Consider a real f.d. vector space $X$ with a scalar product $(x_1, x_2) = x_1^T x_2$. Let $y^T : x \mapsto y^T x = (y, x)$ be a nonzero lin. fun. Short exact sequence $0 \rightarrow K \rightarrow X \rightarrow \mathbb{R}$ where $K = (y^T)^\perp$. Orthogonal splitting $X = K \oplus R_y$.

You want to push forward, or descend, the scalar product on $X$ to the quotient space $y^T : X \rightarrow \mathbb{R}$. This means lifting $c \in \mathbb{R}$ to an elt of $K^\perp = R_y$, i.e. you want $\lambda \in \mathbb{R}$ such that $y^T (\lambda y) = c$, which gives $\lambda = \frac{c}{(y, y)}$, hence the lift is $\lambda y = \frac{cy}{(y, y)}$. Then you restrict the scalar product on $X$ to $K^\perp \cong \mathbb{R}$ which gives $\left( \frac{cy}{(y, y)} \right) \left( \frac{cy}{(y, y)} \right) = \frac{c^2}{(y, y)}$.

Next, do the same for scalar product $x_1^T A x_2 = (x_1, x_2)^T A$ for a pos. def. $A$. Take $c \in \mathbb{R}$ choose $\lambda A^{-1} y$ so that $y^T (\lambda A^{-1} y) = c$ i.e. $c = (y, \lambda A^{-1} y)$ or $\lambda = \frac{c}{(y, A^{-1} y)}$. Then you have the lifting $\lambda A^{-1} y \quad \text{which has norm} \quad \Lambda = \left( \frac{c A^{-1} y}{(y, A^{-1} y)} \right)^T A \left( \frac{c A^{-1} y}{(y, A^{-1} y)} \right) = \frac{c^2}{(y, A^{-1} y)}$.
Signal Flow Graph | Block Diagram equivalent, used to represent graphically the transfer function.

\[ E(s) \xrightarrow{G(s)} C(s) \]

... need also the summing junction ...

\[ E(s) \xrightarrow{G(s)} C(s) \]

... The summing junction for signal flow graph is represented by the branches into a node ...

\[ E(s) = R(s) - H(s)C(s) \quad R(s) \text{ input sig} \]

\[ C(s) = G(s)E(s) \quad C(s) \text{ output sig} \]

\[ E(s) \text{ int. sig} \]

\[ H(s) \text{ trans} \]

There are two different simulation diagrams (type of block diagram or flow graph which realizes a transfer function) e.g. let

\[ G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} \]

CONTROL CANONICAL FORM
Review: $X$ equipped with $(x_1, x_2) = x_1^t x_2$, positive definite, $y^t = (y, -): X \to \mathbb{R}, y \neq 0$. Calculate the induced scalar product in the quotient $X / y^t \to \mathbb{R}$.

You have $X = \text{Ry}^+ \oplus \text{Ry}^-$, orthogonally splitting $\text{Ry}^t = \text{Ker} y^t$. Take $c \in \mathbb{R}$ and lift it into $\text{Ry}$ the orth comp. of $K = \text{Ker} y^t$, say lift $c$ to $x = \lambda y$, where $\lambda$ is determined by $y^t(\lambda y) = c$, i.e. $\lambda = \frac{c}{(y, y)}$. Then the norm of $c$ is the norm of $x$: $(x, x) = \lambda^2 (y, y) = \frac{c^2}{(y, y)}$.

Next you want to link this to adjoining a branch. Here $X = \overline{C}$ is the mode potential space, and the scalar product is induced by $S: \overline{C} \to C$ from the power pos def form $VR^t V$ on $C^!$. To simplify things suppose the graph is a tree, whence $S: \overline{C} \to C^!$. Also $y^t: X \to \mathbb{R}$ is $y \mapsto y_A$, where a ground $O$ has been chosen $\neq A$ and $y_0 = 0, \forall y \in \overline{C}$.

You now to impose the condition $y_A = c$, i.e. pass to an affine hyperplane in $X = \overline{C}$ and also enlarge $C^!$ to $C^! \oplus \mathbb{R}$. 
Try again to understand attaching a branch in order to handle an external emf applied between two nodes, say A and the ground O. Let X be the node potential space $C^0$ equipped with the power form $(x_1, x_2)$ induced from the power form on $C^1$ via the embedding $\phi: C^0 \rightarrow C^1$.

To simplify suppose the graph is a tree, so that $\phi$ is an isomorphism.

Discuss the problem. You have the linear bounded functional $y^t: X \rightarrow \mathbb{R}$ given by $x = y \rightarrow y^t A - y^t O$. You want to fix a value $c \in \mathbb{R}$ and restrict $x$ to the affine hyperplane $y^t x = c$. In this way the $x$ variable has 1 less degree of freedom. So far you haven't changed the number of edges, so you expect to increase $H'$ by 1 dim.

You must understand the linear equations which determine the voltages. Recall the Thévenin situation where the inhomogeneous forcing term is any elt of $C^1$, and one solves $V - RI = E$ with $V, I$ subject to the Kirchhoff conditions.

Discuss because I appear. The point is that $E \in C^1$ is arbitrary while $V \in C^1$ satisfies the Kirchhoff voltage condition.

$$C^0 \rightarrow C^1 \rightarrow H^1$$

It should be true that $E = V + (-RI)$ is the orthogonal splitting of this short exact sequence for the power form on $C^1$. 
I think that you want an analog of the Thue-Vin idea that the inhomogeneous forcing terms should be any elt \( E \) of \( C^1 \). ??

\[ \begin{align*}
X &= C^0 \otimes C^1, \\
X &= \mathbb{R} y^\perp \otimes \mathbb{R} y \\
\text{orthog splitting,}
\end{align*} \]

\[ y \downarrow \]

Can you specify what you want \( D \) or need? \( D \)

Go back to the idea that \( X = C^0 \) remains the same after the new edge is added.

Review the problem. You have a Euclidean space \( X = C^0 \otimes C^1 \) with inner product \((x_1, x_2)\) together with an affine constraint \( c = y^\perp x \) which requires \( x \) to be in some affine hyperplane.

You want the minimum power configuration \( x_c \) subject to this constraint and the power \((x_c, x_c)\) as a function of \( c \).

The Lagrange multiplier method seems to solve this problem by adjoining a new variable \( \lambda \).

\[ F = \frac{1}{2}(x, x) + \lambda(c - (y, x)) \]

Note that \( F \) is quadratic in \( x, \lambda \) because \( y \) constant.

Stationary pt: \[
\begin{align*}
\nabla_x F &= x - \lambda y = 0 \\
\nabla_\lambda F &= c - (y, x) = 0
\end{align*}
\]

You can treat \( c \) as a variable, it seems?

So now we have a new idea to develop, starting from the Lagrange multiplier treatment of an affine constraint.
\[ F = \frac{1}{2}(x, x) + \lambda(c - (y, x)) \]  This is a quadratic form on \( X \oplus R \) the space of \((x, \lambda)\). Instead of restricting \( x \) to the affine hyperplane \( c = (y, x) \) and then finding the critical point \& value, you leave \( X \) alone but add the parameter \( \lambda \) to force the constraint.

Start with even \( R \) network \( K \) with \( AA \) nodes \( 0 \) \& \( 0 \) external nodes \( A \) and the ground \( 0 \) \( A \neq 0 \), and let \( L = K \) with a branch attached joining \( A \) and \( 0 \). Then you have square \( R \) more elegant.

\[ I \rightarrow I \quad I \rightarrow K \]
\[ K \rightarrow L \quad I \rightarrow L \]

Assume \( K \) a tree then \( H^\prime K = 0 \) and you have

\[ \begin{align*}
C_0 K \leftrightarrow C_1 K & \rightarrow H^\prime K \\
\uparrow & \uparrow \\
C_0 L \leftrightarrow C_1 L & \rightarrow H^\prime L \\
\uparrow & \uparrow \\
0 & \overset{R}{=} C_1(L, K) = H^\prime(L, K)
\end{align*} \]
So from $K \rightarrow L \rightarrow L/K$ you get where $K$ is a tree

$$
\mathcal{C}K \sim \mathcal{C}^1K \quad 0
$$

$$
\mathcal{C}L \rightarrow \mathcal{C}^1L \rightarrow H'\mathcal{L}
$$

$$
\mathcal{R} = \mathcal{R}
$$

You actually have a canonical isomorphism $\mathcal{C}^1L = (\mathcal{C}^1K)^{\mathcal{R}}$.

What is your aim? To go from the $K$ picture, with the potential at $A$ fixed, and you find the minimum power configuration and power subject to this inhomogeneous constraint, and to proceed to the $L$ picture, which should fit the Thévenin scheme of handling inhomogeneous constraints by an element of $\mathcal{C}^1L$. There is a possible problem with a resistance value for the attached branch, but you hoping that this will be supplied the push forward of the scalar product on $\mathcal{C}^0K$ via the map $\varphi \mapsto \varphi_{\mathcal{A}} : \mathcal{C}^0K \rightarrow \mathcal{R}$.

Everything should follow from the following data $X$ a Euclidean space, $y^t : X \rightarrow \mathcal{R}$ nonzero linear functional (use of same as $y \in X$). Then $\mathcal{C}^0L \sim \mathcal{C}^0K \sim \mathcal{C}^1K$ are all canonically isom to $X$. Let $\hat{X} = \langle X \rangle_{\mathcal{R}}$ and let $\delta_L = \langle \frac{1}{y^t} \rangle : X \rightarrow \hat{X}$, i.e. $\delta_L$ is the graph of $y^t : X \rightarrow \mathcal{R}$. 
What do you need to do? You want $F = \frac{1}{2} x^t x + \lambda (c - y^t x)$ to give rise to a scalar product on $\hat{x}$. Better you want $\hat{x}$ to be $\mathbb{C}' L$, which means you need a scalar product on $\hat{x}$. To be more precise, you want $L$ to be a connected $R$-network like $K$, so that you can handle the inhomogeneous constraint à la Thévenin by a branch emf. Repeat: Given $X$ with scalar product $x^t x$, and a non-zero linear functional $y^t: x \to R$. Think of $X$ as the space of node emfs, and the constraint $c = y^t x$ as fixing the voltage drop between 2 nodes by attaching a battery of emf $= c$ between these nodes.

Picture: You have a tree $K$, and you attach a branch joining the two "external" nodes to obtain a graph $L$ with one loop.

$$\begin{array}{c}
\text{Try } X \xrightarrow{(\frac{1}{y^t})} (x) \ \ \ \text{for } \mathbb{C}' L \to \mathbb{C}' L
\end{array}$$

Is $(\frac{1}{y^t})$ an isometry? $(x) \in \mathbb{R}$ is equipped with the form $(x^t)\begin{pmatrix} I & 0 \\ 0 & \frac{1}{y^t y} \end{pmatrix} (x)$ which has norm

$$\begin{array}{c}
= x^t x + \frac{(y^t x)^2}{y^t y} > x^t x
\end{array}$$

so $(\frac{1}{y^t})$ is not an isometry, although it is if you restrict to the hyperplane $y^t x = 0$.

Let's look at this in a special case, where the tree $K$ is

$$\begin{array}{c}
\text{Then } X = \{ V_A - V_R \} \ \ \ \text{with the power function } \frac{V^2}{2R}
\end{array}$$

We now make the augmented graph
Thevenin voltage source (pure emf) in series with an internal resistance. Next you need to be the

\[ C^0L = C^0K = X = \{ \square \ V_A = V_R \} \]

\[ C^1L = \{ \square \ (V_R) \} \]

\[ E \] here is

\[ X = C^0L \rightarrow C^1L \]

Given \( V_A \) the node potential then \( E : V_A \rightarrow (V_R) \) where \( V_R = V_A \). What is \( E \)? You know the current \( I = \frac{V_A}{R} \) and \( E \) should be \( (E+R)I \), thus

\[ E = (E+R)I = (E+R) \frac{V_A}{R} \]

Note that if \( E=0 \) we get \( E = V_A \), which gives the required constraint.

