anyway ~~$S(z) = p(z)$~~ what's important is the degree. You need to clean this up in your mind. $S(z)$ rational function of z

$$S(z) = c z^k \frac{\prod_{i=1}^{m} (z-\alpha_i)}{\prod_{j=1}^{n} (z-\beta_j)}$$

unique rep.

$\{\alpha_i\} \cap \{\beta_j\} = 0.$

$\alpha_i, \beta_j \in \mathbb{C}^*$

Assume $|S(z)| = 1$ for $|z| = 1$. Then ref. principle says $S(z^*) = S(z)^*$

$$\frac{1}{\overline{S\left(\frac{1}{\bar{z}}\right)}} = S(z).$$

$$\left(\frac{1}{(\bar{z}^{-1})^k}\right)^{-1} = z^k$$

$$\frac{1}{\overline{\bar{z}^{-1} - \alpha_i}} = \frac{1}{z^{-1} - \bar{\alpha}_i}$$

$$S(z^*)^* = \frac{1}{\bar{c}} z^{+k} \frac{\prod_{j=1}^{m} (z^{-1} - \bar{\beta}_j)}{\prod_{i=1}^{n} (z^{-1} - \bar{\alpha}_i)}$$

$$= \frac{1}{\bar{c}} z^{k-m+n} \frac{\prod_{j=1}^{m} (-\bar{\beta}_j)}{\prod_{i=1}^{n} (-\bar{\alpha}_i)} \frac{\prod (z - \beta_j^*)}{\prod (z - \alpha_i^*)}$$

So one conclude that $S(z^*)^* = S(z)$

$\Longrightarrow$ $m = n$ $\{\beta_j\} = \{\alpha_i^*\}$ up to mult.

$$c\bar{c} = \frac{\prod \bar{\beta}_j}{\prod \bar{\alpha}_i} = \frac{\prod \bar{\beta}_j}{\prod \beta_j^*} = \prod \bar{\beta}_j \beta_j$$

So what? Time to understand de Branges. via the moment problem. Begin with a J matrix.

$$\begin{pmatrix} b_0 & a_1 & & \\ a_1 & b_1 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

$a_i > 0$.

s.c. a sequence of polynomials $p_0, p_1, \cdots$

$$\begin{vmatrix} b_0-\lambda & a_1 & & & \\ a_1 & b_1-\lambda & a_2 & & \\ & a_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_n \\ & & & a_n & b_n-\lambda \end{vmatrix}$$

Put $d_{n+1} = \det(J_{n+1} - \lambda I_{n+1})$ ~~$= |J_n - \lambda I|$~~

$$d_{n+1} = (b_n - \lambda) d_n - a_n^2 d_{n-1}$$

$$\underbrace{(-1)^{n+1} d_{n+1}}_{} = (\lambda - b_n)(-1)^n d_n - a_n^2 (-1)^{n-1} d_{n-1}$$

$$\tilde{p}_{n+1} = (\lambda - b_n)\tilde{p}_n - a_n^2 \tilde{p}_{n-1}$$

$$\tilde{p}_n = \lambda^n + \text{lower terms}$$

$$\frac{\tilde{p}_{n+1}}{\tilde{p}_n} = \lambda - b_n - \frac{a_n^2}{\frac{\tilde{p}_n}{\tilde{p}_{n-1}}}$$

$$\tilde{p}_{n+1} + b_n \tilde{p}_n + a_n^2 \tilde{p}_{n-1} = \lambda \tilde{p}_n$$

$$\frac{c_{n+1}}{c_n} p_{n+1} + b_n p_n + a_n^2 \frac{c_{n-1}}{c_n} p_{n-1} = \lambda p_n$$

so set $\frac{c_{n+1}}{c_n} = a_{n+1}$ $\quad \frac{c_n}{c_{n-1}} = a_n$ $\quad \therefore c_n = a_1 \cdots a_n$

So from $\tilde{p}_{n+1} + (b_n - \lambda)\tilde{p}_n + a_n^2 \tilde{p}_{n-1} = \cancel{\lambda \tilde{p}_n}\ 0$

you get $$a_{n+1} \frac{\tilde{p}_{n+1}}{a_1 \cdots a_{n+1}} + \frac{b_n - \lambda}{a_1 \cdots a_n} \tilde{p}_n + a_n \frac{\tilde{p}_{n-1}}{a_1 \cdots a_{n-1}} = 0$$

So $$p_n = \frac{\tilde{p}_n}{a_1 \cdots a_n} = \frac{1}{a_1 \cdots a_n} \det(\lambda I_n - J_n)$$

So far J_{n+1} ~~matrix~~

sequence $p_0 = 1, p_1, \cdots, p_n$

now you get an inner product on F_n such that $p_0, \cdots, p_n$ is orth basis. We have a Hilbert space of polys. Now de Branges somehow writes this inner product as a measure on the line in a canonical way. ~~So if you consider~~ If you started with $d\mu$ on the line w. finite moments, you get the above. What can you say about the possible $d\mu$'s? What can you do

$$\tilde{p}_0 = 1 \qquad \tilde{p}_1 = \lambda - b_1$$

where $\int (\lambda - b_1)\, d\mu = \mu_1 - b_1 = 0$

$$\int \tilde{p}_1^2\, d\mu = \int (\lambda^2 - 2 b_1 \lambda + b_1^2)\, d\mu$$

$$= \mu_2 - b_1^2 = \mu_2 - \mu_1^2 > 0.$$

$a_1^2 = \mu_2 - \mu_1^2$

What you need is essentially the spectral theorem for $A = J_n$

$$\left(p_1, \frac{1}{\lambda - A} p_1\right) = \sum_{n \geq 0} \lambda^{-n-1} \underbrace{(p_1, A^n p_1)}_{\mu_n = \int x^n d\mu}$$

I'm saying that the moment generating function ~~is conf~~ ?

What do you want? I want to reconstruct de Branges theory at least for polys. This means realizing a de Branges space as polynomials ~~in or~~ inside $L^2(\mathbb{R}, d\mu)$ for some measure.

Here are some steps. Given F_n = poly deg $< n$ with scalar product, you can construct orthog monic polyn. $p_1 = 1$, $\tilde{p}_2 = x - b_1$, ..., $\tilde{p}_n$ and the corresp. J-matrix

$$\begin{pmatrix} b_1 & a_1 & & \\ a_1 & \ddots & & a_{n-1} \\ & & a_{n-1} & b_n \end{pmatrix}$$

except for b_n. Choose b_n. Then we ~~can~~ get the s.a. $A = J_n$. We also get $\left(p_1, \frac{1}{\lambda - A} p_1\right)$. Point is that $\frac{1}{\lambda - A} p_1$ can be found by solving the recursion relations backwards.

$$x \overset{u}{p_k} = a_k \overset{u}{p_{k+1}} + b_k \overset{u}{p_k} + a_{k-1} \overset{u}{p_{k-1}}$$

So you get $u(\lambda)$ $\ni$ $(\lambda - A) u(\lambda) = p_1$

$$a_{k-1} u_{k-1} = (x - b_k) u_k - a_k u_{k+1}$$

$$\frac{a_{k-1} u_{k-1}}{u_k} = x - b_k - a_k^2 \Big/ a_k \frac{u_k}{u_{k+1}}$$

~~Then~~ So you have this rational function

$$\rho_1 = \frac{a_1 u_1}{u_2} = x - b_2 - \frac{a_2^2}{\rho_2}$$

Start with $u_n = 1 \quad u_{n+1} = 0$

~~Then~~

$$\begin{pmatrix} \lambda - b_1 & -a_1 & 0 & 0 \\ -a_1 & \lambda - b_2 & -a_2 & \\ & -a_2 & & \\ & & & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} =$$

$$\frac{-a_1 u_1}{u_2} = \lambda - b_2 - \frac{a_2^2}{\lambda - b_3 -} \frac{a_3^2}{\lambda - b_4 -} \cdots = R(\lambda)$$

$$(\lambda - b_1) u_1 - a_1 u_2 = 1 \qquad u_2 = \frac{a_1 u_1}{R}$$

$$(\lambda - b_1) u_1 - a_1^2 \frac{u_1}{R} = 1$$

$$u_1(\lambda) = \frac{1}{\lambda - b_1 -} \frac{a_1^2}{\lambda - b_2 -} \cdots$$

So you now know the relation between the matrix J and the generating function for the moments. What you need is a measure. The point you need is how to get

$$(f(\lambda), g(\lambda)) = \sum_i f(\alpha_i)\, g(\alpha_i)\, \rho_i$$

where $(p_1, \frac{1}{\lambda - A} p_1) = \sum_i \frac{\rho_i}{\lambda - \alpha_i}$

this should be same contour integral trick

$$\sum_i \overline{f(\alpha_i)}\, g(\alpha_i)\, \rho_i = \frac{1}{2\pi i} \oint \bar{f}(\lambda) g(\lambda) \sum \frac{\rho_i}{\lambda - \alpha_i}\, d\lambda$$

Review. Given a Hilbert space of polys.
Better: Given positive definite matrix

Basic idea: ~~Recall that~~ Instead of the s.a. J_n which arises from any real choice for b_n, you want a complex boundary condition at the right end i.e. the recursion relation

$$a_{k-1} u_{k-1} = (\lambda - b_k) u_k - a_k u_{k+1}$$

R_2
$\parallel$

or

$$\frac{a_{k-1} u_{k-1}}{u_k} = \lambda - b_k - a_k^2 \Big/ \frac{a_k u_k}{u_{k+1}}$$

$$\frac{a_1 u_1}{u_2} = \lambda - b_2 - \frac{a_2^2}{\cdots}$$

leading to

$$(\lambda - b_1) u_1 - a_1 u_2 = 1$$

define $(\lambda - A)^{-1} p_1$

$$R_2 = \frac{a_1 u_1}{u_2} \qquad u_2 = \frac{a_1 u_1}{R_2}$$

$$(\lambda - b_1) u_1 - \frac{a_1^2 u_1}{R_2} = 1$$

$$u_1 = \cfrac{1}{\lambda - b_1 - \cfrac{a_1^2}{\lambda - b_2 - }}$$

$$(i\varepsilon^*+A^*)(\lambda\varepsilon-A)(i\varepsilon^*+A^*) = (i\varepsilon^*+A^*)(\lambda\varepsilon-A)(i\varepsilon^*+A^*)$$

probably works, ~~Aff~~ but you probably need $1-pq$ inv. $\Longleftrightarrow$ $1-qp$ invertible

It doesn't work because $i\varepsilon^*+A^*$ has a ~~no~~ nontrivial kernel. Maybe I can find what is needed. With luck

$$v^- = y + (\lambda\varepsilon - A)\left[\qquad\qquad\right]^{-1}(i\varepsilon^*+A^*)y$$

$$= y + (\lambda\varepsilon-A)(i\varepsilon^*+A^*)\left[\quad Q \quad\right]^{-1} y$$

$$= \left[Q + (\lambda\varepsilon-A)(i\varepsilon^*+A^*)\right] Q^{-1} y$$

What is $1-bb^*$

$$bb^* = (-A+i\varepsilon)(1+A^*A)^{-1}(-A^*-i\varepsilon^*)$$

Question: Given $y \in Y$ what is the corresponding function of z in $L^2(S^1)$? We know that

$\widetilde{v^-} = 1$ $\qquad$ $\widetilde{v^+} = S(z)$

So what you want is a ~~clean~~ function of z assoc. to y which is a holom. section of the line bundle with fibre $Y/(az-b)X$

section of the line bundle $Y/(az-b)X$

~~sections of~~

Start again. $Y = aX \oplus V^+ = V^- \oplus bX$ $\qquad$ $\|ax\|^2 = \|bx\|^2$, $az - b$ inj.

Form $Y \subset H$, u with $X = u^{-1}Y \cap Y \underset{b=u}{\overset{a=1}{\rightrightarrows}} Y$

$H = \oplus u^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{V^- \oplus bX} \oplus uV^+$ $\qquad$ $L^2(S^1, V^-) \xrightarrow{\iota} H \longleftarrow L^2(S^1, V^+)$

so ι^* associates to each elt of H a function

$$\tilde\xi(z) = \sum_n z^n \langle v_0^-, u^{-n}\xi\rangle \qquad \xi = v_0^- \qquad \tilde\xi = 1$$

$\xi = v_0^+$ $\qquad$ $\tilde v_0^+(z) = \sum_{n\geq 0} z^n (v_0^-, \underbrace{u^{-n}}_{ab^*} v_0^+)$

$$= \langle v_0^-, (1-zab^*)^{-1} v_0^+ \rangle$$

So for any element y of Y

$$\tilde y(z) = \sum_{n\geq 0} z^n (v_0^-, \underbrace{u^{-n}}_{(ab^*)^n} y) = (v_0^-, (1-zab^*)^{-1} y)$$

Why isometric. This should be connected with the eigenvector equation, and why S is unitary.

So now you have your isom. emb. $Y \hookrightarrow L^2(S^1)$ ~~and you can embed take Notice that~~. If you want of proof you have to use that $g^n y \to 0$

This has to be cleaned up and written down.

We know that $\quad v_0^- \rightsquigarrow \frac{1}{\lambda+i}$

$v_0^+ \rightsquigarrow S(\lambda)\frac{1}{\lambda+i}$

$$Y = H^+ \cap zSH^- = H^+ \cap \left(\frac{-\lambda+i}{\lambda+i}\right) SH^-$$

$$\to (\lambda+i)q H^+ \cap (-\lambda+i)pH^- \quad = \text{polys in } \lambda \text{ degree} \leq n.$$

$$X \overset{1}{\underset{A}{\rightrightarrows}} Y \qquad z = \frac{1+i\lambda}{1-i\lambda} \qquad \lambda = i\frac{1-z}{1+z} \qquad z = \frac{-\lambda+i}{\lambda+i}$$

$$X \overset{a = A+i}{\underset{b = -A+i}{\rightrightarrows}} Y \qquad az - b = (A+i)z - (-A+i) = A(z+1) + i(z-1)$$

$$= (z+1)\left(A - i\frac{1-z}{1+z}\right) = (z+1)(A-\lambda)$$

Get holom line bundle $L_\lambda = Y/(\lambda - A)X$ over $\mathbb{P}^1$, each element of Y gives a sections. We know already that the section v_0^- is nonzero over $|z| \leq 1$ which is the λ UHP, and v_0^+ is nonzero over LHP. This means

$$\begin{tikzcd} X \arrow[r, "\lambda - A"] \arrow[dr] & Y \arrow[d, "(i-A)^*"] \\ & X \end{tikzcd}$$

$(\bar\lambda 1 - A)^*(\lambda 1 - A)$ invertible in X for λ in UHP

$\lambda 1 - A =$

$\lambda 1 - A + (\lambda - i)\varepsilon$

$$(\lambda\varepsilon - A)^*(\lambda\varepsilon - A) = 1 + (\lambda - i)(i\varepsilon - A)^*\varepsilon$$

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[d, "(i\varepsilon - A)^*"] \\ & X \end{tikzcd}$$

$$(i\varepsilon - A)^*(\lambda\varepsilon - A)$$

$$= (i\varepsilon - A)^*(i\varepsilon - A + (\lambda - i)\varepsilon)$$

$$= 1 + (\lambda - i)\underbrace{(i\varepsilon - A)^*\varepsilon}_{-i\varepsilon^*\varepsilon - A^*\varepsilon}$$

$$z + 1 = \frac{2}{1 - i\lambda}$$

$$= 1 + (\lambda - i)(-i) + (i - \lambda)A^*\varepsilon$$

$=$

~~OK so you wi~~ What are we trying to solve? To find the function $f(\lambda)$ such that

$$\underbrace{+f(\lambda)v_0^-}_{v^-} \cancel{+} y \in \cancel{\ } (\lambda - A)X.$$

$$(\lambda\varepsilon - A)x = -y + v^-$$

$$\underbrace{(i\varepsilon - A)^*(\lambda\varepsilon - A)}x = -(i\varepsilon - A)^* y$$

$$(1 + (\lambda - i)(-i - A^*\varepsilon))x$$
$$-\lambda i - (\lambda - i)A^*\varepsilon$$

$$(\lambda i + (\lambda - i)A^*\varepsilon)x = (i\varepsilon - A)^* y$$

$$x = (\lambda i + (\lambda - i)A^*\varepsilon)^{-1}(i\varepsilon - A)^* y$$

$$v^- = y + (\lambda\varepsilon - A)\big(\underbrace{\lambda i + (\lambda - i)A^*\varepsilon}_{-(i\varepsilon - A)^*(\lambda\varepsilon - A)}\big)^{-1}(i\varepsilon - A)^* y$$

~~What We want to write~~

We want to use $1 - pq$ invertible $\Leftrightarrow$ $1 - qp$ invertible

$$\cancel{(i\varepsilon - A)^*(\lambda\varepsilon - A) = (-i\varepsilon^* - A^*)(\lambda\varepsilon - A)}$$
$$\cancel{= -(i\varepsilon^* + A^*)(}$$

$$(i\varepsilon - A)^*(\lambda\varepsilon - A) = (i\varepsilon - A)^*(i\varepsilon - A + (\lambda - i)\varepsilon)$$
$$= 1 + (\lambda - i)(i\varepsilon - A)^*\varepsilon$$
$$= 1 + (\lambda - i)(-i\varepsilon^* - A)^*\varepsilon \qquad \cancel{\varepsilon^*\varepsilon \neq 1.}$$
$$= 1 - (\lambda - i)(i + A^*\varepsilon)$$
$$= 1 - (\lambda i + 1) - (\lambda - i)A^*\varepsilon$$
$$= -\lambda i - (\lambda - i)A^*\varepsilon$$

~~What do we know about~~

$$(i\varepsilon - A)^*(\lambda\varepsilon - A) = (i\varepsilon - A)^*\big(i\varepsilon - A + (\lambda - i)\varepsilon\big)$$

$$= 1 + (\lambda - i)(i\varepsilon - A)^*\varepsilon$$

$$\|x\|^2 = \|Ax\|^2 + \|\varepsilon x\|^2$$

$$= 1 + (\lambda - i)\underbrace{\big(-i(1 - A^*A) - A^*\varepsilon\big)}_{-i + iA^*A - A^*\varepsilon}$$

$$1 = A^*A + \varepsilon^*\varepsilon$$

$$-i + iA^*A - A^*\varepsilon = -i + iA^*(A + i\varepsilon)$$

$$= 1 + (\lambda - i)(-i) + (\lambda - i)iA^*(A + i\varepsilon)$$

$$\cancel{= 1 - \lambda i - 1 + (i\lambda + 1)A^*(A + i\varepsilon)}$$

meaning of A^* is also changed.

$$= -\lambda i + (\lambda - i)iA^*(A + i\varepsilon)$$

$$= 1 - \frac{(\lambda - i)}{\lambda}A^*(A + i\varepsilon)$$

$$X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$$

$\|\varepsilon x\| = \|x\|$ $\quad\therefore\ \varepsilon^*\varepsilon = 1_X$

$$\|(A + i\varepsilon)x\|^2 = \|(A - i\varepsilon)x\|^2$$

$$\|$$

$$\|Ax\|^2 + i(Ax, \varepsilon x) - i(\varepsilon x, Ax) + \|x\|^2$$

$\therefore\ (Ax_1, \varepsilon x_2) = (\varepsilon x_1, Ax_2)$ $\qquad\therefore\ A^*\varepsilon = \varepsilon^*A$

$$z = \frac{1 + i\lambda}{1 - i\lambda} = \frac{-\lambda + i}{\lambda + i}$$

So go Y_0^-

$$V^- = (\cdot X)^\perp = \big((A - i\varepsilon)X\big)^\perp = \operatorname{Ker}(A^* + i\varepsilon^*)$$

$$\begin{tikzcd} & V^- \arrow[d] & \\ X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[r] \arrow[d, "A^* + i\varepsilon^*"] & L_\lambda \\ & X & \end{tikzcd}$$

Start again $\quad X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y \qquad z = \frac{1+i\lambda}{1-i\lambda}, \; \lambda = i\,\frac{1-z}{1+z}$

$$X \;\underset{b \,=\, \cancel{1+iA} \;\; i\varepsilon - A}{\overset{a \,=\, \cancel{1-iA} \;\; i\varepsilon + A}{\rightrightarrows}}$$

$$V^- = \big((i\varepsilon - A)X\big)^{\perp} = \mathrm{Ker}(-i\varepsilon^* - A^*)$$

$$V^+ = \mathrm{Ker}(i\varepsilon^* + A^*)$$

$$\begin{tikzcd}
 & V^- \arrow[d] & \\
X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[r] \arrow[d, "-(i\varepsilon^* + A^*)"] & L_\lambda \\
 & X &
\end{tikzcd}$$

$$\begin{aligned}
&(-i\varepsilon^* - A^*)(i\varepsilon - A) \\
&= \varepsilon^*\varepsilon - i\cancel{A^*\varepsilon} + i\cancel{\varepsilon^* A} + A^*A \\
&= 1 + A^*A .
\end{aligned}$$

$$(\lambda\varepsilon - A)(x) = -y + v^-$$

$$(-i\varepsilon^* - A^*)(\lambda\varepsilon - A)x = (i\varepsilon^* + A^*)y$$

$$\underbrace{\left(-i\lambda + i\varepsilon^* A - \lambda A^*\varepsilon + A^*A\right)}\,x = (i\varepsilon^* + A^*)y$$

$$1 + A^*A - 1 - i\lambda + (i-\lambda)\varepsilon^* A = 1 + A^*A - (\lambda - i)(i + \varepsilon^* A)$$

What is the analogue of ab^* in the unitary setting? $\quad a = i\varepsilon + A$

$$b = i\varepsilon - A : X \longrightarrow Y$$

we want ~~[illegible]~~ ab^*, where b^* is calculated using the scalar product $\|\varepsilon x\|^2 + \|Ax\|^2 = (x, (1+A^*A)x)$ on X.

