y $g^* y = cu + \delta v$

$$\begin{pmatrix} g^* x \\ g^* y \end{pmatrix} = \begin{pmatrix} x \circ g \\ y \circ g \end{pmatrix} = \begin{pmatrix} au + bv \\ cu + \delta v \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$g^*(x\,dy) = (au + bv)(c\,du + \delta\,dv)$$
$$= \underbrace{(u \ \ v) \begin{pmatrix} a \\ b \end{pmatrix} (c \ \ \delta)}_{\begin{pmatrix} ac & a\delta \\ bc & b\delta \end{pmatrix}} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$u\,dv = (u \ \ v) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}$$

$$g^*(x\,dy) - u\,dv = (u \ \ v) \begin{pmatrix} ac & \overset{bc}{\overset{\shortparallel}{a\delta - 1}} \\ bc & b\delta \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix} = d\left\{ \frac{1}{2} \begin{pmatrix} u \\ v \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right\}$$

At this stage it looks like you can replace u, v by x, y so that g is an automorphism.

$$g: M' \longrightarrow M \qquad (g^* x \ \ g^* y) = (au + bv \ \ \ cu + \delta v) = (u \ \ v) \begin{pmatrix} a & c \\ b & \delta \end{pmatrix} \quad ?$$

$$M'' \xrightarrow{g_1} M' \xrightarrow{g} M \qquad (g_1^* g^* x \ \ g_1^* g^* y) = (g_1^* u \ \ g_1^* v) \begin{pmatrix} a & c \\ b & \delta \end{pmatrix}$$

$$= (u \ \ v) \underbrace{\begin{pmatrix} a_1 & c_1 \\ b_1 & \delta_1 \end{pmatrix} \begin{pmatrix} a & c \\ b & \delta \end{pmatrix}}$$

?

Try to use actions of $\mathbb{R}^2$. $G = \{\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \mid a\delta - bc \neq 0\}$

matrix mult. If $g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}$, ~~then~~ which means

$$g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} ac_1 + bc_2 \\ cc_1 + \delta c_2 \end{pmatrix}, \quad \text{then} \quad \begin{aligned} g^* x &= ax + by \\ g^* y &= cx + \delta y \end{aligned}$$

so $g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Let $g'\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a'c_1 + b'c_2 \\ c'c_1 + \delta' c_2 \end{pmatrix}$

$$g'^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & \delta' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Comp. $g'g$ is $g'g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = g'\begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & \delta' \end{pmatrix}\begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$$(g'g)^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & \delta' \end{pmatrix}\begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

TRY AGAIN. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a linear autom of the plane $\mathbb{R}^2$; say

$$g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} ac_1 + bc_2 \\ cc_1 + \delta c_2 \end{pmatrix}$$

$$\begin{aligned} pr_1 g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= pr_1\begin{pmatrix} \quad \end{pmatrix} = ac_1 + bc_2 \\ &= a\, pr_1\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + b\, pr_2\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= (a\, pr_1 + b\, pr_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

$$\boxed{pr_1 g = a pr_1 + b pr_2 \qquad xg = ax + by}$$

$$\begin{aligned} pr_2\, g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} &= pr_2\begin{pmatrix} ac_1 + bc_2 \\ cc_1 + \delta c_2 \end{pmatrix} = cc_1 + \delta c_2 \\ &= c\, pr_1\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \delta\, pr_2\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \end{aligned}$$

$$\boxed{pr_2 g = c pr_1 + \delta pr_2 \qquad yg = cx + \delta y}$$

Again: $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear auto.

$$g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

reformulate in terms of pr_1, pr_2.

$$pr_1\, g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = ac_1 + bc_2 = (a\,pr_1 + b\,pr_2)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$g^* pr_1 = pr_1 g = a\,pr_1 + b\,pr_2$$
$$g^* pr_2 = pr_2 g = c\,pr_1 + \delta\,pr_2$$

$$\boxed{\begin{aligned} xg &= ax + by \\ yg &= cx + \delta y \end{aligned}}$$

$$g^* : \Omega(\mathbb{R}^2) \longrightarrow \Omega(\mathbb{R}^2) \qquad \text{resp mult, } d$$

What is your aim? You have the connection $d + A$ where $A = x\,dy$ $\quad A = pr_1\, dpr_2$. Just exactly what do you want to do? You have g^* is autom of $\Omega(\mathbb{R}^2), d$.

You want g^* to be an autom of $\Omega(\mathbb{R}^2)$, $d + A$. This means

$$\begin{tikzcd} \Omega(\mathbb{R}^2) \arrow[r, "d+A"] \arrow[d, "g^*"'] & \Omega(\mathbb{R}^2) \arrow[d, "g^*"] \\ \Omega(\mathbb{R}^2) \arrow[r, "d+A"] & \Omega(\mathbb{R}^2) \end{tikzcd}$$

$$g^*(d+A) = (d+A)g^* \quad ?$$
$$g^* A = A g^* \quad ?$$
$$A = (g^*)^{-1} A g^* \quad ?$$

$$A = pr_1\, dpr_2 = x\,dy$$

$$g^*(A) = xg\; d(yg) = (ax + by)(c\,dx + \delta\,dy) = (x\ y)\begin{pmatrix} a \\ b \end{pmatrix}(c\ \delta)\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$g^*A - A = (x\ \ y)\begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = d\Big\{\underbrace{\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\varphi}\Big\}$$

How to finish? Maybe you want to combine g^* on $\Omega(\mathbb{R}^2)$ with an invertible function φ in $\Omega^0(\mathbb{R}^2)$

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\qquad g\begin{pmatrix}c_1\\ c_2\end{pmatrix} = \begin{pmatrix}a & b\\ c & \delta\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}$

$pr_1 g = a\,pr_1 + b\,pr_2 \qquad xg = ax + by$

$pr_2 g = c\,pr_1 + \delta\,pr_2 \qquad yg = cx + \delta y$

g^* auto of $\Omega(\mathbb{R}^2)$ $\qquad g^* d = d g^*$

combine g^* with mult by a function (invertible) to get an op commuting with $d+A$.

$$g^*(d+A)\xi = d(g^*\xi) + (g^*A)g^*\xi = (d+g^*A)g^*\xi$$

$$= (d+A+df)g^*\xi = e^{-f}(d+A)e^{f} g^*\xi$$

$$e^{f} g^*(d+A)\xi = (d+A)e^{f} g^*\xi$$

$g^*x = xg = x + ty \qquad A = x\,dy$

$g^*y = yg = y \qquad g^*A = (x+ty)\,dy$

$g^*A - A = ty\,dy = d(\tfrac{1}{2}ty^2)$

$(g^*\psi)(x,y) = \psi(g^*x, g^*y)$

$= \psi(x+ty, y)$

so it seems you have the operator on functions

$\psi \longmapsto e^{\frac{1}{2}ty^2}\psi g = e^{\frac{1}{2}ty^2} e^{ty\partial_x}\psi$

You should work out the rules for dealing with $\psi(x,y)$

Start again $\quad \{\psi \in C^\infty(\mathbb{R}^2) \mid \psi(x,y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)\}$

$$e^{\partial_y}\psi = \psi = e^{2\pi i y + \partial_x}\psi$$

connection $\qquad (2\pi i x f)^{\sim} = \sum_m e^{2\pi i m y}\, 2\pi i (x+m) f(x+m)$

$\partial_y \tilde{f} + 2\pi i x \tilde{f} =$ same

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x\end{matrix}\ ,\ \begin{matrix}\partial_x + 2\pi i y \\ \partial_y\end{matrix}\right] = 0$$

you wanted to check that $e^{aD_x}e^{bD_y}$ preserve the automorphy conditions.

$e^{a\partial_x} e^{b(2\pi i x + \partial_y)}\psi = e^{2\pi i b(x+a)} e^{a\partial_x} e^{b\partial_y}\psi$

eval at x,y $\quad \cancel{e^{2\pi i ab}}\, e^{2\pi i b x}\psi(x+a, y+b)$

Assume $\psi(x, y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)$

Show that $\boxed{\underbrace{e^{2\pi i b x}\psi(x+a, y+b)}_{\psi_1(x,y)}}$ has same property

OK fixed under $y \mapsto y+1$

$x \mapsto x+1$

$$\longrightarrow e^{2\pi i b x} e^{2\pi i b} \psi(x+1+a, y+b)$$

$$= e^{2\pi i b x} e^{2\pi i b} e^{-2\pi i (y+b)} \psi(x+a, y+b)$$

$$= e^{-2\pi i y} e^{2\pi i b x} \psi(x+a, y+b)$$

$$= e^{-2\pi i y} \psi_1(x,y)$$

$$e^{tx\partial_y}\psi(x,y) = \psi(e^{tx\partial_y}x, e^{tx\partial_y}y)$$

$$= \psi(x \;, \; y+tx)$$

$$e^{ty\partial_x}\psi(x,y) = \psi(x+ty, y)$$

$$\boxed{e^{\frac{1}{2}ty^2 + ty\partial_x}\psi(x,y) = e^{\frac{1}{2}ty^2}\psi(x+ty, y)}$$

missing $2\pi i$ S

You should be able to show this ~~___~~ operator

$\psi \longmapsto e^{\pi i t y^2} e^{ty\partial_x}\psi$ ~~___~~ commutes with $d + 2\pi i x dy$

$$e^{+\pi i t y^2} e^{ty\partial_x}(d + 2\pi i x dy)\psi = e(d + 2\pi i (x+ty)dy) e^{ty\partial_x}\psi$$

$$= \left(d + 2\pi i(x+ty)dy \atop - 2\pi i t y\, dy\right) e^{+\pi i t y^2} e^{ty\partial_x}\psi$$

Feb 1, 02 What is the problem? You consider the trivial line bundle over $\mathbb{R}^2$ equipped with the connection $d + 2\pi i x dy$. Fact: given two connections $d+A_1$, $d+A$ on the trivial line bundle over $\mathbb{R}^2$ having the same curvature, there is an isom between them which is unique up to a scalar factor.

Ex. $g: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad g(c_1, c_2) = (ac_1 + bc_2, cc_1 + \delta c_2)$

$x = pr_1$
$y = pr_2$

$$pr_1\, g(c_1,c_2) = ac_1 + bc_2 = (a\,pr_1 + b\,pr_2)(c_1,c_2)$$
$$pr_2\, g(c_1,c_2) = cc_1 + \delta c_2 = (c\,pr_1 + \delta\,pr_2)(c_1,c_2)$$

$$g^* x = xg = ax + by$$
$$g^* y = yg = cx + \delta y$$

$$g^*(x\,dy) = (ax+by)(c\,dx + \delta\,dy)$$
$$= (x \;\; y)\begin{pmatrix} a \\ b \end{pmatrix}(c \;\; \delta)\begin{pmatrix} dx \\ dy \end{pmatrix}$$

$$g^*(x\,dy) - x\,dy = \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & \overset{bc}{\overset{\shortparallel}{a\delta - 1}} \\ bc & b\delta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = d\left\{\frac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\}$$

What's important? You have the trivial line bundle over $\mathbb{R}^2$ equipped with the connection form $x\,dy$. The curvature of the connection, $dx\,dy$, is fixed under g^*,

$$g^*(dx\,dy) = d(ax+by) \wedge d(cx+\delta y) = (a\delta - bd)\,dx\,dy$$

when $a\delta - bd = 1$, $\begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \in SL(2,\mathbb{R})$. Two line bundles equipped with connection having the same curvature are isom, the isom is unique up to scalar factor. You want to construct this ~~spect~~ isomorphism explicitly.

Repeat what you have: $g: \mathbb{R}^2 \to \mathbb{R}^2$ linear diffeom.

$$\begin{pmatrix} xg \\ yg \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$ g extends to the trivial line bundle over $\mathbb{R}^2$

$\mathbb{R}^2 \xrightarrow{g'} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$

$$g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$g\,(c_1 \;\; c_2) = (ac_1 + bc_2 \quad cc_1 + \delta c_2)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} g = \begin{pmatrix} xg \\ yg \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$(gg')^* x = xgg' = (ax + by)g' = a(a'x + b'y) + b(c'x + d'y)$$

$$(gg')^* y = ygg' = (cx + dy)g' = c(a'x + b'y) + d(c'x + d'y)$$

$$xgg' = (aa' + bc')x + (ab' + bd')y$$

$$ygg' = (ca' + dc')x + (cb' + dd')y$$

$$\begin{pmatrix} xgg' \\ ygg' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

So what do you learn?

$$\mathbb{R}^2 \xrightarrow{g'} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$\Omega^\circ(\mathbb{R}^2) \xleftarrow{g'^*} \Omega^\circ(\mathbb{R}^2) \xleftarrow{g^*} \Omega^\circ(\mathbb{R}^2)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} g'^*\begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \longleftarrow \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = g^*\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\therefore \; (gg')^*\begin{pmatrix} x \\ y \end{pmatrix} = g'^*\left(g^*\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Is action of a linear autom $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ of $\mathbb{R}^2$ on a function $\psi(x,y)$
$(g^*\psi)(x,y) \overset{?}{=} \psi(ax+by,\ cx+dy)$? Go over this. Start with

$$g\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}ac_1+bc_2\\cc_1+dc_2\end{pmatrix} = \begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

Then

$$pr_1\, g\begin{pmatrix}c_1\\c_2\end{pmatrix} = ac_1+bc_2 = (a\,pr_1 + b\,pr_2)\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

$$pr_2 \qquad cc_1+dc_2 = (c\,pr_1 + d\,pr_2)\begin{pmatrix}c_1\\c_2\end{pmatrix}$$

Let $x = pr_1 : \mathbb{R}^2 \to \mathbb{R}$, then $g^*x = xg = ax+by$
$y = pr_2 : \mathbb{R}^2 \to \mathbb{R}$ $\qquad g^*y = yg = cx+dy$

Then $g^* : \Omega^0(\mathbb{R}^2) \longleftarrow \Omega^0(\mathbb{R}^2)$ is $g^*\psi(x,y) = \psi(g^*x, g^*y)$
$= \psi(ax+by,\ cx+dy)$. In other words

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}^*\psi(x,y) = \psi(ax+by,\ cx+dy)$$

~~If you write $\psi\begin{pmatrix}x\\y\end{pmatrix}$ for $\psi(x,y)$, this becomes~~

$$\cancel{\begin{pmatrix}a&b\\c&d\end{pmatrix}^*\psi\begin{pmatrix}x\\y\end{pmatrix} = \psi\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right)}$$

~~Then~~ $$\cancel{\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}^*\begin{pmatrix}a&b\\c&d\end{pmatrix}^*\psi\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}^*\psi(}$$

$$\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\psi(ax+by,\ cx+dy)$$

$$= \psi\big(a(a'x+b'y)+b(c'x+d'y),\ c(a'x+b'y)+d(c'x+d'y)\big)$$

$$= \psi\big((aa'+bc')x+(ab'+bd')y,\ (ca'+dc')x+(cb'+dd')y\big)$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix} = \begin{pmatrix}aa'+bc' & ab'+bd'\\ ca'+dc' & cb'+dd'\end{pmatrix}$$

$$(g')^*\big(g^*\psi(x,y)\big) = (g')^*\psi(xg,\ yg) = \psi(xg'g,\ yg'g)$$

$$g^*x = ax + by$$
$$g^*y = cx + dy$$

Forget order

$$g^*\psi(x,y) = \psi(g^*x, g^*y) = \psi(ax+by, cx+dy)$$

$$g'^*\psi(ax+by, cx+dy) =$$
$$= \psi\big(a(a'x+b'y)+b(c'x+d'y),\ c(a'x+b'y)+d(c'x+d'y)\big)$$

$$g'^*\big(g^*\psi(x,y)\big) = g'^*\big(\psi(g^*x, g^*y)\big) = \psi(g^*g'^*x, g^*g'^*y)$$

$(g')^*g^*\psi = \cancel{(g^*\psi)}g' = \psi g g'$ $\quad g^*\psi = \psi g$

$$M'' \xrightarrow{g} M' \xrightarrow{g} M \xrightarrow{\psi}$$

?

$$\begin{tikzcd}
M'' \arrow[r, "g'"] \arrow[d, "g'^*(g^*\psi)"] & M' \arrow[r, "g"] \arrow[d, "g^*\psi = \psi g"] & M \arrow[d, "\psi"] \\
T \arrow[r, equal] & T \arrow[r, equal] & T
\end{tikzcd}$$

$$g'^*(g^*\psi) = (g^*\psi)g'$$
$$\| \qquad\qquad \|$$
$$g'^*(\psi g) \qquad (\psi g)g'$$
$$= (\psi g)g' =$$

$$g^*\psi(x,y) = \psi(g^*x, g^*y)$$
$$g'^*g^*\psi(x,y) = \psi(\underbrace{g'^*g^*x}_{(gg')^*x}, g'^*g^*y)$$

$$gg' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ & \end{pmatrix}$$

$$g^*x = ax + by$$
$$g'^*x = a'x + b'y$$
$$g'^*y = c'x + d'y$$

$$g'^*g^*x = a(a'x+b'y) + b(c'x+d'y) = (aa'+bc')x + (ab'+bd')y$$

It seems to work, but you still are not convinced that putting $\psi(x,y)$ for ψ is legit.

Given $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear autom.

and given $\psi = \psi(x,y) \in \Omega^0(\mathbb{R}^2)$ one has

$g^*\psi = \psi \circ g \in \Omega^0(\mathbb{R}^2)$. Let $g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$,

let $x = pr_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto c_1$

$y = pr_2$ —————— $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto c_2$

$$g^*x = x \circ g = ax + by$$
$$g^*y = y \circ g = cx + dy$$

so $(g^*\psi)(x,y) = \psi(g^*x, g^*y)$
$= \psi(ax+by,\ cx+dy)$.

$$\boxed{g^*\psi(x,y) = \psi(g^*x, g^*y) = \psi(ax+by,\ cx+dy)}$$

You now understand g^* on $\Omega^0(\mathbb{R}^2)$. g^* should extend easily to $\Omega(\mathbb{R}^2)$. typical diffl form of degree 1 is $P(x,y)\,dx + Q(x,y)\,dy$ and

$g^*[$ —————— $]$

$= P(ax+by,\ cx+\delta y)(a\,dx + b\,dy) + Q(ax+by,\ cx+\delta y)(c\,dx + \delta\,dy)$

$= \{P(ax+by,\ cx+\delta y)\,a + Q(ax+by,\ cx+\delta y)\,c\}\,dx$

$+ \{P(ax+by,\ cx+\delta y)\,b + Q(ax+by,\ cx+\delta y)\,\delta\}\,dy$

So how do we handle a connection such as $d + 2\pi i x\,dy$

Philosophy. ~~All the stuff concerning g^*, d~~ So far you have examined g^* on $\Omega^0(\mathbb{R}^2)$ and $\Omega(\mathbb{R}^2)$. An element $\phi(x,y) \in \Omega^0(\mathbb{R}^2)$ can be viewed either as a function, or as a section of the trivial line bundle.

The action of g^* is the same - composition with g.
A connection on the trivial line bundle has the form $d+A : \Omega^0(\mathbb{R}^2) \longrightarrow \Omega^1(\mathbb{R}^2)$ where A is a 1-form

$$g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \qquad g^*(dx\,dy) = (a\,dx + b\,dy)\wedge(c\,dx + \delta\,dy)$$
$$= \underbrace{(a\delta - bc)}_{\text{assume } = 1.} dx\,dy$$

$$g^*(x\,dy) = (ax+by)(c\,dx + \delta\,dy)$$

$$g^*(x\,dy) - x\,dy = ac\,x\,dx + (a\overset{bc}{\overset{\shortparallel}{\delta}}-1)\,x\,dy$$
$$bc\,y\,dx + b\delta\,y\,dy$$
$$= d\,\frac{1}{2}\left(acx^2 + 2bcxy + b\delta y^2\right)$$

Look at.

$$\begin{array}{ccccccc} 0 \longrightarrow & \Omega^0(\mathbb{R}^2) & \xrightarrow{d+A} & \Omega^0(\mathbb{R}^2)\,dx \oplus \Omega^0(\mathbb{R}^2)\,dy & \xrightarrow{d+A} & \Omega^0(\mathbb{R}^2)\,dx\,dy \to 0 \\ & g^* \downarrow & & g^* \downarrow & & g^* \downarrow \\ & \Omega^0(\mathbb{R}^2) & \longrightarrow & \Omega^0(\mathbb{R}^2)\,dx \oplus \Omega^0(\mathbb{R}^2)\,dy & \longrightarrow & \Omega^0(\mathbb{R}^2)\,dx\,dy \to 0 \end{array}$$

$$g^*(d+A) = dg^* + g^*A \qquad ??$$

If you combine g^* with the quadratic for $\frac{1}{2}(acx^2+2bcxy+b\delta y^2)$ you get a desirable operator on sections of the trivial line bundles. desirable means wrt the connection $d+A$.

