Here then. \( X \) compact (Hausdorff), there is an equivalence between f.g. proj modules over \( C(X) \) and v.b. over \( X \). given by \( E/X \iff \Gamma(X,E) \).

**Step 1:** Any \( E \) is a retract (direct summand) of a trivial v.b.
\[ E \cong X \times V \cong E \]
For \( \epsilon \) r.h. maps \( \rho, \gamma \) sat \( \rho \gamma = 1 \).
(Alt form: \( \exists \) a v.b. \( E' \) together with an r.m. \( E \oplus E' = X \times V \)).

**Proof.** Locally \( E \) is trivial \( \Rightarrow \) \( E \) is a retract of a true ball \( \exists U^\mu \) finite open and maps \( E \leftarrow U^\mu \leftarrow U^\mu \times V^\mu \leftarrow U^\mu \).

Let \( 1 = \sum \frac{x^2}{\mu} \) be a part of \( 1 \). Supp \( x^\mu \subset U^\mu \).

\[
\sum x^\mu \gamma^\mu \epsilon^\mu x^\mu = \sum x^2 = 1
\]

**Step 2:** Given a cont. family of projections \( e^\mu \in \text{End}(V) \)
locally \( \exists \) a cont. family of invertible \( \epsilon^\mu \hspace{0.2cm} 8 \frac{e^\mu}{\epsilon^\mu} \in \text{End}(V) \) such that \( u^\mu e^\mu u^{-1} \) is constant.
Replace \( e^2 = e \) by \( F^2 = 1 \)
cont family of involutions fix base pt. \( \epsilon = F \) at base pt. Want to const. \( u \in u^{-1} = F \).

\( g = F \epsilon \)
\( \epsilon g \epsilon^{-1} = \epsilon F = F \)
Assume \( (g^{1/2})^2 = g \)
\( g^{1/2} g^{-1/2} = g^{-1/2} g^{1/2} = \epsilon \)
\( g^{1/2} \epsilon g^{-1/2} = g^{-1/2} \)
\( g^{1/2} g^{-1/2} = \epsilon \)
\( g \)
\[ g = 1 + x \quad g^z = \sum_{n=0}^{\infty} \frac{z(z-1)\cdots(z-n+1)}{n!} x^n \quad \|x\| < 1. \]

Cayley transform
\[ g = \frac{1 + x}{1 - x} = (1 + i)X \]

\[ X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} g = \frac{g - 1}{g + 1} \]

Claim: \( \|g - 1\| \) small \( \Rightarrow \) \( X \) defined and
\[ g^{1/2} \varepsilon^{-1} = \frac{1 - x}{\sqrt{1 - x^2}} = g^{-1/2} \]

\[ g = \frac{1 + iy}{1 - iy} = \frac{1 - y^2 + 2iy}{1 + y^2} \]

\[ g = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} g^{-1} = 1 - \frac{g}{g+1} \]

\[ 1 - X = \frac{2}{g+1} \quad \frac{2}{1 - X} = \frac{g+1}{2} \]

\[ g = \frac{2}{1 - X} - 1 = \frac{1 + x}{1 - x} \]

Problem: if you want to use \( g^{1/2} = \frac{1 + x}{\sqrt{1 - x^2}} \), you must handle the denominator.

The best method is to use the exponential map.

\[ \exp : \text{End}(V) \rightarrow \text{Aut}(V) \]

which on a small enough star-like nbh of 0 respects the group laws in the radial direction + is a diffeos near 0.

This should yield \( g^{1/2} \) with \( \varepsilon g^{1/2} \varepsilon^{-1} = g^{-1/2} \), since square roots will be unique.
Heisenberg Lie alg. basis \( X, Y, H \) relative to \([X, Y] = H, [X, H] = [Y, H] = 0\).

\[
\exp(aX + bY + cH) = e^{aX} e^{bY} e^{\left(-\frac{ab}{2}H\right)}
\]

\[
e^{X+Y} = \frac{X^2 + XY + YX + Y^2}{2}
\]

\[
e^{X} e^{Y} e^{-\frac{1}{2}H} = \frac{X^2 + XY + Y^2}{2} = \frac{XY - YX}{2}
\]

\[
e^{aX} e^{bY} = e^{aX+bY + \frac{ab}{2}H}
\]

\[
e^{bY} e^{aX} = e^{aX+bY - \frac{ab}{2}H}
\]

\[
e^{aX} e^{bY} e^{-aX} e^{-bY} = e^{ab[X, Y]}
\]

3. Think Lie alg. Except you are missing the \( 2\pi i \).

What you forget is the fact that bilinear forms give 2-cocycles. Review group cohomology.

\[
\begin{array}{ccc}
H & i & \rightarrow & \mathbb{E} \xrightarrow{\delta} & G \\
\end{array}
\]

\[
\delta(s(g_1), s(g_2)) = i \delta(f(g_1, g_2))
\]

\[
s(g_1) s(g_2) = f(g_1, g_2) s(g_1 g_2)
\]

\[
(s(g_1) s(g_2)) s(g_3) = f(g_1, g_2) s(g_1 g_2) s(g_3)
\]

\[
= f(g_1, g_2) f(g_1 g_2, g_3) s(g_1 g_2 g_3) = s(g_1) f(g_2, g_3) s(g_2 g_3) = s(g_1) f(g_2, g_3) f(g_1 g_2 g_3) s(g_1 g_2 g_3)
\]

\[
g_1 f(g_2, g_3) - f(g_1 g_2, g_3) + f(g_1, g_2 g_3) - f(g_1, g_2) = 0
\]
\[ f(y, z) - f(x+y, z) + f(x, y+z) - f(x, y) = 0 \]
\[ yz = (x+y)z + x(y+z) - xy = 0 \]

This is true for any bilinear \( f(x, y) \).

\[ f(y, z) - f(x+y, z) = -f(\frac{x}{y}, z) \]
\[ f(x, y+z) - f(x, y) = f(x, z) \]

You are dealing with \( \mathbb{R}^2 \) the group \( \mathbb{R} \), so you need to understand bilinear form on \( \mathbb{R}^2 \) 4-dimensional space.

\[ \mathbb{R}^2 \rightarrow \text{Heis} \rightarrow \mathbb{R}X + \mathbb{R}Y \]

\[ [s(x), s(y)] = f(x, y) + s([x, y]) \]
\[ [s(x), [s(y), s(z)]] = [s(x), f(y, z) + s([y, z])] \]
\[ = f(y, z) + s([y, z]) \]
\[ [s(x), [s(y), s(z)]] = [s(x), f(y, z) + s([y, z])] \]
\[ = -z f(x, y) + f([x, y], z) + s([x, y], z) \]
\[ [s(y), [s(x), s(z)]] = [s(y), f(x, z) + s([x, z])] \]
\[ = y f(x, z) + f(y, [x, z]) + s([y, [x, z]]) \]

If of abelian
\[ x f(y, z) = y f(x, z) - z f(x, y) \]
You reviewed the 2 cocycles for group + Lie alg extensions, especially central extensions of an abelian group + Lie alg.

\[ M \rightarrow E \rightarrow G \]

\[ s(x)s(y) = \frac{1}{2} f_2(x,y)s(xy) \]

\[ f_2(y,x) - f(x,y) + f(x,y) - f(x,y) = 0 \]

- bilinear \( f_2 \) is a 2-cocycle.
- \( f_1 \) is a 1-cocycle.

\[ f_2(x,y) = f_1(y) - f_1(xy) + f_1(x) \]

is deviation of \( f_1 \) from being a group homom.

In the case \( h \mapsto e \mapsto s \) of

\[ [s(x), s(y)] = f_2(x,y) + s(x+y) \]

\[ s'(x) = f_1(x) + s(x) \]

\[ [s'(x), s'(y)] = f_2'(x,y) + s'(x+y) \]

\[ [f_1(x) + s(x), f_1(y) + s(y)] = f_2'(x,y) + f_1(x+y) + s(x+y) \]

\[ f_1(y) - f_1(xy) - f_1(x) = f_2'(x,y) - f_2(x,y) \]

- coboundary of \( f_1 \)
- linear \( f_1 \) central + of abelian then all \( f_1(x) \) are cocycles.
- bilinear \( f_2(x,y) \) are cocycles.

You now would like to construct the line bundle of degree 1 over \( \mathbb{H}^2 \) as a homogeneous space of the Heisenberg group \( K \) by a \( \mathbb{H}^3 \). What's the best way to proceed?
use model
\[ f(x) \mapsto \tilde{f}(x, y) = \sum_{m} e^{2\pi i mx} f(x + m) \]
\[ \tilde{f}(x, y) = \tilde{f}(x, y + 1) \]
\[ \tilde{f}(x, y) = e^{2\pi i y} f(x + y) \]

note that
\[ (\partial_x f)(x) \mapsto (\partial_x \tilde{f})(x, y) \]
\[ (\partial_y \tilde{f})(x, y) = \sum_{m} e^{2\pi i y} \left( 2\pi i m \right) f(x + m) + (2\pi i \tilde{f})(x, y) \]

\[ \tilde{f}(x f)(x, y) = (\partial_y + 2\pi i x) \tilde{f} \]
\[ \tilde{\partial_x f} = \partial_x \tilde{f} \]
\[ \nabla_x = \partial_x \]
\[ \nabla_y = \partial_y + 2\pi i x \]
\[ [\nabla_x, \nabla_y] = 2\pi i \]

So in \( S(\mathbb{R}) \) define
\[ \nabla_x f = \partial_x f \]
\[ \nabla_y f = (2\pi i x) f \]
\[ \tilde{\nabla_x f}(x, y) = \sum_{m} e^{2\pi i m y} 2\pi i (m + x) f(m + x) \]

Let's use the \( S(\mathbb{R}) \) model for space of sections
\[ \tilde{2\pi i x f}(x) = \sum_{m} e^{2\pi i m y} 2\pi i (x + m) f(x + m) \]
\[ \partial_y \tilde{f}(x) = \sum_{m} e^{2\pi i m y} 2\pi i m f(x + m) \]
\[ (\partial_y + 2\pi i x) \tilde{f} = (2\pi i x f) \]

So what can you do?
Question: You want the Heisenberg group to act transitively on the principal $U(1)$ bundle of degree 1 over $\mathbb{T}^2$.

You want a direct approach to this line bundle by first describing it when pulled back to $\mathbb{R}^2$, then you get the line bundle + connection by taking a suitable quotient by $\mathbb{Z}^2$.

Consider $\nabla = d + 2\pi i x dy$ on the trivial line bundle over $\mathbb{R}^2$. Curvature $\nabla^2 = 2\pi i dx dy + \frac{i}{2\pi} \partial_x dx \partial_y$

$= -dx dy$ represents generator of $H^2(\mathbb{R}^2/\mathbb{Z}^2; \mathbb{R})$.

You are looking at $\mathbb{R}^2$ with $\nabla$ connection $\nabla = \partial_x dx + (\partial_y + 2\pi i x) dy$. What's important? holonomy?

There is something called the holonomy group - you fix a basepoint and then do parallel transport along loops, getting a subgroup of autoqs of the fibre over the basepoint. This is a quotient of $\pi_1$ whose curvature = 0.

It should be true the holonomy group in the case of $\mathbb{R}^2$ with $\nabla = d + 2\pi i x dy$ is the circle; not interesting. You would like to recover the Heisenberg group.

Look at $H$ acting on itself by left translation. Canonical $H$ here should be the extension

$$0 \rightarrow \Pi \rightarrow H \rightarrow \mathbb{R} \times \mathbb{R} \rightarrow 0$$

Recall that there is an extension when you have a bilinear map

$$\begin{pmatrix} \mathbb{R} \times \mathbb{R} \\ \mathbb{R} \times \mathbb{R} \end{pmatrix} \rightarrow \Pi \\
\begin{pmatrix} (a \times b) \\ (c \times d) \end{pmatrix} \mapsto (a+c)x \begin{pmatrix} (b+d)y -bc \end{pmatrix}$$
So it seems that you need to understand completely the link between $H$ and the line bundle with connection on $\mathbb{R}^2$. What you need to understand better is the $\text{SL}(2, \mathbb{R})$ action. And maybe it is related to different connection forms (quadratic).

