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School of Mathematics
University of Minnesota
Minneapolis, Minn. 55455 USA

330 Once the spectrum + mult is given you have the operator A, self adjoint. This operator is specified by its graph $\Gamma_A \subset \begin{smallmatrix} H \\ \oplus \\ H \end{smallmatrix}$, spectrum is where it meets $\binom{1}{\lambda} H$.

$$\begin{tikzcd} & H \arrow[d, "\binom{1}{\lambda}"] & \\ \Gamma_A \arrow[r] & H^{\oplus 2} \arrow[r] \arrow[d, "(-\lambda \;\; 1)"] & \Gamma_A^{\perp} \\ & H & \end{tikzcd} \qquad \begin{pmatrix} \xi \\ A\xi \end{pmatrix}$$

maybe look at the form on $H^{\oplus 2}$

$$\lambda \|\xi^+\|^2 + \|\xi^-\|^2 = \left(\begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi^+ \\ \xi^- \end{pmatrix} \right) \qquad \text{NO.}$$

What you need is the proof that $\lambda - A : D_A \to H$ is invertible for $\mathrm{Im}(\lambda) \neq 0$.

$$H \xrightarrow{\binom{1}{A}} H^{\oplus 2}$$

$$\binom{1}{A}^* \binom{1}{A} = 1 + A^2$$

$$H \underset{\text{isom.}}{\xrightarrow{\binom{1}{A}(1+A^2)^{-1/2}}} H^{\oplus 2} \xleftarrow{\binom{-A}{1}(1+A^2)^{-1/2}} H$$

$$(1+A^2)^{-1/2} (-A \;\; 1) \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} -A \\ 1 \end{pmatrix} (1+A^2)^{-1/2}$$

$$= (1+A^2)^{-1/2} (A^2 + \lambda) (1+A^2)^{-1/2} = \frac{\lambda + A^2}{1 + A^2}$$

Different approach. ~~Go back to~~ Go to D.

First of all, need $\cancel{\frac{1}{\omega - \lambda} = \frac{1}{1+i\omega} - \frac{1+i\lambda}{1-i\lambda}}$

331 Compare $\lambda - a$ $\quad z - \alpha$ under C.T.

$$z = \frac{1-(-i\lambda)}{1+(-i\lambda)} = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i} \qquad \lambda = i\,\frac{1-z}{1+z}$$

$$\lambda - a = i\,\frac{1-z}{1+z} - a = \frac{i-iz-a-az}{1+z} = \frac{(i-a)-z(i+a)}{1+z}$$

$$= \frac{i+a}{1+z}\left(\frac{-a+i}{a+i} - z\right) = \left(-\frac{a+i}{z+1}\right)\left(z - \underbrace{\frac{-a+i}{a+i}}_{\alpha}\right)$$

$$\frac{1}{\lambda-a} = \underbrace{\left(-\frac{1+z}{a+i}\right)\frac{1}{z-\alpha}} = \left(-\frac{1+\alpha}{a+i}\right)\frac{1}{z-\alpha} \quad \frac{1}{a+i}$$

pole at $\lambda=a$ $\quad$ pole at $z=\alpha$

zero at $\lambda=\infty$ $\quad$ zero at $z=-1$ $\qquad 1+\alpha = 1+\frac{-a+i}{a+i} = \frac{2i}{a+i}$

$$z-\alpha = \frac{-\lambda+i}{\lambda+i} - \alpha = \frac{-\lambda+i-(\lambda+i)\alpha}{\lambda+i}$$

$$= \frac{-\lambda(1+\alpha)+(i-i\alpha)}{\lambda+i} = (1+\alpha)\,\frac{i\frac{1-\alpha}{1+\alpha}-\lambda}{\lambda+i}$$

$$\lambda+i = i\left(\frac{1-z}{1+z}+1\right) = \frac{2i}{1+z} \qquad \boxed{\lambda+i = \frac{2i}{1+z}}$$

$$\cancel{1+i\lambda = \frac{2}{1+z}}$$

$$z-\alpha = (1+\alpha)\,\frac{a-\lambda}{\lambda+i} = \frac{2i}{a+i}\,\frac{a-\lambda}{\lambda+i}$$

$$\frac{z-\alpha}{z-\beta} = \frac{b+i}{a+i}\;\frac{a-\lambda}{b-\lambda}$$

$$\frac{1}{a-\lambda} = \frac{z+1}{z-\alpha}\cdot\frac{1}{a+i} = \left(\frac{1}{z-\alpha}+k\right)\frac{\quad}{a+i}$$

$$\frac{1}{a-\lambda} = \frac{z+1}{z-\alpha}\,\frac{1}{a+i} \qquad \boxed{\frac{a+i}{a-\lambda} = \frac{z+1}{z-\alpha}}$$

pole $\lambda = a$ pole $z=\alpha$

zero $\lambda = \infty$ zero $z=-1$

value $\frac{1}{a+i}$ $\lambda = -i$ $z=\infty$

$$= \frac{1+\alpha}{z-\alpha} + 1$$

$$\frac{1}{a-\lambda} = \frac{1+\alpha}{a+i}\,\frac{1}{z-\alpha} + \frac{1}{a+i}$$

$$\frac{2i}{(a+i)^2}$$

So now take $\sum \frac{m_k}{a_k-\lambda}$

$$\sum \frac{m_k}{a_k-\lambda} = \sum \frac{2i}{(a_k+i)^2}\,\frac{m_k}{z-\alpha_k} + \sum \frac{m_k}{a_k+i}$$

do I want this?

You are stupid. Try again. Maybe the point is the type of analytic function. Thus given a rational function with poles on the line. There should be an obvious 1-1 corresp. So your mistake involves asking the wrong question. Try again

$$\frac{1}{\lambda-a} \qquad a\in\mathbb{R}$$

real on $\mathbb{R}$ simple pole at a. You want then ~~a function on the disk~~ a rational fn. real valued on the disk, simple pole at $z=\alpha$.

$$\frac{1}{z-\alpha} \qquad \text{guess} \qquad \sum_{n\ge 0}\alpha^{-n}z^n + \sum_{n\ge 1}\alpha^n z^{-n}$$

$$= \frac{1}{1-\alpha^{-1}z} + \frac{\alpha z^{-1}}{1-\alpha z^{-1}} =$$

$$= \frac{1}{1-\bar{\alpha} z} + \frac{\alpha}{z-\alpha}$$

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$$\frac{rz+s}{z-\alpha} = \frac{\bar r z^{-1}+\bar s}{z^{-1}-\alpha^{-1}} = \frac{(\bar r+\bar s z)\alpha}{\alpha - z}$$

$$-(rz+s) = (\bar r+\bar s z)\alpha$$

$$-r = \bar s \alpha \qquad -s = \bar r \alpha \qquad -\bar s = r\bar\alpha$$

$$\cancel{r=1 \qquad s=\alpha}$$

$$r = -\bar s \alpha \qquad \bar r \alpha = \overline{(-\bar s \alpha)}\,\alpha = -s$$

$$\frac{(-\bar s \alpha)z+s}{z-\alpha} =$$

e.g. $\dfrac{1-\alpha z}{z-\alpha}$

$$\frac{s}{z-\alpha} + \frac{\bar s}{z^{-1}-\alpha^{-1}} = \frac{s}{z-\alpha} + \frac{\bar s \alpha z}{\alpha - z}$$

$$= \frac{s-\bar s \alpha z}{z-\alpha}$$ ~~need van. at z=p~~

$$\frac{1}{\lambda - a} = \frac{1}{i\frac{1-z}{1+z} - a} = \frac{1+z}{i(1-z)-a(1+z)} = \frac{1+z}{(i-a)-(i+a)z}$$

$$= \frac{1+z}{z-\frac{i-a}{i+a}}\;\frac{1}{-(i+a)}$$

$\dfrac{s-\bar s \alpha z}{z-\alpha}$ want 0 at $z=-1$. $s+\bar s\alpha = 0.$

$$\frac{s}{\bar s} = -\alpha$$

$$s = \frac{1}{-(i+a)} \qquad \frac{s}{\bar s} = \frac{a-i}{a+i} = -\alpha.$$

334 Stupid. $f(\alpha) = \frac{1}{2\pi i}\int \frac{f(z)}{z-\alpha}\,dz$

What do you learn?

$$\frac{r\cancel{z}+s}{z-\alpha} = \frac{\bar r z^{-1}+\bar s}{z^{-1}-\alpha^{-1}} = \frac{\alpha\bar r + \bar s\alpha z}{\alpha - z}$$

$-r = \bar s\alpha$ $\quad -s = \alpha\bar r$ $\quad -\bar s\alpha = r$

$-\bar s = \bar\alpha^{-1} r$

$$\frac{-\bar s\alpha z + s}{z-\alpha}$$

pole at $z=\alpha$

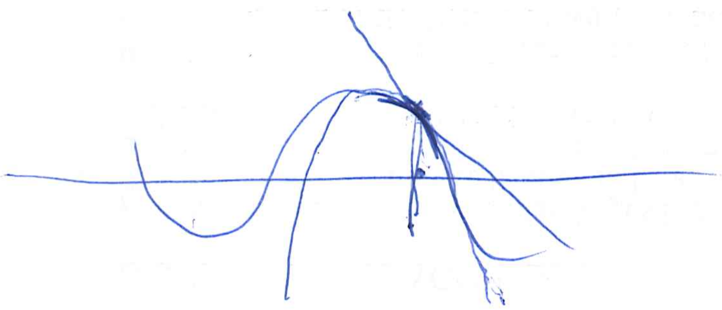
zero at $z = \frac{s}{\bar s\alpha}$ which can be any pt. of circle. Obvious choice is opposite to pole i.e

$\frac{s}{\bar s\alpha} = -\alpha$, if $|s|=1$ $\bar s = s^{-1}$. $s^2 = -\alpha^2$

$s = \pm i\alpha$, so you get

$$\frac{-(-i\bar\alpha)\alpha z + i\alpha}{z-\alpha} = \frac{i(z+\alpha)}{z-\alpha}$$

$$\left(1-\frac{\theta^2}{2}\right)\left(1-\frac{4\theta^2}{2}\right) = 1-\theta^2\left(\frac{1}{2}+2\right)$$



$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

$$\sqrt{1+\sin\theta} = 1 + \frac{x}{2} + \frac{x^2}{8}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!}$$

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$$\frac{r\lambda+s}{\lambda-a} = \frac{\bar r\lambda+\bar s}{\lambda-a}$$

$a\in\mathbb{R}$ $\quad$ r, s real.

to get what corresp. to $\frac{i(z+\alpha)}{z-\alpha}$ you need

how sign change $\alpha\mapsto-\alpha$ affects $a = i\frac{1-\alpha}{1+\alpha}$

$a = i\frac{1-\alpha}{1+\alpha} \longrightarrow i\frac{1+\alpha}{1-\alpha} = -\frac{1}{a}$), so you look for

$$r\frac{\lambda+\frac{1}{a}}{\lambda-a}$$

Understand analytic functions f on D or UHP with $\mathrm{Im}(f)>0$. $\mathrm{Im}(f)$ is a positive harmonic function — $\therefore$ Poisson transform of a measure = radial limit of $\mathrm{Im}(f)\frac{d\theta}{2\pi}$, f unique up to real constant,

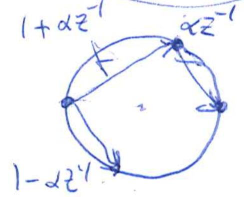
Fix $\alpha=e^{i\phi}$ Want Pick function corresp. to the point measure at α mass 1. Determined up to a real const.

$$\frac{rz-s}{z-\alpha} = \frac{\bar r z^{-1}-\bar s}{z^{-1}-\alpha^{-1}} = \frac{\bar r\alpha-\bar s\alpha z}{\alpha-z} \qquad r=\bar s\alpha$$

Ask for value at 0 to be i

$$\frac{\bar s\alpha z-s}{z-\alpha} \qquad \frac{\bar s\alpha 0-s}{0-\alpha} = \frac{s}{\alpha} = i \qquad s=i\alpha.$$

$$\frac{(-i\bar\alpha)\alpha z-i\alpha}{z-\alpha} = (-i)\frac{z+\alpha}{z-\alpha}$$



$$\frac{1}{\omega-\lambda} - \mathrm{Re}\left(\frac{1}{\omega-i}\right)$$

$$\int\left(\frac{1}{\omega-\lambda}-\frac{\omega}{\omega^2+1}\right)d\mu(\omega) = f(\lambda)$$

make real part vanish at $\lambda=i$

$$f(z) = a_0 + \sum_{n\geq 1} a_n z^n \qquad \text{anal. for } |z| < 1+\varepsilon$$

$$f(e^{i\theta}) = a_0 + \sum_{n\geq 1} a_n e^{+in\theta} \qquad a_n = \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

$$h(\theta) = \operatorname{Im} f(e^{i\theta}) = \operatorname{Im}(a_0) + \sum_{n\geq 1} \frac{1}{2i}\left(a_n e^{in\theta} - \bar{a}_n e^{-in\theta}\right)$$

$$a_n = \int_0^{2\pi} 2i\, h(\theta)\, e^{-in\theta} \frac{d\theta}{2\pi} \qquad n \geq 1.$$

$$i \operatorname{Im}(a_0) = \int_0^{2\pi} i h(\theta) \frac{d\theta}{2\pi}$$

$$f(z) = \operatorname{Re}(a_0) + \int_0^{2\pi} i h(\theta) \underbrace{\left[1 + \sum_{n=1}^{\infty} 2 z^n e^{-in\theta}\right]}_{} \frac{d\theta}{2\pi}$$

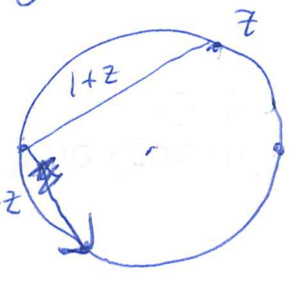
$$1 + 2\frac{z e^{-i\theta}}{1 - z e^{-i\theta}} = \frac{1 + z e^{-i\theta}}{1 - z e^{-i\theta}}$$

$$f(z) = \operatorname{Re} f(0) + i \int_0^{2\pi} \underbrace{\frac{1 + z e^{-i\theta}}{1 - z e^{-i\theta}}}_{} \operatorname{Im} f(e^{i\theta}) \frac{d\theta}{2\pi}$$

$$\frac{(1 + z e^{-i\theta})(1 - \bar{z} e^{i\theta})}{|1 - z e^{-i\theta}|^2} = \frac{1 - |z|^2 + (z e^{-i\theta} - \bar{z} e^{i\theta})}{|1 - z e^{-i\theta}|^2}$$

$$\operatorname{Im} f(z) = \int_0^{2\pi} \underbrace{\frac{1 - |z|^2}{|1 - z e^{-i\theta}|^2}}_{\text{Poisson kernel}} \operatorname{Im} f(e^{i\theta}) \frac{d\theta}{2\pi}$$

$$\cancel{\frac{1}{a - d} = \frac{z + 1}{z - \alpha} \quad \frac{1}{a + i} \quad z = \frac{1 + i\lambda}{1 - i\lambda}}$$



On the circle you know that $i\frac{z+\alpha}{z}$

Keep on trying.

