Review: critical point of $\frac{1}{2}x^TAx$ subject to a linear constraint $y^tx = \mathcal{E}$. Let
$$F = \frac{1}{2}x^TAx + \lambda(c - y^tx)$$
$$dF = x^TAdx + \lambda dy^tx = 0 + dx(c - y^tx)$$

i.e. $Ax = \lambda y$ and $c = y^tx$.

$$\Rightarrow x = \lambda A^{-1}y, \quad c = y^tA^{-1}y, \quad \lambda = \frac{c}{y^tA^{-1}y}, \quad x = \frac{c}{y^tA^{-1}y} A^{-1}y = \frac{1}{2} \frac{c^2}{y^tA^{-1}y}$$

critical value $\frac{1}{2} \frac{c}{y^tA^{-1}y}$

critical point

Review attached to an external emf from "node" $A$ to the ground $0$. This situation can be handled entirely by means of $\mathcal{E}$, the node voltage space, equipped with the power form. In the notation above $x \in \mathcal{E}$, $\frac{1}{2}x^TAx = \text{power form}$, $x \mapsto y^tx$

is $x = \{x(t)\}$. What to say next?

Full phase space picture

$\mathbb{R} \leftarrow \mathbb{C} \leftarrow \mathbb{C}^1 \rightarrow \mathbb{H}^1$ describe what's happening

You have the network $X$ with external emf applied between node $A$ and node $0$. You are confident that the response to this external emf is a mode potential $\phi$ which satisfies a Poisson's equation, more precisely $\phi$ is harmonic away from the nodes $A, 0$. I review the problem. You have a

com $\mathbb{R}$-network $X$ with two $\neq$ modes $A, 0$ specified. You want to fix $V_A, V_0 = 0$ but allow a mode current $i_A$ at $A$ out at $B$. Apparently this works, and you can do a few examples. You even have some ideas of how the current arises via Lagrange multipliers.

Idea: go back to the problem of finding the stationary value of $\frac{1}{2}x^TAx$ subject to the condition $c = y^tx$. This formulation uses only the "voltage" picture. Suppose you use the Lagrange multiplier method:
\[ F = \frac{1}{2} x^T A x + \lambda (c - y^T x), \quad \nabla F = A x - \lambda y = 0 \]
and \[ \nabla F = c - y^T x = 0. \]
So \( x = \lambda A^{-1} y \) and 
\[ c = y^T x = \lambda y^T A^{-1} y. \]
\[
\lambda = \frac{c}{y^T A^{-1} y}, \quad x = \frac{c}{y^T A^{-1} y} A^{-1} y.
\]

Probably what you want is a symplectic interpretation of what's happening: \( y \) is a dual variable to \( x \). Maybe \( c \) and \( \lambda \) are dual, or maybe there's a better interpretation using "affine" ideas. What \( \triangle \) you should be able to do is to find a "duality of \( x^T A x, c - y^T x \) involving "voltage", "current" variables.

Let's review the picture of a connected \( R - \Omega \) network equipped with a pair of nodes \( A = O = \) the ground. Then you have voltage space \( \mathbb{C} \) equipped with a pos. def. quadratic form \( \langle R, \Omega \rangle \), which is the restriction of the power form on \( \mathbb{C} \). You also have a surjection \( \Psi: \mathbb{C} \to \mathbb{R}, \varphi \mapsto \varphi(A) - \varphi(0) = \varphi(A) \). For each \( \varphi \in \mathbb{R}^+, \varphi^{-1}(c) \) is the set of node potentials \( \varphi \in \mathbb{C} \) such that \( \varphi(A) = c \), i.e. such that \( c \) is the voltage drop from \( A \) to \( O \).

This is an inhomogeneous condition. Compare it to a Thévenin condition in which Ohm's Law for an \( R \)-edge: \( V = R I \), is allowed to become inhomogeneous: \( V = +RI + E \)? \( R \) in series with a real battery \( E \) and pure emf. Think of the edge as a pure emf in series with an internal resistance, i.e. a real battery, then you expect the current to flow toward the positive terminal. \( V_{\text{beginning}} = -RI + E = V_{\text{end}} \).
\[ V_a - V_c = RI \quad V_c - V_b = -\varepsilon \]

When you add you get \( V_a - V_b = RI - \varepsilon \).

It looks like you should introduce a new sign convention for the edge: The voltage drop for an emf \( \varepsilon \) is \(-\varepsilon\).
Start again, but avoid the sign difficulties by setting up the linear algebra together with quadratic form. Begin with a connected R-network, linear algebra, linearization:

$$\mathbb{C}^0 \longrightarrow \mathbb{C}^1 \longrightarrow H^1$$

$$\begin{array}{c}
R^{-1} \\
\downarrow \\
\mathbb{C}_0 \leftarrow \mathbb{C}_1 \leftarrow H_1
\end{array}$$

The cochain s.e.s. and the chain s.e.s. are naturally dual. \( R \) is a pos. def. quadratic form on \( \mathbb{C}^1 \), it gives the power of any edge voltage configuration. The quadratic form induces an orthogonal splitting of the cochain s.e.s., and also of the chain s.e.s., these splittings are compatible with the duality.