There remains to find the power function on \( C^1L \). Actually it may be more illuminating to find the splitting of \( E \) into a node potential \( \) and an orthogonal loop voltage. Review. Consider the network:

What are the appropriate circuit equations. You have 2 nodes -1 +1 loop = 2 branches

\[ V_A \rightarrow (V_A) \rightarrow (R) \rightarrow (\epsilon) \]

\[ C^0 \leftrightarrow C^1 \leftrightarrow H \]

In \( C^1 \) you get approx

\[ (V_A) + (RI) = (0) \]

\[ (V_A) + \epsilon I = (E) \]

Add \( (R+\epsilon)I = E \), also you have \( V_A + RI = 0 \)

\[ V_A = R I \]

\[ V_R = V_A - V_0 = RI \]

\[ V_e = V_0 - V_A = \epsilon I - E \]

\[ 0 = (R+\epsilon)I - E \]
\[ \varepsilon I - \varepsilon = V_0 - V_A = -V_B \]

way to think: You start at potential \( V_0 \) before resistor, \( \Rightarrow V_0 = V_B = \varepsilon I \)

\[ V_B - V_A = -\varepsilon \quad \Rightarrow \quad V_0 - V_A = \varepsilon I - \varepsilon. \]

\[
\begin{align*}
\begin{pmatrix}
C_0 & \rightarrow & C_1 \\
V_A & \rightarrow & \begin{pmatrix} V_R \\ V_\varepsilon \end{pmatrix} \\
& & \begin{pmatrix} R_0 \\ 0 \varepsilon \end{pmatrix}
\end{pmatrix}
\end{align*}
\]

\[ (I) \leftarrow I \]

\[ (1-1) \]

\[ C_0 \leftrightarrow C_1 \]

\[ H_1 \]

Equations

\[
\begin{pmatrix} -V_A \\ +V_A \end{pmatrix} + \begin{pmatrix} RI \\ \varepsilon I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \varepsilon \end{pmatrix}
\]

\[ \Rightarrow \quad V_A = RI \quad \text{and} \quad (R+\varepsilon)I = \varepsilon \]

Here seems to be the sign mistake: to add \( V_A \) and \( RI \), Correct is

\[ V_A - V_0 = RI \quad \text{or} \quad V_A - RI = V_0 \]

Look again at

\[
\begin{align*}
\delta V_A & = \begin{pmatrix} V_R \\ V_\varepsilon \end{pmatrix} \\
V_A - V_0 &= RI \\
V_0 - \varepsilon I + \varepsilon &= V_A \\
\text{add} \quad V_0 - V_A &= \varepsilon I - \varepsilon \\
0 &= (R+\varepsilon)I - \varepsilon
\end{align*}
\]
What should be the power function on \( C' \)?

An element of \( C' \) consists of a voltage drop \( V_R \) for the \( R \) branch, and a voltage drop \( V_e \) for the \( e \) branch. How do you handle \( V \)? I think you want to set \( E = 0 \) and get the homogeneous linear system to be nondegenerate. Afterward you introduce \( E \) as an inhomogeneous term.

On the other hand, these might be a simple variational way to handle inhomogeneous terms.

Start again with

\[
\frac{V_R^2}{2R} + \frac{V_e^2}{2E} \quad \text{on} \quad C'
\]

Consider the variational problem of minimizing the power subject to the Kirchhoff constraint \( V_R + V_e = 0 \).

Review: The point of departure seems to be the "voltage picture" of a connected \( R \)-network. This amounts, to a real v.s. \( C' \) of branch voltages, which is equipped with a positive def. scalar product "the power", and also a subspace \( \mathbb{C} \subset C' \) of "conservative" branch voltages; elements of \( \mathbb{C} \) can also be identified with node potentials.

With this data you can form a short exact sequence

\[
\mathbb{C} \rightarrow C' \rightarrow H^1,
\]

and use the scalar product to split this sequence orthogonally, where you have induced scalar products on \( \mathbb{C} \) and \( H^1 \), which combine to give the scalar product on \( C' \).

Then any \( E \in C' \) splits into a node potential \( \Phi \in \mathbb{C} \) and a branch voltage \( E \in C' \).
You can combine the voltage picture with the dual "current picture" consisting of current spaces:

\[ \mathcal{C} \xrightarrow{R} \mathcal{C'} \xrightarrow{s} \mathcal{H'} \]

\[ \mathcal{C}_0 \leftrightarrow \mathcal{C'} \leftrightarrow \mathcal{H}_1 \]

\( H_1 \) is the space of closed currents or "loops", \( R \) is isomorphism: \( C_1 \cong C' \) arising from the scalar product on \( C' \).

Review: 4 Unknowns \( (V_R, I_R, V_e, I_e) \)

\[ \begin{align*}
I_R &= I_e \\
V_e + V_R &= 0 \\
V_R &= RI \\
-V_e &= -\varepsilon I + \varepsilon
\end{align*} \]

\[ V_R = \frac{R}{(R+\varepsilon)}I - \varepsilon \]

The question should be whether these linear equations can be gotten from the voltage picture? From the power form on \( C' \) together with the Kirchhoff voltage constraints.

**IDEA**: Maxwell's Equations \( \partial A = 0, \partial^* A = 0 \) (charge, current) are half homogeneous linear, specifically \( \nabla \cdot B = 0, \epsilon B + \nabla \times E = 0 \). This suggests that the voltage picture with inhomogeneous terms given by voltage sources should be a natural object. It's similar to the Lagrangian approach to a harmonic oscillator, where you work in configuration space with a quadratic \( L = KE - PE + \text{linear term in position} \).
should be allowed inhomogeneous forcing terms. This suggests that the voltage picture should yield all the information about the state of the network. You would like to treat Thevenin emfs in the branches. Is there some variational problem which yield these Thevenin inhomogeneous terms.

\[ C^0 \rightarrow C^1 \rightarrow H^1 \]

\[ C^0 \oplus RI \leftarrow \text{orthogonal splitting for power, } -V+RI \]

Example:

\[ V_A \leftarrow (V_A) = (V_R) \mapsto V_R + V_\varepsilon \]

\[ \varepsilon = (-V_A) + (RI) \]

Variational Problem

\[ F = \frac{V_R^2}{2R} + \frac{V_\varepsilon^2}{2\varepsilon} - \lambda (V_R + V_\varepsilon) \]

Yields equations

\[ \frac{V_R}{R} = \lambda, \quad \frac{V_\varepsilon}{\varepsilon} = \lambda, \quad V_R + V_\varepsilon = 0 \]

It seems right to interpret \( \lambda \) as the current \( I \) and then you get

\[ 0 = V_R + V_\varepsilon = (R + \varepsilon)I, \]

which means you've left out the emf \( \varepsilon \).

So look at

\[ V_R + V_\varepsilon - \varepsilon = 0 \]

So you should have had

\[ F = \frac{V_R^2}{2R} + \frac{V_\varepsilon^2}{2\varepsilon} + \lambda (\varepsilon - V_R - V_\varepsilon) \]

\[ 0 = \frac{V_R}{R} - \lambda = \frac{V_\varepsilon}{\varepsilon} - \lambda, \quad \varepsilon = V_R + V_\varepsilon = (R + \varepsilon)\lambda \]

\[ F = \frac{RA^2}{2} + \frac{\varepsilon A^2}{2} = \frac{R + \varepsilon}{2} \frac{\varepsilon^2}{(R + \varepsilon)^2} = \frac{\varepsilon^2}{2(R + \varepsilon)} \quad \text{critical value} \]
Yesterday you made some progress treating a branch emf as an inhomogeneous constraint using Lagrange multipliers. In more detail, given:

\[
\mathbb{C}^0 \xrightarrow{\mathbb{C}^1} H^1
\]

Together with the power sealer product on \( \mathbb{C}^1 \), and a branch emf \( \mathcal{E} \) (internal emfs on the branches), you get a splitting

\[
\mathbb{C}^0 \oplus (\mathbb{C}^0)^{-1}
\]

Review example. First find the circuit eqns involving both voltages + currents.

\[
\begin{pmatrix}
V_R \\
V_e
\end{pmatrix}, \begin{pmatrix}
I_R \\
I_e
\end{pmatrix}
\]

OHN

\[
V_R = R I_R, \quad V_e = \mathcal{E} I_e
\]

Kirchhoff: \( I_R = I_e \), \( V_R + V_e = 0 \).

You've made a mistake because \( \mathcal{E} \) should occur as part of an inhomogeneous constraint.

\[
\begin{array}{c}
\mathbb{C}^0 \xrightarrow{(1)} \mathbb{C}^1 \xrightarrow{(1)} H^1 \\
V_A \xrightarrow{(1)} \begin{pmatrix}
V_R \\
V_e - \mathcal{E}
\end{pmatrix}
\end{array}
\]

\[
V_o - V_b = \mathcal{E} I = V_e \\
V_b - V_a = -\mathcal{E} \\
V_a - V_d = V_R
\]

\[
F = \frac{V_R^2}{2R} + \frac{V_e^2}{2\mathcal{E}} + \lambda (\mathcal{E} - V_R - V_e)
\]

Variational Method

\[
V_R = \lambda, \quad \frac{V_e}{\mathcal{E}} = \lambda, \quad \mathcal{E} = V_R + V_e = \lambda (R + \mathcal{E})
\]

\[
F = \frac{\lambda^2 R}{2} + \frac{\lambda^2 \mathcal{E}}{2} = \frac{\lambda^2}{2} (R + \mathcal{E}) = \frac{\mathcal{E}^2}{2(R + \mathcal{E})}
\]

i.e. the response the applied \( \mathcal{E} \) is that of a resistance \( R + \mathcal{E} \).
Let's check the calculation by finding the critical point satisfying the constraint $\mathcal{V}_R + \mathcal{V}_e = \mathcal{E}$. Let $\mathcal{V}_R$ be the independent variable, $\mathcal{V}_e = \mathcal{E} - \mathcal{V}_R$. You want the critical point for

$$
\frac{1}{2} \begin{pmatrix} \mathcal{V}_R \\ \mathcal{E} - \mathcal{V}_R \end{pmatrix}^T \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{\mathcal{E}} \end{pmatrix} \begin{pmatrix} \mathcal{V}_R \\ \mathcal{E} - \mathcal{V}_R \end{pmatrix} = 0
$$

which means

$$
\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{\mathcal{E}} \end{pmatrix} \begin{pmatrix} \mathcal{V}_R \\ \mathcal{V}_e \end{pmatrix} = 0
$$

i.e.

$$
\frac{\mathcal{V}_R}{R} = \frac{\mathcal{V}_e}{\mathcal{E}} = \lambda
$$

call this ratio $\lambda$. Then the critical value is

$$
\frac{1}{2} \begin{pmatrix} \lambda \mathcal{R} \\ \lambda \mathcal{E} \end{pmatrix} \begin{pmatrix} \mathcal{R}^{-1} & 0 \\ 0 & \mathcal{E}^{-1} \end{pmatrix} \begin{pmatrix} \lambda \mathcal{R} \\ \lambda \mathcal{E} \end{pmatrix} = \frac{1}{2} \lambda^2 (R + \mathcal{E})
$$

where

$$
\mathcal{E} = \mathcal{V}_R + \mathcal{V}_e = \lambda (R + \mathcal{E})
$$

and so

the critical value in terms of $\mathcal{E}$ is

$$
\frac{1}{2} \frac{\mathcal{E}^2}{R + \mathcal{E}}
$$

Consider a branch $\begin{array}{c} A \rightarrow \mathcal{B} \rightarrow \mathcal{C} \rightarrow \mathcal{D} \end{array}$. A state of the branch is given by a pair $V, I$ satisfying a variant of Ohm's Law:

$$
\begin{align*}
V_A - \mathcal{E}I &= V_B \\
V_A - V_B &= \mathcal{E}I \\
V_B + \mathcal{E} &= 0 \\
V_B - \mathcal{E} &= 0
\end{align*}
$$

Conclude $V_{\text{branch}} = \mathcal{E}I - \mathcal{E}$ state equation.

Now, you want this to arise from a quadratic function of $V$ involving $\mathcal{E}$ as a constant:

$$
F = \frac{V^2}{2\mathcal{E}} + \frac{V\mathcal{E}}{\mathcal{E}}
$$

$$
\frac{dF}{dV} = \frac{V}{\mathcal{E}} + \frac{\mathcal{E}}{\mathcal{E}} = \frac{\mathcal{E}I}{\mathcal{E}} = I
$$
Special case of attaching a branch with emf to handle a constraint.

\[ C^1K = \{ V_R \} \quad \text{with power} \quad V_R^2 \frac{V_R}{2R} \]

\[ C^0K = \{ V_A \} \quad V_R = V_A - V_0 = V_A \]

Your constraint is \(-V_A = E\).

You want the critical point for \( \frac{V_R^2}{2R} \) subject to \( E + V_R = 0 \). The critical point seems to be \( V_R = -E \), critical value is \( \frac{E^2}{2R} \).

Lagrange multi

\[ F = \frac{V_R^2}{2R} - \lambda (E + V_R) \quad \Rightarrow \quad \frac{\partial F}{\partial V_R} = \frac{V_R}{R} - \lambda = 0 \]

\[ \lambda R = V_R = -E \]

\[ F = \frac{V_R^2}{2R} = \frac{E^2}{2R} \]

The other method consists of attaching another branch, then solving the appropriate linear equations, the new point is that for a branch with \( E \) Ohm's law needs to be changed.

\[ A \rightarrow E \rightarrow 0 \quad V = V_A - V_0 = E I - E \]

\[ V = E I - E \quad \text{power} \quad \frac{1}{2E}(V^2 + 2VE) = \frac{1}{2E}[E^2 + (E + I - E) I] \]

\[ V + E = \frac{E}{2} I \]

You have \( \text{power} = \frac{1}{2E}(V + E)^2 = \frac{1}{2} \frac{E}{2} I^2 \).
Next, you want to start with a connected R-network \( K \) with external nodes \( A \neq 0 \), and to understand how to attack a pure end \( E \) branch joining \( A \rightarrow 0 \).

\[
\begin{array}{c}
\text{K} \\
A \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
\text{O}
\end{array}
\]

\[
\phi_0 \rightarrow \phi_A \rightarrow \phi_0 \rightarrow C'K
\]

Here you impose the constraint \( \phi_0 - \phi_A = E \).

The augmented network has a small resistance \( E \) on the \( E \) branch which you want to go to zero.

\[
\phi_0 - \phi_A = E I - E
\]

The power on \( C'K \) should be orthogonal to the power for each branch.