Then $\quad (x, b^*y)_{X'} = (bx, y) = ((i\varepsilon - A)x, y) = (x, (-i\varepsilon^* - A^*)y)_X$

$$(x, b^*y)_{X'} = ((1 + A^*A)x, b^*y)_X$$

$$\begin{aligned}
(x, b^*y)_X &= \left((1+A^*A)^{-1}x, (-i\varepsilon^* - A^*)y\right) \\
&= \left(x, (1+A^*A)^{-1}(-i\varepsilon^* - A^*)y\right).
\end{aligned}$$

So $ab^* = (i\varepsilon + A)(1+A^*A)^{-1}(-i\varepsilon^* - A^*)$

$$(-i\varepsilon^* - A^*)(i\varepsilon + A) = 1 + A^*A$$

$$x = \Big[1+A^*A - (\lambda - i)\underbrace{(i + \varepsilon^* A)}_{\varepsilon^*(i\varepsilon + A)}\Big]^{-1}(i\varepsilon^* + A^*)\,y$$

also $\varepsilon^* A = A^* \varepsilon$

$$v^- = y + (\lambda\varepsilon - A)x$$

What was so easy before:

$$(1 - z b^* a)x = +b^* v^+$$

$$x = \underline{(1 - z b^* a)^{-1}} b^* v^+ = b^*(1 - z ab^*)^{-1} v^+$$

$$(1 - z b^* a) b^* = b^*(1 - z a b^*)$$

maybe do the straight forward thing namely

$$v^- = (1 - bb^*)(1 - zab^*)^{-1} y$$

$$1 - zab^* = 1 - z(i\varepsilon + A)(1 + A^*A)^{-1}(-i\varepsilon^* - A^*)$$

$$\begin{tikzcd} X \arrow[r, shift left, "\varepsilon"] \arrow[r, shift right, "A"'] & Y \end{tikzcd} \qquad z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda + i}{\lambda + i} \qquad \begin{array}{l} a = A + i\varepsilon \\ b = -A + i\varepsilon \end{array} : X \to Y$$

$$V^- = (bX)^\perp = ((-A+i\varepsilon)X)^\perp = \mathrm{Ker}(-A^* - i\varepsilon^*)$$

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[r] \arrow[d, "-i\varepsilon^* - A^*"] & L_\lambda \\ & X & \end{tikzcd}$$

$$(\lambda\varepsilon - A)(x) = -y + v^-$$

$$(i\varepsilon^* + A^*)(\lambda\varepsilon - A)x = (i\varepsilon^* + A^*)y$$

$$v^- = y + (\lambda\varepsilon - A)\underbrace{\big[(i\varepsilon^*+A^*)(\lambda\varepsilon - A)\big]^{-1}}_{\overset{?}{=}\ (i\varepsilon^*+A^*)\left[(\lambda\varepsilon-A)(i\varepsilon^*+A^*)\right]^{-1}}(i\varepsilon^*+A^*)\,y$$

$$y^- = y + (az - b)\left[1 - zb^*a\right]^{-1} b^* y$$

y vanishes to 2nd ord at $z=\infty \iff \exists$

$$\begin{tikzcd}
 & x \arrow[dl, "a"'] \arrow[dr, "b"] & & x' \arrow[dl, "a"'] \\
y & & y' & \\
W \arrow[r, "ba^{-1}"] \arrow[d, hook] & aX \arrow[r, "ba^{-1}"] \arrow[d, hook] & Y & \\
aX \arrow[r, "ba^{-1}"] \arrow[d, hook] & Y & & \\
Y & & &
\end{tikzcd}$$

$$W = \{(x_1, x_2) \mid bx_1 = ax_2\}$$
$$\simeq \{x_1 \mid bx_1 \in aX\} = b^{-1}aX$$
$$\simeq \{x_2 \mid ax_2 \in bX\} = a^{-1}bX$$

$$W = (ba^{-1})^{-1}(aX) \overset{?}{=} ab^{-1}aX$$

It needs some work, but I think it's clear namely, ~~..~~ you have the partially defined operator ba^{-1} and you consider its powers $(ba^{-1})^i$. Ultimately you ~~..~~ end up with a ~~...~~ basis ~~...~~. From $z=0$, $z=\infty$ get complementary flags, so you really find a fairly canonical isom. of Y with polys of degree $\le n$ in z. 1 is the element of Y such that $(ba^{-1})^n 1$ is defined. ?

How does this relate to the scalar product? Now you have a scalar product on Y such that $\|ax\|^2 = \|bx\|^2 \ \forall x$. ba^{-1} is then p. unitary. What can we say about $V^- = (bX)^\perp$ and $V^+ = (aX)^\perp$

aX consists of the polys $\mathbb{C} + \mathbb{C}z + \dots + \mathbb{C}z^{n-1}$
bX ______ $\mathbb{C}z + \dots + \mathbb{C}z^n$

so that p_n is your v_0^+ $\quad$ q_n is your v_0^-

and $S = \frac{p_n}{q_n}$ carries q_n into p_n

Analyzing $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ p.u. Have the canonical basis $\xi_0, u\xi_0, \dots, u^n\xi_0$ up to a nonzero scalar where $\xi_0 \neq 0$ in the line $\ni$ all $(ba^{-1})^k \xi_0$ $0 \leq k \leq n$ defined. So this allows us to identify Y with $\mathbb{C} + \mathbb{C}z + \dots + \mathbb{C}z^n$ if we want. But now look at the orthogonal complements.

From the basis $\xi_0, \dots, u^n\xi_0$ you can extract an orthonormal basis $p_0, \dots, p_n$ by Gram-Schmidt and a "complementary" type of orthonormal basis. $q_n, uq_{n-1}, \dots, u^n q_0$. The idea is to apply Gram-Schmidt to $u^n\xi_0, u^{n-1}\xi_0, \dots, \xi_0$. If we are thinking of $Y = \mathbb{C} + \mathbb{C}z + \dots + \mathbb{C}z^n$, then we want

$$z^i q_{n-i} \in \mathbb{C}z^i + \dots + \mathbb{C}z^n$$

$$z^i q_{n-i} \perp \mathbb{C}z^{i+1} + \dots + \mathbb{C}z^n$$

$$z^i q_{n-i} \text{ to have lower degree term } (>0) z^i$$

This translates to

$$q_{n-i} \in \mathbb{C} + \dots + \mathbb{C}z^{n-i}$$

$$\perp \mathbb{C}z + \dots + \mathbb{C}z^{n-i}$$

$$q_{n-i} \text{ lower term } > 0$$

So it's clear that we get ~~something simple~~ our usual q_n's. So we have two orth bases.

$$V^+ \quad \overbrace{p_0 \;\; \dots \;\; p_n}^{aX}$$

$$V^- \quad \underbrace{q_n \;\; \dots \;\; u^n q_0}_{bX}$$

$$\begin{pmatrix} p_{j+1} \\ q_{j+1} \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_j \\ \bar{h}_j & 1 \end{pmatrix} \begin{pmatrix} u p_j \\ q_j \end{pmatrix}$$

Maybe look at

$$p_j \qquad u^{-j} q_j \qquad ?$$

so what principle should I use?? How to proceed? What is the fundamental problem.

So far you have canonical element ξ in the domain of $(ba^{-1})^j$ for $0 \le j < n$, which allows ident of Y with $\mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n$, $X = \mathbb{C} + \cdots + \mathbb{C}z^{n-1}$, $a = \mathrm{inc}.$, $b = z.$ Then you get two orth sequences

$$p_0 \quad p_1 \quad \cdots\cdots \quad p_n$$

$$q_n \quad zq_{n-1} \quad \cdots\cdots \quad z^n q_0$$

These are ~~[illegible]~~ all polys in ~~[illegible]~~ z given by a recursion fmla. Now comes the crunch.

Maybe you should have done the aboved within the space Y. You now want to map Y isometrically into $L^2(S^1)$. How? You choose a basis element for V^- to become 1.

$$\xi \ , \quad \frac{u\xi + h_0\xi}{k_0}$$

$$\frac{\xi + \bar{h}_0 u\xi}{k_0}, \quad u\xi$$

~~You want to map M~~

The idea now is that there is this unitary extension H, u you can ~~/~~ construct from the partial unitary which can be compared ~~[illegible]~~ mapped into $L^2(S^1)$. Maybe the important point is the association of ~~[illegible]~~ functions on S^1 to elements of Y. ~~You fix~~

Fix $v_0^- = q_n$ perp to $u\xi, \ldots, u^n\xi = bX$

$X \overset{a}{\underset{b}{\rightrightarrows}} Y$ of $\mathcal{O}(u)$ type

Canonical basis $\xi, (ba^{-1})\xi, \quad , (ba^{-1})^n\xi$

up to a nonzero scalar. ~~[illegible]~~ also a canon. basis

$\eta \quad (ab^{-1})\eta \qquad (ab^{-1})^n\eta$ where $\eta = (ba^{-1})^n\xi$.

Apply GS to get orth. basis $p_0, \dots, p_n$

out of the first and $q_n, \dots, z^n q_0$.

$$\begin{cases} p_0, \dots, p_n \\ q_n, \dots, z^{n-1}q_1, z^n q_0 \end{cases}$$ recursion relns.

$V^+ = (aX)^\perp$ ~~[illegible]~~ spanned by p_n

$V^- = (bX)^\perp$ ——— q_n.

Now what isom. embedding of Y in $L^2(S^1)$ ~~with~~

form $H = \cdots \oplus u^{-1}V^- \oplus \underbrace{aX \oplus V^+}_{\substack{\shortparallel \\ V^- \oplus bX}} \oplus uV^+ \oplus \cdots$ with u the shift.

gets ~~[illegible]~~ $\ell^2(\mathbb{Z}, V^-) \longrightarrow H \longleftarrow \ell^2(\mathbb{Z}, V^+)$

Try $n=1$.

p_0, p_1

$q_1, \cancel{u}q_0$

$p_0 = q_0 = \xi$.

we have basis $\xi, u\xi$ for Y $\quad aX = \mathbb{C}\xi \quad bX = \mathbb{C}u\xi$

YES What next?

~~$k_1 p_1 = u p_0$~~

$p_0 = 1 \quad p_1$

$q_1 \quad z = uq_0$

$$k_0 p_1 = z p_0 + h_0 q_0 \qquad k_0 p_1 - h_0 q_0 = z p_0$$

$$k_0 q_1 = q_0 + \bar{h}_0 z p_0 \qquad k_0^2 + |h_0|^2 = 1$$

Start again — $X \underset{b}{\overset{a}{\rightrightarrows}} Y$

$H = \oplus u^{-1}V^- \oplus Y \oplus u V^+ \oplus \cdots\cdots$

$\ell^2(\mathbb{Z}, V^-) \xrightarrow{\iota} H \xleftarrow{j} \ell^2(\mathbb{Z}, V^+)$

$v^+ = bb^* v^+ + \pi^- v^+$

$u^{-1}v^+ = bb^* ab^* v^+ + \pi^- ab^* v^+ + u^{-1}\pi^- v^+$

$u^{-2}v^+ = bb^*(ab^*)^2 v^+ + \pi^-(ab^*)^2 v^+ + u^{-1}\pi^-(ab^*)v^- + u^{-2}\pi^- v^+$

$\therefore\ v^+ = \sum_{n\geq 0} u^n \pi^-(ab^*)^n v^+ = \sum_{n\geq 0} u^n(1-bb^*)(ab^*)^n v^+$

$\tilde{v}^+(z) = (1-bb^*)(1-z\,ab^*)^{-1} v^+$

So you calculate the function on S^1 corresponding to v^+. Maybe work with (v_0^-, v^+)? Go over the principles.

You start with $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ of type $\mathcal{O}(n)$ make H, u, then construct

$$\begin{array}{ccccc} \ell^2(\mathbb{Z}, V^-) & \xleftarrow{i^*} & H & \xrightarrow{j^*} & \ell^2(\mathbb{Z}, V^+) \\ \| & & & & \| \\ L^2(S^1, V^-) & & & & L^2(S^1, V^+) \end{array}$$

what are you missing? You are missing much.

Actually there should be no problem with ι^* and j^*. These should be rigorous. Thus for $\iota^*(v^+)$ you need you need only the inner products

$(u^n v_0^-, v^+) = (v_0^-, \underbrace{u^{-n}v^+}_{bb^*(ab^*)^n v^+ + \pi^-(ab^*)^n v^+ + u\pi^-(ab^*)^{n-1}v^-})$

$= (v_0^-, (ab^*)^n v^+)$

so the function is

$$\sum_{n\geq 0} z^n (u^n v_0^-, v^+) = (v_0^-, (1-zab^*)^{-1} v^+)$$

You need to show that this is unitary

What are you aiming for?? Basically you have this have this mechanism which yields the result you want. The mechanism is that $v_0^- = 1 \in L^2(S^1)$ and $v_0^+ = S(z)$ and that $Y = H^+ \cap zSH^-$. But then you manage to factor $S = \frac{p}{q}$ whence

$$Y = H^+ \cap zSH^- \xrightarrow[\cdot q]{} qH^+ \cap zpH^- = \underbrace{H^+ \cap z^{n+1}H^*}_{\text{polys degree} \leq n}$$

Get $Y = \dfrac{\{\text{polys degree} \leq n\}}{q} \subset L^2(S^1)$

This I understand, but I'm missing the construction of q as q_n and p as p_n. Then things ~~are~~ should be clearer, namely.

~~Important~~ Instead of $y \mapsto \frac{y}{q_n}$ and trying to show this is isometric from Y into $L^2(S^1)$ You want the other direction $L^2(S^1) \longrightarrow H$. Except you want ?? What.

Begin with $L^2(S^1, d\mu)$ and construct p_n, q_n. Consider q_n yes.

$L^2(S^1)$ $\quad L^2(S^1, d\mu)$ $\quad q_n = \frac{1}{k}(q_{n-1} + zp_{n-1})$

$1 \quad q_n$

z^{-1}

$(z^{-1}q_n, q_n)$

You know that $q_n, zq_{n-1}, \dots, z^n q_0$ is an orth basis of Y. You want zq_n.

Let's start with $L^2(S^1, d\mu)$ $d\mu$ inf. support so that this is inf. dim. Then construct p_j, q_j $\forall j$.

$$(q_n, zq_{n-1}) = 0$$

Suppose that $q_n \longrightarrow q_\infty$ i.e. $\sum |h_n^2| < \infty$.

$q_\infty(z)$ analytic for $|z| < 1$. and invertible

$$\int |q_\infty|^2 \, d\mu = 1.$$

Important was $(q_\infty, z^n q_\infty) = 0$ $n > 0$

$$|q_\infty|^2 \, d\mu = \frac{d\theta}{2\pi} \qquad d\mu = \frac{1}{|q_\infty|^2} \frac{d\theta}{2\pi}$$

~~In my opinion this is a very nice paper that you have set. It seems to be about as difficult as Paper 5, while Paper 4 appears harder, but this impression might be due to my poor knowledge of mechanics.~~

~~Concerning alternate questions, I like the circled questions except for number 11, which I feel belongs to the "counting + probability" type of mathematics already well represented in the Paper. If the Algorithm type question 10 is of sufficient quality (this I am unable to judge), then it would seem a better choice.~~

~~Looking forward to more comments about Paper 1~~

~~Sincerely,~~

Let us go on again. $L^2(S^1, d\mu)$ $\xi = 1$

$Y = \mathbb{C}1 + \mathbb{C}z + \cdots + \mathbb{C}z^n$ $\qquad F_j = \mathbb{C}1 + \cdots + \mathbb{C}z^j$

$p_0 = q_0 = 1$

$$\begin{tikzcd}[column sep=small]
 & zF_{j-1} \arrow[dl, dash, "zp_{j-1}"'] \arrow[dr, dash, "q_j"] & \\
zF_{j-2} \arrow[dr, dash, "q_{j-1}"'] & & F_j \arrow[dl, dash, "p_j"] \\
 & F_{j-1} &
\end{tikzcd}$$

$$p_j = \frac{1}{k}\left(zp_{j-1} + h\, q_{j-1}\right)$$

$$q_j = \frac{1}{k}\left(\bar{h}\, p_{j-1} + q_{j-1}\right)$$

$$k = \sqrt{1-|h|^2} \qquad \begin{matrix} \frac{1}{k} & \frac{h}{k} \\ \frac{\bar h}{k} & \frac{1}{k} \end{matrix}$$

$$p_1 = a\, zp_0 + b\, q_1 \qquad a^2 + |b|^2 = 1.$$

$$q_0 = -\bar{b}\, zp_0 + a\, q_1$$

$$p_1 = zp_0\left(a + \frac{b}{a}\bar{b}\right) + \frac{b}{a} q_0 = \frac{1}{a} zp_0 + \frac{b}{a} q_0$$

$$q_1 = \frac{\bar b}{a} zp_0 + \frac{1}{a} q_0$$

$X = F_{n-1}$ $\qquad V^+ = \mathbb{C}p_n$ $\qquad V^- = \mathbb{C}q_n$

eigenvector equation says that

$$(a\lambda - b)x = -v^+ + v^-$$

$$(\lambda - z)x = -S_n(\lambda)\, q_n + p_n$$

so putting $\lambda = z$ gives $S_n(z) = \dfrac{p_n}{q_n}$

$$x_n(\lambda, z) = \frac{-p_n(\lambda)\, q_n(z) + p_n(z)\, q_n(\lambda)}{(\lambda - z)\, q_n(\lambda)}$$

You want to embed Y isometrically into $L^2(S^1, d\theta)$. You want $v_0^- = q_n$ to go into 1 and you want the embedding to commute with mult. by z. Thus use $y \mapsto \dfrac{y}{q_n}$. You have to understand why this is isometric up through degree n.

What do we do now? What's happening is we have $H \longrightarrow L^2(S^1)$

$$\sigma_0^- \longmapsto 1$$
$$\sigma_0^+ \longmapsto S(z)$$

This is the old viewpoint – to proceed further you write $S(z) = \frac{p}{q}$. This morning you made some progress by introducing the orthogonal polys p_n, q_n. What you want is this I think?

given $p_0, p_1, \cdots, p_n$ polys, $p_0 = 1$
$q_0, q_1, \cdots, q_n$, $q_0 = 1$

~~so~~ defined via recursion relations, to show that these sequences are orthonormal for $\|f\|^2 = \int |f|^2 \frac{d\theta}{|q_n|^2 2\pi}$. Among other things you want to know that $\int \frac{d\theta}{|q_n|^2 2\pi}$ is ind of n.

We know that $|p_n| = |q_n|$ and in fact $z^n \overline{p_n(z)} = q_n(z)$ for $z \in S^1$.