Example. $g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+a \\ y+b \end{pmatrix}$

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \qquad pr_1 g = pr_1 + a$$
$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto \begin{pmatrix} c_1+a \\ c_2+b \end{pmatrix} \qquad pr_2 g = pr_2 + b$$

then $g^*(x\,dy) = (x+a)\,dy$
$$g^*(x\,dy) - x\,dy = a\,dy = [d, ay]$$

$$g^*\psi(x,y) = \psi(g^*x, g^*y) = \psi(x+a, y+b).$$

$$g^*(d + 2\pi i\, x\, dy)\psi(x,y) = (d + 2\pi i (x+a) dy)\, g^*\psi(x,y)$$

$$\begin{tikzcd} \Omega^0(\mathbb{R}^2) \arrow[r, "d+"] \arrow[d, "g^*"] & \Omega^1(\mathbb{R}^2) \arrow[d, "g^*"] \\ \Omega^0(\mathbb{R}^2) \arrow[r] & \Omega^1(\mathbb{R}^2) \end{tikzcd}$$

Yesterday: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto \begin{pmatrix} ac_1 + bc_2 \\ cc_1 + dc_2 \end{pmatrix} \xmapsto{x = pr_1} ac_1 + bc_2 = a\cancel{x} + by$

$$g^*(x) = ax + by$$
$$g^*(y) = cx + dy$$

$$g^*\psi(x,y) = \psi(g^*x, g^*y) = \psi(ax+by, cx+dy)$$

g^* extends to $\Omega(\mathbb{R}^2)$ so as to commute with d.

$$d\psi = \partial_x\psi\, dx + \partial_y\psi\, dy$$
$$g^*d\psi = g^*(\partial_x\psi)\cdot(a\,dx + b\,dy) + g^*(\partial_y\psi)\cdot(c\,dx + \delta\, dy)$$
$$= g^*(a\partial_x\psi + c\partial_y\psi)\, dx + g^*(b\partial_x\psi + \delta\partial_y\psi)\, dy$$

$$\partial_x(g^*\psi) = \partial_x\psi(ax+by, cx+\delta y)$$

$$\partial_x g^*\psi(x,y) = \partial_x\psi(ax+by, cx+\delta y)$$
$$= a\psi_x(ax+by, cx+\delta y) + c\psi_y(ax+by, cx+\delta y)$$

$$\partial_x g^*\psi = g^*(a\partial_x + c\partial_y)\psi$$
$$\partial_y g^*\psi = g^*(b\partial_x + \delta\partial_y)\psi$$

$$g^*(d+A)\psi = (d + g^*A)\, g^*\psi = (d + A + dh)\, g^*\psi$$
$$= e^{-h}(d+A)\, e^h g^*\psi$$

$$\boxed{e^h g^*(d+A)\psi = (d+A)\, e^h g^*\psi}$$

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ linear autom.

$$g^*(x) = ax + by$$
$$g^*(y) = cx + dy$$

$$g^*\psi\begin{pmatrix}x\\y\end{pmatrix} = \psi\begin{pmatrix}g^*x\\g^*y\end{pmatrix} = \psi\begin{pmatrix}ax+by\\cx+dy\end{pmatrix} = \psi\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\mathbb{R}^2 \xrightarrow{g'} \mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2$$

$$\cancel{g'^*g^*\psi\begin{pmatrix}x\\y\end{pmatrix}} = \cancel{g^*\psi\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}} = \cancel{\psi\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\begin{pmatrix}g^*x\\g^*y\end{pmatrix}}$$

$$g'^*(g^*\psi) = (g^*\psi)g' \underset{\uparrow}{=} g^*(\psi g') = \psi g' g$$

NOT ASSOC

$$(g^*\psi)g' = (\psi g)g' = \psi\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix} = \psi g g'$$

Something tricky:

$$\mathbb{R}^2 \xrightarrow[\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}]{g'} \mathbb{R}^2 \xrightarrow[\begin{pmatrix}a&b\\c&d\end{pmatrix}]{g} \mathbb{R}^2$$

$$\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}g^*x\\g^*y\end{pmatrix} \longleftarrow \begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\begin{pmatrix}g^*x\\g^*y\end{pmatrix} = g'^*\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\parallel$$

$$\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}(g'g)^*x\\(g'g)^*y\end{pmatrix}$$

To understand this recall that if g acts linearly on a vector space V equipped with basis $e_1, \dots, e_n$ and dual basis $e_1^*, \dots, e_n^*$ then the corresp. matrix for g

$$g_{ij} = e_i^* g e_j$$

$$\sum_i e_i g_{ij} = g e_j$$

is the transpose of the matrix you get by expressing each ge_i in terms of the others

$$ge_\alpha = \sum_\beta c_{\alpha\beta} e_\beta \qquad ge_j = \sum_i c_{ji} e_i \qquad c_{ji} = g_{ij}$$

This explains why the effect of g^* on the basis $\begin{smallmatrix}x\\y\end{smallmatrix}$ is "covariant".

struggle with the mess so far. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $(g^*\psi)(x,y) = \psi(g^*x, g^*y)$
$$= \psi(ax+by, cx+dy).$$

basic argument assuming $ab - c\delta = 1$, $A = 2\pi i x dy$

$$g^*(d+A)\psi = dg^*\psi + g^*A g^*\psi$$
$$= (d + g^*A) g^*\psi \qquad \psi \in \Omega(\mathbb{R}^2)$$
$$= (d + A + dh) g^*\psi$$
$$= e^{-h}(d+A) e^h g^*\psi$$

$$\boxed{\therefore\ e^h g^* \text{ commutes with } d + A.}$$

This seems to give a clean version of the idea that symp. diffeos act on the projective space of sections of "the" line bundle over $\mathbb{R}^2$ with curvature $2\pi i\, dx\, dy$.

There is a projective representation of symp diffeos of $\mathbb{R}^2$ on sections of "the" line bundle over $\mathbb{R}^2$ with curvature $2\pi i\, dx\, dy$.

What's a good way to see, or express, ~~that~~ the operator $e^h g^*$ respects the connection?

$$(e^h g^* \psi)(x,y)$$

$$g^*A - A = (ax+by)(c\,dx + \delta\, dy) - x\,dy$$
$$= \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix}$$
$$h(x,y) = \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\psi(x,y) \longmapsto e^{h(x,y)} \cancel{\psi}(ax+by, cx+\delta y) \qquad \begin{pmatrix} g^*x \\ g^*y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix}$$

you should look at simpler situations, either $g = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, or look at the Lie alg of $SL(2,\mathbb{R})$.

vector fields from $sl(2,\mathbb{R})$

$$x\partial_y, \; y\partial_x, \; x\partial_x - y\partial_y$$

$$\left[x\partial_y, \begin{pmatrix}x\\y\end{pmatrix}\right] = \begin{pmatrix}0\\x\end{pmatrix} = \begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$e^{tx\partial_y}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\\y+tx\end{pmatrix} = \begin{pmatrix}1&0\\t&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$[\mathcal{L}(x\partial_y), d + 2\pi i x dy] = 2\pi i x d[x\partial_y, y] = d(\pi i x^2)$$

$$[\mathcal{L}(x\partial_y) + \pi i x^2, d + 2\pi i x dy] = 0$$

What this should mean is that the operator $\mathcal{L}(x\partial_y) + \pi i x^2$ on sections ψ of the trivial line bundle over $\mathbb{R}^2$, preserves or respects, the connection.

Look in this case for what happens to the two components of the connection $\iota(\partial_x)(d + 2\pi i x dy) = \partial_x$ and $\iota(\partial_y)(d + 2\pi i x dy) = \partial_y + 2\pi i x$.

$$[x\partial_y + \pi i x^2, \partial_x] = -(\partial_y + 2\pi i x)$$

$$[\partial_x, x(\partial_y + \pi i x)] = \partial_y + \pi i x + x(\pi i) \quad \text{OK}$$

$$[x\partial_y + \pi i x^2, \partial_y + 2\pi i x] = 0$$

Yet $\left[\mathcal{L}(x\partial_y) + \pi i x^2, dx\,\partial_x + dy(\partial_y + 2\pi i x)\right] = 0$

$$[\mathcal{L}(x\partial_y) + \pi i x^2, dx\,\partial_x] = -dx(\partial_y + 2\pi i x)$$

$$[\mathcal{L}(x\partial_y) + \pi i x^2, dy(\partial_y + 2\pi i x)] = dx(\partial_y + 2\pi i x)$$

these add to 0 as expected

You encounter again the difficulty about the components of the connection.

Heisenberg group is the central extension of $\mathbb{R}^2$ by the circle assoc. to

$$e^{a\partial_x} e^{b\partial_y} e^{a'\partial_x} e^{b'\partial_y} = e^{(a+a')\partial_x} e^{(b+b')\partial_y} e^{2\pi i b a'}$$

If G is abelian, then a $G \otimes G \longrightarrow M$ will define a central extn of G by M.

$$(m,g)(m',g') = (m+m'+\langle g,g'\rangle, gg')$$

$$c(g_1,g_2) - c(g_0+g_1,g_2) + c(g_0,g_1+g_2) - c(g_0,g_1) = 0$$

$\mathbb{R}^2$ group.

1-coboundary

$$c(g_1) - c(g_0 g_1) + c(g_0)$$

Heisenberg group. V vector space, say 2 diml equipped with a bilinear form B, define product on $\mathbb{C} \times V$ by

$$(c_1,v_1)(c_2,v_2) = (c_1+c_2+B(v_1,v_2), v_1+v_2)$$

V, C abelian groups $\quad B: V \otimes_{\mathbb{Z}} V \longrightarrow C$ bilinear $B(v_1,v_2)$

group law on $C \times V \quad (c_1,v_1)(c_2,v_2) = (c_1+c_2+B(v_1,v_2), v_1+v_2)$

$$((c_1,v_1)(c_2,v_2))(c_3,v_3) = (c_1+c_2+B(v_1,v_2), v_1+v_2)(c_3,v_3)$$

$$= c_1+c_2+B(v_1,v_2)+c_3+B(v_1+v_2,v_3)$$

$$(c_1,v_1)((c_2,v_2)(c_3,v_3)) = (c_1,v_1)(c_2+c_3+B(v_2,v_3), v_2+v_3)$$

$$= (c_1+c_2+c_3+B(v_2,v_3)+B(v_1,v_2+v_3)), v_1+v_2+v_3)$$

$v_1v_2 \quad v_1v_3 \quad v_2v_3$

$v_2v_3 \quad v_1v_2 \quad v_1v_3$

$Q: V \longrightarrow C \qquad \underbrace{Q(v_1+v_2) - Q(v_1) - Q(v_2)}$

Q called quadratic when is bilinear

$$(C \times V)_B$$

Statement $\quad Hom(V\otimes V, C)/\delta Q(V,C) \subset E$

$$\begin{tikzcd} S^2V \arrow[d] & \Lambda^2 V \arrow[d] & & \\ \Gamma_2 V \arrow[r] \arrow[d] & V\otimes V \arrow[r] \arrow[d] & \Lambda^2 V \arrow[r] & 0 \\ V^{(2)} \arrow[r, hook] & S^2 V & & \end{tikzcd}$$

Heis: $(\mathbb{R}\times\mathbb{R}^2)_{2\pi i(x\wedge y)}$ You want to check that $SL(2,\mathbb{R})$ acts on the Heisenberg group.

$$(c_1, v_1)(c_2, v_2) = (c_1+c_2+B(v_1,v_2), v_1+v_2)$$

$$(c_1, gv_1)(c_2, gv_2) = (c_1+c_2+B(gv_1,gv_2), gv_1+gv_2)$$

OK you need g to preserve the bilinear form B

Review. You have this isom. between $\mathcal{S}(\mathbb{R})$ and the space $\mathcal{L}$ of sections of a line bundle L over $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. You have a concrete line bundle over T^2 which comes with a connection, and the whole thing admits translational symmetry. $\mathbb{R}^2$ acts via transl. on $\mathbb{R}^2/\mathbb{Z}^2$

You have line bundle 3 dim, quotient of Heis by $\mathbb{Z}^2$

Forget $\mathbb{Z}^2$. There seems to be a problem. You need to understand why the metaplectic repn of $SL(2,\mathbb{R})$???

$$C\times V \rightsquigarrow C\times V$$

$(c, v)\longmapsto(c, gv)$ homom. if $B(v_1,v_2) = B(gv_1, gv_2)$

the canon choice is for B to be skew symm non deg.

$V=\mathbb{R}^2 \qquad B\left(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}\right) = \begin{pmatrix}x_1\\y_1\end{pmatrix}^t\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix} =$

$=(x_1\ y_1)\begin{pmatrix}0&0\\x_2&0\end{pmatrix} = y_1x_2 \qquad$ like $y\,dx$

skew symmetrize to get $\quad B\left(\begin{pmatrix}x_1\\y_1\end{pmatrix},\begin{pmatrix}x_2\\y_2\end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix}x_1\\y_1\end{pmatrix}^t\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}x_2\\y_2\end{pmatrix}$

like $\frac{1}{2}(-x\,dy+y\,dx)$

Discuss, review the situation. You begin with the isomorphism $\mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{L}$, where $\mathcal{L}$ space of sections of a line bundle L over T^2. On $\mathcal{S}(\mathbb{R})$ one has various operators e.g. $\frac{d}{dx}$, $2\pi i x$, F.T. You should understand the corresponding operators on $\mathcal{L}$.

Now I think you need to pass ~~to~~ a double covering of $SL(2,\mathbb{R})$ in order ~~to~~ to get an action on $\mathcal{S}(\mathbb{R})$ - this is the metaplectic representation.

Review the F.T.

$$\tilde{f}(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m)$$

$$f(x+m) = \sum_m \oint e^{-2\pi i m y}\, \tilde{f}(x,y)\, dy$$

$$e^{2\pi i x y}\tilde{f}(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m)$$

$$\oint e^{2\pi i x y}\tilde{f}(x,y)\,dx = \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)\,dx = \hat{f}(y)$$

Summary: If $F(x,y)\in\mathcal{L}$, then

$$f(x) = \oint F(x,y)\,dy$$

$$\hat{f}(y) = \oint e^{2\pi i x y} F(x,y)\,dx$$

$$\frac{\hbar}{i}\partial_x = \frac{h}{2\pi i}\partial_x$$

satisfy $\hat{f}(y) = \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)\,dx$ i.e. $\hat{f}(y)$ is the F.T. of $f(x)$.

$p = \frac{d}{dx}$ on $\mathcal{S}(\mathbb{R})$ $qp = 2\pi i x \frac{d}{dx}$

$q = 2\pi i x$ on $\mathcal{S}(\mathbb{R})$ Question: You believe that the Lie algebra $sl(2,\mathbb{R})$ acts naturally on $\mathcal{L}$

~~Screwy idea~~ Recall Legendre transform is poor man's F.T. Is there any analog of the L.T. which might involve $\mathbb{Z}$ periodicity?

~~Idea~~: Can the commuting Heisenberg actions

$$\left[\begin{array}{ll} D_x = \partial_x & \partial_x + 2\pi i y \\ D_y = \partial_y + 2\pi i x \;, & \partial_y \end{array}\right] \simeq 0$$

on $C^\infty(\mathbb{R}^2)$ be linked to an isomorphism of $C^\infty(\mathbb{R}^2)$ with a bimodule $\mathcal{L} \otimes \mathcal{L}^\vee$ of finite rank operators?

Your program for now: Recall conjecture that $\mathcal{L}$, which is the space of sections of a line bundle, equipped with connection having constant curvature over T^2 has a natural action of $SL(2,\mathbb{Z})$ (maybe a double covering is needed). You now propose to use the isom. $\mathcal{L} = \mathcal{S}(\mathbb{R})$ to construct an $SL(2,\mathbb{Z})$ action on $\mathcal{L}$.

Suffices to handle $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

First understand F. T.

$$\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m) \qquad \partial_x \tilde{f} = \widetilde{\tfrac{d}{dx} f}$$

$$e^{2\pi i x y} \tilde{f}(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m)$$

$$\partial_y \left\{ e^{2\pi i x y} \tilde{f} \right\} = \sum_m e^{2\pi i (x+m) y}\, 2\pi i (x+m) f(x+m)$$

$$= e^{2\pi i x y} \sum e^{2\pi i m y}\, 2\pi i (x+m) f(x+m) = e^{2\pi i x y}\, \widetilde{2\pi i x f(x)}$$

$(\partial_y + 2\pi i x)\tilde{f} = \widetilde{2\pi i x f}$. From QM the basis "1st order" operators are $f \mapsto d_x f$ and $f \mapsto 2\pi i x f$ $\qquad [d_x, 2\pi i x] = 2\pi i$

Check on the other side of $\sim$ $\qquad [\partial_x, \partial_y + 2\pi i x] = 2\pi i$

Look next at 2nd order ops.

$$\widetilde{d_x d_x f} = \partial_x \widetilde{d_x f} = \partial_x^2 \tilde{f} \qquad \widetilde{2\pi i x\, d_x f} = (\partial_y + 2\pi i x)\partial_x \tilde{f}$$

$$\widetilde{d_x (2\pi i x) f} = \partial_x (\partial_y + 2\pi i x)\tilde{f}$$

So what am I going to do next? Point: quadratic operators generate a Lie alg which should be $sl(2,\mathbb{R}) \oplus \mathbb{R}$.

You have operators D_x, D_y generating a Heisenberg algebra $[D_x, D_y] = 2\pi i$, also the quadratic operators $\frac{1}{2}D_x^2$, $\frac{1}{2}(D_xD_y + D_yD_x)$, $\frac{1}{2}D_y^2$. From an invariant viewpoint you have $\frac{1}{2}(aD_x + bD_y)^2$ squares of linear operators. These should form a Lie alg under (Poisson?) bracket. Recall that the symplectic Lie alg is the space of symmetric bilinear form. Ask for the linear transformation. V is the ~~symmetric~~ vector space equipped with symp. structure, i.e. non deg skew symm. form.

$$\left[\frac{1}{2}D_x^2, \begin{matrix} D_x \\ D_y \end{matrix}\right] = \begin{matrix} 0 \\ D_x \cdot 2\pi i \end{matrix} \qquad \begin{pmatrix} 0 & 0 \\ 2\pi i & 0 \end{pmatrix}\begin{pmatrix} D_x \\ D_y \end{pmatrix}$$

$$\left\{\frac{v^2}{2}, w\right\} = v\{v, w\}$$

Discuss carefully what to do. You have the specific space $\mathcal{S}(\mathbb{R})$ with Heisenberg action

$$D_x f = \frac{d}{dx} f$$
$$D_y f = 2\pi i x f.$$

You know how to exponentiate to get a representation of the associated Lie group.

$$e^{aD_x + bD_y} = e^{-\frac{1}{2}ab\,2\pi i} e^{aD_x} e^{bD_y}$$

$$e^{X+Y} \qquad \frac{X^2 + XY + YX + Y^2}{2}$$

$$e^X e^Y e^{-\frac{1}{2}[X,Y]} \qquad \frac{X^2}{2} + XY + \frac{Y^2}{2} - \frac{[X,Y]}{2}$$

$$\left[\frac{1}{2}D_x^2, aD_x + bD_y\right] = D_x[D_x, bD_y] = 2\pi i b\, D_x$$
$$\left[\frac{1}{2}D_y^2, aD_x + bD_y\right] = D_y[D_y, aD_x] = -2\pi i a D_y$$
$$\left[\frac{1}{2}(D_xD_y + D_yD_x), aD_x + bD_y\right] = D_x(-2\pi i a) + (2\pi i b)D_y = -2\pi i a D_x + 2\pi i b D_y$$
$$\left[\frac{1}{2}D_x^2, \begin{pmatrix} D_x & D_y \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}\right] = \begin{pmatrix} 0 & 2\pi i b D_x \end{pmatrix} = \begin{pmatrix} D_x & D_y \end{pmatrix}\begin{pmatrix} 0 & 2\pi i \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\left[\tfrac{1}{2}D_x^2, \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}\right] = \begin{pmatrix} 0 & 2\pi i D_x\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} 0 & 2\pi i \\ 0 & 0\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}$$

$$\left[\tfrac{1}{2}D_y^2, \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}\right] = \begin{pmatrix} D_y(-2\pi i) & 0\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix} = \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} 0 & 0 \\ -2\pi i & 0\end{pmatrix}\begin{pmatrix} a \\ b\end{pmatrix}$$

$$\left[\tfrac{1}{2}(D_xD_y+D_yD_x), \begin{pmatrix} D_x & D_y\end{pmatrix}\right] = \begin{pmatrix} D_x(-2\pi i) & D_y(2\pi i)\end{pmatrix} = \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} -2\pi i & 0 \\ 0 & 2\pi i\end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}$$

$$\left[\tfrac{1}{2}D_x^2, \tfrac{1}{2}D_y^2\right] = \tfrac{1}{2}\left[\tfrac{1}{2}D_x^2, D_y\right]D_y + \tfrac{1}{2}D_y\left[\tfrac{1}{2}D_x^2, D_y\right]$$
$$= \tfrac{1}{2}D_x(2\pi i)D_y + \tfrac{1}{2}D_y(2\pi i)D_x = 2\pi i\,\tfrac{1}{2}(D_xD_y+D_yD_x)$$

Your aim is to exponentiate $\frac{1}{2}D_x^2$.