Let's describe \( C'L \). All branches except the \( E \) branch are described by the power function \( \frac{1}{2} \frac{V^2}{R} \) whose derivative \( \frac{d}{dV} \left( \frac{1}{2} \frac{V^2}{R} \right) = \frac{V}{R} \) is the current.

Next look at the \( E \) branch where the power function is \( p = \frac{1}{2E} (V + E)^2 \), up to an additive constant, and the derivative \( \frac{d}{dE} = \frac{1}{2E} (V + E) \) is the current.

Begin with \( C^0K = C^0L \), but \( C^0L = C'K \oplus R \) where \( R = \{ V \} \). Then \( s_L : C^0K \rightarrow C^0L \) should have two components, one from \( s_K : C^0K \rightarrow C'K \) and the other should take a node potential \( \phi \in C^0K \) to \( V_e = \phi_0 - \phi_A = E I_e - E \). So you should be getting the usual homogeneous linear circuit equations: \( V = RI \) for all branches in \( K \). For the \( E \) branch you get inhomogeneous term \(-E\).
Begin with a connected R-network K, an "external" node pair A, 0 so that you have \( R \leftarrow \mathbb{C}_K \rightarrow C_K \).

You next attach \( q_A \rightarrow q_0 \leftarrow q \) is a branch joining A to 0 consisting of a pure emf \( E \) in series with a resistance \( \varepsilon \).

This gives an R-network \( L \) having the same node potential space: \( \mathbb{C}_K = \mathbb{C}_L \). What is \( C_{IL} \) and the coboundary map \( \delta_L : \mathbb{C}_L \rightarrow \mathbb{C}_I \)? \( C_{IL} \) is the direct sum over the branches of the branch voltage lines, such a line becomes \( R \) upon orienting the branch. Also there is quadratic form on each branch voltage line given by the power \( \frac{V^2}{R} \).

Program: Today you want to settle the Thévenin business. It should be simple. \( K : \begin{array}{c} A \end{array} \) \( L : \begin{array}{c} E \end{array} \)

You want the response to the applied emf \( E \), i.e., the current \( I \) flowing for each \( E \). Formula

\[
I = \frac{E}{R + \varepsilon}
\]

where \( R \) is equivalent resistance.
Here's the problem:

You have connected $K: \begin{array}{c} A \end{array}$, form $L: \begin{array}{c} A \end{array}\begin{array}{c} \circ \end{array} D$, you want the current response $I_c$ to $E$. Return to old idea of solving the circuit equations with $K$, modified by allowing two node currents at $D(0,A)$.

\[
\begin{array}{c}
R \leftarrow C_0K \rightarrow C'K \\
\mid \quad \mid \\
[4 \leftarrow \frac{1}{4}] \\
R \leftarrow C_0K \leftrightarrow C'K
\end{array}
\]

You perhaps want to arrange this

\[
\begin{array}{c}
C_0K \leftarrow (C'K) \leftarrow (R)
\end{array}
\]

Go over this again to get it clearer. Starting with $K$ and the new node pair $(A,0)$ you know there is a Thevenin equivalent circuit which gives the same relation between the current and voltage associated to this node pair. You expect that this resistance should be used for the new branch in the augmented graph.

**IDEA:** You recall shortening $A,0$ by a zero resistance wire. This gives a different $R$-network form $K-NO$, not unless you shrink the wire to a point, in which case the loop number is unchanged.

So attach the wire - what happens to the circuit equations?
Consider a connected $R$-network $K$ with a given external node pair $(A, 0)$. You should now be able to exhibit the short exact sequence for the augmented graph as follows. Recall that we have the s.e.s. for $K$

$$\bar{C}^0 K \xrightarrow{\delta_K} C' K \xrightarrow{\pi_K} H^1 K$$

and the linear map $f$ given by $f(q) = \varphi_0 - \varphi_A$

Then $\bar{C}^0 K = \bar{C}^0 L$, $C' L = (C' K)$

and $S_L = (S_K) : \bar{C}^0 L \rightarrow (C' K) = C' L$. We want to describe $\pi_L : C' L \rightarrow H^1 L$. Picture

Let's check exactness: 

$$\begin{pmatrix} \pi_K & 0 \\ -f' & 1 \end{pmatrix} \begin{pmatrix} \delta_K \\ f \end{pmatrix} = \begin{pmatrix} \pi_K \delta_K \\ -f' \delta_K + f \end{pmatrix} = 0.$$

On the other hand, let $(V) \in C' K = C' L$ satisfy

$$\pi_L (V) = \begin{pmatrix} \pi_K & 0 \\ -f' & 1 \end{pmatrix} (V) = 0,$$

i.e. $\pi_K V = 0$, $c = +f' V$. By exactness for $K$ one have $V = \delta \varphi$, $\varphi \in \bar{C}^0 K$.

Then $\delta_L \varphi = (\delta_K \varphi) = (V) = (+f' \delta_K \varphi) = (f' V) = (V) = (V) = (c)$

Note: this involves a choice of $f' : C' K \rightarrow R$ such that $f' \delta_K \varphi = f \varphi$. This means that you want a current $f'$ in $K$ such that $\delta f' = [01-1A]$. 
previous work k2. given $V' \overset{\phi}{\rightarrow} V \rightarrow V''$

and $\lambda : V' \rightarrow \mathbb{R}$ construct

$V' \rightarrow V \rightarrow V''$

$W$ is a fibre product

$\lambda' \overset{\phi}{\rightarrow} V$

$\lambda' \downarrow \lambda$

$\mathbb{R} \rightarrow W$

so extending $\lambda'$ to $\lambda : \lambda \mathbb{R} = \lambda'$ gives

a retract of $W$ onto $\mathbb{R}$.

Generalization

$C^0 K \overset{\phi}{\rightarrow} C^1 K \rightarrow H^1 K$

$C^0 L \overset{(\phi,)}{\rightarrow} (C^1 K \oplus \mathbb{U}) \rightarrow H^1 L

U = U$

Hopes: These vector space diagrams fit with Lagrange multipliers.
What you learned yesterday:

\[ \begin{align*}
V' & \xrightarrow{S} V^* \xrightarrow{d} V'' \\
V' & \xrightarrow{\phi} W \\
U & = U
\end{align*} \]

\[ \begin{align*}
C'K & \xrightarrow{\phi} C^K \xrightarrow{H'K} V \\
C'K & \xrightarrow{(d_K)} V \\
U & \xrightarrow{(c_K)} (C^K) \xrightarrow{H'L} U = U
\end{align*} \]

Thus, \( W \) is the cofibre product.

You get a retract of \( W \) onto \( U \) by extending \( j \) to \( X: V \to U \) s.t. \( XS = I \).

Now you need to handle the quadratic form. But first you should understand the Kirchhoff voltage constraints arising from \( (C'K) \xrightarrow{R} H'L \). Put another way, each linear functional on \( H'L \) gives such a constraint. From loops currents in \( K \) you get the voltage law in \( K' \) but there's the constraint given by \( X: C'K \to R \).

Recall \( X_S = \lambda: \phi \mapsto \phi_0 - \phi_A \) viewed as chains: \( \partial X = \lambda \).

Next, you want to introduce the power quadratic form on \( C'L = (C'K)^R \).
The main idea of shorting (θ, α)

\[ c^0K \rightarrow c^1K \rightarrow H^1K \]

\[ c^0L \rightarrow c^1L \rightarrow H^1L \]

\[ c^0M \rightarrow c^1M \rightarrow H^1M \]

Note that \( K \) and \( M \) are honest \( R \)-networks with \( >0 \) resistances for each edge. Take some examples.

\[ K: \quad L: \quad M: \]

So it should be clear that \( K \rightarrow M \) collapses the nodes \((A, 0)\), but doesn't affect the edges of these graphs. Conclude \( C^1M \rightarrow C^1K \) as usual with quadratic form.

It looks like the basic sequence might be

\[ c^0M \rightarrow c^1L \rightarrow H^1K \]

but there seems to be \( 2 \)-dimensional homology: \( \text{Ker} \beta / \text{Im} \alpha \).
IDEA: You are reminded of cohomology with compact support, maybe intersection homology, which can carry positive quadratic forms, non-degenerate at least.

At this point you feel that the quadratic form calculations should be done in the space $C'M \to C'K$ with its positive form.

IDEA: You are also reminded of ignoring nil-modules where the canonical factorization thru the image is used.

$$C'M \to C'\ell \to H'L \to H'K$$

Review yesterday's advance:

The natural map from $K$ to $M$ is bijective on the branches and preserves resistances. Thus $C'K \to C'M$, the isom. respects the power form. You don't see a candidate for the resistance of the attached branch to $L$. You think that stationary point calculations should involve the space $C'K = C'M$ with its pos. definite power form. How to proceed?

IDEA: I think space $\ker \beta / \text{Im} \alpha$

$$C'K \to C'L \to H'K$$
Does this 2-dim space, consisting roughly of an extra mode current (linear in \( \zeta \), for \( K, L \)) and an extra loop voltage for \( L, M \), have a nice interpretation, say symplectic, or involving Lagrange multipliers in some way?

Look at the simplest case:

\[
\begin{array}{c}
\text{K} \\
\downarrow \text{R} \\
\text{L} \\
\downarrow \text{A} \\
\text{M}
\end{array}
\]

\[
0 \xleftarrow{\text{L}} \zeta_0^K \xrightarrow{\text{L}} \zeta_0^L \xrightarrow{\text{H}^L} \zeta_0^M
\]

\[
\text{C}^0_L \xrightarrow{\text{L}} \text{C}^0_L \xrightarrow{\text{H}^L} \text{C}^0_M
\]

\[
\text{C}^0_M \xrightarrow{\text{H}^L} \text{C}^0_M \xrightarrow{\text{H}^L} \text{C}^0_M
\]

Let's focus upon the positive quadratic space \( \text{C}^0 K = \text{C}^1 M \).

This will induce \( \text{pos. def. forms on } \zeta^0 K, \text{H}^L K, \text{C}^0 M, \text{H}^L M \). So you get \( \text{pos. def. forms on } \zeta^0 L = \zeta^0 K \) and \( \text{H}^L L = \text{H}^L M \). So only \( \text{C}^1 L \) is lacking.

Picture

\[
\begin{array}{c}
\text{K} \\
\downarrow \\
\text{L} \\
\downarrow \\
\text{M}
\end{array}
\]

\[
\text{M} = K \downarrow A = 0
\]
The point: \( K, M \) are connected R-networks with \( C^1 M \to C^1 K \) as quadratic spaces. What happens in the Thévenin picture where you have \( E \in C^1 \), \( E \) is a family of branch emf's, being split into \( V \in \tilde{C}^0 \) and \( RI, I \in H_1 ? \) This is some kind of Hodge Decomposition.

Let's go over the situation beginning with 3 short exact sequence

\[ C^0 K \to C^1 K \to H K \]

\[ \text{nullity} = 1 \]

\[ \text{nullity} = 1 \]

\[ C^0 L \to C^1 L \to H^L \]

\[ \text{nullity} = 1 \]

\[ C^0 M \to C^1 M \to H^M \]

What do you want to do?

You have this quadratic space: \( C^1 K \to C^1 M \)

Inside this quadratic space is where you split a Thévenin branch voltage \( E \) into its "harmonic" components. But the splitting depends upon the subspace \( \tilde{C}^0 \) you use. There are 2 candidates, namely: \( \tilde{C}^0 M \to \tilde{C}^0 K \).

Something to do is to understand the two Hodge decompositions:

\[ C^0 K \to C^1 K \to H K \]

\[ \text{nullity} = 1 \]

\[ C^0 M \to C^1 M \to H^M \]
$K: \begin{array}{c}
\bar{C}^0K \rightarrow C^1K \rightarrow H'K \\
\text{cokernel} \uparrow \quad \uparrow S \quad \uparrow \text{nullity} \quad \downarrow 1
\end{array}$

$M: \begin{array}{c}
\bar{C}^0M \rightarrow C^1M \rightarrow H'M \\
\text{It seems that what you have here is a 2 step filtration}
\end{array}$

$0 \subset \bar{C}^0M \subset \bar{C}^0K \subset C^1K$

together with the quadratic form on $C^1K$. Recall that $\bar{C}^0M = \ker \{\bar{C}^0K \rightarrow \mathbb{R}^2\}$ and that you want to minimize the power over $\lambda(\cdot)$ which is an affine hyperplane $\|\bar{C}^0M\|$ in $\bar{C}^0K$.