Guess $H = \mathbb{T} \times \mathbb{R} \times \mathbb{R}$ = unit circle bundle $\mathbb{R}^2$ fiber, $H/\mathbb{T}^2$ unit circle bundle over $\mathbb{T}^2$.

Start with $\mathbb{R}^2$, trivial line bundle $\mathbb{R}^2 \times \mathbb{C}$ equipped with connection $d + 2\pi i \cdot x \, dx$, curvature $2\pi i \cdot dx \wedge dy$.

Vector fields $X = \partial_x$, $Y = \partial_y$ on $\mathbb{R}^2$.

$f(x,y)$ section of trivial line bundle is flat along a curve $x(t), y(t)$ when

$$\frac{df}{dt} + 2\pi i \cdot x \frac{dy}{dt} f = 0$$

linear DE for $f$ over the curve $(x(t), y(t)) = \int_{t_0}^{t} \frac{df}{dt} \, dt$

$$f(t) = e^{\int_{t_0}^{t} 2\pi i \cdot x \, dt} f(t_0)$$

$$\int y \, dx = \int \int dxdy$$

Next look at $\text{SL}(2, \mathbb{R})$ action on $\mathbb{R}^2$. Better first look at different gauge $\omega = dx \wedge dy$

$$xy \, dy + d(xy) = -y \, dx$$

so conjugating $\exp(d + 2\pi i \cdot x \, dx) \exp(-2\pi i \cdot x \, dx) = d + 2\pi i \cdot (d(xy) + x \, dy)$

$$= d - 2\pi i \cdot y \, dx$$

Now comes obvious question: how quadratic functions in $x,y$ yield gauge equivariant connections.
You consider $\mathbb{R}^2$, the trivial line bundle $L = \mathbb{R}^2 \times \mathbb{C}$ over $\mathbb{R}^2$, and you connect $\nabla$ and $\nabla$ with connections on $L$, which means roughly that the curvature is harmonic (translation invariant) and the connection form is $\omega$ to a gauge transform of another.

$$\nabla = d + A = dx^2 + dy (\partial_y + 2\pi i x)$$

What is your idea? $dA = 2\pi i \, dx \, dy$, so your connection form can be?

you've forgotten the idea that a connection is given by a connection form on the principal bundle. Your principal bundle should be $\mathbb{R}^2 \times \mathbb{T}$, coordinates $x, y, \theta$.

$$d\Theta = A = d\theta - 2\pi i \, x \, dx \, dy$$

It shouldn't be difficult. You have the 3-manifold $\mathbb{R}^2 \times \mathbb{T}$, circle bundle over the plane, coordinates $x, y, \theta$, where $\theta \in \mathbb{R}/\mathbb{Z}$, so actually $d\theta$ is well defined; the conn. form is a 1-form on $\mathbb{R}^2 \times \mathbb{T}$ restricting to find. class of the fibre at each $(x, y)$.

$$\Theta = d\theta + P \, dx + Q \, dy$$

whose diff. comes from the base $d\theta = dp \, dx + dq \, dy$

$$= \left( -\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) \, dx \, dy$$

When you do II along a curve you lift the tangent vector to the curve up to a horizontal vector, i.e. in the kernel of $\Theta$.

Consider $\mathbb{R}^2$, the trivial line bundle over it, a volume element, a connection for $\nabla$ whose curvature $dA$ is the volume form $\omega$, is unique up to the differential of a function $df$. Point is that $\omega$ is translation invariant but $A$ is not.

You need to get the formalism straight, because you are
dealing with a connection - This should be written -
\( d + A \), the arbitrariness in the connection is due to the action of gauge transforms: \( g^{-1} (d + A) g \)
equal in the trivial line bundle case to \( A + g^{-1} d g \)
and \( g^{-1} d g = d(\log g) \). Any \( f \) on \( \mathbb{R}^2 \) is log with \( g = e^f \), so you are looking at all 1-form \( A \mod g \). Poincaré lemma says the curvature \( dA \) determines the possible classes of \( d + A \mod \) gauge transfs.

Now look at translation invariance. The invariant d1-forms are spanned by \( dx, dy, dx dy \). Look at \( A \) such that \( dA = dx dy \). Examples: \( A = x dy - y dx \), not translation invariant. \( x dy - (-y dx) = d(xy) \) module \( dx dy \), so there is a unique \( A = P dx + Q dy \), with \( dA = dx dy \).

How can you organize the choices? Since \( P, Q \) are not translation invariant the simplest thing is to require translation invariance module constant.

Let basis \( x dx, y dx, x dy, y dy \).

So it seems that our possible \( A \) is an affine hyperplane in a tensor product \( V \otimes V \) where \( V = \mathbb{R} x + \mathbb{R} y \). You may have \( S^2 V \rightarrow V \otimes V \rightarrow \Lambda^2 V \).

Yes. This looks very reasonable. \( S^2 V \) is the space of quadratic functions, span of \( (ax + by)^2 \).

---

Review. \( \mathbb{R}^2 \) trivial bundle over \( \mathbb{R}^2 \) translation invv.

volume form \( \omega \in \Lambda^2 V \), \( V = \mathbb{R} x + \mathbb{R} y \). Possibly connection forms - are elements of \( V \otimes V \), base \( \{ x, y \} \) and curvature is the map \( V \otimes V \rightarrow \Lambda^2 V \), whose kernel is \( S^2 V \).
Then you fix $w$ and look at its inverse image.

So what to do? You have this connection form $\Gamma x^2 dx$. You can do $\parallel$ transport along curves. Let's try to get the $\parallel$ transport along lines through $0$.

$$x = at, \quad y = bt$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Use vector notation.

$$\begin{pmatrix} \frac{\partial}{\partial t} \sigma(tx, ty) + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ -\frac{1}{2} x' \begin{pmatrix} a \end{pmatrix} + x \end{pmatrix} = 0$$

$$\sigma(tx) = \sigma(0) e^{\frac{1}{2} x' A x' t^2}$$

so the appropriate "exp" map seems to be $\sigma(x) = \sigma(0) e^{\frac{1}{2} x' A x} t^2$.

A should be purely imaginary. Actually it looks like you should calculate $\parallel$ transport along the linear path $x(t) = x_0 + tv$.

$$\begin{pmatrix} \frac{\partial}{\partial t} \sigma(x_0 + tv) + (x_0 + tv)^\ast A \begin{pmatrix} a \\ b \end{pmatrix} \sigma(x_0 + tv) \end{pmatrix} = 0$$

$$\sigma(x_0 + tv) = e^{-\begin{pmatrix} -\frac{1}{2} x_0 \begin{pmatrix} a \end{pmatrix} + tv \end{pmatrix} \begin{pmatrix} x_0 \begin{pmatrix} a \end{pmatrix} + tv \end{pmatrix}^\ast \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}} \cdot \begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$$

No over what you've learned. Consider trivial line bundle over $V = \mathbb{R}^2$ with connection $d + A$, $A \in V^\ast \otimes dV^\ast$. Specify curvatures. The connections of interest are a constant mod $V^\ast \otimes dV^\ast \rightarrow 0$. Parallel transport along a curve $\gamma$ is just $\exp(-\int A)$, you want this for line segments $v_0 + tv$ over $t \leq 1$. 
Try to straighten this out: You have some kind of action of elements \( v \in V \) on \( V \times G \), which should turn out to be a central extension of \( V \) by \( T \). Given \( v \in V \) it acts on \( V \) by translation \( v_0 \mapsto v_0 + v \). Use the linear \( v_0 + tv \) to transport along this path which is \( \exp \{ A(v_0, v) + \frac{1}{2} A(v, v) \} \). The action of \( v \) on \( V \times G \) sends \( (v_0, g) \) to \( (v_0 + v, g) \).

\[ A \] were symmetric

\( V \) 2dual natural R. equipped with \( 0 \neq w \in \Lambda^2 V^* \)

\[ 0 \to \Lambda^2 V^* \to V^* \otimes V^* \to A^2 V^* \to 0 \]

\[ 0 \to \Lambda^2 V^* \to V^* \otimes V^* \to \Lambda^2 V^* \to 0 \]

\[ \Lambda \]

\[ \Lambda \]

To associate to \( v \in V \) a \( \| \) transport differs on \( V \times \mathbb{R} \)

given \( v_0 \in V \) take \( \| \) transports on linear segment \( v_0 + tv \quad 0 \leq t \leq 1 \). This means

\[ 2\pi i \int_0^1 A(v_0 + tv, v) \, dt = 2\pi i \left( A(v_0, v) + \frac{1}{2} A(v, v) \right) \]

forget \( 0 \) which should be irrelevant for \( \| \) transport.

\[ \text{area of } \Delta \]
What seems to happen: Use \( \parallel \) transport along lines thru \( 0 \) to trivialize the line bundle. This means you join \( v \) to \( 0 \) and \( tv \), \( \parallel \) transport gives the phase \( \oint_{0} A(tv, v) \, dt = \frac{1}{2} A(v, v) \) \( \pm 2\pi i \) 

What are you trying to do? You have the line bundle \( V \times \mathbb{C} \) with connection \( d + A \), which allows \( \parallel \) transport along curves. You would like to lift translations on \( V \) to the translation action of \( V \) on itself to the line bundle, but there's an obstruction.

Consider \( v_0 + tv \), \( a.s.t \leq 1 \). The parallel \( \text{transport} \), with the line along this curve is \( \pm 2\pi i \int_{0}^{1} A(v_0 + tv, v) \, dt \)

\[
A(v_0, v) + \frac{1}{2} A(v, v)
\]

If you split \( A \) into Symm. + skew symm. parts then

\[
\text{symm. part } \frac{1}{2} A(v_0, v_0) + A(v_0, v) + \frac{1}{2} A(v, v) \quad \text{yields} \quad \frac{1}{2} A(v_0 + v, v_0 + v)
\]

\[
\text{skew part } \frac{1}{2} (A(v_0, v) - A(v, v_0)) \quad \text{should be area of } \triangle
\]

What can you hope for? to identify \( V \times \mathbb{C} \) with the Heisenberg group.

Let's review \( V \) 2-real and \( v_0 \) \( \mathbb{C} \) trivial line bundle over \( V \), connection \( d + A \), \( A \in \mathbb{V}^{*} \otimes \mathbb{V}^{*} \)

\[
A \, i_v \, dv.
\]

Let \( \parallel \) transport \( 2\pi i \int_{0}^{1} A(v(t), v'(t)) \, dt \)

\[
v(t) = v_0 + tv \quad \int_{0}^{1} A(v_0 + tv, v) \, dt = A(v_0, v) + \frac{1}{2} A(v, v)
\]
Define an action somewhere. Take

$$T_v \left( \begin{array}{c} v_0 \\ c \end{array} \right) = \left( \begin{array}{c} v + v_0 \\ c + A(v_0, v) + \frac{1}{2} A(v, v) \end{array} \right)$$

$$T_{v'} T_{v''} \left( \begin{array}{c} v_0 \\ c \end{array} \right) = T_{v'} \left( \begin{array}{c} v_0 + v' \\ c + A(v_0, v) + \frac{1}{2} A(v, v) \\ + A(v_0 + v', v) + \frac{1}{2} A(v', v) \end{array} \right)$$

$$= \left( \begin{array}{c} v_0 + v'_0 + v'' \\ c + A(v_0, v') + \frac{1}{2} A(v', v') \\ + \frac{1}{2} A(v'_0, v'') + \frac{1}{2} A(v''_0, v'') \end{array} \right)$$

$$T_{v' + v''} \left( \begin{array}{c} v_0 \\ c \end{array} \right) = \left( \begin{array}{c} v_0 + v' + v'' \\ c + A(\overline{v_0}, v'_0, v''_0) + \frac{1}{2} A(v'_0 + v''_0, v''_0) \end{array} \right)$$

The difference is

$$\frac{1}{2} A(v'_0, v'') - \frac{1}{2} A(v'_0, v') - \frac{1}{2} A(v''_0, v')$$

$$= \frac{1}{2} \left( A(v'_0, v'') - A(v''_0, v') \right)$$

You believe that the line bundle is Heisenberg group which you define as $V \times \mathbb{T}$ with a certain product $(v, \xi) \cdot (v', \xi') = (v + v', \xi' + \xi''(v, v') \xi')$. It seems that $\text{SL}(V)$ acts on the Heisenberg group. But then you have

Recall 2-couple $f: V^2 \to \text{abzl. W}$ two act

$$f(v_1, v_3) - f(v_1 + v_2, v_3) + f(v_1, v_2 + v_3) - f(v_1, v_2) = 0$$

$$f(v_1, v_3) - f(v_1, v_3)$$

$$f(v_1, v_3)$$
l-coboundary $(d^{\ast} h)(v_{1},v_{2}) = h(v_{1}) - h(v_{1}v_{2}) + h(v_{2})$

If $h$ is a function on $V$, then $h$ is called quadratic when $h(v_{1}v_{2}) - h(v_{1}) - h(v_{2})$ is bilinear.

