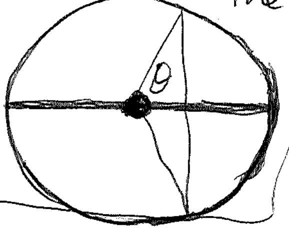
α'' What do you want to know about T?

$T: V_+ \longrightarrow V_-$. The obvious symmetry group acting on the $\left\{ W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \right\}$ is $\begin{pmatrix} U(V_+) & 0 \\ 0 & U(V_-) \end{pmatrix}$. W same as a g $\ni$ $\varepsilon g \varepsilon^{-1} = g^{-1}$. There's a spectrum, the main part being $\ni$ $\cos\theta \in (-1,1)$ and



4 degenerate cases. $F = \pm 1$, $\varepsilon = \pm 1$.

How are you going to replace, improve upon the resolvent $\begin{pmatrix} s & T^* \\ -T & s \end{pmatrix}^{-1} = \left(s - \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \right)^{-1} = \frac{1}{s - X}$, $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

Problem: Given orth repn $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ of F, ε on Euclidean space V, what is the mathematical object you need to describe the associated abstract LC network. Look at a concrete LC network, it has a state space and ~~time~~ evolution.

State space is

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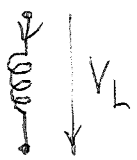
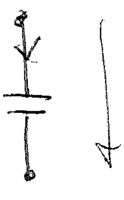
need review of degenerate cases.



I_L V_L $\dot{V}_L = \dot{I}_L$ $\hat{V}_L = s\hat{I}_L - I_L(0)$

$$\begin{array}{ccc} \bar{C}^0 & \xrightarrow{\sim} & C^1_L \\ & & \uparrow s \\ \bar{C}_0 & \xleftarrow{\sim} & C_{1,L} \end{array} \qquad \begin{pmatrix} V_L \\ I_L \end{pmatrix} \qquad \begin{pmatrix} s \\ 1 \end{pmatrix} I_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

intersect $\bar{C}^0 = C^1_L$ with $\begin{pmatrix} s \\ 1 \end{pmatrix} I_L$ can happen when $s = \infty$



I_C V_C

$$\begin{array}{c} \bar{C}^0 = C^1_C \\ \downarrow s \\ \bar{C}_0 = C_{1,C} \end{array}$$

intersect $\bar{C}^0 = C^1_C$ $\begin{pmatrix} 1 \\ s \end{pmatrix} V_C = \begin{pmatrix} * \\ 0 \end{pmatrix}$ $s = 0$

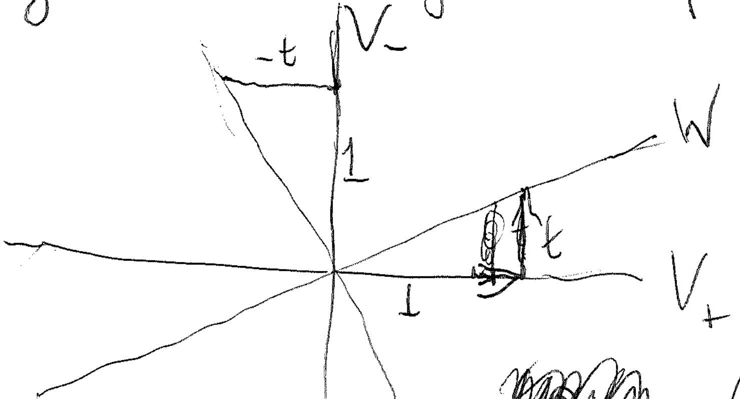
B''

You need the direct sum of simple situations.

Let's look at phases in the s.h.o. cases.

$$W \overset{\alpha}{\hookrightarrow} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad 1 = \alpha^*\alpha = (\alpha_+^* \; \alpha_-^*)\begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix} = \alpha_+^*\alpha_+ + \alpha_-^*\alpha_- = h_+ + h_-$$

What is your aim? $V_\pm = R$ with usual orthogonal structure $(x, x) = x^2$. $\dim_R(W) = 1$. You have $g = F\varepsilon$ an orthogonal transf.



$$F\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Q: Is there a phase ambiguity in this 2dimal case? Given W can view it as the graph of multipl by $t : V_+ \to V_-$. Then you get

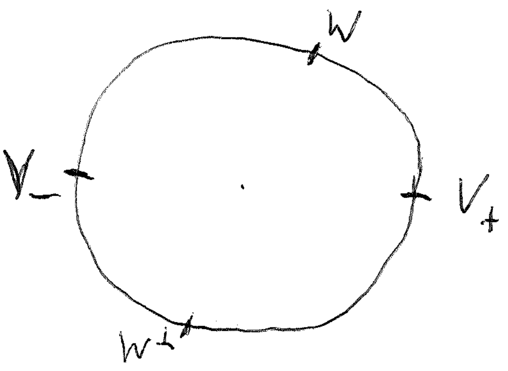
$$F\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}\frac{1}{\sqrt{1+t^2}} = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}\frac{1}{\sqrt{1+t^2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}\varepsilon$$

so $g = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}$ $\qquad g^{1/2} = \pm\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$



γ'' back to the phase business. first recall the details, then write it down. Begin with the complex case: a unitary repn of F, ε on V, s.t. $\frac{g+g^{-1}}{2}$ = scalar operator $\cos\theta$ where $0<\theta<\pi$. Then $V = V_\theta \oplus V_{-\theta}$ where $g = e^{\pm i\theta}$ on $V_{\pm\theta}$. $\varepsilon g \varepsilon^{-1} = g^{-1} \Rightarrow \varepsilon : V_\theta \rightleftarrows V_{-\theta}$

~~Hm.~~ $\varepsilon : V_\theta \xrightarrow{\sim} V_{-\theta}$ is a canon. isom. with inverse ε.

At this point you want to identify the repn. V, F, ε with ~~[illegible]~~ a Hilbert $W \otimes$ irred rep. V_λ.

~~whats the~~ The clearest way to proceed is to choose an orthonormal basis v_i for V_θ, then $v_i, \varepsilon v_i$ $1\le i\le n$ give an orth basis for V. Say $n=1$, so one has a unit vector $v \in V_\theta$ and one $\varepsilon v \in V_{-\theta}$. Relative to this basis

$$\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} \zeta & 0 \\ 0 & \bar\zeta \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} \zeta & \bar\zeta \\ -\zeta & \bar\zeta \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\frac{1}{2} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} \zeta & 0 \\ 0 & \bar\zeta \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\frac{1}{2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

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maybe best is to exhibit I_λ "the" irred. rep. and then ~~construct~~ find $\mathrm{Hom}_{F,\varepsilon}(I_\lambda, V)$

5''

Next – how does T arise? You consider $W \subset \binom{V_+}{V_-}$, better, a unitary reps of F, ε on V such that $\frac{g+g^{-1}}{2}$ has spectrum $\in (-1,1)$. In fact you want " " " $(-1,0]$ for $W = \binom{1}{T} V_+$.

Then $$F\begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

Aim? You want a clean clear picture of an abstract LC network. Can you give a definition? You can define an LC network (assumed connected) to be a connected graph whose edges are labelled with either an $L>0$ or $C>0$.

Associated to an LC network

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook, "\binom{1}{T}"] & C^1_C \oplus C^1_L \arrow[r, two heads, "(-T\ 1)"] \arrow[d, shift right, "Cs"'] & H^1 \\
\bar{C}_0 & C_{1,C} \oplus C_{1,L} \arrow[l, two heads, "(1\ T^*)"'] \arrow[u, shift right, "\frac{1}{Ls}"'] & H_1 \arrow[l, hook', "\binom{-T^*}{1}"']
\end{tikzcd}$$

$-TV_C + V_L = 0$

$I_C + T^* I_L = 0$

are short exact sequences of cochains and chains, which are naturally in duality. How to organize this? First you use $s=1$ to make a pos. def quad form on C^1_C, or to naturally identify $C^1_C = C_{1,C}$. Sim for L

$$\dot{V}_C = I_C \rightsquigarrow \hat{I}_C = \hat{\dot{V}}_C = -V_C(0) + s\hat{V}_C$$

$$\dot{I}_L = V_L \rightsquigarrow \hat{V}_L = \hat{\dot{I}}_L = -I_L(0) + s\hat{I}_L$$

Can you somehow make this meaningful? concrete?

$$s\hat{V}_C + T^*\hat{I}_L = V_C(0)$$
$$-T\hat{V}_C + s\hat{I}_L = I_L(0)$$

Important $s \in \mathbb{C} \cup \infty$ it char vals

Idea: normal form (T replaced by γ)

Ultimate picture should involve an eigenspace decomp of $V = \binom{V_+}{V_-}$

$\mathcal{E}''$

The [illegible] ultimate picture should involve an eigenspace decomposition of the orthogonal representation of $F, \mathcal{E}$ on V, i.e. an orthogonal decomposition of the given repn of $F, \mathcal{E}$ into irreducible representations. ~~[illegible]~~

Essentially this amounts to the eigenspace decomp of the ~~[illegible]~~ symmetric operator $\frac{1}{2}(g+g^{-1})$. What does this look like? The possible spectrum is $[-1, 1]$.

You need to relate the eigenvalue λ for $\frac{1}{2}(g+g^{-1})$ the ~~[illegible]~~ partition of 1 operators: $1 = h_+ + h_- = \alpha_+^* \alpha_+ + \alpha_-^* \alpha_-$ and the frequency variable ω such that $h_+ = \frac{1}{1+\omega^2}$ $h_- = \frac{\omega^2}{1+\omega^2}$.

$$\mathbb{R} \xrightarrow{\binom{1}{\omega}(1+\omega^2)^{-1/2}} \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \xrightarrow[(1+\omega^2)^{-1/2}]{(-\omega \ \ +1)} \mathbb{R}$$

$$\alpha_+ = \frac{1}{\sqrt{1+\omega^2}}, \quad \alpha_- = \frac{\omega}{\sqrt{1+\omega^2}}$$

$$\therefore \quad h_+ = \frac{1}{1+\omega^2} \qquad h_- = \frac{\omega^2}{1+\omega^2}$$

$$F\begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F\begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix}$$

$$X = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$$

$$F\mathcal{E}\begin{pmatrix} 1 & \omega \\ -\omega & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\omega \\ \omega & -1 \end{pmatrix}$$

$$F\mathcal{E}(1-X) = 1+X$$

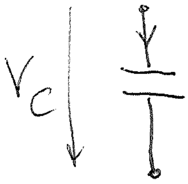
$$\therefore \quad F\mathcal{E} = \frac{1+X}{1-X}$$

5''

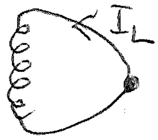
Discuss philosophy. The basic problem seems to be to pass from a point of the Grassmannian i.e. $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$, equivalently an orthogonal (fin dim'l) repn of the infinite dihedral group F, ε, to a linear dynamical system.

Repeat: The problem is to associate to a point $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ of a suitable Grassmannian a first order linear dynamical system of dimension $= \dim(V) =$ number of edges.

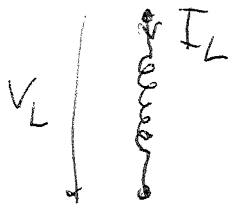
Go over the degenerate cases



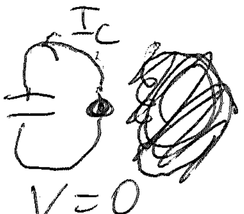
$$\hat{I}_C = \hat{\dot{V}}_C = s\hat{V}_C - V_C(0) \qquad \hat{V}_C = \frac{V_C(0)}{s}$$
$$\overset{\shortparallel}{0}$$



$$0 = \hat{V}_L = \hat{\dot{I}}_L = s\hat{I}_L - I_L(0) \qquad \hat{I}_L = \frac{I_L(0)}{s}$$



$$V_L = \dot{I}_L \quad \therefore \quad V_L = I_L = 0$$
$$I_L = 0$$



$$\dot{V}_C = I_C \qquad V_C = I_C = 0.$$

$$\bar{C}^0 = \begin{pmatrix} C^1_c \\ 0 \end{pmatrix} \Big)\, S \qquad \bar{C}_0 = \begin{pmatrix} C_{1,c} \\ 0 \end{pmatrix}$$

idea for tomorrow use inner product to identify duals

$$W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow W^\perp = H^1$$
$$+$$
$$W \twoheadleftarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \hookleftarrow W^\perp = H_1$$

7'' You want to understand clearly how to start from an orthog repn $\hat{}$ V, F, ε. then double it to get something symplectic, then reduce using constraints to end up with a linear transf on V. Question:

Is there some sort of symplectic quotient in this construction? Generically V_+, V_- have the same dimension (canon iso?)

Is there a recipe to go from F, ε on V to a skew-symmetric linear transformation on V? What about the C.T.? Assume $\pm 1 \notin$ spectrum of $g = F\varepsilon$. Then W should be the graph $\begin{pmatrix} 1 \\ T \end{pmatrix} V_+$ of an invertible operator $T : V_+ \xrightarrow{\sim} V_-$. One has

$$F\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \varepsilon \qquad \begin{matrix} \varepsilon(1-X) \\ '' \\ F(1+X) = (1+X)\varepsilon \end{matrix}$$

$$\Rightarrow F\varepsilon = \frac{1+X}{1-X}.$$

So the recipe should be just the inverse C.T. of $g = F\varepsilon$

$$X = \frac{g-1}{g+1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} g$$

$$g = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X = \frac{X+1}{-X+1} = \frac{1+X}{1-X}.$$

~~So what~~

$$\begin{tikzcd}
W \arrow[r, "\begin{pmatrix} 1 \\ T \end{pmatrix}"] & \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \arrow[r, "(-T\ 1)"] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}"] & W^\perp \\
W & \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \arrow[l, "(1\ T^*)"'] & W^\perp \arrow[l, "\begin{pmatrix} -T^* \\ 1 \end{pmatrix}"']
\end{tikzcd}$$

$$-T\hat{V}_C + \hat{V}_L = 0$$

$$\hat{I}_C + T^*\hat{I}_L = 0$$

$$\hat{I}_C = \hat{\dot{V}}_C = s\hat{V}_C - V_C(0)$$

$$\hat{V}_L = \hat{\dot{I}}_L = s\hat{I}_L - I_L(0)$$

$$s\hat{V}_C + T^*\hat{I}_L = V_C(0)$$

$$-T\hat{V}_C + s\hat{I}_L = I_L(0)$$

$$\left\{ s - \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix} \right\} \begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \begin{pmatrix} V_C(0) \\ I_L(0) \end{pmatrix}$$

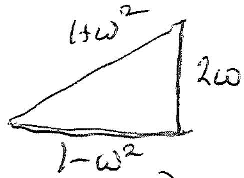
Θ'' Start with a fin. dim Euclidean space V equipped with two reflections F, ε. What's important is the decomposition of V into eigenspaces for the operator $\frac{g+g^{-1}}{2}$, the eigenvalues λ occurring are $\lambda \in [-1,1]$, ~~but~~ except you want to use the parametrization by the variable $\omega \in [0,\infty]$. Relation between ω, λ?

$$X = \begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}$$

$$g = \frac{1+X}{1-X} \qquad g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}} = \begin{pmatrix}1 & -\omega\\ \omega & 1\end{pmatrix}\frac{1}{\sqrt{1+\omega^2}} \quad ?$$

$$\frac{1}{s-X} = \frac{s+X}{s^2-X^2} = \begin{pmatrix}s & -\omega\\ \omega & s\end{pmatrix}\frac{1}{s^2+\omega^2}.$$

$$F(1+X) = (1+X)\varepsilon \qquad F\varepsilon = \frac{1+X}{1-X}$$



$$\frac{1+X}{1-X} = \frac{(1+X)^2}{1-X^2} = \begin{pmatrix}1-\omega^2 & -2\omega\\ 2\omega & 1-\omega^2\end{pmatrix}\frac{1}{1+\omega^2} = \begin{pmatrix}\cos 2\theta & -\sin 2\theta\\ \sin 2\theta & \cos 2\theta\end{pmatrix}$$

where $\tan\theta = \omega$

$$g^{1/2} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

Try again to get this straight $X = \begin{pmatrix}0 & -\omega\\ \omega & 0\end{pmatrix}$ $\omega \in \mathbb{R}$

$$g^{1/2} = \frac{1+X}{\sqrt{1+\omega^2}} =$$

Start again with F, ε two reflections on V. Then $g = F\varepsilon$ is a rotation. Can you assign an angle to g? To say $g = F\varepsilon$ is a rotation is probably meaningless. You have $SO(V) \subset O(V)$, where $SO(V)$ is connected - this is probably what you mean by a rotation. But then $\det(g) = \det(F)\det(\varepsilon)$ obviously depends on the number of signs (dims of ± 1 eigenspaces).

Look at $h = \frac{1}{2}(g+g^{-1})$ which is a symmetric operator on V commuting with F, ε whose spectrum $\subset [-1,1]$. So there is an angle $\theta \in [0,\pi]$ such that $\cos\theta$ is the corresponding eigenvalue of h.

L''   Given F, ε on V you decompose V into the eigenspaces V_λ, $\lambda \in \mathrm{Spec}\{h = \frac{1}{2}(g+g^{-1})\}$. Here $-1 \leq \lambda \leq 1$. This is the basic decomposition to understand.

