What to do? List ways to proceed, or ideas. The basic structure is symplectic, classical mechanics, or rather classical kinematics, you have a 2 dim'l phase space with symplectic form, symmetry group $SL(2,\mathbb{R})$, when quantized you get Weyl algebra & its unique irred repn. which is $L^2(\mathbb{R})$ in various guises, meaning that you need to choose a degree one operator to diagonalize, better a maximal isotropic subspace of phase space. So what you should understand is how to pass between the Hilbert spaces arising from two Lagrangian subspaces. This will probably entail a lot of work. to work with $e^{2\pi i \xi x}$ kernels, where $\xi, x$ are functions on phase spaces.

to keep things simple you keep to $SL(2,\mathbb{R})$.

other idea is stationary phase, doing Gaussian integrals by critical points for the action.

So let's start with a particle with one degree of freedom. kinematics say that state Hilbert space is $L^2(\mathbb{R})$, position op is $q = x$, momentum op is $p = \frac{1}{i}\partial_x$ so that $[p,q] = \frac{1}{i}$

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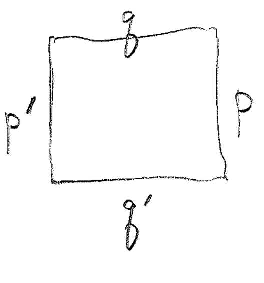
You need the metaplectic action of $SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$ (double cover) on $L^2(\mathbb{R})$. The operators ~~are~~ should be given by imaginary Gaussian kernels, the assoc. quad. form should be easy to obtain via classical mechanics; ~~stationary~~ phase, symplectic transf linked to quadratic form possibly like ~~in a way similar to~~ transfer + scattering matrices.

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$




$$q = cp' + dq' \qquad \boxed{\frac{1}{d}q - \frac{c}{d}p' = q'}$$

$$p = ap' + b\left(-\frac{c}{d}p' + \frac{1}{d}q\right) = \frac{\Delta}{d}p' + \frac{b}{d}q$$

2 diml real vector space $V$ equipped with non deg skew symm form. $A(v, v') = -A(v', v)$. Your aim is to identify a symplectic isom. with a quadratic form.

In gen. begin with $V$, $A$ anti symm non deg. Standard form by choosing complementary Lag. subspaces

$$V = \begin{matrix} X \\ \oplus \\ X^* \end{matrix} \qquad A\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}\right) = \langle x, y' \rangle - \langle x', y \rangle$$

$$\begin{pmatrix} y \\ x \end{pmatrix}^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} = y^* x_1 - y_1^* x \qquad ?$$

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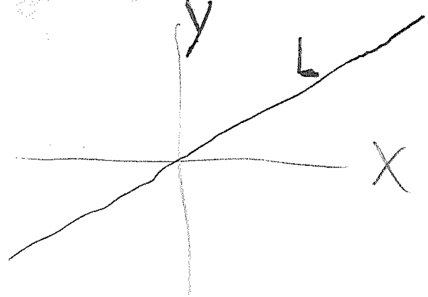
Begin with $V$ vect sp f d $A$ non deg skew symm. form.
Choose polarization $V = X \oplus Y$ $X, Y$ Lagrangian subspaces.




$L \pitchfork Y \longrightarrow L = \begin{pmatrix} 1 \\ T \end{pmatrix} X$

$L$ Lag $\Longleftrightarrow T = T^t$

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Begin with $L^2(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$ with $q = x$, $p = \frac{2\pi}{i}\partial_x$.
Now let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Put $\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$.

Then $[p', q'] = [ap + bq, cp + dq] = (ad - bc)[p, q] = \frac{2\pi}{i}$.

Can you find an intertwining operator between these two representations of CCR? Need simple cases. Now you know that the Lie alg. of $SL(2,\mathbb{R})$ can be identified with quadratic polys.

$\left[\frac{q^2}{2}, p\right] = -\frac{2\pi}{i} q$. How does this work?

$$\left(e^{i\frac{q^2}{2}t} f\right)(x) = e^{i\frac{x^2}{2}t} f(x)$$

$$p\left(e^{i\frac{q^2}{2}t} f\right) = \frac{2\pi}{i}\partial_x\left(e^{i\frac{x^2}{2}t} f\right) = \frac{2\pi}{i}\left(ixt\, e^{i\frac{x^2}{2}t} + e^{i\frac{x^2}{2}t}\partial_x f\right)$$

$$p\, e^{i\frac{q^2}{2}t} f = e^{i\frac{q^2}{2}t}\left(p + \frac{h}{2\pi} t q\right) f$$

continue $\quad [p,q] = \frac{\hbar}{i} \qquad (pf)(x) = \frac{\hbar}{i}\partial_x f$

$$(qf)(x) = xf$$

$$e^{i\frac{q^2}{2}t}\, p\, e^{-i\frac{q^2}{2}t} = p - i[p, \tfrac{q^2}{2}]t = p - i\frac{\hbar}{i}qt = p - \hbar qt = p - (\hbar t) q$$

$$e^{i\frac{q^2}{2}t}\, q\, e^{-i\frac{q^2}{2}t} = q.$$

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$V = \mathbb{R}^2$ equipped with $\omega = 2\pi i\, dx\, dy$.

Aim to construct a line bundle over $\mathbb{R}^2$, which is translation invariant (meaning?)

You have $e^{i(a\partial_x + b\partial_y)}$ acting

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Base manifold $V = \mathbb{R}^2$ Lie group under +

line bundle $V \times \mathbb{C}$ over $V$, trivial line bundle

so sections are fns. $s: V \longrightarrow \mathbb{C}$. $T_{(a,b)}: V \longrightarrow V$

$T_{(a,b)}(x,y) = (a+x, b+y)$.

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Here's a possible way to proceed. Use $\mathcal{S}(\mathbb{R})$ which should be equivalent to smooth sections of the line bundle over the torus. First work out the structure using the $\mathcal{S}(\mathbb{R})$ model.

Structure expected. Real vector space with translational symm.

$\mathcal{S}(\mathbb{R})$ translation operators $\quad e^{isx} \quad e^{it\partial_x}$

$$\left(e^{t\partial_x} f\right)(x) = f(x) + t f'(x) + \frac{t^2}{2!} f''(x) + \cdots = f(t+x)$$

$$\left(e^{isx} f\right)(x) = e^{isx} f(x).$$

$$e^{t\partial_x}\, e^{isx} = \underbrace{e^{[t\partial_x, isx]}}_{e^{its}}\, e^{isx}\, e^{t\partial_x}$$

So where are you next? You have the Heisenberg group: $\mathbb{T} \longrightarrow H \longrightarrow \mathbb{R}^2$. If you restrict to $\mathbb{Z}^2$, then you can lift $\mathbb{Z}^2$ into $H$ and descend.

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Let's begin with $\mathbb{R}^2$ coords $(x, y)$, smooth manifold, Lie group of translations on $\mathbb{R}^2$, Lie alg gen by vector fields $\partial_x$, $\partial_y$ constant coeffs., similarly have constant coeff differential forms, $dx, dy, dx\,dy$ etc.