I need.

Basically you need equality of

$$\frac{1}{\omega-\lambda} - \underbrace{\operatorname{Re}\frac{1}{\omega-i}}_{\frac{\omega}{\omega^2+1}}$$

$$\frac{1}{\omega-\lambda} - \frac{\omega}{\omega^2+1} \qquad \frac{1}{\omega-i} - \frac{\omega}{\omega^2+1}$$

$$\frac{\omega+i}{\omega^2+1}$$

You claim that

$$\frac{1}{a-\lambda} - \frac{a}{a^2+1} = i\,\frac{1+z\alpha^{-1}}{1-z\alpha^{-1}}$$

$$\boxed{\frac{1}{a-\lambda} - \frac{a}{a^2+1} = i\,\frac{\alpha+z}{\alpha-z}} \text{ ? NO}$$

$$\frac{1}{a-\cancel{\lambda}\, i} - \frac{a}{a^2+1} = \frac{i}{a^2+1}$$

$$\frac{1}{a-\lambda} - \frac{a}{a^2+1}$$

real for $\lambda\in\mathbb{R}$
simple pole
value at i
is purely imag.

$\frac{1}{a-\lambda}$ has simple pole at $\lambda=a$
is real on $\mathbb{R}$

value $\frac{1}{a-i} = \frac{a+i}{a^2+1}$

338 Poisson kernel is a measure on the boundary

$$i\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,\frac{d\theta}{2\pi} \overset{?}{=} \left(\frac{1}{x-\lambda}-\frac{x}{x^2+1}\right)\frac{dx}{\pi}$$

relation between x and θ? $\quad e^{i\theta} = \frac{-x+i}{x+i}$

$$e^{i\theta} = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} x$$

$$x = \begin{pmatrix} i & -i \\ -1 & -1 \end{pmatrix} e^{i\theta} = \frac{-ie^{i\theta}+i}{e^{i\theta}+1} = \frac{\frac{e^{i\theta/2}-e^{-i\theta/2}}{2i}}{\frac{e^{i\theta/2}+e^{-i\theta/2}}{2}}$$

$$= \tan(\theta/2).$$

$$\theta/2 = \tan^{-1}(x)$$

$$\frac{d\theta}{2} = \frac{dx}{1+x^2}$$

$$\frac{d\theta}{2\pi} = \frac{2dx}{2\pi(1+x^2)} = \frac{dx}{\pi(1+x^2)}$$

$$z = \frac{1+i\lambda}{1-i\lambda} \qquad \lambda = i\,\frac{1-z}{1+z}$$

$$\frac{1}{x-\lambda} - \frac{x}{x^2+1}$$

pole $\lambda = x$

zero

$$\zeta = \frac{1+ix}{1-ix} \qquad \frac{d\zeta}{\zeta} = \frac{i\,dx}{1+ix} - \frac{-i\,dx}{1-ix} = \frac{2i\,dx}{1+x^2}$$

$$i\,\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{2\pi i\zeta} = \cancel{\frac{1+ix+}{1-ix}}$$

$$\frac{\zeta+z}{\zeta-z} = \frac{\frac{1+ix}{1-ix}+\frac{1+i\lambda}{1-i\lambda}}{\frac{1+ix}{1-ix}-\frac{1+i\lambda}{1-i\lambda}} = \frac{(1+ix)(1-i\lambda)+(1-ix)(1+i\lambda)}{(1+ix)(1-i\lambda)-(1-ix)(1+i\lambda)}$$

$$= \frac{2+2x\lambda}{2(ix-i\lambda)} = \frac{1+x\lambda}{i(x-\lambda)}$$

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$$\frac{1+x\lambda}{i(x-\lambda)}\ \frac{2i\,dx}{(1+x^2)\,2\pi}$$

$$\frac{1}{x-\lambda}-\frac{x}{x^2+1}=\frac{\cancel{x}^2+1-(\cancel{x}-\lambda)x}{(x-\lambda)(1+x^2)}$$

$$i\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,\frac{d\theta}{2\pi}=\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{2\pi\zeta}\qquad \zeta=\frac{1+ix}{1-ix}$$

$$=\frac{\left(\frac{1+ix}{1-ix}\right)+\left(\frac{1+i\lambda}{1-i\lambda}\right)}{\left(\frac{1+ix}{1-ix}\right)-\left(\frac{1+i\lambda}{1-i\lambda}\right)}\left(\frac{i\,dx}{1+ix}-\frac{(-i\,dx)}{1-ix}\right)\frac{1}{2\pi}$$

$$=\frac{(1+ix)(1-i\lambda)+(1-ix)(1+i\lambda)}{(1+ix)(1-i\lambda)-(1-ix)(1+i\lambda)}\left(\frac{2i\,dx}{1+x^2}\right)\frac{1}{2\pi}$$

$$=\frac{2+2x\lambda}{2ix-2i\lambda}\left(\frac{2i\,dx}{1+x^2}\right)\frac{1}{2\pi}$$

$$=\frac{1+x\lambda}{\cancel{i}(x-\lambda)}\,\frac{\cancel{2i}\,dx}{1+x^2}\,\frac{1}{\cancel{2}\pi}=\frac{1+x\lambda}{(x-\lambda)}\,\frac{1}{1+x^2}\,\frac{dx}{\pi}$$

---

take pick fn. $\sum_{n\in\mathbb{Z}}\frac{1}{n-\lambda}$. This should have the form $\left(\xi,\frac{1}{A-\lambda}\xi\right)$ for some s.a. operator A and vector ξ

There are going to be problems here.

Progress. Concept of a Pick function made invariant – want an analytic function on a disk with positive imaginary part, ~~get~~ up to a real constant is same as positive harmonic function in the

340 disk. Question: Is there an intrinsic measure on the boundary?? ~~[illegible]~~

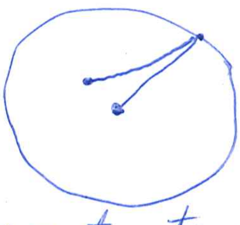
Discussion. Consider a disk in R.S. Use Dirichlet problem solution: cont. function on $\partial D \rightsquigarrow$ harmonic functions $\xrightarrow{\text{eval at int pt}} \mathbb{C}$. To each $z \in D$, get Poisson measure μ_z on ∂D. $\mu_z = \frac{d\theta}{2\pi}$ where $z = 0$.

Question. Given a point ξ on ∂D is there a positive harmonic function on D corresponding to the δ measure at ξ.

Question: Are there rational harmonic functions — Real parts of meromorphic functions? Atiyah's problem?

For the moment focus on the idea that a positive harmonic function has a measure for its boundary values. Why should this be true? because you can convert the Poisson measure to a function using $\frac{d\theta}{2\pi}$

Inner functions. If D is a disk in Riemann sphere we know that it has attached a polarized Hilbert spaces of L^2-section of $\mathcal{O}(-1)$, so you have an intrinsic $H \simeq H^+ \oplus H^-$, hence inner functions should be intrinsically defined. Blashke products seem OKAY, ~~but the~~ singular functions? ~~[illegible]~~ An inner without zeroes has a log which is an analytic function on the disk whose real part is < 0 with radial limits 0 a.e. Radial limits seem independent of base point since geodesics tending



to the same bdry point are ~~[illegible]~~ $\perp$ to bdry. hence very close. I don't see problems. You want to ignore a.e. stuff. Ask about ~~[illegible]~~ analytic function ~~[illegible]~~ except at one point of ∂D which is real on ∂D.

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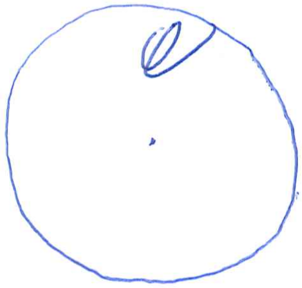

Maybe missing something obvious.

$$\zeta = e^{i\theta} = \frac{1+ix}{1-ix} \qquad z = \frac{1+i\lambda}{1-i\lambda}$$

$$\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}} = \frac{\zeta+z}{\zeta-z} = \frac{\left(\frac{1+ix}{1-ix} + \frac{1+i\lambda}{1-i\lambda}\right)}{\frac{1+ix}{1-ix} - \frac{1+i\lambda}{1-i\lambda}}$$

$$= \frac{(1+ix)(1-i\lambda)+(1-ix)(1+i\lambda)}{(1+ix)(1-i\lambda)-(1-ix)(1+i\lambda)} = \frac{1+x\lambda}{i(x-\lambda)}$$

$$\frac{d\theta}{2\pi} = \frac{d\zeta}{2\pi i\zeta} = \frac{1}{2\pi i}\left(\frac{i\,dx}{1+ix} - \frac{-i\,dx}{1-ix}\right) = \frac{1}{\pi}\frac{dx}{1+x^2}$$

$$i\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,\frac{d\theta}{2\pi} = \frac{1+x\lambda}{x-\lambda}\,\frac{1}{1+x^2}\,\frac{dx}{\pi}$$

$$= \left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right)\frac{dx}{\pi}$$

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actually $z = \frac{1+i\lambda}{1-i\lambda}$

$$\frac{\zeta - z}{\zeta + z} = i\,\frac{x-\lambda}{1+x\lambda}$$ looks interesting

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$$\frac{dz}{2z} = i\,\frac{d\lambda}{1+\lambda^2} \qquad \frac{d\zeta}{2\zeta} = i\,\frac{dx}{1+x^2}$$

what is this kernel you are dealing with

$$i\,\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\,\overset{\mathrm{Im}}{f(\zeta)}\,\frac{d\zeta}{2\pi i\zeta}$$

You need to understand, to write out the details of Poisson formula.

Let's start with $\left(\xi, \frac{1}{\lambda - A}\xi\right) = \int \frac{1}{\lambda - x}\,d\mu(x)$

~~new notation~~

$$i\,\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,\frac{d\theta}{2\pi} = \left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right)\frac{dx}{\pi}$$

where $z = \frac{1+i\lambda}{1-i\lambda}$ $\quad e^{i\theta} = \frac{1+ix}{1-ix}$

What you need to understand is $\sum_{n\in\mathbb{Z}} \frac{1}{\lambda-n}$

from the operator viewpoint.

Consider Hilbert space $L^2(S^1)$ operator $A = z\frac{d}{dz}(z^n) = nz^n$.
self adjoint operator, unitary $u = z$. $u^{-1}Au = A+1$
What is this function $\sum_{n\in\mathbb{Z}} \frac{1}{\lambda-n} = \left(\xi, \frac{1}{\lambda-A}\xi\right)$

where ξ non normalizable state.

We know that $\sum_{n\in\mathbb{Z}} \frac{1}{\lambda-n}$ can be regularized to be ~~an analytic fn a merom~~ an analytic function in the UHP, better it's a merom. function with negative imag. part in the UHP. Regularization unique up to a real constant. You have the following situation – the class of Pick functions on the UHP, i.e. analytic functions in the UHP with pos. imag part, and up to real additive constants these are the same as positive harmonic functions in the UHP, which are the same as measures on the boundary, ??

$$\int \frac{1}{t-\lambda}\, d\mu(t)$$

analytic functions with pos. Im part modulo $\mathbb{R}$
= positive harmonic functions = measures on the bdry
provided you give some point of the disk

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What would you like? You want to relate functions & operators. So given D you have a natural $L^2(\partial D, \mathcal{O}(-1)) = H^2_+ \oplus H^2_-$ intrinsic to D. Action of $\mathrm{Aut}(D) \cong SU(1,1)/\{\pm 1\}$?

Also have intrinsic class of Pick functions, positive harmonic functions on D. $\exists$ Interesting subclass where the Pick functions are real on the boundary except at finitely many sing. pts

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Given a finite Pick function on D, is there an operator interpretation? Let's see if we can construct something invariantly involving $L^2(\partial D, \mathcal{O}(-1))$. What about on S? Cayley transform of the finite Pick function should be a finite Blashke product.

finite Pick function: ~~Blashke product~~

$\exists$ finite Pick function ~~is~~ clear

$f(z)$ rational fn. of z, $\operatorname{Im} f(z) > 0$ for $|z| < 1$.

$= 0$ $=$

< 0 $\longrightarrow > 1$.

poles on $|z| = 1$.

$$\frac{c}{z-\zeta} + \frac{\bar{c}}{z^{-1} - \bar{\zeta}} = \frac{c}{z-\zeta} + \frac{\zeta \bar{c} z}{\zeta - z}$$

$$= \frac{c - \zeta \bar{c} z}{z - \zeta} = \frac{i\zeta + iz}{z - \zeta}$$

$c = i\zeta$

344 Pick fn. on $|z|<1$ has the form

$$* \quad f(z) = \int i \frac{\xi + z}{\xi - z} \, d\mu(\xi)$$

where $d\mu$ is a measure on S^1. Check this

$$i \frac{1+\xi^{-1}z}{1-\xi^{-1}z} = i \underbrace{\boxed{\frac{1-(-\xi^{-1}z)}{1+(-\xi^{-1}z)}}}_{\in RHP} \in UHP.$$

Now the measure $d\mu(\xi)$ leads to a unitary operator + cyclic vector. $H = L^2(S^1, d\mu)$ $u = $ mult by ξ

Maybe $*$ is not invariant, why because if you pass to imaginary parts, then it says that a pos. harmonic function corresp. to a measure on the boundary, and this seems to contradict today's experience

Go over it again. First the solution of Dirichlet problem yields a Poisson kernel which is a measure, or smooth 1-form, on the boundary. So

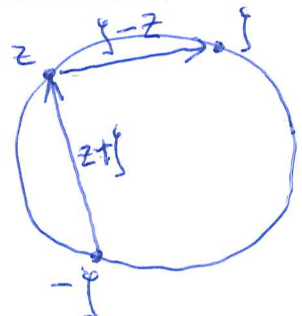
$$f(z) = \int \underbrace{i \frac{\xi + z}{\xi - z} \frac{d\theta}{2\pi}}_{} \overline{\mathrm{Im} f(\xi)}$$

smooth 1-form, which means this kernel extends from being defined on cont functions to distributions

Pick functions $f(z)$ on $|z|<1$ has the form

$$f(z) = \int_0^{2\pi} i \frac{e^{i\theta}+z}{e^{i\theta}-z} \frac{d\mu}{d\theta} \frac{d\theta}{2\pi}$$

$$\frac{1}{i} \frac{z+\xi}{z-\xi} \frac{d\xi}{2\pi i \xi} = \frac{\xi + z}{\xi - z} \frac{d\xi}{2\pi \xi}$$



345 D disk in RS = $\mathbb{P}^1\mathbb{C}$ defined by a hermitian form on 2-dim space T, attached to D have Hilbert space of square integrable sections of $\mathcal{O}(-1)$ over ∂D $\quad H = L^2(\partial D, \mathcal{O}(-1))$, polarization into Hardy spaces of sections extending analytically to D or the opposite disk. Notion of inner function S on D, outgoing subspaces, Blashke products.

Pick function = analytic Poisson transform of a positive distribution on the boundary.