Discuss philosophy: You start with F, ε on V and pass to the Laplace transform of 1st order linear constant coefficient D.E. on V, i.e. e^{tX} where X is a skew symmetric linear operator on V. Except there's degeneracy where $X = \infty$ which needs to be understood.

Instead of $\lambda \in \mathrm{Spec}\{h\} \subset [-1, 1]$ the natural parameter, eigenvalue type parameter is the frequency ω. Actually the frequencies associated to X skew symmetric are of the form $\pm i\omega$. Instead of λ going from 1 to -1 you want ω, or maybe ω^2 to go from 0 to $+\infty$ (or maybe s^2 should run from 0 to $-\infty$).

Let's get the relations straight using the idea that X should be the inverse Cayley transform

$$X = \frac{g-1}{g+1}$$

when $g+1$ is invertible (i.e. $g \neq -1$). Thus you are ignoring the frequency $\omega = \infty$. You know that

$$F\varepsilon = g = \frac{1+X}{1-X}, \quad \text{where } X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

where $T : V_+ \longrightarrow V_-$. To simplify suppose $V_\pm \simeq \mathbb{R}$.

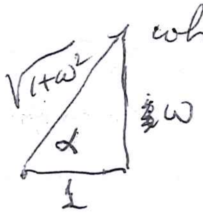
What is the aim? You want to link the angles $\pm\theta$ associated to g to the frequencies $\pm\omega$ associated to the skew symmetric operator X.

Take $X = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. Then $g^{1/2} = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}}$

K''

$$X = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \qquad g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}} = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix} \frac{1}{(1+\omega^2)^{1/2}}$$

so $g^{1/2} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ where $\cos\alpha = \frac{1}{\sqrt{1+\omega^2}}$ $\quad \sin\alpha = \frac{\omega}{\sqrt{1+\omega^2}}$



or $e^{i\alpha} = \frac{1+i\omega}{\sqrt{1+\omega^2}}$

$$g = \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} = \begin{pmatrix} \frac{1-\omega^2}{1+\omega^2} & -\frac{2\omega}{1+\omega^2} \\ \frac{2\omega}{1+\omega^2} & \frac{1-\omega^2}{1+\omega^2} \end{pmatrix}$$

Notice that as ω goes from $-\infty$ to $+\infty$ then $e^{i\alpha}$ goes from $-i$ to $+i$ so $e^{2i\alpha}$ goes from -1 to -1 counterclockwise.

So things are not as nice as you would like.

$$\frac{g+g^{-1}}{2} = \frac{1}{2}\left(\frac{1+X}{1-X} + \frac{1-X}{1+X}\right) = \frac{1+X^2}{1-X^2} = \begin{pmatrix} 1-\omega^2 & \\ & 1-\omega^2 \end{pmatrix} \frac{1}{1+\omega^2}$$

$= \cos(2\alpha)$. It seems that the angles $\pm\theta$ associated to g, equivalently the eigenvalues $e^{\pm i\theta}$ of g, are "double" the angles $\pm\alpha$ belonging to $g^{1/2}$.

Notice: $g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$ looks like the polar decomp of $1+X$. Also maybe $g_s^{1/2} = \frac{s+X}{(s^2-X^2)^{1/2}}$, for $s>0$, but $g_s^{1/2}$ should be defined for $\forall s \notin \mathbb{R}_{\leq 0}$.

next idea might be to look at retracts again

$$W \xleftarrow{(\alpha_+^* \ \alpha_-^*)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W \qquad \underbrace{\alpha_+^*\alpha_+}_{h_+} + \underbrace{\alpha_-^*\alpha_-}_{h_-} = 1_W$$

~~important idea~~ You can reconstruct V_+, V_- by canonical factorization of $h_\pm$ $\quad V_\pm$ = completion of W wrt $(\xi, h_\pm \xi)$.

λ'' Refresh memory about two retract cases

① $W \xleftarrow{(\beta_+\ \beta_-)} \begin{pmatrix}V_+\\ V_-\end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}} W$

$\mathbb{C}$ retract of a $\mathbb{Z}/2$ graded module $h_\pm = \beta_\pm \alpha_\pm$
$h_+ + h_- = 1_W$

② $\begin{pmatrix}W_+\\ W_-\end{pmatrix} \xleftarrow{\begin{pmatrix}\beta_+ & 0\\ 0 & \beta_-\end{pmatrix}} \begin{pmatrix}V\\ V\end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_+ & 0\\ 0 & \alpha_-\end{pmatrix}} \begin{pmatrix}W_+\\ W_-\end{pmatrix}$

$p_\pm = \alpha_\pm \beta_\pm$ two projections on V

$$\begin{pmatrix}0 & \beta_+\alpha_-\\ \beta_-\alpha_+ & 0\end{pmatrix} = \begin{pmatrix}\beta_+ & 0\\ 0 & \beta_-\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_+ & 0\\ 0 & \alpha_-\end{pmatrix}$$

$$\begin{tikzcd} \begin{pmatrix}W_+\\ W_-\end{pmatrix} & V \arrow[l, "\begin{pmatrix}\beta_+\\ \beta_-\end{pmatrix}"'] & \begin{pmatrix}W_+\\ W_-\end{pmatrix} \arrow[l, "(\alpha_+\ \alpha_-)"'] \arrow[ll, bend left, "\begin{pmatrix}1 & \beta_+\alpha_-\\ \beta_-\alpha_+ & 1\end{pmatrix}"] \end{tikzcd}$$

Q: Would it be interesting to have $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ replaced by $\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$? This would yield an odd skew symmetric operator on $\begin{pmatrix}W_+\\ W_-\end{pmatrix}$.

Next project: To introduce random phases, to look for a new quantization for harmonic oscillators. Recall the idea. A harmonic oscillator is a Euclidean space V equipped with a non degenerate skew-symmetric operator X. One has the polar decomp: $X = |X| J$ where $|X|$ gives the frequencies and $J^2 = -1$ is a complex structure on V. Quantization gives the bosonic Fock space $S_{\mathbb{C}} V$. Actually it's more subtle because of metaplectic symmetry, the $\frac{\omega}{2}$ ground state.

You should make a list of ideas.

- V real v.s. with pos. def symm. form and nondegenerate skew-symmetric form, i.e. V Euclidean space equipped with a skew adjoint operator X. V splits orthogonal into 2 planes invariant under X s.t. $X = \begin{pmatrix}0 & -a\\ a & 0\end{pmatrix}$ $0 \neq a$, $|a|$ frequency $|a|$

In case all frequencies are the same, then $O(2n)/U(n)$ = space of complex structures on V = possible square roots of $-I$.

- V complex Hilbert, X non degenerate skew-hermitian. Here the possible phases are $J = iF$, $F = F^* = F^{-1}$. So $\coprod_{0 \le p \le n} Gr_p(\mathbb{C}^n)$ is the space of phases.

- Building idea: $\{A \in End(V) \mid A = A^*, 0 \le A \le 1\}$. Use the eigenvalues to make simplices parametrized by flags.

μ'' • Is there any link between the choice of $g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$ and square roots of $-I$. Notice that this form defines $g^{1/2}$ in terms of the polar decomposition of $1+X$. Similarly $g_s^{1/2} = \frac{s+X}{(s^2-X^2)^{1/2}}$ defined initially for $s>0$, then extended analytically to $\mathbb{C} - \mathbb{R}_{\le 0}$. These represent ~~the~~ variants of the ~~usual~~ polar decomposition recipe.

• Similarity between roots of an irreducible equation and random phases.

• Vague Idea. There seems to be a loss of information involved in passing from an irreducible orthogonal repn. of $F, \mathcal{E}$ on $\mathbb{R}^2$ to the corresponding harmonic oscillator. The oscillator retains only positive type information, e.g. characteristic values, whereas quantization of the oscillator involves lifting to a ~~doubles~~ double covering, that is, undoing the previous step in some way.

• Idea: Eliashberg rigidity thm: a C^0 convergent sequence of C^∞ symplectic transformations is C^∞ convergent. You might be able to use the C^∞ limit to prove decay. In the case $SL_2(\mathbb{R})$ elliptic elements converge to parabolic ones.

• Case $W = \begin{pmatrix}1\\z\end{pmatrix}V_+$, $W^\perp = \begin{pmatrix}-\bar{z}\\1\end{pmatrix}V_-$, the phase of z disappears after conjugation by an element centralizing $\mathcal{E}$.

• It might be useful to have a clean version of the fin diml representation theory (orthogonal, unitary) of $\langle F, \mathcal{E}\rangle$, before looking at phases random. (Do you want $|\sin\theta|$?). Key point is that the endo ring of the irred representation is $\mathbb{R}$ ($\mathbb{C}$), because the group ring maps onto all operator on the irred. repn.

• ζ function of a graph (Bass) $\overset{?}{=}$ char poly of some correspondence maybe from a Kronecker module. Something related to fixpoints of iterates. Examples from geodesic flows?

ν''

Let's try to get the F, ε representation into a canonical form. Fix $\cos\theta \in (-1, 1)$. Then the standard form ~~should be~~ to try is

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{array}{l} \text{on } \mathbb{R}^2 \\ \text{on } \mathbb{C}^2 \end{array}$$

This is an irred $\left\{\begin{array}{l}\text{orth} \\ \text{unitary}\end{array}\right.$ repn.

Something's wrong here because changing θ to $-\theta$ leaves the eigenvalue $\cos\theta$, your invariant for the representation unchanged.

So try indexing the repn by $0 < \theta < \pi$.

You have $G = \langle F, \varepsilon \rangle = \varepsilon \ltimes g \longrightarrow \mathrm{End}(\mathbb{R}^2)$

It should be clear that changing g_θ to $g_{-\theta}$ yields an isomorphic repn., isom being given by ε. But $g_{-\theta}$ is not on the list $0 < \theta < \pi$.

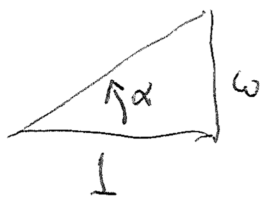
So now where are we?

Take Inv. C.T.

$$g^{1/2} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \alpha = \frac{\theta}{2} \qquad 0 < \alpha < \frac{\pi}{2}$$

$$= \begin{pmatrix} 1 & -\tan\alpha \\ \tan\alpha & 1 \end{pmatrix} \cos\alpha$$

$$= \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}}$$



$$g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}}$$

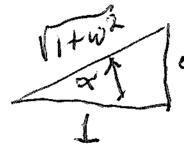
so there is a different angle α around, present.

ξ'' What is the inverse C.T.? $g \longmapsto \frac{g-1}{g+1} = X$

This is how one gets the harmonic oscillator associated to a representation F, ε. Assume $g+1$ invertible define X by $\frac{1+X}{1-X} = g = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X$, so that $X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} g = \frac{g-1}{g+1}$. Then $X^* = \frac{g^{-1}-1}{g^{-1}+1} = -\frac{g-1}{g+1} = -X$ so X is skew-symmetric yielding the 1-parameter gp of orthogonal transf. $\exp(tX)$ with L.T. $= (s-X)^{-1}$.

Now you want to relate the parameters in the F, ε representation, i.e. the angle $\theta \in (0, \pi)$, to the frequency parameter ω for the corresponding s.h.o. So consider $g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{R}^2$.
Let $X = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ be the C.T. of g. One has

$$g^{1/2} = \frac{1+X}{(1-X^2)^{1/2}} = \begin{pmatrix} 1 & -\omega \\ \omega & 1 \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

where 

. It seems that you want ω to range $-\infty < \omega < \infty$, $\omega = \tan\alpha$ $\quad -\pi/2 < \alpha < \frac{\pi}{2}$.
Of course $\theta = 2\alpha$.

Strange situation: On one side you have an orthogonal (or **Unitary**) rep. of $\langle F, \varepsilon \rangle$ which yields naturally the angles $\pm\theta$, spectrum of the orthogonal transf. $g = F\varepsilon$. The eigenvalues of $g = \frac{1+X}{1-X}$. On the other side you have a skew-symmetric operator X with the eigenvalues $\pm i\omega$, where $\omega = \tan\alpha$ and $\theta = 2\alpha$.

You still ~~need~~ want to explain $\frac{s^2+\omega^2}{s(1+\omega^2)} = s\cos^2\alpha + s^{-1}\sin^2\alpha$

0'' • Problem of forced harmonic oscillators. There should be no problem for the case of an arbitrary forcing term which is a function of time with values in the space of dominant variables, e.g. applied voltage source in series with a capacitor, applied current source in series with an inductor. This ~~situation~~ encountered in Thevenin theory (each edge a pure emf in series with an internal resistance)

Let's look for evidence that quantization of a harmonic oscillator involves undoing a squaring process.

Begin by defining a harmonic oscillator to be an orthogonal representation of $\langle F, \varepsilon \rangle$. Then you have natural angles arising: $e^{\pm i\theta}$, the eigenvalues of $g = F\varepsilon$. These are natural phases from the representation viewpoint. Next consider the classical time evolution which is given by the inverse C.T. X of g. The eigenvalues of X give the frequencies of the time evolution. What's natural?

Strange. $\langle F, \varepsilon \rangle \rightsquigarrow$ Spectrum $\{e^{\pm i\theta}\}$

Spectrum $\{\pm i\omega\}$ of X where $\omega = \tan\alpha$ $\quad \alpha = \frac{\theta}{2}$

$-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

So far you have studied the classical mechanics of ~~a~~ connected LC network. You have ended up with a standard "Hamiltonian" picture namely a Euclidean v.s. + skew-symmetric operator X.

The LC network seems to introduce the symplectic structure naturally. The states ~~space~~ are voltages + currents which are naturally dual geometrically: cochains (electrical) + chains (magnetic), and physically dual via the power pairing.

Consider a mechanical harmonic oscillator with two energy types kinetic + potential: $\frac{1}{2}\dot{q}^t m \dot{q}$, $\frac{1}{2}q^t k q$, functions of velocity + position. Then comes the mystery

Π'' of momentum Legendre transform taking the Lagrangian to the Hamiltonian

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quantization: $$H = \frac{p^2}{2m} + \frac{k}{2}q^2 = \underbrace{\left(\frac{-ip}{\sqrt{2m}} + \sqrt{\frac{k}{2}}\,q\right)}_{A^*}\underbrace{\left(\frac{+ip}{\sqrt{2m}} + \sqrt{\frac{k}{2}}\,q\right)}_{A}$$

$$- \underbrace{\left[\frac{-ip}{\sqrt{2m}}, \sqrt{\frac{k}{2}}\,q\right]}$$

$$\frac{1}{2}\sqrt{\frac{k}{m}}(+i)[p,q] = \frac{1}{2}\omega(+i)\frac{\hbar}{i} = +\frac{1}{2}\hbar\omega.$$

$$\therefore \quad H = A^*A + \frac{1}{2}\hbar\omega$$

~~forcing~~ ~~forcing~~

$$[A, A^*] = \hbar\omega$$

The units of A are $(\text{energy})^{1/2}$ it seems.

New program – forced harmonic oscillator. The simplest situation ~~should~~ be when you replace the initial state for the IVP by a "forcing term" which can be any state varying in time.

You had a different idea at some time, which was quantum mechanical. Related to the translations, the Heisenberg group. Time dependent perturbation theory, Dyson's variation of constants. Recall the situation.

You have the quantized s.h.o. Classical phase space is a 2 dim'l real plane and time evolution is given by a skew symmetric like $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. You have chosen a Fock space of ~~states~~. Ground state $|0\rangle$, ann + creation operators a, a^*. Translation operator Heisenberg group.

$$e^{a(\;)\quad a^*(\;)}$$

You have this path in the phase plane of this oscillator, you do time dep. perturbation. I recall that ~~there~~ scattering, namely the effect of the

8''

perturbation on Fock space as you go from $t = -\infty$ to $t = +\infty$ is a translation, which probably amounts the time integral of $\int_{-\infty}^{\infty} dt$ the [redacted] path in the translation plane, <u>plus</u> some phase, which might have an interpretation as a determinant.

The hope is this [redacted] time dependent perturbation picture will illuminate the forced oscillator situation. Let's see if you put into words what might happen. You consider a s.h.o. What is this? Classically it is given by a real 2-plane equipped [redacted] with a pos. def. symmetric form and a non-deg symplectic form. Another version: A Euclidean 2-plane V equipped with an [redacted] invertible skew-symmetric operator X. Apply polar decomp. $X = |X| J$ to get the frequency $\omega = |X|$ and complex structure. Question: [redacted] Does this V, X [redacted] yield a representation of $\langle F, \varepsilon \rangle$ on V?

So it seems possible, likely, that a harmonic oscillator: [redacted] Euclidean space V + [redacted] invertible skew-symmetric operator X is a different object [redacted] than an orthogonal representation of $\langle F, \varepsilon \rangle$ the ∞ dihedral gp on V.

Repeat. <u>Define a harm. osc.</u> to be a Euclidean space V equipped with an invertible skewsymmetric operator X. [redacted] Use polar decomp. $X = |X| J$ to split (V, X) ~~into~~ orthogonally according to the positive eigenvalues ω of $|X|$. These eigenspace have natural complex structures. This reduces to the case where $|X| = \omega > 0$, i.e. a harm osc. with a single frequency.