$\frac{1}{p_n}$ analytic outside S^1

$\frac{1}{q_n}$ —— inside S^1.

$$p_0 = q_0 = 1 \qquad \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} k_1^{-1}(z+h_1) \\ k_1^{-1}(1+\bar{h}_1 z) \end{pmatrix}$$

$$\int \frac{1}{|k_1^{-1}(1+\bar{h}_1 z)|^2} \frac{dz}{2\pi i z} = \int k_1^2 \frac{1}{(1+\bar{h}_1 z)(1+h_1 z^{-1}) z} \frac{dz}{2\pi i}$$

res. at $z = -h_1$

$$= k_1^2 \frac{1}{1+\bar{h}_1(-h_1)} = \frac{k_1^2}{1-|h_1|^2} = 1$$

$$|q_n|^2 = \bar{q}_n q_n = z^{-n} p_n q_n$$

$$(f,g) = \int \frac{\bar{f}g}{p_n q_n z^{-n}} \frac{dz}{2\pi i z} = \int \frac{\bar{f} g z^{n-1}}{p_n q_n} \frac{dz}{2\pi i}$$

$$(z^i, p_n) = \int \underbrace{\frac{z^{-i} z^{n-1}}{q_n}}_{\text{analytic for } |z|<1 \text{ if } 0 \le i < n.} \frac{dz}{2\pi i} = 0$$

$$(z^i, p_{n-1}) = (z^{i+1}, z p_{n-1}) = (z^{i+1}, k_n^{-1}(p_n - h_n q_n)) \quad \text{if } i+1 \le n-1$$

$$k_n \begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} z p_{n-1} \\ q_n \end{pmatrix} \qquad \begin{pmatrix} z p_{n-1} \\ q_{n-1} \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix} \begin{pmatrix} p_n \\ q_n \end{pmatrix}$$

$$(z^i, q_n) = \int \frac{z^{-i} \cancel{q_n}}{p_n \cancel{q_n}} z^{n-1} \frac{dz}{2\pi i} = \int \frac{z^{-i+n}}{z^n} \frac{dz}{2\pi i z} = 0 \quad \text{if } i > 0.$$

Some progress is being made

$$(z^i, p_n) = 0 \qquad i < n$$
$$(z^i, q_n) = 0 \qquad i > 0.$$

$$(z^i, q_{n-1}) = \frac{1}{k_n}(z^i, -\bar{h}_n p_n + q_n) = 0$$
for $0 < i \le n-1$

$$(z^i, q_{n-2}) = \frac{1}{k_{n-1}}(z^i, -\bar{h}_{n-1} p_{n-1} + q_{n-1})$$
$$= 0 \quad \text{for} \quad 0 < i \le n-2$$

General theory gives $L^2(S^1,V^-) \xrightarrow{\iota} H \xleftarrow{j} L^2(S^1,V^+)$
says H can be recovered by $S : L^2(S^1,V^+) \to L^2(S^1,V^-)$

$$S v^+ = (1-bb^*)(1-z\,ab^*)^{-1} v^+$$

$$\iota^* j v^+ = \sum_{n\geq 0} u^n (1-bb^*)(ab^*)^n v^+$$

This S is a contraction operator commuting with z mult.
The general theory does not yield that S is unitary.
~~However~~ This is true when Y is finite-diml. Why?
General fact that if $\gamma^n x$ and $\gamma^{*n} x$ tend to zero for all $x \in X$ (here $\gamma = b^*a$) then ι, j are isos. So this works in finite dims.

My problem: What you did. ~~You construct~~
~~Suppose~~ Here V^+, V^- one-diml. bases v_0^+, v_0^-
~~Identify~~. Ident. $v_0^- = 1$ then $v_0^+ = S(z)$
$S(z) = \left(v_0^-, (1-zab^*)^{-1} v_0^+\right)$. You ~~get~~ have <u>unitary isom.</u>
of H with $L^2(S^1)$ such that $v_0^- \leftrightarrow 1$, $v_0^+ \leftrightarrow S$
$Y = H^+ \cap zSH^-$, ~~so you get~~ Write $S = \frac{p}{q}$, $q =$
$\det(1-zab^*)$, then $Y \xrightarrow{\cdot q} qH^+ \cap zpH^- = H^+ \cap z^{n+1} H^-$

$$= \mathbb{C}1 + \cdots + \mathbb{C}z^n = F_n$$

so we have unitary isom. $Y = \dfrac{F_n}{q} \subset L^2(S^1)$.

Idea: If you start with $X \rightrightarrows Y$ you know there is a canonical line, namely the domain of $(ba^{-1})^n$. You need this line in order get q_n.

I still have problems. Go back

Begin again. Take ~~...~~ $Y = aX \oplus V^+ = V^- \oplus bX$.
choose unit v. $v_0^\pm \in V^\pm$. You $\exists$ isom. embedding
$Y \hookrightarrow L^2(S^1)$ such that $v_0^- \mapsto 1$, $v_0^+ \mapsto S(z)$
where $S(z)$ given by $(az-b)X = -v_0^+ + S(z) v_0^-$

$$\cancel{(az-b)\,x = -v_0}$$

$$(az-b)\,x = -v_0^+ + v^-$$

$$(b^*az-1)\,x = -b^*v_0^+$$

$$x = (1-zb^*a)^{-1}b^*v_0^+ = b^*(1-zab^*)^{-1}v_0^+$$

$$v^- = \left(1-\cancel{zab^*} + (\cancel{a}\cancel{z}-b)b^*\right)(1-zab^*)^{-1}v_0^+$$

$$v^- = (1-bb^*)(1-zab^*)^{-1}v_0^+$$

$$v_0^-(v_0^-,v^-) = \quad S(z) = \left(v_0^-, (1-zab^*)^{-1}v_0^+\right)$$

Define

$$S(z)\,v_0^- = (1-bb^*)(1-zab^*)^{-1}v_0^+ \quad \text{then}$$

So we do have $\underline{(az-b)\,x = -v_0^+ + S(z)\,v_0^-}$

We have isometric emb. ~~such that~~ $Y \hookrightarrow L^2(S^1)$ such that $v_0^- \longmapsto 1$, $v_0^+ \longmapsto S(z)$. Thus

$$\widetilde{v_0^-}(z) = 1 \qquad \widetilde{v_0^+}(z) = S(z) \qquad \forall\, z \in S^1$$

but what do I do with a general $y \in Y$.

You can ~~take~~ write $S(z) = \dfrac{p(z)}{q(z)}$ $\quad q(z) = \det(1-zab^*)$

clear denominator ~~of~~

~~also w~~ Fredholm formulas.

$$(1-zK)^{-1} = \frac{\mathrm{Cof}(1-zK)}{\det(1-zK)} \qquad \text{and } \text{the diagrams etc.}$$

Let's go over this many times.

First the general theory. Start with $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ p.u. ~~of type $O(n)$~~, can construct H, u and isometric embeds.

$$\ell^2(\mathbb{Z}, V^-) \hookrightarrow H \hookleftarrow \ell^2(\mathbb{Z}, V^+)$$

General theory says that provided no bound states only subspace $B \subset X \ni aB = bB$ is zero, then H can be reconstructed from $S : L^2(S^1, V^+) \longrightarrow L^2(S^1, V^-)$ contraction op in the u context. Even so

---

Yesterday's point was based upon

$$L^2(S^1) \xrightarrow{i} H$$

$$i(z^n) = u^n v_0^-$$

so that $\quad i^*(\xi) = \sum_n z^n (z^{+n}, i^*\xi)$

$$= \sum_n z^n (u^{+n} v_0^-, \xi) = \sum_n z^n (v_0^-, u^{-n}\xi)$$

If $\xi \in Y$, then ~~the~~ $u^{-n}\xi \perp v_0^-$ for $-n > 0$

so $\quad i^*(\xi) = \sum_{n \geq 0} z^n (v_0^-, (ab^*)^n \xi) = (v_0^-, (1 - zab^*)^{-1}\xi)$.

This is the basic formula

for $y \in Y$

$$i^*(y)(z) = (v_0^-, (1 - za^*b)^{-1}\xi)$$

~~Anyway, life is hard.~~ You want to interpret as ~~a vector bundle~~ sections of a v.b.

~~The~~ The idea is that elements of Y are holom sections of the line bundle $z \longmapsto Y/(az-b)X$. v_0^- - this section trivializes the bundle over $|z| < 1$

$$(za-b)x = -v^+ + v^-$$

suppose $(za-b)x = v_0^-$ $\quad x \exists.$

then $(zb^*a-1)x = 0$ $\quad \therefore x = 0.$

Review. $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ type $\mathcal{O}(n)$

$\dim X = n$
$\dim Y = n+1$
$az-b$ inj $\forall z.$

eigenvector eqn. $(az-b)x = -v^+ + v^-$

have line bundle $L_z = Y/(az-b)X$ with trivializations over $|z|<1$ and $|z|>1$. Check this.

$$(1-b^*az)x = -b^*v^+$$

$$\begin{tikzcd}
 & & V^- \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "az-b"] \arrow[dr, "-1+z(b^*a)"'] & Y \arrow[r] \arrow[d, "b^*"] & L_z \arrow[r] & 0 \\
 & & X & &
\end{tikzcd}$$

so $V^- \xrightarrow{\sim} L_z$ $\iff$ $1-zb^*a$ invertible on X.

so v_0^- trivializes L over $|z| \le 1.$
v_0^+ ——— L over $|z| \ge 1.$

~~These~~ Using either of these trivs you assign functions to elements of Y. How? Answer is that ~~you~~ the ~~$\tilde{y}(z) v_0^- = y$~~ function $\tilde{y}(z)$ assoc. to $y \in Y$ is defined by $\tilde{y}(z)\, v_0^- - y \in (az-b)X.$

$$\implies b^*y = (1-zb^*a)x$$

$$x = b^*(1-zab^*)^{-1}y$$

$$\tilde{y}(z)v_0^- = y + (az-b)b^*(1-zab^*)^{-1}y$$

$$= (1 - \cancel{zab^*} + \cancel{zab^*} - bb^*)(1-zab^*)^{-1}y$$

$$\tilde{y}(z) = (v_0^-, (1-zab^*)^{-1}y)$$

So translate this to λ case. $X \underset{A}{\overset{\subset 1}{\longrightarrow}} Y$ $\quad z = \frac{1+i\lambda}{1-i\lambda}$

$X \underset{b=1+iA}{\overset{a=1-iA}{\rightrightarrows}} Y \qquad (az-b)X = (\lambda - A)X$ YES!!

again we have the lines $V^\pm = ((1\mp iA)X)^\perp$

$$\begin{array}{ccccccccc} & & & & V^- & & & & \\ & & & & \downarrow & & & & \\ 0 & \longrightarrow & X & \xrightarrow{\lambda - A} & Y & \longrightarrow & L_\lambda & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle (1+iA)^*} & & & & \\ & & & & X & & & & \end{array}$$

$(1+iA)^* \supset (1-iA)$?

$$(1+iA)^*(\lambda - A) = \lambda - A - iA^*(\lambda - A)$$

Possibly better to ~~take~~ use $az-b$ ~~$(az-b)X$~~ inst of $\lambda - A$

I would like somehow to replace $(1-zab^*)^{-1}$ by ~~$\lambda - A$~~ some version of $\lambda - A$. YES

Now ~~...~~ ab^* ~~on bX~~ is a version of ab^{-1} namely its ab^{-1} extended by 0 on V^-.

So we seek an extension of A ~~...~~ to something whose eigenvalues are in ~~say~~ the UHP. Its adjoint would be symmetrically. Then you use $(\lambda - \tilde{A})^{-1}$. So you have

~~ab^*~~
$$(\lambda - A)x' = -v^+ + v^-$$

$$\lambda - A = i\frac{1-z}{1+z} - A = \frac{i(1-z) - A(1+z)}{1+z}$$

$$= \frac{i - A - (i+A)z}{1+z} = i\,\frac{(1+iA) - (1-iA)z}{1+z} = (-i)\frac{az-b}{1+z}$$

Thus $\quad (q_n, z q_{n-1}) = 0$

$$(z^{-1} q_n, q_{n-1}) = 0$$

$$q_n = \frac{1}{k}(q_{n-1} + h z p_{n-1})$$

$$\underbrace{q_n - \frac{h}{k} z p_{n-1}}_{\text{orth}} = \frac{1}{k} q_{n-1}$$

$$\underbrace{z^{-1} q_n - \frac{h}{k} p_{n-1}}_{\text{orth}} = \frac{1}{k} z^{-1} q_{n-1}$$

You are missing something which should be easy.
Given ~~assume~~ a scalar product on $F_n = \mathbb{C}1 + \cdots + \mathbb{C}z^n$ such that $\|x\|^2 = \|zx\|^2$ for $x \in F_{n-1}$
~~form H with shift~~. construct
$p_0, p_1, \ldots, p_n$ by ~~orth~~ GS for $1, z, \ldots, z^n$
$q_n, z q_{n-1}, \ldots, z^n q_0$ GS for $z^n, z^{n-1}, \ldots, 1$

Then $\quad p_n = v_0^+ \qquad q_n = v_0^-$

we also know that
$z^n \overline{p_n(z)} = q_n(z) \qquad z \in S^1$

form H with shift operator u.

$$\oplus u^{-1} V^- \oplus aX \oplus V^+ \oplus \cdots$$

$$\oplus V^- \oplus bX \oplus \cdots$$

Let's look at this more algebraically. You have v_0^-, unit v. gen. V^-. Calculate $L^2(S^1, V^-) \xleftarrow{\iota^*} H \xleftarrow{J} L^2(S^1, V^+)$

~~YES YES YES~~

$$\iota^* J v^+ = \sum_{n \geq 0} u^n \langle v^-, u^{-n} v^+ \rangle$$

$$\iota^* J v^+ = \sum u^k c_k^-$$

$$(u^n v_0^-, \iota^* J v^+) = (u^n v_0^-, v^+) = (v_0^-, u^{-n} v^+).$$

$$u^{-1} = ba^{-1}$$
$$u^{-1} = ab^{-1}$$

$$\therefore \iota^* J v_0^+ = \sum_{n} u^n \underbrace{(v_0^-, u^{-n} v_0^+)}$$

$$= \sum_{n \geq 0} u^n (v_0^-, (ab^*)^n v_0^+)$$

$$= (v_0^-, (1 - z ab^*)^{-1} v_0^+)$$

Start again with $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ of ~~type~~ $\mathcal{O}(n+1)$ 1

Y is a Hilbert space ~~[illegible]~~ $\ni \|ax\|^2 = \|bx\|^2 \quad \forall x$.

~~We then get a canonical filtration~~

Start with $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ of type $\mathcal{O}(n)$ ~~[illegible]~~ what this means is that $az-b$ is injective $\forall z \in \mathbb{P}^1$ and $\dim X = n$ $\dim Y = n+1$. We get a holom line bundle $\mathcal{L}$ over $\mathbb{P}^1$ fibre $Y/(az-b)X$ at z such that $Y = \Gamma(\mathcal{L})$ and $X = \Gamma(\mathcal{L}(-1))$. ~~The line bundle arises by gluing trivial bundles of over the af~~

Note that by construction we have singled out the points $0, \infty$ of $\mathbb{P}^1$ — we really have $\mathbb{P}^1 = z\text{plane} \cup \infty$. Thus we get decreasing filtrations ~~by taking~~ of Y consisting of sections vanishing to various order at a given point. Natural points to take at $z = 0, \infty$.

The section corresp to $y \in Y$ vanishes at $z = \infty$ $\Leftrightarrow$ $y \mapsto 0$ in Y/aX i.e. $y \in aX$. Sim. bX is the space of sections vanishes at $z = 0$. $aX \cap bX =$ sect. vanishing at 0 and ∞. Now you have the isom $ba^{-1} : aX \longrightarrow bX$, takes a section vanishing at ∞ to a section vanishing ~~[illegible]~~ at 0. So this is mult by z. z "$=$" ba^{-1}

Ask for sections vanishing to double order at $z = \infty$ ~~are~~ these are $y \in aX$ and $\ni$ $zy \in aX$

$y = ax \ni ba^{-1}y = bx \in aX$

$$\begin{array}{ccc} & & X \\ & & \downarrow a \\ X & \xrightarrow{b} & Y \end{array}$$

$y \in aX$ $\downarrow a$ $\ni$ $ba^{-1}y \in aX$

$$\begin{array}{ccc} x & \xrightarrow{b} & y' \\ a\downarrow & & \\ y & & \end{array}$$

~~[illegible]~~

I go back to Cramer's rule

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The point is that the inverses of the entries of A^{-1} have a nice description

$$\frac{ad-bc}{d} = a - bd^{-1}c$$

$$\frac{ad-bc}{-b} = c - ab^{-1}d \qquad \text{etc.}$$

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] & Y \arrow[r, "j"] \arrow[d, shift right, "A"'] & Z \arrow[r] & 0 \\ 0 & X' \arrow[l] & Y' \arrow[l, "j'"'] \arrow[u, shift right, "A^{-1}"'] & Z' \arrow[l, "i'"'] & 0 \arrow[l] \end{tikzcd}$$

$$\underbrace{1 - i(j'Ai)^{-1}j'}_{\text{doesn't make sense}} = i'(jA^{-1}i')^{-1}j \; ?$$

$$\begin{tikzcd} & Z' \arrow[d, "i'"] & \\ X \arrow[r, "i"] & Y \arrow[r, "j"] \arrow[d, "j'"] & Z \\ & X' & \end{tikzcd}$$

$$1 - i(j'i)^{-1}j' = i'(ji')^{-1}j$$

$$\begin{tikzcd} X \arrow[r] & Y \arrow[d] \\ X & Y \arrow[l] \end{tikzcd}$$

Y/X ~~confused~~.

$$\begin{tikzcd} & V^- \arrow[d] & \\ X \arrow[r, "az-b"] \arrow[dr, "-1+zb^*a"'] & Y \arrow[r] \arrow[d, "b^*"] & L_z \\ & X & \end{tikzcd}$$

$$y - (az-b)(-1+zb^*a)^{-1} b^* y$$

$$y + (az-b)b^*(1-zab^*)^{-1} y$$

$$(1-z\cancel{a}b^* + (\cancel{az}-b)b^*)(1-zab^*)^{-1} y$$

So what can we do?

$$z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i}$$

$$a = i\varepsilon + A$$
$$b = i\varepsilon - A$$

$$\begin{tikzcd} X \arrow[r, shift left, "\varepsilon"] \arrow[r, shift right, "A"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} & V^- \arrow[d] & \\ X \arrow[r, "\lambda\varepsilon-A"] & Y \arrow[r] \arrow[d, "(i\varepsilon-A)^*"] & L_\lambda \\ & X & \end{tikzcd}$$

$$1 - (\lambda\varepsilon - A)\left[(i\varepsilon-A)^*(\lambda\varepsilon-A)\right]^{-1}(i\varepsilon-A)^*$$

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon-A"] \arrow[d] & \cancel{Y} \arrow[r] \arrow[d, Rightarrow, "1"] & L \arrow[r] & {} \\ X & Y \arrow[l, "(i\varepsilon-A)^*"'] & V^- \arrow[l] \arrow[u] & \end{tikzcd}$$

So maybe whats interesting is the ~~...~~ equivalence of ~~det~~ $(1-zab^*)^{-1}\ \exists \iff (1-zb^*a)^{-1}\ \exists$

$$(1-pq)^{-1} = 1 + p(1-qp)^{-1}q$$

$$(1-pq)(1+p(1-qp)^{-1}q)$$
$$= 1-pq + (1-pq)p(1-qp)^{-1}q = 1.$$
$$p(1-qp)$$

anyway $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ Assume $d^{-1} \exists$.

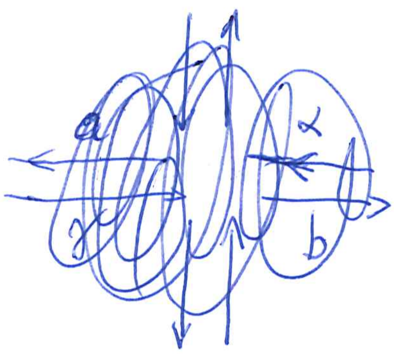
$$\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix} = \begin{pmatrix} a-bd^{-1}c & 0 \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ d^{-1}c & 1 \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -d^{-1}c & 1 \end{pmatrix}\begin{pmatrix} (a-bd^{-1}c)^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\beta"] \arrow[d, shift left, "\delta"] & \bullet \arrow[l, shift left, "c"] \arrow[d, shift left, "a"] \\ \bullet \arrow[u, shift left, "d"] \arrow[r, shift left, "b"] & \bullet \arrow[l, shift left, "\gamma"] \arrow[u, shift left, "\alpha"] \end{tikzcd}$$



$\alpha a + \beta c = 1$

$\alpha b + \beta d = 0$

$\beta = -\alpha b d^{-1}$

$\alpha(a - bd^{-1}c) = 1$

$a\alpha + b\gamma = 1$

$a + b\gamma\alpha^{-1} = \alpha^{-1}$

$a\alpha + b(-d^{-1}c\alpha) = 1$

$(a - bd^{-1}c)\alpha = 1.$

$c\alpha + d\gamma = 0$

$c + d\gamma\alpha^{-1} = 0$

$\gamma\alpha^{-1} = -d^{-1}c$

$\gamma = -d^{-1}c\alpha$

Resolvent. You want to understand why.

Look at your system. First question is what is the relation between $\dfrac{1}{1-zab^*}$ and $\dfrac{1}{1-zb^*a}$

Answer is a coeff. of the resolvent.

$$(1-bb^*)(1-zab^*)^{-1}y \quad \text{any } y.$$

somehow this arises from

$$\begin{tikzcd} & V \arrow[d] & \\ X \arrow[r, "az-b"] & Y \arrow[r] \arrow[d, "b^*"] & L_z \\ & X & \end{tikzcd}$$

$$1-(az-b)(1-zb^*a)^{-1}b^*$$

$$1-(az-b)\left(b^*(az-b)\right)^{-1}b^*$$

$$a - bd^{-1}c$$

This looks like something related to the matrix

$$\begin{pmatrix} 1 & az-b \\ b^* & b^*(az-b) \end{pmatrix} = \begin{pmatrix} 1 \\ b^* \end{pmatrix}\begin{pmatrix} 1 & az-b \end{pmatrix}$$

There seems to be an interesting result here.

~~There's a relation between~~

It seems you have an example of quasi-determinants, ~~A quasi~~ which are matrix elements of the inverse matrix, ~~the~~ a ratio of determinants as in Cramer's rule. For

$A = (a_{ij})$ $\quad a^{11} \simeq a_{21}a_{11}^{-1}a_{12}$?