Next discuss circuit equations. A state of the network consists of a \( V \in \mathbb{C}^0 \) and an \( I \in H_1 \), i.e., edge voltages which come from a node potential and loop currents ( = 1-cycles). By conservation of flux you can replace \( I \) by \( RI \) which \( \in (\mathbb{C}^0)^\perp \), so the splitting \( \mathbb{C}^1 = \mathbb{C}^0 \oplus (\mathbb{C}^0)^\perp \) sets up a 1-1 correspondence between \( \mathbb{C}^1 \) and states of the network:

$$\begin{pmatrix} \mathbb{C}^0 \\ H_1 \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{C}^0 \\ (\mathbb{C}^0)^\perp \end{pmatrix} \rightarrow \mathbb{C}^1$$

So \( \forall \tilde{e} \in \mathbb{C}^1 \) you get \( \tilde{e} = V + RI \) for unique \( V \in \mathbb{C}^0 \) \( I \in H_1 \).

(Signs? \( A \longrightarrow \tilde{e} \longrightarrow B \)).

\( V \) is the voltage drop \( V_A - V_B = RI - \tilde{e} \). Maybe it's simpler to put the emf in the opposite direction, so \( \tilde{e} \) behaves like \( \pm RI \).

Try this: \( V_A - V_B = RI + \tilde{e} \).
Back to starting point: connected R-network

\[ V \in \mathbb{C} \xrightarrow{R} \mathbb{C}' \xrightarrow{H} \mathbb{H} \xrightarrow{RI+\varepsilon} \mathbb{H} \]

\[ \mathbb{C'} \xrightarrow{R} \mathbb{C} \]

Note you would like to discuss an external emf. You have \( V : \mathbb{C} \rightarrow \mathbb{R} \) given, you want to restrict to \( V \in \mathbb{C} \) st., \( \mathbb{H}(V) = c \). So if \( \mathbb{H}(V) = V_A - V_B \) you are fixing the voltage drop from A to B. This condition is inhomogeneous.

How to study this? You have \( \mathbb{C} \) equipped with pos. quad form \( S^R S : \mathbb{C} \rightarrow \mathbb{C} \). Write \( x \) for an element of \( \mathbb{C} \), \( A = S^R S \), and yet for \( y : \mathbb{C} \rightarrow \mathbb{R} \). Want stationary value of \( x^T Ax \) subject to \( y^T x = c \).

So far no real use of \( C' \).

You want to use the augmented network, which should amount to considering \( S: \mathbb{C} \rightarrow \mathbb{C} \)
with \( y : \mathbb{C} \rightarrow \mathbb{R} \) to get

\[ \mathbb{C} \xrightarrow{(S)} \mathbb{C'} \xrightarrow{(R)} \mathbb{C} \]

which is clearly 1-1 as \( S \) is.

Next you need a quadratic form on \( \mathbb{C'} \) which pulls back \( \mathbb{H}(S) \) to the given form on \( \mathbb{C} \), equivalently, you want a quad form on \( \mathbb{C}' \) such that \( \mathbb{H}(S)^T Q(S) \mathbb{H}(S) = S^R S \).
Repeat. You begin with $\mathbb{R}^1$ together with the quadratic form on $C'$. You want to obtain a map $\mathbb{C}^0 \xrightarrow{\delta} (\mathbb{C}'_{\mathbb{R}})$ together with a positive quadratic form on $\mathbb{C}^0$ with appropriate properties. Everything ultimately should result from the quad form on $C'$. Focus upon the data, namely, the filtration

$$0 \subset \text{Ker} \delta < \mathbb{C}^0 \xrightarrow{\delta} \mathbb{C'}$$

which 1.

It seems that $(\text{Ker} \delta)^{\perp}$ is the space $(\mathbb{C}'_{\mathbb{R}})$. Is it contained in the line $\mathbb{C}^0 \cap (\text{Ker} \delta)^{\perp}$?

You should begin with splitting this filtration

$$\text{Ker} \delta \otimes \mathbb{R} \otimes (\mathbb{C}^0)^{\perp}$$

begin again $\mathbb{K} \rightarrow \mathbb{C}^0 \xrightarrow{\delta} \mathbb{C'} \rightarrow H'$

$\mathbb{R} \quad \mathbb{C'} / \mathbb{K}$

What do you mean? You have $\mathbb{C}^0 \xrightarrow{\delta} \mathbb{C'}$, $\delta^* \delta = 1$

Okay, it seems that instead of a 1-step filtration $\mathbb{C}^0 \subset \mathbb{C'}$, you have a 2-step filtration $0 \subset \text{Ker} \delta \subset \mathbb{C}^0 \subset \mathbb{C'}$ and a corresponding chain of quotient spaces of $\mathbb{C}'$, namely

See if you have the answer. Do you do not expect $\mathbb{C}^0$ to change?
You have 2 step filtration of \( C^1 : 0 \leq \text{Ker} \delta \subset C^0 \subset C^1 \) which you split into orthogonal layers. Use the fact that \( \delta \) is isometric: \( \delta^* \delta = 1 \). It should be true that \( \text{Ker} \delta^* = (C^0)^\perp \). Also if \( \gamma \) the quadratic form on \( R \) is the pushforward via \( \delta \) of the quad. form on \( C^0 \) it should be true that \( \gamma \gamma^* = 1 \). You want to compare the above diagram with \( C^0 \xrightarrow{(\delta)} (C^1) \). Ideas: You don't change the mode potential space.