It should make sense: $e^{t\frac{1}{2}D_x^2}$ on $\mathcal{S}(\mathbb{R})$.

You have an i (or $2\pi i$) problem.

OK yes. It seems that instead of $\frac{1}{2}D_x^2$, $\frac{1}{2}D_y^2$, $\frac{1}{2}(D_xD_y+D_yD_x)$ you want $\frac{1}{4\pi i}D_x^2$ $\frac{1}{4\pi i}D_y^2$ $\frac{1}{4\pi i}(D_xD_y+D_yD_x)$. Notice that the purely imaginary phase is correct. D_x is skew adj so $D_x^2 \leq 0$.

Actually since $D_x = \frac{d}{dx}$ a formula for $e^{\frac{t}{4\pi i}D_x^2}$ is given by Fourier transform.

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i xy}\hat{f}(y)\,dy$$

$$\tilde f(x,y) = \sum_m e^{2\pi i m y} f(x+m) \qquad \partial_x \tilde f = \widetilde{\tfrac{d}{dx} f}$$

$$e^{2\pi i x y}\tilde f(x,y) = \sum_m e^{2\pi i (x+m)y} f(x+m)$$

$$e^{-2\pi i x y}\partial_y\left(e^{2\pi i x y}\tilde f(x,y)\right) = \sum_m e^{2\pi i (x+m)y}\, 2\pi i (x+m) f(x+m) = (2\pi i x f)^\sim$$

$$\parallel$$

$$(\partial_y + 2\pi i x)\tilde f. \qquad \text{Problem: } D_x = \tfrac{d}{dx},\ D_y = 2\pi i x \quad \text{on } \mathcal{S}$$

$$\left[\tfrac{1}{2\cdot 2\pi i}D_x^2,\ (D_x\ \ D_y)\right] = (0\ \ D_x) = (D_x\ \ D_y)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\left[\tfrac{1}{2\cdot 2\pi i}D_y^2,\ (D_x\ \ D_y)\right] = (-D_y\ \ 0) = (D_x\ \ D_y)\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}$$

$$e^{\frac{t}{4\pi i}D_y^2} f = e^{\frac{t}{2\cdot 2\pi i}(2\pi i x)^2} f = e^{\pi i t x^2} f$$

$$e^{\pi i t x^2}\left(\tfrac{d}{dx}\quad 2\pi i x\right)e^{-\pi i t x^2} = \left(\tfrac{d}{dx} - 2\pi i t x \quad 2\pi i x\right)$$

$$e^{\pi i t x^2}(D_x\ \ D_y)e^{-\pi i t x^2} = (D_x - tD_y\ \ \ D_y) = (D_x\ \ D_y)\begin{pmatrix}1 & 0\\ -t & 1\end{pmatrix}$$

So what do you need at this point?

Go back to the line bundle L over T^2, $\mathcal{L}$ its space of sections, Θ functions on T^2. $\psi \in \mathcal{L}$ means $\psi \in C^\infty(\mathbb{R}^2)$, $\psi(x,y+1) = \psi(x,y) = e^{2\pi i y}\psi(x+1,y)$. Fix a point p_0 of T^2 coset $x_0+\mathbb{Z}$, $y_0+\mathbb{Z}$. The fiber $L(p_0)$ consists of ψ defined on $(x_0+\mathbb{Z})\times(y_0+\mathbb{Z})$ satisfying the automorph conditions

$$\psi(x_0+m, y_0+n+1) = \psi(x_0+m$$

The fibre $L(p)$ at $p = \pi(x_0,y_0) \in T^2$ consists of $\psi(x,y)$ defined for $\begin{matrix} x \in x_0 + \mathbb{Z} \\ y \in y_0 + \mathbb{Z}\end{matrix}$

sat.
$$\psi(x_0+m, y_0+n) = e^{-2\pi i m y_0}\psi(x_0,y_0)$$

Consider $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$.

$$\begin{array}{ccccc} \mathbb{Z}^2 & \rightarrowtail & \mathbb{R}^2 & \xrightarrow{\pi} & T^2 \\ g\downarrow & & g\downarrow & & g\downarrow \\ \mathbb{Z}^2 & \rightarrowtail & \mathbb{R}^2 & \twoheadrightarrow & T^2 \end{array}$$

$\pi^{-1}(p) = (x_0+\mathbb{Z})\times(y_0+\mathbb{Z})$

$\pi^{-1}(gp) = (ax_0+by_0+\mathbb{Z}, cx_0+dy_0+\mathbb{Z})$

$$g\begin{pmatrix} x_0+m \\ y_0+n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_0+m \\ y_0+n \end{pmatrix} = \begin{pmatrix} ax_0+by_0 \\ cx_0+dy_0 \end{pmatrix} + \begin{pmatrix} am+bn \\ cm+dn \end{pmatrix}$$

$$g(\pi^{-1}p) = g\begin{pmatrix} x_0+\mathbb{Z} \\ y_0+\mathbb{Z} \end{pmatrix} = g\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + g\begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} = g\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} = \pi^{-1}(gp).$$

$$L(p) = \{\psi : \pi^{-1}(p) = (x_0+\mathbb{Z})\times(y_0\times\mathbb{Z}) \to \mathbb{C} \mid \psi(x_0+m, y_0+n) = e^{-2\pi i m y_0}\psi(x_0,y_0)$$

$\downarrow$

$$L(gp) = \{\psi \text{ def'd on } \pi^{-1}(gp) = (ax_0+by_0+\mathbb{Z})\times(cx_0+dy_0+\mathbb{Z})$$

$\mathbb{R}^2 \times_{\chi}^{\mathbb{Z}^2} \mathbb{C}$ not a character.

$\downarrow$

T^2

a section is a $\psi(x,y) \in C^\infty(\mathbb{R}^2)$ sat.

$$\psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)$$

Suppose given $\psi(x,y)$ sat. usual.

pull back to $\psi(ax+by, cx+dy) = \psi_1(x,y)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\| $

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\psi g(x,y)$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{g} & \mathbb{R}^2 \\ \downarrow\pi & & \downarrow\pi \\ T^2 & \xrightarrow{g} & T^2 \end{array}$$

$$\psi_1(x+m, y+n) = \psi(ax+am+by+bn, cx+cm+dy+dn)$$
$$= \psi(ax+by+am+bn, cx+dy+cm+dn)$$

$$\psi_1(x,y) = \psi(x+y, y)$$

$$\psi_1(x+1, y) = \psi(x+y+1, y) = e^{-2\pi i y}\psi(x+y,y) = e^{-2\pi i y}\psi_1(x,y).$$

$$\psi_1(x, y+1) = \psi(x+y+1, y+1) = \psi(x+y+1, y) = e^{-2\pi i y}\psi(x+y,y)$$
$$= e^{-2\pi i y}\psi_1(x,y).$$

so you find that $\boxed{\psi_1(x+1, y-1) = \psi_1(x,y)}$

$$\psi_1(x, y+1) = e^{-2\pi i y}\psi_1(x,y)$$

Feb 4, 02. Given $\psi \in C^\infty(\mathbb{R}^2)$, let $\psi_1(x,y) = \psi(-y,x)$ | 950

Then $\psi_1(x+1,y) = \psi(-y,x+1) = \psi(-y,x) = \psi_1(x,y)$

$\psi_1(x,y+1) = \psi(-y-1,x) = e^{-2\pi i(-1)x}\psi(-y,x) = e^{2\pi ix}\psi_1(x,y)$

Let $\psi_2(x,y) = e^{+2\pi ixy}\psi(x,y)$

$$\psi_2(x+1,y) = e^{2\pi i(x+1)y}\underbrace{\psi(x+1,y)}_{e^{-2\pi iy}\psi(x,y)} = e^{2\pi ixy}\psi(x,y) = \psi_2(x,y)$$

Let $\psi_0(x,y) = e^{-2\pi ixy}\psi_1(x,y)$

$$\psi_0(x,y+1) = e^{-2\pi ixy}e^{-2\pi ix}\psi(-y,x)$$

Start with $\psi \in \mathcal{L} = \left\{ \psi \in C^\infty(\mathbb{R}^2) \;\middle|\; \begin{array}{l} \psi(x,y+1) = \psi(x,y) \\ \psi(x+1,y) = e^{-2\pi iy}\psi(x,y) \end{array} \right\}$

$\psi_1(x,y) = \psi(-y,x)$ $\qquad \psi_1(x+1,y) = \psi(-y,x+1) = \psi(-y,x) = \psi_1(x,y)$

$\psi_1(x,y+1) = \psi(-y-1,x) = e^{-2\pi i(-1)x}\psi(-y,x) = e^{2\pi ix}\psi_1(x,y)$

$$\boxed{\begin{aligned} \psi_1(x,y) &\overset{\text{def}}{=} \psi(-y,x) \\ \psi_1(x+1,y) &= \psi_1(x,y) \\ \psi_1(x,y+1) &= e^{2\pi ix}\psi_1(x,y) \end{aligned}}$$

$\psi_2(x,y) = e^{-2\pi ixy}\psi_1(x,y)$

$$\psi_2(x,y+1) = e^{-2\pi ix(y+1)}\underbrace{\psi_1(x,y+1)}_{e^{2\pi ix}\psi_1(x,y)}$$

$$= e^{-2\pi ixy}\psi_1(x,y) = \psi_2(x,y)$$

$$\psi_2(x+1,y) = e^{-2\pi i(x+1)y}\psi_1(x+1,y) = e^{-2\pi i(x+1)y}\psi_1(x,y) = e^{-2\pi ixy}e^{-2\pi iy}\psi_1(x,y)$$

$\psi(x,y+1) = \psi(x,y) = e^{2\pi iy}\psi(x+1,y)$

$\psi_1(x+1,y) = \psi(-y,x+1) = \psi(-y,x) = \psi_1(x,y)$

$\psi_1(x,y+1) = \psi(-y-1,x) = e^{2\pi ix}\psi(-y,x)$

$$\boxed{\begin{aligned} \psi_1(x,y) &= \psi(-y,x) \\ \psi_1(x+1,y) &= \psi_1(x,y) \\ \psi_1(x,y+1) &= e^{2\pi ix}\psi_1(x,y) \end{aligned}}$$

$\psi_2(x,y) = e^{-2\pi ixy}\psi_1(x,y)$

$\psi_2(x+1,y) = e^{-2\pi ixy}e^{-2\pi iy}\psi_1(x,y)$

$= e^{-2\pi iy}\psi_2(x,y)$

$\psi_2(x,y+1) = e^{-2\pi ixy}e^{-2\pi ix}\psi_1(x,y+1)$

$= e^{-2\pi ixy}\psi_1(x,y)$

$= \psi_2(x,y)$

$$\boxed{\psi(x,y) \longmapsto e^{-2\pi ixy}\psi(-y,x)}$$

$$\psi(-y+mx) = e^{-2\pi i m x}\psi(-y,x) \qquad 95'1$$

$$\psi_2(x,y) = e^{-2\pi i x y}\,\psi(-y,x)$$

$$\psi_2(x+1,y) = e^{-2\pi i (x+1) y}\,\psi(-y,x+1)$$

$$= e^{-2\pi i y}\,e^{-2\pi i x y}\,\psi(-y,x) = e^{-2\pi i y}\,\psi_2(x,y)$$

$$\psi_2(x,y-1) = e^{-2\pi i x y}\,e^{+2\pi i x}\,\underbrace{\psi(-y+1,x)} = e^{-2\pi i x y}\,\psi(-y,x) = \psi_2(x,y)$$

$$\boxed{\psi_0(x+m,y+n) = e^{-2\pi i m y}\,\psi_0(x,y)}$$

put $\psi_1(x,y) = \psi_0(-y,x)$

$$\psi_1(x+m,y+n) = \psi_0(-y-n,x+m) = \psi_0(-y-n,x)$$

$$= e^{-2\pi i(-n)x}\,\psi_0(-y,x)$$

$$\boxed{\psi_1(x+m,y+n) = e^{2\pi i n x}\,\psi_1(x,y)}$$

put $\psi_2(x,y) = e^{-2\pi i x y}\,\psi_1(x,y)$

$$\psi_2(x+m,y+n) = e^{-2\pi i(x+m)(y+n)}\,\psi_1(x+m,y+n)$$

$$= e^{-2\pi i(xy+my+\cancel{nx})}\,e^{\cancel{2\pi i n x}}\,\psi_1(x,y)$$

$$= e^{-2\pi i m y}\,e^{-2\pi i x y}\,\psi_1(x,y)$$

$$\boxed{\psi_2(x+m,y+n) = e^{-2\pi i m y}\,\psi_2(x,y)}$$

Formula $\psi_2(x,y) = e^{-2\pi i x y}\,\psi_1(x,y)$

$$\boxed{\psi_2(x,y) = e^{-2\pi i x y}\,\psi_0(-y,x)}$$

$$g^*A - A = 2\pi i \begin{pmatrix}x\\y\end{pmatrix}^t \begin{pmatrix}ac & bc\\ bc & b\delta\end{pmatrix}\begin{pmatrix}dx\\dy\end{pmatrix} = d\left\{\pi i \begin{pmatrix}x\\y\end{pmatrix}^t \begin{pmatrix}ac & bc\\ bc & b\delta\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right\}$$

$$g\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \quad d\left\{\pi i \begin{pmatrix}x\\y\end{pmatrix}^t \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right\} = d\{2\pi i(-xy)\}$$

$$g\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$-2\pi i x y$

$\pi i y^2$

---

now let's move to another case $\psi_1(x,y) = \psi_0(x+y,y)$ where $\pi i y^2$ might be the adjusting function.

$$\psi_1(x,y) = \psi_0(x+y,y)$$

$$\psi_1(x+m,y+n) = \psi_0(x+y+m+n,y+n) = e^{-2\pi i(m+n)y}\,\psi_1(x,y)$$

IDEA. There's something peculiar about $SL(2,\mathbb{Z})$ being a group of symmetries related to the simple harmonic oscillator.

$$\psi_2(x,y) = e^{\pi i y^2}\psi_1(x,y), \quad \psi_2(x+m,y+n) = e^{\pi i(y+n)^2}\psi_1(x+m,y+n) =$$

$$e^{\pi i(y^2+2yn+n^2)}\, e^{-2\pi i(m+n)y}\,\psi_1(x,y) = e^{\pi i y^2 - 2\pi i m y + \pi i n^2},$$

Start again with the transf. $g: \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, then pull back ψ_0 to $\boxed{\psi_1(x,y) = \psi_0(x+y,y)}$ which satisfies

$$\psi_1(x+m,y+n) = e^{-2\pi i(m+n)y}\psi_1(x,y).$$

You want next to multiply by a suitable $e^{h(y)}$ so as to land back in $\mathcal{L}$. Put $\boxed{\psi_2(x,y) = e^{h(y)}\psi_1(x,y)}$

$$\begin{aligned}\psi_2(x+m,y+n) &= e^{h(y+n)}\psi_1(x+m,y+n) \\ &= e^{h(y+n)}e^{-2\pi i(m+n)y}\psi_1(x,y) \\ &= e^{h(y+n)-h(y)}e^{-2\pi i(m+n)y}\underbrace{e^{h(y)}\psi_1(x,y)}_{\psi_2(x,y)}\end{aligned}$$

you want $h(y+n)-h(y)-2\pi i(m+n)y$ to equal $-2\pi i m y$

$$h(y+n)-h(y) = 2\pi i n y$$

$$h(y+1)-h(y) = 2\pi i y$$

$$h(y+2)-h(y+1) = 2\pi i(y+1)$$

$$h(y+n)-h(y+n-1) = 2\pi i(y+n-1)$$

$$\therefore\ h(y+n)-h(y) = 2\pi i\left(ny+\frac{n(n-1)}{2}\right)$$

Recall $\mathcal{L} = \{\psi \in C^\infty(\mathbb{R}^2) \mid \psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)\}$.

Now we propose to define an action (modulo scalars) by the group $SL(2,\mathbb{Z})$ on $\mathcal{L}$. Use generators $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\boxed{g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$ $\quad g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

Given $\psi_0 \in \mathcal{L}$, put $\psi_1 = \psi_0 g = \psi_0(-y, x)$. Then

$$\psi_1(x+m, y+n) = \psi_0(-y-n, x+m) = e^{-2\pi i(-n)(x+m)}\psi_0(-y, x)$$

$$\boxed{\psi_1(x,y) = \psi_0(-y,x) \qquad \psi_1(x+m, y+n) = e^{2\pi i n x}\psi_1(x,y)}$$

Next put $\psi_2(x,y) = e^{-2\pi i x y}\psi_1(x,y)$

$$\begin{aligned}\psi_2(x+m, y+n) &= e^{-2\pi i(x+m)(y+n)}\psi_1(x+m, y+n) \\ &= e^{-2\pi i(xy+my+\cancel{xn})}\cancel{e^{2\pi i n x}}\psi_1(x,y) \\ &= e^{-2\pi i m y}\left(e^{-2\pi i x y}\psi_1(x,y)\right)\end{aligned}$$

$$\boxed{\psi_2(x+m, y+n) = e^{-2\pi i m y}\psi_2(x,y)}$$

Thus you find $\boxed{\psi_0 \longmapsto \psi_2(x,y) = e^{-2\pi i x y}\psi(-y,x) \text{ carries } \mathcal{L} \text{ into } \mathcal{L}.}$

Next. $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$

Given $\psi_0 \in \mathcal{L}$, put $\psi_1 = \psi_0(x+y, y)$.

$$\psi_1(x+m, y+n) = \psi_0(x+y+m+n, y+n) = e^{-2\pi i(m+n)y}\psi_0(x+y, y) = e^{\boxed{-2\pi i(m+n)y}}\psi_1(x,y)$$

Let $\psi_2(x,y) = e^{h(y)}\psi_1(x,y)$ and try to find h so that $\psi_2 \in \mathcal{L}$

$$\begin{aligned}\psi_2(x+m, y+n) &= e^{h(y+n)}\psi_1(x+m, y+n) = e^{h(y+n)}e^{-2\pi i(m+n)y}\psi_1(x,y) \\ &= e^{h(y+n)-h(y)-2\pi i(m+n)y}\underbrace{e^{h(y)}\psi_1(x,y)} \\ &= e^{h(y+n)-h(y)-2\pi i n y}e^{-2\pi i m y}\psi_2(x,y)\end{aligned}$$

Take $h(y) = 2\pi i \frac{y(y-1)}{2}$

$$h(y+n) - h(y) = \frac{2\pi i}{2}\Big(\underbrace{(y+n)(y+n-1)}_{y^2+2ny+n^2-(y+n)} - \underbrace{y(y-1)}_{-y^2+y}\Big)$$

$$= 2\pi i\left(ny + \frac{n^2-n}{2}\right)$$

$$e^{h(y+n)-h(y)} = e^{2\pi i n y}$$

So conclude $\boxed{\psi_0 \longmapsto \psi_2 = e^{2\pi i \frac{y^2-y}{2}} \psi_0(x+y,y) \quad \text{maps } \mathcal{L} \text{ to } \mathcal{L}}$

Next $g = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. Given $\psi_0 \in \mathcal{L}$ put $\psi_1(x,y) = \psi_0(x,x+y)$

$$\psi_1(x+m, y+n) = \psi_0(x+m, x+y+m+n) = e^{-2\pi i m(x+y)} \psi_0(x,x+y)$$

Let $\psi_2(x,y) = e^{h(x)}\psi_1(x,y)$.