Here is what you want to do. Note that $H = H^+ \oplus H^-$ does not yet have an operator like z on it, i.e. you haven't chosen a coordinatization of D. Actually $D \subset \mathbb{C} \cup \{\infty\}$ is where spectra lie. – a point of D is an eigenvalue. I guess we want smething like $az-b$,

You have used $\mathcal{O}(-1)$ already

One way to proceed is this. Choose a coord. z of D i.e. an isom of D with the unit disk. Then you get unitary operator u of mult. by z (the coord. function) on H. You also get a measure on $\partial D = S^1$ corresp. to the Pick function. The measure gives a gives a Hilbert space together with unitary u and cyclic vector, this triple unique up to isomorphism (unique up to S^1 scalars). If you choose a diff. coord. $\frac{az+b}{\bar{b}z+\bar{a}}$, $|a|^2-|b|^2=1$ then from (H', u, ξ) you get $\left(H', \frac{au+b}{\bar{b}u+\bar{a}}, \xi\right)$ modify by

$$\frac{1}{2\pi i}\, d\log\left(\frac{az+b}{\bar{b}z+\bar{a}}\right) = \frac{1}{2\pi i}\left(\frac{a}{az+b} - \frac{\bar{b}}{\bar{b}z+\bar{a}}\right)dz = \frac{1}{2\pi i}\frac{|a|^2-|b|^2}{(az+b)(\overline{az+b})}\frac{dz}{z}$$

$$= \frac{1}{|az+b|^2}\frac{dz}{2\pi i z}$$

346 See if this works.

$i\,\frac{e^{i\theta}+z}{e^{i\theta}-z}\,\frac{d\theta}{2\pi}$ analytic Poisson kernel. – yields the analytic function with given imag part on ∂D.

Work with unit disk. Suppose given a finite measure on S^1 $\sum_{k=1}^{n} m_k\,\delta(\xi-\xi_k)$ $m_k>0$.

Then you get Pick fn. $\boxed{\sum_{k=1}^{n} m_k\, i\,\frac{\xi_k+z}{\xi_k-z}}$

$$-i\,\frac{\bar{\xi}_k+z^{-1}}{\bar{\xi}_k-z^{-1}} = -i\,\frac{z+\xi_k}{z-\xi_k}$$ real valued for $z\in S^1$.

$$\operatorname{Im}\left(i\,\frac{\xi+z}{\xi-z}\right) = \operatorname{Re}\left(\frac{\xi+z}{\xi-z}\right) = \frac{1}{2}\left(\frac{1+z\xi^{-1}}{1-z\xi^{-1}}+\frac{1+\bar{z}\xi}{1-\bar{z}\xi}\right)$$

$$=\frac{1}{2}\,\frac{1-|z|^2}{|1-z\xi^{-1}|^2}$$ $\times\,\frac{d\theta}{2\pi}$ gives Poisson kernel.

What to do? Start with $d\mu$ on S^1 form analytic Poisson transform $f(z)=\int i\,\frac{\xi+z}{\xi-z}\,d\mu(\xi)$.

This has an obvious operator interpretation. namely $\left(\xi,\ i\,\frac{u+z}{u-z}\,\xi\right) = \left(\xi,\ \frac{1+u^{-1}z}{1-u^{-1}z}\,\xi\right)$ here $z\in D$ number.

I bet this is better than the resolvent coeff. $\left(\xi,\ \frac{1}{z-u}\,\xi\right)$

$$\frac{2-(1-u^{-1}z)}{1-u^{-1}z} = \frac{2}{1-u^{-1}z} - 1$$

347 Given $(E, u, \xi) \rightsquigarrow \left(\xi, i\frac{u+z}{u-z}\xi\right)$

Suppose $z = \dfrac{a\omega+b}{\bar b\omega+\bar a}$ $\quad |a|^2-|b|^2=1.$

$$\frac{u+\frac{a\omega+b}{\bar b\omega+\bar a}}{u-\frac{a\omega+b}{\bar b\omega+\bar a}} = \frac{u(\bar b\omega+\bar a)+a\omega+b}{u(\bar b\omega+\bar a)-a\omega-b}$$

$$= \frac{(u\bar b+a)\omega+u\bar a+b}{(u\bar b-a)\omega+u\bar a-b} = \frac{\omega+\frac{u\bar a+b}{u\bar b+a}}{-\omega+\frac{u\bar a-b}{a-u\bar b}}$$

try again.

$$\frac{rz-s}{1-z\xi^{-1}} \stackrel{?}{=} \frac{\bar r z^{-1}-\bar s}{1-z^{-1}\xi} = \frac{\bar r-\bar s z}{z-\xi}$$

$$\parallel$$

$$\frac{\xi rz-\xi s}{\xi-z} \qquad \bar r=+\xi s \qquad \bar s=+\xi r$$

$$s=+\xi^{-1}\bar r$$

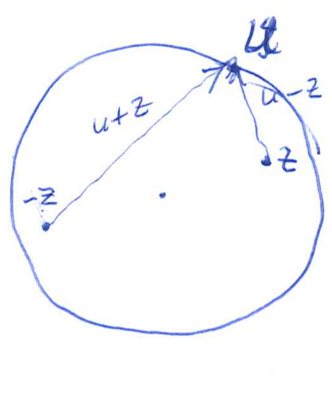
$$\frac{rz-\xi^{-1}\bar r}{1-z\xi^{-1}}$$ purely imag for $z=0$.

$-\xi^{-1}\bar r = i$

$-\xi r=-i \qquad r=\xi^{-1}i$

$$\parallel$$

$$\frac{i\xi^{-1}z+i}{1-z\xi^{-1}} = i\,\frac{1+\xi^{-1}z}{1-\xi^{-1}z}$$



 No more wasting time. Idea somehow — a Pick function, maybe better would be a positive harmonic function on D should determine a Hilbert space Y and some kind of spectral representation of elements of Y as functions on D, ~~ez~~ If true then a point of D determines a point evaluator in Y and you should get a Bergman kernel, reproducing kernel. How might you go from a Pick function to a kernel. $K(w,z)$ holom. in z ante-holom in w, positive (semi) definite as a matrix.

to be more precise.

Let's consider $|z| \leq 1$. Pick kernel $i \frac{\zeta+z}{\zeta-z} \frac{d\zeta}{2\pi i \zeta}$

$$f(z) = \int i \frac{\zeta+z}{\zeta-z} \, d\rho(\zeta) \qquad |z|<1 \Rightarrow \operatorname{Im} f(z) > 0.$$

$$L^2(S^1, d\rho), \quad u = \text{mult by } \zeta, \quad \xi = 1.$$

$$f(z) = \left(\xi, i\frac{u+z}{u-z}\xi\right)$$

this operator is s.a. for $|z|=1$.

$$2 \operatorname{Re}\left(i\frac{u+z}{u-z}\right) = i\frac{u+z}{u-z} - i\frac{u^{-1}+\bar z}{u^{-1}-\bar z}$$

herm. part

$$= i\left(\underbrace{(u+z)(u^{-1}-\bar z) - (u-z)(u^{-1}+\bar z)}_{-u\bar z + zu^{-1} - u\bar z + zu^{-1}}\right)\frac{1}{(u-z)(u^{-1}-\bar z)}$$

$$= 2i\,\frac{zu^{-1} - \bar z u}{(u-z)(u-z)^*}$$

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$$2\,\mathrm{Im}\left(i\,\frac{u+z}{u-z}\right) = \frac{u+z}{u-z} + \left(\frac{u+z}{u-z}\right)^{*}$$

skew-herm part

$$= \frac{2(1-|z|^2)}{(u-z)(u^{-1}-\bar z)}$$

$$\mathrm{Im}\, f(z) = \mathrm{Im}\left(\xi, i\,\frac{u+z}{u-z}\xi\right) = \mathrm{Re}\left(\xi, \frac{u+z}{u-z}\xi\right)$$

$$= \left(\xi, \frac{1-|z|^2}{(u-z)(u^{-1}-\bar z)}\xi\right) = \left\|\frac{\sqrt{1-|z|^2}}{u-z}\xi\right\|^2$$

positive harmonic function on UHP invariant under $\lambda \mapsto \lambda+1$. It should descend via $\lambda \mapsto e^{2\pi i\lambda}$ to a positive harmonic function on the punctured disk $0<|z|<1$. ~~How do I proceed?~~ Take case of Pick functions — analytic maps to UHP, singularity should be removable.

Need Poisson formula on UHP.

$$z = \frac{1+i\lambda}{1-i\lambda} \qquad \zeta = \frac{1+ix}{1-ix}$$

$$\frac{d\zeta}{2\pi i \zeta} = \frac{1}{2\pi i}\left(\frac{i}{1+ix} - \frac{-i}{1-ix}\right)dx = \frac{dx}{\pi(1+x^2)}$$

$$i\,\frac{\zeta+z}{\zeta-z} = i\,\frac{(1+ix)(1-i\lambda)+(1-ix)(1+i\lambda)}{(1+ix)(1-i\lambda)-(1-ix)(1+i\lambda)}$$

$$= i\,\frac{2(1+x\lambda)}{2i(x-\lambda)} = \frac{1+x\lambda}{x-\lambda}$$

$$i\,\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{2\pi i\zeta} = \underbrace{\frac{(1+x\lambda)}{(x-\lambda)(1+x^2)}}\frac{dx}{\pi}$$

$$= \left(\frac{1}{x-\lambda} - \frac{x}{1+x^2}\right)\frac{dx}{\pi}$$

Anyway, if $f(\lambda)$ is a periodic Pick function then what?

$f(\lambda) = g(z)$ g is a Pick function on the disk $|z|<1$.

The poss. sing at $z=0$ is removable. ~~What~~

Positive harmonic function which is periodic. namely $a\,\mathrm{Im}(\lambda)$ with $a>0$. Corresp. analytic function is $a\lambda$ + real const, which is not periodic.

Suppose $h(\lambda)$ positive harmonic function on UHP, which is periodic: $h(\lambda+1) = h(\lambda)$.

Let $f(\lambda)$ be a Pick function with $h = \mathrm{Im}(f)$. f is unique to a real constant $\Rightarrow$ $f(\lambda+1) - f(\lambda) = r \in \mathbb{R}$.

I guess what is going on should be this. $h(\lambda)$ is positive harm. function, so h has unique reps.

$$h(\lambda) = \int \mathrm{Im}\left(\frac{1}{x-\lambda}\right) d\mu(x) + \mu_\infty \mathrm{Im}(\lambda)$$

$$\parallel$$

$$h(\lambda+1) = \underbrace{\int \mathrm{Im}\left(\frac{1}{x-\lambda-1}\right) d\mu(x)}_{\int \mathrm{Im}\left(\frac{1}{x'-\lambda}\right) d\mu(x'+1)} + \mu_\infty \mathrm{Im}(\lambda)$$

$\therefore$ $d\mu(x+1) = d\mu(x)$. So we know that a positive harmonic periodic fun in the UHP is $\mathrm{Im}\left(\underline{\text{periodic Pick fn.}} + \mu_\infty \lambda\right)$ with $\mu_\infty \geq 0$.

So what's the next step? What went wrong with removable singularity? Suppose $h(\lambda) = \mathrm{Im}(\lambda)$. This is periodic so get $h'(z) = h'(e^{2\pi i\lambda}) = \mathrm{Im}(\lambda)$

$$h'(z) = \mathrm{Im}\left(\frac{1}{2\pi i}\log(z)\right)$$

This is indeed harmonic in the punctured disk, and >0

$$= -\frac{1}{2\pi}\log|z|.$$

351 ~~Note~~ Consider the disk case. There's an equivalence between pos. harmonic functions on D and measures on the boundary given by

$$h(z) = \int \frac{1-|z|^2}{|1-z\zeta^{-1}|^2}\, d\mu(\zeta)$$

~~...~~ Start with

$$f(z) = \mathrm{Re}\, f(0) + \int i\, \frac{e^{i\theta}+z}{e^{i\theta}-z}\, \mathrm{Im}\, f(e^{i\theta})\, \frac{d\theta}{2\pi}$$

$$\mathrm{Im}\; i\,\frac{e^{i\theta}+z}{e^{+i\theta}-z} = \frac{1}{2}\left(\frac{\zeta+z}{\zeta-z} + \frac{\zeta^{-1}+\bar z}{\zeta^{-1}-\bar z}\right)$$

$$= \frac{1}{2}\,\frac{(\zeta+z)(\zeta^{-1}-\bar z)+(\zeta-z)(\zeta^{-1}+\bar z)}{|\zeta-z|^2} = \frac{1-|z|^2}{|1-z\zeta^{-1}|^2}$$

Natural to work in $L^2(S^1, d\mu)$

$$h(z) = \left(\xi, \frac{1-|z|^2}{(u-z)(u^*-\bar z)}\,\xi\right) = \left\| \frac{\sqrt{1-|z|^2}}{u-z}\,\xi \right\|^2$$

$$\frac{h(z)}{1-|z|^2} = \left\| \frac{1}{u-z}\,\xi \right\|^2$$

obvious thing to try is $(1-\bar w z)\left(\frac{1}{w-u}\xi, \frac{1}{z-u}\xi\right)$

other ideas. Change a Pick fn. to an S via CT.

$$\left(\frac{1}{w-u}\xi, \frac{1}{z-u}\xi\right)$$

$$= \left(\xi, \frac{1}{\bar w - u^{-1}}\,\frac{1}{z-u}\,\xi\right) = \frac{1}{1-\bar w z}\left(\xi, \left(\frac{z}{u-z} + \frac{1}{u-\bar w^{-1}}\right)\xi\right)$$

$$\frac{u}{\bar w u - 1}\,\frac{1}{z-u} = \frac{A}{u-z} + \frac{B}{u-\bar w^{-1}} \qquad A =$$

$$\frac{u\,\xi}{\bar w u - 1}(-1) = A + (z-u)\cdots \qquad A = \frac{z}{1-\bar w z}$$

$$B = \frac{u(u-\bar w^{-1})}{(\bar w u - 1)(u-z)}\bigg|_{u=\bar w^{-1}} = \frac{\bar w^{-1}}{\bar w^{-1}-z} = \frac{1}{1-\bar w z}$$

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352 Simpler example.

So now what to do? to relate Pick functions to operators. Go to the UHP where a Pick function $f(\lambda)$ has the repn.

$$f(\lambda) = \int_{-\infty}^{\infty} \left(\frac{1}{x-\lambda} - \frac{\lambda}{1+\lambda^2} \right) d\mu(\lambda) + \mu_\infty \lambda \quad \Bigg| \quad \int \frac{1}{1+\lambda^2}\, d\mu(\lambda) < \infty$$

and a pos. harm. function has form

$$h(\lambda) = \int_{-\infty}^{\infty} \frac{\operatorname{Im}(\lambda)}{|x-\lambda|^2}\, d\mu(\lambda) + \mu_\infty \lambda$$

Suppose $\mu_\infty = 0$. Form $L^2(\mathbb{R}, d\mu)$, A = mult by λ, $\xi = 1$. Assume measure bounded.