Fix a volume $\omega = dx\,dy$ and choose a 1-form $\theta$ sat $d\theta = \omega$. $\theta$ is unique up to diffl $df$ ~~a constant~~. Say

$$\theta = P\,dx + Q\,dy \qquad d\theta = \underbrace{dP\,dx + dQ\,dy}_{\left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right)dx\,dy} = dx\,dy$$

What are the solns to $\underbrace{-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}}_{1} = 1$? Look at homog eqn. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, soln $P =$

You want to solve $d\theta = dx\,dy$. homog eqn is $d\theta = 0$ solution is $\theta = df$ $f$ arb. fn. need part soln.

$$\theta = x\,dy \text{ or } -y\,dx \qquad \text{and} \qquad x\,dy - (-y\,dx) = d(xy)$$

Point: $d + \theta$ is a connection on the trivial bundle. curvature is $(d+\theta)^2 = d\theta = dx\,dy$.

$$d + \theta = (\partial_x + P)\,dx + (\partial_y + Q\,dy)$$

What is the meaning of this connection?

Repeat: Begin with linear

Begin again with the Lie group $\mathbb{R}^2$ acting on itself by translation. The infinitesimal action is given by the vector fields $\partial_x, \partial_y$. Next suppose given $\omega = dx\,dy$ symplectic structure invariant under the group action.

$$F(x,y+1) = F(x,y)$$

$$F(x,y) = e^{2\pi i y} F(x+1,y)$$

~~Suppose~~ Can you find a version variant of $e^{2\pi i xy}$ which will give a different line of periodicity

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Start with $F(x,y) = F(x,y+1)$

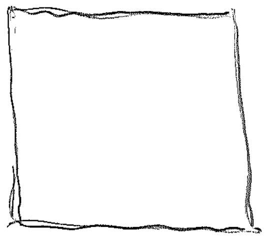
$$F(x,y) = e^{2\pi i y} F(x+1,y) = e^{2\pi i y} e^{2\pi i y} F(x+2,y)$$

$$\cancel{F(x,y) = e^{-2\pi i y} F(x-1,y)}$$

$$\cancel{F(x+1,y) = e^{-2\pi i y} F(x,y)}$$

$$F(x,y) = e^{2\pi i y} F(x+1,y) = \left(e^{2\pi i y}\right)^2 F(x+2,y)$$
$$= \cdots e^{2\pi i y m} F(x+m,y)$$

$$F(x+m,y+n) = e^{-2\pi i y m} F(x,y)$$



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You want to handle any pair of generators for the lattice $\mathbb{Z}^2$.

$$\begin{vmatrix} m & x \\ n & y \end{vmatrix} = \pm 1.$$

Review the ~~tree~~ $SL(2,\mathbb{Z})$ tree

vertex = elt of $\mathbb{P}_1\mathbb{Q} = \mathbb{Q}\cup\infty$

$\pm\binom{m}{n}$ rel. prime $n \geq 0$.

given another $\pm\binom{x}{y}$ rel. prime the pair form a 1-simplex

iff $\begin{vmatrix} x & m \\ y & n \end{vmatrix} = \pm 1$. 2-simplex = three diff lines any 2 lines form a 1-simplex.

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to handle $F$ restricted to a single $\Gamma$ orbit. How to figure it out using ~~principal~~ fibre bundle theory.

$G$ Heisenberg group general by $\frac{1}{i}\partial_x, \frac{1}{i}\partial_y$

Do simplest case. $V = \mathbb{R}^2$, $\Gamma = \mathbb{Z}^2$, $\Gamma$ acts by translations $v \longmapsto v+\gamma$. Then $V$ is a principal $\Gamma$ bundle with bases $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. ~~What is the next point~~ Hence for any $\Gamma$-space $F$ you have an assoc. fiber bundle $V\times^\Gamma F$ whose sections are $f: V \longrightarrow F$ satisfying $f(v\gamma) = (\gamma_*)^{-1}f(v)$

check. $f(v(\gamma_1\gamma_2)) = f((v\gamma_1)\gamma_2) = \gamma_{2,*}^{-1} f(v\gamma_1) = \gamma_{2,*}^{-1}\gamma_{1,*}^{-1} f(v)$

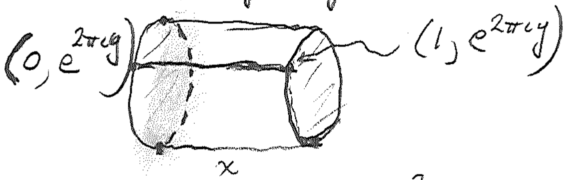
$= (\gamma_{1,*}\gamma_{2,*})^{-1} f(v) = (\gamma_1\gamma_2)_*^{-1} f(v)$.

Want $\underbrace{V\times^\Gamma F}_{E=}$ to be a line bundle over $V/\Gamma$, so $F=\mathbb{C}$ with $\Gamma$ action given by a character, $\chi: \Gamma \to \mathbb{T}$. Such a line bundle is a flat line bundle over the base, which is not what you want.

A connection on a vector bundle allows you to differentiate sections $\nabla: E \longrightarrow T^*\otimes E$, it amounts to $\nabla_x$, $\nabla_y$ derivations

$\nabla(fs) = df\, s + f\nabla s$. wrt functions on $M$. Given a curve in $M$ you can parallel transport vectors along the curve.

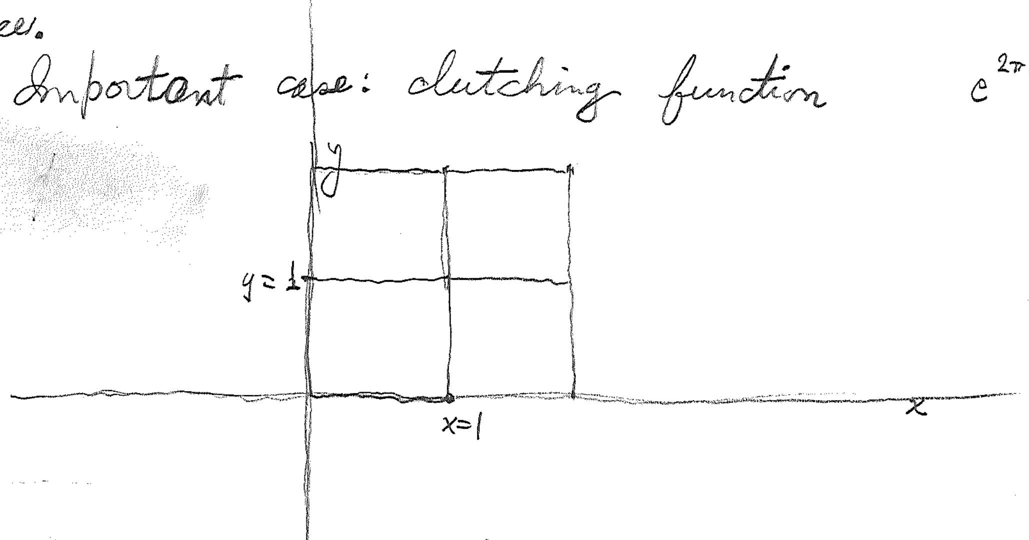
Let's begin with the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. a function on $\mathbb{T}^2$ (cont, smooth) is the same as a function $f(x,y)$ on $\mathbb{R}$ (same type) satisfying the double-period condition $f(x+1,y) = f(x,y) = f(x,y+1)$, i.e. invariance under the group of translations $T_{m,n} : (x,y) \longmapsto (x+m, y+n)$ $m,n \in \mathbb{Z}$. As top space $\mathbb{T}^2$ is the cylinder $[0,1] \times S^1$




modulo $(0, e^{2\pi i y}) \sim (1, e^{2\pi i y})$

describe vector bundle on $\mathbb{T}^2$. restrict to $0 \times S^1$, it becomes trivial since $GL_n(\mathbb{C})$ is connected, ~~by covering htpy you~~ get a trivial bundle over $[0,1] \times S^1$ modulo an identifying $0 \times S^1 \times \mathbb{C}^n \sim 1 \times S^1 \times \mathbb{C}^n$ i.e. an auto $S^1 \times \mathbb{C}^n \to S^1$ i.e. $S^1 \longrightarrow GL_n(\mathbb{C})$ loop. only one invariant - the degree.

