a'

How to deal with ~~[illegible]~~ this partial dynamics?

Return to structure of $\begin{pmatrix} C' \\ C_1 \end{pmatrix} = \begin{pmatrix} C'_C & C'_L \\ C_{1,C} & C_{1,L} \end{pmatrix}$ together

with

$$\boxed{\Gamma_{C,S} = \begin{pmatrix} 1 \\ C_S \end{pmatrix} C'_C \oplus \begin{pmatrix} L_S \\ 1 \end{pmatrix} C_{1,L}} = \left\{ \begin{pmatrix} V_C & L_S I_L \\ C_S V_C & I_L \end{pmatrix} \right\}$$

You would like to simplify the structure by saying that C' ~~[illegible]~~ has the structure of a polarized Euclidean space, i.e. C' is a Euclidean space equipped with an orthogonal splitting $C' = C'_C \overset{\perp}{\oplus} C'_L$. This should amount to making the ~~[illegible]~~ positive forms C, ~~[illegible]~~ L the identity. ~~[illegible]~~ Krein viewpoint?

Let E be a fin. dim Euclidean space, over $\mathbb{R}$ pos. symm. form. $H : E \longrightarrow E'$ $\quad {}^tH = H$

$|\xi|^2 = \langle \xi | H\xi \rangle > 0 \quad \xi \neq 0$

Start again E f.d. $\mathbb{R}$ vector space, E' dual space $H : E \longrightarrow E'$ symm. form i.e.

$\underbrace{\xi_1 \cdot H\xi_2}_{} = \xi_2 \cdot H\xi_1 \qquad {}^tH(\xi_1)(\xi_2) = \xi_1(H\xi_2)$

$H(\xi_2)(\xi_1) =$ [illegible]

b′ Somehow the basic idea is to change variables $\tilde{V}_c = C^{1/2} V_c$? ?

$$\tilde{I}_c = C^{+1/2} I_c = C^{-1/2} C s V_c = C^{1/2} s V_c \ ?$$

$$\begin{pmatrix} C^1 \\ C_1 \end{pmatrix} = \left\{ \begin{pmatrix} V_c \\ I_c \end{pmatrix} \right\} \supset \Gamma_1 = \left\{ \begin{pmatrix} V_c \\ C\,\text{[redacted]}\,V_c \end{pmatrix} \right\} = \begin{pmatrix} 1 \\ C \end{pmatrix} C^1_c$$

$$\Gamma_1 = \left\{ \begin{pmatrix} V_c \\ \underset{\overset{\shortparallel}{I_c}}{C V_c} \end{pmatrix} \right\} = \left\{ \begin{pmatrix} C^{-1/2} V_c \\ C^{+1/2} V_c \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \tilde{V}_c \\ \tilde{I}_c \end{pmatrix} \right\} \ ?$$

$$\Gamma_1 = \begin{pmatrix} 1 \\ C \end{pmatrix} C^1_c = \left\{ \begin{pmatrix} V_c \\ I_c = C V_c \end{pmatrix} \right\} = \left\{ \begin{pmatrix} C^{-1/2} V_c \\ C^{-1/2} C V_c \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} C^{-1/2} V_c \\ C^{1/2} V_c \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \tilde{V}_c \\ \tilde{I}_c \end{pmatrix} \right\}$$ still confused

---

$E \xrightarrow[\sim]{H} E'$ [redacted] > 0.

$\Gamma_H < \begin{pmatrix} E \\ E' \end{pmatrix}$ is Lagrangean ~~iff~~ $H = {}^tH$.

C' Sept 26 Focus upon the splitting

$$\begin{pmatrix} C^1 \\ C_1 \end{pmatrix} = \begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \oplus \underbrace{\Gamma_{C,s}}_{\begin{pmatrix} 1 \\ Cs \end{pmatrix} C^1_C} \oplus \underbrace{\Gamma_{L,s}}_{\begin{pmatrix} Ls \\ 1 \end{pmatrix} C_{1,L}}$$

for generic s. You expect this to hold for $Re(s) \neq 0$. Find a description of the ~~\[illegible\]~~ quotient $\begin{pmatrix} C^1 \\ C_1 \end{pmatrix} / \Gamma_s$, actually you want the map

$$\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \longrightarrow \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} / \Gamma_s$$

to be an isom.

You believe that this is equivalent to nondegeneracy of a quadratic form Q_s depending on s.

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "\iota"] & C^1 \arrow[r, two heads] \arrow[d, "Z_s^{-1}"] & H^1 \\ H_1 \arrow[r] & C_1 \arrow[r, "\iota^*"] & \bar{C}_0 \end{tikzcd}$$

This should be obvious. $Q_s = \iota^* Z_s^{-1} \iota$. Get subquotient thm. You need to understand this ~~carefully~~ well on the "Grassmannian" level. Ultimately your objects are real vector spaces with quadratic forms depending on s.

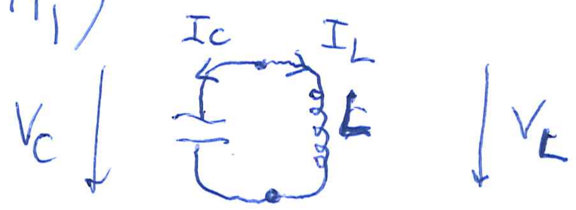
Observe that A_s on $\bar{C}^0$ has sings $0, \infty$ but on H^1 it has lots of interesting poles, which should give interesting oscillations.

d' Aim to

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

$$\bar{C}_0 \twoheadleftarrow C_1 \hookleftarrow H_1$$

back to old problem of why there's free motion on $\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix}$.



$$\left.\begin{aligned} I_C + I_L &= 0 \\ V_C &= V_L \end{aligned}\right\} \text{Kirchhoff}$$

$$CsV_C = I_C$$
$$LsI_L = V_L$$

so in terms of dominant variables V_C, I_L you get

$$CsV_C = I_C = -I_L = -\frac{1}{Ls}V_L = -\frac{1}{Ls}V_C$$

$$\Longrightarrow \left(Cs + \frac{1}{Ls}\right)V_C = 0.$$

You want to go from the symplectic picture to one with ~~Hilbert~~ Euclidean spaces and quadratic forms

Let's begin with

$$\begin{array}{ccccc} \bar{C}^0 & \xrightarrow{\iota} & C^1 & \xrightarrow{j} & H^1 \\ & & \downarrow Z_s^{-1} & & \\ \bar{C}_0 & \xleftarrow{\iota^*} & C_1 & \hookleftarrow & H_1 \end{array}$$

Review the Grass business, i.e. retract of a polarized Hilbert space

$$W \xleftarrow{(\beta_+ \ \beta_-)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W \qquad \beta_+\alpha_+ + \beta_-\alpha_- = 1_W$$

Require $\beta = \alpha^*$ i.e. $\beta_\pm = \alpha_\pm^*$

$$\alpha_+^*\alpha_+ + \alpha_-^*\alpha_- = 1_W$$

e'

Retract

$$W \xleftarrow{(\alpha_+^* \ \alpha_-^*)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W \qquad 1_W = \alpha_+^*\alpha_+ + \alpha_-^*\alpha_-$$

$h_+ = \alpha_+^*\alpha_+$ self-adjoint $0 \le h_+ \le 1$, Suppose $\alpha_+^*\alpha_+ = \lambda$ $0 < \lambda < 1$. Then $\alpha_-^*\alpha_- = 1-\lambda$. You find that ? ? ?

Go back to the Morita equivalence. ~~Retract of~~ Linear retract, $\mathbb{C}$-module retract of a polarized v.s. $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$. A simple ~~version~~ case of GNS

Given the retract:

$$W \xleftarrow{(\beta_+ \ \beta_-)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W \qquad \underbrace{\beta_+\alpha_+}_{h_+} + \underbrace{\beta_-\alpha_-}_{h_-} = 1$$

get W with $h_+ + h_- = 1_W$. Conversely given W ______, can ~~[illegible]~~ define

$$\begin{tikzcd} & V_+ \arrow[dl, "\beta_+"'] & \\ W & & W \arrow[ul, "\alpha_+"'] \arrow[dl, "\alpha_-"] \\ & V_- \arrow[ul, "\beta_-"] & \end{tikzcd}$$

$(\beta_\pm, V_\pm, \alpha_\pm)$ as canonical factorization of $h_\pm$.

~~[illegible]~~ In the Hilbert space situation you define $V_\pm$ via completion of W with $\langle w | h_\pm w \rangle$

f'

$$W \xleftarrow{(\lambda^{1/2}\ (1-\lambda)^{1/2})} \begin{pmatrix} W \\ W \end{pmatrix} \xleftarrow{\begin{pmatrix}\lambda^{1/2} \\ (1-\lambda)^{1/2}\end{pmatrix}} W$$

So you are in the situation where $W = V_+ = V_-$

Not good.

Let's discuss what might happen in general

Recall linear retract of a polarized v.s.

$\beta\alpha = 1$

$$W \xleftarrow{\beta = (\beta_+\ \beta_-)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\alpha = \begin{pmatrix}\alpha_+ \\ \alpha_-\end{pmatrix}} W$$

W has structure $1_W = h_+ + h_-$ $\qquad h_\pm = \beta_\pm \alpha_\pm$.

Given W with this partition, then possible dilations of W to a polarized v.s. corresp to factorizations of h_+, h_-. Hilbert version $\beta = \alpha^*$, have canon. fact $V_\pm$ = completion of W wrt $\langle w | h_\pm w \rangle$

Now you are concerned with not only W but also with $W^\perp = \mathrm{Ker}(\beta) = \mathrm{Coker}(\alpha)$. How to make progress?

Idea: $\Gamma_s \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$ is a correspondence from C^1 to C_1 defined for all $s \in \mathbb{P}_1$? Why? Split into L, C summands.

you have

$\Gamma_{L,s} = \begin{pmatrix} sL \\ 1 \end{pmatrix} C_1$ $\qquad$ L invertible

$\Gamma_{C,s} = \begin{pmatrix} 1 \\ sC \end{pmatrix} C^1$ $\qquad$ C " g′ Problems, Questions, ~~[illegible]~~ You want to review all you know about LC networks.

An LC network is a connected graph such that each edge is either a capacitor (having capacitance $\in (0,\infty)$) or an inductor (having inductance $\in (0,\infty)$).

The Kinematics of such a network depends only on the underlying graph. The vector space of unconstrained states of the network is the direct sum

$$\begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$$

of the v.space of Real valued 1-cochains and 1-chains. An elt. V of C^1 is a family of voltage drops indexed by the edges; an elt I of C_1 is a family of currents indexed by the edges. The natural pairing between 1-cochains and 1-chains yields the power of the state $\begin{pmatrix} V \\ I \end{pmatrix} \in \begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$

$$\text{power} = \sum_{\text{edges } \sigma} V_\sigma I_\sigma = V \cdot I$$

~~[illegible]~~

The ~~[illegible]~~ state space is has a symplectic bilinear form given by

$$\omega\left(\begin{pmatrix} V \\ I \end{pmatrix}, \begin{pmatrix} V' \\ I' \end{pmatrix}\right) = V \cdot I' - I \cdot V'$$

(also) $\exists$ a symmetric form $V \cdot I' + I \cdot V'$ equivs. to power as a quad. form

h' So far have been discussing the unconstrained state space associated [redacted] to the edges of the network.