σ'' Where are you? You are [redacted] studying a general harmonic oscillator and have used the eigenspaces of $|X|$ to split the oscillator into ones with a single frequency. So you've reduced to $X = \omega J$, where $J^2 = -1$ and $\omega > 0$. [redacted]

Q: What can you say about $g = \frac{1+X}{1-X}$? This is something else you've forgotten about: the ungraded case.

So now you might see [redacted] how a harmonic oscillator (V, X) might not arise from an orth repn of $\langle F, \varepsilon\rangle$ on V.

Look at $V = \mathbb{R}^2$ with $X = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. Then $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $F = g\varepsilon = \frac{1+X}{1-X}\varepsilon$ give the repn. you want.

Conj: Given (V, X) a harm. osc. it should be possible (because of X invertible) to find an ε on V such that $\varepsilon X \varepsilon = -X$.

Idea: Is there a role in all this harmonic oscillator stuff, with its symplectic background, for a <u>contact structure</u>

Review $\mathbb{Z}/2$ graded module $\begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ retract of free $\mathbb{Z}/2$ gr mod $\begin{pmatrix} V \\ V \end{pmatrix}$

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{(\beta_+\ \beta_-)} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{(\alpha_+\ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad \beta_\pm \alpha_\pm = 1_{W_\pm} \qquad p_\pm = \alpha_\pm \beta_\pm$$ two proj on V.

$$X = \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$$ odd op on W

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V \xleftarrow{(\alpha_+\ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad V = \mathrm{Im}\,(1_W + X)$$

$$\begin{pmatrix} \beta_+\alpha_+ & \beta_+\alpha_- \\ \beta_-\alpha_+ & \beta_-\alpha_- \end{pmatrix}$$ [redacted]

I''

The idea would be to produce an oscillator type structure. X defined as compress in $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is symmetric. You could make skewsymmetric X by compressing $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ $\begin{pmatrix}1 & -\alpha_+^*\alpha_-\\ \alpha_-^*\alpha_+ & 1\end{pmatrix}$

I can't tell if this is interesting; it might help ~~you~~ to interchange W and V.

$$\begin{pmatrix}V_+\\ V_-\end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_+^* & 0\\ 0 & \alpha_-^*\end{pmatrix}} \begin{pmatrix}W\\ W\end{pmatrix} \xleftarrow{\begin{pmatrix}\alpha_+ & \\ & \alpha_-\end{pmatrix}} \begin{pmatrix}V_+\\ V_-\end{pmatrix}$$

$$X = \begin{pmatrix}\alpha_+^* & 0\\ 0 & \alpha_-^*\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha_+ & 0\\ 0 & \alpha_-\end{pmatrix} = \begin{pmatrix}0 & -\alpha_+^*\alpha_-\\ \alpha_-^*\alpha_+ & 0\end{pmatrix}$$

$$\begin{tikzcd}
\begin{pmatrix}V_+\\ V_-\end{pmatrix} & W \arrow[l, "\begin{pmatrix}\alpha_+^*\\ \alpha_-^*\end{pmatrix}"'] & \begin{pmatrix}V_+\\ V_-\end{pmatrix} \arrow[l, "(\alpha_+ \ \ \alpha_-)"'] \arrow[ll, bend left, "\begin{pmatrix}1_+ & \alpha_+^*\alpha_-\\ \alpha_-^*\alpha_+ & 1_-\end{pmatrix}"]
\end{tikzcd}$$

not $1+X$

Perhaps conj by $\begin{pmatrix}1 & \\ & i\end{pmatrix}$ to? NO.

If X is as above, then you have ~~[illegible]~~ $1 \pm X$ invertible, so you have a C.T. but meaning not clear.

Yesterday you discovered the difference between a general harmonic oscillator and an orth repn of $\langle F, \varepsilon \rangle$ namely the latter is a $\mathbb{Z}/2$-graded version of the former. Recall defn. Harm. osc. is a Euclidean space V equipped with invertible skew symmetric X. In the $\mathbb{Z}/2$ graded version $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ is equipped with ε grading and X is odd $\varepsilon X \varepsilon = -X$. The C.T. $g = \frac{1+X}{1-X}$ is only an orthogonal transf. in the "odd" case and $\varepsilon g \varepsilon^{-1} = g^{-1}$ in the "even" case.

Question: To what does $H = \frac{p^2}{2m} + \frac{k}{2} q^2$ + symplectic volume belong? The flow is $\dot{q} = \frac{p}{m}$, $\dot{p} = -kq$

so $X = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$ $\quad -X^2 = \begin{pmatrix} +m^{-1}k & 0 \\ 0 & -km^{-1} \end{pmatrix}$ $\quad |X| = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$

with $\omega = \sqrt{\frac{k}{m}}$, $\quad J = \begin{pmatrix} 0 & \frac{1}{m\omega} \\ -\frac{k}{\omega} & 0 \end{pmatrix}$ $\quad \frac{k}{\omega} = k\sqrt{\frac{m}{k}} = \sqrt{km}$

$$\frac{1}{m\omega} = \frac{1}{m\sqrt{\frac{k}{m}}} = \frac{1}{\sqrt{km}}$$

$$H = \frac{p^2}{2m} + \frac{k}{2} q^2 = \underbrace{\left(\frac{-ip}{\sqrt{2m}} + \sqrt{\frac{k}{2}}\, q \right)}_{A^*} \underbrace{\left(\frac{ip}{\sqrt{2m}} + \sqrt{\frac{k}{2}}\, q \right)}_{A} + \text{const}$$

$$A^*A = H + \underbrace{\left[\frac{-ip}{\sqrt{2m}}, \sqrt{\frac{k}{2}}\, q \right]}$$

$$\frac{1}{2}\sqrt{\frac{k}{m}} \left[-ip, q \right] = \frac{1}{2}\omega(-i)\frac{\hbar}{i} = \frac{-1}{2}\hbar\omega$$

$$\boxed{H = A^*A + \frac{1}{2}\hbar\omega \qquad [A, A^*] = \hbar\omega}$$

$\dot{x} = Ax + Bu$, $y = Cx + Du$, $H(s) = C\frac{1}{s-A}B + D$ (transfer fn.)

x internal state variables, u = input, y = output

9'' Question: Can you put a mechanical harmonic oscillator into $\mathbb{Z}/2$ graded form? This should be clear from orthogonal decomposition of (V, X) into 2 planes, where X becomes $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. This deserves careful study, to see the possible choices.

<u>IDEA</u>: Review the Artin-Schreier theory of real closed fields, especially Sturm sequences, which are used to calculate the number of real roots.

Yesterday you found, recalled a viewpoint for the quantum forced harmonic oscillator. Classically the forcing term can be any point of phase space depending on time and the motion is (given by solving) $(\partial_t - X)\begin{pmatrix} q \\ p \end{pmatrix} = f(t)$ ($\in$ forcing term.)

Quantum mechanically you probably need to solve this D.E. in the Heisenberg group.

It ought to be interesting to solve this DE using the L.T. It's not immediately clear what this means.

What do you have? A phase space $\mathbb{R}^2$ with skew symmetric operator X giving the time evolution classically. You're given a forcing term which is a smooth path in the phase space with compact support. Maybe the standard way to study this is the time dependent scattering theory. Compare incoming to outgoing. ??

X'' Let's try to understand a bit about whether a mechanical oscillator $\frac{1}{2}p^t m^{-1} p + \frac{1}{2} q^t k q$ has a natural $\mathbb{Z}/2$-grading. Try simple case $\frac{p^2}{2m} + \frac{k}{2}q^2$. Compare with LC case. **Can you really describe what's happening.** In the LC case you have the puzzle of the symplectic quotient preceding the state space. You have

$$W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow W^\perp \qquad\qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} V_+ \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xrightarrow{(-T\ 1)} W^\perp$$

$$\oplus \qquad\qquad\qquad\qquad \Big\downarrow \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$$

$$W' \twoheadleftarrow \begin{pmatrix} V'_+ \\ V'_- \end{pmatrix} \longleftarrow W^0 \qquad\qquad V_+ \xleftarrow{(1\ T^*)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} -T^* \\ 1 \end{pmatrix}} W^+$$

$$V_+$$

$$TV_C = V_L \qquad -T^*\hat{I}_L \qquad \hat{I}_C = \hat{\dot{V}}_C = s\hat{V}_C - V_C(0)$$

$$I_C + T^* I_L = 0 \qquad \hat{V}_L = \hat{\dot{I}}_L = s\hat{I}_L - I_C(0) \qquad \begin{pmatrix} s & T^* \\ -T & s \end{pmatrix}\begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \begin{pmatrix} V_C(0) \\ I_L(0) \end{pmatrix}$$

$$T\hat{V}_C$$

[redacted] symplectic stuff still confusing, however. On the surface you have this ~~...~~ phase space of dim $2e$ cut down to a symplectic ~~quotient~~ of dim e. This looks strange because e doesn't look even, but if $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ is assumed nondegenerate then it's clear.

Look at $H = \frac{1}{2}p^t m^{-1} p + \frac{1}{2} q^t k q$

$$\dot{q} = \frac{\partial H}{\partial p} = m^{-1}p \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq$$

This should fit into the LC scheme.

$$\ddot{q} = m^{-1}\dot{p} = m^{-1}(-k)q \qquad \ddot{q} + (m^{-1}k)q = 0$$

$$m\ddot{q} = -kq \qquad \text{Newton}$$

ψ'' $\quad X = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$ $\qquad \begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2 = \frac{1}{2} \begin{pmatrix} q \\ p \end{pmatrix}^t \underbrace{\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}}$$

$$\eta^t A \xi = \eta^t H X \xi$$

$$A = HX^{-1} = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} 0 & m^{-1} \\ -k & \end{pmatrix} \quad ?$$

A, H are bilinear forms, maps $V \underset{H}{\overset{A}{\rightrightarrows}} V^*$

so X is either $A^{-1}H$ or $H^{-1}A$

symmetric S
antisymm A $: V \longrightarrow V^*$ $\qquad$ You want $AX = H$

phase space

$$A = HX^{-1} = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} 0 & -k^{-1} \\ m & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

What do you want? To see this is the same as what you got from an LC oscillator.

$$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} AX &= H = H^t = X^t A^t = -X^t A \\ HX &= AXX = -X^t AX = -X^t H \end{aligned}}$$

$$H = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \qquad X = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \qquad V \underset{A}{\overset{H}{\rightrightarrows}} V^*$$

$$AX = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}. \qquad AX = H$$

Verify that

X respects the symplectic form A: $\quad X^t A + AX = 0$

symmetric form H: $\quad X^t H + HX = 0$

ω'' Next you want to change to orthonormal bases in the position & momentum spaces. This means writing $k = \kappa^t \kappa$, i.e. $q^t k q = (\kappa q)^t (\kappa q)$, simplest choice is $k^{1/2}$, pos. sqrt. It should be true that $\kappa = g k^{1/2}$ where g ~~can~~ be an arbitrary orthogonal matrix. Do the same in momentum space $m^{-1} = (\mu^{-1})^t (\mu^{-1})$ where $\mu^{-1} = g m^{-1/2}$, g arb. orth. Take simplest choices and chang vbls. $q = k^{-1/2} Q$ so that $q^t k q = Q^t k^{-1/2} k k^{-1/2} Q = Q^t Q$. Sim put $p = m^{+1/2} P$ so that $p^t m^{-1} p = P^t m^{+1/2} m^{-1} m^{+1/2} P = P^t P$.

$$\begin{pmatrix} Q \\ P \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix} = Q^t Q + P^t P$$

Now what? You've made a transf. of coords. from $\begin{pmatrix} q \\ p \end{pmatrix}$ to $\begin{pmatrix} Q \\ P \end{pmatrix}$ which simplifies H, but should make A harder to understand.

$$A = \begin{pmatrix} q_1 \\ p_1 \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q_2 \\ p_2 \end{pmatrix} \qquad \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} k^{-1/2} & 0 \\ 0 & m^{1/2} \end{pmatrix} \begin{pmatrix} Q \\ P \end{pmatrix}$$

$$A = \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}^t \begin{pmatrix} k^{-1/2} & 0 \\ 0 & m^{1/2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} k^{-1/2} & 0 \\ 0 & m^{1/2} \end{pmatrix} \begin{pmatrix} Q_2 \\ P_2 \end{pmatrix}$$

$$= \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}^t \begin{pmatrix} 0 & -k^{-1/2} m^{1/2} \\ m^{1/2} k^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} Q_2 \\ P_2 \end{pmatrix}$$

$$\left(\frac{k}{m}\right)^{1/2} = \omega$$

$$A X = H \qquad X = A^{-1} H = \begin{pmatrix} 0 & m^{-1/2} k^{1/2} \\ -m^{-1/2} k^{1/2} & 0 \end{pmatrix} I$$

So $X = \begin{pmatrix} 0 & m^{-1/2} k^{1/2} \\ -m^{-1/2} k^{1/2} & 0 \end{pmatrix}$ — like $X = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$

α''' There's a problem about $m^{1/2}$, $k^{1/2}$: What do these mean? m and k are symm bilinear forms: $m, k : V \longrightarrow V^*$ so $m^{-1}k$, $k^{-1}m$ are well-defined autos of V.

Review: $H = \frac{1}{2}p^t m^{-1} p + \frac{1}{2} q^t k q$ Hamilt. fn.

Ham. eqns $\dot{q} = \dfrac{\partial H}{\partial p} = m^{-1}p$ $\qquad \dot{p} = -\dfrac{\partial H}{\partial q} = -kq$

m^{-1}, k are pos. def symm bil. forms on p-space and q-space resp. They yield kin + pot energy resp. The Ham flow is a linear operator X on phase space

$$X = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \quad \text{means} \quad \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

V is phase space which splits into V_+ q-space and V_- p-space. So $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ is a polarized Euclidean spaces. What is subtle is the fact that position + momentum spaces are naturally dual.

Stop. Return to abstract harm. osc. situation

Phase space V equipped with H pos. def symm. $V \to V^*$
A nondeg skew-sym $V \to V^*$

[redacted] $H = AX$ defines the Hamiltonian flow on V. i.e. $X = A^{-1}H$. Facts $\begin{cases} X^tA + AX = 0 \\ X^tH + HX = 0. \end{cases}$

$$AX = H = H^t = X^tA^t = -X^tA \Longrightarrow X \text{ skewsymm op on } V \text{ equipped with quad form } X.$$

$$(HX = AXX = -X^tAX = -X^tH)$$

This is the ungraded case. You want to believe that this X is somehow related to a more interesting orthogonal transf. g via the C.T.

β^{14}

Go back to $H = \frac{1}{2}q^t k q + \frac{1}{2}p^t m^{-1} p$ on $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

This is a symmetric bilinear form, pos. def., corresp symm matrix is

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

So $H = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} : \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix}^*$

Now you ~~[illegible]~~ would to forget about k, m^{-1} and instead regard $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ as a polarized Euclidean space.

You can diagonalize k, m^{-1}. Meaning

q stands for a point in V_+

p ——————— ——— V_-

$k : V_+ \longrightarrow V_+^*$ is a linear symm. map $k^t = k$

$$H = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} : \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix}^* \xrightarrow{A^{-1} = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix}}$$

$$X = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} : \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$A^{-1}H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} = X$$

$$-X^2 = -\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} = \begin{pmatrix} m^{-1}k & 0 \\ 0 & km^{-1} \end{pmatrix}$$

$(-X^2)^{1/2} = ?$ Is $-X^2$ symmetric wrt H. YES

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \underbrace{\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}}_{H} \underbrace{\begin{pmatrix} m^{-1}k & 0 \\ 0 & km^{-1} \end{pmatrix}}_{(-X^2)} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} km^{-1} & 0 \\ 0 & m^{-1}k \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$$

g''' Show $m^{-1}k$, km^{-1} <u>symmetric wrt</u> $q^t k q'$, $p^t m^{-1} p'$

$$(m^{-1}kq)^t k q' = q^t k (m^{-1} k q')$$

$$(km^{-1}p)^t m^{-1} p' = p^t m^{-1}(km^{-1}p') \qquad \text{YES}$$

---

check X skew symm wrt H

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \underbrace{\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & km^{-1} \\ -m^{-1}k & 0 \end{pmatrix}}\begin{pmatrix} q' \\ p' \end{pmatrix} \overset{?}{=} \left(\begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}\right)^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}\begin{pmatrix} q' \\ p' \end{pmatrix}$$

$$= \begin{pmatrix} q \\ p \end{pmatrix}\underbrace{\begin{pmatrix} 0 & -k \\ m^{-1} & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}}_{\begin{pmatrix} 0 & -km^{-1} \\ m^{-1}k & 0 \end{pmatrix}}\begin{pmatrix} q' \\ p' \end{pmatrix}$$

different sign

---

m^{-1}, k are pos. def. symm. bilinear forms. Gram-Schmidt process gives orth basis. Choose a ~~basis~~ basis $e_1, \ldots, e_n \in V_+$ then use GS process. This amounts $GL_n R = O_n \times T_n$. Start with k pos. def symm. matrix i.e. make $\mathbb{R}^n$ into a Eucl. space via $q^t k q$. Then ~~~~ Let $g \in GL_n(\mathbb{R})$ be ~~a change~~ another basis, the columns are orthonormal.

$$v_1, \ldots, v_n \in \mathbb{R}^n \quad \text{col. vectors}$$

$$v_i^t k v_j = \delta_{ij} \qquad g = (v_1, \ldots, v_n)$$

$$g^t k g = I_n \quad \text{or} \quad k = (g^{-1})^t g^{-1}$$

δ''' Let $k: V \to V^*$ be pos. def. symm.
Then you get a pos. inner product $q^t k q$ on V
You had some idea about $k^{1/2}$ which doesn't make sense probably. The idea involves constructing $k = K^t K$.