Important case: clutching function $e^{2\pi i y}$




your picture: $\mathbb{R} \times \mathbb{R}/\mathbb{Z}$. ~~No take the cylinder~~

Explain why continuous sections of the line bundle $E$ extend uniquely to cont. functions $\phi(x,y)$ on $\mathbb{R}^2$ satisfying $\phi(x,y) = \phi(x,y+1)$ and $\phi(x+1,y) = e^{-2\pi i y}\phi(x,y)$. A cont. section of $E$ is the same as a cont. function $F(x,y)$ on $\{0 \leq x \leq 1\} \times \{0 \leq y \leq 1\}$ sat.

$F(x,0) = F(x,1) \quad \forall x \in [0,1]$

$F(0,y) = e^{2\pi i y} F(1,y) \quad \forall y \in [0,1]$

O.K.

certainly you have rest. map $\phi(x,y)$ $(x,y) \in \mathbb{R}^2$ to $F(x,y)$ $(x,y) \in [0,1]^2$

Given $F(x,y)$ on $I \times I$. Look first at dim 1.

Is it clear that a cont. $\phi(y)$ on $\mathbb{R}$ sat. $\phi(y) = \phi(y+1)$ is the same as a cont function on $S^1 = \mathbb{R}/\mathbb{Z}$ Yes

but also ——— $I = [0 \le y \le 1]$ sat $\phi(0) = \phi(1)$. This is the ~~statement~~ that topologically $I/0\sim 1 = \mathbb{R}/\mathbb{Z}$. Things become different with $C^1$ functions

Let's simplify and stick to smooth fns. $\phi(x,y)$ with the required conditions $\phi(x,y+1) = \phi(x,y) = e^{2\pi i y}\phi(x+1,y)$.

Look at connections on this line bundle $E$, i.e. covariant deriv. operators $\nabla_x \quad \nabla_y \qquad \nabla : \underset{\sim}{E} \longrightarrow T^* \otimes E \qquad \nabla = dx\, \nabla_x + dy\, \nabla_y$.

You want to construct $\nabla$ to be consistent with period. condition.

$$\nabla = d + A$$

$$\nabla\phi = dx\left(\partial_x\phi + A_x\phi\right) + dy\left(\partial_y\phi + A_y\phi\right)$$

$$\phi(x,y+1) = \phi(x,y)$$

$$\left(\partial_x\phi + A_x\phi\right)(x,y+1) = \left(\partial_x\phi + A_x\phi\right)(x,y)$$

$A_x, A_y$ per. in $y$. because they are fns. on torus.

$$\left(\partial_x\phi + A_x\phi\right)(x+1,y) = \partial_x\phi(x+1,y) + A_x(x+1,y)\phi(x+1,y)$$

$$= e^{-2\pi i y}\left(\partial_x\phi(x,y) + A_x(x+1,y)\phi(x,y)\right)$$

want $A_x(x+1,y) = A_x(x,y)$

$$= e^{-2\pi i y}\left(\partial_x + A_x\right)\phi(x,y).$$

$$\left(\partial_y\phi + A_y\phi\right)(x+1,y) = \partial_y\left(e^{-2\pi i y}\phi(x,y)\right) + e^{-2\pi i y}A_y(x+1,y)\phi(x,y)$$

$$= e^{-2\pi i y}\left\{\left(\partial_y + A_y\right)\phi(x,y) - 2\pi i\,\phi(x,y)\right\}$$

$$\left(\nabla_y\phi\right)(x+1,y) = e^{-2\pi i y}\left\{\left(\nabla_y\phi\right)(x,y) - 2\pi i\,\phi(x,y)\right\}$$

It seems that $\nabla_x = \partial_x \,, \quad \nabla_y = \partial_y$

$$(T_x \phi)(x,y) = \phi(x+1,y)$$
$$(T_y \phi)(x,y) = \phi(x,y+1).$$

you want a connection $\nabla = d + A$ commute with $T_x, T_y$

$$A$$
$$T_y(d\phi + A\phi) = d(T_y\phi) + (T_y A)\, T_y\phi$$
$$= (d+A)(T_y\phi) = (d+A)\phi$$

$$\cancel{T_x(d\phi + A\phi) = d(T_x\phi) + A(T_x\phi)}$$
$$\cancel{= d(e^{-2\pi i y}\phi) + A e^{-2\pi i y}\phi}$$
$$\cancel{=}$$

$$T_x(d\phi + A\phi) = d(T_x\phi) + (T_x A)(T_x\phi)$$
$$= d(e^{-2\pi i y}\phi) + (T_x A)\, e^{-2\pi i y}\phi$$

$$A = dx\, A_x(x,y) + dy\, A_y(x,y)$$

$$T_x A = dx\, A_x(x+1,y) + dy\, A_y(x+1,y)$$

$$d(e^{-2\pi i y}\phi) = -2\pi i\, dy\, e^{-2\pi i y}\phi(x,y) + e^{-2\pi i y}\, d\phi$$

$$(T_x A)e^{-2\pi i y}\phi = \big(dx\, A_x(x+1,[\text{illegible}]y) + dy\, A_y(x+1,y)\big)\, e^{-2\pi i y}\, d\phi$$

$$d + 2\pi i x\, dy = dx(\partial_x) + dy(\underbrace{\partial_y - 2\pi i x}_{\nabla_y})$$

$$\phi(x,y) = \phi(x,y+1) \implies (\underset{\text{also } \nabla_y}{\partial_x}\phi)(x,y) = (\partial_x\phi)(x,y+1)$$

$$\phi(x,y) = e^{2\pi i y}\phi(x+1,y) \implies \text{OK for } \partial_x$$

$$\phi(x,y) = \phi(x,y+1) \implies \partial_x\phi,\ \nabla_y\phi \ \text{ per. in } y$$

$$\phi(x,y) = e^{2\pi i y}\phi(x+1,y) \implies \text{same for } \partial_x\phi$$

$$\cancel{\partial_y\phi(x,y) = \partial_y\left(e^{2\pi i y}\phi(x+1,y)\right) = 2\pi i\, e^{2\pi i y}\,\partial_y\phi(x+1,y)}$$

$$\phi(x+1,y) = e^{-2\pi i y}\phi(x,y)$$

$$(\partial_y\phi)(x+1,y) = e^{-2\pi i y}(\partial_y\phi)(x,y) - 2\pi i\, e^{-2\pi i y}\phi(x,y)$$

$$(\nabla_y\phi)(x,y)$$

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$$(\partial_y + 2\pi i x)\,\phi(x,y) \xmapsto{T_x} \partial_y\phi(x+1,y) + 2\pi i(x+1)\underbrace{\phi(x+1,y)}_{e^{-2\pi i y}\phi(x,y)}$$