~~Next you impose constraints~~

Next introduce constraints given by Kirchhoff's 2 laws: V is 1-coboundary i.e. $V = \delta\phi$, $\phi \in \bar{C}^0$

I is 1-cycle i.e. $\partial I = 0$

Get short [redacted] exact sequence + its dual:

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

$$\bar{C}_0 \twoheadleftarrow C_1 \hookleftarrow H_1$$

Can also write this

$$\underbrace{\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix}}_{\substack{\text{Lag subspace} \\ W.}} \hookrightarrow \underbrace{\begin{pmatrix} C^1 \\ C_1 \end{pmatrix}}_{\substack{\text{symplectic} \\ \text{v. space}}} \twoheadrightarrow \underbrace{\begin{pmatrix} H^1 \\ \bar{C}_0 \end{pmatrix}}_{\substack{\text{dual space} \\ W^*}}$$

to emphasize the symplectic structure. It's not clear that this is useful.

Dynamics: ~~Consider for a motion~~

For a capacitor C [redacted] a time-dependent ~~state~~ $\begin{pmatrix} V(t) \\ I(t) \end{pmatrix}$ satisfies the DE $C\dot{V} = I$, for L, a time dep state satisfies the DE $L\dot{I} = V$.

i' Use ~~L~~.T. to $\overset{\text{replace}}{\text{time}}$ dependent by frequency dependent states. ~~The~~

You have $\partial_t \longmapsto$ mult by s under the L.T.

so you get $\begin{cases} CsV = I & \text{for capacitor edge} \\ LsI = V & \text{for inductor edge} \end{cases}$

Now you can describe a free motion of the LC circuit, namely, it's a state $\begin{pmatrix} V(t) \\ I(t) \end{pmatrix}$ depending on t whose ~~value~~ values lie in the Kirchhoff space, in other words a ~~[illegible]~~ path in the Kirchhoff space ~~constraints~~. Also it should satisfy the dynamical conditions $C\dot{V} = I$, $L\dot{I} = V$ for C (resp. L) edges.

Better to look first for exponential solutions $e^{st}\begin{pmatrix} \hat{V} \\ \hat{I} \end{pmatrix}$, and later worry about completeness.

Actually it shouldn't be hard. If $\begin{pmatrix} V(t) \\ I(t) \end{pmatrix}$ is a solution, then so is $\begin{pmatrix} V(t-t_0) \\ I(t-t_0) \end{pmatrix}$??

Assuming the path $\begin{pmatrix} V(t) \\ I(t) \end{pmatrix}$ is ~~differentiable~~ C^∞ in t, then you can differentiate to get others. ??

j' Recap. The basic object to study is an exponential solution $e^{st}\begin{pmatrix}V\\I\end{pmatrix}$ satisfying the 2 Kirchhoff relations and the dynamical relations:

$V \in \bar{C}^0$ $\qquad CsV_C = I_C \qquad \begin{pmatrix}V_C\\I_C\end{pmatrix} \in \begin{pmatrix}C^1_C\\C_{1C}\end{pmatrix}$

$I \in H_1$ $\qquad LsI_L = V_L \qquad \begin{pmatrix}V_L\\I_L\end{pmatrix} \in \begin{pmatrix}C^1_L\\C_{1L}\end{pmatrix}$

Draw this

$$\bar{C}^0 \hookrightarrow C^1_C \oplus C^1_L \twoheadrightarrow H^1$$

$$\begin{pmatrix}1\\Cs\end{pmatrix}C^1_C \oplus \begin{pmatrix}Ls\\1\end{pmatrix}C_{1L} = \Gamma_s$$

$$\bar{C}_0 \twoheadleftarrow C_{1C} \oplus C_{1L} \hookleftarrow H_1$$

Meaning: An exp soln $e^{st}\begin{pmatrix}V\\I\end{pmatrix}$ amounts to a $V \in \bar{C}^0$, $I \in H_1$ such that the image of V in C^1 and the image of I in H_1 are related by an elt of Γ_s

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] \arrow[d, leftrightarrow, "\Gamma_s"] & H^1 \\
\bar{C}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook']
\end{tikzcd}$$

At this point you understand exp. solutions in terms of the symplectic picture. This is where you have been stuck for a long time.

The way out of the confusion is to restrict to the voltage (cochain) side, since the current (chain) side can be reconstructed by duality. Γ_s is equivalent to a quad form k'

Things you need ~~[illegible]~~ to know

What kind of resolvent you get when $\Gamma_s \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$ is transversal complementary to $\begin{pmatrix} \tilde{C}^0 \\ H_1 \end{pmatrix}$

You need to check this condition is equivalent to the restriction of the energy quadratic form is non degenerate on $\tilde{C}^0$

Review: Given a connected LC network, a graph whose edges are either capacitors or inductors. short exact sequence

$$\tilde{C}^0 \longrightarrow C^1 \longrightarrow H^1$$

$$C^1 = \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} \ni \begin{pmatrix} V_C \\ V_L \end{pmatrix} \qquad {}^tV_C C V_C \qquad {}^tV_L L^{-1} V_L$$

have quadratic form (s dep) $Q_s(V) = s\boxed{C V_C^2} + s^{-1}\boxed{L^{-1} V_L^2}$

Q_s should be equivalent to the correspondence $\Gamma_s \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$

$$\Gamma_{C,s} = \begin{pmatrix} 1 \\ sC \end{pmatrix} C^1_C \quad , \quad \Gamma_{L,s} = \begin{pmatrix} sL \\ 1 \end{pmatrix} C_{1L}$$

Note: this family of correspondences Γ_s is a vector ~~sub~~bundle over $\mathbb{P}^1$ of the trivial bundles with fibre $\begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$.

~~You want to ~~conn~~ link the pictures~~

$$\begin{tikzcd} \tilde{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] & H^1 \\ & \Gamma_s \arrow[u] \arrow[d] & \\ \tilde{C}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook] \end{tikzcd}$$

ℓ' You have Q_s on C^1 which ~~[illegible]~~ ~~quadratic form on an C^1 for~~ generically splits the short exact sequence $\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$ orthogonally + yields quad·form on $\bar{C}^0$, H^1.

---

go over equivalence ?

$$\begin{tikzcd}
 & \bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] & H^1 \\
\Gamma_s \arrow[urr] \arrow[drr] & & & \\
 & \bar{C}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook']
\end{tikzcd}$$

$$\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \cap \Gamma_s = 0 \iff \forall V^{\neq 0} \in C^1 \quad \langle V, \Gamma_s V \rangle \neq 0$$

~~[illegible]~~

You need to calculate

$$\begin{pmatrix} C^1 \\ C_1 \end{pmatrix} \Big/ \Gamma_s \overset{?}{\cong} C^1 \text{ equipped with power quadratic form}$$

Review a little

~~What is the meaning of~~

You believe that $\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \cap \Gamma_s$ is the space of exponential solutions of the 2e linear equations with time behavior e^{st}. This seems obvious.

$$\mathrm{Sing} = \left\{ s \in \mathbb{P}^1 \,\middle|\, \begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \cap \Gamma_s \neq 0 \right\}$$

$$\mathrm{Reg} = \left\{ s \in \mathbb{P}^1 \,\middle|\, \qquad = 0 \right\}$$

For s reg you get a splitting

$$\begin{pmatrix} C^1 \\ C_1 \end{pmatrix} = \begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \oplus \Gamma_s$$

which ~~should~~ might mean that you get an induced m′ quadratic form on $\begin{pmatrix} \tilde{C}^0 \\ H_1 \end{pmatrix}$ © Ala

conn LC network

$$
\begin{tikzcd}
\tilde{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] & H^1 \\
& \Gamma_s \arrow[u] \arrow[d, bend right] & \\
\bar{C}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook]
\end{tikzcd}
$$

$$
\begin{pmatrix} \tilde{C}^0 \\ H_1 \end{pmatrix} \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} \supset \Gamma_s
$$

You believe $\begin{pmatrix} \tilde{C}^0 \\ H_1 \end{pmatrix} \cap \Gamma_s$ is the space of exponential solutions with time dependence e^{st}

$s \in \mathrm{Reg}$ if $\begin{pmatrix} \tilde{C}^0 \\ H_1 \end{pmatrix} \cap \Gamma_s = 0$ in which case one has an isom

$$
\begin{pmatrix} \tilde{C}^0 \\ H_1 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} \Big/ \Gamma_s
$$

Natural question

Given $A : V \longrightarrow V^*$ symmetric, what is

$$
\begin{pmatrix} V \\ V^* \end{pmatrix} \Big/ \begin{pmatrix} 1 \\ A \end{pmatrix} V \qquad \begin{matrix} v \\ v \end{matrix}
$$

n'

$$V^* \xleftarrow{(-A\ 1)} \begin{pmatrix}V\\ V^*\end{pmatrix} \xleftarrow{\begin{pmatrix}1\\ A\end{pmatrix}} V$$

$$\begin{tikzcd}
 & V \arrow[d, hook, "\begin{pmatrix}1\\ 0\end{pmatrix}"'] \arrow[dr, "-A"] & \\
V \arrow[r, "\begin{pmatrix}1\\ A\end{pmatrix}"] \arrow[dr, "A"'] & \begin{pmatrix}V\\ V^*\end{pmatrix} \arrow[r, "(-A\ 1)"] \arrow[d, "(0\ 1)"] & V^* \\
 & V^* &
\end{tikzcd}$$

It looks like $\begin{pmatrix}V\\ V^*\end{pmatrix}\Big/\Gamma_A \xrightarrow[\sim]{(-A\ 1)} V^*$

better would be

$$V \xrightarrow{\begin{pmatrix}1\\ 0\end{pmatrix}} \begin{pmatrix}V\\ V^*\end{pmatrix} \twoheadrightarrow \begin{pmatrix}V\\ V^*\end{pmatrix}\Big/\Gamma_A$$

$$\begin{pmatrix}\bar{C}^0\\ H_1\end{pmatrix} \longrightarrow \begin{pmatrix}C^1\\ C_1\end{pmatrix}\Big/\Gamma_s$$

---

what you want to know is that there is a commutative diag

$$\begin{tikzcd}
V \arrow[d, "A"'] \arrow[dr, "\sim"] & \\
 & \begin{pmatrix}V\\ V^*\end{pmatrix}\Big/\Gamma_A \\
V^* \arrow[ur] &
\end{tikzcd}$$

0′ Given $A : V \longrightarrow V^*$ symm.

To understand $\begin{pmatrix} V \\ V^* \end{pmatrix} \Big/ \Gamma_A .$

$$
\begin{tikzcd}
 & V \arrow[d, "\binom{1}{0}"] & \\
V \arrow[r, "\binom{1}{A}"] & \begin{pmatrix} V \\ V^* \end{pmatrix} \arrow[r, two heads] \arrow[d, "(0\ 1)"] & \begin{pmatrix} V \\ V^* \end{pmatrix} \Big/ \Gamma_A \\
 & V^* &
\end{tikzcd}
$$

$$
V \longleftarrow \begin{pmatrix} V \\ V^* \end{pmatrix} \xleftarrow{\binom{1}{A}} V
$$

but: $(1, 0)$

no obvious complement$_A$ $(0\ A^{-1})$

$$
\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} \Big/ \Gamma_A \xrightarrow[\sim]{(-A\ \ 1)} C_1
$$

in Reg case

You get then an isom $\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \xrightarrow{\sim} C_1$

given by $\begin{pmatrix} V^0 \\ I^\ell \end{pmatrix} \longmapsto -AV^0 + I^\ell .$ In other words

$$
\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] \arrow[d, "\wr"'] & C^1 \arrow[r, two heads] \arrow[d, "A"] & H^1 \\
\bar{E}_0 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook']
\end{tikzcd}
$$

Also have $A^{-1} : C_1 \to C^1$

so you get an induced quad form on H_1

P'

You still don't understand the ~~dynamics~~ motion of an LC network: You want to derive, construct it from the singularities of a frequency dependent quadratic form.