Given \( C^0 \xrightarrow{\delta} C^1 \) together with a pos def form on \( C^1 \), you want the analog of attaching an edge to the network. This is clearly like having \( C^0 \xrightarrow{\delta} (C^1) \). So there's an obvious equivalence between the two diagrams. Note that \( \delta: C^0 \rightarrow R \) can replaced by any 1-diml quotient space \( \delta: C^0 \rightarrow L \) of \( C^0 \). At this stage there is no significance to \( 1 \in R \).

Next consider the quad form \( R^+: C^1 \rightarrow C_0 \) on \( C^1 \). Using this form one gets an splitting of the filtration \( 0 \leq \text{Ker} \delta \subset C^0 \subset C^1 \). There are three layers \( \text{Ker} \delta, C^0/\text{Ker} \delta \rightarrow L, (\delta C^0)^\perp = \text{Ker} \{\delta^*: C^1 \rightarrow C^0\} \).

Look at \( L \).

Go back to the inhomogeneous problem in which an emf is connected between two nodes of a connected R-network.

\[
\begin{align*}
V_A - V_B &= R \text{int} I \\
V_B - V_A &= \varepsilon \\
V - RI &= 0
\end{align*}
\]
Go back to a connected R-network with an emf attached between 2 nodes.

Find the equations determining the state of the network, i.e. the states V, I, for each edge in X, i.e. an elt \((V, I)\) of \((C^1 X)\).

You have 2e variables. The Kirchhoff current constraint is weakened to allow a node current in at A and out at B. You also have the condition \(V_B - V_A = E\) with \(E\) fixed. So the number of equations = the number of unknowns.

Next you should check in detail.

\[ E \in \mathbb{R} \xleftarrow{\delta} C^0 X \xrightarrow{\partial} C^1 X \rightarrow H^1 X \]

| \[ \mathbb{R} \xrightarrow{\gamma^t} C^0 X \leftarrow C^1 X \rightarrow H^1 X \] |

Here \( \gamma^t \) sends \( \varphi \in C^0 X \) to \( \varphi(B) - \varphi(A) \), and \( \gamma^t \) is the 0-current \([B] - [A]\). There's nothing new here it seems.

But let's be careful. You've made a diagram which is the reduced cochain complex of \( X \), together with the external mode information added. What are the appropriate linear (inhomogeneous) equations?

Without the external mode info you have the homogeneous linear system \( V \in C^0 X, I \in H^1 X, V = RI \) which has only the solution \( V = I = 0 \). There is an inhom. version \( V \in C^0(X), I \in H^1 X \), \( V - RI = E_{\text{int}} \) for any \( E_{\text{int}} \in C^1 X \).
But this "internal" system of emfs using edge emfs has 2e equations and 2e unknowns.

Review the situation. X conn R-network, A B two nodes of X, \( \mathbf{C} \): \( \mathbf{G} \rightarrow \mathbf{R} \), \( \mathbf{V} : \mathbf{G} \rightarrow \mathbf{G}(B) - \mathbf{G}(A) \)

\[ \mathbf{G} = \mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}': \quad \mathbf{G} \text{ is a coset for the vector space } \text{Ker} \mathbf{G}. \] You want a stationary point for the power in this coset. This means the variation in directions of \( \text{Ker} \mathbf{G} \) is zero. What does this mean?

Change notation. \( \mathbf{V}' \rightarrow \mathbf{V} \rightarrow \mathbf{V}'' \)

\[ \downarrow \quad \uparrow \quad \downarrow \quad \mathbf{A} = \text{pos def} \]

Let \( \mathbf{V}_0 \in \mathbf{V} \); to minimise \( (\mathbf{V}_0 + \mathbf{v}')^t \mathbf{A} (\mathbf{V}_0 + \mathbf{v}') \) as \( \mathbf{v}' \) ranges over \( \mathbf{V}' \). Variation 1st order is \( (\mathbf{v}')^t \mathbf{A} (\mathbf{v}_0 + \mathbf{v}') = 0 \) i.e. \( \mathbf{v}_0 + \mathbf{v}' \perp \mathbf{V}' \) wrt \( \mathbf{A} \).