So apply this to $\mathbb{R} \to ? \to V$ and you get a Lie group given by $V \times \mathbb{R}$ with mult.

$(v,c) \cdot (v',c') = (v+v', f(v,v') + c+c')$

What about $h$? You get an isom. gp $\mathbb{R}$ from $f' = f + \delta h$ where $h: V \to \mathbb{R}$ is quadratic.

$((v,c) \cdot (v',c')) \mapsto ((v+v', f'(v,v') + c+c'))$

$(v, h(v) + c) \cdot (v', h(v') + c') \mapsto (v+v', f(v,v') + h(v) + h(v') + c+c')$

$f' - f = \delta h$

The other point is the exact sequence

$\mathbb{R}^{2}V^{*} \longrightarrow V^{*} \otimes V^{*} \longrightarrow \Lambda^{2}V^{*}$

$f \quad w$

So there is a canonical choice for $f$, namely its skew symm.

$f(v,v') - \frac{1}{2}(f(v,v') + f(v',v)) = \frac{1}{2}(f(v,v') - f(v',v))$

It should be clear that the group of auto of $V$ preserving $f$ acts on the group, so using $w = skew\ symm\ of f$ yields action of $Aut(V)$

Now that you have $E = V \times \mathbb{R}$, can you relate it to line bundle over $V$ with connection? Eventually you want $\mathbb{R}$ to become $\mathbb{R}/\mathbb{Z} = \mathbb{T}$.

Go back to conn. $D = d + A \quad A \in V^{*} \otimes \text{d}V^{*} = \text{V}^{*} \text{d}V^{*}$
\[ \int_{A} A(v_0 + tv, v) \, dt = A_0(v_0, v) + \frac{1}{2} A(v, v) \]

Let's try to make an action of your \( (V \times \mathbb{R}) \) Heisenberg group on this line bundle.

\[
\begin{align*}
T_{(v, c)} (v', c') &= (v + v', A(v', v) + \frac{1}{2} A(v, v) + c + c') \\
T_{(v, c)} (v, c) &= (v + v', f(v, v') + c + c') \\
\end{align*}
\]

What is the best you can do with \( \parallel \) transport.

\[
T_v (v_0, c_0) = (v_0 + v, A(v_0, v) + \frac{1}{2} A(v, v) + c_0)
\]

Your problem seems to be with basepoint.

The connection form \( A \in V^* \otimes V^* \) appears to require an origin for your 2-plane, i.e. where the elements of \( V^* \) vanish.

Go back to the original geometry, you have a 2-plane \( V \) equipped with a volume 2-form, translation invariant.

Origin is specified when \( A = y \, dx \).
There's so much to understand.

Go back to the Wheel alg., recall this. A principal $G$-bundle $G \to P \to B$, not really relevant since you consider base bundles.

So back to $V$ 2-plane with volume from $\omega \in \Omega^2 V^*$ and a connection $A \in V^* \otimes \Omega^1 V^*$, $dA = \omega$. Consider

$\exp 2\pi i \int A$.

\[ \int_0^1 A(tv_0, v_0) dt = \frac{1}{2} A(v_0, v_0) \quad \text{sin} \quad v, \]

\[ \int_0^1 A(v_0 + tv, v_0 + v) dt = A(v_0, v) + \frac{1}{2} A(v, v) \]

you want to compare $\Rightarrow \frac{1}{2} A(v_0, v_0) + A(v_0, v) + \frac{1}{2} A(v, v)$

with $\frac{1}{2} A(v_0 + v, v_0 + v) = \frac{1}{2} A(v_0, v_0) + \frac{1}{2}(A(v_0, v) + A(v, v))$

$+ \frac{1}{2} A(v_0, v_0)$

difference is $\frac{1}{2}(A(v_0, v) - A(v, v))$. Still got the gap.

Try something else, namely, operators on sections.

$V$, functions on $V$, connection $d + A$, look at flow with velocity $v$, takes $v_0 \to v_0 + tv$, causes $v_0 \to v_0 + tv$, $\partial_v$ is $\partial_v$, but now lift this vector field into the principal bundle, which means you get $\partial_v + A(v_0)$.
A little review

\( f \in L^2(\mathbb{R}) \)

\[ \hat{f}(x, y) = \sum_{m \in \mathbb{Z}} e^{2\pi imy} f(x+m) \quad \text{per in } y \]

\[ e^{2\pi iy} \hat{f}(x, y) = \sum_{m \in \mathbb{Z}} e^{2\pi i(x+m)} y f(x+m) \quad \text{per in } x \]

\( \hat{f}(x, y+1) = \hat{f}(x, y) = e^{2\pi iy} \hat{f}(x+1, y) \)

\( Q = \) periodic \( m x, y \) fn's.

\( M = \{ s(x, y) \} \) as above.

Connection:

\[ \partial_x \hat{f}(x, y) = \frac{\partial}{\partial x} \hat{f}(x, y) \]

\[ (\partial_y \hat{f})(x, y) = \sum_m e^{2\pi imy} 2\pi im f(x+m) \]

\[ \left( \partial \hat{f} \right)(x, y) = \sum_m e^{2\pi imy} 2\pi ix f(x+m) \]

\[ (\partial_y + 2\pi ix) \hat{f} = (2\pi ix f) \]

Suggests \( M \) closed under:

\[ \nabla_x = \partial_x \]

\[ \nabla_y = \partial_y + 2\pi ix \]

\( \partial_y \left( e^{2\pi iy} s(x, y) \right) = \partial_y \left( e^{2\pi iy} s(x+1, y) \right) \)

\[ \nabla_y s(x, y) = \nabla_y \left( e^{2\pi iy} s(x+1, y) \right) \]

\[ = e^{2\pi iy} \left( \left( \partial_y + 2\pi i \right) s(x+1, y) \right) \]

\[ = e^{2\pi iy} \left( \nabla_y s(x+1, y) \right) \]

\[ = e^{2\pi iy} \left( \frac{2\pi (x+1)}{s(x+1, y)} \right) \]

\[ \nabla_y \left( e^{2\pi iy} s(x+1, y) \right) = e^{2\pi iy} \left( \partial_y + 2\pi i + 2\pi ix \right) s(x+1, y) \]

\[ = e^{2\pi iy} \left( \nabla_y s(x+1, y) \right) \]
\[ e^{v \cdot \nabla} = e^{a \nabla_x + b \nabla_y} = e^{a \nabla_x} e^{b \nabla_y} e^{-\frac{1}{2} \lambda \lambda^{*}} \]

\[ (e^{(a,b) \cdot \nabla} s)(x,y) = e^{a \nabla_x + b \nabla_y} e^{b \nabla_x - \pi \lambda b i} s(x,y) \]

\[ = e^{-\pi \lambda b_i} \]

\[ e^{a \nabla_x + b \nabla_y} = e^{b \nabla_y} e^{a \nabla_x - \pi \lambda b i} \]

\[ (e^{a \nabla_x} s)(x,y) = s(x + a, y) \]

\[ b \nabla_y s = b \left( \nabla_y + 2 \pi i x \right) s = b \nabla_y e^{2 \pi i b x} s \]

\[ = e^{2 \pi i b x} s(x, y + b) \]

\[ \text{answer} \quad e^{2 \pi i b x} s(x + a, y + b) \]

\[ \tilde{f}(x,y) = \sum_{m} e^{2 \pi i m y} f(x + m) \]

\[ \nabla \tilde{f} = \tilde{\nabla f} \quad \nabla \tilde{f} = (2 \pi i x f) \]

\[ e^{a \nabla_x + b 2 \pi i x} f = e^{b 2 \pi i x} e^{a \nabla_x} e^{\pi \lambda b i} \left[ e^{2 \pi i b x} f(x + a) \right] \]

Aside from the messy calculations, it seems that you do get an action of the Heisenberg Lie group \( \Gamma(\pi, 2, L) \).
\[ f \in L(R) \quad f \mapsto \tilde{f}(x,y) = \sum_{m \in \mathbb{Z}} e^{2\pi i m y} f(x+m) \]

\[ \tilde{f}(x,y+1) = \tilde{f}(x,y) = e^{2\pi i y} \tilde{f}(x+1,y) \]

\[ e^{2\pi i \chi} \tilde{f}(y,x) = e^{2\pi i (x+1)y} \tilde{f}(x+1,y) \]

\[ L = \text{set of } s(x,y) \text{ on } \mathbb{R}^2 \text{ having these periods, prop.} \]

\[ L \text{ is a module over } C^\infty(\mathbb{T}^2) \]

\[ T = \mathbb{R}/\mathbb{Z} \]

---

\[ \text{idea: could there exist a direct embedding} \]

can you obtain your connection on the line bundle \( L \) over \( T^2 \) as a Grassmannian connection for a direct embedding in trivial rank 1 bundle.

---

what's important now is the action of the Heisenberg group on the line bundle.

but first you need for your lecture to embed \( L \) as a summand of a \( C^\infty(T^1)\oplus^2 \). Choose \( 1 = \chi_0 + \chi_1 \)
idea. A partition $\sum x_\mu^2 = 1$ is appropriate for Hilbert space or situations. Given local isometric embedding $E \xrightarrow{x_\mu} U_\mu \times H_\mu$, then $\sum x_\mu \gamma_\mu : E \to X \times \Omega H_\mu$ should be isometric. Inner product. But a partition $\sum x_\mu x'_\mu = 1$ might fit better the shrinking idea for covering.

so what you want now is local trivialization. Remove $x = 0 \in \mathbb{R}/\mathbb{Z}$. $T^2 \xrightarrow{x} \mathbb{R}/\mathbb{Z}$

$$F(x, y) \quad \text{smooth on} \quad (\mathbb{R} - \mathbb{Z}) \times \mathbb{R} \subset \mathbb{R}^2$$

$$\downarrow$$

$$\mathbb{R}/\mathbb{Z} - \{0\} \times \mathbb{R}/\mathbb{Z} \subset T^2$$

$$F(x, y) \quad \text{smooth function on} \quad \xi(x, y) \quad | \quad x \in \mathbb{R} - \mathbb{Z}$$

$$F(x, y + 1) = F(x, y) = e^{2\pi iy} F(x + 1, y) \quad x \in \mathbb{R} - \mathbb{Z} \quad y \in \mathbb{R}$$

You have $Lu = C(u, C(\mathbb{R}/\mathbb{Z}))$
Let's decide whether the Heisenberg group acts on $L$.

First show $L$ is locally an $A \cong C(\mathbb{T})$ module of rank 1. You need maps $A \to L$ and $L \to A$. Can there exist a map $A \to L$ nonvanishing $W_i$, i.e., $F(x,y) \in L$ nonvanishing. If so, you can take the logarithm and get $G(x,y)$ smooth on $\mathbb{R}^2$ (and at least) satisfying $G(x,y+1) = G(x,y) = 2\pi i y + G(x+1,y)$

Have $e^G$.