So let's study the simple circuit $V_C \downarrow \; I_C \; I_L \; \downarrow V_L$

$$\begin{array}{ccccc} \bar{C}^0 & \hookrightarrow & C^1 & \longrightarrow & H^1 \\ & & \downarrow A_s & & \\ \bar{C}_0 & \twoheadleftarrow & C_1^{*} & \longleftarrow & H_1 \end{array}$$

$\{V_0\}$ $\quad \{(V_C, V_L)\}$ $\quad V^\lambda$

I_0 $\quad \{(I_C, I_L)\}$ $\quad I_\lambda$

Notation $\quad V_0 \quad V_C \quad V_L \quad V_\lambda$

$\qquad\qquad I_0 \quad I_C \quad I_L \quad I_\lambda$

$V_0 \longmapsto (V_0, V_0)$ $\qquad V_C, V_L \mapsto V_\lambda = V_C - V_L$

$$\begin{array}{cc} V_C & V_L \\ \downarrow & \downarrow \\ CsV_C & \frac{1}{Ls}V_L \\ I_C & I_L \end{array}$$

$I_0 = I_C + I_L \longleftarrow (I_C, I_L)$

$\Gamma_s = \binom{1}{A_s} C^1$

$$\left(\iota V \middle| \begin{matrix} I_C \\ I_L \end{matrix}\right) = \left(\begin{matrix} V \\ V \end{matrix} \middle| \begin{matrix} I_C \\ I_L \end{matrix}\right) = \left(V \middle| I_C + I_L\right) = \left(V \middle| \iota^* \begin{pmatrix} I_C \\ I_L \end{pmatrix}\right)$$

$$\left(j I_\lambda \middle| \begin{matrix} V_C \\ V_L \end{matrix}\right) = \left(I_\lambda \middle| j^* \begin{pmatrix} V_C \\ V_L \end{pmatrix}\right) = \left(I_\lambda \middle| V_C - V_L\right)$$

g'

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "\iota"] & C^1 \arrow[r, two heads, "j"] \arrow[d, "A_s"] & H^1 \\ \bar{C}_0 & C_1 \arrow[l, two heads, "\iota^*"'] & H_1 \arrow[l, leftrightarrow, "j^*"'] \end{tikzcd}$$

$\iota V = \begin{pmatrix} V \\ V \end{pmatrix}$

$j\begin{pmatrix} V_C \\ V_L \end{pmatrix} = V_C - V_L$

$j^*(I_\lambda) = \begin{pmatrix} I_C \\ -I_L \end{pmatrix}$

$A_s\begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} Cs V_C \\ \frac{1}{Ls} V_L \end{pmatrix}$

$\iota^*\begin{pmatrix} I_C \\ I_L \end{pmatrix} = I_C + I_L$

V_E

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "\iota"] & C^1 \arrow[r, two heads, "j^*"] \arrow[d, "A_s"] & H^1 \\ \bar{C}_0 & C_1 \arrow[l, two heads, "\iota^*"'] & H_1 \ni I_\lambda \arrow[l, leftrightarrow, "j^*"'] \end{tikzcd}$$

$\iota V = \begin{pmatrix} V \\ V \end{pmatrix}$ $\quad j^*\begin{pmatrix} V_C \\ V_L \end{pmatrix} = V_C - V_L$

$\iota^*\begin{pmatrix} I_C \\ I_L \end{pmatrix} = I_C + I_L$

$jI_\lambda = \begin{pmatrix} I_C \\ -I_L \end{pmatrix}$

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook, "\iota"] & C^1 \arrow[r, two heads, "j^*"] \arrow[d, bend right, "A_s"'] & H^1 \\ \bar{C}_0 & C_1 \arrow[l, two heads, "\iota^*"'] & H_1 \arrow[l, leftrightarrow, "j"'] \end{tikzcd}$$

$\bar{C}^0 = \{V_0 \in \mathbb{C}\}$, $\quad C^1 = \left\{\begin{pmatrix} V_C \\ V_L \end{pmatrix} \in \mathbb{C}^2\right\}$, $\quad H^1 = \{V_\lambda \in \mathbb{C}\}$

$\bar{C}_0 = \{I_0 \in \mathbb{C}\}$, $\quad C_1 = \left\{\begin{pmatrix} I_C \\ I_L \end{pmatrix} \in \mathbb{C}^2\right\}$, $\quad H_1 = \{I_\lambda \in \mathbb{C}\}$

$\iota V_0 = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix}$

$j^*\begin{pmatrix} V_C \\ V_L \end{pmatrix} = V_C - V_L$

$\iota^*\begin{pmatrix} I_C \\ I_L \end{pmatrix} = I_C + I_L$

$jI_\lambda = \begin{pmatrix} I_\lambda \\ -I_\lambda \end{pmatrix}$

$$\left(\iota V_0 \,\middle|\, \begin{pmatrix} I_C \\ I_L \end{pmatrix}\right) = \left(V_0 \,\middle|\, \iota^*\begin{pmatrix} I_C \\ I_L \end{pmatrix}\right) = \left(V_0 \,\middle|\, I_C + I_L\right) = \left(V_0 \,\middle|\, \iota^*\begin{pmatrix} I_C \\ I_L \end{pmatrix}\right)$$

$$\left(jI_\lambda \,\middle|\, \begin{pmatrix} V_C \\ V_L \end{pmatrix}\right) = \left(\begin{pmatrix} I_\lambda \\ -I_\lambda \end{pmatrix} \,\middle|\, \begin{pmatrix} V_C \\ V_L \end{pmatrix}\right) = \left(I_\lambda \,\middle|\, V_C - V_L\right) = \left(I_\lambda \,\middle|\, j^*\begin{pmatrix} V_C \\ V_L \end{pmatrix}\right)$$

$$A_s\begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} Cs V_C \\ (Ls)^{-1} V_L \end{pmatrix} \in C_1$$

r'

Next you want to understand the splitting for s regular.

$$\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \hookrightarrow \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} \hookleftarrow \Gamma_s$$

$$\downarrow$$

$$\begin{pmatrix} H^1 \\ \bar{C}_0 \end{pmatrix}$$

Your aim is to ~~[illegible]~~ prove for generic s that $\begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$ is the direct sum of the subspaces $\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix}$ and Γ_s

Viewpoint? First idea: Have short exact seq

$$\begin{tikzcd} \begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix} \arrow[r, hook] & \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} \arrow[r, two heads] & \begin{pmatrix} H^1 \\ \bar{C}_0 \end{pmatrix} \\ & \Gamma_s \arrow[u, hook] \arrow[ur, "\text{isom?}"'] & \end{tikzcd}$$

and subspace of the total space

Second idea: You have a short exact sequence

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$$

and a quadratic form A_s on C^1. Suppose s real > 0. Then A_s is positive definite on C^1, ~~there's a canonical splitting as a Euclidean space~~

S′ C^1 is a Euclidean space, C^1 splits orthogonally into $\bar{C}^0$ and its orthogonal complement, which is canon isom. to H^1:

$$\bar{C}^0 \xhookrightarrow{i} C^1 \xhookleftarrow{j} (\bar{C}^0)^\perp \simeq H^1$$

$$\begin{tikzcd}
 & (\bar{C}^0)^\perp \arrow[d, hook, "j"] \arrow[dr, dash, "\sim"] & \\
\bar{C}^0 \arrow[r, hook, "i"] & C^1 \arrow[r, two heads] & H^1
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{C}_0 \arrow[dr, dash, "\sim"'] & C_1 \arrow[l, two heads, "i^*"'] \arrow[d, "j^*"] & H_1 \arrow[l, hook'] \\
 & H_1^\perp &
\end{tikzcd}$$

You are still trying to acheive the splitting of $\begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$ into $\begin{pmatrix} \bar{C}^0 \\ H_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ A_s \end{pmatrix} C^1 = \begin{pmatrix} A_s^{-1} \\ 1 \end{pmatrix} C_1$.

Not really, because you are exploring the 2nd idea: Exact seq $\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H^1$ + quadratic form A_s on C^1, which is pos. for $s > 0$. You know the exact seq splits canonically, and further that $\bar{C}^0, H^1$ inherit positive quad. forms compatible with the splitting.

Where do you get a Green's function from all this?

t'

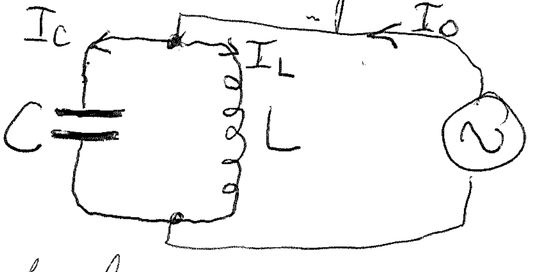
$$\begin{array}{ccc} V_0 & \overset{\iota}{\longmapsto} & \begin{pmatrix} V_0 = V_c \\ V_0 = V_L \end{pmatrix} \\ & & \Big\downarrow A_s \\ \left(Cs + \frac{1}{Ls}\right) V_0 & \overset{\iota^*}{\longleftarrow} & \begin{pmatrix} Cs\,V_0 \\ (Ls)^{-1} V_0 \end{pmatrix} \end{array}$$

$$\begin{array}{ccc} \begin{pmatrix} V_c = \frac{1}{Cs} I_\lambda \\ V_L = -Ls\, I_\lambda \end{pmatrix} & \overset{j^*}{\longrightarrow} & \left(\frac{1}{Cs} + Ls\right) I_\lambda = V_\lambda \\ \Big\uparrow A_s^{-1} & & \\ \begin{pmatrix} I_c = I_\lambda \\ I_L = -I_\lambda \end{pmatrix} & \overset{j}{\longleftarrow} & I_\lambda \end{array}$$

[redacted] Review the situation. [redacted] Consider exact seq + [redacted] pos quad form A_s $(s > 0)$

$$\bar{C}^0 \hookrightarrow C^1 \twoheadrightarrow H_1$$

You want a suitable Green's fn. [redacted] What do you mean? Example. Consider the simple oscillator



$V_0 = V_c = V_L$

and apply emf of frequency s.

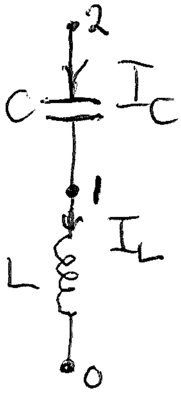
$$I_0 = I_c + I_L$$

$$I_c = Cs\,V_0, \quad I_L = \frac{1}{Ls} V_0$$

$$I_0 = \left(Cs + \frac{1}{Ls}\right) V_0$$

[redacted]

u'



$$\bar{C}^0 \rightleftarrows C^1 = \left\{ \begin{pmatrix} V_c \\ V_L \end{pmatrix} \right\}$$

$$\bar{C}_0 \rightleftarrows C_1 = \left\{ \begin{pmatrix} I_C \\ I_L \end{pmatrix} \right\}$$

$$\partial[C] = [2] - [1]$$
$$\partial[L] = [1] - [0]$$

Yesterdays lesson: the free oscillations, normal modes result from the singularities introduced is pushing the quadratic form A_s ~~down~~ on C^1 down to H^1.