You want to understand the meaning of these graphs. The graphs $K, M$ are connected $\mathbb{R}$-networks with pos def. power, and this leads to an orthogonal splitting:

$C^1K = \bar{C}^0K \oplus (\bar{C}^0K)^\perp$

$V = \lambda - V_A$

$= \lambda I - E$

$V - \lambda I = -E$

$\bar{C}^0K \rightarrow C^1K \rightarrow H'K \quad S \downarrow R$

$\bar{C}^0K \leftarrow C^1K \leftarrow H'K$

Idea last night is to use the Thévenin equivalent resistances associated to the terminals $(A, 0)$. This you understand in the voltage current picture, meaning you see a 1-current $I$ which is closed except at $(A, 0)$ and a 1-voltage $V$ with constraint between $(A, 0)$.
$R \xrightarrow{\phi^t} \mathbb{C}_0 \xrightarrow{\phi} \mathbb{C}_1 \xrightarrow{\phi} \mathbb{C}_0 \xrightarrow{\phi} \mathbb{C}_1$

Idea of Poisson's equation $\Delta \phi = \rho$. Discuss this: you have potential function $\phi$, a norm $\|\partial \phi\|^2$ yielding $d^2\phi$, a Laplacian. $\rho$ change density or O-current, get $\phi \mapsto s^t \phi$ which you want to represent: $\mathbb{R}^2 \xrightarrow{\phi^t} \mathbb{R}^2 \xrightarrow{\phi} \mathbb{R}^2$

Better: $R \xrightarrow{\phi^t} \mathbb{C}_0 \xrightarrow{s^t} \mathbb{C}_1 \xrightarrow{\phi^t} \mathbb{C}_0 \xrightarrow{\phi} \mathbb{C}_1$

Solve $s^t R^t \overline{\mathbf{u}} = \rho$, Poisson's eqn. $\mathbf{u}$ is a node potential, $\mathbf{S} \mathbf{u}$ is its gradient. The collection of branch voltage dots, $R^{-1} \mathbf{S} \mathbf{u}$ is the induced family of branch currents, $s^t R^{-1} \mathbf{S} \mathbf{u}$ is the resulting family of node currents. So when you solve $s^t R^{-1} \mathbf{S} \mathbf{u} = \rho$, where $\mathbf{S} = [A^t - [0 \mathbf{1}]]$, then $\mathbf{u}$ should be harmonic except at $[A, 0]$, harmonic should be a sort of O power condition.

Let's explore "harmonic" further. Recall the integral lattice with 1 ohm branches, look at the variation in the power arising from $\mathbf{S} u_0$:

$\sum_{i=1}^{4} \frac{(u_i - u_0)^2}{2} = \sum_{i=1}^{4} (u_i - u_0) \delta u_i$. This vanish

where $4u_0 = \sum_{i=1}^{4} u_i$ or $u_0 = \frac{1}{4} \sum_{i=1}^{4} u_i$
More generally, consider a \( R \)-network \( K \) and look at \( S(\text{Power}) \) corresponding to \( u_N \) for some node \( N \).

\[
\text{Power} = \frac{1}{2} \sum_b \frac{V_b^2}{R_b},
\]
where \( b \) runs over the branches.

As \( u_N \) varies, what terms in the Power change? You only have to consider \( b \) joining \( N \) to a different node \( M_b \). The contribution to the Power is

\[
S\left( \frac{1}{2} \sum_b \frac{(u_N - u_{M_b})^2}{R_b} \right) = \sum_b \frac{u_N - u_{M_b}}{R_b} \cdot S u_N
\]

Note that

\[
\frac{u_N - u_{M_b}}{R_b} = I_b \quad \text{so} \quad S(\text{Power}) = \sum_b \frac{I_b}{R_b}.
\]

So you see that Stationary power wrt \( u_N \) is the same as the Kirchhoff node current condition at \( N \).

Next go back to Thevenin’s idea of putting emfs in the branches. This means we can have voltage sources present without affecting the Kirchhoff conditions. Therefore we find that the power is stationary wrt variations \( S u_N \) for all nodes \( N \) iff \( u \) is harmonic off the Kirchhoff current condition holds.

This is nice, but you are still missing the Thevenin equivalent resistance of a \( K = \Box \) with an external node \( A \neq \text{ground} \) node \( \Box \).
You need to go over the X Euclidean $y \in X$, $y^*: X \to \mathbb{R}$. $(Ry)^\perp = \text{Ker}(y^*): X \to \mathbb{R}$, $X = Ry \oplus Ry^\perp$.

You want the norm $\|x\|$ induced by push forward wrt $y^*$.

$x = \lambda y$

$\triangleright$ where $y^* = c$.

Try again. $X$ with $\frac{1}{2}\|x\|^2$, $y \in X$, $y^*: X \to \mathbb{R}$ $X$ splits into $Ry$ and $(Ry)^\perp = \text{Ker}(y^*)$. Now let $c \neq 0$, $c \in \mathbb{R}$, and find the $x$ such that $y^*x = c$ and $x \perp (Ry)^\perp$

e.g. $x = \lambda y$. $\Rightarrow$ $c = y^*x = \lambda y^*y \Rightarrow \lambda = \frac{c}{\|y\|^2}$

$x = \frac{c y}{\|y\|^2}$ and $\frac{1}{2}\|x\|^2 = \frac{c^2\|y\|^2}{2\|y\|^4} = \frac{c^2}{2\|y\|^2}$

Still confused...

You want $u \in \mathbb{C}_K$ s.t. $\Delta u = f$, implies a harmonic wave away from $\text{supp}(g)$.

The potential corresponding to the mode current $f$ is $u = \Delta^{-1}f$, and the power function associated to this mode potential and the mode current (O-chain) $p = [A] - [0]$ should be $\frac{1}{2} f^T \Delta^{-1}f$. This checks.

So now you basically understand the Thevenin equivalent resistance, and you can look again at attaching a branch joining nodes $(A, 0)$. Start...
Let's see that $C^1L$ might be a pushout.

$\overline{C^0}K \leftarrow C^1K \rightarrow H^1K$

$\overline{C^0}L \leftarrow C^1L \rightarrow H^1L$

$\overline{C^0}M \leftarrow C^1M \rightarrow H^1M$
Look at \( L \) in the case \( K = \frac{A}{E_0} \), \( L = \frac{A}{E_0} \).

\[ \begin{align*}
\mathcal{C}_0 L & \xrightarrow{S} \mathcal{C}_1 L \rightarrow H'L \\
\{u_A\} & \rightarrow \begin{pmatrix} u_A \\ (1) \end{pmatrix} \rightarrow \begin{pmatrix} V_R \\ V_e \end{pmatrix} \xrightarrow{(1,1)} V_R + V_e = 0
\end{align*} \]

\[ \begin{align*}
I_R - I_e & = 0 \leftrightarrow \begin{pmatrix} I_R \\ I_e \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\mathcal{C}_0 L & \xleftarrow{S} \mathcal{C}_1 L \rightarrow H'L
\end{align*} \]

What are you writing? 4 variables \( V_R, I_R, V_e, I_e \)

Kirkhoff

2 constraints: \( V_R + V_e = 0, I_R = I_e \). Finally Ohm: \( \begin{pmatrix} V_R \\ V_e \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} I_R \\ I_e \end{pmatrix} \)

These four equations have only the zero solution: \( V_e = 0 \Rightarrow V_R = 0 \Rightarrow I_R = 0 \Rightarrow I_e = 0 \).

Do there a Hodge decomposition of some sort?

First notice that by linear algebra, when you have 4 linear homogeneous in 4 unknowns

\[ \begin{align*}
V_R + V_e & = 0 \\
I_R - I_e & = 0 \\
V_R - RI_R & = 0 \\
V_e & = 0
\end{align*} \]

having only the zero solution, then \( AX = Y \) has a unique soln. \( X = A^{-1}Y \) for each \( Y \).

Does this amount to some kind of Hodge decomposition? Ideally, you want a splitting of \( \mathcal{C}_L \) into \( S(\mathcal{C}_0 L) \) and \( \mathcal{C}_L \) complement. But there's...
the degeneracy of the power form to contend with.

Past approach: fact $\varepsilon > 0$ resistance on the branch. The power is $\frac{1}{2} \left( \frac{V^2_R}{R} + \frac{V^2_e}{\varepsilon} \right)$. Alternately use the current picture, where the power is $\frac{1}{2} (RI^2_R + \varepsilon I^2_e)$. What should happen is that the power form on $C' \Lambda$ restricts to a non-degenerate form on $H_1 \Lambda$, namely $\frac{1}{2} (R+\varepsilon) I^2$. The power on $C' \Lambda$ restricts to the form $\frac{1}{2} \left( \frac{u^2_A}{R} + \frac{(-u_A)^2}{\varepsilon} \right) = \frac{1}{2} \left( \frac{1}{R+\varepsilon} \right) u^2_A$, which looks problematic as $\varepsilon \to 0$.

Review. $C_0 \Lambda \quad \rightsquigarrow \quad C' \Lambda \quad \rightsquigarrow \quad H_1 \Lambda$

\[
\begin{pmatrix}
  u_A \\
  (-1)
\end{pmatrix} \begin{pmatrix}
  (u_A)^T \\
  (V_R)
\end{pmatrix} \begin{pmatrix}
  (V_e) \\
  (1)
\end{pmatrix} \quad \implies \quad V_R + V_e = 0
\]

$C_0 \Lambda \quad \overset{\text{sub}}{\longleftarrow} \quad C_1 \Lambda \quad \overset{\text{sub}}{\longleftarrow} \quad H_1 \Lambda$

\[
0 = I_R - I_e \quad \overset{(1-1)}{\implies} \quad \begin{pmatrix}
  I_R \\
  I_e
\end{pmatrix} = \begin{pmatrix}
  (1) \\
  (1)
\end{pmatrix}
\]

The Kirchhoff space is 2-dim. The fact that the Ohm & Kirchhoff spaces are transversal, both are 2-dim and the intersection is 0, means that you get a splitting of $C' \Lambda \oplus C_1 \Lambda$. Also you should get a positive definite form on the Kirchhoff spaces. This needs checking, understanding better.

Review: Begin with a phase space, i.e. the direct sum $(C^1_1)$ of a vector space and its dual. Consider the subspace given by the graph $\frac{1}{2} C^1_1$ of a linear map $T: C^1_1 \to C^1_1$. Such a $T$ is
You have decided to review symplectic conventions concerning "phase space", that is, the direct sum \( (\mathbb{C}^1) \) of a vector space and its dual. Consider a harmonic oscillator: \( \varphi \in \mathbb{C}^1 \) is position \( p \in \mathbb{C}^1 \) is momentum. P.E. \( \mathcal{E} = \frac{1}{2} \varphi^t B \varphi \), K.E. \( = \frac{1}{2} p^t m^{-1} p \). Hamiltonian \( H = \frac{1}{2} (\varphi^t k \varphi) (\varphi^t m^{-1} \varphi) \), the Hamiltonian flow is

\[
\begin{pmatrix}
\frac{d\varphi}{dt} \\
\frac{dp}{dt}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial H}{\partial p} \\
-\frac{\partial H}{\partial \varphi}
\end{pmatrix} = \begin{pmatrix}
0 & m^{-1} \\
-k & 0
\end{pmatrix} \begin{pmatrix}
\varphi \\
p
\end{pmatrix}
\]

You propose to determine the \( \chi \), the generator of Ham. flow sign of the symplectic form \( A \) by \( AX = H \) i.e.

\[
A = \begin{pmatrix}
0 & m^{-1} \\
-k & 0
\end{pmatrix} \Rightarrow A = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

Prove that the Ham. flow respects \( A \) and the symmetric form \( H \). First, recall that an invertible operator \( g \) preserves a bilinear form \( \mathcal{B} : \mathbb{C}^1 \rightarrow \mathbb{C}^1 \) where \( (g^t \varphi)^t \mathcal{B}(g^t \varphi) = \varphi^t g^t B g \varphi \), i.e. \( g^t B g = B \), infinitesimal \( \chi^t B + B \chi = 0 \). Now \( H^t = H = AX \Rightarrow \chi^t A^t = H^t = H = AX \), so \( \chi^t A + A X = 0 \) as \( A^t = -A \). Then \( \chi^t H + H \chi = \chi^t A X + A X \chi = (\chi^t A + A X) \chi = 0 \).

Thus \( \chi \) respects \( A \) and \( H \). Note that interchanging \( \varphi, p \) leads to

\[
A = \begin{pmatrix}
0 & -k \\
m^{-1} & 0
\end{pmatrix} \Rightarrow A = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
Kirchhoff space $K = \mathcal{C}^0 \oplus \mathcal{H}_1$ is a Lagrange invariant subspace of $\mathcal{C}' \oplus \mathcal{C}_1$. Why? The annihilator of $\mathcal{S}\mathcal{C}^0 = \mathcal{C}' \oplus \text{Ker } \mathcal{S}^t$, the annihilator of $\mathcal{S}\mathcal{H}_1 = \mathcal{C}_1 \oplus \text{Ker } \mathcal{S}^t$, so the annihilator of $\mathcal{S}\mathcal{C}^0 + \mathcal{S}\mathcal{H}_1$ is $\text{Ker } \mathcal{S}^t \cap \text{Ker } \mathcal{S}^t = K$.

You have $K \cap \Omega = 0$ as $\Omega = \mathcal{C}^0 \to \mathcal{C}_1 \to \mathcal{H}_1$

\[ \begin{array}{c}
\mathcal{C}^0 \\
\mathcal{C}_1 \\
\mathcal{H}_1
\end{array} \]

\[ \begin{array}{c}
\Omega \\
\mathcal{C}_1
\end{array} \]

Example:

$K$: $V_R + V_e = 0$

$\text{IR} = I_e = 0$

$\Omega$: $V_R - R I_R = 0$

$V_e = 0$

You have $K \oplus \Omega = (\mathcal{C}' \oplus \mathcal{C}_1)$. This splitting of the branch state space should yield for every internal branch end $\theta$ the resulting Kirchhoff state.
In the example you have a state space of 4 variables together with a Kirchhoff space defined by $V_R + V_e = 0$, $I_R = I_e$ and an Ohm space $\Omega$ defined by $V_R = R I_R, V_e = 0$. One has $K \cap \Omega = 0$, so that $K$ and $\Omega$ are complementary: $K \oplus \Omega = \mathbb{R}^4$.