$$\partial_y\left(e^{-2\pi i y}\phi(x,y)\right) = -2\pi i\, e^{-2\pi i y}\phi(x,y) + e^{-2\pi i y}\partial_y\phi(x,y)$$

$$= 2\pi i x\, e^{-2\pi i y}\phi(x,y) + e^{-2\pi i y}\partial_y\phi(x,y)$$

$$= e^{-2\pi i y}\left(\partial_y + 2\pi i x\right)\phi(x,y)$$

Repeat the calculation. Claim

$$\nabla_x = \partial_x \quad , \quad \nabla_y = \partial_y + 2\pi i x$$

preserve $\phi(x,y)$ satisfying $\quad \phi(x,y+1) = \phi(x,y) = e^{2\pi i y}\phi(x+1,y)$

clear for $\partial_x$. Suppose

$$\phi(x,y) = e^{2\pi i y}\,\phi(x+1,y)$$

$$\nabla_y\, \phi(x,y) = \partial_y \phi(x,y) + 2\pi i x\, \phi(x,y)$$

$$\nabla_y\left(e^{2\pi i y}\phi(x+1,y)\right) = 2\pi i\, e^{2\pi i y}\phi(x+1,y) + e^{2\pi i y}\,\partial_y\phi(x+1,y) + 2\pi i x\, e^{2\pi i y}\phi(x+1,y)$$

$$= e^{2\pi i y}\left(2\pi i (x+1)\phi(x+1,y) + \partial_y \phi(x+1,y)\right)$$

$$\boxed{(\nabla_y \phi)(x,y) = e^{2\pi i y}\,(\nabla_y\phi)(x+1,y)}$$

$$\nabla^1 = d + 2\pi i x\, dy \qquad \nabla^2 =$$

$$e^{2\pi i x y}\phi(x,y) \qquad\qquad e^{2\pi i x y}\,\nabla^1\, e^{-2\pi i x y}$$

$$e^{2\pi i x y}\left(d + 2\pi i x\, dy\right)e^{-2\pi i x y} = d - 2\pi i\, d(xy) + 2\pi i x\, dy$$
$$= d - 2\pi i\, y\, dx$$

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Try to understand this better. Begin with $\mathbb{R}^2$ and a connection $\nabla = d + A$ on the trivial line bundle with curvature $dA = 2\pi i\, dx\, dy$. You want $A$ to be as close to a constant coeff form as possible, say $A = x\,dy$

different viewpt. Your $\mathbb{R}^2$ is an affine space, 
no preferred origin. Is there a natural kernel interpretation for sections of the Poincaré line bundle over $\mathbb{T}^2$?

start with objects. $\mathbb{R}^2$ affine vector space $\underbrace{\phantom{xx}}_{}$ with translation operators. have symplectic translation-invariant form
$\omega = dx\,dy$. Choose $A = dx\,A_1 + dy\,A_2$ 1-form on $\mathbb{R}^2$
whose $dA$ is $\omega$:

$$dA = -dx\,dy\,(\partial_y A_1) - dy\,dx\,(\partial_x A_2)$$
$$= dx\,dy\left(\partial_x A_2 - \partial_y A_1\right)$$

to solve $\partial_x A_2 - \partial_y A_1 = 1$. Look at homog. eqn.

$\partial_x B_2 = \partial_y B_1 \implies B_1 = \partial_x V, \quad B_2 = \partial_y V, \quad (B_1, B_2) = $ grad potential $V$. What are the interesting choices for $A$?

Take $A_1 = $ const. $A_2 = x + $ const. given $A = dy\,x$

So what?

Look at $A_2 = 2\pi i x$, $d + A = dx\,(\partial_x) + dy\,(\partial_y + 2\pi i x)$

What should you do at this stage

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go back to $f(x) \in \mathcal{S}(\mathbb{R})$

$$F(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m) \qquad F(x,y+1) = F(x,y)$$

$$e^{2\pi i x y} F(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i (x+m) y} f(x+m) \qquad \text{is periodic in } x$$

$$e^{2\pi i (x+1) y} F(x+1,y) = \cancel{e^{2\pi i x y}} F(x,y)$$

nice model of sections of $E$ over $\mathbb{T}^2$. Find a connection on $\mathcal{S}(\mathbb{R})$ as $C^\infty(\mathbb{T}^2)$. translation $e^{\partial_x}$ $e^{2\pi i x}$

$$\left(e^{\partial_x} f\right)(x) = f(x+1)$$

First define operators $\partial_x$ and $2\pi i X$ on $\mathcal{S}(\mathbb{R})$

$$\left(e^{t\partial_x} f\right)(x) = f(x+t) \qquad \left(e^{2\pi i X s} f\right)(x) = e^{2\pi i s x} f(x).$$

$$\nabla_x F(x,y) = \partial_x F(x,y)$$

$$\partial_y F(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i m\, f(x+m)$$

$$2\pi i x F(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i x\, f(x+m)$$

$$\left(\partial_y + 2\pi i x\right) F(x,y) = \sum_m e^{2\pi i m y}\, 2\pi i (m+x) f(x+m)$$

$$2\pi i x f(x) \longmapsto \sum_m e^{2\pi i m y}\, 2\pi i\, (x+m) f(x+m)$$

So you learn a little

$$\left(\Phi f\right)(x,y) \stackrel{\text{def}}{=} \sum_m e^{2\pi i m y} f(x+m)$$

$$\boxed{\partial_x \Phi f = \Phi \tfrac{d}{dx} f}, \quad \Phi(2\pi i x f) = \sum_m e^{2\pi i m y}\, 2\pi i\, (x+m) f(x+m)$$

$$\partial_y \Phi f = \sum_m e^{2\pi i m y}\, 2\pi i\, m\, f(x+m)$$

$$+ \qquad\qquad +$$

$$2\pi i x \Phi f = \sum_m e^{2\pi i m y}\, 2\pi i\, x\, f(x+m)$$

$$\boxed{\left(\partial_y + 2\pi i x\right)\Phi f = \Phi(2\pi i x f)}$$

$\Phi$ intertwines

$\partial_x$, $\partial_y + 2\pi i x$ and $\frac{d}{dx}$, $2\pi i x$

This yields the connection on your line bundle.

From a connection you get parallel transport along curves.

transform $f$ in $\mathcal{S}(\mathbb{R})$ $\qquad (\Phi f)(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m)$

$$\partial_x \Phi f = \Phi\left(\frac{d}{dx} f\right)$$

$$\left(\partial_y + 2\pi i x\right)\Phi f = \sum_{m\in\mathbb{Z}} e^{2\pi i m y}\, 2\pi i (m+x) f(x+m)$$

$$= \Phi(2\pi i x f)$$

$$\Phi(xf) = \left(\frac{1}{2\pi i}\partial_y + x\right)\Phi(f).$$

Why not define operator directly on $\mathcal{S}(\mathbb{R})$.

$$\nabla_x f = \frac{d}{dx} f \qquad \nabla_y f = 2\pi i x f \qquad [\nabla_x, \nabla_y] = \left[\frac{d}{dx}, 2\pi i x\right]$$

$$= 2\pi i$$

Next compute $e^{s\nabla_x + t\nabla_y} f$

$$XY = YX + H$$

$[X,Y] = \tilde{h}$ scalar

$$e^{X+Y} = 1 + X + Y + \frac{X^2 + XY + YX + Y^2}{2!} + \text{3rd order}$$

$$e^X e^Y = \left(1 + X + \frac{X^2}{2}\right)\left(1 + Y + \frac{Y^2}{2}\right)$$

$$= 1 + X + Y + \frac{X^2 + 2XY + Y^2}{2} + \frac{H}{2}$$

point $\qquad XY + YX = 2YX + H$

$$\cancel{e^{X+tY} = 1 + X + tY + \frac{X^2}{2} + t(XY+YX) + t^2\frac{Y^2}{2}}$$

$$e^{-X} e^{X+Y} e^{-Y} =$$

perturbation expansion

$$e^{-tX} e^{t(X+Y)} \qquad \sim \frac{1}{s+X}$$

pert. exp. and S-matrix. What is the basic idea?