Review formulas. $0 \longrightarrow X \xrightarrow{\iota} Y \xrightarrow{j} Z \longrightarrow 0$

y^tAy quad form on Y, $x^t\iota^tA\iota x$ restriction of y^tAy to X. Assume $\iota^tA\iota$ is nondegenerate i.e. $X \xrightarrow{\sim} X^*$. Then there ~~should be~~ is a push forward quad form j_*A ~~to define~~ whose value at z is the stationary value of A on the affine space $j^{-1}z$.

Pick $y_0 \in j^{-1}z$ $\qquad j^{-1}z = \{ y_0 + \iota x \mid x \in X \}$.

$$(y_0 + \iota x)^t A (y_0 + \iota x) = y_0^t A y_0 + x^t \iota^t A y_0 + y_0^t A \iota x + x^t \iota^t A \iota x$$

$$0 \to X \to Y \to Z \to 0$$

variation $\delta y = y + i\delta x$

$$\cancel{\delta(y^tAy) = \delta x^t i^t A y + y^t A i \delta x = 0}$$

~~if y is a stationary point~~

$$\cancel{\delta(y^tAy) = (y+i\delta x)^t A y + y^t A(y+i\delta x) = 2(y^tAy + y^tAi\delta x)}$$

~~Let y_0~~

$$y + \delta y = y + i\delta x$$

$$\delta(y^tAy) = y^tAi\delta x + (i\delta x)^tAy = 2\,\delta x^t i^t A y$$

Assume zero for all δx $\Rightarrow$ $i^tAy = 0$

says y is $\perp$ to iX.

Suppose $y = y_0 + ix$

$$i^tAy_0 + i^tAix = 0$$

$$\Rightarrow \quad x = -(i^tAi)^{-1} i^tAy_0$$

so the critical point is

$$y = y_0 - i(i^tAi)^{-1}i^tAy_0 \qquad i^tAy = 0$$

W'

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook, "\iota"] & C^1 \arrow[r, two heads, "j^t"] \arrow[d, "A"] & H^1 \\
\bar{C}_0 & C_1 \arrow[l, two heads, "\iota^t"'] & H_1 \arrow[l, hook', "j"']
\end{tikzcd}$$

$y \in C^1$ sat $\iota^t A y = 0$ means $Ay = j\gamma$

$\Longrightarrow$ $y = A^{-1} j \gamma$

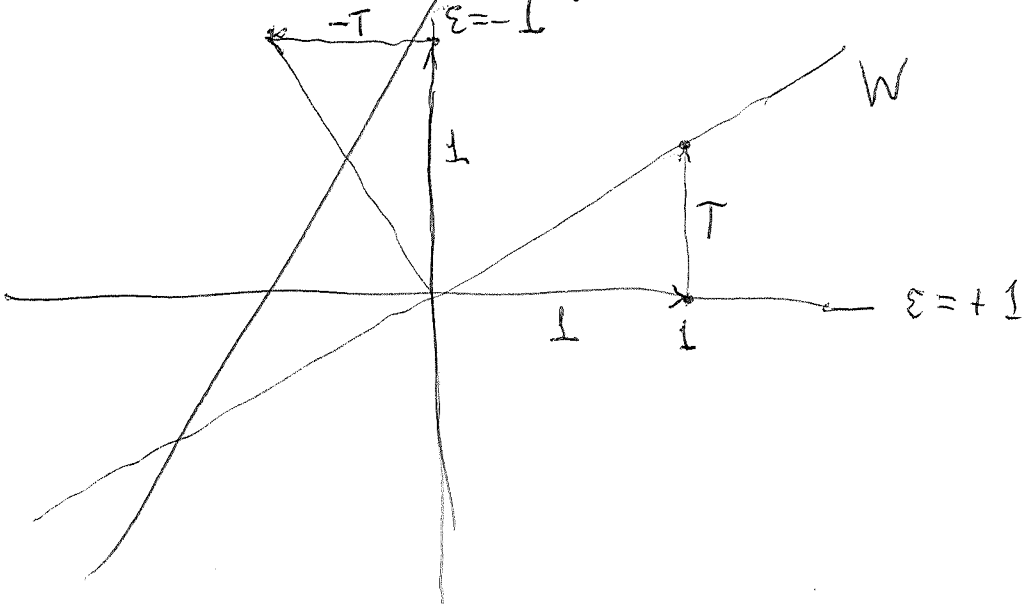
$\Longrightarrow$ $j^t y = \left(j^t A^{-1} j\right) \gamma$

$$W \xhookrightarrow{\iota} \begin{pmatrix} V_+^{*} \\ \\ V_- \end{pmatrix} \xrightarrow{j^*} W^\perp$$

Representation of dihedral group gen by F, ε

$g = F\varepsilon$? Put into words what you want.

You want to ~~split~~ decompose the rep of $\langle F, \varepsilon \rangle$ into irreducibles. What are the irreducibles?



x'

Grassmannian details.

$$\begin{tikzcd} W \arrow[r, hook, "\iota"'] & \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \arrow[l, bend right, "\iota^*"'] \arrow[r, two heads, "j^*"] & W^\perp \arrow[l, bend left, "j"] \end{tikzcd}$$

$$\iota = \begin{pmatrix} \iota_+ \\ \iota_- \end{pmatrix} \qquad j = \begin{pmatrix} j_+ \\ j_- \end{pmatrix} \qquad \begin{aligned} \iota^* j &= 0 \\ j^* \iota &= 0 \end{aligned}$$

$$\iota^*\iota = \begin{pmatrix} \iota_+^* & \iota_-^* \end{pmatrix}\begin{pmatrix} \iota_+ \\ \iota_- \end{pmatrix} = \iota_+^*\iota_+ + \iota_-^*\iota_- = 1_W$$

$$j^*j = \begin{pmatrix} j_+^* & j_-^* \end{pmatrix}\begin{pmatrix} j_+ \\ j_- \end{pmatrix} = j_+^*j_+ + j_-^*j_- = 1_{W^\perp}$$

$$\begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \iota_+ & j_+ \\ \iota_- & j_- \end{pmatrix}} \begin{pmatrix} W \\ W^\perp \end{pmatrix} \xleftarrow{\begin{pmatrix} \iota_+^* & \iota_-^* \\ j_+^* & j_-^* \end{pmatrix}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \quad ?$$

SL_2 stuff

$$W \twoheadleftarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xhookleftarrow{\;j\;} W^\perp$$

Recall the general case

$$\begin{pmatrix} X \\ Y \end{pmatrix} \xleftarrow{\sim} \begin{pmatrix} U \\ V \end{pmatrix}$$

should be the same as ~~an exact~~ a short exact sequence equipped with a contraction.

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] & Y \arrow[r] \arrow[l, bend right] & Z \arrow[r] \arrow[l, bend right] & 0 \end{tikzcd}$$

y'

$$V \longleftarrow \begin{pmatrix} X \\ Y \end{pmatrix} \longleftarrow U$$ [redacted]

TFAE: isom
$$\begin{pmatrix} U \\ V \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} X \\ Y \end{pmatrix} \qquad \begin{pmatrix} a \\ b \end{pmatrix}\begin{pmatrix} h & i \end{pmatrix} + \begin{pmatrix} f \\ g \end{pmatrix}\begin{pmatrix} c & d \end{pmatrix}$$

htpy equiv. of length 1 complexes

Review this quickly. Suppose given a short exact sequence

$$\begin{tikzcd} V \arrow[r, bend right, "\begin{pmatrix} f \\ g \end{pmatrix}"'] & \begin{pmatrix} X \\ Y \end{pmatrix} \arrow[l, "\begin{pmatrix} c & d \end{pmatrix}"'] \arrow[r, bend right, "\begin{pmatrix} h & i \end{pmatrix}"'] & U \arrow[l, "\begin{pmatrix} a \\ b \end{pmatrix}"'] \end{tikzcd}$$

together with a splitting:

$$ca + db = 0 \checkmark \qquad ha + ib = 1_U \checkmark$$
$$hf + ig = 0 \checkmark \qquad cf + dg = 1_V \checkmark$$

$$1_X = fc + ah \qquad 0 = gc + bh$$
$$1_Y = gd + bi \qquad 0 = fd + ai$$

$$\begin{tikzcd} \begin{pmatrix} X \\ Y \end{pmatrix} & \begin{pmatrix} U \\ V \end{pmatrix} \arrow[l, "\begin{pmatrix} a & f \\ b & g \end{pmatrix}"'] & \begin{pmatrix} X \\ Y \end{pmatrix} \arrow[l, "\begin{pmatrix} h & i \\ c & d \end{pmatrix}"'] & \begin{pmatrix} U \\ V \end{pmatrix} \arrow[l, "\begin{pmatrix} a & f \\ b & g \end{pmatrix}"'] \arrow[ll, bend left, "\begin{pmatrix} 1_U & 0 \\ 0 & 1_V \end{pmatrix}"] \end{tikzcd}$$

$$
\begin{tikzcd}
V \arrow[r, bend right, "\begin{pmatrix} f \\ g \end{pmatrix}"'] & \begin{pmatrix} X \\ Y \end{pmatrix} \arrow[l, "(c\ d)"'] \arrow[r, bend right, "(h\ i)"'] & U \arrow[l, "\begin{pmatrix} a \\ b \end{pmatrix}"']
\end{tikzcd}
$$

$ca + db = 0$ ✓

$hf + ig = 0$ ✓

$$1_V = (c\ d)\begin{pmatrix} f \\ g \end{pmatrix} = cf + dg \quad ✓ \qquad 1_U = (h\ i)\begin{pmatrix} a \\ b \end{pmatrix} = ha + ib \quad ✓$$

$$1_{\begin{pmatrix} X \\ Y \end{pmatrix}} = \begin{pmatrix} f \\ g \end{pmatrix}(c\ d) + \begin{pmatrix} a \\ b \end{pmatrix}(h\ i)$$

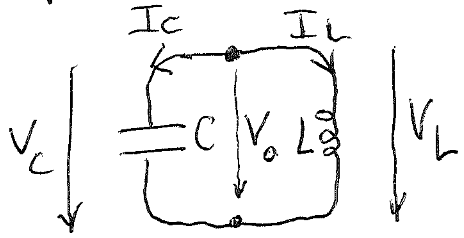
$$\begin{pmatrix} 1_X & 0 \\ 0 & 1_Y \end{pmatrix} = \begin{pmatrix} fc + ah & fd + ai \\ gc + bh & gd + bi \end{pmatrix}$$

$$
\begin{tikzcd}
\begin{pmatrix} X \\ Y \end{pmatrix} & \begin{pmatrix} U \\ V \end{pmatrix} \arrow[l, "\begin{pmatrix} a & f \\ b & g \end{pmatrix}"'] & \begin{pmatrix} X \\ Y \end{pmatrix} \arrow[l, "\begin{pmatrix} h & i \\ c & d \end{pmatrix}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\begin{pmatrix} U \\ V \end{pmatrix} & \begin{pmatrix} X \\ Y \end{pmatrix} \arrow[l, "\begin{pmatrix} h & i \\ c & d \end{pmatrix}"'] & \begin{pmatrix} U \\ V \end{pmatrix} \arrow[l, "\begin{pmatrix} a & f \\ b & g \end{pmatrix}"']
\end{tikzcd}
$$