$$\psi_2(x+m,y+n) = e^{h(x+m)}\psi_1(x+m,y+n) = e^{h(x+m)} e^{-2\pi i m(x+y)}\psi_1(x,y)$$

$$= e^{-2\pi i m(x+y)} e^{h(x+m)-h(x)} \underbrace{e^{h(x)}\psi_1(x,y)}_{\psi_2(x,y)}$$

$$= e^{-2\pi i m y} e^{h(x+m)-h(x)-2\pi i m x}\psi_2(x,y)$$

so take $h(x) = \frac{2\pi i}{2} x(x-1)$

$$h(x+m) - h(x) = \frac{2\pi i}{2}\Big(\underbrace{(x+m)(x+m-1) - x(x-1)}_{\cancel{x^2}+2mx+m^2-\cancel{x}-m-\cancel{x^2}+\cancel{x}}\Big)$$

$$= 2\pi i\left(mx + \frac{m^2-m}{2}\right) = 2\pi i m x$$

Thus $\boxed{\psi_0 \longmapsto e^{2\pi i \frac{x^2-x}{2}}\psi_0(x,x+y) \quad \text{maps } \mathcal{L} \text{ to } \mathcal{L}}$

---

try to understand better - can you replace the autom conditions by a connection?

$$\psi(x+m,y+n) = e^{-2\pi i m y}\psi(x,y)$$

$$e^{2\pi i m y} e^{m\partial_x} e^{n\partial_y} = e^{m(\partial_x + 2\pi i y)} e^{n\partial_y}$$

$$e^{2\pi i m y}\psi(x+m,y)$$

$$\begin{bmatrix} \partial_x & \partial_x + 2\pi i y \\ \partial_y + 2\pi i x\,, & \partial_y \end{bmatrix}$$

Feb 5, 02

$$e^{2\pi i \frac{x^2-x}{2}} e^{x\partial_y} \psi(x,y) \qquad y + x\partial_y y + \frac{x\partial_y^2}{2!} y$$

$$e^{\partial_x + 2\pi i y} e^{2\pi i \frac{x^2-x}{2}} e^{x\partial_y} \psi$$

Repeat yesterday's calculation

$$g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$(g^*\psi)(x,y) = \psi(ax+by, cx+dy)$$

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad (g^*\psi)(x,y) = \psi(-y,x)$$

Set $\psi_2 = e^{-2\pi i x y} \psi_0(-y,x)$

$$\psi_2(x+m, y+n) = e^{-2\pi i (x+m)(y+n)} \psi_0(-y-n, x+m)$$

$$\psi_0(-y-n, x+m) = e^{+2\pi i (+n)(x+m)} \psi_0(-y,x)$$

$$\psi_2(x+m, y+n) = e^{-2\pi i (xy + my + xn + mn) + 2\pi i (nx + nm) + 2\pi i xy} \psi_2(x,y)$$

note m, n do not have to be integers. This is what you were missing.

$$g\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix}$$

$$\psi_1 = g^*\psi_0 = \psi_0(x+ty, y)$$

$$\psi_0(x+a, y+b) = e^{a(\partial_x + 2\pi i y)} e^{b\partial_y} \psi_0(x,y) \quad ??$$

---

connection $\quad \tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m) \qquad \partial_x \tilde{f} = \widetilde{\frac{d}{dx} f}$

$$e^{2\pi i x y} \tilde{f}(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m)$$

$$e^{-2\pi i x y} \partial_y e^{2\pi i x y} \tilde{f}(x,y) = \sum_m e^{2\pi i (\cancel{x+}m) y} 2\pi i (x+m) f(x+m) = \widetilde{2\pi i x f}$$

$$(\partial_y + 2\pi i x)\tilde{f} = \widetilde{2\pi i x f}$$

discuss philosophy. Space $\mathcal{L}$ of sections of L. This is an irred rep of the Heisenberg group. How? via Lie alg action

$D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ $\quad [D_x, D_y] = 2\pi i$. What is the meaning of the commutativity

$$\begin{bmatrix} \partial_x & \partial_x + 2\pi i y \\ \partial_y + 2\pi i x, & \partial_y \end{bmatrix} = 0?$$

What to do? One point to explore is how to employ ~~the~~ the uniqueness of the irred. repn. to construct operators.

Idea from Graeme, Lattice is analogous to a Lagrangian subspace.

Representation theory viewpoint, consequences of uniqueness of repns. of the CCR. Basic symmetries $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ of the affine space $\mathbb{R}^2$ equipped with $2\pi i\, dx dy$. What happens? Let $g \in SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$, g acts on the Heisenberg group, Lie alg; Call this $\mathcal{H}$. Let E be the basic repn. of $\mathcal{H}$, with group action $\mathcal{H} \xrightarrow{\rho} U(E)$. Then $\mathcal{H} \xrightarrow{g} \mathcal{H} \xrightarrow{\rho} U(E)$, $(g^*\rho)$, is another repn irred; uniqueness $\Rightarrow$ $g^*\rho \cong \rho$, and this should yield a proj unitary repn of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on E.

Q: How is this related to the line bundle with constant curvature $2\pi i\,dxdy$ over T^2?? Point is that the space of sections $\mathcal{L}$ gives the basic repn of $\mathcal{H}$.

This is why it's important to generalize your $SL(2,\mathbb{Z})$ action to an $SL(2,\mathbb{R})$ action.

$$\mathcal{L} = \{\psi(x,y) \in C^\infty(\mathbb{R}^2) \mid \psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)\}$$

$$g^*\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} x+ty \\ y\end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} x \\ y\end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & bd\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$$

Given $\psi_0 \in \mathcal{L}$, put $\psi_1(x,y) = \psi_0(x+ty, y)$

$$2\pi i \tfrac{1}{2} t y^2$$

$$\psi_1(x+m, y+n) = \psi_0(x+ty+m+tn, y+n) =$$
$$= e^{-2\pi i m(y+n)}\psi_0(x+t(y+n), y)$$

$$\psi_2(x,y) = e^{h(y)}\psi_1(x,y)$$

$$\psi_2(x+m,y+n) = e^{h(y+n)}\psi_1(x+m,y+n)$$

$$= e^{h(y+n)} \quad e^{-2\pi i m y}\psi_0(x+t(y+n),y)$$

Given $\psi_0 \in \mathcal{L}$, put $\psi_1(x,y) = \psi_0(x+ty,y)$

$$\psi_1(x+m,y+n) = \psi_0\left(x+m+t(y+n),\ y+n\right)$$

$$= e^{-2\pi i(m)(y+n)}\psi_0\left(x+t(y+n),y+n\right) \qquad ?$$

Discuss philosophy. Important point is the uniqueness of (unitary) irred. repn of the CCR, of the basic rep of the Heisenberg group. This seems to be related to the geometric result that there is a unique line bundle together with connection over $\mathbb{R}^2$ having the curvature $2\pi i\,dx\,dy$.

Problem: Is there a link between these two uniqueness results?

Tool for constructing the basic repn. is to choose an appropriate Lagrangian subgroup, which should be maximal abelian

Discuss the problems.

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_x^2,\ (D_x \quad D_y)\right] = (D_x \quad D_y)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_y^2,\ (D_x \quad D_y)\right] = (D_x \quad D_y)\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}(D_xD_y+D_yD_x),\ (D_x \quad D_y)\right] = (D_x \quad D_y)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

on $\mathcal{S}(\mathbb{R})$ $\quad D_x = \partial_x \quad D_y = 2\pi i x.$ and $\frac{1}{4\pi i}D_y^2 = \pi i x^2.$

Philosophy: Similarity between uniqueness of an irred repr of the CCR (Heisenberg group) and uniqueness of a line bundle together with connection over $\mathbb{R}^2$ having constant curvature $2\pi i\, dx\, dy$. Where to start?

Start with the connection $D = d + A$ on the trivial line bundle over $\mathbb{R}^2$, say $A = 2\pi i x\, dy$. Then you get parallel transport isoms between fibres joined by a curve. Consider line segments $(x+ta, y+tb)$ $0 \le t \le 1$.

Formula. Given $\psi(x,y)$ a section, when is it horizontal over the line $(x,y) + t(a,b)$? Amounts to $\frac{d}{dt}\psi(x+ta, y+tb) + 2\pi i(x+ta)b\,\psi(x+ta, y+tb) = 0$?

$$d\psi(x+ta, y+tb) + 2\pi i (x+ta)\, b\, dt\ \psi(x+ta, y+tb) = 0$$

$$\partial_1 \psi(x+ta, y+tb)\, a + 2\pi i (x+ta) b\, \psi(x+ta, y+tb) = 0$$
$$+ \partial_2 \psi(\qquad)\, b$$

so along the line you get

$$a\,\partial_x\psi + b\,\partial_y\psi + 2\pi i x b \psi = 0$$

$$a\,\partial_x\psi + b\left(\partial_y + 2\pi i x\right)\psi = 0$$

Begin again with the trivial line bundle over $\mathbb{R}^2$ tog with the connection $D = d + A = d + 2\pi i x\, dy = dx\, D_x + dy\, D_y$ where $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$, $[D_x, D_y] = 2\pi i$. A connection gives you $\parallel$ transport along curves. Thus given curve $(x(t), y(t))$, can ask for $\psi(t)$ satisfying

$$\frac{d}{dt}\psi(t) + 2\pi i\, x(t)\frac{dy}{dt}(t)\,\psi(t) = 0$$

soln.
$$\psi(t) = e^{-\int 2\pi i\, x(t)\, dy(t)}$$

Put another way given the curve γ, say parametrized by t: $(x(t), y(t))$ $t \in [0, 1]$ you want a section of the trivial line bundle "over" this curve $(x(t), y(t), \psi(t))$ satisfying

$$d\psi + 2\pi i\, x\, dy\, \psi = 0$$

$$\psi(1) = \exp\left(-2\pi i \int_\gamma x\, dy\right) \psi(0)$$

Because the base is $\mathbb{R}^2$ affine space which has translation action by the additive gp $\mathbb{R}^2$, you want what?? There's an obvious curve to use

$$(x,y)(t) = (x(0), y(0)) + t(a,b) \qquad t \in [0,1].$$

Need clarity: You want lifting the vector field on the base to the line bundle. So you take ∂_x, ∂_y on $\mathbb{R}^2$.

FORGET CURVES.! Your line bundle is $\mathbb{R}^2 \times \mathbb{C}$ with coords x, y, z. A section is a map $(x,y) \mapsto (x, y, \psi(x,y))$

Take curve $x = x_0 + t$, $y = y_0$

$$\underbrace{\frac{d}{dt}\psi(x_0+t, y_0)}_{\partial_x \psi(x_0, y_0)} + \cancel{2\pi i (x_0+t)\, dy_0)\, \psi} = 0$$

Take curve $x = x_0$, $y = y_0 + t$

$$\frac{d}{dt}\psi(x_0, y_0+t) + 2\pi i\, x_0\, \cancel{dt}\, \psi(x_0, y_0+t) = 0$$

$$\partial_y \psi(x_0, y_0) + 2\pi i\, x_0\, \psi(x_0, y_0) = 0$$

i.e. $\quad \partial_y + 2\pi i x$

Review. consider the trivial line bundle over $\mathbb{R}^2$ with connection $d + 2\pi i x\, dy$. Explain, clarify how you lift vector fields on the base $\mathbb{R}^2$ to the total space $\mathbb{R}^2 \times \mathbb{C}$. It seems better to introduce the principal bundle $\mathbb{R}^2 \times \mathbb{T}$. Then there is a connection form characterized by properties. The structural group is $\mathbb{T}$, so ~~the connection form is invariant under rotations, it~~ the connection form Θ has values in $\mathrm{Lie}(\mathbb{T}) = i\mathbb{R}$, it is invariant under the (right!) mult of $\mathbb{T}$ on $\mathbb{R}^2 \times \mathbb{T}$, and on each fibre it restricts to the invariant volume form on $\mathbb{T}$ suitably normalized. This volume is perhaps $\frac{dz}{2\pi i z} = \frac{d\theta}{2\pi}$

On $\mathbb{R}^2 \times \mathbb{T}$ you have coords x, y, z where z is restricted to $\mathbb{T}$, i.e. $|z| = 1$. Another viewpoint: the connection form for a vector bundle has values in $\mathrm{End}(V)$, so in skew adj ops on $\mathbb{C}$ in the complex line bundle case.

Crazy ideas today: $\mathbb{Z}/2$ problem for constructing the 2d conformal QFT based on holomorphic functions. You know this gets remedied somehow by using loop groups, because the result is equivalent to the QFT based on holomorphic spinors.

You hope, in pursuing Poisson summation, that $\mathbb{Z}/2$ phenomena will appear somehow to make a supersymmetric gadget.

2nd crazy idea: do grid space theory where the sides of the square are 1-diml fields. Something like this arises for 1-diml scattering, where the transfer matrix and scattering matrix are functions of frequency.

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back to $\mathbb{R}^2 \times \mathbb{T} = \{(x, y, z) \mid x, y \in \mathbb{R},\ |z| = 1\}$

$$\begin{array}{c}\mathbb{R}^2\times\mathbb{T}\\ \downarrow\\ \mathbb{R}^2\end{array}$$

You want a 1-form η on $\mathbb{R}^2 \times \mathbb{T}$ with values in $i\mathbb{R}$ invariant under mult. by $z \in \mathbb{T}$. $\oint \frac{dz}{2\pi i z}\ \cancel{\ } = 1$. To this $\frac{dz}{2\pi i z}$

you can add any 1-form from the bases. e.g.

$A = 2\pi i\, x\, dy$, so $\theta = 2\pi i\, x\, dy + \frac{dz}{2\pi i z}$

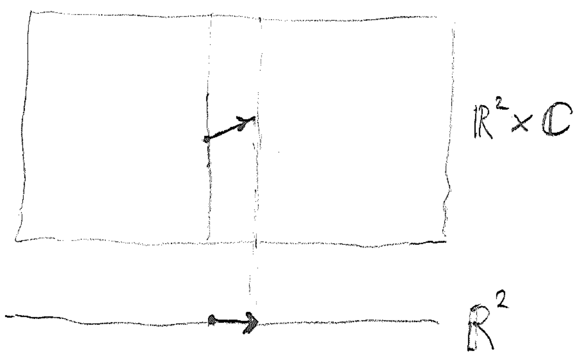
Something looks wrong with the $(2\pi i)$'s. It's clear that you probably want $\theta = \frac{dz}{z} + 2\pi i\, x\, dy$, because the curvature $d\theta = 2\pi i\, dx\, dy$ is to be multiplied by $\frac{i}{2\pi}$.

Go back to $\mathbb{R}^2 \times \mathbb{T} \to \mathbb{R}^2$ $\quad \theta = x\,dy + \frac{dz}{2\pi i z}$

now the i is in the wrong place

Somehow you have to get back on track. Start with $\mathbb{R}^2$ acting on itself by translation, the trivial line bundle $\mathbb{R}^2 \times \mathbb{C} \to \mathbb{R}^2$,

and the connection form $A = 2\pi i\, x\, dy$ having $(d+A)^2 = dA = 2\pi i\, dx\, dy$. The connection $d+A$ should yield a way to lift the vector fields ∂_x, ∂_y on $\mathbb{R}^2$ to operators D_x, D_y on the space of sections of the trivial line bundle satisfying Leibniz. What should this mean?



What you might do in order to understand what's true. You claim that a connection $d+A$ gives a way to lift tangent vectors in the base to tangent vectors in the total space. Hence vector fields in the base lift to vector fields in the total space. So a flow in the base yields a flow in the total space.

Unique lifting of tangent vectors in B to horizontal tangent vectors in the total space E, hence also unique lifting of vector fields in B to horizontal vector fields in E. Ditto flows in the base and horizontal flows in E.

Now you want to work this out explicitly for the connection $d + 2\pi i x dy$ on $\mathbb{R}^2 \times \mathbb{C}$.

$$\begin{array}{c}\mathbb{R}^2\times\mathbb{C}\\ \downarrow \\ \mathbb{R}^2\end{array}$$

Take a point $(x_0, y_0) \in \mathbb{R}^2$ and a tangent vector $a\partial_x + b\partial_y$ at this point, then make a curve yielding this tangent vector e.g. $(x(t), y(t)) = (x_0, y_0) + t(a, b)$. Next take a point (x_0, y_0, z_0) of $\mathbb{R}^2 \times \mathbb{C}$ over (x_0, y_0). You want a curve $(x(t), y(t), z(t))$ to first order over $(x(t), y(t))$ starting at (x_0, y_0, z_0), thus $(x(t), y(t), z(t)) = (x_0, y_0, z_0) + t(a, b, c)$, and c is to be found such that this curve in E is horizontal for the connection. Meaning? Pull back $\begin{array}{c}E\\ \downarrow\\ B\end{array}$ via curve $t \mapsto (x(t), y(t))$, then get $\begin{array}{c}\mathbb{R}\times\mathbb{C}\\ \downarrow\\ \mathbb{R}\end{array}$

curve in $\begin{array}{c}\mathbb{R}\times\mathbb{C}\\ \downarrow\\ \mathbb{R}\end{array}$ is $t, z(t)$ | connection in $d + 2\pi i\, x(t)\, dy(t)$

$d + 2\pi i(x_0 + ta)\, dt\, b$

and flat wrt conn. means

$$d\left(z_0 + tc\right) + 2\pi i\,(x_0 + ta)\, b\, dt = 0$$

$$dt\left(c + 2\pi i x_0 b\right) = 0 \qquad \frac{d}{dt} z(t) + 2\pi i\, x(t) \frac{d}{dt} y(t) = 0$$

Feb 7, 02 Yesterday you realized your problem is again relating different versions of connections. List

$d + A$ operators. Is this limited to vector bundles? No you have $[d + \Theta, -]$ Lie alg valued forms on the principal bundle - note that $(d+\Theta)^2 = [d, \Theta] + \frac{1}{2}[\Theta, \Theta]$ is the super bracket for an odd operator.

lifting of vector fields on the base to the total space

So how do you organize all this in the case of the trivial line bundle over $\mathbb{R}^2$, connection form $A = 2\pi i x\, dy$.

Feb 7, 02. In the example $\mathbb{R}^2$, ∂_x, ∂_y, $A = 2\pi i x dy$, what should happen? The infinitesimal vector fields $a\partial_x + b\partial_y$ should lift via the connection to an operator on sections ψ of the trivial line bundle of the form $a\nabla_x + b\nabla_y$ where $\nabla_{x,y} = \partial_{x,y} + $ 0th order

satisfy $[\,i(a\nabla_x + b\nabla_y), d+A\,] = 0$. $a\nabla_x + b\nabla_y$ should be equivalent to the vector field on the total space of the line bundle which is the lift of $a\partial_x + b\partial_y$. Not sorted yet.