**Question:** What is significant about a splitting of a symplectic vector space into Lagrangian subspaces? Such splittings should be related to symplectic isomorphisms. For example if you choose a linear span $K \supset \mathbb{C}^1$ and the dual $\mathbb{C}^1 \simeq K^* \mathbb{R}$, then you get a symplectic automorphism carrying $\mathbb{C}^1 \cong K$, $\mathbb{C}^1 \cong K^* = \Omega$.

**Example.**

Check $(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})_T$ is a symplectic automorphism.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -T^t + T \end{pmatrix}$$

Let's discuss the splitting in the example at the top.

$$\mathbb{C}^0 \xrightarrow{e = (1)} \mathbb{C}^1 \xrightarrow{\varepsilon = (1, 1)} H^1$$

You know that $\Omega$ and $K$ are complementary, and hence $\Omega \xrightarrow{[H^1]} [\mathbb{C}_0]$

For the purposes of Thevenin theory you want an end $E$ on the $e$ branch, that is, an "input" $(\varepsilon) \in \mathbb{C}^1$. Let's see what it means to split $(\varepsilon)$ into $K$ and $\Omega$ components.

Given $(\varepsilon) \in \mathbb{C}^1$, move it to $(1, 1)(\varepsilon) = E \in H^1$, choose $I \in H_1$ such that $(1, 1)(R \cdot 0)I = E$, i.e. $I = \frac{E}{R + \text{Re}}$. 
Next subtract from \((e)\) the non-conservative part arising from the current \(I:\)

\[
\left(\begin{array}{c}
e \\
0
\end{array}\right) - \left(\begin{array}{c}
R \\
Re
\end{array}\right) \frac{e}{R + Re} = \epsilon \left[\begin{array}{c}
1 \\
0
\end{array}\right] - \left[\begin{array}{c}
R/Re \\
Re/Re
\end{array}\right]
\]

\[
= \epsilon \left[\begin{array}{c}
Re/Re + Re \\
-R/Re + Re
\end{array}\right]
\]

So it seems that we have the decomposition

\[
\left(\begin{array}{c}
e \\
0
\end{array}\right) = \left[\begin{array}{c}
Re \\
-R_e
\end{array}\right] I + \left[\begin{array}{c}
R_e \\
Re
\end{array}\right] I
\]

\[
\epsilon \subset C^0 \\
\leftarrow \quad \quad R \quad I
\]

\[
\text{This is the } \Omega \text{ part.}
\]

The nice thing here is that you can \(\Box\) let \(Re \rightarrow 0\),

where you get \(\left(\begin{array}{c}
e \\
0
\end{array}\right) = \left[\begin{array}{c}
R \cdot I \\
0 \cdot I
\end{array}\right].
\]

\[
\left[\begin{array}{c}
1 \\
0
\end{array}\right] = \left[\begin{array}{c}
R_e \\
-Re
\end{array}\right] \frac{1}{R + Re} + \left[\begin{array}{c}
-R_e \\
R_e
\end{array}\right] \frac{1}{R + Re} \rightarrow \left[\begin{array}{c}
1 \\
0
\end{array}\right] + \left[\begin{array}{c}
0 \\
0
\end{array}\right]
\]

\[
\left[\begin{array}{c}
1 \\
0
\end{array}\right] = \left[\begin{array}{c}
R^{-1} \\
R^{-1} Re^{-1}
\end{array}\right] \frac{1}{R^{-1} + Re^{-1}} + \left[\begin{array}{c}
R_e^{-1} \\
Re^{-1}
\end{array}\right] \frac{1}{R^{-1} + Re^{-1}} \rightarrow \left[\begin{array}{c}
0 \\
-1
\end{array}\right] + \left[\begin{array}{c}
1 \\
1
\end{array}\right]
\]

\[
\left[\begin{array}{c}
0 \\
1
\end{array}\right] = \left[\begin{array}{c}
-R_e^{-1} \\
R_e^{-1}
\end{array}\right] \frac{1}{R^{-1} + Re^{-1}} + \left[\begin{array}{c}
R^{-1} \\
R^{-1} Re^{-1}
\end{array}\right] \frac{1}{R^{-1} + Re^{-1}} \rightarrow \left[\begin{array}{c}
0 \\
1
\end{array}\right] + \left[\begin{array}{c}
0 \\
0
\end{array}\right]
\]

\[
\epsilon \subset C^1 \\
\leftarrow \quad \quad \Omega + \bar{C}_0
\]

Notice that \(\Box\) in the cases \((e) \epsilon C^1, (0) \epsilon C^1\)

\(Re \rightarrow 0\) limit is the same.
Record for later reference the mistake (p. 45) where you say that \( \mathcal{E} \) on the \( \varepsilon \) branch corresponds to an "input" \((\mathcal{E}) \in C'\). Because \( C' = \{(V_e)\} \), the input should be \((\mathcal{E})\).

Questions: What's interesting about two Lagrangean subspaces of a symplectic vs. which are transversal?

The only thing I can see is that one gets a phase space picture for \( S = K \oplus \Omega \), where either \( K \) or \( \Omega \) can be taken to be position (or configuration) space and the other to be momentum. But there's still no dynamics yet.

In our situation, where \( S = \begin{bmatrix} C' \\ C \end{bmatrix} \), \( K = \begin{bmatrix} \mathcal{E} \\ 0 \end{bmatrix} \), and \( \Omega \) is the Ohm correspondence, then the splitting \( S = K \oplus \Omega \) allows us to take any branch state \( \begin{bmatrix} \mathcal{E} \\ C \end{bmatrix} \) and to project it onto a Kirchhoff state using the Ohm relations.

Do you get the desired current response from an external \( \mathcal{E} \) applied between two nodes?

\[
\begin{align*}
K: & \quad V_R + V_e = 0 \\
& \quad I_R = I_e \\
& \quad V_e = \mathcal{E}
\end{align*}
\]

\[
\begin{align*}
\Box & \quad R \\
\end{align*}
\]

\[
\begin{align*}
\Omega & \quad V_R = RI_R \\
\Omega & \quad V_e = \mathcal{E}_R I_e = -\mathcal{E}
\end{align*}
\]

\[
\begin{align*}
0 & = V_R + V_e = (R + R_e)I - \mathcal{E}
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} \mathcal{E} \\ R \end{bmatrix} I \quad + \begin{bmatrix} -R_e \\ R \end{bmatrix} I \\
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} \mathcal{E} \\ R \end{bmatrix} I \quad + \begin{bmatrix} -R_e \\ R \end{bmatrix} I
\end{align*}
\]

Note: the splitting \( \Omega \) has the limit as \( R_e \to \infty \):

\[
\begin{align*}
\begin{bmatrix} \mathcal{E} \\ 0 \end{bmatrix} & \quad + \begin{bmatrix} -1 \end{bmatrix}
\end{align*}
\]

\[
I = \frac{\mathcal{E}}{R + R_e} \text{, so if you let } R_e \to \infty \text{, you get the desired current response in the } \varepsilon \text{ branch.}
\]
You know these are important when all brands have $R > 0$. Observation: When $K \Theta \Omega = [c_1']$, since $K = [c_0'] H_1$ you get $\Omega \rightarrow [H_1'] c_0'$. So it seems that $\Omega$ has a canonical splitting into voltage and current components. You would like to have $\Omega = \begin{bmatrix} \Omega_0 & c_1 \end{bmatrix}$.

Let's display the splitting $K \Theta \Omega = [c_1']$ neatly:

\[
\begin{array}{c}
c_0' \xrightarrow{\delta = (1,1)} c_1' \xrightarrow{\gamma = (1,1)} H_1' \\
H_1' \xrightarrow{\delta_1 = (1,1)} c_1' \xrightarrow{\gamma_1 = (1,1)} c_0 \\
\end{array}
\]

Voltage side:

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} R \\ Re \end{bmatrix} \frac{1}{R+Re} + \begin{bmatrix} -Re \\ Re \end{bmatrix} \frac{1}{R+Re} \\
\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} R \\ Re \end{bmatrix} \frac{1}{R+Re} + \begin{bmatrix} -R \\ R \end{bmatrix} \frac{1}{R+Re} \\
C_1' = C_1 \cap \Omega \\
C_0' = \Omega_0 \\
\]

Current side:

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} R^{-1} \\ -Re^{-1} \end{bmatrix} \frac{1}{R^{-1}+Re^{-1}} + \begin{bmatrix} Re^{-1} \\ R^{-1} \end{bmatrix} \frac{1}{R^{-1}+Re^{-1}} \\
\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -R^{-1} \\ Re^{-1} \end{bmatrix} \frac{1}{R^{-1}+Re^{-1}} + \begin{bmatrix} R^{-1} \\ R^{-1} \end{bmatrix} \frac{1}{R^{-1}+Re^{-1}} \\
C_1' = C_1 \cap \Omega \\
H_1' = \Omega_0 \\
\]
It might help to add the current picture calculations. Begin with \( [1 \, 0] \in C_1 \) move to \( [1 \, -1] \) then \( \begin{pmatrix} 1 & 0 \\ 0 & R_e^{-1} \end{pmatrix} u \) equals 1 \( \Rightarrow u = \frac{1}{R^1 + R_e^{-1}} \). Then remove
\[
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R^1 & \frac{1}{R^1 + R_e^{-1}} \\ -R_e^{-1} & \frac{1}{R^1 + R_e^{-1}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} R_e^{-1} \\ R_e^{-1} \end{pmatrix} \frac{1}{R^1 + R_e^{-1}}
\]
which gives the decomposition
\[
\begin{align*}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} & = \begin{pmatrix} R^1 \\ -R_e^{-1} \end{pmatrix} \frac{1}{R^1 + R_e^{-1}} + \begin{pmatrix} R_e^{-1} \\ R_e^{-1} \end{pmatrix} \frac{1}{R^1 + R_e^{-1}} \\
C_1 & \quad \quad C_1 \cap \Omega \\
\end{align*}
\]

At this point you understand a little better the decomposition of \( [C_1'] \) into \( K \oplus \Omega \), but it's still messy. Here's another approach. Begin with \( [C_1'] \) a 4-dim hyperbolic symplectic space and the Lagrangian subspace \( [C_0] = K \). Let's try to understand the possible \( \Omega \) which are complementary to \( K \), \( \dim \Omega = 2 \).

**Q:** Is it possible that a splitting \( K \oplus \Omega \rightarrow [C_1'] \) with \( K = [C_0] \) yields a splitting of the voltage and the current short exact sequences?

**IDEA:** Is there an analog in the symplectic theory of "retract", where instead of \( [W_+ \left[ V \right] \left[ V \right] \left[ W_- \right] \rightarrow [C_1'] \) a retract of a free Z/2 module you have a retract of a hyperbolic space \( [V_r] \)?
Start with the short exact voltage sequence $\oplus$ the dual short exact current; Then $K$ is a Lagrangian subspace of the symplectic space $S$, and $S/K$ is naturally ism to $K^*$. You want to understand Lagrangian complements $L$ to $K$.

Start again with a different notation. Consider a short exact sequence direct sum with the dual exact sequence arranged as follows:

$$
\begin{bmatrix}
A \\
C^*
\end{bmatrix} \rightarrow \begin{bmatrix}
B \\
B^*
\end{bmatrix} \rightarrow \begin{bmatrix}
C \\
A^*
\end{bmatrix}
$$

Thus we have a hyperbolic symplectic space $S$ together with a special type of Lagrangian subspace compatible with the hyperbolic grading. We follow Lafforgue's convention of Roman + Greek letters: $[b_\beta] \in [B^*]$. The symplectic form on $S$ is $[b_1, b_2]_s = -b_1^t \beta_2 + \beta_1^t b_2$. Let's check that $K$ is isotropic. $[b_1, b_2]_s \in K$ means $(b_1 \in A$ and $(b_2 \in B^* is 0 on $A$. So it's clear. It should now follow that the symplectic form on $S$ induces a nondegenerate pairing between $K$ and $S/K = [C^*]$. Let's use a wedge notation

$$
[b_1, b_2]_s = -b_1^t \beta_2 + \beta_1^t b_2
$$

for the symplectic form on $S$.

Then following $\Lambda^2 B = A \otimes C$ in spirit, the induced pairing between $K$ and $S/K$ should be given by

$$
[a_1, c_2] = -a_1^t a_2 + a_1^t c_2
$$
Next project is to understand properly the Lagrangian complements of $K$ in $S$. You should know that these form an affine space whose vector space is the symmetric maps from $K$ to $S/K$.

Let's consider a special type of $S$, where the hyperbolic structure, i.e. grading into voltage and current, is preserved. This means that you lift $C$ into $B$ and lift $A^*$ into $B^*$ subject to the condition that these lifts $\tilde{C}$, $\tilde{A^*}$ are orthogonal for the symplectic form.