It seems like 8 equations in 8 unknowns

α do [redacted] simple oscillator



Kirchhoff [redacted] $V_C = V_L$

$I_C + I_L = 0$

Ohm $C\dot{V}_C = I_C \,, \quad L\dot{I}_L = V_L$

$$\ddot{V}_C = \frac{1}{C}\dot{I}_C = -\frac{1}{C}\dot{I}_L = -\frac{1}{CL}\dot{V}_L = -\frac{1}{CL}V_C \,, \quad \left(\partial_t^2 + \frac{1}{CL}\right)V_C = 0$$

$$V_0 \longmapsto \begin{pmatrix} V_C \\ V_L \end{pmatrix} = \begin{pmatrix} V_0 \\ V_0 \end{pmatrix}, \quad \begin{pmatrix} V_C \\ V_L \end{pmatrix} \longmapsto V_\lambda = V_C - V_L$$

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] & C^1 \arrow[r, two heads] \arrow[d, shift right, "\begin{pmatrix} Cs & 0 \\ 0 & \frac{1}{Ls} \end{pmatrix}"'] & H^1 \\
\bar{C}_0 & C_1 \arrow[l, two heads] \arrow[u, shift right, "\begin{pmatrix} \frac{1}{Cs} & 0 \\ 0 & Ls \end{pmatrix}"'] & H_1 \arrow[l, hook']
\end{tikzcd}$$

$$I_0 = I_C + I_L \longleftarrow\!\shortmid \begin{pmatrix} I_C \\ I_L \end{pmatrix}, \begin{pmatrix} I_C = I_\lambda \\ I_L = -I_\lambda \end{pmatrix} \longleftarrow\!\shortmid I_\lambda$$

$$\begin{tikzcd}
V_0 \arrow[r, mapsto] & \begin{pmatrix} V_0 \\ V_0 \end{pmatrix} \arrow[d, mapsto] & & \begin{pmatrix} \frac{1}{Cs} I_\lambda \\ -Ls I_\lambda \end{pmatrix} \arrow[r, mapsto] & \left(\frac{1}{Cs} + Ls\right) I_\lambda = V_\lambda \\
\overbrace{\left(Cs + \frac{1}{Ls}\right)V_0}^{I_0} & \begin{pmatrix} CsV_0 \\ \frac{1}{Ls}V_0 \end{pmatrix} \arrow[l, mapsto] & & \begin{pmatrix} I_\lambda \\ -I_\lambda \end{pmatrix} \arrow[u, mapsto] & I_\lambda \arrow[l, mapsto]
\end{tikzcd}$$

[redacted]

Thus $I_\lambda \longmapsto I_\lambda V_\lambda = I_\lambda \left(\frac{1}{Cs} + Ls\right) I_\lambda$ is the quadratic form on H_1. [redacted] Upon applying the isom. $I_\lambda \mapsto \left(\frac{1}{Cs} + Ls\right) I_\lambda = V_\lambda$ one gets the quadratic form $V_\lambda \longmapsto V_\lambda \dfrac{1}{\frac{1}{Cs} + Ls} V_\lambda$ on H^1

β $\quad V$ 2dim Euclidean equipped with orthogonal involutions ε, F. Write $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Then $F\varepsilon = g$ is a rotation through a definite angle θ.

$$F\varepsilon = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$F = \begin{pmatrix} \cos\theta & +\sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

V f.d. Euclidean with F, ε $\qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$

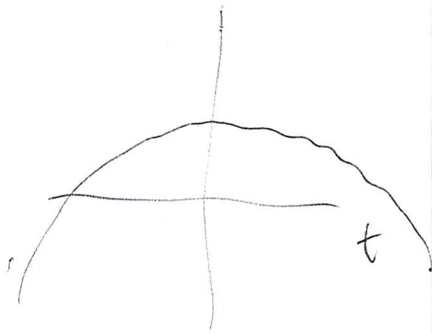
Suppose $\dfrac{F\varepsilon + \varepsilon F}{2} = (\cos\theta)\,\mathrm{Id}$

So you look at

$$\boxed{\begin{matrix} \dfrac{g+g^{-1}}{2} & \dfrac{g-g^{-1}}{2} \\[2ex] \dfrac{g-g^{-1}}{2} & \dfrac{g+g^{-1}}{2} \end{matrix}}$$

Legendre transform.

$$\int e^{-st + F(t)} \qquad \hat{F}(s)$$



critical point occurs at $\quad \dfrac{d}{dt}\big(-st + F(t)\big) = -s + F'(t) = 0$

$F'(t) = s$ $\qquad$ Use $F'(t) = s$ to make t a function of s. Then ~~treat~~ $\hat{F}(s) = -st + F(t)$ as a function of s

$$\frac{d}{ds}\hat{F}(s) = -t \cancel{- s\frac{dt}{ds} + F'(t)\frac{dt}{ds}}$$

Still where do you get dynamics-

Consider general case

$$\begin{array}{ccccc} \bar{C}^0 & \xhookrightarrow{i} & C^1_C \oplus C^1_L & \xrightarrow{j} & H^1 \\ & & \downarrow \qquad \uparrow & & \\ \bar{C}_0 & \xleftarrow{i^t} & C_{1C} \oplus C_{1L} & \xleftarrow{j^t} & H_1 \end{array}$$

Write down equations. variables $V_C \quad I_C$
$V_L \quad I_L$

The point is transversality of the Kirchhoff space K and Γ_s. You know that $K \cap \Gamma_s \neq 0$ means $\exists$ ~~[illegible]~~ normal mode of frequency s. Let's work it out for the simple oscillator. Kirchhoff space is $\begin{pmatrix} V^0 \\ V^0 \end{pmatrix} \oplus I_\lambda$ except that you should express it as the direct sum

$$\left\{ \begin{pmatrix} V_C & V_L \end{pmatrix} \;\middle|\; V_C = V_L \right\} \oplus \left\{ \begin{pmatrix} I_C & I_L \end{pmatrix} \;\middle|\; I_C + I_L = 0 \right\}$$

Γ_s consists of $\left\{ \begin{pmatrix} V_C \\ C s V_C \end{pmatrix} \right\} \oplus \left\{ \begin{pmatrix} V_L \\ \frac{1}{Ls} V_L \end{pmatrix} \right\}$

~~Back to $\langle F, \varepsilon \rangle$ on $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$~~

~~Suppose $V_\pm$ 1 diml over $\mathbb{R}$. Say you have the Euclidean plane $\mathbb{R}^2$ with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and F another involution. $F\varepsilon = g$ is a rotation~~

~~$F\varepsilon = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for a unique θ mod 2π~~

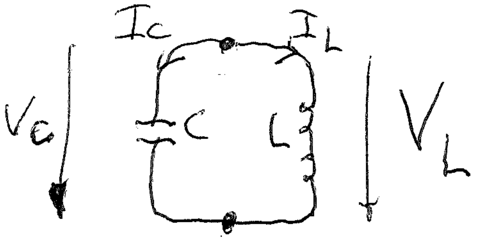
δ Put these conditions together

$$V_C = V_L \qquad I_C + I_L = 0 \qquad C s V_C = I_C, \quad L s I_L = V_L$$

$$C s V_C = C s V_L = C s L s I_L = -C L s^2 I_C = -C L s^2 C s V_C$$

$$(1 + C L s^2) V_C = 0$$

Let's next do the IVP.



$$\partial_t V_C = \frac{1}{C} I_C \qquad \partial_t I_L = \frac{1}{L} V_L$$

$$s \hat{V}_C - V_C(0) = \frac{1}{C} \hat{I}_C = -\frac{1}{C} \hat{I}_L$$

$$s \hat{I}_L - I_L(0) = \frac{1}{L} \hat{V}_L = \frac{1}{L} \hat{V}_C$$

$$\begin{pmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s \end{pmatrix} \begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \begin{pmatrix} V_C(0) \\ I_L(0) \end{pmatrix}$$

$$\begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \frac{1}{s^2 + \frac{1}{LC}} \begin{pmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s \end{pmatrix} \begin{pmatrix} V_C(0) \\ I_L(0) \end{pmatrix}$$

done Oct 8, 2002.

ε Let's next look at the general case.

$$\begin{tikzcd} \bar{C}^0 \arrow[r, hook] & C^1_C \oplus C^1_L \arrow[r, two heads] \arrow[d, "V_C \mapsto CsV_s"'] & H^1 \\ \bar{C}_0 & C_{1C} \oplus C_{1L} \arrow[l, two heads] \arrow[u, "I_L \mapsto LsI_L"'] & H_1 \arrow[l, hook'] \end{tikzcd}$$

$$\dot{V}_C(t) = \frac{1}{C} I_C(t) \quad \Big| \quad \dot{I}_L(t) = \frac{1}{L} V_L(t)$$

$$\boxed{s\hat{V}_C - V_C(0) = \frac{1}{C}\hat{I}_C \quad \Big| \quad s\hat{I}_L - I_L(0) = \frac{1}{L}\hat{V}_L}$$

So you get the equations. ↑ there are e of these, which have to be combined with Kirchhoff's constraints for a total of $2e$ equation.

Main claim is the general case splits orthogonally into simple oscillator types (4 dim'l symp phase space with 2 dim'l constraints.

Idea now is to take a Grassmannian situation

$$\begin{tikzcd} W \arrow[r, hook] & \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \arrow[r, two heads] \arrow[d, "\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}"'] & W^\perp \\ W^* & \begin{pmatrix} V_+^* \\ V_-^* \end{pmatrix} \arrow[l, two heads] & (W^\perp)^* \arrow[l, hook'] \end{tikzcd}$$

Recall yesterday's insight

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] & C^1 \arrow[r] & H^1 \\
& \Gamma_s \arrow[u] \arrow[d] & \\
\bar{C}_0 & C_1 \arrow[l, two heads] \arrow[r] & H_1
\end{tikzcd}$$

The regular case is where the Kirchhoff space and Γ_s are transversal. When this fails i.e. $\mathcal{K} \cap \Gamma_s \neq 0$, then you have normal modes, (free motion).

In the regular case $\mathcal{K} \oplus \Gamma_s \xrightarrow{\sim} C^1 \oplus C_1$

Other point is that the L.T. solves the IVP. The L.T. of $\dot{X}(t)$ is $s\hat{X}(s) - X(0)$, so the initial values of the dominant variables give rise to ~~and~~ inhomogeneous terms.