Get formulas straight first. Philosophy: The connection $d+A$ should enable you to lift the vector fields ∂_x, ∂_y on the base $\mathbb{R}$ to horizontal vector fields on the principal bundle

$$\begin{array}{c}\mathbb{R}^2\times\mathbb{T}\\ \downarrow\\ \mathbb{R}^2\end{array}$$

Call these v.f.

$$X = \partial_x + f_1(x,y)\, 2\pi i z\partial_z \qquad 2\pi i z \partial_z = 2\pi\partial_\theta \qquad z = e^{i\theta}$$
$$Y = \partial_y + f_2(x,y)\, 2\pi i z\partial_z \qquad \frac{dz}{z} = i\,d\theta$$

The connection form Θ on $\mathbb{R}^2\times\mathbb{T}$ should be $\frac{dz}{2\pi i z} +$

Write up the action of $SL(2,\mathbb{Z})$ on the space $\mathcal{L}$ of sections of L. $\mathcal{L} = \{\psi(x,y) \in C^\infty(\mathbb{R}^2) \mid \psi(x+m, y+n) = e^{-2\pi i m y}\psi(x,y)\}$

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} \quad ?$$
$$(x,y) \longmapsto (y,-x)$$

$$g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} c_2 \\ -c_1 \end{pmatrix} \qquad pr_1\, g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = c_2 = pr_2$$

$$g : (x,y) \longmapsto (y,-x) \qquad pr_2\, g\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = -c_1 = -pr_1$$

$$g^*x = y$$
$$g^*y = -x \qquad (g^*\psi)(x,y) = \psi(y,-x).$$

Given $\psi_0 \in \mathcal{L}$ $\qquad \psi_1 = g^*\psi_0 = \psi_0(y,-x)$

$$\psi_1(x+m, y+n) = \psi_0(y+n, -x-m) = \underbrace{e^{+2\pi i n(x+m)}\psi_0(y,-x)}_{\psi_1(x,y)}$$

Now $\quad \psi_1(x+m, y+n) = e^{2\pi i x n}\psi_1(x,y)$.

$$\psi_2(x,y) = e^{-2\pi i x y}\,\psi_1(x,y)$$

$$\psi_2(x+m,y+n) = e^{-2\pi i(x+m)(y+n)}\,\psi_1(x+m,y+n)$$

$$= e^{-2\pi i(xy+my+\cancel{xn})}\,\cancel{e^{2\pi i n x}}\,\psi_1(x,y)$$

$$= e^{-2\pi i m y}\,\underbrace{e^{-2\pi i x y}\,\psi_1(x,y)}_{\psi_2(x,y)}$$

$$\psi_2(x,y) = e^{-2\pi i x y}\,\psi_0(y,-x)$$

$$\psi_2(x+m,y+n) = e^{-2\pi i \underset{xy+my+\cancel{nx}}{(x+m)(y+n)}}\,\underbrace{\psi_0(y+n,-x-\cancel{m})}_{\cancel{e^{2\pi i n x}}\,\psi_0(y,-x)}$$

$$= e^{-2\pi i m y}\,\underbrace{e^{-2\pi i x y}\,\psi_0(y,-x)}_{\psi_2(x,y).}$$

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} g^*x \\ g^*y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix} \qquad (g^*\psi)(x,y) = \psi(x+y,y)$$

Put $\quad \psi_1(x,y) = \psi(x+y,y)$

$$\psi_1(x+m,y+n) = \psi(x+y+m+n, y+\cancel{n})$$

$$= e^{-2\pi i(m+n)y}\,\psi(x+y,y)$$

Put $\quad \psi_2(x,y) = e^{h(y)}\,\psi(x+y,y)$

$$\psi_2(x+m,y+n) = e^{h(y+n)}\,\psi(x+y+m+n, y)$$

$$= e^{-2\pi i y m}\, e^{h(y+n)-h(y)}\, e^{-2\pi i(\cancel{m}+n)y}\,\underbrace{e^{h(y)}\psi(x+y,y)}_{\psi_2(x,y)}$$

$$h(y+\cancel{m}) - h(y) = 2\pi i n y$$

$$h(y) = \frac{y(y-1)}{2} \qquad \frac{1}{2}\left(\cancel{y^2} + 2yn + n^2 - \cancel{y} + n - \cancel{y^2} + \cancel{y}\right) = yn + \frac{n^2-n}{2}$$

$$\psi_1(x,y) = \psi_0(y,-x)$$

$$\psi_1(x+m, y+n) = \psi_0(y+n, -x \cancel{-m})$$

$$= e^{2\pi i n x}\psi_0(y,-x)$$

$$= e^{2\pi i n x}\psi_1(x,y)$$

$$\psi_2(x,y) = e^{h(x,y)}\psi_1(x,y)$$

put $\quad \psi_2(x+m, y+n) = e^{h(x+m,y+n)}\psi_1(x+m,y+n)$

$$\cancel{= e^{h(x+m,y+n) + 2\pi i n x - h(x,y)} e^{h(x,y)}\psi_1(x,y)}$$

~~so $\psi_2 \in \mathcal{L}$ provided~~ $\cancel{h(x+m,y+n) - h(x,y) = -2\pi i n x}$

$$\cancel{-2\pi i(xy + my + nx + mn) + 2\pi i xy}$$

$\psi_2 \in \mathcal{L}$ provided $\quad \psi_2(x+m,y+n) = e^{-2\pi i m y}\psi_2(x,y)$

$$e^{2\pi i m y}\psi_2(x+m,y+n) = e^{2\pi i m y + h(x+m,y+n)}\underbrace{\psi_1(x+m,y+n)}_{e^{2\pi i n x}\psi_1(x,y)}$$

$$= e^{\boxed{2\pi i m y + h(x+m,y+n) + 2\pi i n x - h(x,y)}}\Big(\underbrace{e^{h(x,y)}\psi_1(x,y)}_{\psi_2}\Big)$$

$$-h(x+m,y+n) + h(x,y) = 2\pi i(my + nx)$$

so $\quad h(x,y) = -2\pi i xy \quad$ works.

Iterate: given $\psi_0 \in \mathcal{L}$ let $\psi_1(x,y) = e^{-2\pi i xy}\psi_0(y,-x)$

$$\psi_2(x,y) = e^{-2\pi i xy}\psi_1(y,-x) = \cancel{e^{-2\pi i xy} e^{-2\pi i y(-x)}}\psi_0(-x,-y)$$

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+ty \\ y \end{pmatrix} \qquad \psi_1(x,y) = \psi_0(x+ty, y)$$

$$\psi_1(x+m, y+n) = \psi_0(x+m+ty+tm, y+n)$$

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ tx+y \end{pmatrix}$$

$$\psi_1(x,y) = \psi_0(x, tx+y) \qquad ?$$

$$\psi_1(x+m, y+n) = \psi_0(x+m, t(x+m)+y+n)$$

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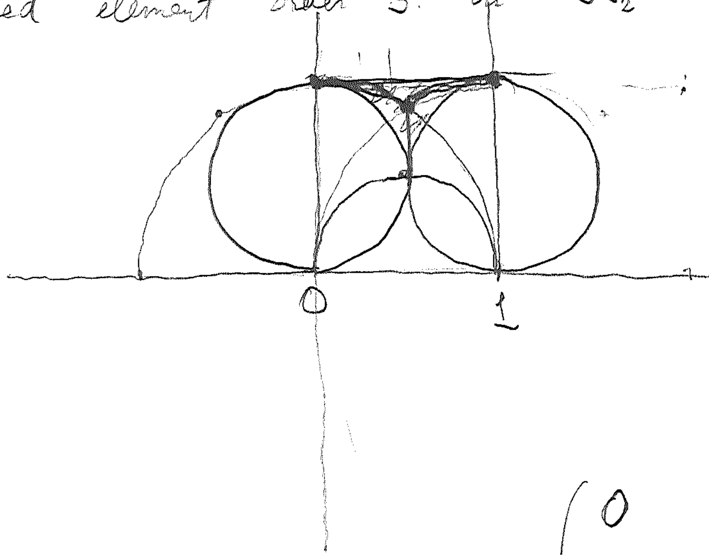
need element order 3. in SL_2

You want order 6.



$$\lambda^2+\lambda+1 = 0$$

You want g to permute

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{matrix} = \\ \infty & 0 & 1 \end{matrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ \phi \end{pmatrix}$$

$$\lambda^2 - \lambda + 1 \qquad \lambda = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$T^2 = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad T^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

You would like to construct an action of $SL(2,\mathbb{Z})$ on $\mathcal{L}$.

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad (g^*\psi)(x,y) = \psi(y, -x+y)$$

put $\boxed{\psi_1(x,y) = \psi_0(y, y-x)}$

$$\psi_1(x+m, y+n) = \psi_0(y+n, y+n-x-m)$$

$$= e^{-2\pi i n(y-x)} \psi_0(y, y-x)$$

$$\boxed{\psi_1(x+m, y+n) = e^{2\pi i n(x-y)} \psi_1(x,y).}$$

put $\boxed{\psi_2(x,y) = e^{h(x,y)} \psi_1(x,y)}$

$$\psi_2(x+m, y+n) = e^{h(x+m,y+n)} \psi_1(x+m, y+n)$$

$$= e^{h(x+m,y+n)} e^{2\pi i n(x-y)} \psi_1(x,y)$$

$$\psi_2(x+m,y+n) = \underbrace{e^{h(x+m,y+n) - h(x,y)} e^{2\pi i n(x-y)}}_{e^{-2\pi i m y}} \underbrace{e^{h(x,y)} \psi_1(x,y)}_{\psi_2(x,y)}$$

$$h(x+m,y+n) - h(x,y) + nx - ny - my$$

$$xy + my + xn + mn - xy + nx - ny - my$$

Express $\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$-\overset{②}{xy} - \overset{③}{xn} - \overset{①}{my} - mn + \overset{②}{xy} + \overset{③}{nx} - ny = -\overset{①}{my}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\psi_2(x,y) = e^{-2\pi i xy}\,\psi_1(x,y)$$

$$\psi_2(x+m,y+n) = e^{-2\pi i(x+m)(y+n)}\,\psi_1(x+m,y+n)$$

$$= e^{-2\pi i(xy+xn+my)}\,e^{2\pi i\{nx-ny\}+2\pi i(xy)}\left(e^{-2\pi i xy}\psi_1(x,y)\right)$$

$$\psi_2(x+m,y+n) = e^{-2\pi i(m+n)y}\,\psi_2(x,y)$$

put $\psi_3(x,y) = e^{k(y)}\,\psi_2(x,y)$

$$\psi_3(x+m,y+n) = e^{k(y+n)}\,\psi_2(x+m,y+n)$$

$$= e^{k(y+n)}\,e^{-2\pi i my}\,e^{-2\pi i ny}\,e^{-k(y)}\,\underbrace{e^{k(y)}\psi_2(x,y)}_{\psi_3(x,y)}$$

$$k(y+n)-k(y) = +2\pi i ny$$

$$k(y) = +2\pi i\,\frac{y^2-y}{2}$$

$$\frac{\overset{\cancel{y^2}+2ny+n^2}{(y+n)^2} - (\cancel{y}+n) - \cancel{y^2}+\cancel{y}}{2} = ny + \frac{n^2-n}{2}$$

So it seems that

too hard

$$\psi_3(x,y) = e^{+2\pi i\frac{y^2-y}{2}}\,\psi_2(x,y)$$

$$= e^{+2\pi i\frac{y^2-y}{2}-2\pi i xy}\,\psi_1(x,y)$$

$$\boxed{\psi_3(x,y) = e^{+2\pi i\frac{y^2-y}{2}-2\pi i xy}\,\psi_0(y,y-x)}$$

is also in $\mathcal{L}$.

$$\psi_3(x+m,y+n) = e^{+2\pi i\left(\frac{(y+m)^2-(y+m)}{2} - (x+m)(y+n)\right)}\,\psi_0(y+n,y+n-x-m)$$

etc.

$$e^{-2\pi i n(y-x)}\,\psi_0(y,y-x) \qquad e^{2\pi i\left(\frac{y^2-y}{2}+xy\right)}\,\psi_3(x,y)$$

$$\frac{(y+n)^2-(y+n)}{2} - \frac{y^2-y}{2} + (xy+my+xn+mn) - xy + ny - nx$$

$$\cancel{y^2}+2ny+n^2-\cancel{y}-n-\cancel{y^2}+\cancel{y}$$

leaving $\psi_3(x+m,y+n) = e^{-2\pi i my}\,\psi_3(x,y)$.

general formula. Given $g \in SL(2,\mathbb{Z})$ ~~it acts~~ should act on $\mathcal{L}$ consistent with its action on $Q = C^\infty(T^2)$ because there is only one line bundle of degree 1

IDEA – similarity between the action of $SU(1,1)$ as diffeos of the circle and the $SL(2,\mathbb{R})$ action you are trying to understand arising from $\frac{1}{2}D_x^2$, $\frac{1}{2}(D_xD_y + D_yD_x)$ etc.

New viewpoint. Begin with similarity:
unique irred repn. of the CCR.
unique line bundle + connection over $\mathbb{R}^2$ with constant curvature

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addition to yesterday's calculations. Do the exponential factors agree?

$$g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} \qquad -x\,dy + g^*(x\,dy) = (ax+by)(c\,dx + \delta\,dy) - x\,dy$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix} \begin{pmatrix} dx \\ dy \end{pmatrix} = d\left\{ \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right\}$$

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix} \simeq \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad -xy$$

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \frac{y^2}{2} \qquad \text{instead got } \frac{y(y-1)}{2}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \frac{x^2}{2} \qquad \frac{x(x-1)}{2}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} \qquad -xy + \frac{y^2}{2} \qquad \frac{y^2-y}{2} - xy$$

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad g^*\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x+y \end{pmatrix}$$

put $\psi_1(x,y) = (g^*\psi_0)(x,y) = \psi_0(y, y-x)$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\psi_1(x+m, y+n) = \psi_0(y+n, y+\cancel{n}-x-\cancel{m})$$

$$= e^{-2\pi i n(y-x)}\psi_0(y, y-x)$$

$$\psi_1(x+m, y+n) = e^{-2\pi i(ny-nx)}\psi_1(x,y)$$

put $\psi_2(x,y) = e^{-2\pi i xy}\psi_1(x,y)$

$$\psi_2(x+m, y+n) = e^{-2\pi i(xy+xn+my)}\psi_1(x+m, y+n)$$

$$= e^{-2\pi i(xy+\cancel{xn}+my+ny-\cancel{nx})}\psi_1(x,y)$$

$$\psi_2(x+m, y+n) = e^{-2\pi i(m+n)y}\psi_2(x,y)$$

put $\psi_3(x,y) = e^{-2\pi i(\frac{y^2-y}{2})}\psi_2(x,y)$

$$\psi_3(x+m, y+n) = e^{-2\pi i(\frac{(y+m)^2-(y+m)}{2} - \frac{y(y-1)}{2})} e^{-2\pi i(\frac{y^2-y}{2})}\psi_2(x,y)$$

$$\psi_3(x+m, y+n) = e^{-2\pi i[\frac{y^2+2yn+n^2-y-n}{2}]}\psi_2(x+m, y+n)$$

$$= e^{-2\pi i[\frac{y^2+2yn+n^2-y-n}{2} + (m+n)y - \frac{y^2-y}{2}]}\psi_3(x,y)$$

put $\psi_3(x,y) = e^{-2\pi i h(y)}\psi_2(x,y)$

$$\psi_3(x+m, y+n) = e^{-2\pi i h(y+n)}\psi_2(x+m, y+n)$$

$$= e^{-2\pi i(h(y+n)-h(y))} e^{-2\pi i(m+n)y}\psi_2(x,y)\, e^{-2\pi i h(y)}$$

$$\psi_3(x+m, y+n) = e^{-2\pi i[h(y+n)-h(y)+(m+n)y]}\psi_3(x,y)$$

$$= e^{-2\pi i my}\psi_3(x,y)\, e^{-2\pi i[h(y+n)-h(y)+ny]}$$

$$\frac{1}{2}\left((y+n)^2 - (\cancel{y}+n) - \cancel{y^2} + \cancel{y}\right) = yn + \frac{n^2-n}{2} \qquad h(y) = -\frac{y^2-y}{2}$$

$$\cancel{y^2} + 2yn + n^2$$

$$\psi_3(x,y) = e^{+2\pi i\left[+\frac{y^2-y}{2} - xy\right]}\psi_0(y, y-x)$$

Let's compute powers. If $\psi \in \mathcal{L}$

test $(T\psi)(x,y) = e^{-2\pi i xy}\psi(y,-x)$ $\qquad \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$$\bigl(T(T\psi)\bigr)(x,y) = e^{-2\pi i xy}(T\psi)(y,-x)$$

$$= e^{-2\pi i xy} e^{-2\pi i y(-x)}\psi(-x,-y) = \psi(-x,-y)$$

$$\psi_1(x,y) = e^{2\pi i\left(\frac{y^2-y}{2} - xy\right)}\psi_0(y, y-x)$$

$$\psi_2(x,y) = e^{2\pi i\left(\frac{y^2-y}{2} - xy\right)}\psi_1(y, y-x)$$

$$\psi_1(y, y-x) = e^{2\pi i\left[\frac{(y-x)^2-(y-x)}{2} - y(y-x)\right]}\psi_0(y-x, \cancel{(y-x)}-y)$$

$$(g^*\psi_0)(x,y) = \psi_0(y, y-x)$$

$$g^*(g^*\psi_0)(x,y) = (g^*\psi_0)(y, y-x) = \psi_0(y-x, -x)$$

$$g^*\psi_0(y-x,-x) = \psi_0(-x, \cancel{-x-(y-x)}) = \psi(-x,-y).$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \cancel{\begin{pmatrix} 0 & 1 \\ -1 & +1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}} =$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x+y \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}(z) = \frac{1}{1-z} = z$$

$$z^2 - z + 1 = 0$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} y \\ -x+y \end{pmatrix} = \begin{pmatrix} -x+y \\ -y-x+y \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} -x+y \\ -x \end{pmatrix} = \begin{pmatrix} -x \\ +x-y-x \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

Feb 8, 02: Consider $\mathbb{R}^2$, the trivial line bundle
$L = \mathbb{R}^2 \times \mathbb{C} \longrightarrow \mathbb{R}^2$ equipped with connection $d + 2\pi i x\,dy$,
A section of L is a smooth fn $\psi(x,y)$. The connection gives a "horizontal" subbundle in the bundle of 1-jets $J_1 L$, i.e. a splitting of

$$0 \longrightarrow T^* \otimes L \longrightarrow J_1 L \longrightarrow L \longrightarrow 0$$

at a point ξ of $\mathbb{R}^2$ the fibre of $J_1 L$ has basis $1, dx, dy$. Given $\psi(x,y)$ diff its 1-jet is

$(j_1\psi)(\xi) \quad \psi + dx\, \partial_x \psi + dy\, \partial_y \psi$ evaluated at ξ

$$\begin{tikzcd} 0 \arrow[r] & T^*\otimes L \arrow[r] \arrow[dr, "id"'] & J_1 L \arrow[r] \arrow[d, "g"] & L \arrow[r] \arrow[l, bend right, "j_1"'] \arrow[dl, "d+A"] & 0 \\ & & T^*\otimes L & & \end{tikzcd}$$

$$g\, j_1(\psi) = (d+A)\psi$$

You have coords. A point of $J_1 L_\xi$ is a function mod 2nd order

$$c_0 + c_1(x - x_0) + c_2(y - y_0)$$

$$j_1\psi_{(x_0,y_0)} = \psi(x_0,y_0) + \frac{\partial\psi}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial\psi}{\partial y}(x_0,y_0)(y-y_0)$$

$$(d+A)\psi = dx\, \partial_x\psi + dy\left(\partial_y\psi + 2\pi i x_0\right)$$

$$g\left(c_0 + c_1(x-x_0) + c_2(y-y_0)\right) =$$

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point of $J_1 L$ over $\xi = (x_0, y_0)$ is $c_0 + c_1(x-x_0) + c_2(y-y_0)$

e.g. $j_1\psi(\xi) = \psi(\xi) + \partial_x\psi(\xi)(x-x_0) + \partial_y\psi(\xi)(y-y_0)$.

$$(d+A)\psi = 2\pi i x\,dy\,\psi + \partial_x\psi\, dx + \partial_y\psi\, dy$$
$$= \partial_x\psi\, dx + (\partial_y + 2\pi i x)\psi\, dy$$

$$g\left(c_0 + c_1(x-x_0) + c_2(y-y_0)\right) = 2\pi i x_0 c_0 + c_1 dx + c_2 dy$$

horizontal subbundle of $J_1 L$ is the kernel of g; it's isom to L. Why are you concerned about horizontal? Because $\mathrm{Ker}(g)$ consists of 1-jets of sections: $\psi + (\partial_x \psi)(x-x_0) + (\partial_y \psi)(y-y_0)$ such that

$(d+A)\psi = 0$ to first order at (x_0, y_0).

$$d + A = dx\, \partial_x + dy\,(\partial_y + 2\pi i x)$$

Important point: operators "commuting" wrt $\partial_x, \partial_y + 2\pi i x$ whatever this means. I think that this should include the parallel transport operators obtained by lifting vector fields on the base which preserve the volume form, i.e. symplectic vector fields.

Ex:

$$\begin{bmatrix} D_x = \partial_x & & \partial_x + 2\pi i y \\ & , & \\ D_y = \partial_y + 2\pi i x & & \partial_y \end{bmatrix} = 0$$

The operators $\partial_x + 2\pi i y$, ∂_y arise in the automorphic condition for sections ψ of $L \to T^2$:

$$e^{s(\partial_x + 2\pi i y)} e^{t\partial_y} \psi(x,y) = e^{s 2\pi i y} e^{s\partial_x} e^{t\partial_y} \psi(x,y)$$
$$= e^{2\pi i s y} \psi(x+s, y+t)$$

Consider $\mathcal{S}(\mathbb{R}^2)$ with the above $\overset{\text{four}}{}$ operator. This amounts to commuting actions of two Heisenberg groups, one arising from D_x, D_y and the other from $\partial_x + 2\pi i y, \partial_y$. You expect that $\mathcal{S}(\mathbb{R})$ splits into the tensor product of the basic representations of these two Heisenberg groups.

Idea: What is the assembly construction for $\mathbb{R}$? The group ring of $\mathbb{R}$ is the C^*-alg of continuous fns. on $\mathbb{R}^\vee = \mathbb{R}$ vanishing at ∞. You need some analog of the universal bundle. This seems to be

$\mathbb{R}$ acting on itself by translation with the base being a point. So $B\mathbb{R} = pt$. Is there something tricky you can do, e.g. involving proper actions, almost periodic functions?