The basic idea is as follows: Given

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] & Y \arrow[r, "j"] \arrow[d, "A"] & Z \arrow[r] & 0 \\ 0 & X' \arrow[l] & Y' \arrow[l, "j'"'] & Z' \arrow[l, "i'"'] & 0 \arrow[l] \end{tikzcd}$$

such that $j'Ai$ is an isomorphism, then there are canonical splittings of the two short exact sequences such that A is the direct sum of $j'Ai : X \to X'$ and an induced map $Z \to Z'$. The induced map should be ~~a kind of~~ the quasi-det. associated to the "matrix coefficient" $j'Ai$ of A

What is the induced map $Z \to Z'$? Answer???

$$A\left(y - i(j'Ai)^{-1}j'Ay\right)$$

$$A - Ai(j'Ai)^{-1}j'A :$$

$$\begin{tikzcd} X \arrow[d, "i"] & & Z' \arrow[d, "i'"] \\ Y \arrow[rr] \arrow[d, "j"] & & Y' \arrow[d, "j'"] \\ Z \arrow[uurr, "\tilde{A}"] \arrow[urr] & & X' \end{tikzcd}$$

$$\boxed{A = Ai(j'Ai)^{-1}j'A + i'\tilde{A}j}$$

So this is the quasi-determinant. So given a matrix $A = \begin{pmatrix} a_{11} & a_{1n} \\ a_{n1} & a_{nn} \end{pmatrix} : Y \xrightarrow{A} Y'$

set a_{kl} and the l gives an $X \xhookrightarrow{i} Y$, the k given a $Y' \to X'$

~~If you delete the [redacted] k-th row mean look at Z'~~
~~l-th column~~

Go to main example

$$\begin{tikzcd} X \arrow[r, "az-b"] \arrow[d, "(b^*az-1)"'] & Y \arrow[r, "j"] \arrow[d, equal] & L_z \arrow[d, "\phi"] \\ X & Y \arrow[l, "b^*"'] & V^- \arrow[l, "i'"'] \end{tikzcd}$$

$$1-(az-b)(b^*az-1)^{-1}b^* = i'\phi j$$

You need to understand how the left side ~~[illegible]~~ becomes $(1-bb^*)(1-zab^*)^{-1}j$

Somehow the quasi-determint in this situation ~~[illegible]~~ $\phi : L_z \longrightarrow V^-$ turns out to ~~be the matrix element~~ connected to the resolvent $(1-zab^*)^{-1}$. Now we know that $(1-zab^*)$ is invertible $\Longleftrightarrow$ $(1-zb^*a)$ is invertible.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "1-zb^*a"'] & Y \arrow[r] \arrow[d, "1-zab^*"] & V^+ \arrow[d, "1"] \\ X \arrow[r, "a"] & Y \arrow[r] & V^+ \end{tikzcd}$$

Review what I know

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "i"] \arrow[d, "s"'] & Y \arrow[r, "j"] \arrow[d, "A"] & Z \arrow[r] & 0 \\ 0 \arrow[r] & X' & Y' \arrow[l, "j'"'] & Z' \arrow[l, "i'"'] & 0 \arrow[l] \end{tikzcd}$$

$$\underbrace{A - Ai(j'Ai)^{-1}j'A}_{\text{quasi-determinant}} = i'\tilde{A}j$$

assume $(j'Ai)^{-1}$ $\exists$

Then you have ~~[illegible]~~

A ~~-~~ $Ai(j'Ai)^{-1}j'A : Y \to Y'$

~~induces~~ factors

$$\begin{tikzcd} X \arrow[d] & Z' \arrow[d, "i'"] \\ Y \arrow[r] \arrow[d, "j"'] & Y' \arrow[d] \\ Z & {} \end{tikzcd}$$

now consider

$$\begin{tikzcd} X \arrow[r, "az-b"] & Y \arrow[r, "\gamma"] \arrow[d, "\parallel"'] & L_z \arrow[d, "\phi"] \\ X & Y \arrow[l, "b^*"] & V^- \arrow[l, "\iota'"] \end{tikzcd}$$

$$\underbrace{1-(az-b)\left(b^*(az-b)\right)^{-1}b^*}_{(1-bb^*)(1-zab^*)^{-1}} = \iota\phi\gamma$$

Notice that ~~$(1-zab^*)^{-1} z (1-zab^*)^{-1}$~~

$$(1-zab^*)^{-1}(az-b)\,b^*b^*$$

$$(1-zab^*)^{-1}(az-b)$$

$$az-b + zab^*(az-b) + z^2(ab^*)^2(az-b)$$

$$= az-b + za(b^*az-1) + z^2ab^*a(b^*az-1)$$

$$= -b.$$

$$(1-zab^*)^{-1}(az-b) = -b.$$

$$b^*(1-zab^*)^{-1}(az-b)$$

$$= (1-zb^*a)^{-1}(b^*az-1) = -1$$

$$-(1-zab^*)b = (az-b)$$

Let's try to get the ~~zeroes~~ radiative modes straight. $z \ni \det(1 - zba^*) = 0$, correspond to what λ?

we know

$1 - zba^*$ inv. $\Longleftrightarrow$ $1 - zab^*$ is.

$$\begin{tikzcd}
 & V^- \arrow[d] \arrow[dr] & \\
X \arrow[r, "az-b"] \arrow[dr, "\sim(1-zb^*a)"'] & Y \arrow[r] \arrow[d, "b^*"] & L_z \\
 & X &
\end{tikzcd}$$

$$\begin{tikzcd}
 & Y \arrow[d, "\binom{q}{1}"] \arrow[dr, "1-pq"] & \\
X \arrow[r, "\binom{1}{p}"] \arrow[dr, "1-qp"'] & X \oplus Y \arrow[r, "(-p\ 1)"] \arrow[d, "(1\ -q)"] & Y \\
 & X &
\end{tikzcd}$$

simplest proof is $\dfrac{1}{\det(1 - zA)} = e^{\sum_{n \geq 1} \frac{z^n}{n} \operatorname{tr}(A^n)}$

reduces to $\operatorname{tr}(ab^*)^n = \operatorname{tr}(b^*a)^n$. Anyway we find zilch. ~~Let us finish~~

Maybe these points are where the section σ_0 assoc to vanishes. Is it true that

It looks to me that if $\det(1 - zab^*)$ is to be essentially equal to $\det(\lambda - B)$, then you want B to be roughly the C.T. of ab^*.

$$ab^* = (i\varepsilon + A)(-i\varepsilon^* - A^*)$$
$$b^*a = (-i\varepsilon^* - A^*)(i\varepsilon + A) = 1 - iA^*\varepsilon - i\varepsilon^* A - A^*A$$

parameter. Alternative — 1-parameter unitary groups. Go back — try to couple to continuous translations.

$$X \xrightarrow{\binom{b}{a}} Y \oplus Y$$

$$1 + i\lambda + (\lambda - i)A^*\varepsilon - (1 + A^*A)$$

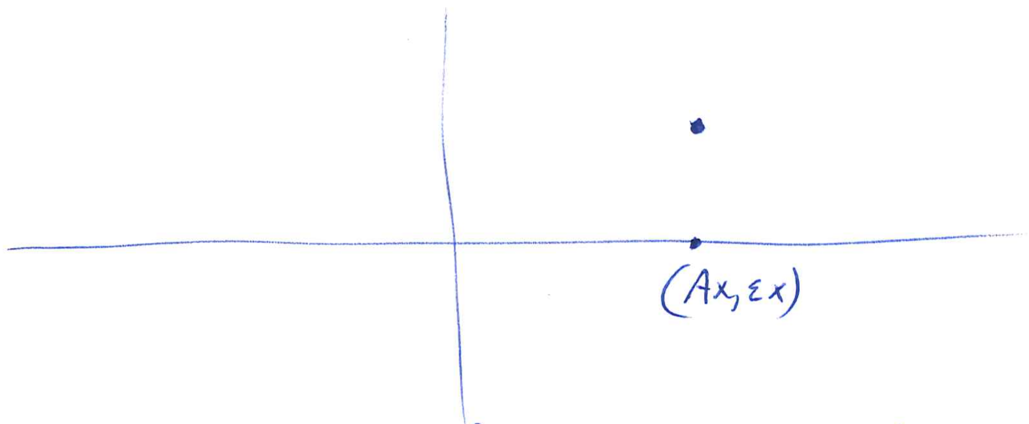
$$(\lambda - i)(+i + A^*\varepsilon) - (1 + A^*A)$$

$$(x, (i\varepsilon^* + A^*)(\lambda\varepsilon - A)x) = (x, (i\lambda + (\lambda - i)A^*\varepsilon - A^*A)x)$$

$$= i\lambda\|x\|^2 + (\lambda - i)(Ax, \varepsilon x) - \|Ax\|^2 = 0$$

$$i\lambda + (\lambda - i)\frac{(Ax, \varepsilon x)}{\|x\|^2} - \frac{\|Ax\|^2}{\|x\|^2} = 0$$

$$(\lambda - i)(i + (Ax, \varepsilon x)) = 1 + \|Ax\|^2 \qquad \|x\| = 1.$$



$$|i + (Ax, \varepsilon x)|^2 = 1 + \|Ax\|^2$$

$$|(Ax, \varepsilon x)| = \|Ax\|$$

now what?? What next? Anyway. Take a break then return to:

Review. $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y \quad z = \frac{-\lambda+i}{\lambda+i} = \frac{1+i\lambda}{1-i\lambda} \quad a = i\varepsilon + A \quad b = i\varepsilon - A$

$$az - b = (i\varepsilon + A)z - (i\varepsilon - A) = i(z-1)\varepsilon + (1+z)A$$
$$= -(1+z)\left(i\frac{1-z}{1+z}\varepsilon - A\right)$$

$i\frac{1-z}{1+z} \to \lambda$

$$\begin{tikzcd} & V^- \arrow[d] & & \\ 0 \arrow[r] & X \arrow[r, "az-b"] & Y \arrow[r] \arrow[d, "b^*"] & L_z \arrow[r] & 0 \\ & & X & & \end{tikzcd}$$

line bundle $Y/(\lambda\varepsilon - A)X = L_\lambda$

v_0^- unit v. sp. V^- gives a ~~nonvan.~~ hol. section of L ~~...~~ nonv over $|z|<1$.

$$(az-b)x = v^- \in V^- \Rightarrow \underbrace{b^*(az-b)}_{-(1-(b^*a)z)}x = 0 \Rightarrow x = 0.$$

Solving $(az-b)x = -y + \hat{y}(z)v_0^-$ gives a function $\vec{y}(z)$ associated to y holom. except at ~~eigenvalues of b~~ roots of $\det(1-zab^*)$

$$\begin{tikzcd} X \arrow[r, "\lambda\varepsilon - A"] & Y \arrow[r] \arrow[d, "b^* = -i\varepsilon^* - A^*"] & L_\lambda \\ & X & \end{tikzcd}$$

$b^*(b-az) = 1 - z\underline{b^*a}$

$$(i\varepsilon^* + A^*)(\lambda\varepsilon - A)x = 0.$$

$i\lambda - i\varepsilon^*A + \lambda A^*\varepsilon - A^*A$ too hard.

You hope to use something like $Y^{\oplus 2}$

Is there a way to bring in λ - the eigenvalue

Now dentist - ~~consider~~

$X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y \qquad z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i} \qquad X \underset{b = i\varepsilon - A}{\overset{a = i\varepsilon + A}{\longrightarrow}} Y$

$$az - b = (i\varepsilon + A)z - (i\varepsilon - A) = i(z-1)\varepsilon + (z+1)A$$

$z+1 = \frac{2i}{\lambda+i}$
$$= -(z+1)\Big(\underbrace{i\frac{1-z}{1+z}}_{\lambda}\varepsilon - A\Big)$$

$\|bx\|^2 = \|x\|^2 + (i\varepsilon x, Ax) + (Ax, \cancel{i\varepsilon x}) + \|Ax\|^2$

$(\varepsilon x, Ax) = (Ax, \varepsilon x) = 0$

$\|ax\|^2 = \|bx\|^2 = \|x\|^2 + \|A\|^2$

$$\begin{tikzcd} X \arrow[r, "(z+1)(\lambda\varepsilon - A)"] & Y \arrow[d, "b^* = -i\varepsilon^* - A^*"] \\ & X \end{tikzcd}$$

$\varepsilon^* A = A^* \varepsilon$

$A^*\varepsilon \quad \varepsilon^* A$

$$\begin{tikzcd} X \arrow[r, "-(az-b)"] \arrow[dr, "1 - zb^*a"'] & Y \arrow[d, "b^*"] \\ & X \end{tikzcd}$$

$$(-i\varepsilon^* - A^*)(\lambda\varepsilon - A)$$
$$= -i\lambda + i\varepsilon^* A - \lambda A^*\varepsilon + A^*A$$

$$b^*a = (-i\varepsilon^* - A^*)(i\varepsilon + A)$$
$$= 1 - i\varepsilon^* A - iA^*\varepsilon + A^*A$$
$$= 1 - 2i\varepsilon^* A + A^*A$$

$$(1 - zb^*a)^{-1} b^* = b^*(1 - z\underline{ab^*})^{-1}$$

$$ab^* = (i\varepsilon + A)(-i\varepsilon^* - A^*)$$

Note $A^*\varepsilon = \varepsilon^* A$ is a s.a. operator on X, but you do not want a self adj operator on Y.

$(\varepsilon^* A)^2 = A^*\varepsilon\varepsilon^* A$

$$b^*a = 1 - 2i\varepsilon^* A + A^*A$$
$$= (1 + A^*A) - 2i\varepsilon^* A$$

$$1 - zb^*a = 1 - \frac{-\lambda+i}{\lambda+i} b^*a$$
$$= \frac{1}{\lambda+i}\big((\lambda+i) - (-\lambda+i)b^*a\big) = \frac{1}{\lambda+i}\big((1+b^*a)\lambda + i(1-b^*a)\big)$$
$$= \frac{1+b^*a}{\lambda+i}\left(\lambda + i\frac{1-b^*a}{1+b^*a}\right).$$

Given $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ $\qquad (\varepsilon x_1, Ax_2) = (Ax_1, \varepsilon x_2)$

Work inside $Y^{\oplus 2}$

$$\left(\begin{pmatrix} 1 \\ \alpha \end{pmatrix} y \,,\, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ 1 \end{pmatrix} y' \right) = (y, (\alpha^* \ 1) \begin{pmatrix} -1 \\ \beta \end{pmatrix} y')$$
$$= -(y, \alpha^* y') + (y, \beta y').$$

The orthagonal for Γ_α for this skew hermitian form is Γ_{α^*}. ~~...~~ So $\alpha = \alpha^* \Longleftrightarrow$ Γ_α is isotropic. Apply now to

$$\left(\begin{pmatrix} \varepsilon \\ A \end{pmatrix} x \,,\, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = -(\varepsilon x, y_2) + (Ax, y_1) = 0$$

If $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \varepsilon \\ A \end{pmatrix} x'$. Then $-(\varepsilon x, Ax') + (Ax, \varepsilon x') = 0$.

So the hermitian condition on $\begin{pmatrix} \varepsilon \\ A \end{pmatrix}$ means that $\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X$ is isotropic. In our example where X is of codim 1 in Y, ~~...~~ the quotient over $\mathbb{C}$

$$\left(\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \right)^0 \Big/ \begin{pmatrix} \varepsilon \\ A \end{pmatrix} X$$

is 2 dim, has skew herm. form

the null lines will yield a circle of hermitian extensions α of $A\varepsilon^{-1}$. The same thing happens with partial unitaries.

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} x \,,\, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} x' \right) = (ax, ax') - (bx, bx') = 0$$
$$\Longleftrightarrow a^* a = b^* b$$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix} x, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = (ax, y_1) - (bx, y_2) = 0$$
$$= (x, a^* y_1 - b^* y_2) = 0.$$

i.e. $\left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \, a^* y_1 = b^* y_2 \right\}$

~~Looking~~ Suppose given $a^* y_1 = b^* y_2$, then ~~[illegible]~~ how can ~~$ba^* y_1 = bb^* y_2$~~ you modify $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ by an $\begin{pmatrix} ax \\ bx \end{pmatrix}$

$$\begin{pmatrix} y_1 + ax \\ y_2 + bx \end{pmatrix}$$

simplest to take $x = -b^* y_2$

$a^* y_1' = a^* y_1 - b^* y_2 = 0$

$$= \begin{pmatrix} y_1 - ab^* y_2 \\ y_2 - bb^* y_2 \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

Then $b^* y_2' = 0$

So we find that the quotient ~~of~~ $\left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^\circ / \begin{pmatrix} a \\ b \end{pmatrix} X$ is the 2 diml space $\begin{pmatrix} V^+ \\ V^- \end{pmatrix}$. We have usual pseudo scalar product, so that the unitary extensions of ~~[illegible]~~ the partial unitary are described by a circle. Look at $\begin{pmatrix} a \\ b \end{pmatrix} X + \begin{pmatrix} v^+ \\ 0 \end{pmatrix}$. This should have the form $\begin{pmatrix} 1 \\ \alpha \end{pmatrix} \boxed{} Y$

$$\alpha(\underbrace{ax + v^+}_{y}) = bx$$

$a^* y =$

$$\begin{aligned} ax + v^+ &= y \quad \Rightarrow x = a^* y \\ bx &= \alpha y \end{aligned}$$

$$\therefore \alpha = ba^*$$

$$\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \underbrace{(\varepsilon x, y_2) \overset{\forall x}{=} (Ax, y_1)}_{\varepsilon^* y_2 = A^* y_1} \right\} \supset \begin{pmatrix} \varepsilon x' \\ Ax' \end{pmatrix}$$

Can remove

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} \varepsilon \varepsilon^* y_1 \\ A\varepsilon^* y_1 \end{pmatrix} = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$$

$\therefore$ Can suppose $\varepsilon^* y_1' = 0$.

Look at $\left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \begin{array}{l} \varepsilon^* y_2 = A^* y_1 \\ \varepsilon^* y_1 = 0. \end{array} \right\}$

$(\varepsilon x, y_2) = (Ax, y_1) \quad \forall x$ has dim. $2(n+1)-n = n+2$

$\Leftrightarrow \quad \varepsilon^* y_2 = A^* y_1$

Certainly $\begin{pmatrix} \mathrm{Ker}(A^*) \\ \mathrm{Ker}(\varepsilon^*) \end{pmatrix} \subset \left(\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X \right)^0 \supset \underbrace{\begin{pmatrix} \varepsilon \\ A \end{pmatrix} X}^{\dim n}$

$$\begin{pmatrix} \varepsilon x \\ Ax \end{pmatrix} \in \begin{pmatrix} \mathrm{Ker}\, A^* \\ \mathrm{Ker}\, \varepsilon^* \end{pmatrix} \quad \text{i.e.} \quad \begin{matrix} A^* \varepsilon x = 0 \\ \varepsilon^* A x = 0. \end{matrix}$$

$$\begin{array}{cc} & n \\ n+1 & \begin{pmatrix} b & a_1 & & & \\ a_1 & & & & \\ & & b_{n-1} & a_{n-1} & \\ & & a_{n-1} & b_n & a_n \\ & & & a_n & \end{pmatrix} \end{array}$$

First the p.u. case $X \underset{b}{\overset{a}{\rightrightarrows}} Y$. Look at $Y^{\oplus 2}$ with ~~herm~~ herm pseudo scalar product $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$
$= \|y_1\|^2 - \|y_2\|^2$

$\begin{pmatrix} b \\ a \end{pmatrix} X \subset Y^{\oplus 2}$ has orth. space

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \underbrace{(bx, y_1) - (ax, y_2) \; \cancel{(y_1, bx) - (y_2, ax)}}_{} = 0 \quad \forall \right\}. \supset$$

$b^* y_1 = a^* y_2$ contains $\begin{pmatrix} b \\ a \end{pmatrix} X$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - \begin{pmatrix} b b^* y_1 \\ \underbrace{a b^* y_1}_{a a^* y_2} \end{pmatrix} = \begin{pmatrix} (1 - b b^*) y_1 \\ (1 - a a^*) y_2 \end{pmatrix}$$

Thus $\left(\begin{pmatrix} b \\ a \end{pmatrix} X \right)^0 = \begin{pmatrix} b \\ a \end{pmatrix} X \oplus \begin{pmatrix} V^- \\ V^+ \end{pmatrix}$ intersection is 0 since $bx \in V^-$ $\Rightarrow b^* b x = 0 \Rightarrow x = 0.$

~~If we can extend ... X ... X~~ Extending a, b to a unitary ~~...~~ amounts to increasing $\begin{pmatrix} b \\ a \end{pmatrix} X$ to $\begin{pmatrix} b' \\ a' \end{pmatrix} X'$

where $b'X' = Y = a'X'$. One chooses an isot. line i.e. a unitary iso between V^+ and V^-. ~~Other~~ Can consider other extensions : self-correspondences extending ba^{-1} on Y.

$$\begin{pmatrix} b \\ a \end{pmatrix} X \subset \ ? \ \subset \left(\begin{pmatrix} b \\ a \end{pmatrix} X \right)^\circ$$

These correspond to lines in $V^- \oplus V^+$. Obvious choice is $\begin{pmatrix} b \\ a \end{pmatrix} X + \begin{pmatrix} 0 \\ V^+ \end{pmatrix} = \left\{ \begin{pmatrix} bx \\ ax + v^+ \end{pmatrix} \right\}$. graph

of $y = ax + v^+ \longmapsto bx = ba^* y$ $\therefore$ graph of ba^*

Now move to $\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X \subset Y^{\oplus 2}$. Recall that $\varepsilon^* A = A^* \varepsilon$, $\varepsilon^* \varepsilon = 1$. Use ~~pseudo~~ pseudo scal. product $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = -(y_1, y_2') + (y_2, y_1')$

Then $\left(\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X \right)^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| (Ax, y_2) = (\varepsilon x, y_1) \right\}$

i.e. $A^* y_2 - \varepsilon^* y_1 = 0$.

contains $\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X$ $\quad A^* \varepsilon x - \varepsilon^* A x = 0$,

Also $\left(\begin{pmatrix} \alpha \\ 1 \end{pmatrix} Y \right)^\circ = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| \underline{(\alpha y, y_2) = (y, y_1) \quad \forall y} \right\}$

$$\alpha^* y_2 = y_1$$

$$= \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix} Y.$$

So extn. of the partial herm. ε, A to a herm. operator should amount to extending $\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X$ to a max isot subspace Γ of $Y^{\oplus 2}$ such that the 2nd proj $\Gamma \xrightarrow{pr_2} Y$ is $\sim$. Our situation $\dim Y = n+1$, $\dim X = n$ $\lambda \varepsilon - A$ always injective. $\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X$ codim 2 in $\left(\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X \right)^\circ$

Obvious solutions of $A^* y_2 = \varepsilon^* y_1$ are $y_1 \in \operatorname{Ker}(\varepsilon^*)$ $y_2 \in \operatorname{Ker}(A^*)$ i.e. $y_1 \in (\varepsilon X)^\perp$ $\quad y_2 \in (AX)^\perp$. ~~Ther~~ You can ask whether $\begin{pmatrix} \operatorname{Ker} \varepsilon^* \\ \operatorname{Ker} A^* \end{pmatrix}$ is complem. to $\left(\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X\right)^{\perp}$ in $\left(\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X\right)^{\circ}$

intersection consists of $\begin{pmatrix} Ax \\ \varepsilon x \end{pmatrix} \ni \begin{cases} \varepsilon^* A x = 0 \\ A^* \varepsilon x = 0. \end{cases}$

Apparently this can happen.