Thing to do next is to decompose the situation into simple oscillators. To go back over the Grassmannian situation

orth repn of $\langle F, \varepsilon \rangle$ on Euclidean space V.

$$V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

You want to go over things you have tried

$$W \xhookrightarrow{\binom{x}{y}} \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$

$\eta \qquad W = \begin{pmatrix} x \\ y \end{pmatrix} \mathbb{R} \qquad W^{\perp} = \begin{pmatrix} -y \\ x \end{pmatrix} \mathbb{R}$

$$F \begin{pmatrix} x & -y \\ y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon$$

$$F = \begin{pmatrix} x & -y \\ y & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{1}{x^2+y^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

---

setup 2 diml case

$$W \hookrightarrow \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \twoheadrightarrow W^{\perp}$$

$$\begin{tikzcd}
\mathbb{R} \arrow[r, bend left, "\binom{x}{y}"] & \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \arrow[r, two heads, "(-y\ +x)"] \arrow[d, "\begin{pmatrix} S & 0 \\ 0 & S^{-1} \end{pmatrix}"'] & \mathbb{R} \\
\mathbb{R} & \begin{pmatrix} \mathbb{R} \\ \mathbb{R} \end{pmatrix} \arrow[l, "(x\ y)"'] & \mathbb{R} \arrow[l, "\binom{-y}{x}"']
\end{tikzcd}$$

$$s\hat{V}_c - V_c(0) = 1$$

transversality means

$$(x\ y)\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \neq 0$$

but you want to ~~solve~~ find the splitting

θ

$$V_0 \longmapsto \begin{pmatrix} xV_0 \\ yV_0 \end{pmatrix} \qquad \begin{pmatrix} V_C \\ V_L \end{pmatrix} \xrightarrow{\quad T \quad} \begin{pmatrix} sV_C \\ s^{-1}V_L \end{pmatrix}$$

equations are

$$\boxed{\begin{aligned} \dot V_C &= I_C \\ \dot I_L &= V_L \\ xI_C + yI_L &= 0 \\ -yV_C + xV_L &= 0 \end{aligned}}$$

Maybe you want normalized variables

Call them V_C, V_L

$$x\dot V_C = xI_C = -yI_L$$

$$x\dot I_L = xV_L = yV_C$$

$$\cancel{\dot V_C = -\frac{y}{x}\dot I_L}$$

$$\cancel{\dot I_L = \frac{y}{x}\dot V_C}$$

$$\boxed{\begin{aligned} \dot V_C &= -\left(\frac{y}{x}\right) I_L \\ \dot I_L &= \left(\frac{y}{x}\right) V_C \end{aligned}}$$

here $\frac{y}{x} = \frac{\sin}{\cos} = \tan$

$$W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+ \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xrightarrow{(-T \;\; 1)} V_-$$

$$\downarrow \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}$$

$$W \xleftarrow{(1 \;\; T^*)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} -T^* \\ 1 \end{pmatrix}} V_-$$

$$\begin{aligned} \dot V_C &= I_C \\ \dot I_L &= V_L \\ -TV_C + V_L &= 0 \\ I_C + T^* I_L &= 0 \end{aligned}$$

$$s\hat V_C - V_C(0) = \hat I_C = -T^* \hat I_L$$

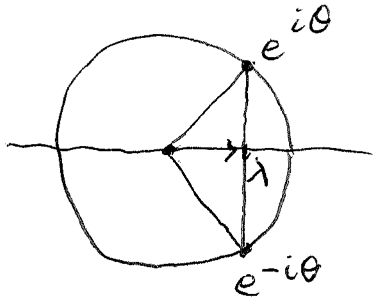
$$s\hat I_L - I_L(0) = \hat V_L = T\hat V_C$$

$$\boxed{\begin{pmatrix} s & T^* \\ -T & s \end{pmatrix} \begin{pmatrix} \hat V_C \\ \hat I_L \end{pmatrix} = \begin{pmatrix} V_C(0) \\ I_L(0) \end{pmatrix}}$$

1 Next you want the splitting of an LC phase space into simple oscillators.

Let $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ be a polarized Hilb space, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. let F be a unitary involution with ± 1 eigenspaces W and $W^\perp$ resp. Then $g = F\varepsilon$ is unitary on V. Also $\frac{g+g^{-1}}{2}$ is selfadjoint and it commutes with both F, ε, so that the representation of F, ε on V decomposes orthogonally $V = \bigoplus_\lambda V_\lambda$ where V_λ is the λ eigenspace of $\frac{g+g^{-1}}{2}$ and $-1 < \lambda < 1$. (For the moment we ignore the ± 1 eigenspaces of g.)

Suppose $V = V_\lambda$ i.e. g satisfies $\frac{g+g^{-1}}{2} = \lambda$ (scalar op). Roots of $\frac{z+z^{-1}}{2} = \lambda$, i.e. $z^2 + 2\lambda z + 1$, are $\lambda \pm \sqrt{\lambda^2 - 1}$, i.e. $\cos\theta \pm i \sin\theta$, where $\cos\theta = \lambda$:



Since $(g - e^{i\theta})(g - e^{-i\theta}) = g^2 - 2\lambda g + 1 = 0$

$$(g - e^{-i\theta}) + (-g + e^{i\theta}) = 2i \sin\theta.$$

So $\frac{g - e^{-i\theta}}{2i\sin\theta}$, $\frac{-g + e^{i\theta}}{2i \sin\theta}$ are annihilating idempotents with sum 1 on V_λ.

K Consider V (Hilb sp.), F, ε invs

assume $\frac{F\varepsilon+\varepsilon F}{2} = \cos\theta$ $0<\theta<\pi$. Put $g = F\varepsilon$.

Then $(g-e^{i\theta})(g-e^{-i\theta}) = 0$, you get

ann. idemp. $e_{-\theta} = \frac{-g+e^{i\theta}}{2i\sin\theta}$ $e_{+\theta} = \frac{g-e^{-i\theta}}{2i\sin\theta}$

Then $V = \begin{pmatrix} V_\theta \\ V_{-\theta}\end{pmatrix}$ $\varepsilon e_\theta \varepsilon = \frac{g^{-1}-e^{-i\theta}}{2i\sin\theta}$?

$$W = \begin{pmatrix}1\\T\end{pmatrix} V_+ \subset \begin{pmatrix}V_+\\V_-\end{pmatrix}$$

$$W^\perp = \begin{pmatrix}-T^*\\1\end{pmatrix} V_- \subset \begin{pmatrix}V_+\\V_-\end{pmatrix}$$

$$F\begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\varepsilon \Rightarrow \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon = g. \qquad g=-1 \iff X=\infty.$$

This is the Cayley transform theory.
It works over $\mathbb{R}$, where g is orthogonal

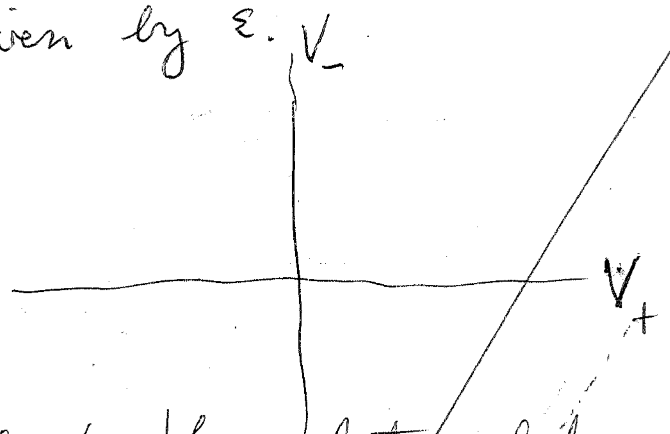
Important idea is how to quantize a harmonic oscillator. On phase space you have a symplectic form ω. You also have a symmetric bilinear form H, the Hamiltonian. These combine to give you a skew symmetric form X.

Quantization uses polar decomp: $X = \overbrace{(-X^2)^{1/2}}^{|X|} J$ $J^2=-1$. $|X|$ gives the energy levels J the quantum kinematics.

λ Program now is make explicit this stuff

Return to V, F, ε. Your aim is decompose into isotypical components and to put these in a canonical form. You want to make any choices explicit. $g = F\varepsilon$ uses the order of F, ε also $\cos\theta$ uses $\theta \in (0, \pi)$.

Hilbert case. Assume $\frac{g+g^{-1}}{2} = \cos\theta$ on V. Then $V = V_\theta \oplus V_{-\theta}$ where $V_\theta = \{\xi \mid g\xi = e^{i\theta}\xi\}$. ~~Also~~ ~~$\varepsilon: V_\theta \cong V_{-\theta}$ isom with inverse ε~~. ε restricted to V_θ is an isom $\varepsilon : V_\theta \xrightarrow{\sim} V_{-\theta}$ with inverse given by ε.



$$g\xi = e^{i\theta}\xi$$

$$g\varepsilon\xi = e^{-i\theta}\varepsilon\xi$$

What's the relation between $V_\theta, V_{-\theta}, V_+, V_-, W, W^\perp$

$$\begin{pmatrix} V_\theta \\ V_{-\theta} \end{pmatrix} \simeq \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \simeq \begin{pmatrix} W \\ W^\perp \end{pmatrix}$$

So you have: $g \quad \varepsilon \quad F$ ~~with giving~~ yielding these. $(g-e^{i\theta})(g-e^{-i\theta}) = g^2 - (2\cos\theta)g + 1 = 0$

$\Longleftrightarrow \quad g + g^{-1} - 2\cos\theta = 0$.

$$(g - e^{-i\theta}) + (-g + e^{+i\theta}) = +2i\sin\theta$$

$$\frac{g - e^{-i\theta}}{2i\sin\theta} + \frac{e^{i\theta} - g}{2i\sin\theta} = 1$$

μ $\qquad \pi_\theta = \dfrac{g - e^{-i\theta}}{2i\sin\theta} \qquad \pi_{-\theta} = \dfrac{e^{i\theta} - g}{2i\sin\theta}$

Review: Aim? Given V repn of F, ε sat $\dfrac{F\varepsilon + \varepsilon F}{2} = \cos\theta$ $\quad \theta \in (0, \pi)$. You want a canonical picture for V. What might this be? V splits into irreducibles which are all isomorphic, and the endom. ring is $\mathbb{C}$.

IDEA. similarity between roots of an irreducible equation and random phases

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Back to irreducible unitary repns of the inf dihedral group. Should be induced from a character on the infinite cyclic group by Mackey theory. Get

$$g \longmapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \varepsilon \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{on } \mathbb{C}^2.$$

image of $\langle F, \varepsilon \rangle$ in $M_2(\mathbb{C})$ generates. Endo ring of the rep is $\mathbb{C}$

Next consider orthogonal repns of F, ε on a Euclidean space. ~~g orthogonal~~

$$g = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{on } \mathbb{R}^2$$

Again F, ε generate $M_2(\mathbb{R})$ $\therefore$ irred

So for any unitary repn V of F, ε with $\frac{1}{2}(F\varepsilon + \varepsilon F) = \cos\theta$ one has canonical isom

$$V \simeq P \otimes \mathrm{Hom}_{F,\varepsilon}(P, V)$$

unique up to a scalar factor of $| \, | = 1$.

You now need to link these ideas from repn theory to other things.