Start again with $H = \frac{1}{2} p^t m^{-1} p + \frac{1}{2} q^t k q$, the Hamiltonian fn. on phase space of $\begin{pmatrix} q \\ p \end{pmatrix} \in \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$.

Hamilton's eqns. of motion:

$$\dot{q} = \frac{\partial H}{\partial p} = m^{-1} p \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq$$

so the infinitesimal generator of the motion is linear transformation

$$X \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$$

X is an odd linear tranf. on phase space

$$H = \begin{pmatrix} m^{-1} & 0 \\ 0 & k \end{pmatrix} : \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow \begin{pmatrix} V_+^* \\ V_-^* \end{pmatrix}$$

H is an even symmetric bilinear for on phase space

$\boxed{\text{WAIT;}}$ For X to be defined you have to interpret k as a map $k: V_+ \to V_-$ and m^{-1} as a map $m^{-1}: V_- \to V_+$ which means that you need to identify V_- with V_+^* and V_+ with V_-^*

pq $\quad gr \frac{cm^2}{sec^2} sec \quad$ erg sec $\quad \exp \frac{itH}{\hbar}$

ε''' try to understand better duality between

position + momentum $\quad qp \quad$ cm gr $\frac{cm}{sec}$ = erg sec

Repeat. mechanical oscillator. phase space $= \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \ni \begin{pmatrix} q \\ p \end{pmatrix}$

Important fact is that V_+, V_- are naturally in duality which means you are given a pairing $q, p \mapsto q^t p$ -

$$V_+ \xrightarrow{\sim} V_-^* \quad ,$$

$$(q, p) \longmapsto q^t p$$

also

$$V_- \longrightarrow V_+^* \qquad q \longmapsto (p \mapsto q^t p) \quad ?$$

$$p \longmapsto (q \mapsto p^t q)$$

So what? What did you learn? Phase space is $\begin{pmatrix} V_+ \\ V_+^* \end{pmatrix}$. Now $k : V_+ \xrightarrow{\sim} V_+^*$

??

As usual you're confused by symplectic stuff!

Apparently what happens is that V_+ and V_- are related by duality, so $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ is naturally symplectic. ~~But you have pos. def.~~ On V you have 2 structures A, H

$$A : V \longrightarrow V^* \qquad \text{anti symm.}$$
$$H : V \longrightarrow V^* \qquad \text{symm pos}$$

Review In the LC case you start with a representation of the dihedral group $\langle g, \varepsilon \rangle$ and obtain the time evolution of the network from the Inv C.T. X of g, which is a skew symm operator odd wrt ε : $\varepsilon X \varepsilon^{-1} = -X$, 5''' There is an ungraded version of the preceding repn of $\langle g,\varepsilon\rangle$, where you start with an orthogonal transf g and obtain the time evolution of a harmonic oscillator from the ICT X of g.

In both of these cases one should assume that $-1 \notin sp(g)$, so that X is well-defined, and also that $1 \notin sp(g)$, so that X is invertible.

Now look at a "mechanical" harmonic oscillator. This means you have a phase space $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ which is the direct sum of position space V_+ and momentum space V_-, which are equipped with a duality pairing allowing one to define the symplectic form

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = -q^t p' + p^t q'$$

$p^t q = q^t p$

Finally you are given the Hamiltonian which is the symmetric bilinear form

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$$

the direct sum of the potential energy on position space and the kinetic energy on momentum space (up to $\frac{1}{2}$).

The question now is whether the mechanical harm. oscillator is $\mathbb{Z}/2$ graded. This seems true because the anti-symmetric form $A = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$ is odd wrt $\varepsilon \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ -p \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad \therefore\ \varepsilon A = -A\varepsilon$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

η'' Also $H = \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}$ is even and

so $X = A^{-1}H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$

is odd. ~~[illegible]~~

~~[illegible]~~ mech. harm. oscillator, phase space = pos. space $\oplus$ mom. space

~~[illegible]~~ $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$, equipped with a duality isom.

~~[illegible]~~ $V \times V \longrightarrow \mathbb{R}$

$$\begin{pmatrix} q \\ p \end{pmatrix}, \begin{pmatrix} q' \\ p' \end{pmatrix} \longmapsto q^t p' + p^t q'$$

~~So it's clear~~ notation confused. $V_+ \times V_- \longrightarrow \mathbb{R}$

$(q, p) \longmapsto q^t p$. What is a dual pair P, A, Q

P right A-module pairing $(q, p) \longmapsto \langle q, p \rangle$ bilinear

Q left ______. ~~left A~~, right A bilinear.

$q^t p$ notation OK when thinking of p, q as column vectors. One has $q^t p = p^t q$. Ideally you ~~have~~ would like to have maps. $\mathbb{R} \xrightarrow{q} V_+$, $\mathbb{R} \xrightarrow{p} V_-$

~~[illegible]~~ $\mathbb{R} \xleftarrow{q^t} V_+^* = V_- \xleftarrow{p} \mathbb{R}$

Then you have OK $\mathbb{R} \xleftarrow{p^t} V_-^* = V_+ \xleftarrow{q} \mathbb{R}$

H symmetric ~~pos. def.~~ bil form $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix}$

Hamiltonian fn. $\frac{1}{2}p^t m^{-1} p + \frac{1}{2}q^t k q$

A anti-symm. nondeg $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q' \\ p' \end{pmatrix} = -q^t p' + p^t q'$

$AX = H \qquad X = A^{-1}H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$

$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix}^{\#} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} \qquad \begin{aligned} \dot{q} &= m^{-1}p \\ \dot{p} &= -kq \end{aligned}$

Θ'''

~~... is a kind an ungraded Question. This should consist of a ... space~~

Now that you have a $\mathbb{Z}/2$-grading on the mechanical harmonic oscillator you want to choose orth bases for V_+, V_-. You want $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ to be just a polarized Euclid-space. You want to make H standard.

$$\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = q^t k q' + p^t m^{-1} p' \overset{?}{=} Q^t Q' + P^t P'$$

linear change of variable $\quad Q = gq \qquad Q^t Q = q^t g^t g q$

$g^t g = k$. $\quad$ Sim $\quad P = hp \qquad P^t P = p^t h^t h p = p^t m^{-1} p$

$h^t h = m^{-1}$. The problem here is that you have not respected the duality between V_+ and V_-, which is given by

$$p^t q = P^t (h^t)^{-1} g^{-1} Q \qquad T^t = (g^t)^{-1} h^{-1}$$
$$q^t p = Q^t (g^t)^{-1} h^{-1} P \qquad T = (h^t)^{-1} g^{-1}$$

Duality is
$$p^t q = P^t T Q$$
$$q^t p = Q^t T^t P$$

$$A = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = -q^t p' + p^t q'$$

$$= -Q^t T^t P' + P^t T Q' = \begin{pmatrix} Q \\ P \end{pmatrix}^t \underbrace{\begin{pmatrix} 0 & -T^t \\ T & 0 \end{pmatrix}}_{\text{this should be } X \text{ as you know}} \begin{pmatrix} Q' \\ P' \end{pmatrix}$$

sign wrong as $X = A^{-1} H = A^{-1}$

Pretty idea. von Neumann used the C.T. to construct the operators on Hilbert space generating time evolution (also translation in position, momentum). You have been looking at time evolution for a harmonic oscillator, a fin dim situation, but still you need a C.T. to handle time it seems.

i''' Next you want an ungraded version. You would like descents to appear. Start with Euclidean space + invertible skew symmetric operator X. Maybe this point of departure is unwise, since all you can think of doing is polar decomp. of X leading to the usual positive frequencies + complex structure J.

Example of descent maybe $\mathbb{R} \to \mathbb{C}$ that you have encountered; $J^2 = -1$. Start with V Euclidean dim $2n$, then the space of complex structures is $O(2n)/U(n)$, and if V Hilbert dim n then $J^2 = -1$ sames as $F^2 = 1$, so you get full $Grass_{\mathbb{C}}(V)$.

Other idea. To define symplectic structure on a vector space using the hyperbolic construction $\begin{pmatrix} V \\ V^* \end{pmatrix}$, like Groth defines λ-rings.

Let V_+ be a f.d. vector space with dual V_-, on $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ you have $H = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = q^t p' + p^t q'$

symmetric nondeg bilinear form and $A = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$ $= -q^t p' + p^t q'$ antisymmetric bilinear form. Then

$$X = A^{-1} H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$ seems to

be the main natural operator around which can be obtained from the data provided.

Notice that off diagonal 1's are the canonical isos $q^t p'$ from V_- to V_+^* ($\mathbb{R} \xrightarrow{p'} V_-$, $V_+ \xrightarrow{q^t} \mathbb{R}$)

and $p^t q' : V_+ \xrightarrow{\sim} V_-^*$. ($\mathbb{R} \xrightarrow{q'} V_+$, $V_- \xrightarrow{p^t} \mathbb{R}$)

Next suppose V_+ equipped with extra structure like a symmetric or anti- bilinear form.

K''' At the moment you have reviewed the hyperbolic space construction $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ where $V_- = V_+^*$, the isom. canon $V_+ \xrightarrow{\sim} V_-^*$ given by $p^t q$

$$V_- \xrightarrow{\sim} V_+^* \quad \text{———} \quad q^t p$$

The canonical symmetric bilinear form

$$H = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$$

and anti-sym.

$$A = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$$

$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $X = \varepsilon$!!!

Next. You want very much to handle an ungraded harmonic oscillator, i.e. a vector [redacted] space W equipped with a pos. def symm. form $H : W \longrightarrow W^*$
an anti-symm nondeg form $A : W \longrightarrow W^*$

The tool to use you hope, is the hyperbolic space $\begin{pmatrix} V_+ \\ V_- \end{pmatrix} = \begin{pmatrix} W \\ W^* \end{pmatrix}$. [redacted] Maps you have

$$\begin{pmatrix} W \\ W^* \end{pmatrix} \longleftarrow \begin{pmatrix} W \\ W^* \end{pmatrix}$$

Start again with W a real v.s. equipped with $H : W \longrightarrow W^*$ pos. def symm. and $A : W \longrightarrow W^*$ nondeg anti symm. $X = A^{-1}H$ skew symm. operator on W, wrt H. Q: What consequences of this structure: A, H, X on W are there for the hyperbolic space $V = \begin{pmatrix} W \\ W^* \end{pmatrix}$? Idea: Infinitesimal symplectic transf on V are in 1-1 corresp with symmetric ~~operators~~ forms. by the graph construction. Thus given $B : W \longrightarrow W^*$ any bilinear form, one gets a subspace $\Gamma_B = \begin{pmatrix} 1 \\ B \end{pmatrix} W$ of V. It should be true that Γ_B is Lagrangian $\iff B = B^t$.

λ''' $\begin{pmatrix}1\\B\end{pmatrix}q = \begin{pmatrix}q\\Bq\end{pmatrix}$ $\begin{pmatrix}q\\Bq\end{pmatrix}^t \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}q'\\Bq'\end{pmatrix} = -q^t B q' + \underbrace{(Bq)^t q'}_{q^t B^t q'}$

is $=0$ $\forall q, q' \iff -B + B^t = 0$. Yes.

Also you have $\begin{pmatrix}C\\1\end{pmatrix}W^*$ is Lagrangian ($C: W^* \to W$)

when? $\begin{pmatrix}C\\1\end{pmatrix}p = \begin{pmatrix}Cp\\p\end{pmatrix}$ $0 = \begin{pmatrix}Cp\\p\end{pmatrix}^t \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}\begin{pmatrix}Cp'\\p'\end{pmatrix} = \begin{matrix}-p^t C^t p'\\+p^t C p'\end{matrix}$

$\forall p, p' \iff -C^t + C = 0$. I guess you might like complementary Lagrangian subspaces: $\Gamma_B \oplus \Gamma'_C \simeq V$.

$$\begin{pmatrix}1\\B\end{pmatrix}W \oplus \begin{pmatrix}C\\1\end{pmatrix}W^* \xrightarrow[?]{\sim} \begin{pmatrix}W\\W^*\end{pmatrix}$$

~~[illegible]~~ Let's try a different approach

Start with V equipped with H pos def symm
A nondeg antisymm.

Then you get $X = A^{-1}H$ skew symm nondeg op on V.

Then form $X = |X| J$ where $J^2 = -1$.

Consider V complex with H pos def herm. bil.

IDEA you've forgotten that for a complex vector space V equipped with hermitean inner product, then the real part of this product (v, v') is a pos. def symmetric bilinear form and the imaginary part is a symplectic form. I think it's true that this oscillator has ~~frequency~~ a single frequency $= 1$.

Unfinished: Relation between an ungraded harmonic oscillator W and the hyperbolic space $\begin{pmatrix}W\\W^*\end{pmatrix} = V$. The idea might be clearer in the case of quadratic forms, where you have the Witt group defined by orthogonal direct sum with hyperbolic spaces set $= 0$.

μ''' Let's return to LC networks with external nodes. [redacted] Recall that at some point you obtained a kind of response function from a subquotient of a polarized Euclidean space.

[redacted]

Classical response from a harmonic oscillator.

Review the mechanical oscillator:

Phase space $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \ni \begin{pmatrix} q \\ p \end{pmatrix}$ where $V_- \xrightarrow{\sim} V_+^*$, $p \longmapsto (q \mapsto q^t p)$

and $V_+ \xrightarrow{\sim} V_-^*$, $q \longmapsto (p \longmapsto p^t q)$

Symplectic form A: $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$ where the 1 has to be interpreted ~~via~~ duality pairing

Symm. form H: $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$

Symplectic flow $X = A^{-1}H = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix}$

i.e. $\dot{q} = m^{-1} p$, $\dot{p} = -kq$. ~~[illegible]~~

~~Next change coordinates~~ Next change variables. You assume H pos. def., choose orth bases in V_+, V_- so k, m^{-1} become diagonal "I". Let $q = gQ$, $p = hP$

where $q^t k q = Q^t (g^t k g) Q = Q^t Q$.

$p^t m^{-1} p = P^t (h^t m^{-1} h) P = P^t P$.

duality pairing $q^t p = Q^t g^t h P \qquad p^t q = P^t h^t g Q$

$$A = \left[\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}\begin{pmatrix} Q \\ P \end{pmatrix}\right]^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}\begin{pmatrix} Q' \\ P' \end{pmatrix}\right]$$

$$\begin{pmatrix} g^t & 0 \\ 0 & h^t \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} = \begin{pmatrix} 0 & -g^t h \\ h^t g & 0 \end{pmatrix}$$

v'''

Repeat: $q = gQ$, $p = hP$ $\qquad q^t k q = Q^t \underbrace{(g^t k g)}_{I} Q$

$p^t m^{-1} p = P^t \underbrace{h^t m^{-1} h}_{I} P$, Then $q^t p = Q^t g^t h P$

$$A = \begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$$

$$= \begin{pmatrix} Q \\ P \end{pmatrix}^t \underbrace{\begin{pmatrix} g^t & 0 \\ 0 & h^t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}}_{\begin{pmatrix} 0 & -g^t h \\ h^t g & 0 \end{pmatrix}} \begin{pmatrix} Q' \\ P' \end{pmatrix}$$

$$X = A^{-1} H = \begin{pmatrix} 0 & (h^t g)^{-1} \\ -(g^t h)^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -T^t \\ T & 0 \end{pmatrix}$$ where $T = -(g^t h)^{-1}$

$$H = \tfrac{1}{2} q^2 + \tfrac{1}{2} p^2$$

$$\dot q = \frac{\partial H}{\partial p} = p, \quad \dot p = -\frac{\partial H}{\partial q} = -q$$

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[redacted]

[redacted] Discuss an LC network, closed, connected where closed means no "external" nodes.

[redacted] Response to an applied force on a harmonic oscillator seems to be a quadratic form depending on S – it seems non dynamic. e.g.

$m\ddot{x} + \varepsilon\dot{x} + kx = f(t) = Ae^{st}$ yields $x = Be^{st}$ where $B = \dfrac{A}{m\omega^2 + \varepsilon\omega + k}$. Steady-state as opposed to transient

Program: You have this image picture of [redacted] a closed connected LC network given by an orthogonal repn of F, Σ. You want [redacted] the analog for a conn. LC network with external nodes, which you might call an open LC network. Hopefully the external variables, i.e. the [redacted] external state space, supports a response function which is a quadratic form depending on S.

[redacted]

Now list possible lines of enquiry:

(1) Terminology open + closed networks suggests there is [redacted] a cobordism analogy. Also cobordism involves quadratic forms. Harmonic oscillators involve both A, H bilinear forms.