Have $F(x,y+1) = F(x,y) = e^{2\pi i y} F(x+1,y)$. Assume $F(x,y) \neq 0 \quad \forall x,y \in \mathbb{R}^2 \implies \int G(x,y) = e^G = F$. $G$ continuous smooth etc. Then $e^{G(x,y+1) - G(x,y)} = 1$

$\Rightarrow$ \begin{align*}
G(x,y+1) - G(x,y) &= 2\pi i k \\
G(x,y) - G(x+1,y) &= 2\pi i y + 2\pi i k
\end{align*}

Apply $\partial_x$. \begin{align*}
(\partial_x G)(x,y) \quad &\text{doubly periodic} \\
(\partial_y G)(x,y) \quad &\text{periodic in } y \\
(\partial_y G)(x,y) - (\partial_y G)(x+1,y) &= 2\pi i dy
\end{align*}

\[
\int_{(0,1)}^{(0,1)} dG = G(1,1) - G(0,1)
\]

$F$ non-vanishing

\[
\frac{dF}{F}(x,y+1) = \frac{dF}{F}(x,y) = 2\pi i dy + \frac{dF}{F}(x+1,y)
\]
Assume \( L \) has a non-vanishing section \( F(x,y) \neq 0 \) on \( IR^2 \). View \( x \mapsto F(x,y) \) as a path in the loop space of \( C^* \). The degree of the path is independent of \( x \)

\[
\oint d \log F(x,y) = \oint F(x,y)^{-1} \partial_y F(x,y) \, dy
\]

divided by \( 2\pi i \).

But \( d \log F(x,y) = d \log (e^{2\pi i y} F(x+1,y)) \)

\[
= 2\pi i \, dy + d \log F(x+1,y)
\]

\[
\oint = 2\pi i + \oint'
\]

Question: Since \( L \) is supposed to have a connection with some sort of translational invariance properties, might there be a way to exploit translation invariance to reconstruct a direct embedding of \( L \) into a trivial bundle, possibly infinite dimensional?

Related ideas: Mumford's theory of \( \Theta \) functions involving Heisenberg groups.

Earlier question of direct embedding into a trivial bundle such that the YM connection is the Grassmannian form.

\( A = C^\infty(T^2) = \) smooth periodic on \( IR^2 \)

\[
0 \leq x \leq b, \quad y \in IR
\]

I often use interval in \( IR \) of lines

\[
<1 \quad (I \times Z) \times IR
\]

often in...
Can you set this up without referring to open sets?

Go back to the commutation, etc.

\[ F(x, y + 1) = F(x, y) = e^{2\pi i y} F(x + 1, y) \]

\[ F(x, y) = \sum_{m \in \mathbb{Z}} e^{2\pi i m y} f(x + m) \]

\[ f_n(x) = \int e^{-2\pi i m y} F(x, y) dy = \int e^{-2\pi i m y + 2\pi i y} F(x + 1, y) dy \]

\[ = f_{n-1}(x + 1) = \ldots = f_0(x + m). \]

\[ F \in \mathcal{L} \implies \partial_x F \in \mathcal{L} \quad \text{in fact} \quad F = \tilde{\mathcal{F}} \implies \partial_x F = \tilde{\mathcal{F}} \partial_x \]

\[ F \in \mathcal{L} \implies e^{2\pi i x y} F(x, y) = e^{2\pi i (x+1)y} F(x+1, y) \]

\[ \frac{e^{-2\pi i x y}}{\partial_y} \frac{e^{2\pi i x y}}{\partial_y} F(x, y) = e^{-2\pi i x y} \partial_y e^{2\pi i x y} e^{2\pi i y} F(x, y) \]

\[ (\partial_y + 2\pi i x) F(x, y) = \left( \partial_y + 2\pi i x \right) e^{2\pi i y} F(x + 1, y) \]

\[ = e^{2\pi i y} \left( \partial_y + 2\pi i (x + 1) \right) F(x + 1, y) \]

\[ \nabla_y = \partial_y + 2\pi i x \quad \text{preserves } \mathcal{L} \]

\[ \nabla_y f(x, y) = \sum_{m \in \mathbb{Z}} (\partial_y + 2\pi i x) \left( e^{2\pi i m y} f(x + m) \right) \]

\[ = \sum_{m} e^{2\pi i m y} \left( \frac{\partial_y + 2\pi i (x+m)}{2\pi (x f)}(x+m) \right) \]

\[ \nabla_y \tilde{f} = 2\pi i x \tilde{f} \]
It is clear that \( \partial_x = \partial_x \), \( \partial_y = \partial_y + 2\pi i \partial_x \)
define a connection on the module \( \mathcal{L} \) over the
ring \( \mathcal{A} = C^\infty(T^2) \).

\[
\begin{align*}
\mathcal{D}(f \xi) &= df \xi + f D \xi \\
\mathcal{D}_x(f \xi) &= (\partial_x f) \xi + f \mathcal{D}_x \xi \\
D &= d + 2\pi i x dy \\
D^2 &= 2\pi i dx dy
\end{align*}
\]

A connection on the base allows us to lift vector fields in the base to transverse vector fields in the total space.

\( \partial_x, \partial_y \) on \( T^2 \)

\[
\exp(v \cdot D)
\]

degrees Mumford \( \Theta \) funs. I think he considered finite
Heisenberg groups, that is, the canonical central extension
of \( A \times A \) by appropriate \( \mu \) (roots of 1), where \( A \) is
a finite subgroup of an abelian variety.

It seems that Mumford has a good noncomm analogue
of Mumford's theory.

\( \partial_x \) lifts to \( \mathcal{D}_x = \partial_x \) in \( L \)

\[
\begin{align*}
\partial_y \mathcal{D}_y &= \partial_y + 2\pi i \partial_x \\
e^{2\pi i xy} \partial_y e^{2\pi i xy} &= e^{-2\pi i xy} \sum_m e^{2\pi i (x+m) y} \int e^{2\pi i (x+m) y} \frac{df(x+m)}{dx} d\xi_m (x+m) f(x+m) \\
\mathcal{D}_y \tilde{f} &= (2\pi i x \tilde{f})^2
\end{align*}
\]
Next you want a vector \( v = (i) e \in \mathbb{R}^2 \) 

\[
\text{translation on } f(x, y) \mapsto f(x+a, y+b)
\]

\[
(e^{aD_x + bD_y} f)(x, y) \equiv \phi(t, x, y) = e^{t(aD_x + bD_y)} f
\]

\[
\frac{\partial}{\partial t} \phi = (aD_x + bD_y) \phi
\]

\[
\phi|_{t=0} = f
\]

So you have a lot of work left.

\[
e^{(aD_x + bD_y)} F(x, y) = e^{bD_y} e^{aD_x} e^{\frac{1}{2}[aD_x, bD_y]} F
\]

\[
= e^{\pi i ab} e^{2\pi i bx} e^{b2\pi i y} F(x+a, y+b)
\]

\[
e^{(aD_x + bD_y)} F(x, y) = e^{\pi i ab} e^{2\pi i bx} F(x+a, y+b)
\]

Bascially you should work with \( e^{aD_x} e^{bD_y} \) 

A typical element of the \( \text{H} \)-group is \( e^x e^y e^{c_1} e^{c_2} \) \( \text{H} \) and the product is 

\[
(e^x e^y e^{c_1} e^{c_2})(e^{a_1} e^{b_1} e^{c_1} e^{c_2}) = e^x e^y e^{c_1} e^{c_2}
\]

Consider now \( e^{aD_x} e^{bD_y} f(x) = e^{aD_x} e^{b2\pi i x} f(x) = e^{2\pi i (x+a)} f(x+a) \) 

\[
e^{bD_y} e^{aD_x} f(x) = e^{bD_y} f(x+a) = e^{2\pi i bx} f(x+a)
\]

\[
e^{aD_x} e^{bD_y} = e^{[aD_x, bD_y]} e^{bD_y} e^{aD_x} ?
\]

Program. Work out the action of the Heisenberg group on \( L \)

Question: How can you find the stabilizes of a point, say \((0,0) \in \mathbb{R}^2 \).
1. If \( f(x) \in \mathcal{S}(\mathbb{R}) \), then

\[
\hat{f}(x, y) = \sum_m e^{2\pi i m y} f(x+m) \in \mathcal{L}
\]

where

\[\mathcal{L} = \left\{ F(x, y) \in C^\infty(\mathbb{R}^2) \mid F(x, y+1) = F(x, y) \right\} \]

\[F(x, y) = e^{2\pi i y} F(x+1, y)\]

2. Conversely, if \( F \in \mathcal{S}(\mathbb{R}) \) Lebesgue space of rapidly dec. smooth funs on \( \mathbb{R} \),

\[
\mathcal{L} = \left\{ F(x, y) \in C^\infty(\mathbb{R}^2) \mid F(x, y+1) = F(x, y) = e^{2\pi i y} F(x+1, y) \right\}
\]

then \( f \in \mathcal{S}(\mathbb{R}) \Rightarrow \hat{f}(x, y) = \sum_m e^{2\pi i m y} f(x+m) \in \mathcal{L} \)

\[
F \in \mathcal{L} \Rightarrow \hat{F}(x) = \int F(x, y) dy \in \mathcal{S}(\mathbb{R})
\]

and these two maps are inverses of each other.

\( L \) is a module over \( C^\infty(T^2) \). Claim it's a fg proj.

\( A \)-module, i.e. a direct summand (retract) of \( A \) free \( fg \) \( A \)-module.

Consider open \( U \subset T^2 \)

\[
L(U) = \left\{ F \in C^\infty(\mathbb{R}^2) \mid \text{same per cond.} \right\}
\]

e.g. \( U = \left\{ (x+\mathbb{Z}, y+\mathbb{Z}) \mid x \in \mathbb{Z} \right\} \)

\[
\pi^{-1} U = \bigsqcup_{n \in \mathbb{Z}} (n_1, n_1+1) \times \mathbb{R}/\mathbb{Z}
\]

\[
F \in C^\infty(\pi^{-1}U)
\]

\[
F(x, y) \qquad x \in (n, n+1), y \in \mathbb{R}/\mathbb{Z}
\]

\[
L(\pi^{-1} \{ x+\mathbb{Z} + \mathbb{Z} \}) \Rightarrow \{ F(x, y) \in C^\infty(0,1) \times \mathbb{R}/\mathbb{Z} \} \]
Idea about the transform \( f(x) \mapsto \hat{f}(x,y) \), what might it mean to localize at a point \((x,y) \in \mathbb{T}^2\) with position \(x\) and momentum \(y\). You are

reminded of Hormander’s partition of unity, rather his localization theory behind Fourier integral operators.

Another idea — coherent states: coherent state representation is complete but not orthogonal.

Connection on \( L \) you seek operators \( D_x, D_y \) on \( \mathcal{A} \) in the sense that Liouville holds:

\[
D_x (\psi) = \partial_x \psi + f D_y \psi \quad \text{also for } D_y
\]

because of \( \mathcal{A} \), \( \mathbb{R} \) has \( \mathbb{Z} \) you can look for \( D_x, D_y \) on \( L \):

\[
\tilde{f}(x,y) = \sum_m e^{2\pi i m y} f(x+m)
\]

Clearly if \( \partial_x \tilde{f} = \frac{d}{dx} \tilde{f} \):

\[
\tilde{f}(x,y) \quad \text{and} \quad \tilde{f}(x,y) = \sum_m e^{2\pi i m y} 2\pi i m f(x+m)
\]

\[
2\pi i x \tilde{f}(x,y) = \sum_m e^{2\pi i m y} 2\pi i x f(x+m)
\]

\[
(\partial_y + 2\pi i x) \tilde{f}(x,y) = \sum_m e^{2\pi i m y} 2\pi i (x+m)f(x+m) = (2\pi i x f)
\]

\[
D_y \psi = (\partial_y + 2\pi i x) \psi
\]

\[
[D_x, D_y] = 2\pi i
\]

so now you have good control over the connection

\[
e^{aD_x e^{bD_y} \psi(x,y)} =
\]

IDEA: you get a representation of the Heisenberg group on \( L^2 \) this should be irreducible, at least the \( L^2 \) version is.