The main piece of information you have is

$$\frac{2i\operatorname{Im}(\lambda)}{|x-\lambda|^2} = \frac{1}{x-\lambda} - \frac{1}{x-\bar\lambda}$$

$$\frac{2\operatorname{Im}(\lambda)}{|x-\lambda|^2} = \frac{i}{\lambda - x} + \frac{\lambda}{x-\lambda}$$

$$2h(\lambda) = \left(\xi, \frac{i}{\lambda - A}\xi\right)$$

$$h(\lambda) = \left(\xi, \frac{\operatorname{Im}(\lambda)}{(A-\bar\lambda)(A-\lambda)}\xi\right)$$

$$= \left\| \frac{\operatorname{Im}(\lambda)^{1/2}}{A-\lambda}\xi \right\|^2$$

How do you proceed to understand this situation

You have a choice between finite measures and smooth ones. In view of the radial ~~[illegible]~~ scaling argument you probably should first handle smooth measures. So you begin with a positive harmonic function on $|z| < 1+\varepsilon$.

~~[illegible]~~ Consider $|z| < 1$. A positive harmonic function on the disk is equivalent to a measure on the boundary via the formula

$$h(z) = \int \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi), \quad \text{and}$$

a measure is equivalent to ~~a positive d~~ its moments which form a pos. def fn on $\mathbb{Z}$.

353 This should clarify things a lot. In fact you get the analytic function you've been missing in the circle case. ~~Now how to~~

Given a collection $\mu_n = \int \xi^n d\mu$ $n \in \mathbb{Z}$ of moments. There is an easy equivalence between measures on S^1; pos. def. fns on $\mathbb{Z}$, positive harmonic functions h on $|z|<1$, (H, u, ξ). Apparently $h(z)$ gives the norm squared roughly of some element in H depending ~~on~~ $z \in D$, so there should be an extension to a Bergman type kernel. Yes.

First handle smooth, better: analytic case on the boundary $\rho \frac{d\theta}{2\pi}$ where ρ is real analytic and >0 on S^1.

Work things out. Take measure $\rho \frac{d\theta}{2\pi}$

$$f(z) = \sum_{n \geq 0} a_n z^n \qquad a_n = \int f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi}$$

$$2i \operatorname{Im} f(e^{i\theta}) = a_0 - \bar{a}_0 + \sum_{n \geq 1} \left(a_n e^{in\theta} - \bar{a}_n e^{-in\theta} \right)$$

$$2i \int \operatorname{Im} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} = \begin{matrix} a_n & n \geq 1 \\ a_0 - \bar{a}_0 & n = 0. \end{matrix}$$

$\operatorname{Re}(a_0) + i \operatorname{Im}(a_0)$

$$f(z) = a_0 + \sum_{n \geq 1} 2i \int \operatorname{Im} f(e^{i\theta}) e^{-in\theta} z^n \frac{d\theta}{2\pi}$$

$$i \operatorname{Im}(a_0) = \frac{a_0 - \bar{a}_0}{2} = \int i \operatorname{Im} f(e^{i\theta}) \frac{d\theta}{2\pi}$$

$$f(z) = \operatorname{Re}(f(0)) + i \int \underbrace{\left(1 + 2 \sum_{n \geq 1} (e^{-i\theta} z)^n \right)}_{\frac{e^{i\theta} + z}{e^{i\theta} - z}} \operatorname{Im} f(e^{i\theta}) \frac{d\theta}{2\pi}$$

$$\operatorname{Im} f(z) = \int \frac{1-|z|^2}{|e^{i\theta}-z|^2} \underbrace{\operatorname{Im} f(e^{i\theta})}_{\rho} \frac{d\theta}{2\pi}$$

$$\frac{1}{2}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z} + \frac{e^{-i\theta}+\bar{z}}{e^{-i\theta}-\bar{z}}\right) = \frac{1-|z|^2}{|e^{i\theta}-z|^2}$$

Question: Suppose given moments μ_n satisfying pos. cond. ie pos. def function on $\mathbb{Z}$, then how do you see that the corresponding $f(z)$ has pos. imag. part.

$$\operatorname{Im} f(e^{i\theta}) = \sum \mu_n e^{in\theta} \qquad \text{formal}$$

$$= \mu_0 + \sum_{n\geq 1} \underbrace{\left(\mu_n e^{in\theta} + \bar{\mu}_n e^{-in\theta}\right)}$$

extends to $\mu_n z^n + \bar{\mu}_n \bar{z}^n$

So you have obvious harmonic extension of a real function on S^1. Can you find Poisson kernel this way.

$$e^{in\theta} \longmapsto z^n \qquad n \geq 0$$

$$\longmapsto \bar{z}^{-n} \qquad n \leq 0.$$

$$\sum_{n\geq 0} z^n e^{-in\theta} + \sum_{n\geq 1} \bar{z}^{-n} e^{in\theta} = \frac{1}{1-ze^{-i\theta}} + \frac{\bar{z}e^{+i\theta}}{1-\bar{z}e^{i\theta}}$$

$$= \frac{1-|z|^2}{|1-ze^{-i\theta}|^2}$$

circular arg. equivalence of

- pos. harm function on $|z|<1$.
- pos. def function $\{\mu_n\}$ on $\mathbb{Z}$
- measure on $|z|=1$.

355 So how do you proceed? Start with $\{\mu_n\}$

$\ni$ μ_{i-j} hermitian pos. semi def

get hermitian scalar prod. on $\mathbb{C}[z,z^{-1}]$ $\ni$ mult. by z

is unitary. $(z^j, z^k) = \mu_{j-k}$ $\qquad \|\sum c_j z^j\|^2 = \sum_{j,k} \bar{c}_j \mu_{j-k} c_k \geq 0$

Positive def. function μ_j In part. $\begin{pmatrix} \mu_0 & \mu_n \\ \bar{\mu}_n & \mu_0 \end{pmatrix}$ pos semi

$\mu_0^2 - |\mu_n|^2 \geq 0$.

~~Assume that~~ How do you get a harm. fn.

$$2\pi \frac{d\mu}{d\theta} = \sum \mu_n z^n \qquad \text{Let } \mu_n = \int \bar{z}^{-n} d\mu$$

Appropriate real harm. fn. is

$$\boxed{\sum_{n\geq 0} \mu_n z^n + \sum_{n \geq 1} \mu_{-n} \bar{z}^n}$$

~~As~~ boundary values correct. e.g. $\mu_n = \zeta^{-n}$ $d\mu$ δ-function at $e^{i\theta} = \zeta$.

$$\sum_{n\geq 0} \zeta^{-n} z^n + \sum_{n\geq 1} \zeta^n \bar{z}^n = \frac{1}{1-\zeta^{-1}z} + \frac{\zeta \bar{z}}{1-\zeta\bar{z}} = \frac{1-|z|^2}{|1-\zeta^{-1}z|^2}$$

~~Now~~ But you have this Hilbert space $L^2(S^1, d\mu)$ obtained by completing $\mathbb{C}[\zeta, \zeta^{-1}]$ as above. And in this Hilbert space you have ~~[illegible]~~ ζ is a boundary variable

z int. pt

$$\frac{1}{z-\zeta} = -\frac{1}{\zeta}\left(\frac{1}{1-z\zeta^{-1}}\right)$$

$$= -\sum_{n\geq 0} z^n \zeta^{-n-1}$$

Motivation $h(z) = \int \frac{1-|z|^2}{|z-\zeta|^2} d\mu = \left\| \frac{(1-|z|^2)^{1/2}}{z-\zeta} \right\|^2$

So try to calculate $\left\| \frac{1}{z-\zeta} \right\|^2$ in the Hilbert space obtained by completing $\mathbb{C}[\zeta,\zeta^{-1}]$.

$$\frac{1}{\bar{z}-\zeta^{-1}}\,\frac{1}{z-\zeta} = \frac{1}{\zeta-z}\,\frac{\zeta}{1-\zeta\bar{z}}$$

$$= \frac{A}{\zeta-z} + \frac{B}{1-\zeta\bar{z}}$$

$$A = \frac{z}{1-|z|^2} \qquad B = \frac{1}{\bar{z}^{-1}-z}\,\bar{z}^{-1} = \frac{1}{1-|z|^2}$$

$$\frac{1}{(\bar{z}-\zeta^{-1})(z-\zeta)} = \frac{1}{1-|z|^2}\left(\frac{z}{\zeta-z} + \frac{1}{1-\zeta\bar{z}}\right)$$

$$\left\langle \frac{1}{(\zeta^{-1}-\bar{z})(\zeta-z)} \right\rangle = \frac{1}{1-|z|^2}$$

You are try to show that the harmonic function

$$\cancel{h(z) = \sum_{n\geq 0} \bar{z}^n\zeta^{-n} + \sum_{n\geq 1} \bar{z}^{n}\zeta^{n} = \frac{1-|z|^2}{|1-\bar{\zeta}z|}}$$

$$h(z) = \underbrace{\sum_{n\geq 0}\mu_n z^n + \sum_{n\geq 1}\mu_{-n}\bar{z}^n} \qquad \text{is positive}$$

This should be true because $h(z) = \left\| \frac{(1-|z|^2)^{1/2}}{z-\zeta} \right\|^2$ in the associated Hilbert space. Reverse the derivation namely

$$\frac{1}{2}\left(\frac{z+\zeta}{z-\zeta} + \frac{\bar{z}+\zeta^{-1}}{\bar{z}-\zeta^{-1}}\right)$$

$$= \frac{|z|^2-1}{|z-\zeta|^2}$$

clear.

$$\frac{1}{2}\left(\frac{\zeta+z}{\zeta-z} + \frac{\zeta^{-1}+\bar{z}}{\zeta^{-1}-\bar{z}}\right) = \frac{1-|z|^2}{|\zeta-z|^2}$$

357 You have

$$h(z) = \sum_{n\geq 0} \mu_n z^n + \sum_{n\geq 1} \mu_{-n} \bar{z}^n$$

$$= \sum_{n\geq 0} \langle \zeta^{-n} \rangle z^n + \sum_{n\geq 1} \langle \zeta^n \rangle \bar{z}^n$$

$$= \left\langle \frac{1}{1-z\zeta^{-1}} + \frac{\zeta\bar{z}}{1-\zeta\bar{z}} \right\rangle$$

$$= \left\langle \frac{1-|z|^2}{|1-z\zeta^{-1}|^2} \right\rangle = \left(\frac{1}{1-z\zeta^{-1}}, \frac{1}{1-z\zeta^{-1}} \right)(1-|z|^2)$$

What is $\left(\frac{1}{1-w\zeta^{-1}}, \frac{1}{1-z\zeta^{-1}} \right)$?

$$\frac{1}{1-\bar{w}\zeta} \frac{1}{1-z\zeta^{-1}}$$

$$\frac{1}{1-\bar{w}\zeta} + \frac{z\zeta^{-1}}{1-z\zeta^{-1}} = \frac{1-\bar{w}z}{(1-\bar{w}\zeta)(1-z\zeta^{-1})}$$

$$\left\langle \frac{1}{1-\bar{w}\zeta} \right\rangle$$

Recall $f(z) = \int i\frac{\zeta+z}{\zeta-z}\, d\mu$

$$f(z) = \left\langle i\frac{\zeta+z}{\zeta-z} \right\rangle = \left\langle i\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} \right\rangle$$

$$=$$

$$-\frac{1}{2} + \frac{1}{1-\bar{w}\zeta} = \frac{-1+\bar{w}\zeta+2}{2(1-\bar{w}\zeta)} = \frac{1}{2}\frac{1+\bar{w}\zeta}{1-\bar{w}\zeta}, \text{ has } \langle\ \rangle = \frac{i}{2}\overline{f(w)}$$

$$\frac{1}{2} + \frac{z\zeta^{-1}}{1-z\zeta^{-1}} = \frac{1-z\zeta^{-1}+2z\zeta^{-1}}{2(1-z\zeta^{-1})} = \frac{1}{2}\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} \text{ has } \langle\ \rangle = \frac{1}{2i}f(z)$$

seems like $(1-\bar{w}z)\left(\frac{1}{1-w\zeta^{-1}}, \frac{1}{1-z\zeta^{-1}} \right) = \frac{1}{2i}\left(f(z) - \overline{f(w)}\right)$

$$\boxed{\left(\frac{1}{1-w\zeta^{-1}}, \frac{1}{1-z\zeta^{-1}} \right) = \frac{1}{2i}\frac{f(z)-\overline{f(w)}}{1-\bar{w}z}}$$

358 $d\mu$ measure on S^1

$f(z) = \int i \frac{\zeta + z}{\zeta - z}\, d\mu$ Pick function

$h(z) = \operatorname{Im} f(z) = \int \frac{1-|z|^2}{|1-z\zeta^{-1}|^2}\, d\mu$ positive harm fn.

$$\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} = 1 + \frac{2z\zeta^{-1}}{1-z\zeta^{-1}}$$

I want to calculate inner wrt $d\mu$ of $\frac{1}{1-w\zeta^{-1}}$ and $\frac{1}{1-z\zeta^{-1}}$

$$\underbrace{\int \frac{1-\bar{w}z}{(1-\bar{w}\zeta)(1-z\zeta^{-1})}}\, d\mu$$

$\|$

$$\frac{A}{1-\bar{w}\zeta} + \frac{B'}{1-z\zeta^{-1}}$$

$B =$

$$\cancel{\frac{1-\bar{w}z}{(1-\bar{w}\zeta)(1-z\zeta^{-1})} = \frac{A}{1-\bar{w}\zeta} + \frac{B}{1-z\zeta^{-1}}}$$

$$\cancel{1-\bar{w}z = A(1-z\zeta^{-1}) + B(1-\bar{w}\zeta)}$$

?

set $$\frac{1-\bar{w}z}{1-z\zeta^{-1}} = A + B\,\frac{1-\bar{w}\zeta}{1-z\zeta^{-1}}$$

$$\underbrace{-\frac{1}{2} + \frac{1}{1-\bar{w}\zeta}} + \frac{z\zeta^{-1}}{1-z\zeta^{-1}} + \frac{1}{2} = \frac{1-\bar{w}z}{(1-\bar{w}\zeta)(1-z\zeta^{-1})}$$

$//$

$$\frac{1}{2}\left(\frac{1+\bar{w}\zeta}{1-\bar{w}\zeta} + \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\right)$$

$$\int \overline{\frac{1+\bar{w}\zeta}{1-\bar{w}\zeta}}\, d\mu = \int \frac{1+w\zeta^{-1}}{1-w\zeta^{-1}}\, d\mu = \frac{1}{i} f(w)$$

$$\frac{1}{2i}\left(f(z) - \overline{f(w)}\right) \Big/ 1-\bar{w}z.$$

359 Examples. $f(z) = i$ $\qquad h(z) = 1$

Your calculations with $\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}$ are too hard.

$$\boxed{\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} = -1 + \frac{2}{1-z\zeta^{-1}}}$$

so up to constants you deal with the resolvent $\frac{1}{1-z\zeta^{-1}}$

$$f(z) = \int i\left(-1+\frac{2}{1-z\zeta^{-1}}\right) d\mu$$

$$f(z) - \overline{f(w)} = \int \left[\left(-i + \frac{2i}{1-z\zeta^{-1}}\right) - \left(i + \frac{-2i}{1-\bar w \zeta}\right)\right] d\mu$$

$$= 2i \int \underbrace{\left[-1 + \frac{1}{1-z\zeta^{-1}} + \frac{1}{1-\bar w\zeta}\right]}_{\sum_{n\geq 0} z^n \zeta^{-n} + \sum_{n\geq 1} \bar w^n \zeta^n} d\mu$$

$$\frac{1}{1-z\zeta^{-1}} + \frac{\bar w \zeta}{1-\bar w\zeta} = \frac{1-\bar w z}{(1-\bar w \zeta)(1-z\zeta^{-1})}$$

$$\frac{1}{2i} \frac{f(z)-\overline{f(w)}}{1-\bar w z} = \int \overline{\frac{1}{1-w\zeta^{-1}}} \frac{1}{1-z\zeta^{-1}} d\mu$$

This should be the point evaluator for a Hilbert space of analytic functions on the disk.