Take $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \left(\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X\right)^{\circ}$ i.e. $A^* y_2 = \varepsilon^* y_1$ ....

What is $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$= -(y_1, y_2) + (y_2, y_1)$? Certainly

It seems that $\begin{pmatrix} \operatorname{Ker} \varepsilon^* \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \operatorname{Ker} A^* \end{pmatrix}$ are isotropic lines. So add to $\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X$

$$\begin{pmatrix} A \\ \varepsilon \end{pmatrix} X + \begin{pmatrix} 0 \\ \operatorname{Ker} A^* \end{pmatrix} \ni \begin{pmatrix} Ax \\ \varepsilon x + k \end{pmatrix} \qquad A^* k = 0.$$

Provided $\varepsilon X + \underbrace{\operatorname{Ker} A^*}_{(AX)^{\perp}} = Y$ This is the wrong approach.

Go back to $\begin{pmatrix} b \\ a \end{pmatrix} X \subset \left(\begin{pmatrix} b \\ a \end{pmatrix} X\right)^{\circ} = \begin{pmatrix} b \\ a \end{pmatrix} X \oplus \begin{pmatrix} V^- \\ V^+ \end{pmatrix}$

The canonical choice is $\begin{pmatrix} b \\ a \end{pmatrix} X + \begin{pmatrix} 0 \\ V^+ \end{pmatrix}$ which will be the graph of the operator ba^*

At this point you hope that working in $Y^{\oplus 2}$ might help. I want to check that

$$\begin{array}{ccc} \begin{pmatrix} A \\ \varepsilon \end{pmatrix} X & \subset & Y^{\oplus 2} \\ & & \Big\downarrow \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} \\ \begin{pmatrix} b \\ a \end{pmatrix} X & \subset & Y^{\oplus 2} \end{array}$$

So what????

$$\left(\begin{pmatrix}-1 & i\\ 1 & i\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}\right)^{*}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}-1 & i\\ 1 & i\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix}-1 & 1\\ -i & -i\end{pmatrix}\begin{pmatrix}-1 & +i\\ -1 & -i\end{pmatrix} = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

The question. What do you add to $\begin{pmatrix}A\\ \varepsilon\end{pmatrix}X \subset Y^{\oplus 2}$ to make it a graph? Maybe go back to J-matrices.

$$n+1\left\{\underbrace{\overbrace{\left(\begin{array}{ccccc} b_1 & a_1 & & & \\ a_1 & b_2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & a_{n-1} & b_n \\ \hline & & & & a_n \end{array}\right)}^{n}}_{A}\right. \qquad \underbrace{\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\varepsilon}$$

$\varepsilon^* A$ is the main minor.

Go over what you are doing?

Look at hermitian extensions of A. You add in a unit vector – last column. To go further you have to give b_{n+1} any real no. for self adjointness.

It looks like there is an operator on Y where $b_{n+1} = 0$. Besides $\varepsilon^* A = A^*\varepsilon$ on X, there should be something like $\frac{1}{2}(A\varepsilon^* + \varepsilon^* A)$ on Y

Note $Y = \Gamma(\mathcal{O}(n))$ so $\mathbb{P}Y$ is the space of divisors of degree n in $\mathbb{P}^1$. This means that any line in Y is determined by zero set. It should now be clear that

~~Let's consider something new. Let's consider something new. Anyway, let's treat the stuff again from~~

What do we learn? That $y \longmapsto (\lambda - T)^{-1} y$ gives an isom. embed of Y into functions on $\mathbb{R}$ with $\|f\|^2 = \int \frac{d\lambda}{\pi}\left(f, \frac{T-T^*}{2i} f\right)$. You want $\frac{T-T^*}{2i} = |\,\rangle\langle\,|$. $\frac{T-T^*}{2i}$ is positive + rank 1 $\therefore$ $|y_0\rangle\langle y_0|$. How do I manage this? Also want T to have eigenv. in LHP. If $Ty = \lambda y$ then

$$\left(y, \frac{T-T^*}{2i} y\right) = \frac{(y, Ty) - (Ty, y)}{2i} = \frac{\lambda - \bar\lambda}{2i} \|y\|^2$$

I think we showed

$$\int_{-\infty}^{\infty} \frac{d\lambda}{\pi} \left((\lambda - T)^{-1} y, \left(\frac{T-T^*}{2i}\right)(\lambda - T)^{-1} y \right) = \|y\|^2$$

provided eigenvalues of T in the UHP.

so $\frac{T-T^*}{2i} > 0 \implies \mathrm{Spec}(T)$ in UHP.

$$\tilde{y}(\lambda) = \left(v_0, (\lambda - T)^{-1} y\right)$$

$$\widetilde{Ty}(\lambda) = \left(v_0, (\lambda-T)^{-1} Ty\right) = \left(v_0, (\lambda - T)^{-1}(T - \lambda + \lambda) y\right)$$

$$= \lambda\, \tilde{y}(\lambda) - (v_0, y).$$

So it looks like you want v_0 to be $\perp$ to εX

So what. $\tilde{y}(\lambda) = (v_0, (\lambda - T)^{-1} y)$

$$\widetilde{Ay}(\cancel{A}) = (v_0, (\lambda-T)^{-1} A y)$$
$$= (v_0, (\lambda-T)^{-1}(\lambda y + (A-\lambda) y))$$
$$= \lambda \tilde{y}(\lambda) + (v_0, (\lambda-T)^{-1}(A-\lambda) y)$$

want 2nd term to vanish, using $y \in \varepsilon X = \mathcal{D}_A$ and that T extends A. If T extends A on $\mathcal{D}_A$, then $Ay = Ty$ and the 2nd term is (v_0, y), so we need $v_0 \perp \varepsilon X = \mathcal{D}_A$.

---

continue $\displaystyle \frac{1}{\lambda - T} - \frac{1}{\lambda - T^*} = \frac{1}{\lambda - T^*}\Big(\underbrace{\lambda - T^* - (\lambda - T)}_{T-T^*}\Big)\frac{1}{\lambda - T}$

$$\cancel{\int_{-\infty}^{\infty} \|(\lambda-T)^{-1}y\|^2 \frac{d\lambda}{\pi}} \qquad \int_{-\infty}^{\infty} \left((\lambda-T)^{-1}y, \left(\frac{T-T^*}{2i}\right)(\lambda-T)^{-1}y\right)\frac{d\lambda}{\pi}$$

$$= \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi i}\Big((y, (\lambda-T)^{-1}y) - (y, (\lambda-T^*)^{-1}y)\Big) = \|y\|^2 \quad \text{if } sp(T) \subset UHP$$

this is very nice. to see if it can be added

$$\tilde{T}y(\lambda) = (v_0, (\lambda-T)^{-1}Ty) = (v_0, (\lambda-T)^{-1}(T-\lambda+\lambda)y)$$
$$= (v_0, y) + \lambda \tilde{y}(\lambda). \qquad \text{L.T.}$$

Given $X \overset{\varepsilon}{\underset{A}{\rightrightarrows}} Y$ of type $O(n)$ is there a natural choice of T

~~Let's start again~~

SUNDAY 2:45

$$\partial_x E + \partial_t I = 0$$
$$\partial_x I + \partial_t E = 0$$

$$E_0 = -L\,\partial_t I_0$$

$$(\partial_x + \partial_t)(E+I) = 0$$
$$(\partial_x - \partial_t)(E-I) = 0$$
$$E+I = f(x-t)$$
$$E-I = g(x+t)$$

~~Anyway can go~~

Use L.T.

$$(\partial_x + s)(\hat{E}+\hat{I}) = E_{t=0} + I_{t=0} = f(x)$$
$$(\partial_x - s)(\hat{E}-\hat{I}) = -E_{t=0} + I_{t=0} = g(x)$$

$$(\hat{E}+\hat{I})(x,s) = \hat{A}e^{-sx} + \int_0^x e^{-sx+sy} f(y)\,dy$$
$$(\hat{E}-\hat{I})(x,s) = \hat{B}e^{sx} + \int_0^x e^{s(x-y)} g(y)\,dy$$

$$(\hat{E}+\hat{I})(0,s) = \hat{A}$$
$$(\hat{E}-\hat{I})(0,s) = \hat{B}$$

$$\hat{E}(0,s) = \frac{\hat{A}+\hat{B}}{2}$$
$$\hat{I}(0,s) = \frac{\hat{A}-\hat{B}}{2}$$

$$\partial_t I_{x=0} + L^{-1} E_{x=0} = 0$$

$$s\hat{I}(0,s) + L^{-1}\hat{E}(0,s) = I(0,0)$$

$$(E+I)(x,t) = \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} e^{st}\left(\hat{A}e^{-sx} + \int_0^x e^{-s(x-y)} f(y)\,dy\right) ds$$

$$= \underbrace{A(t-x)}_{t>x} + \int_0^x \delta(t-x+y)\, f(y)\,dy \qquad 0 < x-t < x$$

$$= \underbrace{A(t-x)}_{t>x} + \begin{cases} f(x-t) & x > t > 0 \\ 0 & \text{otherwise} \end{cases}$$

construct H corresponding to ~~the~~ an inductance coupled to trans. line.

$$\left.\begin{array}{l}\partial_x E + \partial_t I = 0\\ \partial_x I + \partial_t E = 0\\ E_{x=0} = -L\,\partial_t I_{x=0}\end{array}\right\}$$ wave equation | equations of motion

~~[illegible]~~ Your aim is to construct a Hilbert space with one param. unitary group which solves or yields the solution to these equation. ~~[illegible]~~ ~~[illegible]~~ (Solving equations of motion means producing a space of ~~[illegible]~~ states with time flow). $\int_0^\infty e^{i\lambda t} f(t)\,dt$ $s=-i\lambda$

perform F.T. in time $\hat{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$ $=$

~~$\int_0^\infty e^{i\lambda t} f(t)\,dt$ ~~

$$f(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}\hat{f}(s)\,ds = \int_{-\infty}^{\infty} e^{-i\lambda t}\hat{f}(-i\lambda)\frac{d\lambda}{2\pi}$$

$$\partial_x E + sI = 0 \qquad (\partial_x + s)(E+I) = 0 \qquad E+I = Ae^{-sx}$$
$$\partial_x I + sE = 0 \qquad (\partial_x - s)(E-I) = 0 \qquad E-I = Be^{sx}$$
$$E_{x=0} = -Ls\,I_{x=0} \qquad \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}E_0\\ I_0\end{pmatrix} = \begin{pmatrix}A\\ B\end{pmatrix}$$

$$S = \frac{A}{B} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}(-Ls) = \frac{-Ls+1}{-Ls-1} = \frac{Ls-1}{Ls+1}.$$

So you have transformed the search for a Hilbert space consisting of ~~[illegible]~~ functions ~~$E(x), I(x)$~~ $\begin{pmatrix}E(x)\\ I(x)\end{pmatrix}$ to ~~[illegible]~~ constructing a space of functions of λ. Specifically a state is a pair $\hat{A}(-i\lambda), \hat{B}(-i\lambda)$ such that $S(\lambda)B(\lambda) = A(\lambda)$. Get two rep of $\hat{H}$ as $L^2(\mathbb{R}, \frac{d\lambda}{2\pi})$ namely incoming rep. $\begin{pmatrix}A\\ B\end{pmatrix} \longmapsto B$ and outgoing rep $\begin{pmatrix}A\\ B\end{pmatrix} \longmapsto A$. Why the terminology. $(E+I)(x,t) =$

$$\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi} e^{-i\lambda t} A(\lambda) e^{-(-i\lambda)x} = \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi} e^{i\lambda(x-t)} A(\lambda) = f(x-t)$$

~~[illegible]~~

$$\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi} e^{-i\lambda t} B(\lambda) e^{(-i\lambda)x} = \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi} e^{-i\lambda(x+t)} B(\lambda) = g(x+t)$$

So now have the frequency picture of H as $L^2(\mathbb{R}, \frac{d\lambda}{2\pi})$ in two ways related by $S(-i\lambda)$. Need $H = H^- \oplus X \oplus H^+$. YOUR AIM: You want ~~to see~~ the geometric picture of H split into three parts and how they are attached together. The latter involves glueing partial hermitian operators together.

Let's mimic ~~what~~ the discrete case. We use the incoming representation, which means $H = L^2(\mathbb{R}, \frac{d\omega}{2\pi})$, the incoming subspace is the negative Hardy space i.e. L^2 functions analytic in LHP, and the outgoing space is SH^+, where H^+ is the positive Hardy space. If correct $H^- \cap SH^+$ should be 1 diml. Recall $S(-i\lambda)$

$$= \frac{-i\lambda L - 1}{-i\lambda L + 1} = \frac{\lambda L - i}{\lambda L + i}.$$

It's possible you have half planes in the wrong place. ~~What ...~~ Ask whether $SH^+ \subset H^+$? does it preserve analyticity in the UHP yes. pole occurs at $\lambda L + i = 0$ $\lambda = -L^{-1}i$ $X = H^+ \ominus SH^+$ should be generated by the evaluator as $L^{-1}i$ which is

$$\frac{i}{\lambda - (\overline{L^{-1}i})} = \frac{i}{\lambda + L^{-1}i}$$

What else? ~~want~~

What seems to be necessary is that ~~not~~ you have $H = H^- \oplus X \oplus SH^+$, but you also need the graph of $T = \lambda$ inside $H \oplus H$. If you ~~have~~ have the subspaces $\mathcal{D}_{T^-} \oplus \mathcal{D}_{T_X} \oplus \mathcal{D}_{T_{SH^+}}$ of $H^{\oplus 2}$ you need to ~~make~~ increase it suitably to get the graph of a self-adjoint operator

Example. Look at $\mathcal{D}_\lambda$ on $H^+ = H^{2+}(\mathbb{R}, \frac{d\lambda}{2\pi})$

You have $\begin{pmatrix} f \\ \lambda f \end{pmatrix} \in (H^+)^2$. These are $f(\lambda)$ anal. UHP $\ni \int (1+|\lambda|^2)|f|^2 \frac{d\lambda}{2\pi} < \infty$.

Adjoint graph should be 1-codim more.

$$\begin{pmatrix} g \\ g' \end{pmatrix} \in H^{\oplus 2} \qquad (\lambda f, g) = (f, g') \qquad \forall\, f \in \mathcal{D}_\lambda$$

Ultimately you look at $\lambda \pm i$

Study $A = \frac{1}{i}\partial_x$ on $L^2(\mathbb{R}_{>0})$, same as mult by λ on Hardy space. $H^2_+(\mathbb{R}, \frac{d\lambda}{2\pi})$. Consider $H^{\oplus 2}$

$$\mathcal{D}_A \xhookrightarrow{\begin{pmatrix} 1 \\ A \end{pmatrix}} H^{\oplus 2} \qquad \Gamma_A^\perp = \left\{ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \middle| (\xi_1, f) + (\xi_2, Af) = 0 \ \ \forall f \in \mathcal{D}_A \right\}$$

~~Define $A^* = \{\xi_2 \in H \mid \ldots\}$~~

A^* has domain $\{\xi \in H \mid \exists\, \xi' \in H$ such that $(x, \xi') = (Ax, \xi)$ $\forall x \in \mathcal{D}_A\}$

$$\Gamma_{A^*} = \left\{ \begin{pmatrix} \xi \\ \xi' \end{pmatrix} \middle| \left\langle \begin{pmatrix} \xi \\ \xi' \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ Ax \end{pmatrix} \right\rangle = 0 \right\} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Gamma_A^\perp$$

Let's work with the Hardy space. You have

$$\mathcal{D}_A = \left\{ f \in H_+ \middle| \int (1 + |\lambda|^2) |f|^2 \frac{d\lambda}{2\pi} < \infty \right\}.$$

$$\mathcal{D}_A \overset{\lambda + i}{\underset{\lambda - i}{\rightrightarrows}} H_+ \quad \text{isometries.}$$

So go back to H use $L^2(\mathbb{R}, \frac{d\omega}{2\pi})$ model with usual H^- for incoming space and SH^+ for the outgoing spaces although you may prefer to use $A =$ mult by ω, it seems you have to replace A by $u = \frac{1 + iA}{1 - iA} = \frac{-A + i}{A + i}$. It seems that the graph C.T. technique for self-adjoint operators is not affected by whether we use a ~~[illegible]~~ continuous or discrete ~~[illegible]~~ transmission line.

Let's analyze the wave equations using L.T.

$$\partial_x E + \partial_t I = 0 \qquad \text{Put } \hat{E}(x,s) = \int_0^\infty e^{-st} E(x,t)\,dt \quad \hat{I}$$

$$\partial_x I + \partial_t E = 0 \qquad (\partial_x + s)(\hat{E} + \hat{I}) = E(x,0) + I(x,0)$$

$$E_{x=0} = -L\,\partial_t I_{x=0} \qquad (\partial_x - s)(\hat{E} - \hat{I}) = (E - I)(x,0)$$

$$\partial_t I(0,t) = -L^{-1} E(0,t) \qquad s\hat{I}(0,s) = -L^{-1}\hat{E}(0,s) + \hat{I}(0,0)$$

$$(\partial_x + s)(\hat{E} + \hat{I}) = (E+I)(x,0)$$

$$\hat{E} + \hat{I} = e^{-xs}\int_0^x e^{ys}(E+I)(y,0)\,dy + e^{-xs}(\hat{E}+\hat{I})(0,s)$$

~~How do we ... At some point you ...~~

General case. Connect a LC 1-port to a transmission line. Again use frequency analysis, time dependence $e^{st} = e^{-i\omega t}$.

$$(E+I)(x,s) = A(s)\,e^{-xs} = A(-i\omega)\,e^{+i\omega x}$$

$$(E-I)(x,s) = B(s)\,e^{xs} = B(-i\omega)\,e^{-i\omega x}$$

$$(E+I)(x,t) = \int \frac{d\omega}{2\pi} A(-i\omega)\,e^{i\omega(x-t)} = f(x-t)$$

$$(E-I)(x,t) = \int \frac{d\omega}{2\pi} B(-i\omega)\,e^{-i\omega(x+t)} = g(x+t)$$

use incoming rep. States are $B(-i\omega) = \phi(\omega)$ L^2 norm.