Let $\begin{bmatrix} A & C \\ C^* & A^* \end{bmatrix}$ be the hyperbolic symplectic space with the symplectic form $\begin{bmatrix} a_1 & c_1 \\ \gamma_1 & x_1 \end{bmatrix} \wedge \begin{bmatrix} a_2 & c_2 \\ \gamma_2 & x_2 \end{bmatrix} = -a_1^tx_2 - c_1^ty_2 + \gamma_1^tx_2 + a_2^tx_2$

There is an equivalence between Lagrangian subspaces $L$ of $S$ complementary to $K$ and symmetric maps $K \leftarrow \Lambda$. Look at the maps preserving the horizontal grading $A \leftarrow C$. The graph of this map consists of elements $\begin{bmatrix} u(c_1) & c_1 \vline & u(c_2) & c_2 \\ v(\alpha_1) & x_1 \vline & v(\alpha_2) & x_2 \end{bmatrix} = \begin{bmatrix} -u(c_1)^tx_2 & -c_1^tx_2 \\ -c_1^tv(\alpha_2) & -x_2^tv(\alpha_2) + x_2^tv(\alpha_2) + x_1^tu(c_2) + x_1^tu(c_2) \end{bmatrix}$.

$u(c_1)^tx_1 = (uc_1)^tx_1 = c_1^tx_1$, etc.

So you find that the graph is Lagrangian iff $u + v = 0$. Now you want to know that this is equivalent to $u, v$ being symmetric.
\[
\begin{bmatrix}
B \\
B^*
\end{bmatrix} = \begin{bmatrix}
A \\
C^*
\end{bmatrix} \oplus \begin{bmatrix}
A^*
\end{bmatrix}
\]

\[
S = K \oplus \Lambda
\]

\(S\) is a hyperbolic vs. when equipped with the skew-symmetric pairing, denoted by \(\wedge\).

\[
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} \wedge \begin{bmatrix}
b_1' \\
b_2'
\end{bmatrix} = \begin{bmatrix}
b_1^t & 0 & 1
\end{bmatrix} \begin{bmatrix}
b_2
\end{bmatrix} = b_1^t b_2 - b_2^t b_1
\]

Thus in terms of the components of \(S\) one has

\[
\begin{bmatrix}
a_1 + c_1 \\
a_2 + c_2
\end{bmatrix} \wedge \begin{bmatrix}
a_2 + c_2 \\
a_1 + c_1
\end{bmatrix} = \begin{bmatrix}
(a_1 + c_1)^t \alpha_2 + (a_1 + c_1)^t \alpha_2
\end{bmatrix} = a_1^t \alpha_2 + c_1^t \alpha_2 - a_2^t \alpha_1 - c_2^t \alpha_1
\]

\[
\begin{bmatrix}
a_1 \\
a_2
\end{bmatrix} \wedge \begin{bmatrix}
a_2 \\
a_1
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix} \wedge \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix} - \begin{bmatrix}
\alpha_2 \\
\alpha_1
\end{bmatrix} \wedge \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix}
\]

Note the 2x2 determinant pattern.

Next we determine the Lagrangian subspaces of \(S\) which are complementary to \(K\). By:

\(\Gamma_T\) are the graphs of maps of maps from \(\Lambda\) to \(K\):

\[
[A]^T \begin{bmatrix}
\Lambda \\
\Lambda^*
\end{bmatrix} \begin{bmatrix}
C \\
A^*
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

But \(\Gamma_T\) Lagrangian should be equivalent to \(T\) symmetric.

provided you identify \(K\) with the dual of \(\Lambda\) appropriately.
The basic pairing between $K$ and $\Lambda$ should be

$$\begin{bmatrix} a \end{bmatrix} \cdot \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} a^t \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} = a^t x - y^t c$$

Let $T: \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} u & w \end{bmatrix} \begin{bmatrix} c \end{bmatrix} \mapsto \begin{bmatrix} c \end{bmatrix} \cdot \begin{bmatrix} A^t \end{bmatrix}$. You want to understand when $T$ is symmetric

with the pairing above, that is, when

$$\begin{bmatrix} c' \end{bmatrix}^t \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u & w \end{bmatrix} \begin{bmatrix} c \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} u & w \end{bmatrix} \begin{bmatrix} c \end{bmatrix}$$

is equal to $\begin{bmatrix} c' \end{bmatrix} \begin{bmatrix} v' & -u' \end{bmatrix} \begin{bmatrix} c \end{bmatrix} \Rightarrow$ YES.

---

$T: A \begin{bmatrix} u & w \end{bmatrix} \begin{bmatrix} c \end{bmatrix} \mapsto \begin{bmatrix} u c + u' d & c \end{bmatrix}$

$$\begin{bmatrix} u c_1 + u' d_1 & c_1 \\ V' c_1 + V d_1 & d_1 \end{bmatrix} \mapsto \begin{bmatrix} u c_2 + u' d_2 & c_2 \\ V' c_2 + V d_2 & d_2 \end{bmatrix}$$

$$= (u c_1)^t x_2 + (u' d_1)^t x_2 + c_1 V' c_2 + c_1 V d_2$$

$$- (V' c_1)^t c_2 - (V d_1)^t c_2 - d_1 u c_2 - d_1 u' d_2$$

$$\begin{aligned}
&= c_1 u^t x_2 + d_1 u'^t x_2 + c_1 v' c_2 + c_1 v d_2 \\
&- c_1 v t c_2 - d_1 v'^t c_2 - d_1 u c_2 - d_1 u' d_2
\end{aligned}$$

So $T$ isotropic means $\begin{bmatrix} v' & v \\ -u & -u' \end{bmatrix}$ symmetric

But $T$ isotropic should be equivalent to $T$ symmetric which means that you take two
\[ [c_1] \wedge [u', u'] [c_2] = [c_1]^t [-1, 0] [v', v] [c_2] \]

\[ [c_1] \wedge [v', v] [c_2] = [c_1]^t [v', v] [c_2] = [c_1]^t [-u', -u'] [c_1] \]

which indeed is symmetric under \(1 \leftrightarrow 2\) iff \([v', v]\) is symmetric.

Look at the special case where \(T = [u', u']\)

respects the voltage-current grading, that is \(u' = 0\)

and \(v' = 0\). Then

\[ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} = \begin{bmatrix} 0 & v \\ -u & 0 \end{bmatrix} = \begin{bmatrix} 0 & v \\ -u & 0 \end{bmatrix}^t \]

iff \(-u = v^t\). In this case

\[ T : \begin{bmatrix} A \\ C^* \end{bmatrix} \overset{[u', u']}{\underset{[0, -u^t]}{\longleftrightarrow}} \begin{bmatrix} C \\ A^* \end{bmatrix} \]

Review the problem: You consider an abstract version of a network consisting of a short exact sequence of ftd. \(M\) vs. and the dual sequence:

\[ A \longrightarrow B \longrightarrow C \]

\[ C^* \longrightarrow B^* \longrightarrow A^* \]

Put \(S = [B^*], K = [A^*], S/K = [C^*] \). \(S\) is the hyperbolic symplectic space associated to the v.s. \(V\) with skew-form

\[ [\beta_1] \wedge [\beta_2] = [b_1]^t \begin{bmatrix} 0 & I \\ -1 & 0 \end{bmatrix} [b_2] = b_1 b_2 - b_2 b_1. \]
X5. $K$ is a Lagrangian subspace $S$ and the quotient $S/K$ is isomorphic to $K^*$. 

Next, you need another version of the Ohm's law relations between voltage and current for the branches. The typical example is a positive symmetric operator $R^1: B \to B^*$, but you want to handle degenerate situations. Now the graphs $[R^1]B = [1][K]B^*$ coincide and yield a Lagrangian subspace $S$, which is complementary to $K$.

The appropriate substitute for the Ohm relations seems to be a Lagrangian subspace $\Omega \subset S$ such that $\Omega \to S \to S/K$ is an isomorphism. Then you get $S = K \oplus \Omega$, a kind of Hodge decomposition.

You need to understand such an $\Omega$ better. Once you choose a "basepoint" Lagrangian complement $\Lambda$ to $K$: $S = K \oplus \Lambda$, then another one $\Omega$ is specified by an symmetric operator $T: \Lambda \to K$ which is in a suitable sense.

---

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^*$</td>
<td>$B^*$</td>
<td>$C^*$</td>
</tr>
</tbody>
</table>

$K \to S \to S/K = K^*$

IDEA: In these anything cohomologically interesting about the choice of basepoint, mod 2 maybe because of the symmetry given by $\times$?
$45$

$$K \rightarrow S \rightarrow K^*$$

$$K \leftrightarrow S^* \rightarrow K^*$$?

Go back to the starting point:

$$\begin{bmatrix} A \\ C^* \end{bmatrix} \rightarrow \begin{bmatrix} B \\ B^* \end{bmatrix} \rightarrow \begin{bmatrix} C \\ A^* \end{bmatrix}$$

$$K \leftrightarrow S \quad S/K = K^*$$

Aim: To understand Lagrangian $\Omega \subset S$ s.t. $\Omega$ is a complement to $K$: $S = K \oplus \Omega$.

There should be an important class of such $\Omega$, namely those $\Omega \subset \begin{bmatrix} B \\ B^* \end{bmatrix}$ which respect the voltage-current grading, i.e. $\Omega = \begin{bmatrix} \Omega \cap B \\ \Omega \cap B^* \end{bmatrix}$. Such an $\Omega$ should arise by choosing a splitting of the s.e.s. $A \rightarrow B \rightarrow C$, i.e. a lifting $\tilde{C} \subset B$ of $C$, and then $\tilde{C} = \Omega \cap B$.

Using duality the splitting chosen should yield a splitting $C$ the dual s.e.s., i.e. a lifting $\tilde{A}^* \subset B^*$ of $A^*$, and then $\tilde{A}^* = \Omega \cap B^*$.

Let's fix a homogeneous $\Omega$ as above, call it $\Lambda$, and let's work with the corresponding splitting

$$S = K \oplus \Lambda = \begin{bmatrix} A \\ C^* \end{bmatrix} \oplus \begin{bmatrix} C \\ A^* \end{bmatrix}$$

The homogeneous $\Omega$ to consider are graphs of operators $\begin{bmatrix} A & 0 \\ C^* & A^* \end{bmatrix}$ satisfying $\begin{bmatrix} 0 & u' \\ v & 0 \end{bmatrix}$, i.e. $u': A \rightarrow A^*$ and $v': C^* \rightarrow C$, symmetric.
So it now seems that you have a complete analogue of the picture occurring in a $K$-network with $>0$ resistances for the branches. Namely, any $H$ complementary to $K$ will yield compatible splittings of the voltage and current s.e.s.

\[
\begin{align*}
A & \hookrightarrow B \hookrightarrow C \\
C^* & \hookrightarrow B^* \hookrightarrow A^*
\end{align*}
\]

You also seem to get symmetric maps
\[
C^* \xleftarrow{\sim} C, \quad A \xleftarrow{\sim} A^*
\]

which means Lagrangian subspaces in the hyperbolic spaces $[C^K]$ and $[A^K]$.  

**IDEA.** In studying $[A^K] \xrightarrow{S} [B^K] \xrightarrow{S/K = K} [C^K]$, you can choose a pos. def. scalar product on $B$. Then you have $B \rightarrow B^*, \; b \mapsto b^t$, and inclusion iso.

\[
A \rightarrow A^*, \quad A^\perp = C = C^*.
\]

So you have a Euclidean picture
\[
\begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \oplus \begin{bmatrix} C \\ A \end{bmatrix}
\]

where the skew form is given by a skew-symmetric operator in "4 dimensions".
Return to our abstract network:

\[
\begin{bmatrix}
A^* \\
C^*
\end{bmatrix} \leftrightarrow \begin{bmatrix}
B^* \\
B^*
\end{bmatrix} \rightarrow \begin{bmatrix}
C^* \\
A^*
\end{bmatrix}
\]

\[K \quad S \quad S/K = K^*\]

and suppose given a splitting compatible with duality, so that

\[
\begin{bmatrix}
B^* \\
B^*
\end{bmatrix} = \begin{bmatrix}
A^* \\
C^*
\end{bmatrix} \oplus \begin{bmatrix}
C^* \\
A^*
\end{bmatrix}
\]

\[S = K \oplus K^*\]

So \( S \) has two splittings into complementary Lagrangian subspaces. Review the formula for the symplectic form in \( S \):

\[
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} \wedge \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = \begin{bmatrix}
b_1 \end{bmatrix}^t \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} = b_1^t b_2 - b_2^t b_1
\]

\[
\begin{bmatrix}
a_1 + c_1 \\
\gamma_1 + x_1
\end{bmatrix} \wedge \begin{bmatrix}
a_2 + c_2 \\
\gamma_2 + x_2
\end{bmatrix} = \begin{bmatrix}
a_1 \wedge [c_2] \\
\gamma_1 \wedge [x_2]
\end{bmatrix} + \begin{bmatrix}
c_1 \wedge [a_2] \\
x_1 \wedge [\gamma_2]
\end{bmatrix}
\]

\[= a_1^t a_2 - c_1^t c_2 + c_1^t x_2 - \gamma_1^t c_2 + c_1^t \gamma_2 - a_1^t c_2\]

Here, you are writing:

\[S = \begin{bmatrix}
A^* \\
C^*
\end{bmatrix} = \begin{bmatrix}
A^* \\
C^*
\end{bmatrix} \oplus \begin{bmatrix}
A^* \\
C^*
\end{bmatrix} \rightarrow \begin{bmatrix}
K \\
K^*
\end{bmatrix}\]
You can also write
\[
S = \begin{bmatrix} B \\ B^* \end{bmatrix} = \begin{bmatrix} A \\ C^* \\ C \end{bmatrix} = \begin{bmatrix} c^t \\ a^t \\ x^t \\ x^t \end{bmatrix}
\]

\[
b_1^t b_2 - b_1^t b_2 = \begin{bmatrix} a^t \\ c^t \\ x^t \\ x^t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \begin{bmatrix} a_2 \\ c_2 \\ x_2 \\ x_2 \end{bmatrix}
\]

IDEA: This "4-dim" picture reminds you about the two hermitian forms arising in the wave equation.