Example. $d\mu = \frac{d\theta}{2\pi}$ $\qquad f(z) = \int i\left(-1+\frac{2}{1-z\zeta^{-1}}\right)\frac{d\theta}{2\pi} = i$

get $\frac{1}{1-\bar w z}$ which is the pt. eval. for Hardy space

$$\int \overline{\frac{1}{1-w\bar\zeta}} f(\zeta) \frac{d\theta}{2\pi} = \int \frac{1}{\zeta - w} f(\zeta) \frac{d\zeta}{2\pi i} = f(w).$$

~~What about~~ What about a de Branges type measure? Szego better

$\rho \frac{d\theta}{2\pi}$ $\qquad \rho = |\text{analytic fn}|^2$

$\log \rho = g + \bar g$ $\qquad \rho = |e^g|^2.$

 Can always do this. Can you calculate the Pick function assoc. to such a measure? Alternative take a rational function with poles outside circle S^1

Alt: Rational Pick function poles outside S^1.

~~for $|z|<$~~

Start with $i\dfrac{1+z\zeta^{-1}}{1-z\zeta^{-1}}$ $\quad \zeta \in S^1$ ~~$|z|<1$~~ then this is a Pick function with a pole at ζ

Replace z by rz $\quad 0<r<1$. You still have Pick fn.

Best thing to do maybe is to ~~...~~ suppose $L^2(S^1, d\mu)$ arises from a contraction? Simplest case C = mult by c of mod <1 on $\mathbb{C}$. Then

$$d\mu = \rho \frac{d\theta}{2\pi} \qquad \rho = \sum_{n\geq 0} c^n \zeta^{-n} + \sum_{n>0} \bar{c}^n \zeta^n = \frac{1-|c|^2}{|1-c\zeta^{-1}|^2}$$

$$i\underbrace{\left(\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\right)}_{\text{anal in } D}\left(\frac{1}{\zeta-c\cancel{\zeta^{-1}}} + \underbrace{\frac{\bar{c}\cancel{\zeta}}{1-\bar{c}\zeta}}_{\text{anal. in } D}\right)\frac{d\zeta}{2\pi i\cancel{\zeta}}$$

<u>NO</u>

$$f(z) = i\,\frac{1+zc^{-1}}{1-zc^{-1}} = i\,\frac{c+z}{c-z}$$

$$\rho\frac{d\theta}{2\pi} = \left(\frac{1}{1-c\zeta^{-1}} + \frac{\bar{c}\zeta}{1-\bar{c}\zeta}\right)\frac{d\zeta}{2\pi i\zeta} = \left(\frac{1}{\zeta-c} + \frac{\bar{c}}{1-\bar{c}\zeta}\right)\frac{d\zeta}{2\pi i}$$

$$\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}} = \cancel{\ldots} \; \frac{\zeta+z}{\zeta-z} = \frac{\zeta-z+2z}{\zeta-z} = 1+\frac{2z}{\zeta-z}$$

$$\int\left(1+\frac{2z}{\zeta-z}\right)\left(\frac{1}{\zeta-c}+\frac{\bar{c}}{1-\bar{c}\zeta}\right)\frac{d\zeta}{2\pi i}$$

$$\underbrace{2z\left(\frac{1}{z-c}+\frac{\bar{c}}{1-\bar{c}z}\right)}_{\text{Res at } z} + \underbrace{\left(1+\frac{2z}{c-z}\right)}_{\text{res at } c} = 1+\frac{2z\bar{c}}{1-z\bar{c}}$$

$$= \frac{1+\bar{c}z}{1-\bar{c}z} = \frac{1}{i}f(z)$$

361 $\quad f(z) = i\,\dfrac{1+\bar{c}z}{1-\bar{c}z}$ Try to be more general.

The idea is that give $d\mu$ you get a Hilbert space ~~consisting of~~ ~~generated by the functions~~ $L^2(S^1, d\mu)$ which contains the functions $\dfrac{1}{1-z\zeta^{-1}}$ for $|z|<1$.

These functions lie in $H^2_-(S^1, d\mu)$ ~~You have a half space.~~ and they probably generate $H^2_-(S^1, d\mu)$.

Sort out the details. ~~in part~~

Organize. Need more examples of Pick functions equivalently measures on S^1. For example which rational functions $f(z)$ when restricted to $|z|<1$ are analytic and have positive imaginary part.

Example $\quad i\,\dfrac{1+\bar{c}z}{1-\bar{c}z}$ with $|c|<1$.

This maps $|z|<1$ into UHP. ~~Can also take~~ ~~Use RHP closed under sums and products~~ Can take positive real combinations. Also higher degree maps. You should be able to classify degree n rational functions preserving the unit circle and unit disk. Same as Blaschke products of degree n. finit. supp. measures.

Take $d\mu = \rho \dfrac{d\theta}{2\pi}$ where ρ smooth >0.

Then $\rho = |g(e^{i\theta})|^2$ where $g(z)$ analytic inv. in D

$$L^2(S^1, d\mu) \xrightarrow{\sim} L^2(S^1, d\theta)$$

$$\frac{f}{g} \longleftarrow\!\!| \; f$$

$$\int \left|\frac{f}{g}\right|^2 d\mu = \int \left|\frac{f}{g}\right|^2 \rho \frac{d\theta}{2\pi} = \int |f|^2 \frac{d\theta}{2\pi}$$

362 preserves H^2. ~~to~~

$$f(z) = \int \overline{e_z}\, f\, d\mu = \int \overline{e_z(\zeta)}\, f(\zeta)\, \overline{g(\zeta)}\, g(\zeta)\, \frac{d\theta}{2\pi}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\int \overline{\frac{1}{1-\bar z \zeta}}\, f(\zeta)\, \frac{d\theta}{2\pi} \qquad\qquad \int \overline{e_z(\zeta) g(\zeta)}\, f(\zeta)\, g(\zeta)\, \frac{d\theta}{2\pi}$$

$$\|$$

$$\frac{1}{g(z)} \int \overline{\frac{1}{1-\bar z \zeta}}\, f(\zeta) g(\zeta) \frac{d\theta}{2\pi} = \int \overline{\frac{1}{g(z)} \frac{1}{1-\bar z \zeta}}\, f(\zeta) g(\zeta) \frac{d\theta}{2\pi}$$

$$e_z(\zeta) g(\zeta) = \frac{1}{g(z)} \frac{1}{1-\bar z \zeta}$$

$$e_z(\zeta) = \frac{1}{g(z)} \frac{1}{1-\bar z \zeta} \frac{1}{g(\zeta)}$$

$$\int \overline{g(z)^{-1} \frac{1}{1-\bar z \zeta} g(\zeta)^{-1}}\, f(\zeta)\, \overline{g(\zeta)}\, g(\zeta)\, \frac{d\theta}{2\pi}$$

$$= \int g(z)^{-1} \frac{1}{1-z\zeta^{-1}} \cancel{\overline{g(\zeta)}^{-1}}\, f(\zeta)\, \cancel{\overline{g(\zeta)}}\, g(\zeta) \frac{d\theta}{2\pi} \quad \frac{d\zeta}{2\pi i \zeta}$$

$$= \frac{1}{g(z)} \int \frac{1}{\zeta - z} f(\zeta) g(\zeta) \frac{d\zeta}{2\pi i} = \frac{1}{g(z)} f(z) g(z) = f(z).$$

Write differently $\rho = \dfrac{1}{|E(\zeta)|^2}$

$$e_z(\zeta) = \frac{\overline{E(z)}\, E(\zeta)}{1 - \bar z \zeta}$$

seems to be the pt. evaluator in $H^2(S^1, d\mu)$ $\quad d\mu = \frac{1}{|E|^2} \frac{d\theta}{2\pi}$

$$\int \overline{e_z(\zeta)}\, \frac{1}{1-\bar{w}\zeta}\, d\mu \overset{?}{=} \frac{1}{1-\bar{w}z}$$

$$\|$$

$$\frac{E(z)}{1-z\zeta^{-1}}\, \frac{1}{1-\bar{w}\zeta}\, \frac{1}{E(\zeta)}\, \frac{d\theta}{2\pi} \qquad \text{OK.}$$

You may need E inverted

You need to connect Pick functions and scattering fns. Go over scattering. Begin with c a contraction say $\|c\| < 1$. Positive definition function

$$d\mu = \sum_{n\geq 0} \zeta^{-n} c^n + \sum_{n \geq 1} \zeta^n (c^*)^n = \sum_{n\in\mathbb{Z}} \mu_n \zeta^{-n}$$

$$= (1-\zeta^{-1}c)^{-1} + \zeta c^*(1-\zeta c^*)^{-1}$$

$$= (1-\zeta^{-1}c)^{-1}\left(1-\zeta c^* + (1-\zeta^{-1}c)\zeta c^*\right)(1-\zeta c^*)^{-1}$$

$$= (1-\zeta^{-1}c)^{-1}(1-cc^*)(1-\zeta c^*)^{-1} \quad \underset{=}{\text{also}} \quad (1-\zeta c^*)^{-1}(1-c^*c)(1-\zeta^{-1}c)^{-1}$$

and $L^2(S^1, d\mu)$ is a dilation of c.

$(\xi, z^l \xi') = \int z$ [redacted] $l\left(\xi \sum_n \mu_n \zeta^{-n} \xi'\right) \frac{d\theta}{2\pi}$

$$= (\xi, \mu_l \xi') = (\xi, c^l \xi') \qquad l \geq 0.$$

OK. You dilate Y, c to get H but then you get the wings; namely.

You have elements $\sum_{n\in\mathbb{Z}} \zeta^n y_n \qquad \zeta y - cy$

$$(y_1, \zeta y - cy) = 0 \qquad (\zeta y - \cancel{cy}, \zeta y - cy) = \|y\|^2 - \|cy\|^2$$

364. You find embeddings

$$L^2(S^1, \underbrace{\overline{(1-cc^*)^{1/2}Y}}_{V^-}) \hookrightarrow H \hookleftarrow L^2(S^1, \underbrace{\overline{(1-c^*c)^{1/2}Y}}_{V^+})$$

You want to understand an $S(z)$ general scattering function ~~analytic~~ analytic in $|z|<1$ bdd by 1. You want a Hilbert space interpretation of ~~the~~ the "~~kernel~~" "kernel" $\frac{1-\overline{S(w)}S(z)}{1-\bar{w}z}$. I think you understand this when S is a unitary scattering fn.

Look at the graph of S.

Try again. Form $S^2 \times S^2$ and look at open submanifold $D \times D$. Look at the graph of S. Some sort of divisor, ~~$D \times D$~~.

Divide out by bound states. You have the dilation of $S: L^2(S^1, V^+) \longrightarrow L^2(S^1, V^-)$

You want to find an analytic map

$z \longmapsto \xi_z$ from D to a Hilbert space

such that
$$(\xi_w, \xi_z) = \frac{1-\overline{S(w)}S(z)}{1-\bar{w}z}$$
~~Analogy~~

$$\left(\frac{1}{1-w\zeta^{-1}}, \frac{1}{1-z\zeta^{-1}}\right) = \sum_{n\geq 0} \bar{w}^n z^n = \frac{1}{1-\bar{w}z}$$

$$\left(\frac{S(w)}{1-w\zeta^{-1}}, \frac{S(z)}{1-z\zeta^{-1}}\right) = \frac{\overline{S(w)}S(z)}{1-\bar{w}z}$$

Note this does not require $|S(z)|\leq 1$

You should be almost there?

 So you want to find

$$\frac{S(\zeta)}{1-\omega\zeta^{-1}} = \frac{S(\zeta)-S(\omega)}{1-\omega\zeta^{-1}} \quad ?$$

Be systematic. Use $S(\zeta)$ cont. of an $L^2(S^1)$ form dilation which will be an H with

$$\begin{tikzcd} L^2(S^1) \arrow[r, hook, "a"] \arrow[rr, bend right, "S"'] & H \arrow[r, "b^*"] & L^2(S^1) \end{tikzcd}$$

H obtained by completing $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad y_i \in L^2(S^1)$

$$\|ay_1+by_2\|^2 = \|y_1\|^2+\|y_2\|^2+(Sy_1,y_2)+(y_2,Sy_1)$$
$$= \|Sy_1+y_2\|^2 + \|(1-S^*S)^{1/2}y_1\|^2$$
$$= \|y_1+S^*y_2\|^2 + \|(1-SS^*)^{1/2}y_2\|^2$$

~~what~~ the old problem of recovering X, c from S.

$$\cdots \oplus u^{-1}V^- \oplus V^- \oplus X \oplus V^+ \oplus uV^+ \oplus \cdots$$

Review this problem again. Start with (X,c) construct dilation $H = L^2(S^1, d\mu)$ where $d\mu$ is the pos. def. function on $\mathbb{Z}$ with values in $\mathcal{L}(X)$

$$\int \zeta^n d\mu = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0. \end{cases}$$

structure of H.
$$2\pi \frac{d\mu}{d\theta} = \sum_{n\geq 0} \zeta^{-n}c^n + \sum_{n\geq 1}\zeta^n (c^*)^n$$
$$= \frac{1}{1-c\zeta^{-1}} + \frac{\zeta c^*}{1-c^*\zeta} = \text{etc.}$$

Structure of H amounts to the ~~dec~~ isom

$$H = \cdots \oplus V^- \oplus X \oplus V^+ \oplus uV^+ \oplus \cdots$$

+ scattering consequences

~~Assuming no bound states~~

From this decomp. you get $L^2(S^1, V^\pm) \hookrightarrow H$

and S. Then

Start with $S(z)$ analytic $|S(z)| \leq 1$ on D. You want the corresponding X, c. [redacted] Have contr. $S: H^2_+ \to H^2_+$ which you dilate somehow.

$$\begin{tikzcd}
H^2_+ \arrow[r, hook, "a"] \arrow[dr, "S"'] & W \arrow[r, two heads] \arrow[d, "b^*"] & X \\
& H^2_+ &
\end{tikzcd}$$

$$W = X \oplus \underbrace{V^+ \oplus uV^+ \oplus \cdots}_{aH^2_+}$$

$$\cup$$

$$W \supset \underbrace{V^- \oplus uV^- \oplus \cdots}_{bH^2_+}$$

$y_1, y_2 \in H^2_+$

You have $W =$ completion of $ay_1 + by_2$ [redacted]

with $\|ay_1 + by_2\|^2 = \|Sy_1 + y_2\|^2 + \|(1-S^*S)^{1/2} y_1\|^2$

$$= \|y_1 + S^* y_2\|^2 + \|(1-SS^*)^{1/2} y_2\|^2$$

Do you see a contraction on X?

$$\begin{tikzcd}
X \oplus aH^2_+ \arrow[d, "b^*"] \\
H^2_+
\end{tikzcd}$$

Another point.