What about \( L^\infty \) completions?
There should be an $L^2$ norm on $L$, such that the representation of the Heisenberg group is unitary, just because the transport should preserve norm square of a section, and volume is translation invariant. You are hoping that $L$ is a kind of tensor product of $C_0^\infty(\mathbb{R}/\mathbb{Z})$ with $\mathcal{C}_0^\infty(\mathbb{Z})$, rapidly decreasing functions on $\mathbb{Z}$. If so, then $L$ should have an interpretation as kernels.

$L$ is some twisted version of $C^0(\mathbb{T}) \otimes C^0(\mathbb{T})$, better is the idea that $L$ is the line bundle over $C^0(\mathbb{T}^2)$, where the first circle $\mathbb{T}$ is $\mathbb{R}/\mathbb{Z}$ and the second circle is the character group of $\mathbb{Z}$. $L$ is a $K$-theory correspondence linking the classifying space $B\mathbb{Z} \cong \mathbb{R}/\mathbb{Z}$ with the group ring $\mathbb{C}[\mathbb{Z}]$ which is $C(\mathbb{Z}) \cong C(\mathbb{T})$.

Also known as Poincare duality.

What's important: $L$ is an irreducible rep of Heisenberg algebra to construct $L$ a complex line bundle over $\mathbb{T}^2$ equipped with a connection, such that $L$ can be identified with the space of sections of $L$ over $\mathbb{T}^2$.

The first attempt is to define the fibre of $L$ over a point $(x,y) \in \mathbb{T}^2$ and the space $L$ of all sections modulo sections vanishing at $(x,y)$. This is an intrinsic definition, and it should be clear that you get a locally trivial fibre bundle of complex lines. Local triviality comes from non-vanishing $\psi \in L^\wedge$ at $(x,y) \in \mathbb{T}^2$.

The operators $D_x, D_y$ on $L$, which cover $dx, dy$ and should define was a Lie rep. of Heisenberg on $L$, covering the translation action.

Let's now do this construction carefully. You have a
specific A module $L$, consisting of all
smooth $\psi(x,y)$ in $\mathbb{R}^2$ sat.
$\psi(x,y+1) = e^{2\pi i y} \psi(x+1,y)$.  

Introduce an equiv rel. $\psi \sim \psi'$ iff equal at $(x,y)$
But it should be possible to say what $L((x,y))$ is.
$L(U) = \{ \psi(x,y) \in C^\infty(U) \mid \text{ automorphy conditions hold} \}$
$L((x_0,y_0)) = \{ \psi(x,y) \text{ functions } \pi^{-1}(x_0,y_0) \}$

$L = \mathbb{R}^2 \times \mathbb{Z}^2$

\[ L((x_0+Z,y_0+Z)) = \{ \psi(x,y) \text{ on } (x_0+Z,y_0+Z) \text{ sat. period. cond.} \]
\[ \psi(x,y+1) = e^{2\pi i y} \psi(x+1,y) \]

This looks like a flat vb.

As progress was made on going from $\mathbb{L}$ to $L$.

$L(U) = \text{ smooth functions } f(x,y)$ sat.
$f(x+m,y+n) = e^{2\pi i (m,y)} f(x,y)$

$L(U) = \{ \psi(x,y) \text{ on } (x_0+Z,y_0+Z) \mid \psi(x_0+m,y_0+n) = e^{2\pi i (m,y)} \psi(x_0,y_0) \}$

1-gets for $L$  \[ J_1L \rightarrow \mathbb{L} \otimes T^* \rightarrow J_1L \rightarrow L \rightarrow 0 \]

$J_1L((x_0,y_0)) = \{ \psi \in L \mid \psi \text{ vanishes to 2nd order at } (x_0,y_0) \}$

$J_1L \cong \text{ sections of } J_1L \text{ over } T^2$
Let's forget the periodicity conditions for the moment and focus on what happens over $\mathbb{R}^2$. Then perhaps it would better to review holom. function picture for the harmonic oscillator (compare).

\[ |f|^2 = \int \int f(x)^2 e^{-|z|^2} \frac{dx dy}{\pi} \]

\[ d\bar{z} = dx + idy \quad d\bar{z} = dx - idy \]

\[ \frac{d\bar{z} d\bar{z}^*}{2\pi} = +2i \frac{dx dy}{\pi} \]

\[ [a, a^*] = 1, \quad a |0 \rangle = 0 \]

\[ \langle 0 | a^n a^* n | 0 \rangle = n! \]

\[ \langle f | f \rangle = \int \int |f(x)^2 e^{-|z|^2} \frac{dx dy}{\pi} \]

Reproducing kernel. Indeed \( f(x) = \langle K(z, x) | f \rangle \)

\[ f(x) = \int e^{\omega \bar{z}} f(z) \frac{dx dy}{\pi} \]

Idea: action of $\mathbb{Z}^2$ on $\mathcal{S}(\mathbb{R})$, one generator \( f(x) \rightarrow f(x+1) \), the other \( f(x) \rightarrow e^{2\pi i x} f(x) \); the quotient by the first sends \( f(x) \) to the periodic function \( \sum f(x+m) \), the quotient by the second sends \( f(x) \) to the measure \( \sum_{n \in \mathbb{Z}} \delta(x-n) f(x) \) supported on \( \mathbb{Z} \). Both lie outside $\mathcal{S}(\mathbb{R})$.

Can you take the quotient of $\mathcal{S}(\mathbb{R})$ by $\mathbb{Z}^2$?
Program for today: Wed Jan 23, 02. Let a good understanding of the line bundle \( L \) over \( T^2 \) its connection and the action of the Heisenberg group.

You have a representation the group \( \mathbb{Z}^2 \) in \( \mathfrak{L} \). Ideally, this is equivalent to a module structure for \( \mathfrak{L} \) over the group ring of \( \mathbb{Z}^2 \) which is the ring \( \mathcal{C} \) of functions on \( T^2 \). The quotient of \( \mathfrak{L} \) the group \( \mathbb{Z}^2 \) should be \( \mathfrak{L}/\mathfrak{I} \mathfrak{L} \) where \( \mathfrak{I} \) is the augmentation ideal in the group ring \( \mathcal{A} \), in other words the fibre \( L(0,0) \).

OK now you have the problem of making sense out of the statement that

\[
\begin{align*}
\mathfrak{L}(\mathbb{R}) &\rightarrow C^\infty(\mathbb{R}/\mathbb{Z}) \\
\mathfrak{L}(\mathbb{R}) &\rightarrow C^\infty(\mathbb{R}/\mathbb{Z})
\end{align*}
\]

is the quotient of the \( \mathbb{Z} \)-translation action on \( \mathfrak{L}(\mathbb{R}) \).

---

For \( \psi \in C^\infty(\mathbb{R}^2) \) \( g(y-1, y) \psi(x,y) \leq \psi(x,y) \)

satisfies \( \psi(x+1, y+1) = \psi(x,y) \)

Question: direct embedding of \( \mathbb{L} \) into an (infinite dimensional?) trivial bundle over \( T^2 \) such that the Grassmannian connection yields the desired connection.

To understand \( L \) well enough to describe its connection, start with \( L = \{ \psi \text{ smooth} \mid \psi(x+m, y+n) = e^{-2\pi i m y} \psi(x,y) \} \)

\[
D_x = \partial_x \\
D_y = \partial_y + 2\pi i x
\]

Go back to the case without \( \mathbb{Z}^2 \) action. \( C^\infty(\mathbb{R}^2) \) with the operators \( D_x \) and \( D_y \).
How should you proceed? It's clear that when you say $L$ is $C^{\infty}(\mathbb{R}^2)$ without $Z^2$ conditions, that you have the sections of the trivial line bundle.

So it should be then a matter of algebra identifying $L$ with the Heisenberg group.

\[ e^{2\pi i c} e^{ibD_y} e^{aD_x} \psi(x, y) = e^{2\pi i c} e^{b(d_y+2\pi i x)} e^{aD_x} \psi(x, y) \]

\[ \psi(x+a, y) \]

\[ = e^{2\pi i c} e^{2\pi i bD_y} \psi(x+a, y+b) \]

Check that

\[ D_x \left( \frac{\partial}{\partial y} \right) \psi(x, y) = \frac{\partial}{\partial y} \left[ f(x, y) \psi(x, y) + f(x, y) D_x \psi \right] \]

\[ e^{aD_x} (f \psi) = (e^{aD_x} f)(e^{aD_x} \psi) \]

\[ D^*_x (f \psi) = (\partial x f) \psi + f D_x \psi \]

\[ \frac{a^n}{n!} D^n_x (f \psi) = \frac{a^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \partial_x^k f D_x^{n-k} \psi \]

\[ = \sum_{k+l=n} \frac{a^k}{k!} \partial_x^k f \frac{a^l}{l!} D_x^l \psi \]

What's next?

\[ L(x_0 + 2, y_0 + 2) = \{ \psi(x, y) \text{ on } (x_0 + 2) \times (y_0 + 2) \} \]

\[ \psi(x+a, y+b) = e^{2\pi im_0} \psi(x, y) \]

\[ \Rightarrow a, b \in \mathbb{Z} \]

\[ e^{2\pi i c} e^{2\pi i b} \psi(x+a, y+b) \]

\[ e^{2\pi i c} \psi(x, y) \]
Look at $A = C^\infty(\mathbb{R}^2)$, $\mathcal{L} = C^\infty(\mathbb{R}^2)$.

\[ e^{aD_x}e^{bD_y}f(x,y) = f(x+a, y+b). \]
\[ e^{bD_y}e^{aD_x}\psi(x,y) = e^{2\pi ibx}\psi(x+a, y+b) \]

**Idea:** You want to view the isomorphism $\mathcal{L}(\mathbb{R}) \cong \mathcal{L}$ as providing a way to localize an $\text{F}(a) \in \mathcal{L}(\mathbb{R})$ into pieces with approximately defined time + frequency.

This may not work because position would be a point in the circle's momentum.

**Usual Heisenberg:** $2\pi i x$, $2\pi i y$

**Question:** Suppose you take a partition of unity on the $\mathbb{R}$-circle and form the product to get a partition of unity on $\mathbb{T}^2$. Is there an interesting description of what it means for an element of $\mathcal{L}(\mathbb{R})$ to be supported in a neighborhood of a point of $\mathbb{T}^2$?

Think of time and frequency for a discrete signal, time is measured in seconds, frequency in either radians per second or there's an obvious duality (pairing) which is dimensionless. The $x$ coord gives the phase shift of the discrete signal, the $y$ coord is really the frequency, this is clear from

\[ \sum_{m} e^{2\pi imy} f(x+m) \]

the discrete signal $m \mapsto f(x+m)$, $x$ is phase shift.