The confusion here may be due to the fact that the coset \( \mathbf{C} \) generates a vector space of dimension equal to 1, higher. So when you look for \( \mathbf{V}' \rightarrow \mathbf{V} \rightarrow \mathbf{V}'' \) the \( \mathbf{V}' \) is \( \text{Ker} \mathbf{G} \).

minimise \( \frac{1}{2} \mathbf{x}^t \mathbf{A} \mathbf{x} \) subject to \( \mathbf{c} = \mathbf{y}^t \)

\[ F = \frac{1}{2} \mathbf{x}^t \mathbf{A} \mathbf{x} + \lambda (\mathbf{c} - \mathbf{y}^t \mathbf{x}) \quad \quad \partial_x F = \mathbf{A} \mathbf{x} - \lambda \mathbf{y} = \mathbf{0} \]

Symplectic viewpoint

\[ \partial_y F = \mathbf{c} - \mathbf{y}^t \mathbf{x} = \mathbf{0} \]
Review the Legendre Transform. Consider phase space for the real line with double coordinates \( x, \dot{x} \). Let \( F \) be a (suitable) function of \( x \), e.g., \( F = \frac{1}{2a}x^2 \). Form the function \( \dot{x} - F \) on phase space and perform the following push-forward to the \( \dot{x} \) line. Let \( \hat{F} \) be the function of \( \dot{x} \) which gives the critical value of \( \dot{x} - F \) as a function of \( x \). When \( F = \frac{1}{2a}x^2 \) the critical point of \( \dot{x} - \frac{1}{2a}x^2 \) is \( \dot{x} = \frac{x}{a} \), so \( a \dot{x}^2 \leq \frac{1}{2a} \dot{x}^2 \).

General \( F \) case. \( \hat{F}(x, \dot{x}) = x \dot{x} - F(x) \), the critical points are \( \dot{x} \) such that \( \frac{\partial}{\partial \dot{x}} \dot{x} - F = 0 \), so that \( x \) becomes a function of \( \dot{x} \) via the IFT. Then \( \hat{F} = x \dot{x} - F \) is a function of \( \dot{x} \), well-defined at least locally. Then

\[
\frac{d\hat{F}}{d\dot{x}} = x + \dot{x} \frac{dx}{d\dot{x}} - \frac{dF}{dx} \frac{dx}{d\dot{x}} = x
\]
Go back to consider R-network X with external nodes pair A, B. You want the response to an emf $E_a$ attached to these nodes. This is an inhomogeneous linear system of equations.

Where to start? With the linearization of $X$:

$$\mathcal{C}X \quad \mathcal{C}'X$$

This means:

**Question:** Is there a link between Lagrange multipliers and Legendre transform?

Look at the situation you have, namely a connected R-network with an inhomogeneous linear condition $V_B - V_A = E_a$ in the space $\mathcal{C}X$ of node potentials.

Lagrange multipliers method enables you to handle the constraint with?

Repeat constraint $V_B - V_A = E_a$ means

Set up intelligently. Basic object is a vs. with positive quadratic form. Operations of restriction to a subspace, push forward to a quotient space. At some stages you might have dilatation, e.g. in the CL case.

But now you want inhomogeneous constraints, i.e. fixing value(s) of voltage variable(s).
Let's consider a simple situation.

\[ \mathbb{R}^n \times \mathbb{R}^m \xrightarrow{\varphi} \mathbb{R} \quad \text{linear map} \neq 0, \ K = \ker \varphi. \]

Aim: to link Lagrange multi. & Legendre Transform.

You start with \( X \) equipped with \( \frac{1}{2} x^t A x \) Do L.T. \[ \begin{array}{c}
\forall x - \frac{1}{2} x^t A x \\
\text{critical point } x \text{ corres. to } \lambda \text{ is } \lambda^t = x^t A, \quad \lambda = A x \\
x = A^{-1} \lambda \end{array} \]

\[ \frac{1}{2} (A^{-1} \lambda)^t A (A^{-1} \lambda) = \frac{1}{2} \lambda^t A^{-1} \lambda. \]

But what about Lagrange multipliers?

\[ F = \frac{1}{2} x^t A x + x (c - y^t x) \quad c, y \text{ fixed} \]

\[ \begin{array}{c}
\nabla_x F = A x - \lambda y = 0 \\
\nabla_\lambda F = c - y^t x = 0 \\
x^t A x = \lambda c \\
= \frac{c^2}{y^t A^{-1} y} \end{array} \]

Connected R-network: \( \mathbb{C}^0 \xrightarrow{\mathbb{D}} \mathbb{C}' \) with pos. quad form \( R' \)

Consider an inhomogeneous linear problem, where you constrain \( \varphi \in \mathbb{C}^0 \) to satisfy \( \varphi(B) - \varphi(A) = \varepsilon \).

\[ \begin{array}{c}
\varphi(A) - \varphi(B) = R \cdot \mathbb{I} \\
\varphi(B) - \varphi(A) = \varepsilon. \end{array} \]
Consider a quadratic space \( W \rightarrow W^* \) via a linear functional \( f : W \rightarrow \mathbb{R} \). The kernel of \( f \) is \( \ker(f) \rightarrow W \rightarrow \mathbb{R} \). Interesting is the critical value of \( f \) on the hyperplane \( f^{-1}(x) \) \( x \in \mathbb{R} \), which should be a quadratic function of \( x \).