(2) Consider a connected LC network equipped with an input node and ground nodes. You have some ideas about attaching a voltage source to these nodes. [redacted] For instance, how the external node changes the Kirchhoff constraints. You should study this situation using the polarized Euclidean space version of a closed LC network. One problem [redacted] might be that the dominant vbls (V_c, I_L) have [redacted] to be [redacted] related to the applied voltage V_a. Maybe all you have to do

$0'''$ is to project V_a onto the space of dominant vbls, better: to express V_a in terms of (V_C, I_L), then use this as a forcing term for the oscillator.

Is there a K-theory aspect to quadratic forms? Answer should be yes, because there's a Witt group generated by iso classes of quadratic spaces, with relations saying that the Witt class respects direct sums and that hyperbolic quadratic spaces have Witt inv = 0.

Ex. Work over a field K containing $1/2$. You want to take any quadratic space and show it's a summand of a hyperbolic space. The latter have the form $V = \begin{pmatrix} W \\ W^* \end{pmatrix}$ with quad form $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} = q^t p + p^t q$.

Consider $W = K$ with quadratic form qaq where $a \in K$ is nonzero. Is it possible to find $b \in K$ such that $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$ is hyperbolic. $b = -a$ should work.

The quad form when $b = -a$ is $a(q^2 - p^2) = a(q+p)(q-p)$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q - p \\ q + p \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & +1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} \ ?$$

Diagonalize $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find c s.t.

Pick a vector with norm $\neq 0$

$$v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 1 \end{pmatrix} \qquad \underbrace{\begin{pmatrix} c \\ 1 \end{pmatrix}}_{v_2}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{v_1} = c \ ?$$

Start again $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = q^2 - p^2$. Now put $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 1 \end{pmatrix}$

$$e_1 \cdot e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \qquad e_1 \cdot e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^t \begin{pmatrix} c & 0 \\ 0 & -1 \end{pmatrix} = c$$

$$e_2 \cdot e_2 = \begin{pmatrix} c \\ 1 \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} c \\ 1 \end{pmatrix} = c^2 - 1.$$

get $\begin{pmatrix} 1 & c \\ c & c^2 - 1 \end{pmatrix}$ note det $= c^2 - 1 - c^2 = -1 = (1)(-1)$. ✓

What does this mean? You are trying to show that the quad form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is hyperbolic.

Π''' $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = q^2 - p^2$ is hyperbolic means

$\exists \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2)$ s.t. $\begin{pmatrix} a & \cancel{b} \\ \cancel{b} & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -c \\ b & -d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \underbrace{\begin{pmatrix} a^2 - c^2 & ab - cd \\ ba - dc & b^2 - d^2 \end{pmatrix}}_{} \stackrel{?}{=} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$(ab)^2 - (ad)^2 - (bc)^2 + \cancel{(cd)^2}$

$\underline{\cancel{(ab)^2} + 2abcd + \cancel{(cd)^2}}$

$-(ad - bc)^2$

$\det = -(ad-bc)^2$

$a = 1,\ c = +1,\ b = \pm 1,\ d = -1$

$a = b = c = 1,\ d = -1$

$ab - cd = 1 + 1$

$\begin{pmatrix} 1 - (-1)^2 & (-1) - (-1)1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & +1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & +2 \\ +2 & 0 \end{pmatrix} \ ?$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

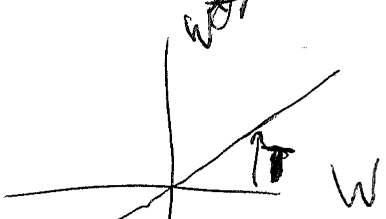
So now you ~~know~~ should know that on $\begin{pmatrix} W \\ W^* \end{pmatrix}$ for $\begin{pmatrix} & s \\ -s^{-1} & \end{pmatrix}$ is hyperbolic. ?

Next. Suppose given $s : W \xrightarrow{\sim} W^*$ symm.

Here should be the idea. Given W form $\begin{pmatrix} W \\ W^* \end{pmatrix}$ with $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}$

$\S'''$ Point: A maximal isotropic subspace should yield a hyperbolic form.

Consider an



$\Gamma_T = \begin{pmatrix} 1 \\ T \end{pmatrix} W$ a complement to W^*

Γ_T isotropic $\Leftrightarrow T+T^t=0$

$$\left(\begin{pmatrix} 1 \\ T \end{pmatrix} w\right)^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T \end{pmatrix} w' = 0$$

$$\underbrace{w^t \begin{pmatrix} 1 & T^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ T \end{pmatrix} w'}_{\begin{pmatrix} T \\ 1 \end{pmatrix}} = w(T+T^t)w'$$

you want to recall the skew symm operators = Lie orth gp.

The question bothering you is whether a nondeg quadratic space $W \xrightarrow{S} W^*$ can be embedded in the hyperbolic space $V = \begin{pmatrix} W \\ W^* \end{pmatrix}$ equipped w $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix}$

So you want a graph $\begin{pmatrix} 1 \\ S \end{pmatrix} : W \longrightarrow V$. For this to be an isometric embedding you need

$$\begin{pmatrix} q \\ Sq \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ Sq' \end{pmatrix} = q^t S q' + q^t S^t q'$$

so up to 2^{-1} it's clear.

Does $\exists$ a complement? $\begin{pmatrix} S^{-1} \\ 1 \end{pmatrix} W^*$

$$\begin{pmatrix} S^{-1}p \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} S^{-1}p' \\ p' \end{pmatrix} = p^t (S^{-1})^t p' + p^t S^{-1} p'$$

σ''' Repeat, W quad space: $W \xrightarrow[\sim]{S} W^*$ $\quad S = S^t$.

Form $V = \begin{pmatrix} W \\ W^* \end{pmatrix}$ quad form $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q^\circ \\ p^\circ \end{pmatrix} = q^t p^\circ + p^t q^\circ$

Embeddings $q \mapsto \begin{pmatrix} q \\ Sq \end{pmatrix}$, $\quad p \mapsto \begin{pmatrix} S^{-1}p \\ p \end{pmatrix}$

$W \hookrightarrow V \qquad W^* \hookrightarrow V$

Check first that $q \mapsto \begin{pmatrix} q \\ Sq \end{pmatrix}$ ~~preserve~~ respect quad form

$$\begin{pmatrix} q \\ Sq \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ Sq \end{pmatrix} = \begin{pmatrix} q^t & q^t S^t \end{pmatrix} \begin{pmatrix} Sq \\ q \end{pmatrix} = q^t S q + q^t S^t q$$
$$= 2 q^t S q$$

Next ~~compt~~ compute orth space.

$$\begin{pmatrix} q' \\ p' \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q \\ Sq \end{pmatrix} = q'^t S q + p'^t q = 0 \quad \forall q$$
$$\Longrightarrow q'^t S + p'^t = 0$$
$$\Rightarrow (Sq')^t + p'^t = 0$$

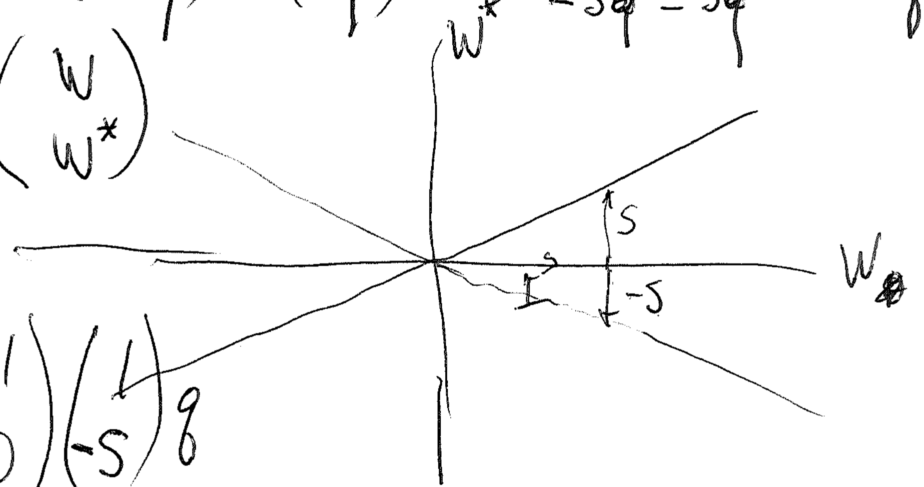
$\therefore p' = -Sq'$.

So it seems that in V the subspaces $\begin{pmatrix} 1 \\ S \end{pmatrix} W_+$ and $\begin{pmatrix} 1 \\ -S \end{pmatrix} W_+$ are orthogonal $\quad \begin{pmatrix} 1 \\ -S \end{pmatrix}^t \begin{pmatrix} S \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -S^t \end{pmatrix} \begin{pmatrix} S \\ 1 \end{pmatrix} = S - S^t$

complementary because $\begin{pmatrix} q \\ -Sq \end{pmatrix} = \begin{pmatrix} q' \\ Sq' \end{pmatrix} \Longrightarrow q' = q$, $\; -Sq = Sq \Longrightarrow Sq = 0$, $q = 0$

Next ~~what~~

$\begin{pmatrix} 1 \\ S \end{pmatrix} W \subset \begin{pmatrix} W \\ W^* \end{pmatrix}$



$$q^t \begin{pmatrix} 1 & -S^t \end{pmatrix} \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -S \end{pmatrix}} q$$

$$q^t \begin{pmatrix} 1 & -S^t \end{pmatrix} \begin{pmatrix} -Sq \\ q \end{pmatrix} = q^t(-Sq - S^t q) = -2 q^t S q$$

III

[redacted] Begin with $S : W \longrightarrow W^*$ S symmetric

Form $V = \begin{pmatrix} W \\ W^* \end{pmatrix}$ with $\begin{pmatrix} q \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} q' \\ p' \end{pmatrix} = q^t p' + p^t q'$.

Look at $W \xrightarrow{\binom{1}{S}} \begin{pmatrix} W \\ W^* \end{pmatrix}$ $\left(\begin{pmatrix} 1 \\ S \end{pmatrix} q \right)^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ S \end{pmatrix} q$

$$= q^t (1 \ \ S^t) \begin{pmatrix} S \\ 1 \end{pmatrix} q = q^t (S + S^t) q \qquad W \xrightarrow{\binom{1}{S}} \begin{pmatrix} W \\ W^* \end{pmatrix} \xleftarrow{\binom{1}{-S}} W$$

$$\left(\begin{pmatrix} 1 \\ S \end{pmatrix} q \right)^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -S \end{pmatrix} q' = q^t (1 \ \ S^t) \begin{pmatrix} -S \\ 1 \end{pmatrix} q' = q^t (-S + S^t) q$$

Next take $A : W \longrightarrow W^*$ antisymm $\qquad \begin{matrix} -2A \\ \| \\ -A + A^t \end{matrix}$

$$\left(\begin{pmatrix} 1 \\ A \end{pmatrix} q \right)^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ A \end{pmatrix} q = q^t (1 \ \ A^t) \begin{pmatrix} -A \\ 1 \end{pmatrix} q = q^t (\ldots) q$$

$$\left(\begin{pmatrix} 1 \\ -A \end{pmatrix} q \right)^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -A \end{pmatrix} q = q^t \overbrace{(A - A^t)}^{2A} q$$

$$\left(\begin{pmatrix} 1 \\ -A \end{pmatrix} q \right)^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ A \end{pmatrix} q = q^t (1 \ -A^t) \begin{pmatrix} -A \\ 1 \end{pmatrix} q = q^t \overbrace{(-A - A^t)}^{0} q$$

$$q^t (1 \ \ S^t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ S \end{pmatrix} q' = q^t (1 \ \ S^t) \begin{pmatrix} S \\ 1 \end{pmatrix} q = q^t (S + S^t) q'$$

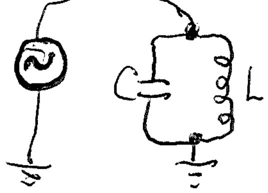
$$q^t (1 \ \ S^t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -S \end{pmatrix} q' = q^t (-S + S^t) q' = 0$$

$$q^t (1 \ -S^t) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -S \end{pmatrix} q' = q^t (1 \ -S^t) \begin{pmatrix} -S \\ 1 \end{pmatrix} q = q^t (-S - S^t) q'$$

[redacted] IDEA: When you look at response quantum of the harmonic oscillator it might be useful to mix A and S in some fashion. For instance the pencil $A + zS$.

For the moment you need to treat the harmonic oscillator with V_{ap} at an external node.

U'''



dominant obls - V_C, I_L

$v \quad l \quad e$

$2-1+2=3$

$$V_{ap} = V_C = V_L \qquad C\partial_t V_C = I_C$$

$$I_{ap} = I_C + I_L \qquad L\partial_t I_L = V_L$$

steady state [redacted] time dep e^{st}

$$Cs\hat{V}_C = \hat{I}_C$$
$$Ls\hat{I}_L = \hat{V}_L$$

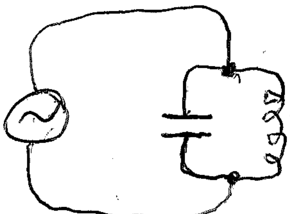
[redacted]

$$Cs\,\hat{V}_{ap} = \hat{I}_C$$

$$\frac{1}{Ls}\hat{V}_{ap} = \hat{I}_L$$

$$\left(Cs + \frac{1}{Ls}\right)\hat{V}_{ap} = \hat{I}_C + \hat{I}_L = \hat{I}_{ap} \qquad \therefore \frac{\hat{V}_{ap}}{\hat{I}_{ap}} = \frac{1}{Cs + \frac{1}{Ls}}$$

STUPID IDEA: [redacted] Non uniquess of decomposition of a p-torsion R-module, R PID, e.g. Jordan normal forms, could this lead to an interesting random field?



$v \quad l \quad e$

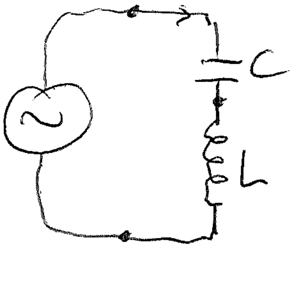
$2-1+2=3$ [redacted]

Questions What are the solutions for this network? You want time dependent solns. You expect there to be [redacted] a kind of steady state solution caused by $V_{ap}(t)$ - assume $V_{ap}=0$ for $t \ll 0$ - with a transient solution superimposed. The general solution should be a particular solution + the general solution of the homogeneous eqn.

[redacted]

$v \;{-1}\; l \quad e$

$3 \;\; {-1} \;\; 1 \;\; 3$



$$V_{ap} = V_C + V_L \qquad I_{ab} = I_C = I_L$$

$$\frac{\hat{V}_{ab}}{\hat{I}_{ab}} = \frac{\hat{V}_C}{\hat{I}_C} + \frac{\hat{V}_L}{\hat{I}_L} = \frac{1}{Cs} + Ls$$

φ''' So how are you going to handle an applied voltage source between two nodes. It seems that you need to add an edge to the graph. The problem [redacted] then is how to treat this new edge. [redacted] What happens from the polarized Euclidean viewpt. $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longrightarrow W^\perp$

W is the space of node [redacted] potentials, functions on the nodes mod constants. The external pair of nodes gives a linear fnl on W.

Idea: Use what you've learned about embedding a quadratic space into a hyperbolic one.

Consider  as this moves in time you get ~~an edge~~ a varying node voltage. In general ~~you have a~~ for each linear functional on W you

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Focus upon the [redacted] cobordism idea

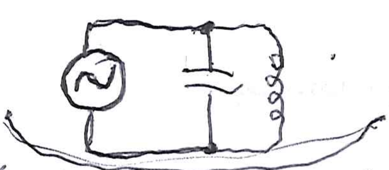
Observation: When you form [redacted] the spaces of cochains + chains

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

you are forming hyperbolic quadratic spaces:

$$\bar{C}_0 \twoheadleftarrow C_1 \hookleftarrow H_1 \qquad \begin{pmatrix} V \\ V^* \end{pmatrix} \qquad \underbrace{\begin{pmatrix} g \\ p \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g' \\ p' \end{pmatrix}}_{\text{(symmetric bilinear)}}$$

You really have the "power" quadratic form. So what about the pos def. [redacted] quadratic form $C^1 \xrightarrow{S} C_1$ related to the dynamics? Answer: The original quadratic space C^1, S is embedded in the hyperbolic quadratic space via $\Gamma_S = \begin{pmatrix} 1 \\ S \end{pmatrix} C^1$ with the orthogonal complement Γ_{-S}.

x''' You have to understand an applied voltage source . By attaching the source to the two nodes, you are weakening the Kirchhoff constraints[redacted]. Ideas to pursue: cobordism in the context of quadratic forms. Let's begin with the closed connected [redacted] LC network: 
$\overset{v}{2}-\overset{\ell}{1}+\overset{e}{1}=\overset{c}{2}$. [redacted] State space is 4 dim'l (without Kirchhoff constraints; call this the edge state space.

[redacted]

You need to link your understanding of a closed LC network (conn.) to a voltage source applied between external nodes. Why does this seem difficult? (i) the graph for the applied voltage source acquires an extra edge which seems to have its own voltage + current states making a 2-dimensional phase space, [redacted] (ii) you want to treat the applied voltage as a forcing term, the possible forcing terms correspond to dominant edges,

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Start again, need review [redacted] of something, maybe list ideas,

- cobordism idea: Think of a graph with external nodes as a singular 1-manifold with boundary[redacted].