$\mu^t = e^{-i\omega t}$. Solve as before to get $S = \frac{A}{B} = \frac{Z-1}{Z+1}$

Go back: The point is that these LC circuit + trans. line are "real". So if we have 1-port with imped. $Z(-i\omega)$, then

You want to consider a scattering situation ~~[illegible]~~ with continuous time. So I start with an X, β an operator on X $\ni$ $\mathrm{Im}(\beta) = \frac{\beta - \beta^*}{2i} \leq 0$, and $\mathrm{Im}(\beta)$ should have rank 1. Then I get

$$H = L^2(\mathbb{R}, \mu) \frac{d\omega}{2\pi} \qquad \mu(\omega) = \frac{i}{\omega - \beta^*} - \frac{i}{\omega - \beta} = \frac{1}{\omega - \beta^*}(-2\mathrm{Im}\beta)\frac{1}{\omega - \beta}$$

together with incoming + outgoing reprs.

$$L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \xleftarrow[\left(\sqrt{-2\mathrm{Im}\beta}\,\frac{1}{\omega - \beta^*}\right)^*]{\overset{in}{\sim}} H \xrightarrow[\left(\sqrt{-2\mathrm{Im}\beta}\,\frac{1}{\omega - \beta}\right)]{\overset{out}{\sim}} L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right)$$

scattering is approx. $(-2\mathrm{Im}\beta)^{1/2} \frac{1}{\omega - \beta^*} (\omega - \beta)(-2\mathrm{Im}\beta)^{-1/2}$

$$f \in H = L^2\left(\mathbb{R}, \mu \frac{d\omega}{2\pi}\right) \qquad \|f\|^2 = \int \left(f, \frac{1}{\omega - \beta^*}(-2\mathrm{Im}\beta)\frac{1}{\omega - \beta} f\right)\frac{d\omega}{2\pi}$$

$$= \int \left\| \sqrt{-2\mathrm{Im}\beta}\, \frac{1}{\omega - \beta} f \right\|^2 \frac{d\omega}{2\pi}$$

The scattering is

$$S(\omega) = \boxed{(-2\mathrm{Im}\beta)^{1/2} \frac{1}{\omega - \beta} (\omega - \beta^*)(-2\mathrm{Im}\beta)^{-1/2}}$$

Note β has spect. in LHP so S analytic in UHP.

Here's what you want. You want to take

$$H = L^2\left(\mathbb{R}, \frac{d\omega}{2\pi}\right) \qquad e^{-i\alpha t} = \text{mult by } e^{-i\omega t}$$

Next you take an $S(\omega) = c \prod_{j=1}^{n} \frac{\omega - \omega_j}{\omega - \omega_j^*}$

Then put $X = H^+ \cap SH^-$ $\qquad H = H^- \oplus X \oplus SH^+$

$X \xrightarrow{\sim} H^+/SH^+$. Need to check ~~to~~ g

$j^* e^{-i\alpha t} j = e^{-i\beta t}$ for $t \geq 0$

OKAY. So assume ~~$g^* e^{-i\omega A} g = e^{i\omega B}$ ... $\omega > 0$.~~

$$g^* u^t_{t \geq 0} g = g^t. \qquad u^t = e^{-iAt} \quad \text{all } t \quad A \text{ s.a.} \qquad g^t = e^{-iBt} \quad t \geq 0.$$

~~...~~ $sp(B) \subset LHP$

$Im(B) < 0$

Then what?

You want a measure on the line.

$$(x', e^{-iAt} x) = \begin{matrix} (x', e^{-iBt} x) & t \geq 0 \\ (x', e^{-iB^*t} x) & t \leq 0 \end{matrix}$$

You want

$$\int e^{-i\omega t} \mu \frac{d\omega}{2\pi} = \begin{cases} e^{-iBt} & t \geq 0 \\ e^{-iB^*t} & t \leq 0 \end{cases}$$

$$\mu = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \begin{cases} e^{-iBt} & t \geq 0 \\ e^{-iB^*t} & t \leq 0 \end{cases}$$

$$g^* e^{-iAt} g = e^{-iBt} \quad t \geq 0$$

$$\Rightarrow g^* \frac{1}{\omega - A} g = \frac{1}{\omega - B}$$

$$\int_0^\infty e^{i\omega t} e^{-iAt} dt = \frac{1}{-i\omega + iA}$$

for $Im(\omega) > 0$

but then maybe for other ω by analytic cont.

$$= \frac{1}{i(B-\omega)} + \frac{1}{i(\omega - B^*)}$$

$$= \frac{i}{\omega - B} + \frac{-i}{\omega - B^*} = \frac{1}{\omega - B^*}\left(-i(\omega - B) + i(\omega - B^*)\right)\frac{1}{\omega - B}$$

$$= \frac{1}{\omega - B^*} \underbrace{i(B - B^*)}_{-2\,Im(B)} \frac{1}{\omega - B}$$

assume ≥ 0. $\qquad -i\omega = s$

---

let's move back to $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$

$$z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda + \varepsilon}{\lambda + i}$$

$$X \underset{b = i\varepsilon - A}{\overset{a = i\varepsilon + A}{\rightrightarrows}} Y$$

$$\frac{1}{-i\omega + iB} = \frac{i}{\omega - B}$$

$$\int_0^\infty e^{+i\omega t} e^{-iBt} dt = \frac{1}{-i\omega + iB}$$

Continuous case - how do I get isom. embedding.

Start with β on X $\quad -2\,\mathrm{Im}(\beta) = (-2)\frac{\beta-\beta^*}{2i} = |\xi\rangle\langle\xi|$ Ass.

~~Then~~ You want to assoc. to x a function $x(\omega)$

$x(\omega) = (\xi | \frac{1}{\omega-\beta} x)$. What's happening?

$$\int \frac{d\omega}{2\pi} |x(\omega)|^2 = \int \frac{d\omega}{2\pi} \left(\frac{1}{\omega-\beta}x, \xi\right)\left(\xi, \frac{1}{\omega-\beta}x\right)$$

$$= \int \frac{d\omega}{2\pi} \left(\frac{1}{\omega-\beta}x, \underbrace{(-2)\frac{\beta-\beta^*}{2i}}_{i(\beta-\beta^*)} \frac{1}{\omega-\beta}x\right)$$

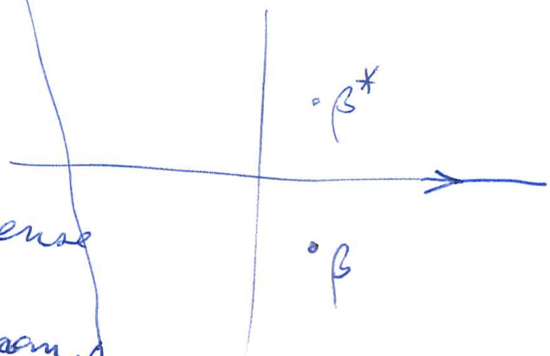
$$= \int \frac{d\omega}{2\pi} \left(x, \frac{1}{\omega-\beta^*}\, i(\beta-\beta^*) \frac{1}{\omega-\beta} x\right)$$

$$\frac{i}{\omega-\beta} + \frac{-i}{\omega-\beta^*} = \frac{1}{\omega-\beta^*}\left((-i)(\omega-\beta) + i(\omega-\beta^*)\right)\frac{1}{\omega-\beta}$$

$$= \frac{1}{\omega-\beta^*}\, i(\beta-\beta^*)\frac{1}{\omega-\beta}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(x, \left(\frac{+i}{\omega-\beta} + \frac{-i}{\omega-\beta^*}\right) x\right)$$

?



It seems the only way to make sense of this is to keep the quadratic denom. so that you can add the semi-circle  and use contour

Review a little. Given β on X with spectrum in LHP, $\mathrm{Im}(\beta) = \dfrac{\beta-\beta^*}{2i} \le 0$

then $x \longmapsto \tilde{x}(\omega) = \left(i(\beta-\beta^*)\right)^{1/2}\dfrac{1}{\omega-\beta}x$ is an isometric embedding of X in $L^2(\mathbb{R})$.

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\tilde{x}(\omega),\tilde{x}(\omega)\right) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(x,\frac{1}{\omega-\beta^*}\,i(\beta-\beta^*)\frac{1}{\omega-\beta}x\right)$$

$$= \int = \frac{1}{2\pi}\left(x,\ 2\pi i\ i(\beta-\beta^*)\frac{1}{\beta^*-\beta}x\right)$$

$$= \|x\|^2$$

Consider $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ as usual.

$$\begin{tikzcd}
 & & Y^- \arrow[d] & & \\
0 \arrow[r] & X \arrow[r, "az-b"] & Y \arrow[r] \arrow[d, "b^*"] & Y/(az-b) \arrow[r] & 0 \\
 & & X & &
\end{tikzcd}$$

We have here an example of a quasi-det. of the id.

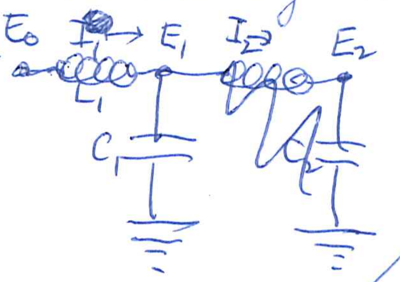
$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "az-b"] & Y \arrow[r] \arrow[d] & Z \arrow[r] & 0 \\
0 & X' \arrow[l] & Y' \arrow[l, "b^*"'] & Z' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

There is an induced map $Z \to Z'$ when $b^*(az-b)$ is an isom.

$$y \longmapsto y - (az-b)\left(b^*(az-b)\right)^{-1}b^*y$$

Let's try something naive

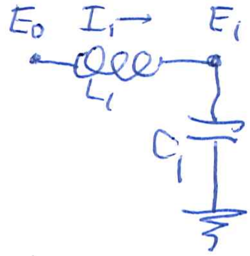
Study



$$E_0 = L_1 s I_1 + E_1$$
$$I_1 = C_1 s E_1 + I_2$$
$$E_1 = L_2 s I_2 \; E_2$$

$$I_{n-1} = C_{n-1} s E_{n-1} + I_n$$
$$E_{n-1} = L_n s I_n + E_n$$
$$I_n = C_n s E_n$$

Start again



$$E_0 = L_1 s I_1 + E_1$$
$$I_1 = C_1 s E_1$$

Can you make this into a J matrix. First replace $E_0, I_1, E_1, I_2, E_2, \dots, I_n, E_n$ by $u_0, iu_1, \dots, i^{2n} u_{2n}$. replace $L_1, C_1, \dots, L_n, C_n$ by $\alpha_1, \dots, \alpha_n > 0$

$$u_0 = \alpha_1 s i u_1 + {}^{(-1)} u_2$$
$$u_1 = \alpha_2 i s u_2 + {}^{(-1)} u_3$$

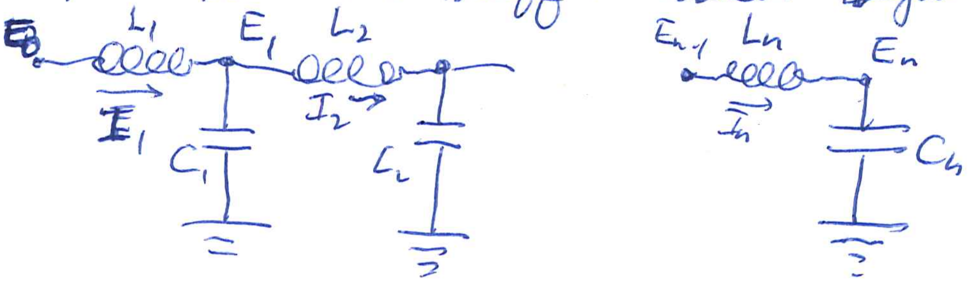
$$u_0 - \alpha_1 i s u_1 + u_2 = 0$$
$$u_0 + u_2 = \alpha_1 \omega u_1$$
$$u_1 + u_3 = \alpha_2 \omega u_2$$

$$\omega \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_n \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & & \\ & 1 & & 1 & \\ & & \ddots & & \ddots \\ & & & & 1 \\ & & & 1 & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ \vdots \\ u_n \end{pmatrix}$$

---

back to symplectic stuff. We start with a polarized real Hilbert space H and get a family of quadratic form $Q_s(\xi) = s\|\xi_+\|^2 + s^{-1}\|\xi_-\|^2$. Alternative work in symp. space $H \oplus H^*$ and the Lag. subspace Γ_{Q_s}. We have a symp. quotient defined by the isotropic space $W \oplus V^\circ$ annihilator $V \oplus W^\circ$ and symp. quotient $V/W \oplus \underbrace{W^\circ/V^\circ}_{(V/W)^*}$

I have to make a serious attempt to connect LC 1-ports to other stuff. ~~Bega~~ Begin with ladder.



graph has $n+2$ vertices, $2n$ edges, $n-1$ loops.

$$v - e + l = \cancel{n}+2 - \cancel{2n} + \cancel{n} - 1 = 1$$

equations of motion with time dependence e^{st} yield ~~eqn~~ eigenvector equations

$$E_0 = L_1 s I_1 + E_1$$
$$I_1 = C_1 s E_1 + I_2$$

$2n$ equations in $2n+1$ variables $E_0, I_1, E_1, \dots, I_n, E_n$ ~~[illegible]~~ K-module of type $\mathcal{O}(2n)$. ~~[illegible]~~

$$E_{n-1} = L_n s I_n + E_n$$
$$I_n = C_n s E_n$$

How does this relate to $H = H^+ \oplus H^-$

$$\begin{array}{ccc} V & \hookrightarrow & H \\ \downarrow & & \\ V/W & & \end{array}$$

Should I be doing something symplectically?

<u>Idea:</u> Take a symplectic vector space $V \oplus V^*$

$$\omega\left(\binom{v}{\lambda}, \binom{v'}{\lambda'}\right) = (\lambda, v') - (\lambda', v)$$

Note that V, V^* are Lagrangian; ask when $\Gamma_A = \binom{1}{A} V$ ~~A~~ $\subset V \oplus V^*$ is ~~Lag~~ isotr $(Av, v') = (Av', v)$, i.e. $A : V \to V^*$ is symm. quadratic form. Ask about symplectic quotients.

$$(E - I)(x,t) = \underset{t+x>0}{B(t+x)} + \int_0^{} \delta(t+x-y)\, g(y)\, dy \quad 0 < x+t < x$$

$$= \underset{t+x>0.}{B(x+t)} + \begin{cases} g(x+t) & -x < t < 0 \\ 0 & \text{otherwise} \end{cases}$$

Let's try a different approach to solving

$$\partial_x E + \partial_t I = 0 \qquad E(x,t) \qquad x \geq 0 \qquad \forall t$$

$$\partial_x I + \partial_t E = 0 \qquad I(x,t)$$

$$E_{x=0} = -L\, \partial_t I_{x=0}$$

Look for inductance coupled to resistance.

$$(\partial_x + s)(\hat{E} + \hat{I}) = 0 \qquad \hat{E} + \hat{I} = Ae^{-sx}$$

$$(\partial_x - s)(\hat{E} - \hat{I}) = 0 \qquad \hat{E} - \hat{I} = Be^{+sx}$$

$$E + I = Ae^{-sx+st}$$

$$E - I = Be^{sx+st}$$

$$E_{x=0} + I_{x=0} = Ae^{st}$$

$$E_{x=0} - I_{x=0} = Be^{st}$$

$$E_{x=0} = \tfrac{1}{2}(A+B)e^{st}$$

$$I_{x=0} = \tfrac{1}{2}(A-B)e^{st}$$

$$-L\,\partial_t I_{x=0} = -L \cdot \tfrac{1}{2}(A-B)\, s\, e^{st} = \tfrac{1}{2}(A+B)\, e^{st}$$

$$Ls(B-A) = A+B$$

$$Ls = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \frac{A}{B} \qquad \frac{A}{B} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} Ls = \frac{Ls-1}{Ls+1}$$

couple finite LC circuit to a trans. line.

$$H = H^- \oplus X \oplus H^+$$

look on the level of the eigenvectors.

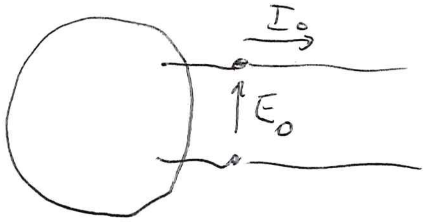
transmission line yields eigenvectors

$$\partial_x E + sI = 0 \qquad \partial_x (E+I) + \partial_t (E+I) = 0$$

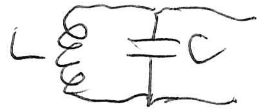
$$\partial_x I + sE = 0 \qquad (E+I)(x,t) = f(x-t$$

$$(\partial_x + \cancel{\partial_x} s)(E+I) = 0 \qquad (E+I)(x) = Ae^{-sx}$$

$$(\partial_x - s)(E-I) = 0 \qquad (E-I)(x) = Be^{sx}$$

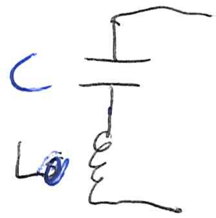


$$Z(s) = \frac{-E_0}{I_0} \qquad E$$



$$Z = \frac{1}{\frac{1}{Ls} + Cs} \simeq \frac{Ls}{LCs^2+1}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$



$$Z = Ls + \frac{1}{Cs} = \frac{LCs^2+1}{Cs}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix}$$

$$S = \frac{A}{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -Z \end{pmatrix} = \frac{-Z+1}{-Z-1} = \frac{Z-1}{Z+1}$$

$$= \frac{LCs^2 - Cs + 1}{LCs^2 + Cs + 1}$$

$$\frac{Cs \pm \sqrt{C^2s^2 - 4LCs^2}}{2LCs^2}$$

$$\frac{C \pm \sqrt{C^2 - 4LC}}{2LCs}$$

There should be a Hilbert space with 1-par. unitary group assoc to LC port coupled to transmission line. You should be able to describe it by glueing as well as via the incoming + outgoing representations. Let's begin by trying to obtain the latter. These representations involve $L^2(\mathbb{R})$

Discuss the example. $Z = Ls$ $\quad S = \frac{A}{B} = \frac{Z-1}{Z+1} = \frac{Ls-1}{Ls+1}$

$s = -i\lambda$ so that $Re(s) > 0 \longleftrightarrow Im(\lambda) > 0$.

$$\begin{cases} (\partial_x + s)(E+I) = 0 & E+I = Ae^{-xs} \\ (\partial_x - s)(E-I) = 0 & E-I = Be^{xs} \\ E_{x=0} = -Ls\, I_{x=0} & \end{cases} \qquad \frac{A}{B} = \frac{Ls-1}{Ls+1}$$

$$\frac{A}{B} = \left(\frac{E+I}{E-I}\right)_{x=0} = \begin{pmatrix} 1 & 1 \\ +1 & -1 \end{pmatrix}(-Ls) = \frac{+Ls - 1}{+Ls+1}$$

this is the eigenvector equation. It gives a holom. line bundle over the s plane - actually $\mathbb{P}^1$. Somewhere there is the Hilbert space H obtained by glueing, and two representations of it

$$L^2(\mathbb{R}) \longrightarrow H \longleftarrow L^2(\mathbb{R})$$

related by S

Start again: You have a LC 1-port conn. to a tr. line. This should yield an H with 1-parameter unitary group u^t. You should be able to describe H by glueing: $H = H^- \oplus X \oplus H^+$, where u^t is a kind of shift $(u_t f)(x) = f(x-t)$ on H^+ and $(u_t g)(x) = g(x+t)$ in H^-. You should also have incoming + outgoing representations

$$L^2(\mathbb{R}) \longrightarrow H \longleftarrow L^2(\mathbb{R})$$

by functions of the frequency parameter $\lambda = +is$

formulas

$$\left\{\begin{array}{l}(\partial_x + s)(E+I) = 0 \\ (\partial_x - s)(E-I) = 0 \\ E_{x=0} = -LsI_{x=0}\end{array}\right. \qquad \left\{\begin{array}{l} E+I = A(s)e^{-xs} \\ E-I = B(s)e^{xs} \\ \delta(s) = \dfrac{A(s)}{B(s)} = \begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}(-Ls) = \dfrac{+Ls-1}{+Ls+1}\end{array}\right.$$

eigenv. eqns. solution to the eigenvector equation.

So for each s you get a 1-diml space of solutions, namely $\mathbb{C}\begin{pmatrix}Ls-1\\ Ls+1\end{pmatrix}$. Get a holom. line bundle over $\mathbb{P}^1$ type $\mathcal{O}(1)$. ~~Next I want the end~~

Next you want the incoming + outgoing repns.