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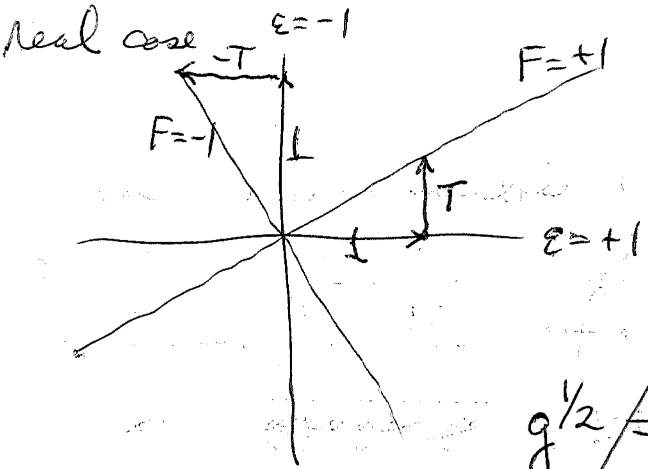
real case



$$F \overbrace{\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}}^{1+X} = (1+X)\varepsilon = \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon = g.$$

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix} \frac{1}{(1-T^*T)^{1/2}} & -T^* \frac{1}{(1+TT^*)^{1/2}} \\ T \frac{1}{(1-T^*T)^{1/2}} & \frac{1}{(1+TT^*)^{1/2}} \end{pmatrix}$$

Consider real case V, F, ε $\quad W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$, $W^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} V_-$

$$F\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}\varepsilon \qquad F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$\frac{1+X}{1-X} = F\varepsilon = g. \qquad \text{Assume} \quad \frac{g+g^{-1}}{2} = \cos\theta \qquad \theta \in (0,\pi)$$

$$\cos\theta = \left(\frac{1+X}{1-X} + \frac{1-X}{1+X}\right)\frac{1}{2} = \frac{1+X^2}{1-X^2} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X^2 \Rightarrow X^2 = \frac{\cos\theta - 1}{\cos\theta + 1}$$

$$-X^2 = \frac{1-\cos\theta}{1+\cos\theta} = \frac{\sin^2(\theta/2)}{\cos^2(\theta/2)} \quad \text{but} \quad -X^2 = -\begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}\begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$

$$-X^2 = \begin{pmatrix} T^*T & 0 \\ 0 & TT^* \end{pmatrix} \qquad \therefore \; -X^2 = \left(\frac{\sin\theta/2}{\cos\theta/2}\right)^2$$

ξ something happens here

with the phase of T. T^*T and $T^*T = \left(\tan \theta/2\right)^2$

so the char values of T are all $= \left|\tan\frac{\theta}{2}\right|$ which

ranges over $(0,\infty)$ for $\theta \in (0,\pi)$.

The eigenvalues of $\begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ are purely

imaginary.

You have seen that the LC circuit

resolvent is $\begin{pmatrix} S & T^* \\ -T & S \end{pmatrix}^{-1}$ which leads to

singularities of $\dfrac{S}{S^2+\omega^2}$.

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{1-T^*T}} & 0 \\ 0 & \frac{1}{\sqrt{1-TT^*}} \end{pmatrix}$$

Something here should contain the phase J of

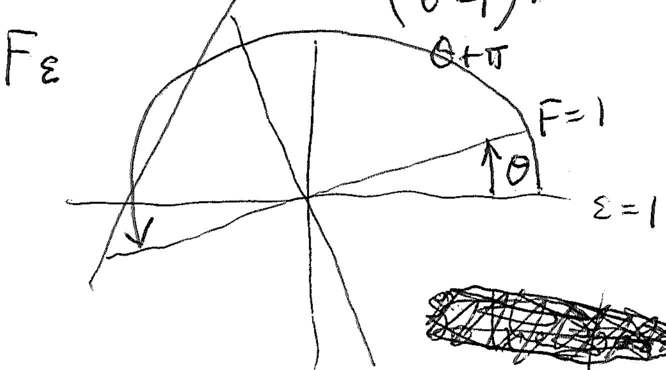
$X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$, namely, replace X by tX and

let $t \longrightarrow \infty$.

Aim 12 Oct is to sort out irred F,ε reps.

Idea: There is this angle θ whose double is intrinsic? Let V be an irred orthogonal repn of F,ε on $\mathbb{R}^2$

such that $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. What are the possibilities for F?



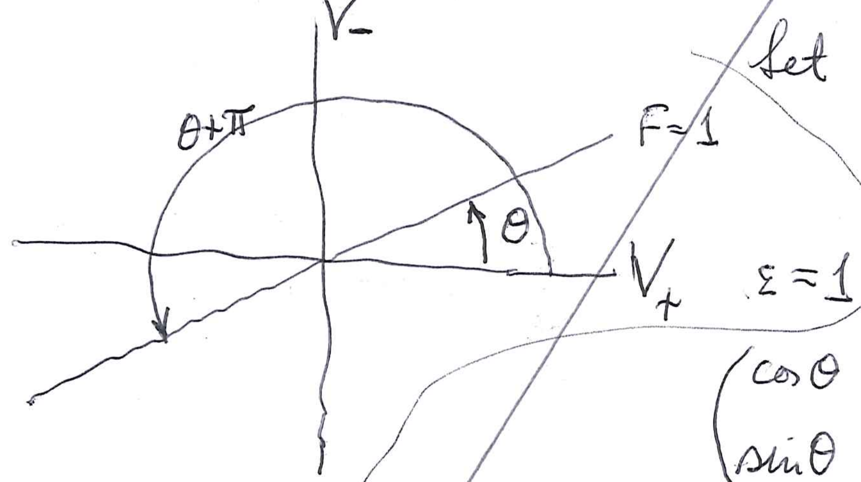
Put $g^{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$$g^{1/2}\varepsilon g^{-1/2} = g\varepsilon = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

Possible $F = g^{1/2} \varepsilon g^{-1/2}$

Vague Idea: ✓ There [redacted] seems to be a loss of information involved in passing from an irreducible orthogonal representation of F, ε on $\mathbb{R}^2$ to the corresponding harmonic oscillator. The oscillator retains only positive type information, e.g. characteristic values. Now quantization of the oscillator involves lifting to a double covering, that is, undoing [redacted] the previous step in some way.

Start with $V = \mathbb{R}^2$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and F another orthogonal involution.



Set $g^{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Then $g^{\frac{1}{2}} \varepsilon g^{-1/2}$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

Simpler would have been $g^{1/2} \varepsilon g^{-1/2} = g\varepsilon$

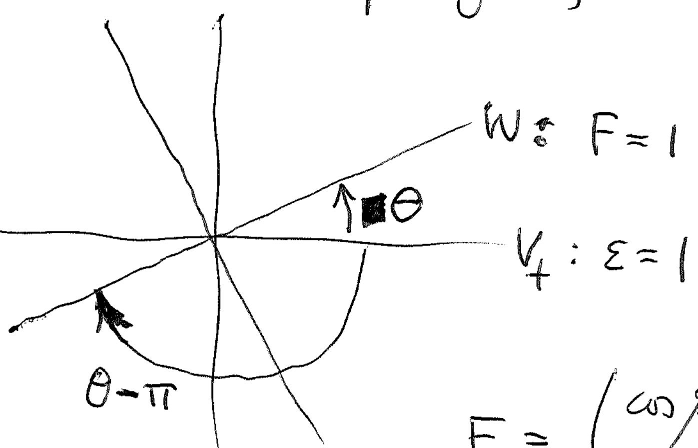
$$= \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} \varepsilon = F$$

Let's go over this again. You are looking at [redacted] irred orth reps of the infinite dihedral group $\langle F, \varepsilon \rangle$ on $\mathbb{R}^2$ such that $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $g = F\varepsilon$ is an orthogonal [redacted] transformation of $\det = +1$ (orientation-preserving) so it's a rotation $g = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$, get two values

π for $g^{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$?

Confused again. Given orth rep of F, ε on $\mathbb{R}^2$ s.t. $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $g = F\varepsilon$ is a rotation, i.e. $g \in SO(2, \mathbb{R})$. There are 2 square roots of g namely $g^{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, $\begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = g^{-1/2}$?

Given orth rep of F, ε on $\mathbb{R}^2$ $\quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$$g^{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$g^{1/2}\varepsilon g^{-1/2} = g\varepsilon = F$$

$$F = \begin{pmatrix} \cos 2\theta & +\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \qquad -\pi/2 < \theta < \pi/2$$

$$\frac{1+F}{2} = \begin{pmatrix} \cos^2\theta & +\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$$

Also $\quad W = \begin{pmatrix} 1 \\ t \end{pmatrix} V_+ \qquad W^{\perp} = \begin{pmatrix} -t \\ 1 \end{pmatrix} V_-$

$$F \begin{pmatrix} 1 & [redacted] & -t \\ t & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \varepsilon \implies \frac{1+X}{1-X} = F\varepsilon = g$$

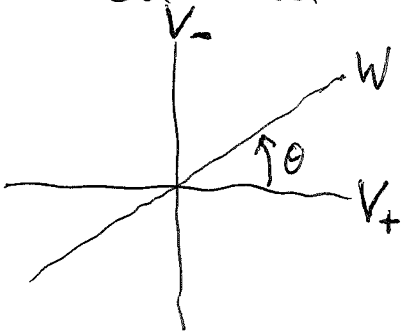
$$1+X \qquad 1+X \qquad \implies \frac{1+X}{\sqrt{1-X^2}} = g^{1/2}$$

$$g^{1/2} = \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \frac{1}{\sqrt{1+t^2}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad t = \tan\theta$$

$$-\infty < t < \infty$$

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

ρ Q: Classify irred orthogonal reps of F, ε on $\mathbb{R}^2$ such that $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. A:



$$F = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \varepsilon \begin{pmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\boxed{F = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}}$$

Q: Classify irreducible unitary reps of F, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{C}^2$. Let $W = \begin{pmatrix} 1 \\ z \end{pmatrix} V_+$, $W^\perp = \begin{pmatrix} -\bar{z} \\ 1 \end{pmatrix} V_-$,

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix} 1 & -\bar{z} \\ z & 1 \end{pmatrix} \frac{1}{\sqrt{1+|z|^2}}$$ as $X = \begin{pmatrix} 0 & -\bar{z} \\ z & 0 \end{pmatrix} \Rightarrow$

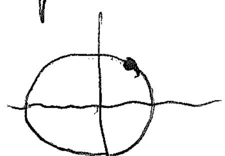
$$1 - X^2 = \begin{pmatrix} 1+|z|^2 & 0 \\ 0 & 1+|z|^2 \end{pmatrix}$$

If $z = re^{i\varphi}$, then

$$g^{1/2} = \begin{pmatrix} \cos\theta & e^{-i\varphi}(-\sin\theta) \\ e^{i\varphi}\sin\theta & \cos\theta \end{pmatrix}$$ where $\frac{1}{\sqrt{1+r^2}} = \cos\theta$, $\frac{r}{\sqrt{1+r^2}} = \sin\theta$

so $$F = \begin{pmatrix} \cos 2\theta & e^{-i\varphi}(-\sin 2\theta) \\ e^{i\varphi}\sin 2\theta & \cos 2\theta \end{pmatrix}$$

Let V be a unitary repn of F, ε such that $\frac{1}{2}(g+g^{-1}) = \frac{1}{2}(F\varepsilon + \varepsilon F)$ is the scalar operator λI, where $\lambda = \cos 2\theta \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$



Try to show that $V = V_{2\theta} \oplus V_{-2\theta}$, where $V_{\pm 2\theta} = \operatorname{Ker}(g - e^{\pm 2i\theta})$. You have

$$(g - e^{-2i\theta})(g - e^{2i\theta}) = g^2 - 2\cos(2\theta) g + 1$$
$$= g^2 - 2\frac{g+g^{-1}}{2} g + 1 = 0$$