Idea: Think of a limiting process involving the assembly construction for $\mathbb{Z}$. The torus should have the x circle getting tighter and the y circle getting longer: $\mathbb{R}/\mathbb{Z}L^{-1} \times \mathbb{R}/L\mathbb{Z}$; as $L \to 0$ this seems to approach $pt \times \mathbb{R}$. I think you want to focus on the whether the assembly module has a limit, some sort of K-type class.

Ideas that come to mind: 1) Lattices of volume 1 in $\mathbb{R}^2$ degenerating to a Lagrangian subspace (i.e. any line in $\mathbb{R}^2$)

2) Super Lie gp $\mathbb{R}^{1|1}$ whose Lie superalg is $\mathbb{R}X + \mathbb{R}X^2$ the free Lie superalgebra generated by an odd element X.

3) The assembly module is a module over the tensor product of functions of x "$\in$ classifying space", and the group algebra of $G = \mathbb{R}$ which by FT is functions $y \in$ dual group $\hat{G} = \mathbb{R}$. It would seem that the only functions of x surviving the limit should be $1, \delta_x$ where δx should be paired with the infinitesimal translation ∂_x on the group alg.) meaning mult by $2\pi i y$ on functions of y. So you might be able to find an interesting summand of a free module over $\Omega(\mathbb{R})$

Explore possible link between uniqueness of irred repn. of CCR and the uniqueness of line bundle + connection over $\mathbb{R}^2$ such that curvature is translation invariant volume. Consider over $\mathbb{R}^2$ the trivial line bundle $\mathbb{R}^2 \times \mathbb{C} \xrightarrow{pr_1} \mathbb{R}^2$, conn. form $A = 2\pi i x\,dy$, curvature $(d+A)^2 = dA = 2\pi i\, dx\,dy$.

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix}\;,\;\begin{matrix} \partial_x + 2\pi i y \\ \partial_y \end{matrix}\right] = 0$$

D_x, D_y generate a Heisenberg group representation on $\mathcal{S}(\mathbb{R}^2)$, same for $\partial_x + 2\pi i y$, ∂_y, these actions commute so one expects $\mathcal{S}(\mathbb{R}^2) \simeq \mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R})$, tensor product of the two basic repns of the two Heisenberg groups. You expect that $\partial_x + 2\pi i y$; ∂_y, because they commute with $d+A$, should be parallel transport operators on the trivial line bundle with connection $d+A$. Evidence for this comes from the fact that translations

$$e^{m(\partial_x + 2\pi i y)} e^{n(\partial_y)} \psi(x,y) = e^{2\pi i m y} \psi(x+m, y+n)$$

with integral components yield the automorphic conditions for $\mathcal{L}$

The four operators $\begin{matrix} \partial_x & \partial_x + 2\pi i y \\ \partial_y + 2\pi i x & \partial_y \end{matrix}$ generate the same space as $\partial_x, \partial_y, 2\pi i x, 2\pi i y$ so $\mathcal{S}(\mathbb{R}^2)$ is an irred rep.

Apply uniqueness of the basic repn of CCR in simple cases. Keep examples simple. $\left[\begin{pmatrix} 0 & +1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$D_x = \partial_x \qquad [D_x, D_y] = 2\pi i.$$
$$D_y = \partial_y + 2\pi i x$$

Generators for $SL(2,\mathbb{R})$. Lie alg. basis $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$D_x = \frac{d}{dx} \qquad D_y = 2\pi i x$$

Let's try implementing the symmetry on $\mathcal{S}(\mathbb{R})$.

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_x^2, \begin{pmatrix} D_x \\ D_y \end{pmatrix}\right] = \begin{pmatrix} 0 \\ +D_x \end{pmatrix} \qquad \left[\frac{1}{2\pi i}\frac{1}{2}(D_xD_y + D_yD_x), \begin{pmatrix} D_x \\ D_y \end{pmatrix}\right] = \begin{pmatrix} -D_x \\ D_y \end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_y^2, \begin{pmatrix} D_x \\ D_y \end{pmatrix}\right] = \begin{pmatrix} -D_y \\ 0 \end{pmatrix} \qquad (D_x \ D_y)\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = (0 \ -D_x)$$

$$\frac{1}{2\pi i}\frac{1}{2}D_y^2 = \frac{1}{2(2\pi i)}(2\pi i x)^2 = \pi i x^2$$

So $f(x) \longmapsto e^{-t\pi i x^2} f(x)$ should be the unitary transf on $L^2(\mathbb{R})$ implementing $\begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \in SL(2,\mathbb{R})$

Check this $e^{t\pi i x^2} \partial_x e^{-t\pi i x^2} f(x)$?

$$[\pi i x^2, \partial_x] = \partial_x - 2\pi i x$$

$$\cancel{\left[\frac{1}{2}D_x^2, \frac{1}{2}D_y^2\right] = \left[D_x, \frac{1}{2}D_y^2\right]D_x}$$

$$[D_x^2, D_y^2] = [D_x^2, D_y]D_y + D_y[D_x^2, D_y]$$

$$= \left([D_x, D_y]D_x + D_x[D_x, D_y]\right)D_y$$

$$+ D_y\left([D_x, D_y]D_x + D_x[D_x, D_y]\right)$$

$$= 2\pi i\, 2D_x D_y + 2\pi i\, 2 D_y D_x$$

$$\left[\tfrac{1}{2}D_x^2, \tfrac{1}{2}D_y^2\right] = 2\pi i \cdot 2\,(D_x D_y + D_y D_x)$$

D_x, D_y are skew adjoint ?

D_x^2

but what about

$\partial_x, \; 2\pi i x$

$$\frac{1}{2}\left(\frac{1}{2\pi i}\partial_x\right)^2 + \frac{1}{2}(x)^2$$

$$\frac{1}{2\pi i}(\partial_x x + x\partial_x)$$

you're being stupid, you are studying the Lie alg of the symplectic group $SL_2(\mathbb{R})$, which you know can be identified with the space of real quadratic forms in dim 2. What do you know about such things. Sylvester signature. $++$, $+-$, $--$

Look at $SU(1,1)$

Let us try for a better understanding. What is your essential goal? To understand properly the similarity between the uniqueness of the basic repn of the CCR and the uniqueness of the line bundle + connection over $\mathbb{R}^2$ having translation invariant curvature $2\pi i\, dx\, dy$

You have a model for the basic repn. of the CCR namely smooth sections of the line bundle when descended appropriately to T^2.

You would like to construct the metaplectic $SL(2,\mathbb{R})$ repn on the basic Heisenberg rep using sections

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Feb 10, 02 Problem: Does $SL(2,\mathbb{Z})$ act on $\mathcal{L}$?

Go back to
$$D_x = \partial_x \qquad [D_x, D_y] = 2\pi i$$
$$D_y = \partial_y + 2\pi i x$$

other set $\partial_x + 2\pi i y$, ∂_y commuting with D_x, D_y

Think of these operators acting on $\mathcal{S}(\mathbb{R}^2)$, put them together to get the basic Heisenberg rep 2 degrees of freedom. Use the second set, rather

$$e^{m(\partial_x + 2\pi i y)} e^{n\partial_y} \psi(x,y) = e^{2\pi i m y} \psi(x+m, y+n)$$

to take quotient by $\mathbb{Z}^2$ on $\mathbb{R}^2$

Puzzle about quotient of $\mathcal{S}(\mathbb{R})$ by $\mathbb{Z}^2$, which acts by $e^{m(\partial_x) + n(2\pi i x)}$, making $\mathcal{S}(\mathbb{R})$ into a module over $C^\infty(T^2)$, the module of sections of $L \to T^2$. Do spectral analysis, this means "decompose" $\mathcal{S}(\mathbb{R})$ into irred reps of $\mathbb{Z}^2$, equivalently look at the fibers of L.

Spectral analysis should associate to each element of $\mathcal{S}(\mathbb{R})$ a transform which is a section of L over T^2. What is the puzzle? The fibre of L at $(\bar{x},\bar{y}) \in T^2$ should be

$$\mathcal{S}(\mathbb{R}) / I_{\bar{x}\bar{y}}\, \mathcal{S}(\mathbb{R}) \qquad I_{\bar{x}\bar{y}} = \text{smooth functions on } T^2 \text{ vanishing at } \bar{x}\bar{y}.$$

This appears to be a quotient by an action of the group $\mathbb{Z}^2$. But our equivalence $\mathcal{S}(\mathbb{R}) \simeq \mathcal{L}$ is a quotient by an action of $\mathbb{Z}$:

$$\mathcal{F}(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m) = \sum_m e^{m(\partial_x + 2\pi i y)} f(x).$$

Explore the commuting Heisenberg representations

$$\begin{bmatrix} D_x = \partial_x & , & \partial_x + 2\pi i y \\ D_y = \partial_y + 2\pi i x & , & \partial_y \end{bmatrix} = 0.$$

$$e^{-2\pi i x y}\partial_y e^{2\pi i x y} = \partial_y + 2\pi i x$$
$$e^{-2\pi i x y}\partial_x e^{2\pi i x y} = \partial_x + 2\pi i y$$

Look at these operator on $\mathcal{S}(\mathbb{R}^2)$. Another basis is $\partial_x, 2\pi i x, \partial_y, 2\pi i y$ so you have the basic Heisenberg rep of dim 2. Aim: To construct the natural action (projective) of $SL(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R})$ which arises from $SL(2,\mathbb{R})$ acting on the Heisenberg group (inf. generators $\frac{d}{dx}, 2\pi i x$) and the uniqueness of the basic repn.

Let's study the "classical" approach based on Gaussians and F.T. $f(x) \in \mathcal{S}$ $\quad \hat{f}(y) = \int e^{-2\pi i x y} f(x)\, dx$

$f(x) = \int e^{2\pi i x y} \hat{f}(y)\, dy$. $\quad Q =$ quadratic in x, y

$T_Q f = \int e^{iQ(x,y)} \hat{f}(y)\, dy$?

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contact transf. $\quad L = L(q, \dot{q})$

$$\delta \int_a^b L(q,\dot{q})\, dt = \int_a^b \left(\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} \right) dt = 0$$

$$= \left[\frac{\partial L}{\partial \dot{q}} \delta q \right]_a^b + \int_a^b \left(-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \right) \delta q \, dt = 0$$

Lagrange eqns. $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \qquad L(q,\dot{q})$

form Legendre transf. $\quad p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q}) \qquad p\dot{q} - L = H(q,p)$

Start with Lagrange $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$

Put $p = \frac{\partial L}{\partial \dot{q}}(q,\dot{q})$ $\dot{q}$ function of p, q.

$$H(q,p) = p\dot{q} - L(q,\dot{q})$$

$$\frac{\partial H}{\partial q} = \cancel{p\frac{\partial \dot{q}}{\partial q}} - \frac{\partial L}{\partial q} - \cancel{\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q}} = -\frac{d}{dt}p$$

$$\frac{\partial H}{\partial p} = \dot{q} + \cancel{p\frac{\partial \dot{q}}{\partial p}} - \cancel{\frac{\partial L}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p}}$$

This stuff concerns dynamics but really you want to understand all quadratic Hamiltonians.

Let's see if there is something you can accomplish with the commuting Heisenberg operators

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix}, \begin{matrix} \partial_x + 2\pi i y \\ \partial_y \end{matrix}\right] = 0$$

These operators act on $C^\infty(\mathbb{R}^2)$ and exponentiate nicely to translation operators with imaginary exponential multipliers. local operators defined on germs.

Now restrict to $\mathcal{S}(\mathbb{R}^2)$. You think that these commuting Heisenberg repns on $\mathcal{S}(\mathbb{R}^2)$ should lead to an isomorphism (non obvious) $\mathcal{S}(\mathbb{R}) \otimes \mathcal{S}(\mathbb{R}) \xrightarrow{\sim} \mathcal{S}(\mathbb{R}^2)$ where $\frac{d}{dx}$, $2\pi i x$ on the first $\mathcal{S}(\mathbb{R})$ are D_x, D_y and the second " is $\partial_x + 2\pi i y, \partial_y$.

Why should this be true? Apply $\sum_{m,n} e^{m(\partial_x + 2\pi i y)} e^{n\partial_y}$.

This should somehow amount to a "projection" operator, projection onto $\mathbb{C}$ summing, trace. What you've done should be a 2 diml version of $\mathcal{S}(\mathbb{R}) \longrightarrow C^\infty(T)$ $f(x) \longmapsto \sum_{m\in\mathbb{Z}} f(x+m)$

~~A basic fact about the Heisenberg Lie alg: $[X,Y]=H$, $[X,H]=[Y,H]=0$ is that it ...~~

A basic fact about a Heisenberg repn, e.g. $[D_x, D_y] = 2\pi i$ is that adjoining squares of linear operators $aD_x + bD_y$ yields a Lie algebra

Feb 11, 02

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_x^2, (D_x \quad D_y)\right] = (0 \quad D_x) = (D_x \quad D_y)\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_y^2, (D_x \quad D_y)\right] = (-D_y \quad 0) = (D_x \quad D_y)\begin{pmatrix}0 & 0\\ -1 & 0\end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}(D_xD_y + D_yD_x), (D_x \quad D_y)\right] = (-D_x \quad D_y) = (D_x \quad D_y)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$$

$$\frac{1}{2\pi i}\frac{1}{2}\left[D_x^2, D_y^2\right] = \frac{1}{2\pi i}\frac{1}{2}\left([D_x^2, D_y]D_y + D_y[D_x^2, D_y]\right) = D_xD_y + D_yD_x$$

Look at $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$ the components of the connection $d + 2\pi i x dy$ on the trivial line bundle over $\mathbb{R}^2$. You know how to exponentiate $aD_x + bD_y$, using the existence of flows for vector fields. But what about exponentiating quadratic operators, precisely you want e^{itQ} Q real ~~[illegible]~~ symmetric bilinear forms, function of D_x, D_y

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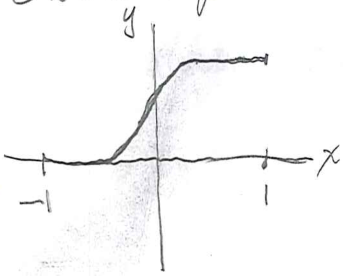
Idea Degeneration of a volume 1 lattice to a line. Can you find an analog of the unitary equivalence between $L^2(S^1)$ and $L^2(\mathbb{R})$ you obtained by making the constant grid space continuous in one direction. Recall that your pictures of $L^2(S^1)$ and $L^2(\mathbb{R})$ involved spinors - sections of $\mathcal{O}(-1)$ over $\mathbb{RP}^1$'s in $\mathbb{CP}_1$. You expect the metaplectic picture to be harder.

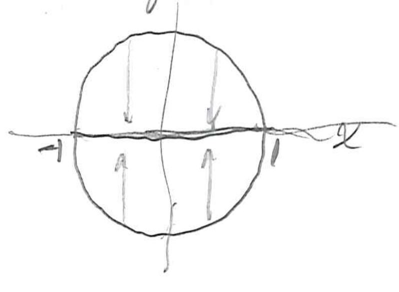
Feb 11, 02 continue with the analogy between uniqueness of the basic rep of the CCR (or Heisenberg group) and the uniqueness of the line bundle + connection over $\mathbb{R}^2$ with curvature $2\pi i\, dx dy$.

You are beginning to think that the latter can be viewed ~~as~~ ~~maybe~~ as a space of kernels for operators on the basic repn, or maybe that the line bundle picture is geometric quantization.

Obvious partition of 1 on the circle: Pull back the function



on $[-1, 1]$ via the x coord: call this χ_1, then let $\chi_{-1} = 1 - \chi_1$.



Go back to $D = d + A = d + 2\pi i x dy$ on the trivial bundle over $\mathbb{R}^2$. Proof that $SL(2,\mathbb{R})$ acts projectively ~~at least~~ on the line bundle + connection.

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad x \downarrow\downarrow y \quad \mathbb{R} \qquad \begin{aligned} g^*x &= xg = ax+by \\ g^*y &= yg = cx+dy \end{aligned}$$

$$g^*(d+A)\psi = g^*(d\psi) + g^*(A\psi) = d\, g^*\psi + g^*A\, g^*\psi \overset{\shortparallel}{=} (d+g^*A)\, g^*\psi$$

$$g^*A - A = 2\pi i (ax+by)(c\,dx + \delta\,dy) - 2\pi i\, x\,dy$$

$$\overset{\shortparallel}{} (x \ \ y)\begin{pmatrix} ac & a\delta - 1 \\ bc & b\delta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix} = d\left\{ \underbrace{\tfrac{1}{2}\begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{h} \right\}$$

$$g^*\left[(d+A)\psi\right] = (d + A + dh)\, g^*\psi = e^{-h}(d+A)e^{h} g^*\psi$$

$$\boxed{(e^h g^*)(d+A) = (d+A)(e^h g^*)}$$

$$\boxed{(e^h g^*)\psi(x,y) = e^{h(x,y)}\psi(ax+by,\ cx+\delta y)}$$

this is an operator on $C^\infty(\mathbb{R}^2)$. There is an extension of it to $\Omega(\mathbb{R}^2)$, to $\Omega^1(\mathbb{R}) = \Omega^0(\mathbb{R})\,dx + \Omega^0(\mathbb{R})\,dy$

Feb 12, 02 Work out examples. Infinitesimal action

$$g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix} = \begin{pmatrix} 1+\delta a & \delta b \\ \delta c & 1-\delta a \end{pmatrix} = \begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1-\varepsilon a \end{pmatrix}$$

$$\psi\big((1+\varepsilon a)x + \varepsilon b y,\ \varepsilon c x + (1-\varepsilon a)y\big)$$

$$= \psi\big(x + \varepsilon(ax+by),\ y + \varepsilon(cx - ay)\big)$$

$$= \psi(x,y) + \varepsilon\,\partial_x\psi(x,y)(ax+by) + \varepsilon\,\partial_y\psi(x,y)(cx-ay)$$

$$= \psi(x,y) + \varepsilon\Big[ax\partial_x\psi(x,y) + by\partial_x\psi(x,y) + c\,x\partial_y\psi(x,y) - ay\partial_y\psi(x,y)\Big]$$

$$= \psi(x,y) + \varepsilon\big(a(x\partial_x - y\partial_y) + by\partial_x + c\,x\partial_y\big)\psi(x,y)$$

$$(1+\varepsilon h)(1+\varepsilon X)\psi(x,y) \doteq \psi(x,y) + \varepsilon(h+X)\psi(x,y)$$

$$\frac{1}{\varepsilon}\;\frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}\begin{pmatrix}(1+\varepsilon a)\varepsilon c & \varepsilon b\varepsilon c\\ \varepsilon b\varepsilon c & \varepsilon b(1-\varepsilon a)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\begin{pmatrix}x\\y\end{pmatrix}\begin{pmatrix}(1+\varepsilon a)c & \varepsilon bc\\ \varepsilon bc & b(1-\varepsilon a)\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\to \frac{1}{2}(cx^2+by^2)$$

So it should be true that

$$x\partial_x - y\partial_y\,,\quad y\partial_x + \pi i y^2\,,\quad x\partial_y + \pi i x^2$$

commute with $d+A$

$$[x\partial_x - y\partial_y,\ \cancel{d} + 2\pi i x\,dy] = 2\pi i x\,dy + 2\pi i x(-y) = 0$$

$$[y\partial_x + \pi i y^2,\ d + 2\pi i x\,dy] = 2\pi i y\,dy - 2\pi i y\,dy = 0$$

$$[x\partial_y + \pi i x^2,\ d + 2\pi i x\,dy] = 2\pi i x\,dx - 2\pi i x\,dx = 0$$

$$[x\partial_x - y\partial_y,\ y\partial_x + \pi i y^2] = [x\partial_x, y\partial_{\cancel{x}}] - [y\partial_y, y\partial_x] - [y\partial_y, \pi i y^2]$$
$$= -2(y\partial_x + \pi i y^2)$$

$$[x\partial_x - y\partial_y,\ x\partial_y + \pi i x^2] = 2(x\partial_y + \pi i x^2)$$

$$[x\partial_y + \pi i x^2,\ y\partial_x + \pi i y^2] = x\partial_x - y\partial_y$$

Discuss the situation. You are looking at $\mathbb{R}^2$, the trivial line bundle over $\mathbb{R}$, the connection form $A = 2\pi i x\,dy$, with curvature $2\pi i\,dx\,dy$. Have $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ acting by diffeos. on $\mathbb{R}^2$ preserving curvature, you can lift this action to the line bundle up to scalars, so you get a central extension of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ by $\mathbb{T}$ on this line bundle + connection covering the action below on $\mathbb{R}^2$

You have two commuting Heisenberg repns.

$$\begin{bmatrix} D_x = \partial_x & & \partial_x + 2\pi i y \\ & , & \\ D_y = \partial_y + 2\pi i x & & \partial_y \end{bmatrix} = 0$$

You need understanding. ~~The main object is~~

IDEA: What is the UHP analog of what you are doing? You can again consider the trivial line bundle ~~with~~ over the UHP equipped with connection form having constant curvature. Note that there is a Riem. metric which is preserved. It seems you have only $SL(2,\mathbb{R})$ acting, no group $\mathbb{R}^2$ of translations.