Recall: \( x \mapsto W \rightarrow Y \) with \( g > 0 \) in \( W \).

The split sequence gets given by

\[
\begin{pmatrix}
  x \\
  y
\end{pmatrix}
= \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  x \\
  y
\end{pmatrix} = 0 \quad \forall x \in \mathbb{R}^2
\]

\[
X^\perp = \left\{ \begin{pmatrix}
  x \\
  y
\end{pmatrix} \mid (8x)^t (a \ b) (x) = 0 \right\} = \begin{pmatrix}
  -a^{\#} \\
  b
\end{pmatrix}
\]

\[
\begin{pmatrix}
  -b^{\#} & a^{-1} \\
  c & d
\end{pmatrix}
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
= \begin{pmatrix}
  -b^{\#} a^{-1} & 1 \\
  -ca^{-1} b & d
\end{pmatrix}
\]

\[
x^t a x + y^t (d - ca^{-1} b) y
\]
\[
\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \\ \end{pmatrix} \begin{pmatrix} a & b \\ c & d \\ \end{pmatrix} \begin{pmatrix} 1 \\ -a^{-1}b \\ \end{pmatrix} \\
= \begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \\ \end{pmatrix} \begin{pmatrix} a & 0 \\ c & -ca^{-1}b+d \\ \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -ca^{-1}b+d \\ \end{pmatrix}
\]

Let's see if we can apply this. Go back to

\[
K = \mathbb{K}/ \mathbb{C}^0 \rightarrow \mathbb{C}^0 \rightarrow \mathbb{C}^1
\]

\[
\begin{array}{c}
\downarrow \\
\mathbb{R}
\end{array}
\]

You want to split \( \mathbb{C}^0 \) into \( X \oplus Y \)

You are probably being stupid because the norm on these spaces \( K, \mathbb{C}^0, \mathbb{R} \) come from the scalar product on \( \mathbb{C}^1 \). Think Euclidean!

Let's review problems & ideas. Maybe another example, say a tree e.g.

\[
\begin{array}{c}
\downarrow \\
\uparrow
\end{array}
\]
Consider \( \mathbb{C}^X \rightarrow c^1X \) and \( \delta \psi = \begin{pmatrix} (\psi(A) - \psi(B)) \\ \psi(B) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \)

\[
\frac{1}{2R_1} (\psi_A - \psi_B)^2 + \frac{1}{2R_2} \psi_B^2 \quad \text{power form on } \mathbb{C}^X.
\]

Next you want \( \gamma : \mathbb{C}^X \rightarrow \mathbb{R} \)

\( \psi \mapsto \psi_A \)

\[ K = \ker \gamma = \{ \psi | \psi_A = 0 \} \]

\[ K \rightarrow \mathbb{C}^X \quad \psi \mapsto \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \]

\[ \begin{bmatrix} \psi_A \\ \psi_B \end{bmatrix} \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} \psi_A \\ \psi_B \end{bmatrix} \]

Power (\( \psi \)) = \frac{1}{2R_1} (\psi_A - \psi_B)^2 + \frac{1}{2R_2} \psi_B^2

Power (\( \psi \)) restricted to \( \ker \gamma \)

\[ \frac{1}{2R_1} \psi_B^2 + \frac{1}{2R_2} \psi_B^2 = \frac{1}{2} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \psi_B^2 \]

\[ \text{orth comp of } \begin{pmatrix} 0 \\ \psi_B \end{pmatrix} \text{ is: } \begin{pmatrix} -\frac{1}{R_1} \psi_A + \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \psi_B = 0 \end{pmatrix} \]

\[ \frac{\psi_B - \psi_A}{R_1} + \frac{\psi_B}{R_2} = 0 \]

\[ \frac{\psi_A - \psi_B}{R_1} = \frac{\psi_B}{R_2} \]
\[ \begin{bmatrix} \Phi_A \\ \Phi_B \end{bmatrix} \begin{bmatrix} \frac{\Phi_A - \Phi_B}{R_1} \\ 0 \end{bmatrix} = \frac{\Phi_A (\Phi_A - \Phi_B)}{R_1} = \frac{\Phi_A \Phi_B}{R_2} \]

Repeat the calculation.

\[ \mathcal{C} \ni (\Phi_A, \Phi_B) \rightarrow (\Phi_A - \Phi_B) \in \mathcal{C}^1 \]

\[ \| (\Phi_A - \Phi_B) \|_2^2 = \frac{1}{R_1} (\Phi_A - \Phi_B)^2 + \frac{1}{R_2} \Phi_B^2 \]

\[ = \begin{bmatrix} \Phi_A \\ \Phi_B \end{bmatrix} \begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ \frac{1}{R_1} & \frac{1}{R_1 + \frac{1}{R_2}} \end{bmatrix} \begin{bmatrix} \Phi_A \\ \Phi_B \end{bmatrix} \]

norm squared on \( \mathcal{C}^1 \).