**Question:** Do the elements $\text{F}(xy) \in \mathcal{L}$ admit a natural interpretation as kernels?
Next look at the case where $\mathbb{Z}^2$ is ignored. 

$$A = \{ f(x, y) \in C^\infty(\mathbb{R}^2) \}$$

$$L = \{ \psi(x, y) \in C^\infty(\mathbb{R}^2) \}$$

Connection $D_x = \partial_x$, $D_y = \partial_y + 2\pi i y$ on $L$ means librate 

$$\delta_x^L(f\psi) = \frac{\partial f}{\partial x} \psi + f \frac{\partial \psi}{\partial x}$$

Recall 

$$e^{aD_x} e^{bD_y} f(x, y) = f(x+a, y+b)$$

$$e^{aD_x} e^{bD_y} \psi(x, y) = e^{aD_x} e^{bD_y} e^{2\pi i bx} \psi(x, y)$$

$$= e^{aD_x} e^{2\pi i bx} \psi(x, y+b)$$

$$= e^{2\pi i bx(x+a)} \psi(x+a, y+b)$$

$$e^{aD_x} e^{bD_y} \psi(x, y) = e^{2\pi i ab} e^{2\pi i bx} \psi(x+a, y+b)$$

Check 

$$e^{aD_x} e^{bD_y} f = e^{[aD_x, bD_y]} e^{bD_y} e^{aD_x} f = e^{2\pi i ab} e^{bD_y} e^{aD_x} f$$

$$= e^{2\pi i ab} e^{2\pi i bx} \psi(x+a, y+b)$$

How to handle $\psi$?

$$e^{2\pi i(y+b)} e^{2\pi i b(x+1)} \psi(x+a, y+b) = e^{2\pi i y} e^{2\pi i bx} \psi(x+a, y+b)$$

$$e^{2\pi i b(x+1)} \psi(x+1, y+b) = e^{2\pi i bx} e^{-2\pi i y} \psi(x+a, y+b)$$

$$e^{-2\pi i(y+b)} \psi(x+a, y+b)$$
\[
\psi'(x, y) = e^{2\pi i b x} \psi(x + a, y + b)
\]
\[
\psi'(x+1, y) = e^{2\pi i b x + 2\pi i b} \psi(x + a + 1, y + b)
\]
\[
e^{2\pi i y} \psi'(x+1, y) = e^{2\pi i b x + 2\pi i y} \psi(x + a, y + b)
\]

\[
\psi(x, y+1) = \begin{pmatrix} \psi(x, y) \\ e^{2\pi i y} \psi(x+1, y) \end{pmatrix}
\]
\[
\partial_y \psi(x, y) = e^{2\pi i y} \left( 2\pi i \psi(x+1, y) + \partial_y \psi(x+1, y) \right)
\]
\[
\left( \partial_y + 2\pi i x \right) \psi(x, y) = e^{2\pi i y} \psi(x+1, y)
\]
\[
= e^{2\pi i y} \left( 2\pi i \psi(x+1) \psi(x+1, y) + \partial_y \psi(x+1, y) \right)
\]

Assume \( \psi(x, y) = \psi(x, y+1) \)

\[
\exp^{2\pi i x y} \psi(x, y) = e^{2\pi i (x+1)} \psi(x, y+1)
\]

\[
e^{bD_y} \psi(x, y) = e^{2\pi i b x} e^{b \partial_y} \psi(x, y)
\]
\[
\psi_1(x, y) = e^{2\pi i b x} \psi(x, y+b)
\]
\[
e^{2\pi i y} \psi_1(x+1, y) = e^{2\pi i y} \exp^{2\pi i b(x+1)} \psi(x+1, y+b)
\]
\[
= e^{2\pi i b x} \psi(x, y+b) = \psi_1(x, y)
\]

\[
\psi(x, y) = \sum_{n \in \mathbb{Z}} \exp^{2\pi i y} f(x + \imath n)
\]

\[
\int |\psi(x, y)|^2 \, dy = \sum_n |f(x + \imath n)|^2 \quad \therefore \|\psi\|_{L^2}^2 = \|f\|_{L^2}^2
\]
IDEA

You might expect that a direct embedding of $L$ into a twist bundle could yield a simple diffeomorphism of order 1 on $N$.

The transformation $R^2 \rightarrow T^2$ should be possible to use to triangulate $\mathbb{R}^2$ and $(0,1)$ and $(1,0)$ from $0$.

The triangulation has to contain the set $\mathbb{R}^2$ of $\mathbb{Z}$-lines.

You ought to arrange what comes under $L$ to be a $T^2$ action on $T^2$.

It is obvious that if you turn the fiber over $L$ over the point $x$ then the action of $T^2$ on $(\mathbb{R} + \mathbb{Z}) \times (\mathbb{R} + \mathbb{Z})$ satisfies the action condition.

The line at 1-pie.

You might have this by acting on $T^2$.

IDEA

twist bundle
For \( \psi \), you have
\[
\psi(x+m, y+n) = e^{-2\pi i n y} \psi(x,y)
\]
\[
\phi(x,y) = e^{2\pi i x y} \psi(x,y) = e^{2\pi i x y} e^{2\pi i x y} \psi(x+1,y) = \phi(x+1,y)
\]
\[
\phi(x,y+1) = e^{2\pi i x (y+1)} \psi(x,y+1) = e^{2\pi i x} \psi(x,y+1)
\]
\[
\phi(x,y) = \phi(x,y)
\]
\[
\phi(x,y+1) = e^{2\pi i x} \phi(x,y)
\]
\[
\phi(x+m, y+n) = e^{2\pi i m x} \phi(x,y)
\]

\[
\psi_1(x,y) = e^{2\pi i b x} \psi(x+a, y+b)
\]

\[
\psi_1(x+1,y) = e^{2\pi i b x + 2\pi i b} \psi(x+a+1, y+b)
\]

\[
= e^{-2\pi i b y} \psi(x+a, y+b)
\]

\[
= e^{-2\pi i b y} (e^{2\pi i b x} \psi(x+a, y+b)) = e^{-2\pi i b y} \psi_1(x,y)
\]

\[
e^{2\pi i x y + 2\pi i b x} \psi(x+a, y+b)
\]

\[
e^{2\pi i (x+a)(y+b)} \psi(x+a, y+b)
\]

\[
e^{-2\pi i a(y+b)}
\]

proposed to transform by a simple element of \( SL(2, \mathbb{Z}) \).
what ideas? \( \mathbb{Z}^2 \otimes \mathbb{R}/\mathbb{Z} = T^2 \), you want a coordinate free approach; start with a real 2-plane \( V \) and a lattice \( \Gamma \). Suppose \( V \) oriented.
Start with a real 2-plane \( V \), volume element, and a lattice \( \Gamma \) such that \( V/\Gamma \) has volume 1. Choose as basis for \( \Gamma \): \( \mathbf{e}_1, \mathbf{e}_2 \) such that \( \mathbf{e}_1 \mathbf{e}_2 = \text{vol} \). Then you get a standard picture.
In general, given \( V, \text{vol}, \gamma_1, \gamma_2 = \text{vol} \), then your lattice \( \mathbb{Z} \gamma_1 + n \gamma_2 \). You can think of if you pick \( \gamma_1 \) first, then \( \gamma_2 \) is on the "positive" side.

The usual conventions require periodicity \( \text{Im} \gamma_1 \), especially if you complex plane you take \( \gamma_1 = 1, \gamma_2 \in \text{UHP} \).

Something is wrong here. Volume = \( \text{Im} \gamma_1 \). You are introducing a conformal structure on \( V \), it shouldn't be part of the structure at this point. So begin with:

\[ \begin{array}{ccc}
& & \infty \\
0 \quad | & \quad 1 \\
\end{array} \]

This is your basic geometry. What's important?

Recall that given complementary lines in the lattice \( \Gamma \), there are two directions to go in the triangulations.
What you would like to do is to replace
the condition defining \( L \):
\[
\psi(x+m, y+n) = e^{-2\pi i m y} \psi(x, y)
\]

Let \( \phi(x, y) = \psi(x+y, y) \)
\[
\phi(x, y) = \psi(1, y) \psi(x)
\]

\[
\phi(x+1, y) = \psi(x+1+y, y) = e^{-2\pi i y} \psi(x+y, y) = e^{-2\pi i y} \phi(x, y)
\]

\[
\phi(x, y+1) = \psi(x+y, y+1) = e^{2\pi i y} \psi(x+y, y) = e^{2\pi i y} \phi(x, y)
\]

so you find that \( \phi(x+1, y) = \phi(x, y+1) \) so it seems
that \( \phi(x, y) \) is fixed under \( (x, y) \to (x+m, y-n) \)

\[
\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

\[
\phi(x, y) = \psi(x+y, y) = e^{2\pi i y} \psi(x+y, y+1) = e^{2\pi i y} \phi(x, y+1)
\]

\[
= e^{2\pi i y} (e^{2\pi i y} \phi(x, y+2))
\]

\[
\phi(x, y) = \psi(x+y, y) = e^{2\pi i y} \psi(x+y+1, y) = e^{2\pi i y} \phi(x+1, y)
\]

\[
\phi(x+1, y) = \psi(x+1+y, y) = e^{-2\pi i y} \psi(x+y, y) = e^{-2\pi i y} \phi(x, y)
\]

\[
\phi(x, y) = e^{2\pi i y} \phi(x+1, y) = (e^{2\pi i y})^2 \phi(x+2, y)
\]

\[
\phi(x, y) = (e^{2\pi i y})^2 \phi(x, y+2)
\]
\[ \psi(x, y+1) = \psi(x, y), \quad \psi(x+1, y) = e^{-2\pi i y} \psi(x, y) \]

\[ \psi(x+m, y+n) = e^{-2\pi i m y} \psi(x, y) \]

Put

\[ \phi(x, y) = \psi(x, y, y) \]

\[ \phi(x+1, y) = \psi(x+1, y, y) = e^{-2\pi i y} \psi(x, y, y) = e^{-2\pi i y} \phi(x, y) \]

\[ \phi(x+m, y) = e^{-2\pi i m y} \phi(x, y) \]

\[ \phi(x, y+1) = \psi(x, y+1, y+1) = e^{-2\pi i y} \psi(x, y+1, y) = e^{-2\pi i y} \phi(x, y) \]

\[ \phi(x, y+n) = e^{-2\pi i (n-1) y} \phi(x, y+n-1) \]

\[ \phi(x, y+n) = \phi(x, y+n-1) + e^{-2\pi i (y+n-1)} \phi(x, y+n-1) \]

So it seems that you get

\[ \phi(x+m, y) = e^{-2\pi i m y} \phi(x, y) \]

\[ \phi(x, y+n) = e^{-2\pi i n y} \phi(x, y) \]

\[ \phi(x+m, y+n) = e^{-2\pi i n y} \phi(x+m, y) = e^{-2\pi i m y} e^{-2\pi i n y} \phi(x, y) \]

\[ \phi(x+m, y+n) = e^{-2\pi i (m+n) y} \phi(x, y) \]

\[ \psi(x+1, y+1) = \psi(x+1, y) = e^{-2\pi i y} \psi(x, y) \]
Discuss, examine, what happens with the action of \( SL(2, \mathbb{Z}) \). At the moment you have a picture of the sheaf bundle \( \mathcal{L} \) given by its space of sections \( \mathcal{L} \). Maybe it might be a good idea to get a clean description of \( \mathcal{L} \). Go back to the idea of being things working over the plane, ignoring the automorphism conditions.

Begin with \( \mathcal{L} = \mathcal{E} \circ (\mathbb{R}^2) \). \( \mathcal{D}_x = \partial_x \), \( \mathcal{D}_y = \partial_y + \frac{2\pi i}{x} \).

Then, \( e^{\partial_x} e^{b \partial_y} \psi(x, y) = e^{\partial_x} e^{b(2\pi i x)} e^{b \partial_y} \psi(x, y) = e^{\partial_x} e^{2\pi i b x} \psi(x, y) = e^{2\pi i b(x+y)} \phi(x+a, y+b) \).

\( e^{b \partial_y} e^{\partial_x} \psi(x, y) = e^{2\pi i b x} \psi(x+a, y+b) \).

\( e^{\partial_x} e^{b \partial_y} \psi = e^{2\pi i b} e^{b \partial_y} e^{\partial_x} \psi \).

Make this intrinsic. \( V \), real 2-plane, connection \( D \) on the trivial line bundle \( \mathcal{L} \) over \( V \), \( D = d + A \).

In your case, \( V = \mathbb{R}^2 \), \( A = 2\pi i x dy \), but you have already seen that it's nice to use the gauge transform.

\( \psi(x, y) \rightarrow e^{2\pi i x y} \psi(x, y) \).

\( e^{2\pi i x y} (d + 2\pi i x dy) e^{-2\pi i x y} = d - 2\pi i x dy \).

The arbitrariness in \( A \) is the differential of a quadratic function + linear function (?).

\( e^{-\frac{1}{2}} (d + A) e^{\frac{1}{2} \psi} = d + (\xi dx + A) \).