Review the [redacted] situation for a closed connected LC network. You have a phase space, or state space consisting of the voltage drop and current for each edges. A state is given by coordinates $(V, I) \in C^1 \times C_1$; its power is $V \cdot I$, [redacted] the duality pairing between 1-cochain and 1-chains.

ψ''' In order to describe the motion by a first order DE you need to introduce phase space. This is not correct. Things are strange because the dynamics on the edges are like motion of a particle with constant velocity. The point is that only after restricting to Kirchhoff states do you get a flow.

Here's [redacted] what needs to be understood: To obtain the motion of the [redacted] closed LC circuit you solve the I.V.P. using the dominant variables (V_C, I_L), which are system of coords on the Kirchhoff space. This means working in the hyperbolic space $C^1 \times C_1$. [redacted] But to understand what's happening you [redacted] calculate the flow in $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ which is C^1 equipped with the power quadratic form + polarization

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Program: [redacted] To understand better the [redacted] "mechanical" harmonic oscillator:

$$\begin{pmatrix} S & T^* \\ -T & S \end{pmatrix} \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix} = \begin{pmatrix} q(0) \\ p(0) \end{pmatrix}$$

arising from an LC network which is suitably nondegenerate. The puzzle: You start with a polarized Euclidean space C^1 A), then construct [redacted] the "phase space" $\begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$ which has 4 components $\begin{pmatrix} V_C & V_L \\ I_C & I_L \end{pmatrix}$, symplectic structure. There should be in all of this a symplectic quotient yielding a mechanical harmonic oscillator.

ω''' Make attempt to understand constraints.
Would it help to look at a resistance network?

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] \arrow[d, "R^{-1}"] & H^1 \\ \bar{C}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook'] \end{tikzcd}$$

What are the natural questions?

- R^{-1} yields a pos. def inner product on C^1: the power dissipated in a given edge voltage configuration

This positive definite form on C^1 induces one on $\bar{C}^0$, the space of node potentials, and one on H^1. It seems that the pos. def form on H^1 is more **natural** as a pos. def. form on H_1, the space of loop currents.

You are looking at Hodge theory for a graph. The only result you can see is the orthogonal decomposition. There are "harmonic" 1-cochains and 1-chains

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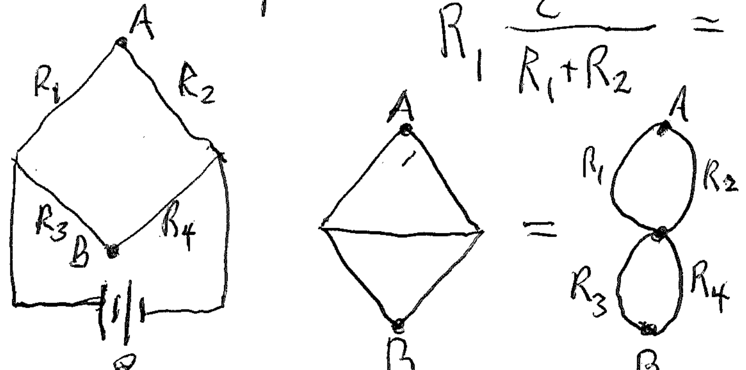
Thevenin example



$$R_1 \frac{\mathcal{E}}{R_1+R_2} = \varphi(A) \qquad R_3 \frac{\mathcal{E}}{R_3+R_4} = \varphi(B)$$

$$\mathcal{E}_0 = \varphi(A) - \varphi(B) = \left(\frac{R_1}{R_1+R_2} - \frac{R_3}{R_3+R_4}\right)\mathcal{E}$$

$$R_0 = \frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_3+R_4}$$

$\mathcal{E}_0, R_0$ are the e.m.f. and internal resistance of the 1-port with nodes A, B.

9''

You've reviewed different pictures of a vector space equipped with 2 splittings: 0) repn of $\mathbb{Z}/2 * \mathbb{Z}/2$

1) Short exact sequence with splitting. $\mathbb{Z}/2 \rtimes \mathbb{Z}$

2) homotopy equivalence between length 1 complexes

Let's now restrict to f.dim. - you want the spectral theory understood clearly.

Begin with classifying irreducibles.

$$g = F\varepsilon \qquad \frac{g+g^{-1}}{2} = \cos\theta \quad \text{if } g = e^{i\theta}$$

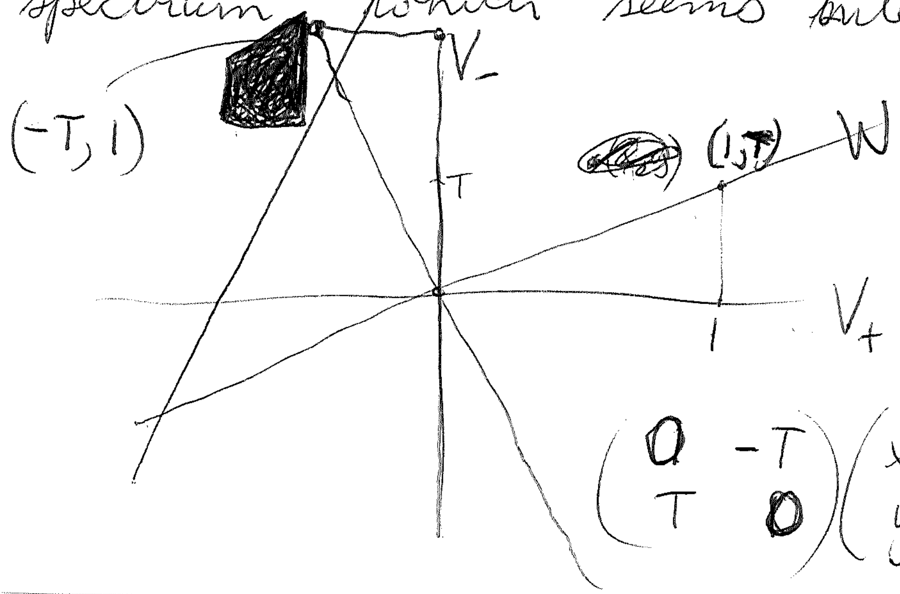
Try

$$W \xleftarrow{\beta = (\beta_+\ \beta_-)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\alpha = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W, \quad 1_W - \beta\alpha = h_+ + h_-$$

W should be the $F = +1$ eigenspace.

$$p = \alpha\beta \qquad p^2 = p. \qquad p = \frac{F+1}{2}$$

It's probably misleading to look first at the linear retract of a polarized vector space, because this situation suppresses part of the spectrum which seems interesting.



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ T \end{pmatrix} = \begin{pmatrix} -T \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -Ty \\ Tx \end{pmatrix}$$

b''

So you consider this V finite diml repn of [redacted] $\langle F, \varepsilon \rangle$. Let $g v_0 = z v_0$ $\quad z \in \mathbb{C}^\times$

Then $g \varepsilon v_0 = \varepsilon g^{-1} v_0 = z^{-1} \varepsilon v_0$

$$\varepsilon (a v_0 + b \varepsilon v_0) = b v_0 + a \varepsilon v_0$$
$$g (a v_0 + b \varepsilon v_0) = a z v_0 + b z^{-1} \varepsilon v_0$$

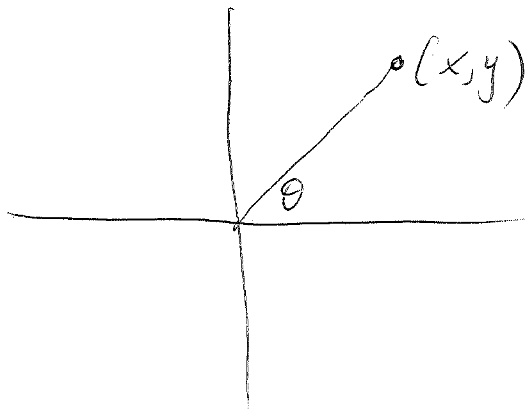
[redacted]

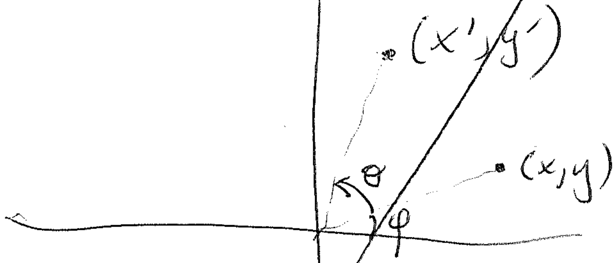
$$\therefore \quad g \longmapsto \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \qquad \varepsilon \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$F = g\varepsilon \longmapsto \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$$

Something is wrong with this representation because ~~[illegible]~~ the good representation should be defined over $\mathbb{R}$. All the irreducible fin. diml. representations should be characters or 2 diml real repns.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$





$$(x', y') = r\left(\cos(\theta+\varphi) + i \sin(\theta+\varphi)\right)$$

$$x' + i y' = e^{i\theta}(x + i y) = (\cos\theta + i \sin\theta)(x + i y)$$

$$= \left((\cos\theta) x - (\sin\theta) y\right) + i\left((\sin\theta) x + (\cos\theta) y\right)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

C'' virred f.d reps of $\langle F, \varepsilon \rangle$

$$g = F\varepsilon \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \varepsilon \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F \mapsto g\varepsilon = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\cancel{\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos^2\theta - \sin^2\theta \\ 2\sin\theta\cos\theta \end{pmatrix} = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}}$$

$$\frac{F+1}{2} = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & \sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} = \begin{pmatrix} \cancel{2}\cos^2\left(\frac{\theta}{2}\right) & \cancel{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \\ \cancel{2}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) & \cancel{2}\sin^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$1 + \cos 2\alpha = \cos^2\alpha - \cancel{\sin^2\alpha} + \cos^2\alpha + \cancel{\sin^2\alpha} = 2\cos^2\alpha$$

so $\dfrac{F+1}{2}$ projects onto $\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}\mathbb{R}$ $\quad \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \sin\frac{\theta}{2} & -\cos\frac{\theta}{2} \end{pmatrix}$

$$\frac{1-F}{2} = \frac{1}{2}\begin{pmatrix} +1-\cos\theta & -\sin\theta \\ -\sin\theta & +1+\cos\theta \end{pmatrix} = \frac{1}{\cancel{2}}\begin{pmatrix} +\cancel{2}\sin^2\frac{\theta}{2} & -\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ -\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2} & +\cancel{2}\cos^2\frac{\theta}{2} \end{pmatrix}$$

dihedral group $\langle F, \varepsilon \rangle = \langle \varepsilon \rangle \ltimes g \qquad g = F\varepsilon$

represent on $\mathbb{R}^2$ by

$$g \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \varepsilon \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F = g\varepsilon = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \qquad \begin{matrix} \mathrm{tr} = 0 \\ \det = -1 \end{matrix}$$

$$\frac{F+1}{2} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\theta/2\cos\theta/2 \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \sin^2\theta/2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}$$

d''

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} W \\ W^\perp \end{pmatrix} \xleftarrow{\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} W \\ W^\perp \end{pmatrix}$$

8 operators 8 relations. You are looking at a representation of the inf dihedral group $\langle \varepsilon, F \rangle$, a unitary orthogonal? representation. You want to decompose it into irreducibles. What's the center of the group alg?

$$\frac{F\varepsilon + \varepsilon F}{2} = \frac{g+g^{-1}}{2} \qquad (F+\varepsilon)^2 = F\varepsilon + \varepsilon F + 2$$
$$(F-\varepsilon)^2 = 2 - F\varepsilon - \varepsilon F$$

What is the partition

$$1_W = a^*a + c^*c \qquad a = \frac{1+\varepsilon}{2}\,\frac{1+F}{2}$$

$$a^*a = \frac{1+F}{2}\,\frac{1+\varepsilon}{2}\,\cancel{\frac{1+\varepsilon}{2}}\,\frac{1+F}{2}$$

$$c^*c = \frac{1+F}{2}\,\frac{1-\varepsilon}{2}\,\frac{1+F}{2}$$

You should be able to assume $\frac{F\varepsilon+\varepsilon F}{2}$ is a scalar operator $\in (-1,1)$, and then deduce everything else.

$$g = F\varepsilon = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad g^{-1} = \varepsilon F = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

So imagine a $\underbrace{\text{}}_{}$ f.d. real vector space Z equipped with involutions F, ε such that $\frac{F\varepsilon + \varepsilon F}{2}$ is the scalar operator λ. ~~This is absurd.~~

This is absurd: You are given two involutions F, ε on a f.d. complex v.s. say complex to begin.

e''

~~H f.d. Hilbert space, F ε two~~

E f.d. Euclidean space, F + ε two orthogonal involutions (for an involution F orthogonal means $F^* = F^{-1} = F$ i.e. F is symmetric.

$g = F\varepsilon$ orthogonal transf on E

one knows E splits orthogonally into 2 planes stable under g, but why should they be stable under ε.

Go back to complex Hilbert space H with 2 unitary involutions F, ε.

Better, prove the decomposition into 2diml rotations for a real orthogonal matrix. So take an orthog matrix, look at its characteristic equation. Eigenvalues should lie in S^1.

To understand why an orthog transf splits into 2diml rotations.

E Euclidean space i.e. real v.s. + pos. def. symmetric form

g orthog transf. $g^* = g^{-1}$, where $g^* =$ transpose

Complexify – choose an orth basis for E, then the orthog matrix g is unitary in $M_n(\mathbb{C})$. Pick an eigenvector $gv = \xi v$ $v \in \mathbb{C}^n$

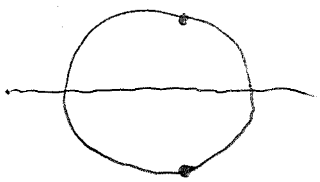
$\therefore$ $g\bar{v} = \bar{\xi}\bar{v}$

get 2 plane basis $\frac{v+\bar{v}}{2}$, $\frac{v-\bar{v}}{2i}$ Should get 2diml rotation.

f'' What happens next? You started with a ~~?~~ Euclidean space + 2 orth invs F, ε. Put $g = F\varepsilon$ orth. ??

Maybe go to the complex case first.

H fd Hilb $\quad F, \varepsilon$ unitary s.a. inv. $\quad g = F\varepsilon$



because $\varepsilon g \varepsilon^{-1} = g^{-1}$ the spectrum of g is symmetric about real axis.
You can split H, F, ε into eigenspaces for the s.a. operator $\frac{g+g^{-1}}{2} = \frac{F\varepsilon + \varepsilon F}{2}$

$$V_\theta = \{\xi \in H \mid g\xi = e^{i\theta}\xi\} \qquad \varepsilon g \xi = \varepsilon e^{i\theta}\xi$$
$$V_{-\theta} = \{\xi \in H \mid g\xi = e^{-i\theta}\xi\} \qquad g^{-1}(\varepsilon\xi) = e^{i\theta}(\varepsilon\xi)$$

$$\therefore \quad \varepsilon(V_\theta) = V_{-\theta}, \quad \varepsilon(V_{-\theta}) = V_\theta$$

$$H = \begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix} \qquad g \longmapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \varepsilon \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & -e^{i\theta} \\ e^{-i\theta} & e^{-i\theta} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

g''

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{i\theta}+e^{-i\theta} & i(e^{i\theta}-e^{-i\theta}) \\ \frac{e^{i\theta}-e^{-i\theta}}{i} & e^{i\theta}+e^{-i\theta} \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

left mult. by $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ then right mult by $\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\cos\theta - e^{i\theta} = \frac{e^{i\theta}+e^{-i\theta}}{2} - e^{i\theta} = \frac{-e^{i\theta}+e^{-i\theta}}{2}$$

$$= -i\sin\theta$$

$$\begin{pmatrix} -i\sin\theta & -\sin\theta \\ \sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} 1 \\ \ \end{pmatrix}$$

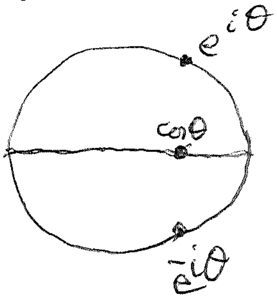
~~$\sin\theta$~~ $i\cos\theta + \sin\theta$

$$\boxed{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}}$$

$$\begin{matrix} e^{i\theta} & e^{-i\theta} \\ -ie^{i\theta} & +ie^{-i\theta} \end{matrix}$$

b''

~~[illegible]~~ Repeat H fin. dim. Hilb with 2 unitary inv. F, ε. $g = F\varepsilon$ unitary since $g^{-1} = \varepsilon g \varepsilon^{-1}$ spectrum of g (which is a divisor supported on S^1) is symmetric about real axis. In fact ε sets up isom $V_\theta \xrightarrow{\sim} V_{-\theta}$ of



Decompose H according to the eigenvalues of $\frac{1}{2}(g+g^{-1}) = \frac{1}{2}(F\varepsilon + \varepsilon F)$.