The next point is the map \( R : \mathcal{C}^1 \rightarrow \mathbb{R} \)

\( \Phi_i \rightarrow \Phi_A \)

\( \text{Ker} R = \{ \Phi \in \mathcal{C}^1 | \Phi_A = 0 \} = \{ (0) \} \in \mathcal{C}^1 \}

\( (\text{Ker} R)^\perp = \{ (\Phi_B) \in \mathcal{C}^1 \} | -\frac{\Phi_A}{R_1} + (\frac{1}{R_1 + \frac{1}{R_2}}) \Phi_B = 0 \} \)

\[ \frac{\Phi_A - \Phi_B}{R_1} = \frac{\Phi_B}{R_2} \]

\[ \frac{\Phi_A}{R_1} = (\frac{1}{R_1 + \frac{1}{R_2}}) \Phi_B \]

\[ \Phi_A = R_1 \left( \frac{1}{R_1 + \frac{1}{R_2}} \right) \Phi_B \]

norm squared on \( (\text{Ker} R)^\perp \)

\[ \frac{\Phi_A (\Phi_A - \Phi_B)}{R_1} = \frac{\Phi_A \Phi_B}{R_2}. \]

Positive?

\[ \frac{\Phi_A \Phi_B}{R_2} = \frac{\Phi_B}{R_2} R_1 \left( \frac{1}{R_1 + \frac{1}{R_2}} \right) \Phi_B > 0 \]
\[ \frac{\varphi_A - \varphi_B}{R_1} = \frac{\varphi_B}{R_2} \]

\[ \frac{\varphi_A}{R_1} = \frac{\varphi_B}{R_1} + \frac{\varphi_B}{R_2} = \frac{R_2 + R_1}{R_1 R_2} \frac{\varphi_B}{R_2} \]

\[ \varphi_A = \frac{R_1 + R_2}{R_2} \varphi_B \]

\[ \frac{\varphi_A \varphi_B}{R_2} = \frac{R_1 + R_2}{R_2^2} \varphi_B^2 \geq 0 \]

OK, this seems to work, but the situation is still opaque. What's a way to increase understanding?

Idea: Finding the critical point, i.e., the orthogonal complement to \( \ker \varphi \) in \( C^0 \) somehow introduces the current condition \( \frac{\varphi_A - \varphi_B}{R_1} = \frac{\varphi_B}{R_2} \), i.e., \( I_1 = I_2 \)

This is the Kirchhoff current condition at the node B.

Assume you have a conn. R-network equipped with a ground \( 0 \) and a node \( A = 0 \). Do you attach an emf \( \varepsilon \) from \( 0 \) to \( A \). Your problem is to calculate the state of the network with the attached emf. Difficulties:

You feel that it's enough to work with voltages, i.e., \( \varphi \in C^0 \) and the positive definite forms induced via \( \varphi : C^0 \to C' \). But currents pop up naturally.
How should you handle the inhomogeneous condition \( \Phi_A = \Phi_0 + \Phi_a \)?

Start again: conn. R-network w 2 nodes A, O with attached Emf \( E_a \). To calculate the state of the network. Inhomogeneous condition \( \Phi_A - \Phi_0 = E_a \).

You have \( V-1 \) variables \( \Phi_N \), \( N \) mode \( \neq 0 \).

Do you have a positive sign? form on \( E_0 \) which has dim \( V-1 \).

Old problem - augmented graph

Yesterday's example

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A R1 B R2 C
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Program: Category of quadratic spaces

**Objects** are vector spaces over \( \mathbb{R} \) equipped with positive quad form. Q-category arising from induced quad form on subquotients.

IDEA: L-version involving complexes, which perhaps generalizes what you are doing with cochains in a graph.

Your program should be to understand why the dual framework of chains on the graph arises naturally in the calculations. This is physics philosophy, 

For R networks there is only statics and no dynamics, but CL networks have dynamics via Cayley Transform.
Start with a quadratic space and review the pushing the quadratic form to a quotient space. Take yesterday's example:

\[ \mathbb{C}^0 \ni (\varphi_A, \varphi_B) \mapsto ((\varphi_A - \varphi_B) \varphi_B) \in \mathbb{C}^1 \]

The power form on \( \mathbb{C}^1 \) is:

\[
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \mapsto \frac{V_1^2}{R_1} + \frac{V_2^2}{R_2} \quad \text{better}
\]

The transition matrix is:

\[
\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{R_1} & 0 \\ 0 & \frac{1}{R_2} \end{pmatrix} \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}, \text{ restrict to }
\begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix}
\]

\[
\begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_2} \end{pmatrix} \begin{pmatrix} \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{R_1} & \frac{1}{R_2} \\ \frac{1}{R_1} & -\frac{1}{R_1} \end{pmatrix}
\]

Review the situation: \( \mathbb{C}^0 \) or power form and \( \mathbb{C}^0 \to \mathbb{R} \)

\[ \gamma : \varphi \mapsto \varphi_A \quad \text{K is P \to } \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \varphi_B \]