To solve $e^{tH}\partial_t \psi = (H+V)\psi$ $\quad \cancel{\partial_t\left(e^{-tH}\psi\right) =}$

$$\partial_t\left(e^{-tH}\psi\right) = \cancel{-He^{-tH}\psi} + e^{-tH}(\cancel{H}+V)\psi$$

$$e^{-tH}\psi = \int_?^t e^{-tH} V \psi$$

745 a ~~to~~ 849
return to math
after tax stuff

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$$\frac{1}{\lambda-H-V} = \frac{1}{\lambda-H} + \frac{1}{\lambda-H} V \frac{1}{\lambda-H} + \cdots$$

$$\left[V, \frac{1}{\lambda-H}\right] = + \frac{1}{\lambda-H}\left(\underbrace{+[V,H]}\right)\frac{1}{\lambda-H}$$

$$V\frac{1}{\lambda-H} = \frac{1}{\lambda-H}V + \frac{1}{\lambda-H}[V,H]\frac{1}{\lambda-H}$$

---

$$e^X Y e^{-X} = Y + [X,Y]$$

$$\cancel{e^X Y^2 e^{-X} = Y^2 + 2YH + H^2}$$

$$e^X e^Y e^{-X} = e^Y e^H \qquad e^X e^Y e^{-X} e^{-Y} = e^{[X,Y]}$$

$$e^{-X} e^{X+Y} \qquad e^{-tX} e^{t(X+Y)}$$

---

$[X,Y] = H$ sat. $[X,H] = [Y,H] = 0$

$$e^{-tX} e^{t(X+Y)} = u(t) \qquad u'(t) = \cancel{e^{-tX}(-X)e^{t(X+Y)}} + e^{-tX}(X+Y)e^{t(X+Y)}$$

$$= e^{-tX} Y e^{t(X+Y)}$$

$$= e^{-tX} Y e^{tX} u(t)$$

$$= (Y \mp tH)u(t)$$

$$u(t) = e^{tY \mp \frac{t^2}{2}H}$$

$$\boxed{e^{-tX} Y e^{tX} = \sum \frac{t^n}{n!} \mathrm{ad}(-X)^n Y = e^{-t\,\mathrm{ad}(X)} = Y - tH}$$

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}H} \qquad e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]}$$

Prop. $X, Y$ commute with $H = [X,Y]$ $\Rightarrow$

$$e^{X+Y} = e^X e^Y e^{-\frac{1}{2}H}$$

$$\frac{XY+YX}{2} \qquad XY - \frac{1}{2}(XY-YX)$$

apply to $f(x) \in \mathcal{S}(\mathbb{R})$ $\quad X = \partial_x \quad Y = 2\pi i x$

$$e^{aX+bY} f(x) = e^{aX} e^{bY} e^{-\frac{ab}{2}2\pi i} f(x)$$

$$= e^{-ab\pi i} e^{aX} e^{2\pi i b x} f(x)$$

$$= e^{-\pi i a b} e^{2\pi i b(x+a)} f(x+a)$$

$$\boxed{e^{aX+bY} f(x) = e^{\pi i a b} e^{2\pi i b x} f(x+a)}$$

Interesting point: sign occurs even for $a, b \in \mathbb{Z}$

---

Repeat. $u(t) = e^{-tX} e^{t(X+Y)}$, $u'(t) = e^{-tX}(-X+X+Y)e^{t(X+Y)} =$

$$\underbrace{e^{-tX} Y e^{tX}}_{} u(t).$$

$$\sum_{n\geq 0} \frac{(-t)^n}{n!} \operatorname{ad}(X)^n Y = Y - tH. \quad \therefore \quad u'(t) = (Y - tH)u(t)$$

since $[Y,H] = 0$ $\quad \partial_t\left(e^{tY} e^{-\frac{t^2}{2}H}\right) = \underbrace{e^{tY}(Y - tH)}_{} e^{-\frac{t^2}{2}H}$

so $u(t) = e^{tY} e^{-\frac{t^2}{2}H}$ $\quad \therefore \quad e^{-tX} e^{t(X+Y)} = e^{tY} e^{-\frac{t^2}{2}H}$ so

$$e^{t(X+Y)} = e^{tX} e^{tY} e^{-\frac{t^2}{2}H}$$

---

Now you want to look at $SL(2,\mathbb{Z})$ acting on $\mathbb{R}^2/\mathbb{Z}^2$, with the aim of lifting this action to the line bundle of degree 1. You need the right viewpoint. Perhaps the Chern-Weil theory is relevant. I feel that because the group is descrete this should be irrelevant.

Let's review the torus picture. Start with the principal bundle $\mathbb{Z}^2 \rightarrowtail \mathbb{R}^2 \twoheadrightarrow (\mathbb{R}/\mathbb{Z})^2$. You get flat line bundles from characters of $\mathbb{Z}^2$. But these are topologically trivial since $(\mathbb{Z}^2)^\wedge \simeq \mathbb{T}^2$ is connected. You want the line bundle of degree 1. Its construction seems to involve, introduce a connection with constant curvature. But the problem of describing the line bundle $\mathcal{L}$ of degree 1 over $\mathbb{T}^2$ should be purely topological. Your aim is to understand whether there is an action of $SL(2,\mathbb{Z})$ on this $\mathcal{L}$ which covers that action of $SL(2,\mathbb{Z})$ on $\mathbb{T}^2$. You might have to use the metaplectic extension.

Start with $\Gamma(\mathcal{L})$ = space of $\phi(x,y)$ smooth on $\mathbb{R}^2$ satisfying $\phi(x,y+1) = \phi(x,y) = e^{2\pi i y}\phi(x+1,y)$. Write as

$$\left.\begin{aligned}\phi(x,y+1) &= \phi(x,y)\\ \phi(x+1,y) &= e^{-2\pi i y}\phi(x,y)\end{aligned}\right\} \Longrightarrow \phi(x+m,y+n) = e^{-2\pi i m y}\phi(x,y)$$

$\Gamma(\mathcal{L})$ is a module over $\mathcal{A} = C^\infty(\mathbb{T}^2) = \Gamma(\mathcal{O})$ = ring of doubly-periodic smooth fns. $f(x,y) = f(x+1,y) = f(x,y+1)$.

You should be able to show that $\Gamma(\mathcal{L})$ is a summand of $\mathcal{A}^{\oplus 2}$ using a partition of unity which is constant in $y$ and corresponds to an open covering of the $x$ circle by two intervals.