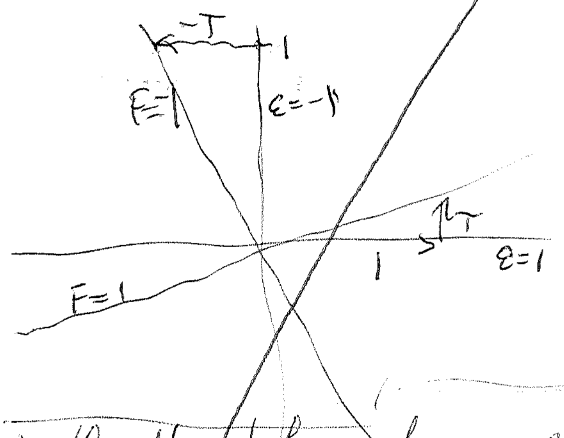
~~[illegible]~~ Can assume $\frac{1}{2}(g+g^{-1}) = \cancel{\text{scalar}}\,(\cos\theta)\,\mathrm{Id}$

Then $H = \begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix}$ $\quad g \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\quad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

You need to understand the real version, how it [redacted] arises. Suppose given E f.d. Euclidean with 2 orth inv. F, ε, ~~[illegible]~~ Can decompose E according to $\frac{1}{2}(g+g^{-1})$ which is symmetric

endo ring of ~~[illegible]~~ an irreducible rep. $\mathbb{R}, \mathbb{C}, \mathbb{H}$?

What you would like?



$$F\begin{pmatrix} 1 & -T \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T \\ T & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon$$

What's the puzzle? You're given E, F, ε such that $\frac{1}{2}(F\varepsilon + \varepsilon F)$ is λI, $-1 < \lambda < 1$. Is there a canonical form for F.

c''

~~This~~ Go back to $W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \twoheadrightarrow W^\perp$

~~So we have the same space~~ View this as the space $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ equipped with involution $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and also with the involution F such that $F = +1$ on W, $F = -1$ on $W^\perp$. To simplify suppose $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$ where $T : V_+ \to V_-$.

The problem is to decompose ~~to the~~ the dihedral representation given by $(V, \cancel{\ \ } F, \varepsilon)$. into isotypical components. You believe that this should be ~~[illegible]~~ related to the characteristic values of the operator T, i.e. the eigenvalues of $(T^*T)^{1/2}$ and $(TT^*)^{1/2}$. Why?

~~[illegible]~~ Is there a phase quantity you're overlooking?

$$\frac{1}{2}(F\varepsilon + \varepsilon F) = \frac{1}{2}(g + g^{-1})$$ centralizes F, ε.

~~For the eigenvalue λ of~~ So the rep (V, F, ε) splits ~~[illegible]~~ into eigenspaces for $\frac{1}{2}(g+g^{-1})$: $(V_\lambda, F, \varepsilon)$

$$1 + X = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \qquad F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon$$

$$\frac{1}{2}(F\varepsilon + \varepsilon F) = \frac{1}{2}\left(\frac{1+X}{1-X} + \frac{1-X}{1+X}\right)$$

$$= \frac{1}{2}\,\frac{1 + \cancel{2X} + X^2 + 1 - \cancel{2X} + X^2}{1 - X^2} = \frac{1+X^2}{1-X^2}$$

j″ Repeat. $W \hookrightarrow \begin{pmatrix}V_+\\V_-\end{pmatrix} \twoheadrightarrow W^\perp$. $V=\begin{pmatrix}V_+\\V_-\end{pmatrix}$

$\varepsilon = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ on V, $F=1$ on W, -1 on $W^\perp$. Assume $W=\begin{pmatrix}1\\T\end{pmatrix}V_+$ where $T: V_+\to V_-$. Then $W^\perp = \begin{pmatrix}-T^*\\1\end{pmatrix}V_-$

$$F\begin{pmatrix}1&-T^*\\T&1\end{pmatrix} = \begin{pmatrix}1&-T^*\\T&1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$ put $X=\begin{pmatrix}0&-T^*\\T&0\end{pmatrix}$

$$F(1+X)=(1+X)\varepsilon = \varepsilon(1-X) \Rightarrow \frac{1+X}{1-X}=F\varepsilon$$

You know $\frac12(F\varepsilon+\varepsilon F) = \frac12(g+g^*)$ is hermitian and central, so that (V,F,ε) splits into eigenspaces $(V_\lambda, F_\lambda, \varepsilon)$ where $\frac12(g+g^{-1})=\lambda$ on V_λ.

You have $\frac12(g+g^{-1}) = \frac12\left(\frac{1+X}{1-X}+\frac{1-X}{1+X}\right)=$

$$\frac12\left(\frac{1+\cancel{2X}+X^2+1-\cancel{2X}+X^2}{1-X^2}\right) = \frac{1+X^2}{1-X^2} \qquad X^2 = \begin{pmatrix}-T^*T&0\\0&-TT^*\end{pmatrix}$$

$$\frac{1+X^2}{1-X^2} = \begin{pmatrix}\frac{1-T^*T}{1+T^*T}&0\\0&\frac{1-TT^*}{1+TT^*}\end{pmatrix} = \begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}$$

$$\frac{1-\mu}{1+\mu} = \begin{pmatrix}-1&1\\1&1\end{pmatrix}\mu = \lambda \qquad -1\le\lambda\le 1$$

$$\mu = \begin{pmatrix}1&-1\\-1&-1\end{pmatrix}(\lambda) = \frac{-\lambda+1}{\lambda+1} = \frac{1-\lambda}{1+\lambda}$$

2'' V fd Hilb with F, ε $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$g = F\varepsilon$ is unitary, $\frac{1}{2}(g+g^{-1})$ is herm., let $\xi \in V$ be a unit vector which is an eigenvector for $\frac{1}{2}(g+g^{-1})$: $\frac{1}{2}(g\xi + g^{-1}\xi) = \lambda \xi$

$-1 \leq \lambda \leq 1$. ~~Assume $\lambda \in (-1, 1)$. Let $T = \mathbb{C}g\xi + \mathbb{C}g^{-1}\xi$.~~

~~Claim T stable under F, ε~~ Because F, ε, $\frac{1}{2}(g+g^{-1})$ commute with λ the λ-eigenspace V_λ of $\frac{1}{2}(g+g^{-1})$ is stable under F, ε. Let $\xi \in V_\lambda$ be a unit eigenvector for g: $g\xi = e^{i\theta}\xi$. Then $g^{-1}\varepsilon\xi = e^{i\theta}\varepsilon\xi$

$g(\varepsilon\xi) = e^{-i\theta}\varepsilon\xi$. $\begin{pmatrix} \mathbb{C}\xi \\ \mathbb{C}\varepsilon\xi \end{pmatrix} = \mathbb{C}\xi + \mathbb{C}\varepsilon\xi$ stable under ε, g with

$g(a\xi + b\varepsilon\xi) = ae^{i\theta}\xi + be^{-i\theta}\varepsilon\xi$ $g \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

V fd Hilb with F, ε, $g = F\varepsilon$ $V = \bigoplus V_\theta$

$\sigma V_\theta = V_{-\theta}$

~~s.h.o. case~~

$W \xhookrightarrow{\alpha} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \twoheadrightarrow W^\perp$ s.h.o. case

$\alpha = \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}$ $\alpha^*\alpha = (\alpha_+^* \ \alpha_-^*)\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \overbrace{\alpha_+^*\alpha_+}^{h_+} + \overbrace{\alpha_-^*\alpha_-}^{h_-} = 1$

h_+, h_- tell you how energy is distributed between C, L edges.

Example. $\bar{C}^0 \xhookrightarrow{\alpha} \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} \twoheadrightarrow H^1$

$\alpha(V_0) = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix}$ $\begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} = \left\{ \begin{pmatrix} V_C \\ V_L \end{pmatrix} \right\}$ $\left\| \begin{pmatrix} V_C \\ V_L \end{pmatrix} \right\|^2 = CV_C^2 + \frac{1}{L}V_L^2$

$\|\alpha(V_0)\|^2 = \left\| \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} \right\| = (C + L^{-1})V_0^2$

$\|\alpha_+ V_0\|^2 = CV_0^2$ $\|\alpha_- V_0\|^2 = L^{-1}V_0^2$

ℓ''

Try again.

$$\bar{C}^0 \xhookrightarrow{\alpha} \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix}$$

$$V_0 \longmapsto \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} \qquad \|\alpha V_0\|^2 = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix}^* \begin{pmatrix} C & \\ & L^{-1} \end{pmatrix} \begin{pmatrix} V_0 \\ V_0 \end{pmatrix}$$

$$= CV_0^2 + L^{-1}V_0^2$$

$$\left(\alpha V_0 \,\middle|\, \begin{pmatrix} V_C \\ V_L \end{pmatrix} \right) = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} \begin{pmatrix} C & \\ & L^{-1} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix} = V_0 C V_C + V_0 L^{-1} V_L$$

$$= V_0 (C V_C + L^{-1} V_L).$$

$$\therefore \ \alpha^* = \begin{pmatrix} C & L^{-1} \end{pmatrix}$$

$$\bar{C}^0 \xhookrightarrow{\alpha} \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} \qquad \begin{pmatrix} V_C \\ V_L \end{pmatrix} \qquad \left\| \begin{pmatrix} V_C \\ V_L \end{pmatrix} \right\|^2 = \begin{pmatrix} V_C \\ V_L \end{pmatrix}^* \begin{pmatrix} C & 0 \\ 0 & L^{-1} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix}$$

$$= V_C C V_C + V_L L^{-1} V_L$$

C

$$\bar{C}^0 \xhookrightarrow{\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} \xrightarrow{\begin{pmatrix} C^{1/2} & \\ & L^{-1/2} \end{pmatrix}} \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix}$$

$$\begin{pmatrix} V_C \\ V_L \end{pmatrix} \longmapsto \begin{pmatrix} C^{+1/2} V_C \\ L^{-1/2} V_L \end{pmatrix}$$

$$\bar{C}^0 \xrightarrow{\begin{pmatrix} C^{1/2} \\ L^{-1/2} \end{pmatrix}} \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix}$$

$$V_0 \longmapsto \begin{pmatrix} C^{1/2} \\ L^{-1/2} \end{pmatrix} V_0 \longmapsto CV_0^2 + L^{-1}V_0^2$$

m''

Return to $\bar{C}^\circ \hookrightarrow C^1_C \oplus C^1_L$ with the pos quad form

$$\begin{pmatrix} V_C \\ V_L \end{pmatrix}^* \begin{pmatrix} C & 0 \\ 0 & L^{-1} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix} = V_C^* C V_C + V_L^* L^{-1} V_L$$

on C^1. Aim to split into eigenspaces. But first you want the dot product in a standard form.

~~Put~~ $\cancel{V_C = C^{-1/2} X_+ \,,\; V_L = L^{+1/2} X_-}$

$$\cancel{\begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} C^{-1/2} & 0 \\ 0 & L^{+1/2} \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix}}$$

~~Then~~ $$\cancel{\begin{pmatrix} V_C' \\ V_L' \end{pmatrix}^* \begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} X_+' \\ X_-' \end{pmatrix}^* \begin{pmatrix} C^{-1/2} & 0 \\ 0 & L^{+1/2} \end{pmatrix} \begin{pmatrix} C^{-1/2} & 0 \\ 0 & L^{+1/2} \end{pmatrix} \begin{pmatrix} X_+ \\ X_- \end{pmatrix}}$$

Put $X_+ = C^{+1/2} V_C$, $X_- = L^{-1/2} V_L$

$$\begin{pmatrix} V_C \\ V_L \end{pmatrix}^* \begin{pmatrix} C & \\ & L^{-1} \end{pmatrix} \begin{pmatrix} V_C \\ V_L \end{pmatrix} = X_+^* X_+ + X_-^* X_-$$

$$\begin{tikzcd} \bar{C}^\circ \arrow[r, hook] \arrow[d, "S"', "\wr"] & C^1_C \oplus C^1_L \arrow[d, "\wr"] & \begin{pmatrix} V_C \\ V_L \end{pmatrix} \arrow[d, mapsto] \\ W \arrow[r, hook, "\alpha"] & \begin{pmatrix} V_+ \\ V_- \end{pmatrix} & \begin{pmatrix} X_+ = C^{1/2} V_C \\ X_- = L^{-1/2} V_L \end{pmatrix} \end{tikzcd}$$

In the s.h.o. case $\bar{C}^\circ = \{ V_0 \}$ $\quad \alpha V_0 = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} \mapsto \begin{pmatrix} C^{1/2} V_0 \\ L^{-1/2} V_0 \end{pmatrix}$

. . .

n''

$$\begin{array}{ccc}
\vec{C}^{0} & & C^1_C \oplus C^1_L \\
\| & & \| \\
\mathbb{R} & \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} & \begin{pmatrix}\mathbb{R}\\ \mathbb{R}\end{pmatrix} \\
\| & & \Big\downarrow \begin{pmatrix} C^{1/2} & 0 \\ 0 & L^{-1/2}\end{pmatrix} \\
\mathbb{R} & \xrightarrow{\alpha = \begin{pmatrix}C^{1/2}\\ L^{-1/2}\end{pmatrix}} & \begin{pmatrix}\mathbb{R}\\ \mathbb{R}\end{pmatrix}
\end{array}$$

so you have the line

$$\frac{x}{y} = \frac{C^{1/2}}{L^{-1/2}} = \sqrt{LC} \qquad y = \frac{x}{\sqrt{CL}}$$

You want α isometric.

$$\alpha^*\alpha = \left(C^{1/2}\ L^{-1/2}\right)\begin{pmatrix}C^{1/2}\\ L^{-1/2}\end{pmatrix} = C + L^{-1}$$

so you change α to $\tilde{\alpha} = \dfrac{1}{\sqrt{C+L^{-1}}}\begin{pmatrix}C^{1/2}\\ L^{-1/2}\end{pmatrix}$

$$\tilde{\alpha} = \begin{pmatrix}\sqrt{\dfrac{C}{C+L^{-1}}} \\[2ex] \sqrt{\dfrac{L^{-1}}{C+L^{-1}}}\end{pmatrix} = \begin{pmatrix}\sqrt{\dfrac{CL}{CL+1}} \\[2ex] \sqrt{\dfrac{1}{CL+1}}\end{pmatrix}$$

What does all this mean?

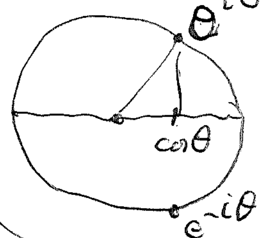
Keep on refreshing your memory. Reps of $\langle F, \varepsilon \rangle$

V f.d. Hilb $\quad V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = +1$

on W, -1 on $W^\perp$. $\quad g = F\varepsilon$ unitary,

$V_\theta = \operatorname{Ker}(g - e^{i\theta}) \qquad g\xi = e^{i\theta}\xi \Rightarrow g^{-1}\varepsilon\xi = e^{i\theta}\varepsilon\xi$

$\therefore\ \varepsilon : V_\theta \xrightarrow{\sim} V_{-\theta},$



What can you say if $\cos\theta \in (-1, 1)$?

$V = \begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix}$?

0'' Consider f.d. unitary repn V of $\langle F, \varepsilon \rangle$. Have

$V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $g = F\varepsilon$ unitary

let $V_\theta = \{ \xi \in V \mid g\xi = e^{i\theta}\xi \}$ $\Rightarrow$ $\varepsilon V_\theta = V_{-\theta}$

$\Rightarrow$ $V_\theta + V_{-\theta}$ stable under ε, g $\therefore$ under F

Restrict $e^{i\theta} \neq \pm 1$, say

$0 < \theta < \pi$. $\Rightarrow V = \begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix}$ $g \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

You want to conclude that if $V = V_\theta \oplus V_{-\theta}$ then have isom.

$$(V, F, \varepsilon) \simeq \begin{pmatrix} V_+ \\ V_- \end{pmatrix}, \ \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ F = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}$$

$$\frac{1}{2i} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2i} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} i & -e^{i\theta} \\ e^{-i\theta} i & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

p'' $\quad \frac{1}{2i}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}g&0\\0&g^{-1}\end{pmatrix}\begin{pmatrix}i&-1\\i&1\end{pmatrix}$

$$= \frac{1}{2i}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}gi&-g\\g^{-1}i&+g^{-1}\end{pmatrix} = \frac{1}{2i}\begin{pmatrix}(g+g^{-1})i & -g+g^{-1}\\ g-g^{-1} & ig+ig^{-1}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{g+g^{-1}}{2} & -\frac{g-g^{-1}}{2i}\\ \frac{g-g^{-1}}{2i} & \frac{g+g^{-1}}{2}\end{pmatrix}$$

You have a vague feeling that the phase of $\frac{g-g^{-1}}{2}$ might be significant

---

Repeat.

Given a unitary f.d. rep V of $\langle F, \varepsilon\rangle$, get

$V=\begin{pmatrix}V_+\\V_-\end{pmatrix}$, $\varepsilon = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$, put $g=F\varepsilon$

Look at $F=1$ subspace W of V. Assume $g+1$ invertible. Then it should be true that

$W = \begin{pmatrix}1\\T\end{pmatrix}V_+ \qquad W^\perp = \begin{pmatrix}-T^*\\1\end{pmatrix}V_- \qquad g=\begin{pmatrix}1&1\\-1&1\end{pmatrix}X$

$F\begin{pmatrix}1&-T^*\\T&1\end{pmatrix} = \begin{pmatrix}1&-T^*\\T&1\end{pmatrix}\varepsilon \qquad F\varepsilon = \frac{1+X}{1-X}$

$X = \begin{pmatrix}1&-1\\1&1\end{pmatrix}g = \frac{g-1}{g+1}$

$$\frac{g+g^{-1}}{2} = \frac12\left(\frac{1+2X+X^2+1-2X+X^2}{1-X^2}\right) = \frac{1+X^2}{1-X^2}$$

$$\frac{g-g^{-1}}{2i} = \frac{1}{2i}\left(\frac{\cancel{1}+2X+\cancel{X^2}-\cancel{1}+2X-\cancel{X^2}}{1-X^2}\right) = \frac{1}{i}\,\frac{2X}{1-X^2}$$

$$g+1 = \frac{1+X}{1-X}+1 = \frac{2}{1-X}$$

8'' It's [redacted] time to organize the possibilities

Given a f.d. unitary rep of $\langle F, \varepsilon \rangle$ on V.

$$V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Put $g = F\varepsilon$ which is unitary. Let $V_\theta = \{\xi \in V \mid g\xi = e^{i\theta}\xi\}$. Since $\varepsilon : V_\theta \leftrightarrows V_{-\theta}$, $V_\theta + V_{-\theta}$ is stable under $\langle g, \varepsilon \rangle = \langle F, \varepsilon \rangle$.