K's restriction of P to K \( \subset \mathbb{C}^0 \)

\[ K^{-1} = \left\{ \begin{pmatrix} \varphi_A \\ \varphi_B \end{pmatrix} \middle| \frac{\varphi_A}{R_1} = \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \varphi_B \right\} \]

\[ \varphi_A = R_1 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \varphi_B \]

\[ \varphi_A \left( \frac{1}{R_1} \varphi_A - \frac{1}{R_1} \varphi_B \right) = \varphi_A \left( \varphi_A - \varphi_B \right) \]

\[ \frac{\varphi_A^2}{R_1^2} \left( R_1 - \frac{1}{\frac{R_1}{R_1 + \frac{1}{R_2}}} \right) = \frac{\varphi_A^2}{R_1^2} \left( \frac{R_2}{R_1 + \frac{1}{R_2}} + \frac{R_2}{R_1 + \frac{1}{R_2}} \right) = \frac{\varphi_A^2}{R_1^2} \left( \frac{R_2}{R_1 + \frac{1}{R_2}} \right) \]
This calculation is awkward, and it might tell you how to do things simpler by passing to phase space.

To begin with:

\[ V' \rightarrow V \rightarrow V'' \]

Let's begin with a Lagrange multipliers example. Take a vector space with quadratic form \( \frac{1}{2} x^t A x \) \( A : X \rightarrow X^* \) and a nonzero linear functional \( y^t x \) \( y \in X^* \).

You want the push-forward quadratic form \( \Gamma \) for a Lagrange transform case, namely let \( c \in \mathbb{R} \)

\[ F = \frac{1}{2} x^t A x + \lambda(c - y^t x) \]

\[ \nabla_x F = A x - A y = 0 \quad \frac{\partial F}{\partial \lambda} = c - y^t x = 0 \]

(Notice that one has two new variables \( y, c \) here which is a puzzle.)

Continue with Lagrange method, which should mean to eliminate the variables \( x, \lambda \). (Note: \( F \) has variables \( x, \lambda \) hence \( n+1 \) real variables \( n = \dim X \).) \( c, y \) are constants.

\[ \nabla_x F = 0 \quad \nabla_{\lambda} F = 0 \] are \( n+1 \) eqns.

Use \( A x = \lambda y \) to get \( x = \lambda A^{-1} y \) and

\( c = y^t (\lambda A^{-1} y) = \lambda y^t A^{-1} y \). Thus \( x \) has been eliminated and also \( \lambda \); \( \lambda = \frac{c}{y^t A^{-1} y} \) so we get the point

\[ x = \frac{c}{y^t A^{-1} y} A^{-1} y \]

and critical value

\[ \frac{1}{2} x^t A x = \frac{1}{2} x^t \frac{c}{y^t A^{-1} y} A^{-1} y = \frac{1}{2} \frac{c^2}{y^t A^{-1} y} \]
So you've just done Lagrange multiplier method, but not Legendre transform, which should proceed as follows: Consider for each \( y \in \mathbb{R}^n \)

\[ y^t x - \frac{1}{2} x^t A x \]

and find its critical point and critical value.

Critical point:

\[ y^t x - x^t A = 0 \quad \text{or} \quad A x = y, \quad x = A^{-1} y \]

Critical value:

\[ y^t A^{-1} y - \frac{1}{2} (A^{-1} y)^t A (A^{-1} y) = \frac{1}{2} y^t A^{-1} y \]

So the Legendre transform \( \frac{1}{2} x^t A x \) is \( \frac{1}{2} y^t A^{-1} y \).

Is there any relation to the push forward of \( A \) via \( y \)?

Repeat. \( X \) is equipped with positive definite \( x^t A x \). Let \( y \in \mathbb{R}^n \) \( y \neq 0 \), so that \( f: x \mapsto y^t x, \quad X \rightarrow \mathbb{R} \)

is onto. One has push forward \( f_*(A) \) defined by restricting \( A \) to \( (\ker y)^\perp \), and then using \( (\ker y)^\perp \rightarrow \mathbb{R} \).

\( X \) becomes Euclidean space, \( \exists ! \) \( x_0 \) such that \( x_0^t A x = y^t x \quad \forall x \)

Maybe \( X = \mathbb{R}^n \) column vectors, then scalar product is \( \langle y, x \rangle = y^t x \). Given \( A = A^t > 0 \) and \( y \in \mathbb{R}^n, \ y \neq 0 \), get \( \langle y, \cdot \rangle: \mathbb{R} \rightarrow \mathbb{R} \) onto.

\[ K = \{ x \in \mathbb{R}^n \mid y^t x = 0 \} \]

Want to minimize \( \frac{1}{2} x^t A x \) on \( \{ x \mid c = y^t x \} \).
$X = \mathbb{R}^n$ column vectors equipped with usual scalar product $(x, y) = x^t y = \sum_i x_i y_i$ and norm $\|x\| = (x, x)^{1/2}$. Consider a nonzero linear functional $\xi$ on $X$, i.e. $\xi: X \to \mathbb{R}$ is linear and onto.