Action of $SL(2,\mathbb{Z})$ on $\mathcal{A} = \Gamma(\mathcal{O})$ should be

$$f(x,y) \longmapsto f\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right) = f(ax+by,\ cx+dy)$$

$$f(x,y) = f\begin{pmatrix}x\\y\end{pmatrix}$$

Check $f\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x+m\\y+n\end{pmatrix}\right) = f\begin{pmatrix}ax+by+\underbrace{am+bn}_{\in\mathbb{Z}}\\ cx+dy+\underbrace{cm+dn}_{\in\mathbb{Z}}\end{pmatrix}$

generators for $SL(2,\mathbb{Z})$? $\begin{pmatrix}1&1\\ &1\end{pmatrix}$ $\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

this you should be able to handle.

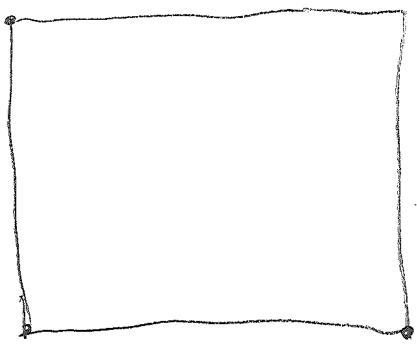
Action of $SL_2(2,\mathbb{Z})$ on $a = \Gamma(\theta)$ is simply

$$f\begin{pmatrix}x\\y\end{pmatrix} \longmapsto f\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\right) = f\begin{pmatrix}ax+by\\cx+dy\end{pmatrix}$$

Take $\begin{pmatrix}a&b\\c&d\end{pmatrix} = \begin{pmatrix}1&1\\&1\end{pmatrix}$, get $f\begin{pmatrix}x\\y\end{pmatrix} \longmapsto f\begin{pmatrix}x+y\\y\end{pmatrix}$

Try same thing with $\phi$. The ~~basic~~ idea seems to be to keep one periodicity direction fixed.

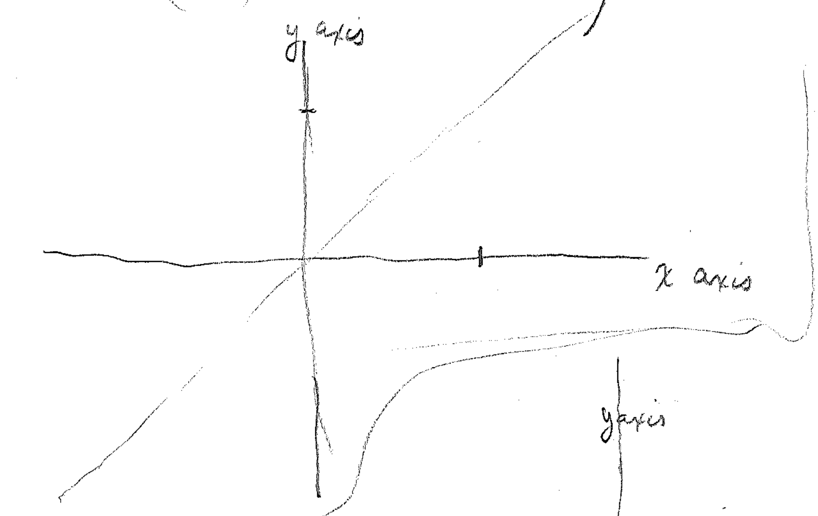
Go back to $\phi(x, y+1) = \phi(x,y) = e^{2\pi i y}\phi(x+1,y)$



Let $\psi(x,y) = \phi(x+y,y)$

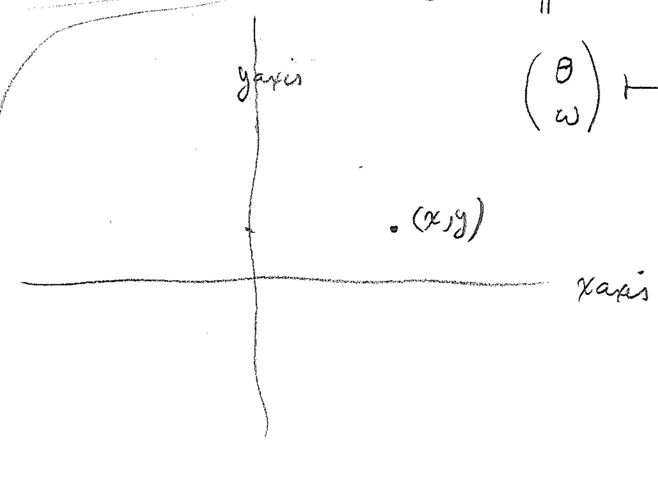
consider $\begin{pmatrix}1&1\\&1\end{pmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ $\quad \begin{pmatrix}x\\y\end{pmatrix} \quad \begin{pmatrix}x+y\\y\end{pmatrix}$ ?

$$\begin{pmatrix}\mathbb{T}\\\mathbb{T}\end{pmatrix} \longrightarrow \begin{pmatrix}\mathbb{T}\\\mathbb{T}\end{pmatrix} \qquad \begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x+y\\y\end{pmatrix} \quad ?$$




definitely you have $\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}x+y\\y\end{pmatrix}$ from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ inducing

$$\mathbb{R}^2/\mathbb{Z}^2 \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$$
$$\mathbb{T}^2 \longrightarrow \mathbb{T}^2$$
$$\begin{pmatrix}\theta\\\omega\end{pmatrix} \longmapsto$$



Last night's viewpoint (11 Jan 02). Consider a 2 torus $T$, more precisely, a compact connected abelian Lie group of dim 2. Then $T$ can be described canonically as $V/\Gamma$, where $V$ is a 2 diml real vector space and $\Gamma$ is a lattice in $V$ (free abelian group of rank 2 generating $V$). One has $\Gamma = \pi_1(T,0)$, $V = \mathrm{Lie}(T)$ = universal covering of $T$ $= \Gamma \otimes_{\mathbb{Z}} \mathbb{R}$. Upshot: a 2-torus $T$ is equivalent to a free abelian group $\Gamma$ of rank 2. $\Gamma = \mathrm{Hom}_{\text{Lie gbs}}(\mathbb{T}, T)$.

$\mathrm{Aut}(T) = \mathrm{Aut}(\Gamma) \simeq GL(2,\mathbb{Z})$.

Look next at line bundles over $T$; these are described by elements of $H^2(T,\mathbb{Z}) \simeq \mathbb{Z}$. Choosing an isom amounts to orienting $T$, then the degree of a line bundle is defined, and there is a unique line bundle up to isomorphism for each degree.

Problem: to see if there is a canonical construction of the line bundle of degree 1; call it $\mathcal{L}$. You think this should mean rigidifying $\mathcal{L}$, that is, equipping $\mathcal{L}$ with extra structure so that it has no automorphisms. You propose to rigidify with a connection and a point in the fibre of $\mathcal{L}$ over $0 \in T$. This should be closely related to YM.