Back to the trivial bundle over $\mathbb{R}^2$ with the commuting Heisenberg reps.

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix} , \begin{matrix} \partial_x + 2\pi i y \\ \partial_y \end{matrix}\right] = 0$$

on functions on $\mathbb{R}^2$. These operators can be exponentiated. You think that can also exponentiate the quadratic operators from both sides.

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_x^2 , (D_x \ D_y)\right] = (0 \ D_x) = (D_x \ D_y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}(D_xD_y + D_yD_x), (D_x \ D_y)\right] = (-D_x \ D_y) = (D_x \ D_y)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\left[\frac{1}{2\pi i}\frac{1}{2}D_y^2 , (D_x \ D_y)\right] = (-D_y \ 0) = (D_x \ D_y)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

What is happening??

Source of confusion: D_x, D_y are the components of the connection $d+A$. You want to understand operators $e^h g^*$ which preserve, respect the connection: $(d+A)(e^h g^*) = (e^h g^*)(d+A)$ so the operators $e^h g^*$ should be obtained from $\begin{matrix}\partial_x + 2\pi i y \\ \partial_y\end{matrix}$ and quadratics. $\left(a(\partial_x + 2\pi i y) + b\partial_y\right)^2$

Feb. 13, 02 You have ~~to~~ clean up what you've done. You have a geometric quantization picture, where the symmetries are "geometric" autos, i.e. given a diffeom of the base followed by a gauge transf. It seems that you get 2 commuting geometric projective actions of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ in this way. This is not surprising because you have $Sp(4,\mathbb{R}) \ltimes \mathbb{R}^4$ acting on $\mathcal{S}(\mathbb{R}^4)$. Question: Look at the metaplectic representation of $SL(2,\mathbb{R}) = Sp(2,\mathbb{R})$ on $\mathcal{S}(\mathbb{R})$. Is there a natural class of elements of $SL(2,\mathbb{R})$ whose effect on $\mathcal{S}(\mathbb{R})$ are geometric? Exs

$$e^{-\pi i t x^2}\begin{pmatrix} \frac{d}{dx} & 2\pi i x \end{pmatrix} e^{\pi i t x^2} = \begin{pmatrix} \frac{d}{dx} + 2\pi i t x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{d}{dx} & 2\pi i x \end{pmatrix}\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

This is geometric: Id diffeom on $\mathbb{R}$ + gauge transf.

$$e^{-tx\frac{d}{dx}}\begin{pmatrix} \frac{d}{dx} & 2\pi i x \end{pmatrix} e^{+tx\frac{d}{dx}} = \ ?$$

$$e^{tx\frac{d}{dx}} f(x) = f\left(e^{tx\frac{d}{dx}} x\right) = f(e^t x)$$

$$e^{tx\frac{d}{dx}} x = x + tx\frac{d}{dx}x + \frac{t^2}{2}\left(x\frac{d}{dx}\right)^2 x + \cdots = x + tx + \frac{t^2}{2}x + \cdots = e^t x$$

$$e^{-tx\frac{d}{dx}}(2\pi i x)e^{tx\frac{d}{dx}} f(x) = e^{-tx\frac{d}{dx}}\left(2\pi i x f(e^t x)\right) = 2\pi i e^{-t} x f(x)$$

$$e^{-tx\frac{d}{dx}} \frac{d}{dx} e^{tx\frac{d}{dx}} f(x) = e^{-tx\frac{d}{dx}} \frac{d}{dx}\left(f(e^t x)\right) = e^{-tx\frac{d}{dx}}\left(\frac{df}{dx}(e^t x)\, e^t\right)$$

$$= e^t \frac{df}{dx}(x) = e^t \frac{d}{dx} f(x)$$

$$e^{-tx\frac{d}{dx}}\begin{pmatrix} \frac{d}{dx} & 2\pi i x \end{pmatrix} e^{tx\frac{d}{dx}} = \begin{pmatrix} e^t\frac{d}{dx} & e^{-t} 2\pi i x \end{pmatrix} = \begin{pmatrix} \frac{d}{dx} & 2\pi i x \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

So what comes next? You have found that it might be possible to have two commuting geometric projective actions of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on the sections of the line bundle.

Repeat calculation so far. $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \longmapsto \begin{pmatrix} ac_1 + bc_2 \\ cc_1 + dc_2 \end{pmatrix}$

$$g^*(x) = ax + by \qquad g^*\psi(x,y) = \psi(ax+by,\ cx+dy)$$
$$g^*(y) = cx + dy \qquad A = d + 2\pi i x\,dy \qquad dA = 2\pi i\, dx\,dy$$

$$g^*(d + A)\psi = d\, g^*\psi + (g^*A)(g^*\psi)$$
$$= (d + A + dh)\, g^*\psi$$
$$e^h g^*\big((d+A)\psi\big) = (d+A)\left(e^h g^*\psi\right)$$

Somehow you have to organize the operators properly. Why not begin with the translation ops.

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix} \quad , \quad \begin{matrix} \partial_x + 2\pi i y \\ \partial_y \end{matrix}\right] = 0$$

What you see here is 4 diffle operators on sections ψ of the trivial linear bundle $\mathbb{R}^2 \times \mathbb{C} \longrightarrow \mathbb{R}^2$; each is a vector field on the base $\mathbb{R}^2$ lifted 'horizontally' + a mult. op. You have 2 different liftings of ∂_x, ∂_y. You have $[D_x, D_y] = 2\pi i$ $\qquad [\nabla_x, \nabla_y] = -2\pi i$.

Problem: To construct proj repn of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ extending each of these.

Aim: There should be two commuting projective representations of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on sections of L which both lift the obvious repn of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ on $\mathbb{R}^2$. This must be wrong ? ?

Hope: The problem might be resolved by keeping track of left and right actions. Let's make precise the Lie algebras. These consist of diff'l operator acting on sections of the trivial line bundle i.e. C^∞ functions $\psi(x,y)$ on $\mathbb{R}^2$

$$[D_x, D_y] = 2\pi i \quad \left[\begin{array}{ll} D_x = \partial_x & \nabla_x = \partial_x + 2\pi i y \\ D_y = \partial_y + 2\pi i x & \nabla_y = \partial_y \\ \frac{1}{4\pi i} D_x^2 & , \quad \frac{1}{4\pi i}\nabla_x^2 \\ \frac{1}{4\pi i} D_y^2 & \frac{1}{4\pi i}\nabla_y^2 \\ \frac{1}{4\pi i}(D_x D_y + D_y D_x) & \frac{1}{4\pi i}(\nabla_x \nabla_y + \nabla_y \nabla_x) \end{array}\right] = 0 \quad [\nabla_x, \nabla_y] = -2\pi i$$

$$\left[\frac{1}{4\pi i} D_x^2, (D_x \ \ D_y)\right] = (0 \ \ D_x) = (D_x \ \ D_y)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\frac{1}{4\pi i} D_y^2, (D_x \ \ D_y)\right] = (-D_y \ \ 0) = (D_x \ \ D_y)\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$\left[\frac{1}{4\pi i}(D_x D_y + D_y D_x), (D_x \ \ D_y)\right] = (-D_x \ \ +D_y) = (D_x \ \ D_y)\begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

$$\left[\frac{1}{4\pi i}\nabla_x^2, (\nabla_x \ \ \nabla_y)\right] = (0 \ \ -\nabla_x) = (\nabla_x \ \ \nabla_y)\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\left[\frac{1}{4\pi i}\nabla_y^2, (\nabla_x \ \ \nabla_y)\right] = (\nabla_y \ \ 0) = (\nabla_x \ \ \nabla_y)\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\left[\frac{1}{4\pi i}(\nabla_x \nabla_y + \nabla_y \nabla_x), (\nabla_x \ \ \nabla_y)\right] = (\nabla_x \ \ -\nabla_y) = (\nabla_x \ \ \nabla_y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\left[\frac{1}{4\pi i} D_x^2, \frac{1}{4\pi i} D_y^2\right] = \frac{1}{4\pi i}(D_x D_y + D_y D_x)$$

$$\left[\frac{1}{4\pi i}\nabla_x^2, \frac{1}{4\pi i}\nabla_y^2\right] = -\frac{1}{4\pi i}(\nabla_x \nabla_y + \nabla_y \nabla_x)$$

IDEA: The construction of the metaplectic representation of $SL(2,\mathbb{R})$ should be related to the proof of Matsumoto's thm. on K_2F. I think what's involved in the construction of the universal central extn of $SL(2,\mathbb{R})$ is to take the obvious "1-parameter" subgroups generating $SL(2,\mathbb{R})$, manufacture a candidate for a covering group which is a set X with Bruhat decomposition, left and right action by the 1-parameter subgroups and a map $X \longrightarrow SL(2,\mathbb{R})$. The key point is then that the left + right actions commute.

Before getting involved with Bruhat decomposition, try first to make sense out of the Lie alg situation. You have two Lie subalgebras of the differential operators on $C^\infty(\mathbb{R}^2)$ both of dimension 5, and which commute. Moreover what

---

$$\left[\frac{1}{4\pi i}(D_xD_y+D_yD_x),\ \frac{1}{4\pi i}D_x^2\right] = -2\left(\frac{1}{4\pi i}D_x^2\right)$$

$$\left[\frac{1}{4\pi i}(D_xD_y+D_yD_x),\ \frac{1}{4\pi i}D_y^2\right] = 2\left(\frac{1}{4\pi i}D_y^2\right)$$

$$\left[\frac{1}{4\pi i}(\nabla_x\nabla_y+\nabla_y\nabla_x),\ \frac{1}{4\pi i}\nabla_x^2\right] = 2\left(\frac{1}{4\pi i}\nabla_x^2\right)$$

$$\left[\frac{1}{4\pi i}(\nabla_x\nabla_y+\nabla_y\nabla_x),\ \frac{1}{4\pi i}\nabla_y^2\right] = -2\left(\frac{1}{4\pi i}\nabla_y^2\right)$$

This shows that the map

$$\frac{1}{4\pi i}D_x^2 \longmapsto \frac{1}{4\pi i}\nabla_x^2$$

$$\frac{1}{4\pi i}D_y^2 \longmapsto \frac{1}{4\pi i}\nabla_y^2$$

$$\frac{1}{4\pi i}(D_xD_y+D_yD_x) \longmapsto \frac{1}{4\pi i}(\nabla_x\nabla_y+\nabla_y\nabla_x)$$

is an anti homomorphism of Lie algebras, i.e. like $g\mapsto g^{-1}$ for a group.

This strengthens your feeling that there is some gadget around with a left action of the D's and a right action of the ∇'s

Feb 14, 02

Make an effort to get the diffeomorphism picture, the Lie group picture, based on the action of $SL(2,\mathbb{R})\ltimes\mathbb{R}^2$ on $\mathbb{R}^2$. There are two actions of $GL(2,\mathbb{R})$ on $\mathbb{R}^2$ namely $g\mapsto g,\ (g^t)^{-1}$. In fact there are more because you can also conjugate.

$$g^{-1}=\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1}=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\qquad (g^t)^{-1}=\begin{pmatrix}d&-c\\-b&a\end{pmatrix}$$

Consider $\left\{g\in GL(2,\mathbb{R})\,\middle|\, g^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}g=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right\}$

$$g^t=\begin{pmatrix}0&1\\-1&0\end{pmatrix}g^{-1}\begin{pmatrix}0&-1\\1&0\end{pmatrix}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\frac{1}{\det}$$

$$=\begin{pmatrix}-c&a\\-d&b\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\frac{1}{\det}=\begin{pmatrix}a&c\\b&d\end{pmatrix}\frac{1}{\det}$$

What are you doing? The condition $g^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}g=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ means $g\in Sp_2(2,\mathbb{R})$; the above calc gives $g^t=g^t\frac{1}{\det}$. Alt.

$$1=\begin{pmatrix}0&-1\\+1&0\end{pmatrix}g^t\begin{pmatrix}0&1\\-1&0\end{pmatrix}g=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}a&c\\b&d\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

$$=\begin{pmatrix}-b&-d\\a&c\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}g=\begin{pmatrix}d&-b\\-c&a\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}$$

So it looks like there is only one reasonable way to get $SL(2,\mathbb{R})$ to act on $\mathbb{R}^2$, unless you want to conjugate.

Given $g=\begin{pmatrix}a&b\\c&d\end{pmatrix}:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ you get g^* on $\Omega(\mathbb{R}^2)$

$(g^*\psi)(x,y)=\psi(ax+by,\,cx+dy)$ for $\psi\in\Omega^0(\mathbb{R}^2)$

$d(g^*\psi)=g^*(d\psi)$ $\qquad g^*(x\,dy)=(ax+by)\,d(cx+\delta y)$

you want to find h_g so that $e^{h_g}g^*$ commutes with D_x, D_y also $\nabla_x\ \nabla_y$

Recall difficulty of understanding $d+A$ in terms of its components when there's a change of variables. Given $g \in SL(2,\mathbb{R})$, you can find a function h, in fact a quadratic function of x,y. such that

$$\boxed{e^h g^*((d+A)\psi) = (d+A)(e^h g^*\psi)}$$

What does this mean in terms of the components of

$$d + A = dx(\partial_x) + dy(\partial_y + 2\pi i x)$$

$$d\psi + A\psi = dx(\partial_x\psi) + dy(\partial_y\psi + 2\pi i x\psi)$$

$$g^*(d\psi + A\psi) = d(ax+by)\, g^*(\partial_x\psi) + d(cx+\delta y)\, g^*(\partial_y\psi + 2\pi i x\psi)$$

You want to know $g^*(\partial_x\psi)$ and $g^*(\partial_y\psi)$. This should involve the chain rule. The answer should involve linear combinations (using a,b,c,δ) of $\partial_x(g^*\psi)$, $\partial_y(g^*\psi)$.

$$\partial_x(g^*\psi)(x,y) = \partial_x\big(\psi(ax+by,\, cx+\delta y)\big)$$

$$= \underbrace{\partial_1\psi(ax+by,\, cx+\delta y)}_{g^*(\partial_x\psi(x,y))}\, a + \underbrace{\partial_2\psi(ax+by,\, cx+\delta y)}_{g^*(\partial_y\psi(x,y))}\, c$$

$$\boxed{\partial_x(g^*\psi) = g^*((a\partial_x + c\partial_y)\psi)} \quad ①$$

$$\partial_y(g^*\psi) = g^*((b\partial_x + \delta\partial_y)\psi) \quad ②$$

OK.

$$d(g^*\psi) = dx\, \partial_x(g^*\psi) + dy\, \partial_y(g^*\psi)$$

$$g^*(d\psi) = g^*(dx)\, g^*(\partial_x\psi) + g^*(dy)\, g^*(\partial_y\psi)$$

$$= (a\,dx + b\,dy)\, g^*(\partial_x\psi) + (c\,dx + \delta\,dy)\, g^*(\partial_y\psi)$$

$$= dx\, \boxed{g^*(a\partial_x\psi + c\partial_y\psi)}^{①} + dy\, g^*(b\partial_x\psi + \delta\partial_y\psi)^{②}$$

Given $g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix}$ $\quad (g^*\psi)(x,y) = \psi(ax+by,\ cx+\delta y)$

$$\partial_x(g^*\psi) = (\partial_x\psi)(ax+by,\ cx+\delta y)\,a + (\partial_y\psi)(ax+by,\ cx+\delta y)\,c$$
$$= g^*((a\partial_x + c\partial_y)\psi)(x,y)$$

$$\partial_x(g^*\psi) = g^*(a\partial_x\psi + c\partial_y\psi)$$
$$\partial_y(g^*\psi) = g^*(b\partial_x\psi + \delta\partial_y\psi)$$

$$d(g^*\psi) = dx\,(a\ g^*(\partial_x\psi) + c\,g^*(\partial_y\psi)) + dy\,(b\,g^*(\partial_x\psi) + \delta\,g^*(\partial_y\psi))$$
$$= d(ax+by)\,g^*(\partial_x\psi) + d(cx+\delta y)\,g^*(\partial_y\psi)$$
$$= g^*(dx\ \partial_x\psi + dy\ \partial_y\psi) = g^*(d\psi)$$

Next you want to understand the semi direct prod $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

$$g = \begin{pmatrix} a & b \\ c & \delta \end{pmatrix},\ v = \begin{pmatrix} s \\ t \end{pmatrix} \qquad \begin{aligned} g^*x &= ax+by+s \\ g^*y &= cx+\delta y+t \end{aligned}$$

$$\psi(x,y) \mapsto \psi(ax+by+s,\ cx+\delta y+t)$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \xmapsto{g} \begin{pmatrix} ac_1+bc_2 \\ cc_1+\delta c_2 \end{pmatrix} \xmapsto{v} \begin{pmatrix} ac_1+bc_2+s \\ cc_1+\delta c_2+t \end{pmatrix}$$

$$(vg)^*\psi = \psi v g = g^*(v^*\psi)$$

So far you have looked at the operators on $C^\infty(\mathbb{R}^2)$ coming from $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$. One puzzle

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \xrightarrow{g} \begin{pmatrix} ac_1+bc_2 \\ cc_1+dc_2 \end{pmatrix} \xmapsto{g'} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} ac_1+bc_2 \\ cc_1+dc_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} a'a+b'c & a'b+b'd \\ c'a+d'c & c'b+d'd \end{pmatrix}}_{g'g}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

$$xg' = a'x+b'y$$
$$xg'g = a'(ax+by) + b'(cx+dy)$$
$$(g'g)^*(x) = a'\,g^*x + b'\,g^*y$$

$$(g'g)^*(x) = a'g^*x + b'g^*y$$
$$= g^*(a'x + b'y) = g^*(g'^*x)$$

So it should be clear how to compute with these diffeos on $\mathbb{R}^2$. Next step is to handle operators on the trivial line bundle over $\mathbb{R}^2$ respecting the given connection $d + 2\pi i x dy$.

You want to lift translation $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x+s \\ y+t \end{pmatrix}$

$$g^* x = x + s = e^{s\partial_x} x \qquad g^* = e^{s\partial_x + t\partial_y}$$
$$g^* y = y + t = e^{t\partial_y} y$$

Want $e^h g^*$ to preserve $d + 2\pi i x dy$

$$g^*A = 2\pi i(x+s)\,dy \qquad g^*A - A = 2\pi i s\,dy = d(\overbrace{2\pi i s y}^{h})$$

$$g^*(d\psi + A\psi) = d(g^*\psi) + (g^*A)(g^*\psi)$$
$$= (d + A + dh)\,g^*\psi = e^{-h}(d+A)e^h g^*\psi$$

$\therefore$ you get $\quad e^{2\pi i s y}\psi(x+s, y+t) = e^{2\pi i s y} e^{s\partial_x} e^{t\partial_y}$
$$= e^{s(\partial_x + 2\pi i y)} e^{t\partial_y}$$

Infinitesimally you get the diff ops $\nabla_x = \partial_x + 2\pi i y$, $\nabla_y = \partial_y$ commuting with $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$

So far have taken g to be a translation $g^*(x \;\; y) = (x+s \;\; y+t)$. Then you found h so that $e^h g^*$ preserves $d + A = d + 2\pi i x dy$.