How to study? Begin with the 2-plane \( V = \mathbb{R}^2 \) and a nonzero volume form \( \omega \). Choose a connection on the trivial line bundle \( d + P dx + Q dy \) with curvature \( \omega = 2\pi i dx dy \) e.g. \( d + 2\pi i x dy \). Subtract...
\[ A = P \, dx + (Q - 2\pi i \, x) \, dy \]
\[ dA = (2_y P - 2\pi i) \, dx \, dy \]
\[ d(\text{Pdx + Qdy}) = (-2_y P + 2_x Q) \, dx \, dy \]

Repeat, \( V = \mathbb{R}^2 \) with volume \( 2\pi i \, dx \, dy \)

\[ \text{curvature } D = d + A = d + \text{Pdx + Qdy} \rightarrow dA = (-2_y P + 2_x Q) \, dx \, dy \]

By Poincaré lemma
\[ A = 2\pi i \, x \, dy + df \]
where \( f(x, y) \) is unique up to an additive constant.

Repeat \( V = \mathbb{R}^2 \) with volume \( 2\pi i \, dx \, dy \), consider a connection \( D = d + A \) on the trivial line bundle \( V \times \mathbb{C} \) with curvature \( D^2 = dA = 2\pi i \, dx \, dy \), e.g. \( A = 2\pi i \, x \, dy \).

Let \( A_1 = \text{Pdx + Qdy} \) be another connection form with same curvature. Poincaré lemma \( \Rightarrow \ A_1 = 2\pi i \, x \, dy + df \) where \( f \) is unique up to an add. const.

Philosophy here: \( \omega \) is translation invariant, so you would like \( A \) to be translation invariant, impossible, so do the best you can. From the 2 examples, \( A = 2\pi i \, x \, dy \) and \(-2\pi i \, y \, dx \), you have translation invariance in the \( x \) and \( y \) directions resp. So it becomes clear that you want to choose a line (one dual subspace of \( V \)) and require \( A \) to be invariant under translations parallel to this line.

\[ A = P(x, y) \, dx + Q(x, y) \, dy \]
\[ l = R(r, s) \]
\[ P(x + tr, y + ts) \, dx + Q(x + tr, y + ts) \, dy \]

so \( P, Q \) must be constant along cosets of \( l \)

i.e. they are functions on \( V/l \)

So it seems you have a natural class of connections to pin down generalizing your two examples
\[-2\pi i y \, dx, 2\pi i x \, dy\]
Recall exact sequence
\[ \mathbb{R}^2 \xrightarrow{\omega} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \xrightarrow{d} \mathbb{R}^2 \]
looking at \( \omega \) which is acted by \( SL(V,\omega) \)

- y do Question: Does \( -y \) dx have an intrinsic
meaning? This is a 1-form on \( V \)

\[ \omega = dz - dy \]

You want to understand 1-forms \( A \) on \( V \) such that
\( dA = \omega \). Given such a 1-form \( A_0 \), Poincaré's lemma says any other has the form \( A = A_0 + df \), unique modulo constants. Consider translations on \( V \): \( (x,y) \rightarrow (x+6,y+6) \)

These preserve \( \omega \), hence translations act on \( \{ A \mid dA = \omega \} \).

It seems you want to choose \( A_0 \) so that its orbit under the translation group is as small as possible. For example if \( A_0 = xdy \), then \( A_0 \) is preserved under \( y \rightarrow y+6 \), so the orbit is \( \{ (x+6,y) \mid x \in \mathbb{R}^2 \} \), 1-dime affine line.

Next look at the action of \( SL(2,\mathbb{R}) \), and maybe \( SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 \)

You should have an action of \( SL(2,\mathbb{R}) \ltimes \mathbb{R}^2 \) on the
space of connections \( dA \) on the trivial line bundle \( \mathbb{R}^2 \times \mathbb{C} \)? having curvature \( \omega \)

\[ \mathbb{R}^2 \xrightarrow{d\theta + \frac{1}{2}(dx+2\pi i dy)} \mathbb{R}^2 \]

you probably want to replace \( SL(2,\mathbb{R}) \) with its Lie alg. 
IDEA: \( SL(2,\mathbb{R}) \) operates on the circle \( \mathbb{P}^1(\mathbb{R}) \)
Go back to the trivial line bundle over $\mathbb{R}^2$ whose sections are just smooth functions $\lambda(x, y)$.

This time, you won't translate and symplectic linear transformations on $\mathbb{R}^2$. The Lie alg of $SL(2, \mathbb{R})$ should give rise to vector fields on $\mathbb{R}^2$: $\lambda(x, y) \partial_x$, $\lambda(x, y) \partial_y$.

$$\left(\begin{array}{c}
\partial_x \\
\partial_y
\end{array}\right)$$

basis of $\text{End}(V) =$ endos of trace 0

Lie of translations have form $\partial_x$, $\partial_y$.

$$[\lambda \partial_y, \lambda \partial_x] = \lambda [\partial_y, \partial_x] + \lambda^2 [\partial_y, \partial_x]$$

$$= \lambda \partial_x - \lambda \partial_y$$

Now take the connection $\text{d} + 2\pi i x \text{d}y$ on $L = \mathbb{R}^2 \times \mathbb{C}$.

A vector field? Start with $\mathbb{R}^2$ equipped with the infinitesimal translation action. The Lie alg (abelian) with basis $\partial_x, \partial_y$ (vector fields). Then enlarge the action to include the infinitesimal action of $SL(2, \mathbb{R})$. So then you have 5 vector fields $\partial_x, \partial_y, x \partial_x, y \partial_x, x \partial_y - y \partial_y$.

on $\mathbb{R}^2$. It might be true the stabilizers of the connection is the largest Lie subalgebra such that the lift whose lift to $L$ preserves bracket.

Review. You have brought in $SL(2, \mathbb{R})$. Begin with $\mathbb{R}^2$ and volume $\omega = \text{d}x \text{d}y$, symmetry group $SL(2, \mathbb{R}) \times \mathbb{R}^2$.

Describe the action by the Lie alg. basis $\partial_x, \partial_y, x \partial_x, y \partial_x, x \partial_y - y \partial_y$.

Now look at connections on the trivial line bundle over $\mathbb{R}^2$.

What is the meaning of $[D_0, x]_\omega$? Probably this is $L_x - L_x A$. The lift of $L_x$.

$[D_0, x]_\omega = [d + A] \omega = L_x - L_x A$
Look at
\[ \frac{\partial}{\partial x} D - D \frac{\partial}{\partial x} = X + \frac{(2\pi i)^{\gamma}}{L_x(x+y)} \]
\[ X = \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial x} + (2\pi i)^{\gamma} \quad \frac{\partial}{\partial y} \quad D_x \quad 0 \]
\[ x \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \quad x \frac{\partial}{\partial x} + (2\pi i)^{\gamma} \quad x \frac{\partial}{\partial y} \quad -2\pi i x \quad -2\pi i x \]
\[ x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \quad -2\pi i x \quad 0 \]

So \(-\frac{\partial}{\partial x}\) is the obstruction to \(X\) preserving the
connection \(A\).

need a discussion about what to expect, the problem
is that the symmetries of the original situation \(\mathbb{R}^2, \omega\)
do not preserve \(A\). Let's start with \(\mathbb{R}^2, \omega\) and a
choice for \(A\), say \(2\pi i x \, dy\). Take a symmetry of \((\mathbb{R}^2, \omega)\)
say a translation \((x,y) \mapsto (x+a, y+b)\). \(A\) is not preserved
\(d + 2\pi i x \, dy \mapsto d + 2\pi i(x+a) \, dy\), but you can combine this
symmetry with a gauge transformation
\[
(2\pi i x \, dy + 2\pi i a \, dy) \mapsto e^{2\pi i a y} d + 2\pi i x \, dy
\]
to preserve the connection. Thus you can lift translations
on \((\mathbb{R}^2, \omega)\) to autos of the line bundle + connection
\((\mathbb{R}^2 \times \mathbb{R}, d + 2\pi i x \, dy)\). The gauge transform is unique up to a
phase factor, so there is a central extension of \(\mathbb{R}^2\) acting on \(L\).

Find formula
\[
\psi(x,y) \mapsto e^{2\pi i a y} \psi(x+a,y+b)
\]
\[
(\partial_y + 2\pi i x) \psi(x,y) \mapsto e^{2\pi i a y} (\partial_y + 2\pi i(x+a)) \psi(x+a,y+b)
\]
\[
\psi(x, y) = e^{a x} e^{b y} \psi(x, y)
\]
\[
= e^{a x} e^{2\pi i b x} \psi(x, y + b) = e^{2\pi i b(x + a)} \psi(x + a, y + b)
\]
\[
\psi \in \mathcal{C}^\infty(\mathbb{R}^2)
\]

\[
D \psi = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} + 2\pi i x \frac{\partial \psi}{\partial y} + 2\pi i y \frac{\partial \psi}{\partial x}
\]

\[
T_{a,b} \psi(x, y) = \psi(x+a, y+b) = e^{a x} e^{b y} \psi(x, y)
\]

\[
(x, y) \mapsto (x+a, y+b)
\]

\[
D \psi (x, y) = \frac{\partial \psi}{\partial x} \psi(x, y) + \frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial x} + 2\pi i x \right) \psi(x, y)
\]

\[
D \psi (x+a, y+b) = \frac{\partial \psi}{\partial x} \psi(x+a, y+b) + \frac{\partial \psi}{\partial y} \left( \frac{\partial \psi}{\partial x} + 2\pi i x + 2\pi i a \right) \psi(x+a, y+b)
\]

So your notation is confused. You have to keep straight

\[
D_y \psi(x+a, y+b) = (\frac{\partial \psi}{\partial y}) (x+a, y+b) + 2\pi i x \psi(x+a, y+b)
\]

and

\[
(\frac{\partial \psi}{\partial y}) (x+a, y+b) = (\frac{\partial \psi}{\partial y}) (x+a, y+b) + 2\pi i (x+a) \psi(x+a, y+b)
\]

Maybe you should be working with operators on functions and evaluation at points \( \langle x, y \rangle \). Operators should act on \( \mathcal{C}^\infty(\mathbb{R}) \), possibly \( \Omega(\mathbb{R}) \).

First you need the connection.

Start again with the manifold \( \mathbb{R}^2 \) and "vol" \( 2\pi i \, dx \wedge dy \).

Have symmetries from \( \text{SL}(2, \mathbb{R}) \times \mathbb{R}^2 \), but we will look first at translations. Want connected with curvature \( A \), but \( d + A \) will not be translation invariant. Possible to correct by using a gauge transform. Example. \( A = 2\pi i x \, dy \)

\[
e^{a x} e^{b y} x \, dy = e^{a x} x \, dy e^{b y}
\]

\[
= (x + a) \, dy \, e^{a x} e^{b y}
\]

\[
e^{a x} e^{b y} (a + 2\pi i x \, dy) = (a + 2\pi i x + 2\pi i a) \, e^{a x} e^{b y}
\]

\[
e^{-2\pi i a x}
\]
\[ e^{a \partial_x} e^{b \partial_y} (d + 2\pi \text{i} x \text{d}y) = (d + 2\pi \text{i} x \text{d}y + 2\pi \text{i} a \text{d}y) e^{a \partial_x} e^{b \partial_y} \]

so \[ e^{2\pi \text{i} a y} e^{a \partial_x} e^{b \partial_y} \] leaves \( d + 2\pi \text{i} x \text{d}y \) fixed.