---

Let's avoid connections until necessary. Instead, let's identify $C(\mathbb{T}^2)$ with $C(\mathbb{T}, C(\mathbb{T}))$ = the ring of continuous functions $F(x)$ for $x \in \mathbb{R}/\mathbb{Z}$ with values in the ring of cont. functions of $y \in \mathbb{R}/\mathbb{Z}$. Then $C(\mathbb{T}^2)$ can be viewed as the ring of cont. functions $F(x)$ for $x \in [0,1]$ with values in cont fns $y \in \mathbb{R}/\mathbb{Z} \longmapsto F(x,y)$, ~~Thus $C(\mathbb{T}^2)$ is the family of all cont sections over the $x$ circle with values in the ring $C(\mathbb{T})$ for~~ satisfying the periodicity cond. $F(0,y) = e^{2\pi i y}F(1,y)$

Somehow you want to describe the vertical structure ($x$ fixed) as a trivial fibre bundle with fibre $\mathbb{C}(\mathbb{T})$ $y$ direction with the ends identified via the clutching function $e^{2\pi i y}$

---

Start again with the basic problem. You have the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ which is a manifold, in fact Lie gp, on which $GL(2,\mathbb{Z})$ acts as Lie group autos. How $GL(2,\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\mathbb{R}^2, \mathbb{Z}^2$ via

$$(x, y) \longmapsto (ax+by, cx+dy)$$

The isomorphism classes of complex line bundles over $\mathbb{T}^2$ is $H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}$, so if $\mathbb{T}^2$ is oriented there is a unique line bundle of given degree up to isomorphism.

It's clear that restricting to the subgp $SL(2,\mathbb{Z})$ preserves orientation (hence degree), so the action of $SL(2,\mathbb{Z})$ on $\mathbb{R}^2/\mathbb{Z}^2$ preserves iso classes of line bundle. Better: Given $g \in SL(2,\mathbb{Z})$, then $g: T \longrightarrow T$ is a Lie gp auto preserving orientation, and you have pullback $g^*$ on functions $f$: $f \mapsto g^*f = f \circ g$ and also ________________ line vector bundles. Since degree is preserved and there is a unique line bundle of each degree up to isom. you have $\theta_g : g^*\mathcal{L} \simeq \mathcal{L}$ $\forall g$. Question: Can you choose $\theta_g$ so that you get an action of $SL(2,\mathbb{Z})$ on $\mathcal{L}$ covering the action on the base $T = \mathbb{T}^2$?

Calculate using the picture for the space of sections of $\mathcal{L}$ namely function $\phi(x,y)$ on $\mathbb{R}^2$ satisfying the periodicity conditions $\phi(x,y) = \phi(x, y+1)$ and $\phi(x,y) = e^{2\pi i y}\phi(x+1, y)$.

Question: How do you know this space of functions is the space of sections of a line bundle over $\mathbb{T}^2$? Embed in a trivial bundle of rank 2.

today to understand line bundle

$\mathcal{a}$ ring of $f(x,y)$ $\quad f(x+m, y+n) = f(x,y)$

$\mathcal{s}$ mod of $s(x,y)$ $\quad s(x+m, y+n) = e^{-2\pi i m y} s(x,y)$

to embed the module $\mathcal{s}$ as a retract of a free module

ignore $y$ direction - look at $f(x)$, $s(x)$ $\quad f(x+m) = f(x)$ $\quad s(x+m) = q^{-m} s(x)$.

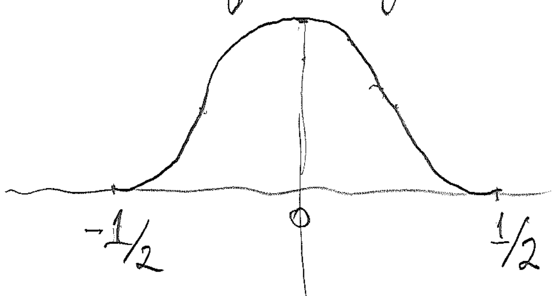
---

$\mathcal{a}$ = ring of continuous $f(x)$ $x \in \mathbb{R}$ periodic $f(x+m) = f(x)$

$\mathcal{s}$ = space of ——— $s(x)$ ——— sat $s(x+m) = q^{-m} s(x)$.

$\mathcal{s}$ is an $\mathcal{a}$-module. You want to construct $\mathcal{a}$-module maps $\mathcal{s} \xrightarrow{\alpha} \mathcal{a} \oplus \mathcal{a} \xrightarrow{\beta} \mathcal{s}$ $\quad \beta\alpha = 1$. Look first at $\beta$.

Partition of unity on $\mathbb{R}/\mathbb{Z}$ e.g.




$$\cos(\pi x)^2 + \sin(\pi x)^2 = 1$$

$$\frac{1+\cos(2\pi x)}{2} + \frac{1-\cos(2\pi x)}{2} = 1$$

There still seems to be a problem unless $\cos(\pi x)$, $\sin(\pi x)$ which have period 2 are replaced by $|\cos(\pi x)|$ and $|\sin(\pi x)|$

---

Consider principal $\mathbb{Z}$-bundle $\begin{array}{c}\mathbb{R}\\ \downarrow \pi \\ \mathbb{R}/\mathbb{Z}\end{array}$, let $\mathbb{Z}$ act on $\mathbb{C}$ by $n \mapsto$ mult by $q^n$

form. $\begin{array}{c}\mathbb{R} \times^{\mathbb{Z}} \mathbb{C}_{(q)}\\ \downarrow \pi \\ \mathbb{R}/\mathbb{Z}\end{array}$ get complex line bundle $\mathcal{L}_q$

$$\Gamma(\pi U, \mathcal{L}_q) = \left\{ f : U \to \mathbb{C} \text{ cont.} \;\middle|\; f(x+m) = q^{-m} f(x) \quad x \in U \right\}$$

suppose $U = (a, b)$, $0 < b-a < 1$ $\quad \pi : U \xrightarrow{\sim} \pi U$

Begin with $T = \mathbb{R}^2/\mathbb{Z}^2$, $\mathcal{a} =$ smooth cont fns $f(x,y)$ fixed under $(x,y) \mapsto (x+m, y+n)$. 856

$\mathcal{S} = \Gamma(T, \mathcal{L}) =$ smooth cont fns $s(x,y)$ sat $s(x, y+1) = s(x,y)$
$s(x+1, y) = e^{-2\pi i y} s(x,y)$

$$s(x+2, y) = e^{-2\pi i y} s(x+1, y) = \left(e^{-2\pi i y}\right)^2 s(x,y)$$

~~$s(x+m, y)$~~ $\boxed{s(x+m, y+n) = e^{-2\pi i m y} s(x,y)}$

change variables $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "g"] \arrow[d, "\pi"] & \mathbb{R}^2 \arrow[d, "\pi"] \\ \mathbb{T}^2 \arrow[r, "g"] & \mathbb{T}^2 \end{tikzcd}$$

When you have a fibre bundle you should have pull back. You know that $g^*\mathcal{L} \simeq \mathcal{L}$ for $g \in SL(2,\mathbb{Z})$, and want to find an explicit isom. (hopefully compatible with composition)

What is $g^*\mathcal{L}$?