At the moment you need to work on your conjecture in the case of the symplectic space \( S = [B^*] \) and the Lagrangian subspace \( K = [C^*] \), that a complementary Lagrangian subspace \( \Omega \) to \( K \) is equivalent to the following:

1) Splittings of the voltage and current s.e.s. compatible with duality:
\[
\begin{bmatrix} B \\ B^* \end{bmatrix} = \begin{bmatrix} A \\ C^* \end{bmatrix} \oplus \begin{bmatrix} C \\ A^* \end{bmatrix}
\]

2) Symmetric maps \( C \leftarrow C \rightarrow A \leftarrow A^* \).

Let's begin with the (voltage) s.e.s. together with a splitting:
\[
\begin{array}{c}
A \\ B \\ C
\end{array}
\]

\[
\begin{array}{c}
A \\ B \\ C
\end{array}
\]
Now dualize to get the (current) s.e.s. with the splitting:

\[ \Delta^* \]
\[ \rightarrow \]
\[ \rightarrow \]
\[ \Gamma^* \]

\[ C^* \rightarrow B^* \rightarrow A^* \]

You would like to put these two splitting diagrams together

\[ \Gamma^* \]
\[ \Delta^* \]
\[ \rightarrow \]
\[ \rightarrow \]

\[ A \]
\[ \rightarrow \]
\[ B \]
\[ \rightarrow \]
\[ C \]

\[ C^* \]
\[ \rightarrow \]
\[ B^* \]
\[ \rightarrow \]
\[ A^* \]

\[ \Delta^* \]
\[ \rightarrow \]
\[ \Gamma^* \]

What does this mean? You have the hyperbolic space of B with the complementary Lagrangian subspaces \[ [A] \] and \[ [\Gamma^*] \].

So far you have respected the voltage-current grading, but you should be able to add symmetric maps \[ C^* \leftarrow C \], \[ A \leftarrow A^* \]?
Now you should go back to the example where you've adjoint a pure emf branch. Review the equations. There are 2 edges hence 4 variables $V_R, V_e, I_R, I_e$ subject to Kirchhoff conditions $V_R + V_e = 0, I_R = I_e$ and Ohm's Law conditions. For the $R$ edge you have $V_R = R I_R$, but the $e$ edge might cause problems.

To proceed put in a small resistance $R_e = \varepsilon$ on the $e$ edge. Then Ohm says: $\varepsilon = \varepsilon I + R I$. The problem is to understand the Ohm's conditions using Lagrangian subspaces.

How do you handle things? It's probably a bad idea to introduce $\varepsilon$. Instead consider the voltage + current s.e.s.

$$C^0 \overset{C}{\rightarrow} \overset{H}{C'} \rightarrow H^1$$

$$H \overset{C_1}{\rightarrow} \overset{\bar{C}_0}{C_1}$$

You need a Lagrangian complement $\Omega$ to the Kirchhoff space $K$. A branch with resistance $R$ is described by a hyperbolic plane $\{ [V] | I \}$ equipped with the Lagrangian subspace $V = R I$. This makes sense for $R = 0$ and $R = \infty$, but you
We need to make $\Omega \cap K = 0$.

The equations are $V_R + V_e = 0$, $I_R = I_e$ and

for $\Omega$ are $V_R = RIR$, $V_e = 0$. ... only the zero solution, so $\Omega \cap K = 0$.

Next you want the corresponding splitting of $S = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = K + \Omega$

Now you want to take $\begin{bmatrix} e_1 \\ -e_2 \end{bmatrix} \in C^1$ and split it into $K$ and $\Omega$ components. Actually you should perhaps take $\begin{bmatrix} e_1 \\ -e_2 \end{bmatrix}$ in $C^1$. You push this into $\begin{bmatrix} e_1 \\ -e_2 \end{bmatrix} \in [H'_{C^0}].$ To find $\omega \in \Omega$ with the same image in $[H'_{C^0}].$ This means finding $I$ such that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}[R_{C^0}]\begin{bmatrix} 1 \\ 1 \end{bmatrix}I = \begin{bmatrix} R_{C^0} \\ 0 \end{bmatrix} = RI$ is equal to $-\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$. So $I = -\frac{e_2}{R}$ and $\begin{bmatrix} e_1 \\ -e_2 \end{bmatrix} - (R_{C^0}I) = \begin{bmatrix} -RI \end{bmatrix}$, which is the image of a $\psi \in C^0$. (Perhaps $\epsilon$ should be $-\epsilon$ above).
What you want here is the splittings of the two s.e.s. and also quadratic forms on $H^1$ and $\bar{C}_0$. So far you have split \[
\begin{bmatrix}
0 \\
-\varepsilon
\end{bmatrix}
\] into \[
\begin{bmatrix}
B & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
H_1 \\
\varepsilon
\end{bmatrix}
+ \begin{bmatrix}
-\varepsilon \\
\varepsilon
\end{bmatrix}
= \begin{bmatrix}
-\varepsilon \\
0
\end{bmatrix}
+ \begin{bmatrix}
-\varepsilon \\
\varepsilon
\end{bmatrix}
\]

This is confusing. You need an intelligent way to handle the splitting $S = K \oplus \Omega$. You know that $\Omega$ amounts to a homogeneous splitting with the voltage-current grading together in quadratic forms on the "twangs".

Picture

Notice that in this situation you get a splitting of the voltage s.e.s. given by \[
I \mapsto \begin{bmatrix}
I_1 \\
0
\end{bmatrix}
\mapsto \begin{bmatrix}
B & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
I_1 \\
0
\end{bmatrix}
= \begin{bmatrix}
RI
\end{bmatrix}
\]
from $H_1$ to $C^1$. There doesn't seem to be a similar map $\bar{C}_0 \to C^1$, however, there is a unique splitting of the current s.e.s. compatible with the voltage splitting and duality.

Notice that if $V \in H^1$, then \[
\begin{bmatrix}
R \\
0
\end{bmatrix}
\]

\[
R^{-1} V = \begin{bmatrix}
V \\
0
\end{bmatrix} \in C^1
\]
gives the desired lifting of $H^1$ into $C^1$. Similarly

\[
R^{-1} [R \Omega] : C^1 \to H^1 \to H_1
\]

\[
\begin{bmatrix}
I_{KR} \\
IR
\end{bmatrix}
\]

\[
\begin{bmatrix}
I_{R} \\
I_R
\end{bmatrix}
\]
is a retract of $C^1$ onto $H_1$. Similarly

\[
\begin{bmatrix}
I_R \\
I_R
\end{bmatrix}
\]
So the picture of the splittings of the voltage + current splittings becomes

\[
\begin{align*}
C^0 & \xrightarrow{[-1]} C \xrightarrow{[1 \ 1]} H^1 \\
H_1 & \xrightarrow{[1 \ 0]} C \xrightarrow{[1 \ -1]} C^0
\end{align*}
\]

Unfortunately the voltage + current splittings described here are not expressed in the same form - for voltage you have a subspace, for current you have a quotient space.

Better:

\[
\begin{align*}
C^0 & \xrightarrow{[1 \ -1]} C \xrightarrow{[1 \ 1]} H_1 \\
H_1 & \xrightarrow{[1 \ 0]} C \xrightarrow{[1 \ -1]} C^0
\end{align*}
\]

\[
\begin{bmatrix}1 & 0 \ -1 \end{bmatrix} + \begin{bmatrix}1 & 1 \ 0 \ 1\end{bmatrix} = \begin{bmatrix}1 & 0 \ 0 & 1\end{bmatrix}
\]

\[
\begin{bmatrix}1 & 1 \ 0 \ -1\end{bmatrix} + \begin{bmatrix}0 & 0 \ 1 \ 1\end{bmatrix} = \begin{bmatrix}1 & 0 \ 0 & 1\end{bmatrix}
\]

\[
\begin{bmatrix}1 & 1 \ 0 \ -1\end{bmatrix} + \begin{bmatrix}0 & 0 \ 1 \ 1\end{bmatrix} = \begin{bmatrix}1 & 0 \ 0 & 1\end{bmatrix}
\]
Also record the shorter picture

\[
\begin{align*}
\mathcal{C} & \xleftarrow{[-1]} \mathcal{C}' \xrightarrow{[0]} \mathcal{H}' \\
\mathcal{H} & \xleftarrow{[1]} \mathcal{C} \xrightarrow{[0]} \mathcal{C}_0
\end{align*}
\]

Let's review the situation in the simple case.

- State space \([\mathcal{C}' \mathcal{C}]\) variables \(V_R, V_e, I_R, I_e\)
- Kirchhoff: \(V_R + V_e = 0, \quad I_R = I_e\)
- Ohm: \(V_e = 0, \quad V_R = R I_R\)

So \(K \cap \Omega = 0\), assuming \(R \neq 0\). You believe that
\(\Omega\) yields splitting of the voltage + current short exact sequence, compatible with duality, and also symmetric bilinear forms

\[
\mathcal{H} \leftarrow \mathcal{H}' \quad \text{and} \quad \mathcal{C}_0 \leftarrow \mathcal{C}_0.
\]

It seems the former is the inverse of the obvious map \(R: \mathcal{H} \rightarrow \mathcal{H}'\). Is there an obvious candidate for the latter? Notice that the map goes from currents to node potentials.

**Hodge Decomposition**

\[
S = K \oplus \Omega,
\]

The important thing is the splitting, that is, the projection operators on \(S\). This is the output; it's related to Hadamard's finite part and the Birkhoff decomposition for loop groups. Connecting with scattering, stuff you did with grid spaces, Weierstrass-Hopf?.

**IDEA:** Grothendieck motif used to define \(\lambda\)-ring, where one has a triple involving the free \(\lambda\)-ring generated by \(A\). Questions is whether you can do something similar with Lie symplectic spaces and the hyperbolic functor.
IDEA which seems familiar: Work with Euclidean spaces instead of dual pairs \( V, V^* \). More precisely, given a dual pair \( V, V^* \) choose a positive scalar product \( \langle \cdot, \cdot \rangle : V \rightarrow V^* \), and then handle bilinear forms as operators. Thus a symplectic vector space is represented by a Euclidean space with invertible skew-symmetric operator. Then choose the Euclidean structure better (by polar decomp) you get a Euclidean space with complex structure \( J \).

The hyperbolic symplectic space associated a Euclidean space \( V \) is the complexification \( \mathbb{C} \otimes_{\mathbb{R}} V \) of \( V \).

Q: What are Lagrangian subspaces of \( V, J \)? The symplectic form should be the imaginary part of the hermitian scalar product. Thus a real subspace \( L \subset V \) is isotropic when the hermitian form is real on \( L \).

\[ \text{Return to} \quad \begin{array}{cccc}
\mathbb{C} & \mathbb{R} & \mathbb{C} & 1 \\
\mathbb{C} & 1 & \mathbb{C} & \mathbb{C} \\
1 & \mathbb{C} & \mathbb{C} & 1 \\
\mathbb{R} & \mathbb{C} & 1 & 1 \\
\end{array} \]

\[ \text{S/K = K^*} \]
Hodge decomposition $S = K \oplus \Omega$

$$S = \begin{bmatrix} C' \end{bmatrix} \oplus \begin{bmatrix} V_R & V_e \\ I_R & I_e \end{bmatrix}, \quad K = \begin{bmatrix} C_0 \\ H_1 \end{bmatrix} \oplus \begin{bmatrix} H' \\ E_0 \end{bmatrix}, \quad S/K = \begin{bmatrix} H' \\ C_0 \end{bmatrix} = K^{*3} \rho \begin{bmatrix} u \\ J \end{bmatrix}$$

Exact sequence

$$\begin{array}{c}
\begin{bmatrix} C_0 \\ H_1 \end{bmatrix} \xrightarrow{i} \begin{bmatrix} C' \end{bmatrix} \xrightarrow{\pi} \begin{bmatrix} H' \\ C_0 \end{bmatrix}
\end{array}$$

where

$$i = \begin{bmatrix} I & -I \\ I & I \end{bmatrix}, \quad \pi = \begin{bmatrix} V_R & V_e \\ I_R & I_e \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The next one has $\Omega$ which is the graph of the resistance $[R \ O \ E] : C_1 \to C'$; this is symmetric so its graph $\Omega$ is Lagrangian. Thus $\Omega$ has the elements

$$\begin{bmatrix} V_R & V_e \\ I_R & I_e \end{bmatrix} = \begin{bmatrix} R & 0 \\ O & E \end{bmatrix} \begin{bmatrix} I_R \\ I_e \end{bmatrix} = \begin{bmatrix} R & I_R \\ E & I_e \end{bmatrix}$$

for $\begin{bmatrix} I_R \\ I_e \end{bmatrix} \in C_1$.