Set hyperplane $K = \{x \mid \xi(x) = 0\} = \xi^\perp$ when you identify $\xi \in X^*$ with the vector $\xi \in X$ such that $\xi(x) = (\xi, x) = \xi^t x$.

You have an orthogonal splitting

$$X = K \oplus \mathbb{R} \xi$$

What is the scalar product on $\mathbb{R} \xi$? Ans.

$$(c \xi, c' \xi) = cc' \|\xi\|^2.$$ 

Next let's do the same calculation with the scalar product $(x, y)_A = x^t A y$ where $A = A^t > 0$.

Let $\xi: X \to \mathbb{R}$, $\xi(x) = \xi^t x$ be a nonzero linear functional on $X$. Write $\xi^t x = \xi^t A^t A \xi = (A^{-1} \xi, x)_A$ i.e. you represent the linear form $\xi$ by the $A$-scalar product with $A^{-1} \xi$. One has an $A$-orthogonal splitting

$$X = K \oplus \mathbb{R} (A^{-1} \xi)$$

$K = \{x \mid x^t A A^{-1} \xi = 0\}$

Ker $\xi$

Now restrict $(\cdot, \cdot)_A$ to $\mathbb{R} (A^{-1} \xi)$.

$$\langle c A^{-1} \xi, c' A^{-1} \xi \rangle_A = (c A^{-1} \xi, A c' A^{-1} \xi) = cc' (\xi, A^{-1} \xi)$$
4.3 Repeat what you did. First take the setting $X = \mathbb{R}^n$ with scalar product $(x, y) = x^t y$. Identify a linear dual $\tilde{x}: X \to \mathbb{R}$ with the vector $\tilde{x}$ such that $\tilde{x}(x) = (\tilde{x}, x) = \tilde{x}^t x$, and drop the $n$. Thus you have $X \sim X^*$ sending $y$ to $y^t = (x \mapsto y^t x)$.

Better: If $y \in X$, then $x \mapsto y^t x$ is a lin ful in $X$, and one gets via $X \to X^*$, $y \mapsto y^t$.

Now let $X \to \mathbb{R}$, $x \mapsto y^t x$ be a nonzero linear ful. One has ortho splitting

$$X = K \oplus R_y$$

where $K = \ker y^t = y^\perp$

The push-forward scalar product on $\mathbb{R}y$ is the restriction of $(x, x')$ to the orth comp of $K$, i.e. $\mathbb{R}y$. $	herefore$

$$(c y, c' y) = c c' \|y\|^2$$

Now consider $X = \mathbb{R}^n$ with scalar product $(x, y)_A = (x, A x')$ where $A$ pos. def. Let $y^t \in \mathbb{R}^n$ be a lin ful. Then $y^t x = (y, x) = (y, A^t A x) = (A^{-1} y, x)_A$. So $y^t$ is represented for the $A$-scalar prod by $A^{-1} y$. Next get $A$-orth splitting

$$X = K \oplus \mathbb{R} A^{-1} y$$

where $K = \ker y^t$

The push forward scalar product on $\mathbb{R} A^{-1} y$ is $(c A^{-1} y, c' A^{-1} y)_A = c c' (A^{-1} y, A A^{-1} y) = c c' (y, A^{-1} y)$.
Now look at Legendre F.

\[ L = y^T x - \frac{1}{2} x^T A x \]

Let \( y \) be fixed. Then \( L \) has a unique critical point when \( y^T x - x^T A = 0 \), i.e., \( A x = y \iff x = A^{-1} y \)

and the critical value is

\[ L = y^T A^{-1} y - \frac{1}{2} (A^{-1} y)^T A A^{-1} y = \frac{1}{2} y^T A^{-1} y \]

Let's now understand why \( F = \frac{1}{2} x^T A x + \lambda (c - y^T x) \)
yields something different. What you should have done earlier is to restrict \( \frac{1}{2} x^T A x \) to the hyperplane \( c = y^T x \), then found the critical point.

\( n = 1 \). \( c = y^T x \), \( x = \frac{c}{y^T} \), \( F \) yields \( F = \frac{1}{2} \frac{c^2 A}{y^2} = \frac{1}{2} \frac{c^2}{y^T A^{-1} y} \)

\( \nabla F = A x - \lambda y \equiv 0 \), \( \nabla F = c - y^T x = 0 \)

\[ x = \frac{\lambda y}{A} = \frac{c}{y^T} \quad F = \frac{1}{2} \left( \frac{c}{y^T} \right)^2 A \]

Review the calculation

\( \nabla F = A x - \lambda y = 0 \quad \nabla F = c - y^T x = 0 \)

\[ x = A^{-1} y \quad y^T x = A y^T A^{-1} y = c \quad \lambda = \frac{c}{y^T A^{-1} y} \]

\[ x = \frac{c A^{-1} y}{y^T A^{-1} y} \quad F = \frac{1}{2} \left( \frac{y^T A^{-1} c A (c A^{-1} y)}{(y^T A^{-1} y)^2} \right) = \frac{1}{2} \frac{c^2}{y^T A^{-1} y} \]