Replace $e^{i\theta}$ by ζ, so V_ζ = g-eigenspace eigenvalue ζ

First case $\zeta = \zeta^{-1}$ $\quad \zeta = \pm 1$. Then $V_\zeta = V_{\zeta^{-1}}$ etc.

Otherwise $V = \begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix}$, $F \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$, $\varepsilon \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Now put back $\zeta = e^{i\theta}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$\cancel{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \quad \end{pmatrix}}$$

$$\frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \frac{1}{2i} \begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2i} ($$

r'' $\frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$

$$= \frac{1}{2i}\begin{pmatrix} i & -1 \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2i}\begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

V unit f.d. rep of $\langle F, \varepsilon \rangle$ $\qquad V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$F\varepsilon = g$ unitary $\qquad V_\theta = \{\xi \in V \mid g\xi = e^{i\theta}\xi\}$

$\varepsilon : V_\theta \overset{\sim}{\longleftrightarrow} V_{-\theta}$ $\qquad \begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix}$ stable under ε, g

$g \longmapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\qquad \varepsilon \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad$?

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where to start. Think of ε fixed, and $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and

$$g_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

5''

Now start again. Study unitary repn (V, F, ε). Write $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ with $\varepsilon \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Put $g = F\varepsilon$. g is unitary.

Put $V_\theta = \{ \xi \in V \mid g\xi = e^{i\theta}\xi \}$. $\varepsilon : V_\theta \xrightarrow{\sim} V_{-\theta}$

Let ξ be a unit vector in V_θ. Then $\mathbb{C}\xi + \mathbb{C}\varepsilon\xi \subset V$ stable under ε, g, hence also F. irred rep. for $\theta \in (0, \pi)$. ~~If $e^{i\theta} = \pm 1$,~~ ~~then $\xi, \varepsilon\xi$~~

Suppose $g\xi = \xi$. i.e. $\xi \in V_0$ then $\varepsilon\xi \in V_0$. So g fixes $\xi, \varepsilon\xi$. ~~If g~~ ~~want an irred repn~~ So $g = 1$ and $F = \varepsilon = \pm 1$ on $\mathbb{C}\xi + \mathbb{C}\varepsilon\xi$, get 2 characters. Sim. for $\theta = \pi$

$$g\xi = -\xi \implies g^{-1}\varepsilon\xi = -\varepsilon\xi \implies g\begin{pmatrix} \xi \\ \varepsilon\xi \end{pmatrix} = \begin{pmatrix} -\xi \\ -\varepsilon\xi \end{pmatrix}, \quad g = -1$$

So you have irred 2 diml reps.

$$g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{on } \mathbb{C}^2$$

$$F = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$$

which you can conj to

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} \cos & \sin \\ \sin & -\cos \end{pmatrix}$$

$$\begin{pmatrix} \frac{g+g^{-1}}{2} & -\frac{g-g^{-1}}{2i} \\ \frac{g-g^{-1}}{2i} & \frac{g+g^{-1}}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

$$\begin{pmatrix} g & g^{-1} \\ \frac{g}{i} & ig^{-1} \end{pmatrix}$$

t'' Given V f.d. unitary repn of $\langle F, \varepsilon \rangle$

Splitting $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$g = F\varepsilon$

$$\begin{pmatrix} \frac{1}{2}(g+g^{-1}) & -\frac{g-g^{-1}}{2i} \\ \frac{g-g^{-1}}{2i} & \frac{1}{2}(g+g^{-1}) \end{pmatrix} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

Is this well defined on V?

You are given on V two unitary operators $\varepsilon : \varepsilon^2 = 1$, and g s.t. $\varepsilon g \varepsilon^{-1} = g^{-1}$. From ε you get

$$\mathrm{Hom}(V, V) = \begin{matrix} \mathrm{Hom}(V_+, V_+) & \mathrm{Hom}(V_+, V_-) \\ \mathrm{Hom}(V_-, V_+) & \mathrm{Hom} \end{matrix}$$

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \frac{1+\varepsilon}{2} \\ \frac{1-\varepsilon}{2} \end{pmatrix} g \begin{pmatrix} \frac{1+\varepsilon}{2} & \frac{1-\varepsilon}{2} \end{pmatrix}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$\frac{1+\varepsilon}{2}\, g\, \frac{1+\varepsilon}{2} = \cancel{\frac{g + \varepsilon g + g\varepsilon + g^{-1}}{4}}$$

It seems that you have too much.

Take V to be irreducible $V = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$ $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$g = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$

$$F = \begin{pmatrix} \frac{1}{2}(g+g^{-1}) & \frac{g-g^{-1}}{2i} \\ \frac{g-g^{-1}}{2i} & -\frac{1}{2}(g+g^{-1}) \end{pmatrix}$$

$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

$\frac{1+\cos\theta}{2} = \cos^2\frac{\theta}{2}$

$\frac{1-\cos\theta}{2} = \sin^2\frac{\theta}{2}$

$g = \frac{1+X}{1-X}$ $g^{1/2} = \frac{1+X}{\sqrt{1-X^2}}$

u''

$$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon = g \qquad F = \frac{1+X}{1-X}\varepsilon$$

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} \qquad g^{1/2}\varepsilon g^{-1/2} = g\varepsilon = F$$

If
$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad \text{then} \quad F = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\frac{1+F}{2} = \begin{pmatrix} \cos^2\frac{\theta}{2} & \sin\theta/2\,\cos\theta/2 \\ \cos\frac{\theta}{2}\sin\frac{\theta}{2} & \sin\theta/2\,\sin\theta/2 \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \end{pmatrix}$$

$$\frac{1+F}{2} = \begin{pmatrix} \frac{1}{2}\left(1+\frac{1}{2}(g+g^{-1})\right) & \frac{1}{2}\frac{g-g^{-1}}{2i} \\ \frac{1}{2}\frac{g-g^{-1}}{2i} & \frac{1}{2}\left(1-\frac{1}{2}(g+g^{-1})\right) \end{pmatrix} \begin{pmatrix} \frac{g^{1/2}+g^{-1/2}}{2} \\ \frac{g^{1/2}-g^{-1/2}}{2i} \end{pmatrix} \begin{pmatrix} \frac{g^{1/2}+g^{-1/2}}{2} & \frac{g^{1/2}-g^{-1/2}}{2i} \end{pmatrix}$$

$$\frac{g^{1/2}+g^{-1/2}}{2} = \frac{1+X+1-X}{2(1-X^2)^{1/2}} = \frac{1}{\sqrt{1-X^2}}$$

$$\frac{g^{1/2}-g^{-1/2}}{2i} = \frac{1}{2i}\frac{1+X-1+X}{(1-X^2)^{1/2}} = \frac{1}{i}\frac{X}{\sqrt{1-X^2}}$$

$$\begin{pmatrix} 1 \\ \frac{1}{i}X \end{pmatrix} \frac{1}{1-X^2} \begin{pmatrix} 1 & \frac{1}{i}X \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ T \end{pmatrix} \quad 1$$

V'' So what ~~...~~ are you doing.

Repn unitary of $\langle F, \varepsilon \rangle$ on V, write $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, ~~...~~ have $g = F\varepsilon$ unitary

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\quad g \quad} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$g = \underbrace{\frac{g+g^{-1}}{2}}_{\text{even}} + i \underbrace{\frac{g-g^{-1}}{2i}}_{\text{odd}}$$

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \iota_+^* \\ \iota_-^* \end{pmatrix}} V \xleftarrow{\quad g \quad} V \xleftarrow{(\iota_+ \ \iota_-)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$$\begin{pmatrix} \iota_+^* g \iota_+ & \iota_+^* g \iota_- \\ \iota_-^* g \iota_+ & \iota_-^* g \iota_- \end{pmatrix}$$

ι_+

~~...~~ $\iota_\pm$ $\iota_\pm^*$ should satisf orthog relns. + comp

$$\begin{pmatrix} \iota_+^* \\ \iota_-^* \end{pmatrix} (\iota_+ \ \iota_-) = \begin{pmatrix} 1_+ & 0 \\ 0 & 1_- \end{pmatrix}$$

$$(\iota_+ \ \iota_-) \begin{pmatrix} \iota_+^* \\ \iota_-^* \end{pmatrix} = 1_V$$

~~...~~ Try again. Unitary repn of $\langle F, \varepsilon \rangle$ on V. Assume $\frac{g+g^{-1}}{2} = (\cos\theta)\mathrm{Id}$ where $\theta \in (0, \pi)$. Does it follow that $\frac{g-g^{-1}}{2i} = (\sin\theta)\mathrm{Id}$?

Seems not. Let's understand this well.

$$V = \mathbb{C}^2 \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} \cos\theta & +\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

W''

Consider again a unitary repn of $\langle F, \varepsilon \rangle$ on V finite dimensional. Use ε to get splitting

$V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ with $\varepsilon \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. ~~Let~~ Let

$V = W \oplus W^\perp$ where $F = +1$ on W
$\qquad\qquad\qquad\qquad\quad = -1$ on $W^\perp$

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow[\begin{pmatrix} a & b \\ c & d \end{pmatrix}]{} \begin{pmatrix} W \\ W^\perp \end{pmatrix} \longleftarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \longleftarrow \begin{pmatrix} W \\ W^\perp \end{pmatrix} \ ?$$

$$W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \twoheadrightarrow W^\perp$$

$$\longrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

Consider $V = \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}$ $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, let W be a line in V, $\quad W = \begin{pmatrix} 1 \\ T \end{pmatrix} \mathbb{C} \quad T \in \mathbb{C}$

$$W^\perp = \begin{pmatrix} -\bar{T} \\ 1 \end{pmatrix} \mathbb{C}$$

$F = \begin{cases} 1 & \text{on } W \\ -1 & \text{on } W^\perp \end{cases} \qquad F \begin{pmatrix} 1 & -\bar{T} \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -\bar{T} \\ T & 1 \end{pmatrix} \varepsilon$

$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X) \qquad \dfrac{1+X}{1-X} = F\varepsilon$

Why not simplify by doing real case.

x'' $\quad V = \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, W a line in V

$W = \mathbb{R}\begin{pmatrix} x \\ y \end{pmatrix}$ where $x^2+y^2=1$. $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} -x \\ -y \end{pmatrix}$ yield the same line. $W^\perp = \begin{pmatrix} -y \\ x \end{pmatrix}\mathbb{R}$

$$F\begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}\varepsilon = \varepsilon\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} = F\varepsilon = g$$

$$\parallel$$

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\frac{1}{x^2+y^2}$$

$$\frac{1+F}{2} = \begin{pmatrix} x \\ y \end{pmatrix}\frac{1}{x^2+y^2}\begin{pmatrix} x & y \end{pmatrix}$$

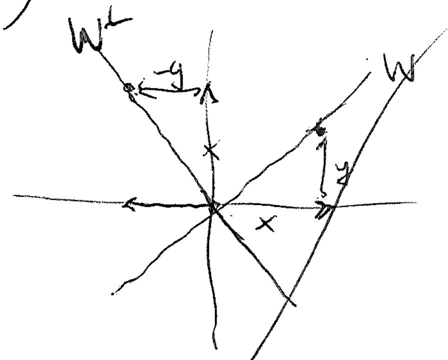
$$\frac{1-F}{2} = \begin{pmatrix} -y \\ x \end{pmatrix}\frac{1}{x^2+y^2}\begin{pmatrix} -y & x \end{pmatrix}$$

$$g = \begin{pmatrix} x^2-y^2 & -2xy \\ 2xy & x^2-y^2 \end{pmatrix}\frac{1}{x^2+y^2}$$

$$F = \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2} & \frac{+2xy}{x^2+y^2} \\ \frac{2xy}{x^2+y^2} & \frac{-x^2+y^2}{x^2+y^2} \end{pmatrix}$$

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x^2-y^2}{x^2+y^2}x + \frac{2xy}{x^2+y^2}y \\ \frac{2x^2y}{x^2+y^2} + \frac{-x^2+y^2}{x^2+y^2}y \end{pmatrix} = \frac{1}{x^2+y^2}\begin{pmatrix} x^3-y^2x+2xy^2 \\ 2x^2y-x^2y+y^3 \end{pmatrix}$$

y'' $\quad V = \begin{pmatrix}\mathbb{R}\\ \mathbb{R}\end{pmatrix} \quad \varepsilon = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ W line in V



$$W = \begin{pmatrix}x\\ y\end{pmatrix}\mathbb{R}, \quad W^\perp = \begin{pmatrix}-y\\ x\end{pmatrix}\mathbb{R}$$

What exactly do you want?

You have

$$W \xrightarrow{i} \begin{pmatrix}\text{[redacted]}\ \mathbb{R}\\ \mathbb{R}\end{pmatrix} \xleftarrow{j} W^\perp$$

On $V = \begin{pmatrix}\mathbb{R}\\ \mathbb{R}\end{pmatrix}$ you have the quadratic form

$$\begin{pmatrix}x\\ y\end{pmatrix}^* \begin{pmatrix}s & \\ & s^{-1}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = sx^2 + s^{-1}y^2 = A_s\begin{pmatrix}x\\ y\end{pmatrix}$$

[redacted] Let us equip W with the inner product induced from V, also $W^\perp$. This is just the restriction of A_s for $s=1$.

unit vector gen. W is

$$\frac{1}{\sqrt{x^2+y^2}}\begin{pmatrix}x\\ y\end{pmatrix}.$$

Restriction of A_s is

$$A_s\left(\frac{t}{\sqrt{x^2+y^2}}\begin{pmatrix}x\\ y\end{pmatrix}\right) = \frac{t^2}{x^2+y^2}\left(sx^2 + s^{-1}y^2\right)$$

What's important here is that you have

$$\frac{x^2}{x^2+y^2}\, s + \frac{y^2}{x^2+y^2}\, s^{-1}$$

are ≥ 0 + add to 1

$$\frac{1}{1+\left(\frac{y}{x}\right)^2}\, s + \frac{\left(\frac{y}{x}\right)^2}{1+\left(\frac{y}{x}\right)^2}\, s^{-1}$$

You recall introducing a frequency variable ω. Something like

$$\frac{s^2+\omega^2}{s(1+\omega^2)} = \frac{1}{1+\omega^2}\, s + \frac{\omega^2}{1+\omega^2}\, s^{-1}$$

Z″ So you learn that the C.T. picture of the Grassmannian is related to the frequency variable s.

For example: Given $W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ better $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+ \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

then you should an isometric

$$V_+ \xhookrightarrow{\begin{pmatrix} 1 \\ T \end{pmatrix}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad (1 \ T^*)\begin{pmatrix} 1 \\ T \end{pmatrix} = 1 + T^*T$$

$$V_+ \xhookrightarrow{\begin{pmatrix} 1 \\ T \end{pmatrix}(1+T^*T)^{-1/2}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad \begin{aligned} \alpha_+ &= (1+T^*T)^{-1/2} \\ \alpha_- &= T(1+T^*T)^{-1/2} \end{aligned}$$

check
$$\alpha_+^*\alpha_+ = (1+T^*T)^{-1/2}(1+T^*T)^{-1/2} = \frac{1}{1+T^*T}$$
$$\alpha_-^*\alpha_- = (1+T^*T)^{-1/2}T^*T(1+T^*T)^{-1/2} = \frac{T^*T}{1+T^*T}$$

~~What is then~~ The induced quadratic form on $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$ should be $(\alpha_+^* \ \alpha_-^*)\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \frac{s}{1+T^*T} + \frac{1}{s}\frac{T^*T}{1+T^*T}$

$$= \frac{s^2 + T^*T}{s(1+T^*T)}$$ think of T as $\tan\theta$

Then $$\begin{pmatrix} 1 \\ \tan\theta \end{pmatrix}\frac{1}{\sec\theta} = \begin{pmatrix} 1 \\ \tan\theta \end{pmatrix}\cos\theta = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Question. Given $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, F

$g = F\varepsilon$

$$\begin{pmatrix} \frac{g+g^{-1}}{2} & -\frac{g-g^{-1}}{2i} \\ \frac{g-g^{-1}}{2i} & \frac{g+g^{-1}}{2} \end{pmatrix}$$ this should have an obvious meaning on $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

Restrict to case $\frac{g+g^{-1}}{2} = (\cos\theta)\,Id$

Question Anything special about $F\varepsilon$ rather than εF