$X \oplus V^+$

$V^- \oplus uX \oplus uV^+$

Let's try some more. You begin with X, c

and construct $\underbrace{X \oplus V^+ \oplus uV^+ \oplus \cdots}_{X \oplus H^2_+ \otimes V^+} = W$ with $u^* u = 1$.

$$\cup$$

$$H^2 \otimes V^-$$

 ~~So you seem to ...~~

Discussion. You want to extend what you did for unitary S, namely: Given suitable X, c get an isom embedding ε of X into H^+ such that the orthogonal comp $H^+ \ominus \varepsilon X$ is outgoing, whence of the form SH^+ with S inner. So conservative X, c concrete model as holom. functions on D and you calculated the point evaluator.

Recently you examined Pick functions ~~&~~ Pos. harm. function on D equivalent to a measure on S^1, get a Hilbert space of analytic functions on the disk, formula for point evaluator. What is the abstract char. (H, u, ξ)?

~~S~~imple examples. $h(z) = 1 \qquad f(z) = i$

$$\frac{1}{2i}\,\frac{f(z) - \overline{f(w)}}{1 - \bar{w} z} = \frac{1}{1 - \bar{w} z} \qquad \text{pt. eval. for } H^+$$

$$f(z) = i\,\frac{1-z}{1+z} \qquad h(z) = \frac{1 - |z|^2}{|1+z|^2}$$

measure has 1 pt suppose

so the Hilbert space should be 1-dim.

UHP version $\quad f(\lambda) = \lambda \qquad \dfrac{f(\lambda) - \overline{f(\mu)}}{\lambda - \bar{\mu}} = 1$

~~Anyway what's happening~~?

You should do scattering in general. Begin with (X, c), form $W = X \oplus V^+ \oplus uV^+ \oplus \cdots$

specifically you complete $\mathbb{C}[u] \otimes X$ wrt scalar product

$$\Big\| \sum_{k=0}^{\infty} u^k x_k \Big\|^2 = \cdots \qquad (x_k, u^\ell x_\ell) = (x_k, (c^*)^{k-\ell} x_\ell)$$

e.g. $\quad \|x_0 + u x_1\|^2 = \|x_0 + c x_1\|^2 + \|(1 - c^*c)^{1/2} x_1\|^2$

$$= \|c^* x_0 + x_1\|^2 + \|(1 - cc^*)^{1/2} x_0\|^2$$

V^+ appears as $\{(u - c) x_0\}$

V^- —— $\{(1 - uc^*) x_0\}$.

$n \geq 1 \quad ((1 - uc^*) x, u^n (1 - uc^*) x_0) = (x, u^n (1 - uc^*) x_0) \quad ?$

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$n \geq 1$ $\quad ((1-uc^*)x_0^{0}, u^n x) = (x^0, u^n x) - (uc^* x_0, u^n x)$
$= (x^0, c^n x) - \left(c^* x_0, \frac{u^{n-1}}{c^{n-1}} x\right) = 0$

So ~~[illegible]~~ this W contains $H^2 \otimes V^-$

$$W = X \oplus H^2 \otimes V^+ \supset H^2 \otimes V^-$$

In the case where $X \hookrightarrow H^2 \otimes V^-$ via $(1-cc^*)^{1/2} \frac{1}{1-zc^*}$
i.e. $(c^*)^n x \mapsto 0$ all $x \in X$, then ~~$W = X \oplus S \; H^2 \oplus$~~
we have $H^2 \otimes V^+ \simeq S(H^2 \otimes V^-)$.

Anyway it should basically be clear. You construct $W \simeq X \oplus H^2 \otimes V^+$ from X, c also construct $H^2 \otimes V^- \hookrightarrow W$. Then you have

$$\begin{tikzcd} & W \arrow[dr, "j_-^*"] & \\ H^2 \otimes V^+ \arrow[ur, hook, "j_+"] \arrow[rr, "S"'] & & H^2 \otimes V^- \end{tikzcd}$$

$$\| j_+ y_1 + j_- y_2 \|^2 = \| \overbrace{S y_1 + y_2}^{j_-^*(j_+ y_1 + j_- y_2)} \|^2 + \| (1 - S^* S)^{1/2} y_1 \|^2$$

if $S^* S = 1$, then j_- is an isom.

$$\| j_+ y_1 + j_- y_2 \|^2 = \| \underbrace{y_1 + S^* y_2}_{j_+^*(j_+ y_1 + j_- y_2)} \|^2 + \| (1 - SS^*) y_2 \|^2$$

$X = \mathrm{Ker}(j_+^*)$ You are trying to recover (X, c)

Basically you ~~[illegible]~~ want to embed X in a Hilbert space of functions, but it seems you need two functions, a graph type embedding. There are two functions

$$x \longmapsto (1-cc^*)^{1/2} \frac{1}{1 - \zeta c^*} x$$

Maybe you should try to generate with $V^+ \oplus V^-$.

Start with (X, c). Get pos. def. function of $\mathbb{Z}$ values in $\mathcal{L}(X)$. $\sim 1 + \sum_{n\geq 0} \zeta^{-n} c^n + \sum_{n\geq 0} \zeta^n (c^*)^n$

$$= \frac{\zeta^{-1}c}{1-\zeta^{-1}c} + \frac{1}{1-\zeta c^*} = \frac{1}{1-\zeta^{-1}c}(1-cc^*)\frac{1}{1-\zeta c^*}$$

$$= \frac{1}{1-\zeta c^*}(1-c^*c)\frac{1}{1-\bar{\zeta} c}$$

Get dilation $L^2(S^1, d\mu)$.

Aim to take an $S(z) \in \mathcal{B}$ and to show that $\frac{1-\overline{S(w)}S(z)}{1-\bar{w}z}$ is positive definite. I know this is true for S inner ~~and~~ so by approximation (Schur) it's true in general.

Go over the proof when S is inner. $X = H^+ \ominus SH^+$

Start with $\left(\frac{1}{1-\bar{z}\zeta}, f\right) = f(z)$ in H^+.

so

$$\cancel{\left(S(\zeta)\frac{1}{1-\bar{z}\zeta}, Sf\right) = S(z)f(z) \text{ in } SH^+}$$

$$\left(S(\zeta)\frac{1}{1-\bar{z}\zeta}, Sf\right) = \left(\frac{1}{1-\bar{z}\zeta}, f\right) = f(z).$$

$$\Big(\underbrace{\overline{S(z)}\, S(\zeta)\frac{1}{1-\bar{z}\zeta}}_{\text{point evaluator for } SH^+}, Sf\Big) = (Sf)(z).$$

So $\overline{S(z)}\, S\frac{1}{1-\bar{z}\zeta}$ is the kernel of the proj of H^+ onto SH^+

$\therefore \frac{1-\overline{S(z)}S(\zeta)}{1-\bar{z}\zeta}$ ——————— of H^+ ont X.

370 In these arguments you use the fact that the elements $\frac{1}{1-\bar{z}\zeta}$ ~~are~~ span H^+.

Extrapolate. Look at the interpolation problem $f(a_i) = b_i$ $i=1,\dots,n$. $f \in \mathcal{B}$. Corresp. linear functional $\left(\frac{1}{1-\bar{a}_i z}, f\right) = b_i$ You ~~are~~ want a

Given $a_1,\dots,a_n$ dist. points in D. Form

$$H^+ = \left(\sum_{j=1}^{n} \mathbb{C}\frac{1}{1-\bar{a}_j z}\right) \oplus \underbrace{\left(\prod_{j=1}^{n} \frac{z-a_j}{1-\bar{a}_j z}\right) H^+}_{\text{these functions vanish at } a_1,\dots,a_n} \qquad \text{orth d.s.}$$

there is a unique elt f here such that $f(a_j) = b_j$ $\forall j$

Go back to the non unitary case. A contraction c^{n} yields $L^2(S^1, d\mu_c)$

$$\int z^n d\mu_c = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0 \end{cases} \qquad 2\pi \frac{d\mu_c}{d\theta} = \sum_{n\geq 0} \zeta^{-n} c^n + \sum_{n\geq 1} \zeta^n c^{*n}$$

$$= \frac{1}{1-\zeta^{-1}c}(1-cc^*)\frac{1}{1-\zeta c^*} = \frac{1}{1-\zeta c^*}(1-c^*c)\frac{1}{1-\zeta^{-1}c}$$

Actually why not bring this inside the unit circle.

$$\sum_{n\geq 0} \bar{w}^n c^n + \sum_{n\geq 1} z^n c^{*n} = \frac{1}{1-\bar{w}c} + \frac{zc^*}{1-zc^*}$$

$$= \frac{1}{1-\bar{w}c}\left(1-zc^* + (1-\bar{w}c)zc^*\right)\frac{1}{1-zc^*}$$

$$= \frac{1}{1-\bar{w}c}\underbrace{\left(1-\bar{w}cc^*z\right)}_{1-\bar{w}z+\bar{w}(1-cc^*)z}\frac{1}{1-zc^*}$$

371 You have some understanding of why

$$\frac{1-\overline{S(w)}\,S(z)}{1-\bar{w}z} \text{ is a pos. semi-def matrix in}$$

the case that S is inner: $S^*S = 1$. Proof amounts to this Given $z_1, \ldots, z_n \in D$ and $t_1, \ldots, t_n \in \mathbb{C}$.

$$\left(\frac{1-\overline{S(z)}S(\zeta)}{1-\bar{z}\zeta}, \frac{1-\overline{S(w)}S(\zeta)}{1-\bar{w}\zeta} \right)$$

$$= \left(\frac{1}{1-\bar{z}\zeta}, \frac{1-\overline{S(w)}S(\zeta)}{1-\bar{w}\zeta} \right) - \left(\frac{\overline{S(z)}S(\zeta)}{1-\bar{z}\zeta}, \frac{\cancel{1}}{1-\bar{w}\zeta} \right)$$

$$+ \left(\frac{\overline{S(z)}S(\zeta)}{1-\bar{z}\zeta}, \frac{\overline{S(w)}S(\zeta)}{1-\bar{w}\zeta} \right)$$

$$= \frac{1-\overline{S(w)}S(z)}{1-\bar{w}z} - \overline{\frac{\overline{S(z)}S(w)}{1-\bar{z}w}} + S(z)\underbrace{\left(S\frac{1}{1-\bar{z}\zeta}, S\frac{1}{1-\bar{w}\zeta} \right)}_{\frac{1}{1-\bar{w}z} - \left(\frac{1}{1-\bar{z}\zeta}, (1-S^*S)\frac{1}{1-\bar{w}\zeta} \right)}\overline{S(w)}$$

$$= \frac{1-\overline{S(w)}S(z)}{1-\bar{w}z} - \left((1-S^*S)^{1/2}\frac{1}{1-\bar{z}\zeta}, (1-S^*S)^{1/2}\frac{1}{1-\bar{w}\zeta} \right)$$

It looks like it works. IDEA - you seem to be ~~play~~ playing with quadratic, better hermitian forms depending on a parameter, ~~again~~ along with filtrations again.

372 There are lots of things to correlate, But in any case go over what you just did Working inside the Hardy space. You introduce the element $K_{\bar z} = \dfrac{1-\overline{S(z)}\,S(\zeta)}{1-\bar z \zeta} \in H^+$ and you compute that

$$(K_{\bar z}, K_{\bar w}) + \left((1-S^*S)^{1/2} \frac{\overline{S(z)}}{1-\bar z \zeta}, (1-S^*S)^{1/2} \frac{\overline{S(w)}}{1-\bar w \zeta} \right) = \frac{1-\overline{S(w)}\,S(z)}{1-\bar w z}$$

$$\| $$

$$\left(\frac{1-\overline{S(z)}\,S}{1-\bar z \zeta}, \frac{1-\overline{S(w)}\,S}{1-\bar w \zeta} \right) \cancel{+ \left(\frac{1-\overline{S(z)}\,S}{1-\bar z \zeta}, \frac{1}{1-\bar w \zeta} (1-\overline{S(w)}\,S) \right)}$$

$$\| $$

$$\left(\frac{1}{1-\bar z \zeta}, \frac{1-\overline{S(w)}\,S}{1-\bar w \zeta} \right) - \left(\frac{\overline{S(z)}\,S}{1-\bar z \zeta}, \frac{1}{1-\bar w \zeta} \right) + \left(\frac{\overline{S(z)}\,S}{1-\bar z \zeta}, \frac{\overline{S(w)}\,S}{1-\bar w \zeta} \right)$$

$$\| $$

$$\frac{1-\overline{S(w)}\,S(z)}{1-\bar w z} - \frac{\overline{\overline{S(z)}\,S(w)}}{1-\bar z w} + S(z) \left(\frac{1}{1-\bar z \zeta}, S^*S \frac{1}{1-\bar w \zeta} \right) \overline{S(w)}$$

$$S(z) \left(\frac{1}{1-\bar z \zeta}, (1-S^*S) \frac{1}{1-\bar w \zeta} \right) \overline{S(w)}$$

$$\| \quad \frac{S(z)\overline{S(w)}}{1-\bar w z}$$

This seems to work and establishes the ~~main~~ ~~res~~ fact that $\dfrac{1-\overline{S(w)}\,S(z)}{1-\bar w z}$ is positive definite.

Anyway you would like to link this calculation to the Hilbert space stuff. Is it possible to interpret $K_{\bar z} = \dfrac{1-\overline{S(z)}\,S}{1-\bar z \zeta}$ as a point evaluator?

Let's start with $S : H^+ \longrightarrow H^+$ $\qquad |S(z)| \leq 1.$

then dilate

$$\begin{tikzcd}
 & (1-SS^*)^{1/2}H^+ \arrow[d] & \\
H^+ \arrow[r, hook] \arrow[dr, "S"'] & W \arrow[d] & X \arrow[l, hook'] \; = \; \overline{(1-S^*S)^{1/2}H^+} \\
 & H^+ &
\end{tikzcd}$$

This is puzzling. Try a simple example. Take S = constant of modulus < 1.

$$K_{\bar z} = \frac{1-|S|^2}{1-\bar z \zeta} \qquad K_{\bar w} = \frac{1-|S|^2}{1-\bar w \zeta}$$

$$\left(K_{\bar z}, K_{\bar w}\right) = \frac{(1-|S|^2)^2}{1-\bar w z}$$

$$\left((1-|S|^2)^{1/2}\frac{\bar S}{1-\bar z \zeta},\ (1-|S|^2)^{1/2}\frac{\bar S}{1-\bar w \zeta} \right) = \frac{(1-|S|^2)|S|^2}{1-\bar w z}$$

sum $\quad (1-|S|^2)^2 + (1-|S|^2)|S|^2 = 1-|S|^2$

It seems that we have a pair depending on z

$$K_{\bar z} = \frac{1-\overline{S(z)}\,S(\zeta)}{1-\bar z \zeta} \in H^+$$

$$L_{\bar z} = \frac{\overline{S(z)}}{1-\bar z \zeta} \in X$$

374 I think you need more examples.

Let's try interpolation — given $a_1, \dots, a_n \in D$ distinct and a set of values $b_1, \dots, b_n \in D$, to solve

$S(a_j) = b_j$ Nec. cond. is $\dfrac{1-\bar{b}_j b_k}{1-\bar{a}_j a_k}$ is ≥ 0.