The restriction of $\pi$ to $\Omega$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} R & 0 \\ O & E \end{bmatrix} \begin{bmatrix} I_R \\ I_e \end{bmatrix} = \begin{bmatrix} R & E \\ 1 \end{bmatrix} \begin{bmatrix} I_R \\ I_e \end{bmatrix} = \begin{bmatrix} u \\ J \end{bmatrix}$$

which has inverse

$$\begin{bmatrix} I_R \\ I_e \end{bmatrix} = \frac{1}{+R + E} \begin{bmatrix} +1 + E \\ +1 - R \end{bmatrix} \begin{bmatrix} u \\ J \end{bmatrix}$$

Then one has the lifting of $[u/J] \in S/K$ into $\Omega$ given by

$$\begin{bmatrix} R & 0 \\ O & E \end{bmatrix} \frac{1}{+R + E} \begin{bmatrix} 1 & E \\ 1 & -R \end{bmatrix} \begin{bmatrix} u \\ J \end{bmatrix}$$

and we conclude that
is the projection operator on $S$ with kernel $K$ and image $\Omega$.

Review the network $A$, states are $\begin{pmatrix} V_R \\ V_e \\ I_R \\ I_e \end{pmatrix}$, Kirchhoff constraints: $V_R + V_e = 0$, $I_R - I_e = 0$

define a Lagrangean subspace $K = \left\{ \begin{bmatrix} 0 \\ \mathbf{E} \end{bmatrix} \right\} \subset S$

such that $S/K = K^* = \left\{ \begin{bmatrix} H^* \end{bmatrix} \right\}$. The Ohm subspace $\Omega$ of $S$ is given by $V_R = R I_R$, $V_e = 0$. It is the graph of the resistance map $\begin{pmatrix} I_R \\ I_e \end{pmatrix} \mapsto \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} I_R \\ I_e \end{pmatrix}$.

Thus $\Omega \equiv \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} I_R \\ I_e \end{pmatrix}$ from $C_i$ to $C'$. Thus $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I_R \\ I_e \end{pmatrix} = \begin{pmatrix} R I_R \\ I_e \end{pmatrix}$ for $I_R, I_e$.

$\Omega$ is a Lagrangean subspace of $S$ because the resistance map $C_i \rightarrow C'$ is symmetric.

Assume $R \neq 0$. Then the relations defining $K \oplus \Omega$ imply that $K \cap \Omega = 0$, hence $S = K \oplus \Omega$. This can be viewed (or called) a Hodge decomposition where the positivity of the resistance map has been relaxed. (Maybe better to say it is an analog or generalization of the Hodge decomposition where the resistance map is $> 0$.)

Next you want to see how this decomposition leads to splittings of the voltage and current $1$-forms. You also want to find the symmetric maps $H^1 \rightarrow H_1$ and $C_0 \rightarrow C_0$. Associated
Let's construct the Hodge decomposition by showing that $\Omega \subset S \overset{\pi}{\longrightarrow} S/K$ is an isom. $\pi$ can be viewed as the Kirchhoff constraint map:

\[
\begin{bmatrix} V_R^e \\ V_e \\ I_R \\ I_e \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_R^e \\ V_e \\ I_R \\ I_e \end{bmatrix} = \begin{bmatrix} u \\ J \end{bmatrix} \in H^0 \subset \Omega.
\]

Restricting to $\Omega$ gives

\[
\begin{bmatrix} I_R \\ I_e \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} R I_R \\ 0 \\ I_R \\ I_e \end{bmatrix} = \begin{bmatrix} u \end{bmatrix} = \begin{bmatrix} u \\ J \end{bmatrix} = \begin{bmatrix} u \\ J \end{bmatrix} \in \Omega.
\]

Then

\[
\begin{bmatrix} R & 0 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \overset{R \neq 0, \text{ since}\ R \neq 0, \text{ showing}}{\longrightarrow} \begin{bmatrix} u \end{bmatrix} \in \Omega.
\]

Picture:

\[
\begin{array}{ccc}
\mathcal{C}_0 & \longrightarrow & \mathcal{C}' \longrightarrow \mathcal{H}^1 \overset{\Phi}{\longrightarrow} \mathcal{U} \\
\uparrow & & \uparrow \\
\Omega \sim & \longrightarrow & \begin{bmatrix} \mathcal{H}^1 \\ \mathcal{C}_0 \end{bmatrix} \overset{\Phi}{\longrightarrow} \begin{bmatrix} u \\ 0 \end{bmatrix} \\
\downarrow & & \uparrow \\
\mathcal{H}^1 & \longrightarrow & \mathcal{C}_0 \overset{\Phi}{\longrightarrow} \mathcal{C}_0 \\
\end{array}
\]

Left $\mathcal{H}^1$ and $\mathcal{C}_0$ into $\Omega$ in the way indicated, using the splitting of $\Omega$ given the isom. $\Omega \sim \begin{bmatrix} \mathcal{H}^1 \\ \mathcal{C}_0 \end{bmatrix}$. This amounts to

\[
\begin{bmatrix} u \\ J \end{bmatrix} \rightarrow \frac{1}{R} \begin{bmatrix} 1 & 0 \\ 1 & -R \end{bmatrix} \begin{bmatrix} u \\ J \end{bmatrix} = \begin{bmatrix} u \\ J \end{bmatrix} \in \Omega.
\]
\[ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} R & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{R} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} R \\ 0 \\ 0 \end{bmatrix} \frac{1}{R} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \]

Thus we get the lifting of \( u \in H' \) to \( \frac{1}{R} u \in S \)

and the lifting of \( J \) in \( C_0 \) to \( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in S \). Take the former and project it into \( C^i \) to get \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \in C^i \).

Take the latter and project it into \( C_1 \) to get \( \begin{bmatrix} 0 \\ -1 \end{bmatrix} J \in C_1 \).

Notice this checks with \( \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} u = u, \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} J = J \).

Apparently you get the following splittings for the voltage + current s.e.s.

\[ \begin{array}{ccc}
C^0 & \overset{[1]}{\longrightarrow} & C^i \\
\downarrow & & \downarrow \\
H' & \overset{[0]}{\longrightarrow} & C_1
\end{array} \]

These liftings should be \( u, v \) in

\[ \begin{bmatrix} u' \\ u'' \end{bmatrix} : \begin{bmatrix} C^0 \\ H' \end{bmatrix} \overset{[0]}{\longrightarrow} \begin{bmatrix} \mathbb{C} \\ C^0 \end{bmatrix} \]

The symmetric maps \( u', v' \) should be obtained by taking the current part of \( \begin{bmatrix} R \\ 1 \end{bmatrix} \frac{1}{R} \) which is \( \begin{bmatrix} 1 \\ \frac{1}{R} \end{bmatrix} \frac{1}{R} \) and the voltage part of \( \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \frac{1}{R} \) which is zero. Thus \( u' = 0 \) and it looks like \( v' = \frac{1}{R} : H' \rightarrow H_1 \).
Q: Can you interpret a quadratic space in "triple" terms using the hyperbolic functor?

Recall

\[ H(V) = \begin{bmatrix} V^* \\ V \end{bmatrix} \in \begin{bmatrix} \mathcal{V} \\ \varphi \end{bmatrix} \begin{bmatrix} \mathcal{O} \\ 0 \end{bmatrix} \]

\[ H(V)^* = \begin{bmatrix} V^* \\ V \end{bmatrix}^* = \begin{bmatrix} V^* \end{bmatrix} \in \begin{bmatrix} \mathcal{V}^* \\ \varphi \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix} \]

So that the symmetric form on \( H(V) \) is

\[
\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \varphi_1^t \\ \varphi_2^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \varphi_2 \\ \varphi_1 \end{bmatrix} = \varphi_1^t \varphi_2 + \varphi_2^t \varphi_1
\]

(\text{It's probably clearer to define the symmetric form of } H(V) \text{ via})

\[ H(V) = \begin{bmatrix} V^* \\ V \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V^* \\ V \end{bmatrix} = \begin{bmatrix} V^* \end{bmatrix}^* = H(V)^* \ ? \]

Suppose \( T: V \rightarrow V^* \) is symmetric. Then

\[
\begin{array}{ccc}
V & \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} & H(V) \in \begin{bmatrix} V^* \\ V \end{bmatrix} \\
2T & \downarrow & \downarrow S \\
2T^* & \xrightarrow{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} & H(V)^* \in \begin{bmatrix} V^* \\ V \end{bmatrix}
\end{array}
\]

\[
\text{If } 2T \text{ is invertible, it looks like the quadratic space } (V, T) \text{ is a retract of the hyperbolic space } (H(V), [0, 0]).
\]

Certainly you should know how to use the invertibility of \( 2T \) to split \( H(V) \) into \( V \) and an orthogonal complement.
Calculate the Hodge decomposition of the vector space $V_R^4$ defined by Kirchhoff's equations $V_R + V_e = I_R - I_e = 0$ and Ohm's law $V_R = R I_R$, $V_e = 0$. Let $[C_0]$ have coordinates $[U]$

\[ K \overset{\pi}{\longrightarrow} S/K = K^* \]

$\Omega$ is the graph of $[T_R] \mapsto [R 0] [T_R] = [R I_R]$, so

\[ \Omega = \{ [R 0] [T_R] \mid R 00 [T_R] \} \]

which has inverse $[U] \mapsto \frac{1}{+R} [+1 0 7] [I_R]$, so the projection of $S$ onto $\Omega$ with kernel $K$ is $\pi : S \rightarrow S/K$, where

\[ \begin{bmatrix} R & 0 \\ 0 & 0 \\ 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \frac{1}{R} \begin{bmatrix} R & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \]

\[ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ R^{-1} & 0 & 0 \\ LR^{-1} & R^{-1} & -1 \end{bmatrix} \]

On the other hand, the product $e(1-e) = 0$, so you have the desired decomposition.
Next find the response to the forcing term given by \( V_e = -\mathcal{E}, \ V_R = I_R = I_e = 0 \).

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-R^t & -R^t & 1 & 0 \\
-R^t & -R^t & 1 & 0
\end{pmatrix}
\begin{pmatrix}
\mathcal{E} \\
-\mathcal{E} \\
R^t \mathcal{E} \\
R^t \mathcal{E}
\end{pmatrix}
= \begin{pmatrix}
V_R \\
V_e \\
I_R \\
I_e
\end{pmatrix}
\]

which is what you expected.

Idea that hyperbolic functor might give rise to a kind (or variant) of a triple. The free quadratic spaces should be the hyperbolic spaces.

You know that an embedding of a quadratic space \((V, T; V \rightarrow V^*)\), \(T\) symmetric + Koszul, into the hyperbolic space \(H(W)\) is described by \([\mathcal{E}] : V \rightarrow \left[\frac{W}{W^*}\right]\) such that

\[
\begin{pmatrix}
[\mathcal{E}] \\
\delta \mathcal{E}
\end{pmatrix}
\begin{pmatrix}
[w] \\
[w^*]
\end{pmatrix}
\]

commutes, i.e.

\[
\begin{pmatrix}
[x^t, \beta^t] \\
[x, \beta]
\end{pmatrix} = x^t \beta + \beta^t x
\]

Thus the embedding yields a map \(h = \beta^t x : V \rightarrow V^*\) such that \(h + h^t = T\). Conversely, given such an \(h\) we can choose a factorization \(V \xrightarrow{\psi} W\) and then put \(\beta = x^t \psi : V \rightarrow W^*\) such that \(h = \beta^t x\), \(h + h^t = T\).

An \(h : V \rightarrow V^*\) s.t. \(h + h^t = T\) has the form \(h = \frac{1}{2}(T + X)\), where \(X : V \rightarrow V^*\) is skew-symmetric. The simplest case is \(X = 0\), \(h = \frac{1}{2} T\), but nontrivial \(X\) should yield bigger \(W\)'s.

All the above reminds you of retracts of a free \(Z/2\) module. **QUESTION:** Can you do something...
about the Cayley transform in the quadratic phase situation that you couldn’t handle before. “Before” refers to your attempt to link the theory of retracts of a free \(2\times 2\) module \([V]\), where you encountered an odd operator \(X\), to the Cayley transform.

March 31, 03. IDEA about the Inverse Cayley Transform.

An Abstract LC network gives rise to a unitary representation of the infinite dihedral group \(\langle F, \varepsilon \rangle\). You would like to express all of LC network theory in terms of these representations. An important example of this theory is the dynamics, which is given by the resolvent \((s-X)^{-1}\), where \(X\) is the I.C.T. of \(g = F\varepsilon\).

\(X\) is skew-symmetric + odd \(\varepsilon\).

May 2, 03: A form of \(\langle F, \varepsilon \rangle\) \(\Rightarrow\) \(g = F\varepsilon = \frac{1+X}{s-X}\)

\(A, H \Rightarrow X = A^{-1}H\)

Good case is where \(F, \varepsilon\) are “transversal” where you have

\[
\begin{pmatrix}
F^{-1} & V_-
\end{pmatrix}
\begin{pmatrix}
1 & -T^*
\end{pmatrix} = 1 + X, \quad F(1+X) = (1+X)\varepsilon
\]

\[\Rightarrow \quad F\varepsilon(1-X) = 1 + X\]

Puzzle: How do dynamics arise. Algebraically it seems you replace \(1\) by the Laplace transform variable \(s\).