Point An elt ξ of H is equivalent to the trajectory at ξ. $\xi = \cancel{(E(x,t), I(x,t))}$ $\begin{pmatrix}(E+I)(x,t)\\ (E-I)(x,t)\end{pmatrix} = \begin{pmatrix}f(x-t)\\ g(x+t)\end{pmatrix}$

so the outgoing repn is $\begin{pmatrix}E\\ I\end{pmatrix}(x,t) \longmapsto (E+I)(x,t) = f(x-t)$

time dependent should be e^{st} $A(s)e^{-xs}$ so that

$$f(x-t) = \int_{-i\infty}^{i\infty} A(s)e^{-s(x-t)}\frac{ds}{2\pi i} = \int_{\infty}^{-\infty} A(-i\lambda)\, e^{i\lambda(x-t)}\frac{d\lambda}{2\pi}$$

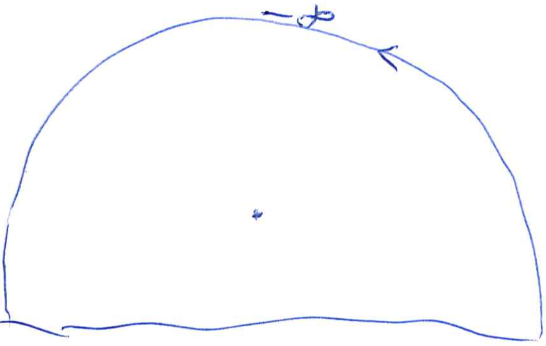
So for you get that H is roughly pairs $\begin{pmatrix}\hat{f}(\lambda)\\ \hat{g}(\lambda)\end{pmatrix}$ in $L^2(\mathbb{R})$ $\ni$ $S\hat{g} = \hat{f}$.

Program. First analyze $H^+ \subset L^2(\mathbb{R})$. These are certain ~~analytic~~ functions $f(\lambda)$ on the line which admit analytic extensions to the UHP s.a. op mult by λ, so you don't know what $f(\lambda)$ are in H^+. You want $f(\lambda) = \int_0^\infty e^{i\lambda x}\phi(x)\,dx$

$$|f(\lambda)|^2 \le \left(\int_0^\infty |e^{i\lambda x}|^2 dx\right)\|\phi\|^2 \qquad \int_0^\infty e^{-2\,\mathrm{Im}(\lambda)x}dx = \frac{1}{2\,\mathrm{Im}(\lambda)}$$

$|f(\lambda)| \leqslant \|f\| \left(\frac{1}{2\,\mathrm{Im}(\lambda)}\right)^{1/2}$. Point ~~eval~~ evaluators?

$$\int_{-\infty}^{\infty} \overline{\frac{1}{\lambda+i}}\, f(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \frac{f(\lambda)}{\lambda - i}\, d\lambda = 2\pi i f(i)$$



$$\int_{0}^{\pi} \left|f(Re^{i\theta})\right| \, d\theta$$

OKAY if f bdd

spinors. $\quad x' = \frac{ax+b}{cx+d} \qquad dx' = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2}\,dx = \frac{dx}{(cx+d)^2}$

Thus define $f(x)\,dx^{1/2} \longmapsto f\left(\frac{ax+b}{cx+d}\right)\frac{1}{cx+d}\,dx^{1/2}$
and you get a unitary rep of $SL_2(\mathbb{R})$.

$$z \longmapsto z' = \frac{\bar{d}z + \bar{c}}{cz+d}$$

$$dz' = \frac{(cz+d)\,\bar{d} - (\bar{d}z+\bar{c})\,c}{(cz+d)^2}\,dz = \frac{dz}{(cz+d)^2}$$

$z = re^{i\theta} \qquad \log z = \log r + i\theta \qquad \frac{dz}{z} = \frac{dr}{r} + i\,d\theta$

$$\therefore d\theta = \mathrm{Im}\left(\frac{dz}{z}\right) \longmapsto \mathrm{Im}\left(\frac{cz+d}{az+b}\,\frac{dz}{(cz+d)^2}\right) = \mathrm{Im}\left(\frac{dz}{(\bar{d}z+\bar{c})(cz+d)}\right)$$

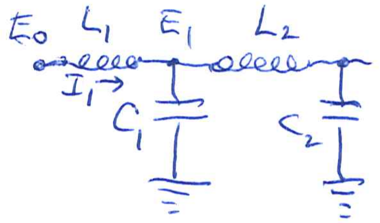
$$= \mathrm{Im}\left(\frac{1}{(\bar{d}+\bar{c}\bar{z})(cz+d)}\,\frac{dz}{z}\right) = \frac{1}{|cz+d|^2}\,d\theta$$

$$f(z)\,d\theta^{1/2} \longmapsto f\left(\frac{\bar{d}z+\bar{c}}{cz+d}\right)\frac{1}{|cz+d|}\,d\theta^{1/2}$$

NO way.

OK.

different approach. LC network coupled to a transmission line.



$$Z = L_1 s + \cfrac{1}{C_1 s + \cfrac{1}{L_2 s +}}$$

$$\begin{pmatrix} L_1 s & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} C_1 s & 1 \\ 1 & 0 \end{pmatrix}$$

$$E_0 - E_1 = L_1 s I_1$$

$$I_1 - I_2 = C_1 s E_1$$

$$\begin{pmatrix} E_0 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & L_1 s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & L_1 s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C_1 s & 1 \end{pmatrix}}^{\parallel}\begin{pmatrix} E_1 \\ I_2 \end{pmatrix}$$

$$-\partial_x E = l s I \qquad \partial_x E + \partial_t I = 0 \qquad l = c = 1$$

$$-\partial_x I = c s E \qquad \partial_x I + \partial_t E = 0$$

$$\partial_x^2 E - \partial_t^2 E = 0.$$

You propose to couple a transmission line to a partial hermitian operator of type $O(n)$.

To solve

$$\partial_x E + s I = 0$$
$$\partial_x I + s E = 0 \qquad \text{for } x \geq 0.$$

$$\partial_x (E + I) + s(E + I) = 0 \qquad E + I = A e^{-sx} \qquad s = -i\lambda$$

$$\partial_x (E - I) - s(E - I) = 0 \qquad E - I = B e^{sx}$$

You want $Z_{x=0}(s) = \dfrac{E_{x=0}}{I_{x=0}} = \dfrac{A+B}{A-B}$

Need decaying solution. For each s there is a 2 dim'l space of eigenfunctions $\begin{pmatrix} A \\ B \end{pmatrix}$, and there is a unique line where the eigenfunction decays. namely e^{-sx}. so like unit resistance

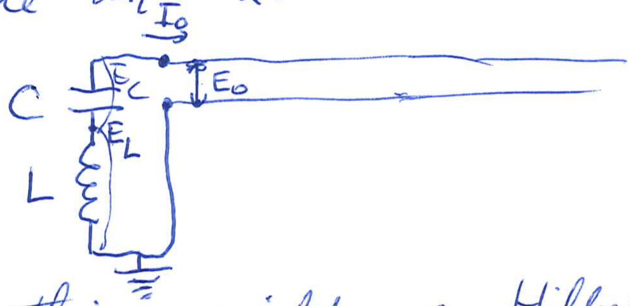
What is going on? You are trying to get a different type of Hilbert space ~~containing~~ where there is a 1-parameter unitary group instead of a unitary operator.

There should be a construction analogous to $\cdots \oplus u^{-1}V^- \oplus aX \oplus V^+ \oplus uV^+ \oplus$
How? Instead of $V^+ \oplus uV^+ \oplus \cdots$ you want $L^2(0,\infty)$ and $\frac{1}{i}\partial_x$ on the other side you want $L^2(-\infty,0)$ and $\frac{1}{i}\partial_x$. You need to attach these things together. You can put

$H = L^2(R_-) \oplus X \oplus L^2(R_+)$ but then you must give
$\mathcal{D}$ = domain of A and the maps $\mathcal{D} \underset{A}{\overset{\varepsilon}{\rightrightarrows}} H$.

First you need $L^2(R_+) \underset{\frac{1}{i}\partial_x}{\overset{\varepsilon}{\leftleftarrows}} \mathcal{D}_+$. What's the point here form CT. In general $D_A \xhookrightarrow{(\varepsilon,A)} H \oplus H \hookleftarrow D_{A^*}$

Couple an LC circuit to a transmission line.



this thing yields a Hilbert space (states with the energy norm) and time evolution, a 1-param. unitary gp. Should have form $H = H^- \oplus X \oplus H^+$. Let's look at eigenvectors.

$$\left.\begin{aligned} \partial_x E + \partial_t I &= 0 \\ \partial_x I + \partial_t E &= 0 \end{aligned}\right\} \text{ wave equation for the transmission line.}$$

$$E_C + E_L = E_{x=0}$$

$$E_L = -L\frac{dI_0}{dt} \qquad C\frac{dE_C}{dt} = -I_0$$

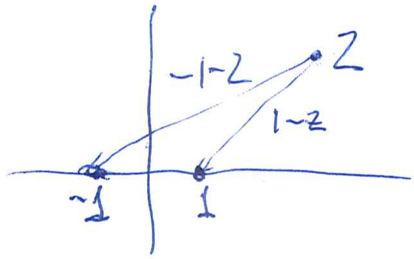
Assume time dependence e^{st}

$$\partial_x E + sI = 0 \qquad (\partial_x + s)(E+I) = 0 \qquad E_x + I_x = Ae^{-sx}$$

$$\partial_x I + sE = 0 \qquad (\partial_x - s)(E-I) = 0 \qquad E_x - I_x = Be^{sx}$$

$$E_L = -Ls I_0 \qquad CsE_C = -I_0$$

$$E_0 = E_C + E_L = -\left(\frac{1}{Cs} + Ls\right) I_0$$



$$E_0 + I_0 = A \qquad E_0 - I_0 = B$$

$$S = \frac{A}{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \left(-\left(\frac{LCs^2+1}{Cs} \right) \right)$$

$$= \frac{\left(-\frac{LCs^2+1}{Cs}\right) + 1}{\left(-\frac{LCs^2+1}{Cs}\right) - 1} = \frac{LCs^2 - Cs + 1}{LCs^2 + Cs + 1}$$

Now what you are doing, should work in general, namely if the LC circuit has impedance $Z = \frac{E}{I}$ then you want $-Z = \frac{E_0}{I_0}$

hence $\frac{A}{B} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (-Z) = \frac{Z-1}{Z+1}$

The main point I think is that this has very little to do with one parameter unitary groups. Ultimately you take your $Z = \frac{E}{I}$ for the LC circuit and the scattering is $S = \frac{Z-1}{Z+1}$ or $\boxed{\frac{1-Z}{1+Z}}$ if we change the sign of B. Thus $Re(Z) > 0$ corresp to $|S| < 1$.

The next business concerns representing elts of Y by functions. There are tricky points. You have H constructed in theory, but in practice are you using εX, where are $V^\pm$?

$$\Big\|\sum_{n\geq 0} u^n x_n\Big\|^2 - \Big\|\sum_{n\geq 0} g^n x_n\Big\|^2 = \Big\|\sum_{n\geq 0} u^n x_{n+1}\Big\|^2 - \Big\|g\sum_{n\geq 0} g^n x_{n+1}\Big\|^2$$

$$\Big\|\sum_{n\geq 0} u^n x_{n+1}\Big\|^2 - \Big\|\sum_{n\geq 0} g^n x_{n+1}\Big\|^2 = \Big\|\sum_{n\geq 0} u^n x_{n+2}\Big\| - \Big\|g\sum_{n\geq 0} g^n x_{n+2}\Big\|^2$$

$$\therefore \Big\|\sum_{n\geq 0} u^n x_n\Big\|^2 = \Big\|\sum_{n\geq 0} g^n x_n\Big\|^2 + \Big\|(1-g^*g)^{1/2}\sum_{n\geq 0} g^n x_{n+1}\Big\|^2$$
$$+ \Big\|(1-g^*g)^{1/2}\sum_{n\geq 0} g^n x_{n+2}\Big\|^2 + \cdots$$

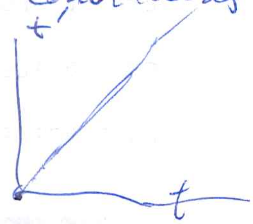
Anyway you could try the same

$$\Big\|\int_0^\infty u^t x(t)\,dt\Big\|^2$$

Yes it should now be possible to formally treat the ~~centen~~ one parameter unitary group case

Start with X, g^t $\qquad g^t$ 1-param. grp of contractions

$j: X \hookrightarrow H$, u^t, $\qquad j^* u^t j = g^t$



$$\Big\|\int_0^\infty u^t x_t\,dt\Big\|^2 = \int dt\,dt'\,(u^t x_t, u^{t'} x_{t'})$$

$$= \int_{t\geq t'} dt\,dt'\,(g^{t-t'} x_t, x_{t'}) + \int_{t\leq t'} dt\,dt'\,(x_t, g^{(t'-t)} x_{t'})$$

$$= \int_0^\infty dt' \int_0^\infty dt\,(g^t x_{t+t'}, x_{t'}) + \int_0^\infty dt \int_0^\infty dt'\,(x_t, g^{t'} x_{t'+t})$$

$$= \int_0^\infty dt' \Big(\int_0^\infty dt\, g^t x_{t+t'}, x_{t'}\Big) + \Big(x_{t'}, \int_0^\infty g^t x_{t+t'}\,dt'\Big)$$

$$\begin{matrix} b_1-\lambda & a_1 & \\ a_1 & b_2-\lambda & a_2 \\ & a_2 & \end{matrix}$$

$$(b_1-\lambda)u_1 + a_1 u_2 = 0 \qquad 0 = \lambda - b_1 - a_1\frac{u_2}{u_1}$$

$$a_1 u_1 + (b_2-\lambda)u_2 + a_2 u_3 = 0 \qquad a_1\frac{u_1}{u_2} = \lambda - b_2 - a_2\frac{u_3}{u_2}$$

$$\lambda - b_1 - \cfrac{a_1^2}{\lambda - b_2 - \cfrac{a_2^2}{\lambda - b_3 -}}$$

$$Z = L_1 s + \frac{1}{C_1 s +}\,\frac{1}{|L_2 s +}\,\frac{1}{|C_2 s +}$$

~~Markets~~ The problem: Given a partial herm. of $X \underset{A}{\overset{\varepsilon}{\rightleftarrows}} Y$ of type $O(n)$ I can complete it to a hermitian operator. Yesterday I started exploring the 1-parameter group version. Why not look at the X, γ^t version?

First review the discrete case. Given X a Hilbert space, γ a contraction operator on X. You look for $X \xrightarrow{j} H$, u $\quad j^* u^n j = \gamma^n \quad n \geq 0$.

$$\Big\|\sum_{n\geq 0} u^n x_n\Big\|^2 = \|x_0\|^2 + \Big(x_0, \sum_{n\geq 1}\gamma^n x_n\Big) + \Big(\sum_{n\geq 1}\gamma^n x_n, x_0\Big) + \Big\|\sum_{n\geq 1} u^{n-1}x_n\Big\|^2$$

$$= \Big\|\sum_{n\geq 0}\gamma^n x_n\Big\|^2 - \Big\|\sum_{n\geq 1}\gamma^n x_n$$

Set $$y_t = \int_0^\infty \gamma^{t'} x_{t+t'}\,dt'$$

$$\left\| \int_0^\infty u^t x_t\,dt \right\|^2 = \int dt' \big((y_{t'}, x_{t'}) + (x_{t'}, y_{t'}) \big)$$

~~So what??~~ $\exists$ all kinds of questions.

Suppose $\dim(X) = 1$. Want $u^t = e^{iAt}$

try to get continuous case straight

$$X \xrightarrow[j]{} H, \quad u^t \qquad j^* u^t j = \gamma^t \qquad t \geq 0.$$

e.g. $\dim(X) = 1$ $\quad |\gamma| < 1$. (What about $\gamma = 0$?)

You seek a positive definite function on the group $\mathbb{R}$ namely $(x, u^t x)$ function of t, should be F.T. of measure

$$(x, u^t x) = \int e^{i\lambda t}\,d\mu(\lambda) \qquad \mu'(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}(x, u^t x)\,dt$$

$$\| \quad (x, \gamma^t x) \quad t \geq 0$$

$$(x, (\gamma^*)^{-t} x) \quad t \leq 0$$

$$\mu'(\lambda) = \int_{-\infty}^{0} e^{-i\lambda t} \gamma^{*-t}\,dt + \int_0^\infty e^{-i\lambda t} \gamma^t\,dt$$

so we need $\log(\gamma)$ Yes.

$$\gamma = e^\beta \qquad \mathrm{Re}(\beta) < 0.$$

$$\int_0^\infty e^{-i\lambda t} e^{\beta t}\,dt = \frac{1}{i\lambda - \beta}$$

$$\int_{-\infty}^{0} e^{-i\lambda t} e^{-\beta^* t}\,dt = \frac{1}{-i\lambda - \beta^*}$$

$E = LsI \qquad Z = \frac{E}{I} = Ls = -iL\omega$

$Z = -1$ should give

Yes β should arise by put reseslance $R=1$ across the 1-port. Then $e^{-i\beta t}$ should be ~~a~~ contractive indicating decay of energy.

$$\frac{E}{I} = Ls = -1$$

equations of motion are $E = L\frac{dI}{dt} \qquad E = -I$

$L\frac{dE}{dt} + E = 0 \qquad E = \text{const}\, e^{-L^{-1}t}$. So you have a clear picture of β in the case of a LC 1-port.

So what? ~~Nomore~~ How does this relate to (A, Σ). Explain how an LC 1-port yields a partial hermitian operator, or how a partial hermitian operator yields the ~~suitab~~ appropriate response function Z.

This involves ~~recall~~ reviewing earlier stuff about LC circuits. ~~the eqn~~ the equivalence between rational matrix functions $Z(s) = \sum \frac{a_\omega(1+\omega^2)}{s^2+\omega^2}$ with $a_\omega \geq 0$ $\sum a_\omega = 1$ and subquotients of a polarized Hilb. space $H = H^+ \oplus H^-$. ~~The~~

This stuff is related to GR quasi-determinant. Idea

On $H = H^+ \oplus H^-$ you have $s\|\xi^+\|^2 + s^{-1}\|\xi^-\|^2$.

So work with real vector spaces and quadratic forms. ~~The~~ You have this pos. def. quadr. form on H and it induces a q.f. on any subquotient. and that's how you ~~generate~~ obtain a $Z(s)$.

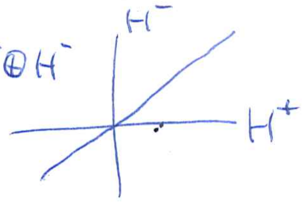
Basic idea

$$V/W \qquad \begin{array}{c} H \\ \downarrow \\ H/W \end{array}$$

Recall Grass stuff. $W \subset H^+ \oplus H^-$.

W is the graph of a corresp from H^+ to H^-

Claim canonical (spectral) splitting into "characteric values". For each $0 \le \omega < \infty$ you have $W = \binom{1}{\omega} H^+ \subset H^+ \oplus H^-$



isom. by $\begin{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \\ \frac{\omega}{\sqrt{1+\omega^2}} \end{pmatrix} : H^+ \longrightarrow \begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix}$ isom.

decompose $W = \oplus W_\omega$ $\quad H^\pm = \oplus H^\pm_\omega$

$\begin{pmatrix} \iota^+ \\ \iota^- \end{pmatrix} : W \longrightarrow \begin{matrix} H^+ \\ \oplus \\ H^- \end{matrix}$ is direct sum of $\iota_\omega = \begin{pmatrix} \iota^+_\omega \\ \iota^-_\omega \end{pmatrix} : W_\omega \longrightarrow \begin{matrix} H^+_\omega \\ \oplus \\ H^-_\omega \end{matrix}$

$$\iota_\omega^* (s\pi^+ \oplus s^{-1}\pi^-)\iota_\omega = s\frac{1}{1+\omega^2} + s^{-1}\frac{\omega^2}{1+\omega^2} = \frac{s+s^{-1}\omega^2}{1+\omega^2}$$

Thus the g.f on W is $\bigoplus \frac{s+s^{-1}\omega^2}{1+\omega^2}\pi_\omega$ on $\bigoplus_\omega W_\omega$ ..

So this is the way a $Z(s)$ arises – a subquotient of a polarized Hilbert space. Question: How is this related to partial unitaries, scattering etc. You physically connect an LC port to a transmission line to obtain a Hilbert space & one parameter unitary group.

Is there a link between LC circuits and partial ~~unitaries~~ real skew-symmetric ops? Maybe the problem lives at the wave equation end. Physically you're given a time evolution on a linear space of states. So whence comes the scalar product – some notion of energy.