Apparently there's a solution with S a Blashke product.

$$\frac{S(z)-b_1}{1-\bar{b}_1 S(z)}$$ vanishes when $z = a_1$ need $|b_1| < 1$.

$$\frac{S(z)-b_1}{1-\bar{b}_1 S(z)} = \frac{z-a_1}{1-\bar{a}_1 z} S_1(z)$$

$$S_1(z) = \frac{1-\bar{a}_1 z}{z-a_1} \frac{S(z)-b_1}{1-\bar{b}_1 S(z)}$$

$$S_1(a_j) = \frac{1-\bar{a}_1 a_j}{a_j - a_1} \frac{b_j - b_1}{1-\bar{b}_1 b_j}$$

Interpolation problem goes as follows: given $a_1, \dots, a_n$ dist. and $b_1, \dots, b_n$. Then on $H^+/\Pi(z-a_i)H^+$ you have an operator mult. by b_j on the a_j eigenspace. You want this to be a contraction I believe

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operator $S \dfrac{1}{1-\bar{a}_i z} =$

There's an obvious gap namely the condition that $\dfrac{1-\bar{b}_j b_k}{1-\bar{a}_j a_k}$ is ≥ 0 has to be put in a form invariant under autos of source and target disk. Once this done ~~the~~ you can assume $a_1 = b_1 = 0$. Then $S(z) = z S_1(z)$

$S_1(a_j) = \dfrac{b_j}{a_j}$ $\qquad \dfrac{1 - \bar{b}_j \bar{a}_j^{-1} b_k a_k^{-1}}{1-\bar{a}_j a_k}$

375 Review positivity of $\frac{1-\overline{S(w)}\,S(z)}{1-\bar w z}$. Based on a formula inside H^+.

$$K_{\bar z} = \frac{1-\overline{S(z)}\,S(\zeta)}{1-\bar z \zeta} \qquad L_{\bar z} = \frac{\overline{S(z)}}{1-\bar z \zeta}$$

anti-holom. maps from D to H^+

$$(K_{\bar z}, K_{\bar w}) + \underbrace{(L_{\bar z}, (1-S^*S)L_{\bar w})}_{(L_{\bar z}, L_{\bar w}) - (SL_{\bar z}, SL_{\bar w})}$$

$$K_{\bar z} = \frac{1}{1-\bar z \zeta} - SL_{\bar z}$$

$$(K_{\bar z}, K_{\bar w}) = \left(\frac{1}{1-\bar z\zeta} - SL_{\bar z}, \frac{1}{1-\bar w \zeta} - SL_{\bar w}\right)$$

$$= \frac{1}{1-\bar w z} - S(z)\frac{\overline{S(w)}}{1-\bar w z} - \overline{\frac{S(w)\overline{S(z)}}{1-\bar z w}} + (SL_{\bar z}, SL_{\bar w})$$

$$(K_{\bar z}, K_{\bar w}) + (L_{\bar z}, L_{\bar w}) - (SL_{\bar z}, SL_{\bar w})$$

$$= \frac{1}{1-\bar w z} - 2\frac{S(z)\overline{S(w)}}{1-\bar w z} + \frac{S(z)\overline{S(w)}}{1-\bar w z} = \frac{1-\overline{S(w)}\,S(z)}{1-\bar w z}$$

The calculation is straightforward. But what is its meaning? Should involve dilation

$$W = X \oplus H^2 \otimes V_+$$

$$\begin{tikzcd} K_{\bar z} \in H^2\otimes V_+ \arrow[r, leftrightarrow, "a"] \arrow[dr, bend right, "b^*a = S"'] & W \arrow[r, leftrightarrow] & X \\ & H^2 \otimes V_- \arrow[u, "b"'] & \end{tikzcd}$$

376 ~~[illegible]~~ Start with $S(z) \in \mathcal{B}$

Get contraction $f \mapsto Sf$ from H^2 to itself.

W = completion of $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in (H^2)^{\oplus 2}$ ~~under~~ wrt

$$\|ay_1 + by_2\|^2 = \|Sy_1 + y_2\|^2 + \|(1-S^*S)^{1/2}y_1\|^2$$
$$= \|y_1 + S^*y_2\|^2 + \|(1-SS^*)^{1/2}y_2\|^2$$

X = completion of H^2 wrt $\|y\|_X^2 = \|(1-S^*S)^{1/2}y\|^2$

Then $W \longleftarrow H^2 \oplus X$ $\quad$ u = mult by ζ is isometry.

$(b, (a-bS)\pi^{-1})$

$$\pi : H^2 \longrightarrow X$$ canonical map

$\ni \|\pi(y)\|_X^2 = \|y\|^2 - \|Sy\|^2$

$$\|(a-bS)y\|^2 = \|y\|^2 - (ay, bSy) - (\underbrace{bSy}, \overset{b^*}{a}y) + \|Sy\|^2 =$$

~~[illegible]~~

There seems to be a antiholom. map

$$bK_{\bar z} - (a-bS)L_{\bar z}$$

from D to W. Apply b^* to get $K_{\bar z}$, apply a^* to get $S^*K_{\bar z} - (1-S^*S)L_{\bar z}$

$$bK_{\bar z} - (a-bS)L_{\bar z} = b\underbrace{\left(K_{\bar z} + SL_{\bar z}\right)}_{\frac{1}{1-\bar z\zeta}} - aL_{\bar z}$$

377 $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ $\pi^{-s}\Gamma(s) = \int_0^\infty e^{-\pi t} t^s \frac{dt}{t}$

$$n^{-s}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right) = 2\int_0^\infty e^{-\pi n^2 t^2} t^{s} \frac{dt}{t}$$

~~$t\phi(t) \sim t$ $t\to+\infty$~~

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \underbrace{\left(2\sum_{n=1}^{\infty} e^{-\pi n^2 t^2}\right)}_{(\phi(t)-1)} t^s \frac{dt}{t}$$

$$\phi(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}$$

~~$\phi\left(\frac{1}{t}\right) = \frac{1}{t}\phi(t)$~~

~~$\phi(t) \sim 1$ as $t\to+\infty$~~

$$\phi(t) \sim 1 \qquad \text{as } t \to +\infty$$

$$\phi(t) \sim \frac{1}{t} \qquad \text{as } t \to 0 \qquad \left(-t\frac{d}{dt}+1\right)\left(t\frac{d}{dt}\right)$$

$$\int \left(-t\frac{d}{dt}+1\right)\left(t\frac{d}{dt}\right)(\phi(t)-1)\, t^{s-1}\,dt = \int (\phi(t)-1)\, s(s-1)\, t^s \frac{dt}{t}$$

$$\|$$

~~$\int\left(-t\frac{d}{dt}+1\right)$~~ $\int \left(t\frac{d}{dt}\right)\left(t\frac{d}{dt}+1\right)(\phi(t)-1)$

Better $t = e^x$ $\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_{-\infty}^{\infty} \underbrace{(\phi(e^x)-1)} e^{sx}\,dx$

~ 1 $x\to+\infty$

$\sim e^{-x}$ $x\to-\infty$

$$\int_{-\infty}^{\infty}\left(\frac{d}{dx}+1\right)\frac{d}{dx}(\phi(e^x)-1)\, e^{sx}\,dx = \int_{-\infty}^{\infty}(\phi(e^x)-1)\underbrace{\left(-\frac{d}{dx}\left(-\frac{d}{dx}+1\right)e^{sx}\right)}_{(-s)(1-s)e^{sx}} dx$$

Your question is ~~c~~ can you get the Riemann-Siegel asymptotic fmla.

The following objects seem to be equivalent:

1. Pick function modulo real constants on D (unit disk)
2. positive harmonic functions on D
3. positive definite function on the group $\mathbb{Z}$
4. measure on $S^1 = \partial D$
5. cyclic unitary representation of $\mathbb{Z}$: (H, u, ξ)
6. sequence h_n, $n \geq 1$ in D (in finite case a finite sequence $h_1, \dots, h_{n-1}$ in D and $h_n \in S^1$.
7. bdd analytic function $S(z)$ on D sup norm ≤ 1.

~~If you have a problem~~

~~Add~~ Given the measure $d\mu$ on S^1 equivalently the pos. def fn. $\mu_n = \int \xi^n \, d\mu$ $\qquad d\mu = \sum \mu_n \xi^{-n} \frac{d\theta}{2\pi}$

you get p_n, q_n.

$$p_n \in (\xi^n + P_{n-1}) \cap P_{n-1}^{\perp}$$

$$q_n \in (1 + \xi P_{n-1}) \cap \xi P_{n-1}^{\perp}$$

$$q_n = \xi^n \overline{p_n(\bar{\xi}^{-1})}$$

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$$

What's maybe happening is that you have

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

and you **begin with** $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. $\qquad S_n = \frac{p_n}{q_n}$

You want the simplest examples. What are the simplest measures? What do you know?

A finite support $d\mu$ yields a finite Schur sequence

If $d\mu$ has n point supp, then $P_{n-1} = \mathbb{C} + \mathbb{C}z + \dots + \mathbb{C}z^{n-1} \xrightarrow{\sim} L^2(S^1, d\mu)$

$zp_{n-1} \in P_{n-1}$ $\qquad zp_{n-1} + h_n q_{n-1} = 0$

$p_{n-1} \in (z^{n-1} + P_{n-2}) \cap (P_{n-2})^\perp$ $\qquad q_{n-1} + \bar{h}_n z p_{n-1} = 0$

$zp_{n-1} \in zP_{n-2}^\perp = \mathbb{C} q_{n-1}.$

$$\begin{pmatrix} 1 & h_n \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \cdots\cdots \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

You've learned that for an n-pt supp measure corresponds to finite Schur sequence of length n. I guess you can rearrange the recursion relations so as to yield a

$$S(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \bar{h}_1 \\ h_1 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\Big(\cdots\Big)\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \hline h_n \end{pmatrix}$$

Work out in more detail.

Let $d\mu$ be a delta measure at $\zeta = k$

Then $p_0 = q_0 = 1$ and $zp_0 = kq_0$

$$\underbrace{\begin{pmatrix} 1 & h_1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{z + h_1} = 0$$

$h_1 = -k$

You are trying to connect a measure on S^1 to an S, the idea being that the orthog poly sequence yields $\{h_n\}$ which gives S via Schur expansion. Alternative: Remove cyclic vector from the domain of u to get a partial unitary. Similar to adjoining the projection

 ~~Philosophy seems to be this~~

Pick function $\int i \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}\, d\mu$

Use Fourier coefficients. $d\mu = \sum \mu_n \zeta^{-n} \frac{d\theta}{2\pi}$
where μ_n is a pos. def. function on $\mathbb{Z}$.
$\sum \mu_n \zeta^{-n}$ on S^1 extends to the harmonic function

$$\sum_{n\geq 0} \mu_n \bar{z}^n + \sum_{n\geq 1} \bar{\mu}_n z^n$$ Call this $h(z)$

$$h(z) = \underbrace{\left(-\frac{\mu_0}{2} + \sum_{n\geq 0} \mu_n \bar{z}^n\right)}_{-\frac{1}{2i}\overline{f(z)}} + \underbrace{\left(-\frac{\mu_0}{2} + \sum_{n\geq 0} \mu_{-n} z^n\right)}_{+\frac{1}{2i} f(z)}$$

$h = \operatorname{Im} f$ where

$$f = i\left(-\mu_0 + 2\sum \mu_{-n} z^n\right)$$

$$= \int i \underbrace{\left(-1 + \sum_{n=0}^{\infty} 2\zeta^{-n} z^n\right)}_{-1 + \frac{2}{1-z\zeta^{-1}} = \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}} d\mu$$

philosophy: Count dims. Consider a ~~cyclic~~ cyclic unit rep. of dim n. Same as an n-point measure on S^1 real dim $2n$. The possible operators form a space of real dim n. n-fold symm. product of the circle.

possible scalar products on P_{n-1}. ~~The go~~
give moments $\mu_0, \mu_1, \dots, \mu_{n-1}$ Probably dim $2n-1$; equiv. to $\mu_0 = \|1\|^2, h_1, \dots, h_{n-1}$

381 Review the operator interpretation of Pick functions

$$d\mu = \sum \mu_n \zeta^{-n} \frac{d\theta}{2\pi} \qquad \int \zeta^n d\mu = \mu_n$$

$\sum \mu_n \zeta^{-n}$ extends to harmonic function on D

$$h(z) = \sum_{n\geq 0} \mu_n \bar z^n + \sum_{n\geq 1} \mu_{-n} z^n$$

$$= \underbrace{\left(-\frac{\mu_0}{2} + \sum_{n\geq 0} \mu_n \bar z^n\right)}_{-\frac{\overline{f(z)}}{2i}} + \underbrace{\left(-\frac{\mu_0}{2} + \sum_{n\geq 0} \mu_{-n} z^n\right)}_{\frac{f(z)}{2i}}$$

$$\therefore h(z) = \operatorname{Im} f(z) \qquad f(z) = i\left(-\mu_0 + 2\sum_{n\geq 1} \mu_{-n} z^n\right)$$

$$f(z) = \int i\underbrace{\left(-1 + 2\sum_{n\geq 0} \zeta^{-n} z^n\right)}\, d\mu$$

$$-1 + \frac{2}{1-z\zeta^{-1}} = \frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}$$

You have Hilbert space $L^2(S^1, d\mu) \supset H^2(S^1, d\mu)$

Look at $\frac{1}{1-\bar z \zeta}$ ~~preoperator~~

$$\left(\frac{1}{1-\bar z\zeta}, \frac{1}{1-\bar w\zeta}\right) = \left(1, \frac{1}{1-z\zeta^{-1}}\frac{1}{1-\bar w\zeta}\right),$$

$$\frac{1}{1-z\zeta^{-1}} + \frac{\bar w\zeta}{1-\bar w\zeta} = \frac{1-z\bar w}{(1-z\zeta^{-1})(1-\bar w\zeta)}$$

$$\frac{1}{1-z\bar w}\left(1, \underbrace{\left(\frac{1}{1-z\zeta^{-1}} - \frac{1}{2}\right)}_{\frac{1}{2i}\int i\frac{1+z\zeta^{-1}}{1-z\zeta^{-1}}d\mu} + \underbrace{\left(\frac{1}{1-\bar w\zeta} - \frac{1}{2}\right)}_{\frac{1}{2i}\int i\left(\frac{1+\bar w\zeta}{1-\bar w\zeta}\right)d\mu}\right) \quad \text{"} \quad \frac{f(z)}{2i} - \frac{\overline{f(w)}}{2i}$$