You have to find out what to do. It seems that for each $g \in SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ you are going to get two ways to lift it. There are two extended Heisenberg representations which commute

Take $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$; consider g^* on $\Omega(\mathbb{R}^2)$ and find fn. h such that $e^h g^*$ commutes with $d + A$, $A = 2\pi i\, x\, dy$

You have
$$g^*(x) = ax + by$$
$$g^*(y) = cx + \delta y$$
$$g^*A - A = 2\pi i\left[(ax+by)(c\,dx + \delta\, dy) - x\,dy\right]$$
$$(x \;\; y)\begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} dx \\ dy \end{pmatrix}$$
$$h = \pi i \begin{pmatrix} x \\ y \end{pmatrix}^t \begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$
$$dh = g^*A - A$$
$$g^*A = A + dh$$

so you have the operator
$$\psi(x,y) \longmapsto e^{\pi i \binom{x}{y}^t (\quad) \binom{x}{y}}\, \psi(ax+by,\, cx+\delta y)$$

does this operator respect the 2-dim'l space spanned by ∇_x, ∇_y
$$\left[\begin{matrix} D_x = \partial_x & & \nabla_x = \partial_x + 2\pi i y \\ & , & \\ D_y = \partial_y + 2\pi i x & & \nabla_y = \partial_y \end{matrix}\right]$$

What did you do this morning for the diffeo g? Behavior of g^* and ∂_x, ∂_y.
$$g^*\psi(x,y) = \psi(ax+by,\, cx+\delta y)$$
$$\partial_x(g^*\psi) = (a\partial_x\psi + c\partial_y\psi)(ax+by,\, cx+\delta y)$$

dx $\boxed{\partial_x(g^*\psi) = a\, g^*(\partial_x\psi) + c\, g^*(\partial_y\psi)}$

dy $\boxed{\partial_y(g^*\psi) = b\, g^*(\partial_x\psi) + \delta\, g^*(\partial_y\psi)}$

$$d(g^*\psi) = d(g^*x)\, g^*(\partial_x\psi) + d(g^*y)\, g^*(\partial_x\psi) = g^*(dx\,\partial_x\psi + dy\,\partial_y\psi)$$

Let's review the situation. You have the trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ over $\mathbb{R}^2$ equipped with connection
$$\begin{array}{c}\mathbb{R}^2\times\mathbb{C}\\ \downarrow\\ \mathbb{R}^2\end{array}$$
$d+A$, $A = 2\pi i x\,dy$. The components of $d+A$ are $D_x = \partial_x$ and $D_y = \partial_y + 2\pi i x$, which satisfy $[D_x, D_y] = 2\pi i$, so you get a repn of the Heisenberg Lie alg w basis $D_x, D_y, 2\pi i$ as first order differential operators on $\mathbb{R}^2$.
There is another Heisenberg Lie alg w basis $\nabla_x = \partial_x + 2\pi i y$, $\nabla_y = \partial_y$. Point is that $D_x D_y$ commute with ∇_x, ∇_y.

<u>Problem</u>: What is the significance of ∇_x, ∇_y?

Feb 15, 02 Key point: Any diffeo g of $\mathbb{R}^2$ preserving the volume can be lifted to the line bundle so as to respect the connection, uniquely up to a scalar factor. The lifting has the form $e^h g^*$. It should be possible to describe this as some kind of diffeomorphism of the line bundle. The lifting depends on the connection, the D and ∇ liftings should result from different conn. forms. Key fact is that the two liftings of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ commute, because the Lie algebra action of $SL(2,\mathbb{R})$ is represented by quadratic expressions in the translation generators.

Check various things. Instead of $2\pi i x\,dy$ consider $2\pi i(x\,dy + df)$ so that

$$\begin{aligned} d + 2\pi i x\,dy &= dx(\partial_x) + dy(\partial_y + 2\pi i x) \\ + 2\pi i\,df &= 2\pi i(dx\,\partial_x f + dy\,\partial_y f) \end{aligned}$$

get the two components: $\partial_x + 2\pi i \partial_x f$, $\partial_y + 2\pi i x + 2\pi i \partial_y f$

$f = -xy$ $\qquad \partial_x - 2\pi i y$, $\quad \partial_y$

$f = -\frac{1}{2}xy$ $\qquad \partial_x - \pi i y$, $\quad \partial_y + \pi i x$

Let's sort out gravity + gauge operators

$$e^h g^* \xrightarrow{\text{inf.}} (1+\varepsilon h)(1+\varepsilon X)$$

e^h is a matrix function, X is a vector field.

infinitesimal operators $X + h$. What you want is how to interpret $X+h$ as a vector field on the principal bundle

Start again with $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ acting on $\mathbb{R}^2$

$$g^*\psi(x,y) = \psi(g^*x, g^*y) = \psi(ax+by+s,\ cx+\delta y+t)$$

$$\frac{1}{2\pi i}\begin{pmatrix} g^*A \\ -A \end{pmatrix} = \cancel{g}^* x\, dg^*y = (ax+by\overset{+s}{})(c\,dx+\delta\,dy)$$
$$= - \quad x\,dy$$

$$\frac{1}{2\pi i}(g^*A - A) = d\left\{\frac{1}{2}(x\ \ y)\begin{pmatrix} ac & bc \\ bc & b\delta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right\} + s\,d(cx+\delta y) \qquad ?$$

Describe the situation. (1) You have the Lie gp $SL(2,\mathbb{R})\ltimes\mathbb{R}^2$ acting by diffeomorphisms of $\mathbb{R}^2$: $g = \left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}, \begin{pmatrix} s \\ t\end{pmatrix}\right)$

$$(g^*\psi)(x,y) = \psi(ax+by+s,\ cx+dy+t)$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\cdot} \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} \xmapsto{+\begin{pmatrix} s \\ t\end{pmatrix}+} \begin{pmatrix} ax+by+s \\ cx+dy+t \end{pmatrix}$$

$$\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}x\right)^*\left(\begin{pmatrix} s \\ t\end{pmatrix}+\right)^*\psi(x,y) = \left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}x\right)^*\psi(s+x,\ t+y)$$

$$= \psi(s+ax+by,\ t+cx+dy)$$

$$g = \left(\begin{pmatrix} s \\ t\end{pmatrix}+\right)\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}x\right)$$

(2) unique (up to constant scalar factor) way to lift g^* to an operator $e^h g^*$ which commutes with the connection $d+A$

Let's restrict attention to $\begin{pmatrix} s \\ t\end{pmatrix} \in \mathbb{R}^2$

$$g^*\begin{pmatrix} x \\ y\end{pmatrix} = \begin{pmatrix} s+x \\ t+y\end{pmatrix} \qquad g^* = e^{s\partial_x + t\partial_y} \qquad g^*A = 2\pi i(s+x)\,d(t+y)$$
$$A = 2\pi i\, x\,dy$$

$$g^*A - A = 2\pi i s\,dy = d(\overset{h}{2\pi i ys})$$

$$\left(e^{h}(g^*\psi)\right) = e^{2\pi i ys}\, e^{s\partial_x}\, e^{t\partial_y}\, \psi(s+x, t+y)$$

$$= e^{s(\partial_x + 2\pi i y)}\, e^{t\partial_y}\,\psi(x,y)$$

as a check you get

$$\left[\begin{matrix} D_x = \partial_x \\ D_y = \partial_y + 2\pi i x \end{matrix}\ ,\ \begin{matrix} \nabla_x = \partial_x + 2\pi i y \\ \nabla_y = \partial_y \end{matrix}\right] = 0$$

What should come next? You should take other g in $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and find the 2 liftings and check they commute.

$$D_x = \partial_x \qquad \frac{1}{4\pi i} D_y^2 = \frac{1}{4\pi i}\partial_y^2 + x\partial_y - \pi i x^2$$

$$D_y = \partial_y + 2\pi i x \qquad \frac{1}{4\pi i}\left(\partial_x(\partial_y + 2\pi i x) + (\partial_y + 2\pi i x)\partial_x\right)$$

$$[D_x, D_y] = 2\pi i$$

$$\left[\frac{1}{4\pi i}D_x^2, \begin{pmatrix} D_x & D_y\end{pmatrix}\right] = \begin{pmatrix} 0 & D_x\end{pmatrix} = \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$$

$$\left[\frac{1}{4\pi i}D_y^2, \begin{pmatrix} D_x & D_y\end{pmatrix}\right] = \begin{pmatrix} -D_y & 0\end{pmatrix} = \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix}$$

$$\left[\frac{1}{4\pi i}(D_xD_y + D_yD_x), \begin{pmatrix} D_x & D_y\end{pmatrix}\right] = \begin{pmatrix} -D_x & D_y\end{pmatrix} = \begin{pmatrix} D_x & D_y\end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1\end{pmatrix}$$

$$\left[\frac{1}{4\pi i}(D_xD_y + D_yD_x), \frac{1}{4\pi i}D_x^2\right] = -2\frac{1}{4\pi i}D_x^2$$

$$\left[\frac{1}{4\pi i}(D_xD_y + D_yD_x), \frac{1}{4\pi i}D_y^2\right] = +2\frac{1}{4\pi i}D_y^2$$

$$\left[\frac{1}{4\pi i}D_x^2, \frac{1}{4\pi i}D_y^2\right] = \frac{1}{4\pi i}(D_xD_y + D_yD_x)$$

$$\left(\partial_x(g^*\psi)\right)(x,y) = \partial_x \psi(ax+by,\ cx+\bar{d}y)$$
$$= a(\partial_x\psi)(ax+by,\ cx+\bar{d}y) + c(\partial_y\psi)(ax+by,\ cx+\bar{d}y)$$
$$= \left(a\, g^*(\partial_x\psi) + c\, g^*(\partial_y\psi)\right)(x,y)$$

$\partial_x(g^*\psi) = a\, g^*(\partial_x\psi) + c\, g^*(\partial_y\psi)$	$g^*(dx) = a\,dx + b\,dy$
$\partial_y(g^*\psi) = b\, g^*(\partial_x\psi) + \bar{d}\, g^*(\partial_y\psi)$	$g^*(dy) = c\,dx + \bar{d}\,dy$

$$d(g^*\psi) = dx\,\partial_x(g^*\psi) + dy\,\partial_y(g^*\psi) = (a\,dx + b\,dy)\,g^*(\partial_x\psi) + (c\,dx + \bar{d}\,dy)\,g^*(\partial_y\psi)$$
$$= d(g^*x)\,g^*(\partial_x\psi) + d(g^*y)\,g^*(dy) = g^*(dx\,\partial_x\psi + dy\,\partial_y\psi)$$
$$= g^*(d\psi).$$

Go back to $d+A = d + 2\pi i\, x\,dy$, and find a fn. h such that $(e^h g^*)(d+A) = (d+A)(e^h g^*)$

$$g^*\left((d+A)\psi\right) = d(g^*\psi) + (g^*A)(g^*\psi)$$
$$= (d + g^*A)(g^*\psi)$$

$$d\left(g^*A - A\right) = g^*(dA) - dA = 0, \quad \text{so } \exists!\ h \text{ up to}$$

an add. constant such that $d(2\pi i\, h) = g^*A - A$

$$\frac{1}{2\pi i}\left(g^*A - A\right) = (g^*x)\,d(g^*y) - x\,dy$$
$$= (ax+by)(c\,dx + \bar{d}\,dy) - x\,dy$$
$$= ac\,x\,dx + (a\bar{d}-1)\,x\,dy + bc\,y\,dx + b\bar{d}\,y\,dy$$
$$= \begin{pmatrix} x & y\end{pmatrix}\begin{pmatrix} ac & bc \\ bc & b\bar{d}\end{pmatrix}\begin{pmatrix} dx \\ dy\end{pmatrix} = d\Bigg\{\underbrace{\frac{1}{2}\begin{pmatrix} x \\ y\end{pmatrix}^t\begin{pmatrix} ac & bc \\ bc & b\bar{d}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}}_{\frac{1}{2\pi i}h}\Bigg\}$$

Next make this infinitesimal
$$g = \begin{pmatrix} 1+\varepsilon a & \varepsilon b \\ \varepsilon c & 1-\varepsilon a\end{pmatrix} \qquad \det(g) = 1 + O(\varepsilon^2)$$
$$h = \pi i \begin{pmatrix} x \\ y\end{pmatrix}^t \begin{pmatrix} \varepsilon c & 0 \\ 0 & \varepsilon b\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$$

$$(g^*\psi)(x,y) = \psi(x+\varepsilon ax+\varepsilon by,\ \varepsilon cx+y-\varepsilon ay)$$
$$= \psi(x,y) + \varepsilon\left[(ax+by)\partial_x\psi + (cx-ay)\partial_y\psi\right]$$

get vector field on $\mathbb{R}^2$.

$$X = a(x\partial_x - y\partial_y) + by\partial_x\psi + cx\partial_y\psi$$
$$h_X = \pi i(cx^2+by^2)$$

Check that $(\mathcal{L}(X)+h_X)(d+A) = (d+A)(\mathcal{L}(X)+h_X)$

$$\mathcal{L}(\underbrace{x\partial_x - y\partial_y}_{X_1})(d+A)\psi = d\,\mathcal{L}(X_1)\psi + (\mathcal{L}(X_1)A)\psi + A\,\mathcal{L}(X_1)\psi$$

$$\frac{1}{2\pi i}\mathcal{L}(X_1)A = \mathcal{L}(X_1)\,x\,dy = \underbrace{[x\partial_x - y\partial_y, x]}_{x}dy + x\,d\underbrace{[x\partial_x - y\partial_y, y]}_{-y}$$
$$= x\,dy - x\,dy = 0$$

$$\frac{1}{2\pi i}[\mathcal{L}(y\partial_x), d+A] = [\mathcal{L}(y\partial_x), x\,dy] = [y\partial_x, x]dy + \cancel{x\,d[y\partial_x, y]} = y\,dy$$

$$\frac{1}{2\pi i}[\pi i y^2, d+A] = \frac{1}{2\pi i}\pi i[y^2, d] = -y\,dy$$

$$\therefore [\mathcal{L}(y\partial_x) + \pi i y^2,\ d+A] = 0.$$

$$\frac{1}{2\pi i}[\mathcal{L}(x\partial_y), d+A] = [\mathcal{L}(x\partial_y), x\,dy] = \cancel{[x\partial_y, x]}dy + x\,d\underbrace{[x\partial_y, y]}$$

$x\,dx$ $\parallel$

$$\frac{1}{2\pi i}[\pi i x^2, d+A] = \frac{1}{2}[x^2, d] = -x\,dx$$

$$\therefore [\mathcal{L}(x\partial_y) + \pi i x^2,\ d+A] = 0$$

$$g^*\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}s+x\\t+y\end{pmatrix} \qquad g^*(\underset{-x\,dy}{x\,dy}) = (s+\cancel{x})\,dy = d(sy) \qquad h = 2\pi i\,sy$$

$$\left(e^h(g^*\psi)\right)(x,y) = e^{2\pi i sy}\psi(s+x, t+y) = \left(e^{s(2\pi i y+\partial_x)}e^{t\partial_y}\psi\right)(x,y)$$

infinitesimally you get $\mathcal{L}(\partial_x) + 2\pi i y$, $\mathcal{L}(\partial_y)$

Things to understand

1) Is the operator $\mathcal{L}(X) + h_X$ the horizontal lift of the vector field X in some sense? Horizontal means with respect to the connection on the trivial line bundle. The horizontal lift should be defined for any vector field, what is the significance of X preserving the volume?

Review the connection business for the principal bundle which should be $P = \mathbb{T} \times \mathbb{R}^2$, let's recall the structure to fix the ideas. You look at the action of $\mathbb{T}$ given by multiplication, and restrict to invariant differential forms $\Omega(P)^{\mathbb{T}}$. Atiyah short exact sequence, a splitting of which is a connection $0 \to P \times^G \mathfrak{g} \to (T_P)^G \twoheadrightarrow T_B \to 0$.

$B = \mathbb{R}^2$ $\quad P = \mathbb{R}^2 \times \mathbb{T}$ three coords x, y, θ defined up to 2π

vector fields $\partial_x, \partial_y, \partial_\varphi$ dual to 1-forms $dx, dy, d\varphi$

What does a connection form look like? Restriction to the fibre should be $i\, d\varphi$. $\quad d + A = dx(\partial_x) + dy(\partial_y + 2\pi i x)$.

You need to link the connection form Θ on P to the operator $d + A$. The latter should come from $d + \Theta$ over P using a section of P.

IDEA Could the dilogarithm be lurking around somewhere? $x\,dy$ when $x, y \in \mathbb{R}/\mathbb{Z}$ similar to $\log f \; d\log g$. Could you encounter $\int \log(1-f)\, df$ somewhere? Bloch did something like this. Also you know this pairing $\int \log f \; d\log g$ is important for the LT QFT.

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two commuting Heisenberg repns.

$$\begin{bmatrix} D_x = \partial_x & , & \nabla_x = \partial_x + 2\pi i y \\ D_y = \partial_y + 2\pi i x & & \nabla_y = \partial_y \end{bmatrix}$$

you know that ∇_x, ∇_y are the lifts of the vector fields ∂_x, ∂_y on $\mathbb{R}^2$ to operators $\mathcal{L}(\partial_x) + h_1$, $\mathcal{L}(\partial_y) + h_2$

commuting with $d+A$.

Summarizing some opinions. You have pretty much failed to get anything by lifting the action of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ to the line bundle ~~so~~ as to preserve the connection $d+A$. At the moment you seem to have three projective representations of $G = SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$, namely, No 1) generated by $D_x = \partial_x$, $D_y = \partial_y + 2\pi i x$. The basis of $Lie(G)$ given by $\partial_x, \partial_y, x\partial_x - y\partial_y, y\partial_x, x\partial_y$ acts via $D_x, D_y, \frac{1}{4\pi i}(D_x D_y + D_y D_x), \frac{1}{4\pi i} D_x^2, \frac{1}{4\pi i} D_y^2$ roughly

No 2) acts via same sort of operators, with D_x, D_y replaced by ∇_x, ∇_y.

No 3) $\nabla_x = \partial_x + 2\pi i y$, $\nabla_y = \partial_y$, $x\partial_x - y\partial_y$, $y\partial_x + \pi i y^2$, $x\partial_y + \pi i x^2$.

Ultimately you may need to get the order + signs straight.

Look at the representation theory. To look at $\mathcal{S}(\mathbb{R}^2)$ with $\partial_x, \partial_y, 2\pi i x, 2\pi i y$ then you want to change to $\partial_x, \partial_y + 2\pi i x, \partial_x + 2\pi i y, \partial_y$. The aim is to find a tensor product decomposition

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Feb 17, 02 Write up the three actions of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$

First look at the 2nd Heisenberg rep $\nabla_x = \partial_x + 2\pi i y$, $\nabla_y = \partial_y$ on $\mathcal{S}(\mathbb{R}^2)$. A rep of the Heisenberg group is Morita equivalent to a vector space. $\Gamma \tilde{\times} \check{\Gamma}$. So in principal you should be able to use eigenfunction expansion for ∇_y to split off $\mathcal{S}(\mathbb{R}_y)$ as a tensor factor

Two processes

$$\psi \longmapsto \sum_{m,n} e^{m\nabla_x} e^{n\nabla_y} \psi$$

$$\psi \longmapsto \psi(\ ,y)$$

Question: Recall the construction of the Heisenberg group as the set $\mathbb{T} \times \mathbb{R}^2$ of triples $\left(\zeta, e^{sX}, e^{tY}\right)$ with $\zeta \in \mathbb{T}$, $(s,t) \in \mathbb{R}^2$. equipped with the product

$$\left(\zeta_1 e^{s_1 X} e^{t_1 Y}\right)\left(\zeta_2 e^{s_2 X} e^{t_2 Y}\right) = \zeta_1 \zeta_2 e^{-2\pi i t_1 s_2} e^{(s_1+s_2)X} e^{(t_1+t_2)Y}$$

This should be a special case of the following: G loc. compact, abelian, $G^\vee$ Pontryagin dual of G, $\mathrm{Heis}(G) = \mathbb{T} \times G \times G^\vee =$ the group extension

$$\mathbb{T} \rightarrowtail \mathrm{Heis}(G) \twoheadrightarrow G \times G^\vee$$

defined by the 2-cocycle $(g_1 + \chi_1, g_2 + \chi_2) \longmapsto \langle \chi_1, g_2 \rangle$

The question is how is the Heisenberg group for $\mathbb{R}$ related to the principal $\mathbb{T}$ bundle over $\mathbb{R}^2$ having curvature $2\pi i\, dx\, dy$? Similarity: You choose a bilinear form with skew symm. $2\pi i\, dx\, dy$ in order to construct the Heisenberg group. This bilinear form is a 2-cocycle. Same choice is made to get a connection

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Repeat. Consider over $\mathbb{R}^2$ "the" (hermitian complex) line bundle with connection whose curvature is $2\pi i\, dx\, dy$. Just means a 1-form A on $\mathbb{R}^2$ with values in $i\mathbb{R}$ such that $dA = 2\pi i\, dx dy$. Any ~~two~~ are related by a gauge transf. $e^{2\pi i h}$ with h real 'quadratic form'. Still not clear.

Start again with $\mathbb{R}^2$ and the (translation & $SL(2,\mathbb{R})$ invariant) curvature $2\pi i\, dx dy$. You need to pick a 1-form A values in $i\mathbb{R}$ such that $dA = 2\pi i\, dx\, dy$. Such A should be acted upon by $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$. Possibilities? Pick $A_0 = 2\pi i\, x\, dy$ then ~~dy~~ any A differs from A_0 by df $\quad f$ any $\in C^\infty(\mathbb{R}^2)$.