$C^\infty(\mathbb{R}^2) \supset g^*\mathcal{S}$ $\qquad \mathcal{S}$ $\qquad \mathcal{a} \otimes_{\underset{g^*}{\leftarrow} \mathcal{a}} \mathcal{S}$

$$\mathcal{a} \xleftarrow{g^*} \mathcal{a}$$

$$\begin{tikzcd} s & & sg^{-1} \\ C^\infty(\mathbb{R}^2) & & C^\infty(\mathbb{R}^2) \arrow[ll, "g^*"'] \\ C^\infty(\mathbb{T}^2) \arrow[u, "\pi^*"] & & C^\infty(\mathbb{T}^2) \arrow[ll, "g^*"'] \arrow[u, "\pi^*"'] \\ f & & fg^{-1} \end{tikzcd}$$

$$(sg^{-1})(x,y) = s(dx - by, -cx + ay)$$

$$\cancel{sg^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = s\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = s\left\{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ n \end{pmatrix}\right\}}$$

$$sg^{-1}\begin{pmatrix} x \\ y \end{pmatrix} = sg^{-1}\begin{pmatrix} x \\ y+n \end{pmatrix} = sg^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$$

Try variables. A section of $\mathcal{L}$ is a function $s(x,y)$ on the $x,y$ plane sat. $s(x+m, y+n) = e^{-2\pi i m y} s(x,y)$

A function in $\mathcal{A}$ is a function $f(x,y)$ on the $x,y$ plane sat $f(x+m, y+n) = f(x,y)$. Now change ind variables

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Think of $f$ as a variable depending on $x$ and $y$ then it becomes a variable dep on $u$ and $v$, and is double periodic, since $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \longmapsto \begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{Z} \oplus \mathbb{Z}$

Now try $s$ is a variable dep on $x,y$ so it become dep on $u,v$. ~~$s(x+1,y)=$~~ $s(x,y+1) = s(x,y)$.

$s$ is unchanged by $\begin{matrix} x \longmapsto x \\ y \longmapsto y+1 \end{matrix}$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y+1 \end{pmatrix}$

by $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} b \\ d \end{pmatrix}$. WRONG

---

next $s(x+1,y) = e^{-2\pi i y} s(x,y)$

introduce $\tilde{s}(x,y) = e^{2\pi i xy} s(x,y)$. Then $\tilde{s}(x+1,y) = \tilde{s}(x,y)$.

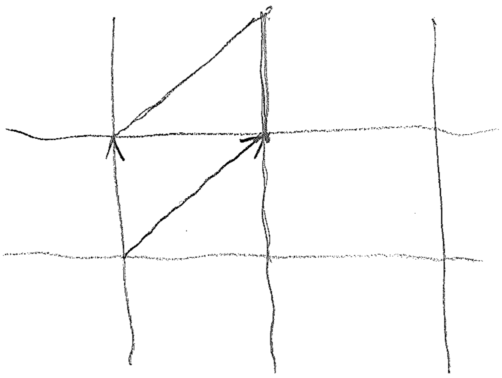
the variable $\tilde{s}$ is unchanged by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

i.e. $\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a \\ c \end{pmatrix}$

$$\tilde{s} = e^{2\pi i (au+bv)(cu+dv)} s(u,v)$$

Try a different approach, namely $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$s(x,y+1) = s(x,y) = e^{2\pi i y} s(x+1,y)$$
$$s(x+1,y) = e^{-2\pi i y} s(x,y)$$



$$s(x+1,y+1) = e^{-2\pi i (y+1)} s(x,y+1)$$

---

Calculate what happens in the direction of (1, 1)

$$s(x+m,y+m) = e^{-2\pi i m(y+m)} s(x,y+m) \qquad ?$$

Start again
$$s(x,y+1) = s(x,y)$$
$$s(x+1,y) = e^{-2\pi i y} s(x,y)$$

$$s(x+m,y+n) = s(x+m,y) = e^{-2\pi i m y} s(x,y)$$

---

An element of $\mathcal{A}$ is a variable $f$ dependent on $x,y$ which is unchanged under $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Change ind. variables to $u,v$ via $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

then $f$ unchanged under

$$\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} d \\ -c \end{pmatrix}$$

$$\text{"} \qquad \text{"} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -b \\ a \end{pmatrix}$$

$s$ dep on $x,y$ unchanged under $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\tilde{s} = e^{2\pi i xy} s$ unchanged under $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

change ind vbls to $\begin{pmatrix} u \\ v \end{pmatrix}$ via $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

whence $s$ unchanged under $\begin{pmatrix} u \\ v \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} -b \\ a \end{pmatrix}}$

$\tilde{s} = e^{2\pi i xy} s$ unchanged under $\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} d \\ -c \end{pmatrix}$

$$xy = (au+bv)(cu+dv)$$

$$\boxed{\tilde{s}(u,v) = e^{2\pi i (au+bu)(cu+dv)} s(u,v) \quad \text{inv. under } \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} d \\ -c \end{pmatrix}}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$s(x,y) = \sum e^{2\pi i m y} \phi(x+m) \qquad \begin{aligned} x &= u+v \\ y &= v \end{aligned}$$

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discuss

$$s(x+1, y+1) = s(x+1, y) = e^{-2\pi i y} s(x,y).$$

$$g(x+1, y+1) - g(x,y) = -y$$

$$-g(x,y) = \tfrac{1}{2} y(y-1)$$

$$\boxed{e^{\pi i y(y-1)} s(x,y) = \tilde{s}(x,y).}$$

$$\tilde{s}(x+1, y+1) = e^{\pi i (y+1)(y)} s(x+1, y+1) = \underbrace{e^{\pi i (y+1) y} e^{-2\pi i y}}_{} s(x,y)$$

$$e^{\pi i (y^2 - y)} = e^{\pi i y^2 + \pi i y - 2\pi i y}$$

$$\therefore \boxed{\tilde{s}(x+1, y+1) = \tilde{s}(x,y)}$$

Look at the discrete situation, namely, functions $s(x,y)$ defined for say $x,y \in \mathbb{Z}^2$ subject to $s(x,y+1) = s(x,y) = \underbrace{e^{+2\pi i y}}_{1} s(x+1,y)$. In this case $s$ is constant. Suppose that $(x,y) \in$ some coset of $\mathbb{R}^2/\mathbb{Z}^2$. Maybe work

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$$f(x) \longmapsto \tilde{f}(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+m) \quad \text{periodic in } y$$

$$e^{2\pi i x y}\tilde{f}(x,y) = \sum_m e^{2\pi i (x+m) y} f(x+m) \quad \text{per. in } x$$

$$e^{2\pi i \cancel{(x+1)} y}\tilde{f}(x+1,y) = \cancel{e^{2\pi i x y}}\, s(x,y)$$

$$\nabla_x = \partial_x, \qquad e^{2\pi i x y}(2\pi i x)\tilde{f} + e^{2\pi i x y}\partial_y \tilde{f} = 2\pi i\, \widetilde{xf(x)}$$

$$\partial_y\left(e^{2\pi i x y}\tilde{f}(x,y)\right) = e^{2\pi i x y}\left(\underbrace{(\partial_y + 2\pi i x)}\tilde{f}\right) = \sum_m e^{2\pi i m y} 2\pi i (m+x) f(x+m)$$

$$\boxed{\begin{aligned}\nabla_x &= \partial_x \\ \nabla_y &= \partial_y + 2\pi i x\end{aligned}}$$

gauge transf. $\mathbb{T}^2 \longrightarrow \mathbb{T}$ are there any?

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Look at Serre theorem over the circle. Begin with

$$\Gamma(\mathbb{R}/\mathbb{Z}, E) = \{ f : \mathbb{R} \to V \mid f(x+m) = g^{-m} f(x) \} \qquad g \in \mathrm{Aut}(V)$$

cont. functions same as $f : [0,1] \longrightarrow V$ $\quad f(1) = g^{-1} f(0)$.

$\chi_0(x)$ $\quad f : (-\varepsilon, 1+\varepsilon)$ $\quad f(x+1) = g^{-1} f(x)$

$-\varepsilon < x < \varepsilon$