382 $$\boxed{\left(\frac{1}{1-\bar z \zeta}, \frac{1}{1-\bar w \zeta}\right) = \frac{1}{2i}\,\frac{f(z)-\overline{f(w)}}{1-z\bar w}}$$

$$\left(\frac{1}{2}, \frac{1}{1-\bar w \zeta}\right) = \frac{1}{2}\,\frac{1}{2i}\left(f(0)-\overline{f(w)}\right)$$

$$\left(\frac{1}{1-\bar z \zeta}, \frac{1}{2}\right) = \frac{1}{2}\,\frac{1}{2i}\left(f(z)-\overline{f(0)}\right)$$

$$\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{\mu_0}{4} \qquad f(0) = 2i\frac{\mu_0}{2}$$

$$2i\left(-\frac{1}{2}+\frac{1}{1-\bar z \zeta}, -\frac{1}{2}+\frac{1}{1-\bar w \zeta}\right) =$$

$$\frac{f(z)-\overline{f(w)}}{1-z\bar w} - \frac{1}{2}\left(i\mu_0 - \overline{f(w)}\right)$$

$$-\frac{1}{2}\left(f(z)+i\mu_0\right) + \frac{i\mu_0}{2} \qquad \frac{f(0)}{2}$$

$$= f(z)\left(\frac{1}{1-z\bar w}-\frac{1}{2}\right) - \overline{f(w)}\left(\frac{1}{1-z\bar w}-\frac{1}{2}\right) - \frac{i\mu_0}{2}$$

set $z = w = 0$.

$$\frac{2i}{4}\mu_0 = i\mu_0\frac{1}{2} - \overline{i\mu_0}\,\frac{1}{2} - \frac{i\mu_0}{2}$$

$$\boxed{\frac{1}{4}\left(\frac{1+\bar z \zeta}{1-\bar z \zeta}, \frac{1+\bar w \zeta}{1-\bar w \zeta}\right) = \frac{f(z)-\overline{f(w)}}{2\cdot 2i}\left(\frac{1+\bar z w}{1-\bar z w}\right) - \frac{\mu_0}{4}}$$

383 $$\boxed{\left(\frac{1+\bar z\zeta}{1-\bar z\zeta}, \frac{1+\bar w\zeta}{1-\bar w\zeta}\right) = \frac{f(z)-\overline{f(w)}}{i}\left(\frac{1+\bar z w}{1-\bar z w}\right) - \frac{f(0)}{i}}$$

$z=w=0$ $$\underset{\mu_0}{\overset{\shortparallel}{(1,1)}} = \frac{i\mu_0+i\mu_0}{i} - \mu_0 \quad ✓$$

This transf does not seem to improve matters. e.g: requires $f(0) \in i\mathbb{R}$

Take a measure $d\mu$ on circle, form corresp. (H, u, ξ) $\quad H = $ Completion of $\mathbb{C}[u,u^{-1}]\xi \ni \sum u^n c_n$

$$(u^k; u^\ell) = \int \zeta^{-k+\ell}\, d\mu = \mu_{\ell-k}$$

positive semi-def. of $\mu_{\ell-k}$ should be equivalent to

$$\sum_{n\geq 0}\mu_n \bar z^n + \sum_{n\geq 0}\mu_{-n} z^n - \mu_0 > 0 \quad \text{for any } |z|<1.$$

corresp. kernel is

$$-1+\sum_{n\geq 0}\zeta^n\bar z^n + \sum_{n\geq 0}\zeta^{-n} z = \frac{\zeta\bar z}{1-\zeta\bar z} + \frac{1}{1-\zeta^{-1} z}$$

$$= \frac{\cancel{\zeta\bar z} - |z|^2 + 1 - \cancel{\zeta\bar z}}{|1-z\zeta^{-1}|^2}$$

Point: Let $$d\mu_r = \underbrace{\left(\sum_{n\geq 0}\mu_n r^n e^{-in\theta} + \sum_{n>0}\mu_{-n} r^n e^{in\theta}\right)}_{\mu_r(e^{i\theta})}\frac{d\theta}{2\pi}$$

Then positivity of the harmonic function $h(z)$ implies $\mu_r(e^{i\theta}) \geq 0$, hence $\int \mu_r \frac{d\theta}{2\pi}$ of L-polys is ≥ 0, whence $(\mu_{\ell-k}) \geq 0$. Preceding was digression

384 Start from cyclic unitary of $\mathbb{Z}$: (H, u, ξ)

Concretely $H = L^2(S^1, d\mu)$ $u =$ mult by $\zeta = e^{i\theta}$, $\xi = 1$.

You propose to change u ~~on~~: $\mathbb{C}1 \longrightarrow \mathbb{C}\zeta$

keeping u fixed on $\mathbb{C}1^\perp$. ~~[illegible]~~

Call the modified u, u_h when $|h| = 1$.

~~Your problem is now to~~

You have u on H and a cyclic line $\mathbb{C}\xi$

$$H = \overset{v^+}{\mathbb{C}\xi} \oplus aX \qquad ua = b.$$

$$= \overset{v^-}{\mathbb{C}u\xi} \oplus bX$$

eigenvector equation

~~$v^+ + ax = v^- + bx$~~

Let $u(h) = \zeta h$

$h = v^+ + ax_1$

$h = v^- + bx_2$

$u(h) = u(v^+) + bx_1$ $\qquad \zeta h = \zeta v^- + \zeta b x_2$

$x_1 = \zeta x_2$

?

$\therefore$ $u(v^+) + \zeta bx =$

Try again. $u(h) = \zeta(h)$

$h = ax_1 + v^+ = v^- + bx$

$u(h) = u(v^+) + bx_1 = \zeta v^- + \zeta bx$

$\therefore$ conclude that $x_1 = \zeta x$ and $\zeta ax + v^+ = v^- + bx$

$$\boxed{(\zeta a - b)x = -v^+ + v^-} \quad \text{as} \quad \boxed{u(v^+) = \zeta v^-}$$

$$\Downarrow$$

$$(1 - \zeta b^* a)x = b^* v^+$$

$$v^- = (1 - bb^*)(1 - \zeta ab^*)^{-1} v^+ \qquad \text{and} \quad |\zeta| < 1.$$

$$u(v^+) = \zeta v^- = \zeta S(\zeta) v^+$$

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pencil of divisors idea.

spectrum = divisor mult. 1

cyclic repn. gives a pencil of divisors

Return to $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix}$

then $\delta_n = \dfrac{p_n}{q_n} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} (z\, \delta_{n-1})$

Consider $d\mu$ $\qquad L^2(S^1, d\mu) = H$

$V^+ = \mathbb{C}1 \qquad H = V^+ \oplus aX = V^- \oplus bX$

$V^- = \mathbb{C}\xi$. You need to find example.

Calcul

Review $\quad H,\ u,\ Y \subset H,\ X = u^{-1}Y \cap Y \ \underset{b=ua}{\overset{a}{\rightrightarrows}}\ Y$

$H = Y^\perp \oplus X \oplus V^+ \qquad \xi = w + x_1 + v^+$

$\quad = Y^\perp \oplus V^- \oplus uX \qquad \xi = w + v^- + ux$

assume $u(\xi) = z\xi \qquad u\xi - z\xi = \cancel{uw} + ux_1 + \cancel{uv^+}$
$\qquad\qquad\qquad\qquad - \cancel{zw} - zux + \cancel{zv^-}$

project onto $uX \qquad w \perp X \Rightarrow uw \perp uX$

$uX = (Y + V^-)^\perp \qquad v^+ \perp X \Rightarrow uv^+ \perp uX$

Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525
24 - 29 St Giles'
Oxford OX1 3LB
Mathematical Institute
Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
UNIVERSITY OF OXFORD

you get $ux_1 = zux. \qquad x_1 = zx$

$zx + v^+ = v^- + ux \qquad \boxed{(z-u)x = -v^+ + v^-}$

386 ~~[illegible]~~ Go over partial unitaries, how they yield a pencil of spectra. $Y = aX \oplus V^+ = V^- \oplus bX$

$a^*a = b^*b = 1$. Eigenvector equation is

$(za-b)x = -v_+ + v_-$ Solution for $|z|<1$ is

$$x = (1-zb^*a)^{-1} b^* v_+ = b^*(1-zab^*)^{-1} v_+$$

$$v_- = (1-bb^*)(1-zab^*)^{-1} v_+ = S(z)\, v_+$$

Note: $c_0 = ba^*$ contraction assoc. to partial unitary, so have $S(z) = (1-c_0c_0^*)^{1/2}(1-zc_0^*)^{-1}$

Now discuss boundary condition $\xi_\pm$ unit vectors in $V^\pm$

$c_h = ba^* + \xi_- h \xi_+^*$

Suppose $y = ax_1 + v^+ = v^- + bx$

satisfies $(z - c_h)y = 0$.

$$c_h y = bx_1 + \xi_- h(\xi_+^* v_+) = zv_- + zbx$$

$\therefore x_1 = zx$ $\xi_- h(\xi_+, v_+) = zv_-$

also have $v_- = S(z) v_+$ $\boxed{hv_+ = zS(z)v_+}$

here you view $S(z)$ as an op from V_+ to V_-

If you want to treat $S(z)$ and h as numbers, take

$v_+ = \xi_+$ $v_- = \xi_- S(z)$, get $\xi_- h = zS(z)\xi_-$

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525
UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

so the eigenvalue condition is $\boxed{h = zS(z)}$

387 Ann? Go back to determinants. This brings up the problem of analyticity

$$\det(z - c_h) \qquad c_h = ba^* + \xi_- h \xi_+^*$$

$$\delta \log\det(z-c_h) = \operatorname{tr}\left(\frac{-1}{z-c_h}\,\xi_-\,\delta h\,\xi_+^*\right)$$

$$= -\,\xi_+^*\,\frac{1}{z-c_h}\,\xi_-\,\delta h$$

$$\frac{1}{z-c_h} = \frac{1}{z-c_0-\xi_- h \xi_+^*}$$

$$\xi_+^*\frac{1}{z-c_h}\xi_- = \left(\xi_+^*\frac{1}{z-c_0}\xi_-\right) + \left(\xi_+^*\frac{1}{z-c_0}\xi_-\right) h \overbrace{\left(\xi_+^*\frac{1}{z-c_0}\xi_-\right)}^{f} + \cdots$$

$$= f\,\frac{1}{1-hf}$$

$$-\delta\log\det(z-c_h) = f\,\frac{1}{1-hf}\,\delta h = -\delta\log(1-hf)$$

$$\therefore\ \det(z-c_h) = (1-hf)\cdot\det(z-c_0)$$

What is $\xi_+^*\dfrac{1}{z-ba^*}\xi_-$? this is the scattering ~~matrix~~ for $|z|>1$.

What can you do?

Review equiv. between contraction, partial unitaries, and scattering operators.

UNIVERSITY OF OXFORD
Mathematical Institute
24 - 29 St Giles'
Oxford OX1 3LB

Professor D. Quillen, FRS
Waynflete Professor of Pure Mathematics
Telephone: Direct line (0865) 2-73560
Enquiries (0865) 2-73525

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c contraction on X

$H =$ completion of $\bigoplus_{n\in\mathbb{Z}} u^n X$ wrt.

$$(u^m x_m, u^n x_n) = (x_m, u^{n-m} x_n)$$

$$(x'_m, u^k x'') = \begin{cases} (x'_m, c^k x'') & k \geq 0 \\ (x'_m, (c^*)^{-k} x'') & k \leq 0. \end{cases}$$

$$H^2 = L^2(S^1, d\mu) \qquad d\mu = \underbrace{\sum_n \zeta^{-n} \mu_n}\ \frac{d\theta}{2\pi}$$

$$\mu_n = \begin{cases} c^n & n \geq 0 \\ (c^*)^{-n} & n \leq 0 \end{cases} \qquad \frac{1}{1 - c\zeta^{-1}} + \frac{c^*\zeta}{1 - c^*\zeta} \quad \text{etc.}$$

Example. $c =$ mult by a on one diml space X.

$$d\mu = \frac{1 - |a|^2}{|1 - a\zeta^{-1}|^2} \frac{d\theta}{2\pi}$$

Take $S(z)$ Blashke product of degree n.

$$Y = H^+ \ominus \zeta S(\zeta) H^+$$

$$X = H^+ \ominus S(\zeta) H^+$$

$$\begin{array}{ccc} H^+ & \xleftarrow{aX} & S(\zeta)H^+ \\ \mathbb{C}1 = V^- \uparrow & & \downarrow V^+ = \mathbb{C}S \\ \zeta H^+ & \xleftrightarrow[bX]{} & \zeta S(\zeta) H^+ \end{array}$$

You should have an isometric embedding

$$Y \longrightarrow H^+$$

$$y \longmapsto \underbrace{(1 - bb^*)(1 - z b^*)^{-1} y}_{\tilde{y}(z)}$$

$$(za - b)x = -y + \tilde{y}(z) 1$$

$$(z - \zeta) x(\zeta) = -y(\zeta) + \tilde{y}(z)$$

Let $\zeta = z$ get $\tilde{y}(z) = y(z)$

389 At this point you would like to go beyond the partial unitary to ~~a st~~

Consider $Y = aX \oplus V_+ = V_- \oplus bX$ of type $\mathcal{O}(n)$ as usual, ~~defined by~~ $\xi_\pm$ unit vectors in $V_\pm$.

$$(az-b)x = = -\xi_+ + S(z)\xi_-$$

$$W = \begin{pmatrix} a \\ b \end{pmatrix} X \subset W^0 = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{matrix} V^+ \\ \oplus \\ V^- \end{matrix} \subset \begin{matrix} Y \\ \oplus \\ Y \end{matrix}$$

$$\begin{pmatrix} y \\ zy \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} x + \begin{pmatrix} v^+ \\ v^- \end{pmatrix}$$

$$z(ax + v^+) = bx + v^-$$

$$(za-b)x = -zv^+ + v^-$$

$$\Rightarrow \quad S(z) z v^+ = v^-$$

so if $v^+ = \xi_+$ then

$$\xi_-\left(\xi_-^* S(z) z \xi_+\right) = v^-$$

If $v^+ = \xi^+$, then

$$v^- = z(\xi_-, S(z)\xi_+)\xi_-$$

suppose give $V = \begin{pmatrix} 1 \\ c \end{pmatrix} Y \qquad W \subset V \subset W^0$

$$V = \begin{pmatrix} a \\ b \end{pmatrix} X + \mathbb{C}\begin{pmatrix} \xi_+ \\ h\xi_- \end{pmatrix}$$

Then

$$\begin{pmatrix} y \\ zy \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} x + \begin{pmatrix} \xi_+ \\ h\xi_- \end{pmatrix}$$

$$z(ax + \xi_+) = bx + h\xi_-$$

$$(za-b)x = -z\xi_+ + h\xi_- \qquad = -z\xi_+ + zS(z)\xi_-$$

$$\therefore \quad \boxed{h = zS(z)}$$

~~effective~~ characteristic equation for c

$$W^0 \cap \begin{pmatrix} 1 \\ z \end{pmatrix} Y \ni \begin{pmatrix} ax + v^+ \\ bx + v^- \end{pmatrix} = \begin{pmatrix} y \\ zy \end{pmatrix}$$

$$(za-b)x = -zv^+ + v^- = -z\xi^+ + zS(z)\xi^-$$