Leave this for the moment and return to examples. You want to understand in the case of an LC 1-port the ~~[illegible]~~ β operator on X (there should also be one on Y)

$X \underset{b}{\overset{a}{\rightrightarrows}} Y$ type $O(u)$. $Y = aX \oplus V^+ = V^- \oplus bX$

eigenvector eqn. $(az-b)x = -v^+ + v^-$

better – get holom. line bundle $Y/(az-b)X = L_z$ over $\mathbb{P}^1$.

trivialize over $|z| \le 1$ by solving $(az-b)x = -y + \tilde{y}(z) v_0^-$

solution is
$$x = (1-zb^*a)^{-1} b^* y$$
$$\tilde{y}(z) = (v_0^-, (1-zab^*)^{-1} y).$$

Next you want $y \mapsto \tilde{y}(z)$ to be isometric embedding of Y into $L^2(S^1)$. Here use

$$\int \frac{d\theta}{2\pi} |\tilde{y}(z)|^2 = \int \frac{d\theta}{2\pi} \left((1-zab^*)^{-1} y, (1-b^*b)(1-zab^*)^{-1} y \right)$$

$$= \int \frac{d\theta}{2\pi} \left(y, \underbrace{(1-z^{-1}ba^*)^{-1}(1-b^*b)(1-zab^*)^{-1}}\, y \right)$$

$$(1-z^{-1}ba^*)^{-1}(1-zab^*) \qquad (1-z^{-1}ba^*)(1-zab^*)^{-1}$$

$\gamma = ab^*$

$1-\gamma^*\gamma =$

$1 - b\cancel{a^*}ab^*$

$$(1-z^{-1}\gamma^*)^{-1}(1-\gamma^*\gamma)(1-z\gamma)^{-1}$$

$$(1-z^{-1}\gamma^*)^{-1}\left((1-z^{-1}\gamma^*)z\gamma + (1-z\gamma)\right)(1-z\gamma)^{-1}$$

$$(1-z^{-1}\gamma^*) + z^{-1}\gamma^*(1-z\gamma)$$

$$z\gamma(1-z\gamma)^{-1} + (1-z^{-1}\gamma^*)^{-1} = \cdots z^2\gamma^2 + z\gamma + 1 + z^{-1}\gamma^* +$$

$$(1-z\gamma)^{-1} + (1-z^{-1}\gamma^*)^{-1} z^{-1}\gamma^* =$$

$$\int \frac{dz}{2\pi i z} \left(y, z\gamma(1-z\gamma)^{-1} y + (1-z^{-1}\gamma^*)^{-1} y \right)$$

Let's try $\mathcal{D}_A = \{ f \in H_+ \mid \int (1+\lambda^2)|f|^2 \frac{d\lambda}{2\pi} < \infty \}$.

$\mathcal{D}_A \underset{\lambda - i}{\overset{\lambda+i}{\rightrightarrows}} H_+$ isometric, $\lambda + i$ should be ~~onto~~ an isom. unitary equiv. $(\lambda - i)\mathcal{D}_A$ codim 1.

OK go back to glueing $H_- \oplus X \oplus H_+$.

What do I find? I guess H_+ contains interesting lines probably also H_-. Suppose $X = 0$. Then you have $z = \frac{1+i\lambda}{1-i\lambda} = \frac{-\lambda+i}{\lambda+i}$, so I basically have the orthonormal basis

$$z^n \frac{1}{\lambda+i} = \frac{(-\lambda+i)^n}{(\lambda+i)^{n+1}} \quad n \geq 0 \text{ for } H_+$$

$X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ type $O(n)$ sp $\beta \subset$ LHP (or Y)

The idea: given β on X $\ni$ $\frac{\beta - \beta^*}{2i} \leq 0$

$$\tilde{x}(\omega) = \left(i(\beta - \beta^*)\right)^{1/2} \frac{1}{\omega - \beta} x$$

$$\int \frac{d\omega}{2\pi} |\tilde{x}(\omega)|^2 = \left(x, \int \frac{d\omega}{2\pi} \frac{1}{\omega - \beta^*} i(\beta-\beta^*) \frac{1}{\omega - \beta} x \right) = \|x\|^2$$

complete contour in the ~~LHP~~ UHP, simple pole

This will give an isometric embedding ~~Xon~~ into functions of ω. If we then $q(\omega) = \det(\omega - \beta)$, then $q\tilde{x}$ is a poly of degree $n-1$, $\|x\|^2 = \int |q\tilde{x}|^2 \frac{d\omega}{|q|^2 2\pi}$

Now the problem is to find β on X (also on Y), the q for Y should be $q(\omega + i)$ probably)

~~Now the problem for X On X~~

Need an example. $\dim(X) = 1$. To what extent does an LC 1-port yield a $X \underset{A}{\overset{\varepsilon}{\rightarrow}} Y$? e.g. $L\xi$

Start with equations of motion.

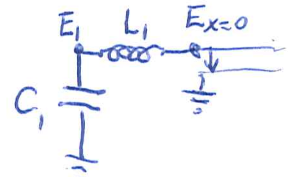
$$\partial_x E + \partial_t I = 0 \qquad (\partial_x + s)(E+I) = 0 \qquad E+I = C(s)e^{-sx}$$

$$\partial_x I + \partial_t E = 0 \qquad (\partial_x - s)(E-I) = 0 \qquad E-I = D(s)e^{sx}$$

$$E_{x=0} = -L\partial_t I_{x=0} \qquad E_{x=0} = -Ls\, I_{x=0} \qquad -Ls = \frac{E}{I}\Big|_{x=0} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} C \\ D \end{pmatrix}$$

$$\frac{C}{D} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}(-Ls) = \frac{Ls-1}{Ls+1} = \frac{-i\omega L - 1}{-i\omega L + 1} = \frac{\omega - iL^{-1}}{\omega + iL^{-1}}$$

more involved system



$$Cs E_1 = -I_{x=0} = I_1$$

$$\underbrace{E_{x=0}}_{E_0} - E_1 = L_1 s(-I_{x=0}) = L_1 s I_1 \qquad Z = \frac{E_0}{I_1} = L_1 s + \frac{1}{Cs}$$

$$\frac{C(s)}{D(s)} = \frac{Ls + \frac{1}{Cs} - 1}{Ls + \frac{1}{Cs} + 1} = \frac{LCs^2 - Cs + 1}{LCs^2 + Cs + 1} \qquad s = \frac{C \pm \sqrt{C^2 - 4LC}}{2LC}$$

have positive real part

Note that these equations do not immediately yield a scalar product on states, which is preserved by time evolution. But maybe it's easy in the ladder case.

What to do? For an LC port, the ~~response~~ impedance function gives the polar data so it seems you have an obvious skew-symmetric operator. Impedance ∞ means 0 applied current, so you are getting the free oscillations of the e.g. $Z = \frac{1}{Cs + \frac{1}{Ls}} = \frac{Ls}{CLs^2+1}$

So it seems one does have a way to get eigenvalues for a LC port. Basically an LC port gives frequencies for the two basic boundary types: $I=0$ or $E=0$. Can I do this for a general $X \xrightarrow[A]{\varepsilon} Y$?

It is possible to explain ~~examp~~ exactly what is an LC 1-port. Same as $Z(s) = \sum \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega$

$a_\omega > 0$ fin. many $0 \le \omega \le \infty$.

$$Z(s) = L_1 s + \cfrac{1}{C_1 s +}$$

Consider $Z(s)$ a rational function of the appropriate type. Form $S(s) = \frac{Z-1}{Z+1}$ e.g. $\frac{Ls-1}{Ls+1}$ $\frac{1-Cs}{1+Cs}$

from S you get a Hilbert space $X = H^+ \cap SH^-$

~~The~~ Note that S appears to belong to a unitary ~~[illegible]~~ op., Z appears to belong to a real skew-symmetric operator, Z is like the resolvent

---

Start again with $X \underset{A}{\overset{\varepsilon}{\rightrightarrows}} Y$ of type $O(n)$. ~~[illegible]~~

There's a canonical splitting of Y into ~~lines~~ a direct sum of. Think of Y as holom. sections of $O(n)$, each $y \neq 0$ has n zeroes.

Get a line by specifying section vanishing to order k at $\lambda = 0$, $n-k$ at $\lambda = \infty$. $L_\lambda = Y/(\varepsilon\lambda - A)X$. Thus if $y \neq 0$ vanishes to order n at ∞, this means

$y, (A\varepsilon^{-1})y, \dots, (A\varepsilon^{-1})^n y$ are all defined. Now ~~[illegible]~~

orth this sequence to get $p_0, \dots, p_n$ and a J.-mat.

$$\begin{matrix} b_1 & a_1 & \\ a_1 & b_2 & \\ & a_2 & \end{matrix}$$

Example

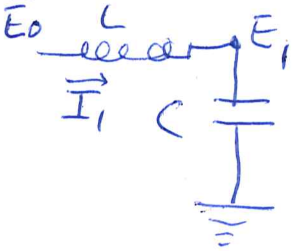
$$\begin{pmatrix} 0 & a_1 & \\ a_1 & 0 & \\ & a_2 & \end{pmatrix}$$

$$0 = (b_1 - \lambda) u_1 + a_1 u_2 \qquad \frac{a_1 u_2}{u_1} = \lambda - b_1$$

$$0 = a_1 u_1 + (b_2 - \lambda) u_2 + a_2 u_3 \qquad \frac{a_2 u_3}{u_2} = \lambda - b_2 - \frac{a_1^2}{a_1 u_2 / u_1}$$

$$0 = a_2 u_2 + (b_3 - \lambda) u_3 + a_3 u_4 \qquad \frac{a_3 u_4}{u_3} = \lambda - b_3 - \frac{a_2^2}{a_2 u_3 / u_2}$$

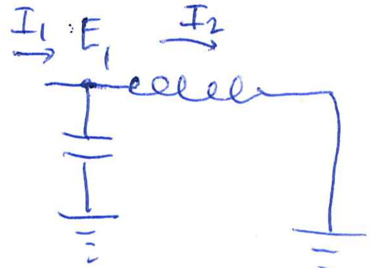
Anyway electrically? Because you are using the electrical picture you need to find the scalar product on the space of states. You need to fit LC circuit into the theory about partial operators. Example



$$E_0 - E_1 = L s I_1 \qquad E_0 - L s I_1 - E_1 = 0$$

$$I_1 = C s E_1 \qquad I_1 - C s E_1 = 0 .$$

$$\begin{pmatrix} E_0 \\ 0 \end{pmatrix} = s \begin{pmatrix} L & \\ & C \end{pmatrix} \begin{pmatrix} I_1 \\ E_1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} I_1 \\ E_1 \end{pmatrix}$$



$$E_1 = L s I_2 \qquad I_1 - I_2 = C s E_1$$

$$\begin{pmatrix} I_1 \\ 0 \end{pmatrix} = s \begin{pmatrix} C & \\ & L \end{pmatrix} \begin{pmatrix} E_1 \\ I_2 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E_1 \\ I_2 \end{pmatrix}$$

Question What is the natural scalar product on the state space of an LC network

For a discrete string there is an obvious energy - kinetic energy of the masses and potential energy of the string segments. I would like to understand an LC circuit. You have a graph with L, C edges, connected graph. State: Voltage $E \in \bar{C}^0$ Current $I \in C_1$

$$0 \longrightarrow \bar{C}^0 \xrightarrow{\delta} C^1 \longrightarrow H^1 \longrightarrow 0$$

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \longleftarrow H_1 \longleftarrow 0$$

Def Symp space T

$$T^{2n} \supset W^0 \downarrow W^0/W.$$

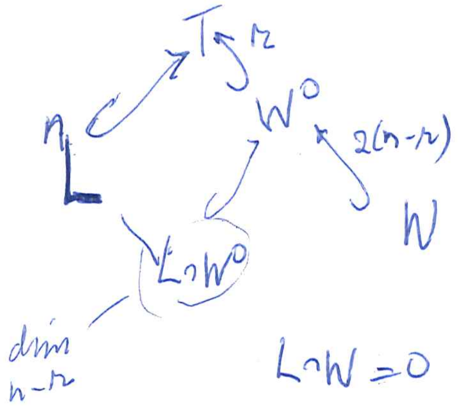
$\dim W = r < n.$

$\dim W^0 = 2n - r$

$\dim(W^0/W) = 2(n-r)$

T symplectic space $\underset{\text{Lagrangian}}{L} \subset T$

You want to get an induced Lagrangian subspace inside W^0/W. Obvious thing to do is to intersect ~~[illegible]~~ $L \cap W^0$ has dimension $n-r$



If you assume

$L \cap W = 0$

$\underset{n}{L} + \underset{2n-r}{W^0} = \underset{2n}{T} \Rightarrow \dim L \cap W^0 = n-r$

Then $\underbrace{L \cap W^0}_{n-r} \hookrightarrow \underbrace{W^0/W}_{2(n-r)}$

$$\begin{tikzcd} V \arrow[r] & H \arrow[r] \arrow[d, "Q"] & Z \\ V^* & H^* \arrow[l] & Z^* \arrow[l] \end{tikzcd}$$

$T/W^0 = V^*$

$Z^* = V^0$

$T = H \oplus H^*$

$L = \Gamma_Q$

$W = V \subset H$

$W^0 = H \oplus Z^*$

$W^0/W = H/V \oplus V^0$

~~first condition is $L + W^0 = T$~~

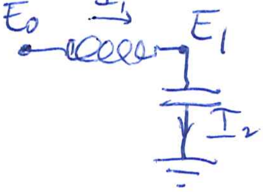
condition that $L \cap W = 0$ should be OK.

$\Gamma_Q = \begin{pmatrix} 1 \\ Q \end{pmatrix} H$ $\quad V$

If $\begin{pmatrix} v \\ 0 \end{pmatrix} \in L \Leftrightarrow Q(v) = 0 \Rightarrow Q \restriction V$ deg.

But if you have a rational map from $\mathbb{P}^1$ to $\mathbb{P}^1(V)$ is extends to a regular map. So you seem to get a line bundle.

Example.



$$H = C_1 = \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

$$H^* = C_1^\vee = \begin{pmatrix} E_0 - E_1 \\ E_1 \end{pmatrix}$$

$$\begin{pmatrix} Ls & \\ & \frac{1}{Cs} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} E_0 - E_1 \\ E_1 \end{pmatrix} \quad \text{so} \quad Q_s\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = LsI_1^2 + \frac{1}{Cs} I_2^2$$

subquotient $\quad W \subset V \subset H \qquad W$ where $I_1 = I_2$, so $V = H$
no loops?

So calculate the line $W^\perp$ wrt Q_s.

$$W^\perp = \left\{ \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \middle| \begin{pmatrix} I \\ I \end{pmatrix}^t \begin{pmatrix} Ls & \\ & (Cs)^{-1} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = 0 \quad \forall I \right\}$$

$$= \left\{ \begin{pmatrix} I_1 \\ I_2 \end{pmatrix} \middle| LsI_1 + \frac{1}{Cs} I_2 = 0 \right\} \qquad ?$$

Let's use coords I_1, E_1 to describe H, so that

$$Q_s\begin{pmatrix} I_1 \\ E_1 \end{pmatrix} = LsI_1^2 + CsE_1^2$$

What is W? where $I_2 = I_1 \qquad I_2 = CsE_1$

$W =$ where $I_1 = CsE_1$

$$= \left\{ \begin{pmatrix} I_1 \\ E_1 \end{pmatrix} \middle| I_1 = CsE_1 \right\}$$

$$W^\perp = \left\{ \begin{pmatrix} I_1 \\ E_1 \end{pmatrix} \middle| \underbrace{\begin{pmatrix} +Cs \\ 1 \end{pmatrix} \begin{pmatrix} Ls & 0 \\ 0 & Cs \end{pmatrix} \begin{pmatrix} I_1 \\ E_1 \end{pmatrix}}_{+LCs^2 I_1 + CsE_1 = 0} = 0 \right\}$$

$$+LCs^2 I_1 + CsE_1 = 0 \Rightarrow LsI_1 + E_1 = 0$$

$$= \left\{ \begin{pmatrix} I_1 \\ E_1 \end{pmatrix} \middle| \right.$$

Next. Suppose you have a rational function

$\dim(V/W) = 1.$ $\qquad Z(s) = \frac{a_0}{s} + \sum \frac{s(1+\omega_j^2)}{s^2+\omega_j^2} a_{\omega_j} + a_\infty s$

Then you can recover the Hilbert space somehow.

$$Z(s) = \sum_{0\le\omega\le\infty} \frac{s(1+\omega^2)}{s^2+\omega^2} a_\omega \qquad a_\omega \ge 0, \ \sum a_\omega = 1$$

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & H \\ V/W \arrow[r, hook, "(\iota_\omega)"] & \bigoplus_\omega H_\omega \arrow[u] \end{tikzcd}$$

ξ

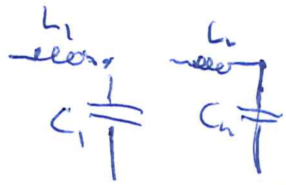
$s\pi^+ \oplus s^{-1}\pi^-$

$\bigoplus_\omega \frac{s+s^{-1}\omega^2}{1+\omega^2}\pi_\omega$

$Z(s)$

$\bigoplus \frac{s(1+\omega^2)}{s^2+\omega^2}\pi_\omega$

Assume V/W has dim 1, then where is the line bundle



$$Z(s) = L_1 s + \frac{1}{C_1 s +} \frac{1}{|L_2 s +} \ \frac{1}{|L_{n-1}s+} \frac{1}{|C_n s}$$

Start with $V \hookrightarrow H^+ \oplus H^-$ subspace

amounts to $0 \le a \le 1$ on V. $\qquad H^+ = \sqrt{a}\,V, \quad H^- = \sqrt{1-a}\,V$

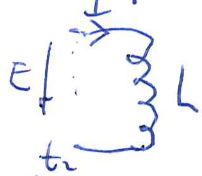
A self adj. v_0 cyclic $\qquad \xi \longmapsto (v_0, (\lambda - A)^{-1}\xi)$

$$\xi \mapsto (v_0, (\lambda-A)^{-1}\xi) = \tilde{\xi}(\lambda)$$

$$I = C\partial_t E$$

$E_0 \quad I \quad E_1$ $\qquad E_0 - E_1 = LsI \qquad \int EI = \left[\tfrac{1}{2}CE^2\right]_{t_1}^{t_2}$

Power $\quad E = L\partial_t I$



energy in $\displaystyle\int_{t_1}^{t_2} EI\,dt = \int_{t_1}^{t_2} LI\partial_t I\,dt = \left[\tfrac{1}{2}LI^2\right]_{t_1}^{t_2}$

$$X \xrightarrow{a} Y$$
$$b \downarrow$$
$$Y$$

What's missing a bit is the

$$z = \frac{-\lambda + i}{\lambda + i} \qquad a = i\varepsilon + A$$
$$\lambda = i\frac{1-z}{1+z} \qquad b = i\varepsilon - A$$

$$X \xrightarrow{\varepsilon\lambda - A} Y$$
$$\downarrow -i\varepsilon^* - A^*$$
$$X$$

$$\varepsilon i(1-z) - (1+z)A$$
$$= (-i\varepsilon - A)z + \varepsilon i - A$$

$$-(i\varepsilon^* + A^*)(\varepsilon\lambda - A)$$

Look for radiation states
decay mode.

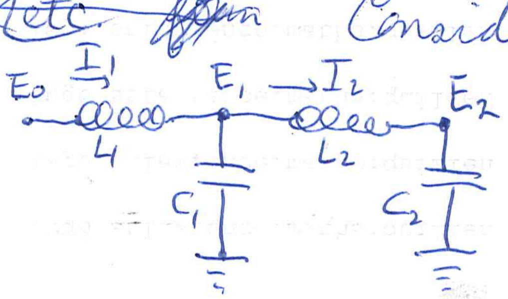
$$(E+I) = A e^{-s(x-t)}$$
$$(E-I) = B e^{s(x+t)}$$

$$e^{st}$$

$$S = A/B$$

$$\begin{pmatrix} +1 & -1 \\ +1 & +1 \end{pmatrix}\begin{pmatrix} Z \end{pmatrix} = S = \frac{2-1}{2+1}$$

You need an example. Is it possible for me to ~~L_1, C etc. from~~ Consider a ladder network



L_1 may be zero
C_2 may be inf.

~~A state of this system is a collection of~~

Your aim is to find a Hilbert space. The network gives a set of DE's.

$$E_0 - E_1 = L_1 \partial_t I_1 \qquad \frac{E_0}{I_1} = L_1 s + \frac{1}{I_1/E_1}$$
$$I_1 - I_2 = C_1 \partial_t E_1 \qquad \frac{I_1}{E_1} = C_1 s + \frac{1}{E_1/I_2}$$
$$E_1 - E_2 = L_2 \partial_t I_2 \qquad E_1/I_2 = L_2 s + \frac{1}{I_2/E_2}$$
$$I_2 = C_2 \partial_t E_2 \qquad I_2/E_2 = C_2 s$$

You ~~have~~ have a state consisting of ~~~~
5 ~~~~ real quantities E_0, I_1, E_1, I_2, E_2
and you have 4 DE's. E_0 can be assigned
Here I have ~~~~ X of dim 4, Y of dim 5.

What do you want? You want some kind of natural inner product on X and Y

The idea - write the linear equations as a J-matrix

$$
\begin{aligned}
E_0 &= L_1 s I_1 + E_1 & \qquad & E_0 - L_1 s I_1 - E_1 \\
I_1 &= C_1 s E_1 + I_2 & & I_1 - C_1 s E_1 - I_2 \\
E_1 &= L_2 s I_2 + E_2 & & E_1 - L_2 s I_2 - E_2 \\
I_2 &= C_2 s E_2 & & I_2 - C_2 s E_2
\end{aligned}
$$

$$
s \begin{pmatrix} L_1 & & & \\ & C_1 & & \\ & & L_2 & \\ & & & C_2 \end{pmatrix} \begin{pmatrix} I_1 \\ E_1 \\ I_2 \\ E_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}}_{\text{skew symmetric}} \begin{pmatrix} I_1 \\ E_1 \\ I_2 \\ E_2 \end{pmatrix}
$$