This means that \( \partial_x + 2\pi \text{i} y, \partial_y \) commute with \( d + 2\pi \text{i} x \text{d}y \):

\[ [\partial_x + 2\pi \text{i} y, \partial_y + 2\pi \text{i} x] = 2\pi \text{i} - 2\pi \text{i} = 0 \]

You have reached a difficulty, you did not expect to worry about left + right when dealing with symmetries of \( \mathbb{L} \). Go back and review.

\[ \mathcal{A} = C^\infty(T^2) = \{ f : C^\infty(R^2) \mid e^{a \partial_x} e^{b \partial_y} f = f \} \]

for \( a, b \in \mathbb{Z} \)

\[ \mathcal{L} = \{ \psi \in C^\infty(R^2) \mid \psi(x, y+1) = \psi(x, y) = e^{2\pi \text{i} y} \psi(x, y) \}

e^{b \partial_y} \psi = \psi \quad b \in \mathbb{Z}

e^{a (\partial_x + 2\pi \text{i} y)} \psi = \psi \quad a \in \mathbb{Z} \]

begin with \( R^2 \) with infinitesimal translations \( a \partial_x + b \partial_y \) and 2-form \( \omega = 2\pi \text{i} \text{d}x \text{d}y \) which is translation invariant:

\[ \mathcal{L} \] (the 
\begin{align*}
\text{line bundle with curvature } \omega, \quad \text{e.g.} \quad d + 2\pi \text{i} x \text{d}y. \quad \text{Choose a connection on the trivial connection.}
\end{align*}

\[ D(\psi) = df \psi + f D\psi \]

\[ D(\psi) = df \psi + f D\psi \]

Idea: the connection is not translation invariant, but this can be corrected by a gauge transform:

\[ \mathcal{L}(\partial_x) (d + 2\pi \text{i} x \text{d}y) = 2\pi \text{i} \text{d}y \]

\[ \mathcal{L}(\partial_x + b \partial_y) (\text{d} + 2\pi \text{i} x \text{d}y) = 2\pi \text{i} \text{a d}y \]

\[ [d + 2\pi \text{i} x \text{d}y, 2\pi \text{i} a y] = 2\pi \text{i} a \text{d}y \]
\[ \left( (x + \beta) (d + 2\pi i x dy) \right) (d + 2\pi i x dy) = 0 \]

You seem to need practice. Start with the Lie group \( \mathbb{R}^2 \), the invariant vector field \( \partial_x, \partial_y \) giving translation \( \langle x, y \rangle e^{\alpha \partial_x + \beta \partial_y} x = \langle x + \alpha, y + \beta \rangle x \).

Fix \( A = 2\pi i x dy \) a 1-form such that \( D = d + A \) is a connection on the trivial line bundle, whose curvature is \( dA = 2\pi i dy \cdot dy = 0 \). Now \( \omega = 0 \) is translation invariant, but \( A \) isn't:

\[
\begin{align*}
\left( \partial_x \partial_y (x + \alpha) \partial_y \right) e^{\partial_x + \partial_y} & = (x + \alpha) e^{\partial_x + \partial_y} \\
\left( \partial_x \partial_y (x + \alpha) \partial_y \right) e^{\partial_x + \partial_y} & = (x + \alpha) dy \\
\left( \partial_x \partial_y (d + 2\pi i x dy) \right) e^{\partial_x + \partial_y} & = d + 2\pi i (x + \alpha) dy \\
\left( \partial_x \partial_y (d + 2\pi i x dy) \right) e^{\partial_x + \partial_y} & = d + 2\pi i (\alpha + x) dy
\end{align*}
\]

Therefore you find:

\[
\left( 2\pi i y \partial_x \partial_y \right) e^{\partial_x + \partial_y} (d + 2\pi i x dy) = (d + 2\pi i x dy) e^{2\pi i y \partial_x \partial_y}
\]

Although the connection \( d + 2\pi i x dy \) commutes with \( d + 2\pi i x dy \):

\[
[\partial_x + 2\pi i y, \partial_y] = -2\pi i
\]

What's strange at this point is that:

\[
\begin{bmatrix}
\partial_x + 2\pi i y \\
\partial_y
\end{bmatrix}
\begin{bmatrix}
\partial_x \\
\partial_y
\end{bmatrix}
= 0
\]

These generate a Heisenberg algebra. So one should be a left action and the other should be a right action.
Next case \( SL(2, \mathbb{R}) \) acting. Ref. \( \{ x \partial_y, y \partial_x \} = x \partial_x - y \partial_y \)
\( x \partial_y, y \partial_x, x \partial_x - y \partial_y \) are vector fields on \( \mathbb{R}^2 \) fixing \( 0 \).

\[
[ x \partial_y, \; d + 2 \pi i x \partial y ]
\]

\[
e^{t x \partial_y}
\]

Everything should be an operator on \( \Omega^*(\mathbb{R}^2) \). A vector field is first order differential operator on \( \Omega^*(\mathbb{R}^2) \), but it has a natural extension to all forms. As \( L(x) = [d, l(x)] \) so it seems that \( [x \partial_y, \; ] \) means \( L(x \partial_y) \)

\[
L(x \partial_y) (x \partial y) = (L(x \partial_y) x) \partial y + x (L(x \partial_y) \partial y)
\]

\[
= x d L(x \partial_y) y = x \partial x
\]

\[
L(x \partial_y) x \partial y = d (l(x \partial_y) x \partial y) + i(x \partial_y) x \partial y d y
\]

\[
= d (x^2) - d x = x \partial x
\]

\[
L(y \partial_x) (x \partial y) = y \partial y + x \partial 0 = y \partial y
\]

\[
(d (l(y \partial_x) + l(y \partial_x) d)) (x \partial y) = l(y \partial_x) x \partial y d y = y \partial y
\]

\[
L(x \partial_x) (x \partial y) = x \partial y
\]

\[
L(y \partial_x) (x \partial y) = 0 \partial y + x \partial (l(y \partial_x) \partial y) = x \partial y
\]

\[
L(x \partial_x - y \partial_y) (x \partial y) = 0
\]

Check.

\[
L(x \partial_y) L(y \partial_x) (x \partial y) = L(x \partial_y) y \partial y = x \partial y + y \partial x
\]

\[
L(y \partial_x) L(x \partial_y) (x \partial y) = L(y \partial_x) x \partial x = y \partial x + x \partial y
\]

Summarize

\[
L(x \partial_y) (x \partial y) = x \partial x
\]

\[
L(y \partial_x) (x \partial y) = y \partial y
\]

\[
L(x \partial_x - y \partial_y) (x \partial y) = 0
\]

Now you apparently the vector field \( x \partial_x - y \partial_y \) respects \( D = d + 2 \pi i x \partial y \)
Good question: What is \( e^{(x\partial_x - y\partial_y)} \) applied to \( f \in \Omega(\mathbb{R}^2) \)?

\[
e^{tx\partial_x} x = \sum_{n=0}^\infty \frac{(tx)^n}{n!} x = e^{t x}
\]

\[
e^{t(x\partial_x - y\partial_y)} y = \begin{bmatrix} e^{tx} \\ e^{ty} \end{bmatrix}
\]

\( \text{sl}(2,\mathbb{R}) \) acts on \( \Omega(\mathbb{R}^2) \) \( x \mapsto L(x) \)

It has been \( x^2 > y^2 \), \( x\partial_x - y\partial_y \)

\[
L(x\partial_y)(d + 2\pi i xdy) = 2\pi i \left( L(x\partial_y)(xdy) \right)
\]

\[
x \left( L(x\partial_y) \right) dxdy + y \left( L(x\partial_y) \right) dydxdy
\]

\[
L(y\partial_x)(d + 2\pi i xdy) = 2\pi i \left[ L(y\partial_x) dxdy + xL(y\partial_x) dydxdy \right]
\]

\[
\begin{align*}
L(x\partial_y)(d + 2\pi i xdy) &= 2\pi i (xdx) \\
L(y\partial_x)(d + 2\pi i xdy) &= 2\pi i (ydy)
\end{align*}
\]

\[
dL(x\partial_y) xdy + i(x\partial_y) dxdy
\]

\[
d(x^2) \otimes - xdx = xdy.
\]

So what you want to adjust by a gauge transform.

\[
e ^{2\pi i x^2} (d + 2\pi i xdy) e^{-2\pi i x^2} = d + 2\pi i xdy + 2\pi i \partial_y (xdy)
\]

\[
e ^{-2\pi i x^2} (d + 2\pi i xdy) e^{2\pi i x^2} = d + 2\pi i xdy + 2\pi i xdx
\]

\[
L(x\partial_y)(d + 2\pi i xdy) + \left[ \pi i x^2, d + 2\pi i xdy \right] = 0
\]

\[
2\pi i (xdx) - \pi i [dy^2] = -2\pi i xdx.
\]

\[
L(y\partial_x)(d + 2\pi i xdy) + \left[ \pi i y^2, d + 2\pi i xdy \right] = 0
\]

\[
L(x\partial_y - y\partial_y)(d + 2\pi i xdy) = 0
\]
Review. Plane $\mathbb{R}^2$, action of $SL(2, \mathbb{R}) \times \mathbb{R}^2$

Lie algebra $\partial_x, \partial_y, x\partial_y, y\partial_x, x\partial_x - y\partial_y$

$\omega = 2\pi i \, dx \wedge dy$ invariant

Connection $d + x \partial y$ is not invariant but can be corrected by gauge transform.

\[
\begin{align*}
[x \partial_y, \partial_x] &= -\partial_y \\
[x \partial_y, \partial_y] &= 0 \\
[x \partial_x, \partial_y] &= x \\
[y \partial_x, \partial_x] &= 0 \\
[y \partial_x, \partial_y] &= y
\end{align*}
\]

\[
\begin{align*}
\text{ad}(x \partial_y)(x) &= 0 \\
\text{ad}(y \partial_x)(y) &= 0
\end{align*}
\]

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]

Now to calculate. Manifold $\mathbb{R}^2$, to be playing with operators on $C^\infty(\mathbb{R}^2)$, more generally, $\Omega(\mathbb{R}^2)$. As usual for

If $X = f \partial_x + g \partial_y$ is a vector field on $\mathbb{R}^2$, then it acts on $\Omega(\mathbb{R}^2)$ by $\mathcal{L}(X) = d \circ (X) + (X) \circ d$

trivial (complex) line bundle over $\mathbb{R}^2$

connection $d + A \quad A \in \Omega^1(\mathbb{R}^2)$

\[
\text{Describe your aim. Take } A = xdy \\
\text{d}(a \partial_x + b \partial_y) \, dx \wedge dy = d \circ (a \partial_x + b \partial_y) \, dx \wedge dy = d \circ (a \, dy - b \, dx) = 0.
\]
\[ L(\alpha \partial_x + b \partial_y) (d + x \partial_y) = L(\alpha \partial_x + b \partial_y) d \partial_y + L(x \partial_x) \partial_y \]
\[ = d (\partial_x) + x \partial_y - b \partial_x = \partial_y \\
= e^{a \partial_y} (d + x \partial_y) e^{-a \partial_y} = d - \partial_y + x \partial_y \\
[ L(\alpha \partial_x + b \partial_y) + A \partial_x (e^{a \partial_y}) ] (d + x \partial_y) = d + x \partial_y \]

 Aim: To link with what you did before.

section \( L = \{ \psi \in C^\infty (\mathbb{R}^2) \mid \text{autm. cond.} \} \)?  \textbf{Ex.}  
\( f \in L(\mathbb{R}) \quad \tilde{f} = \sum_{m \in \mathbb{Z}} e^{m (\partial_x + 2\pi i y)} f \)

\( \tilde{f}(x,y) = \sum_{m} e^{2\pi i x y} f(x+m) \)
\( = \sum_{m} e^{2\pi i x y} e^{m \partial_x} f(x) = \sum_{m} e^{m (\partial_x + 2\pi i y)} f(x). \)

\[ e^{\partial_y} \tilde{f} = f \]

Don't worry about notation for \( L(\mathbb{R}) \), rather get the automorphy condition straight.

\[ e^{2\pi i x y} \tilde{f} = e^{2\pi i x y} f \]
\[ e^{2\pi i (x+1) y} e^{\partial_x} f = e^{2\pi i x y} f \]
\[ e^{2\pi i y} e^{\partial_x} f = e^{2\pi i y} f \]
\[ e^{2\pi i y} \partial_x \tilde{f} = \tilde{f} \]

Almost there.

\( L = \{ \psi \in C^\infty (\mathbb{R}^2) \mid e^{\partial_y} \psi = \psi, \quad e^{2\pi i y} \partial_x \psi = \psi \} \)

\[ \tilde{f} = \sum_{m} e^{2\pi i m y} e^{m \partial_x} f = \sum_{m} e^{m (\partial_x + 2\pi i y)} f \]

\[ \partial_x \tilde{f} = \tilde{f} \]
\[ \partial_y \tilde{f} = \sum_{m} e^{m \partial_x} e^{m 2\pi i y} (2\pi i y) f = (2\pi i x) \tilde{f} \]