σ $$\frac{g-e^{-2i\theta}}{2i\sin 2\theta} + \frac{-g+e^{2i\theta}}{2i\sin 2\theta} = 1 \qquad \text{OK}$$

so any $\xi \in V$ splits

$$\underbrace{\frac{g-e^{-2i\theta}}{2i\sin 2\theta}\xi}_{\substack{\text{killed by}\\ g-e^{2i\theta}}} + \underbrace{\frac{-g+e^{2i\theta}}{2i\sin 2\theta}\xi}_{\substack{\text{killed by}\\ g-e^{-2i\theta}}} = \xi$$

Also $$\varepsilon\, \frac{g-e^{-2i\theta}}{2i\sin 2\theta}\, \varepsilon^{-1} = \frac{g^{-1}-e^{-2i\theta}}{2i\sin 2\theta} \qquad \frac{1+\varepsilon}{2}V \quad \frac{1-\varepsilon}{2}V$$

So $$V = V_{2\theta} \oplus V_{-2\theta} \cong V_+ \oplus V_-$$

Let $\xi \in V_{2\theta} \Rightarrow g\xi = e^{2i\theta}\xi \Longrightarrow g^{-1}\varepsilon\xi = e^{2i\theta}\varepsilon\xi$
$\Longrightarrow e^{-2i\theta}\varepsilon\xi = g\varepsilon\xi \Longrightarrow \varepsilon\xi \in V_{-2\theta}$. Sim
$\xi \in V_{-2\theta} \Longrightarrow \varepsilon\xi \in V_{2\theta}$. You now have the rep

$$V = \begin{pmatrix} V_{2\theta} \\ V_{-2\theta} \end{pmatrix} \qquad g \longmapsto \begin{pmatrix} e^{2i\theta} & 0 \\ 0 & e^{-2i\theta} \end{pmatrix} \qquad \varepsilon \longmapsto \begin{pmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{pmatrix}$$

So perhaps the point is that all of the subspaces $V_{2\theta}, V_{-2\theta}, V_+, V_-$ are canonically isom.

There is ~~a complex vector space a vector space~~ subspace Y together with an isom

$$\mathbb{C}^2 \otimes Y \xrightarrow{\sim} V \qquad \text{compatible with } F, \varepsilon$$

τ

V a unitary rep of F, ε such that
$\frac{1}{2}(F\varepsilon + \varepsilon F) = \frac{g+g^{-1}}{2}$ is a scalar operator λ.

then $\lambda \in [-1,1]$, say $\lambda \in (-1,1)$

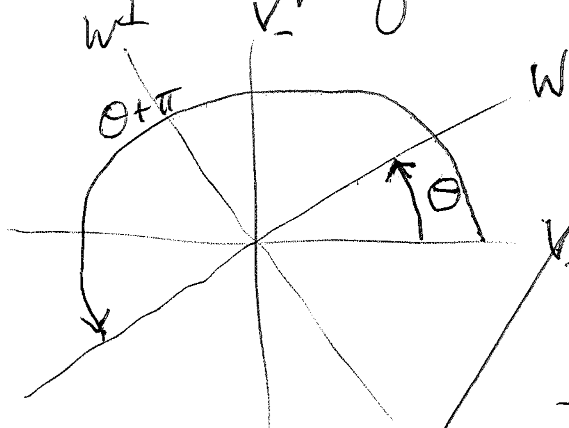
$$\frac{g+g^{-1}}{2} = \lambda \implies g^2 - 2\lambda g + 1 = 0$$

eigenvalues $g = \lambda \pm \sqrt{\lambda^2-1} = \cos 2\theta \pm i \sin 2\theta$

Then let $(g - e^{2i\theta})(g - e^{-2i\theta}) = 0$

$$V = V_{2\theta} \oplus V_{-2\theta} \simeq V_+ \oplus V_-$$

V orth rep of F, ε on $\mathbb{R}^2$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



$$g^{1/2} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$F = g^{1/2} \varepsilon g^{-1/2} = g\varepsilon = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

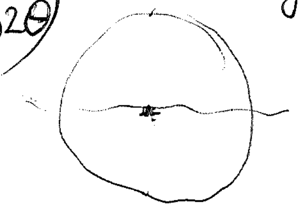
$$\frac{1+F}{2} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix}$$

$$g = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix} \qquad \frac{g+g^{-1}}{2} = \cos 2\theta$$

Q: What are the irred reps. $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$F = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix} \qquad \frac{g+g^{-1}}{2} = \cos 2\theta \qquad -\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$$

gives all real lines



U

$$W = \begin{pmatrix}1\\T\end{pmatrix}V_+ \qquad W^\perp = \begin{pmatrix}-T^*\\1\end{pmatrix}V_-$$

$$X = \begin{pmatrix}0 & -T^*\\ T & 0\end{pmatrix} \qquad -X^2 = \begin{pmatrix}+T^*T & 0\\ 0 & +TT^*\end{pmatrix}$$

$$F(1+X) = (1+X)\varepsilon = \varepsilon(1-X)$$

$$g = F\varepsilon = \frac{1+X}{1-X}$$

$$g^{1/2} = \frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix}1 & -T^*\\ T & 1\end{pmatrix}\begin{pmatrix}1+T^*T & 0\\ 0 & 1+TT^*\end{pmatrix}^{-1/2}$$

Look at irred unitary reps. in $\mathbb{C}^2$.

$$W = \begin{pmatrix}1\\z\end{pmatrix}V_+ \qquad W^\perp = \begin{pmatrix}-\bar z\\1\end{pmatrix}V_-$$

$$g^{1/2} = \begin{pmatrix}1 & -\bar z\\ z & 1\end{pmatrix}(1+|z|^2)^{-1/2}$$

$$\boxed{F = g^{1/2}\varepsilon g^{-1/2}} \qquad g = \begin{pmatrix}1 & -\bar z\\ z & 1\end{pmatrix}^2 \frac{1}{1+|z|^2}$$

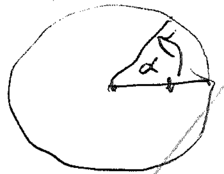
$$= \begin{pmatrix}1-|z|^2 & -2\bar z\\ 2z & 1-|z|^2\end{pmatrix}\frac{1}{1+|z|^2} = \begin{pmatrix}\frac{1-|z|^2}{1+|z|^2} & \frac{-2\bar z}{1+|z|^2}\\ \frac{2z}{1+|z|^2} & \frac{1-|z|^2}{1+|z|^2}\end{pmatrix}$$

$$F = g\varepsilon = \begin{pmatrix}\frac{1-|z|^2}{1+|z|^2} & \frac{2\bar z}{1+|z|^2}\\ \frac{2z}{1+|z|^2} & -\frac{1-|z|^2}{1+|z|^2}\end{pmatrix} = \begin{pmatrix}\cos 2\theta & -e^{-i\varphi}\sin 2\theta\\ e^{i\varphi}\sin 2\theta & -\cos 2\theta\end{pmatrix}$$

Maybe what's involved is that the phase $e^{i\varphi}$ in z disappears because of something commuting with ε?

4 F, ε unitary rep on V given $\cancel{V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \quad \varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

Assume $\frac{1}{2}(F\varepsilon + \varepsilon F)$ = scalar $\cos\alpha$



$\alpha \in (0, \pi)$

$\cancel{\frac{g+g^{-1}}{2} = \cos\alpha}$ $\qquad -1 < \cos\alpha < 1$

$g^2 - (2\cos\alpha)g + 1 = 0$ $\qquad$ eigenv. of $g = \cos\alpha \pm \sqrt{\cos^2\alpha - 1}$

$g + g^{-1} = 2\cos\alpha$ $\qquad = \cos\alpha \pm i\sin\alpha$

$\therefore$ get decomp. $V = V_\alpha \oplus V_{-\alpha}$ $\qquad V_{\pm\alpha} = \mathrm{Ker}(g - e^{\pm i\alpha})$

At this point you have

$$\begin{matrix} V_\alpha \\ \oplus \\ V_{-\alpha} \end{matrix} \cong \begin{matrix} V_+ \\ \oplus \\ V_- \end{matrix}$$

How is ↗ related to the F decomp. $V = \begin{pmatrix} W \\ W^\perp \end{pmatrix}$?

Idea: You have 3 decompositions of V into

$$\begin{matrix} V_\alpha \\ \oplus \\ V_{-\alpha} \end{matrix} \qquad \begin{matrix} W \\ \oplus \\ W^\perp \end{matrix} \qquad \begin{matrix} V_+ \\ \oplus \\ V_- \end{matrix}$$

Program. You still want to understand a general unitary repn V of F, ε such that $\frac{1}{2}(F\varepsilon + \varepsilon F)$ is a scalar op. You know that ↗ is hermitian of $\| \; \| \leq 1$, so you have $\frac{1}{2}(F\varepsilon + \varepsilon F) = \cos\theta$ with $0 < \theta < \pi$ (ignore $g = \pm 1$ for the moment). Then you have

$$\frac{1}{2}(g + g^{-1}) - \cos\theta = 0 \qquad g^2 - 2g\cos\theta + 1 = 0$$

g has eigenvalues $\cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$

$$(g - e^{i\theta})(g - e^{-i\theta}) = g^2 + g(-2\cos\theta) + 1 = 0$$

X What is [redacted] the algebra of operators, the image of the group algebra in End(V)? Should be generated by F, ε or g, ε.

You believe the unique irred repn of F, ε with $\frac{1}{2}(g+g^{-1}) = \cos\theta$ is given by $\mathbb{C}^2$ with $g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

The problem is to find, calculate the g eigenspaces. Perhaps look at $\mathbb{P}^1$

Repeat: V unitary rep of F, ε such that $\frac{F\varepsilon + \varepsilon F}{2} = \cos\theta$. Then you have a splitting $V = V_\zeta \oplus V_{\zeta^{-1}}$ $\zeta = e^{i\theta}$ such that [redacted] $g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ on $\begin{pmatrix} V_\zeta \\ V_{\zeta^{-1}} \end{pmatrix}$. What can you do. Choose a basis for V_ζ apply ε to get a basis for $\varepsilon V_\zeta = V_{\zeta^{-1}}$. Then you have a rep relative to this basis for both ε and g.

Something happens in the real case which might be interesting. Take $V = \mathbb{R}^2$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. You have $g = F\varepsilon$ an orthogonal rotation $g = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ $\frac{g+g^{-1}}{2} = \cos\alpha$.

Start again. V unitary repn of F, ε such that $\frac{g+g^{-1}}{2}$ is scalar operator $\cos\theta$ $0 < \theta < \pi$. Then $V = V_{e^{i\theta}} \oplus V_{e^{-i\theta}}$. Choose orth. basis ξ for $V_{e^{i\theta}}$ combine with $\varepsilon\xi_j$ to get orth basis for V $\ni$ $g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ $\varepsilon =$ [redacted] $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

ψ

$$\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}=\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$$

$=$ [redacted] $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. $\quad \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}e^{i\theta}&-e^{i\theta}\\e^{-i\theta}&e^{-i\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta&-i\sin\theta\\ \text{[redacted]}\, i\sin\theta&\cos\theta\end{pmatrix}$ $\quad$ conj by $\begin{pmatrix}1&0\\0&i\end{pmatrix}$

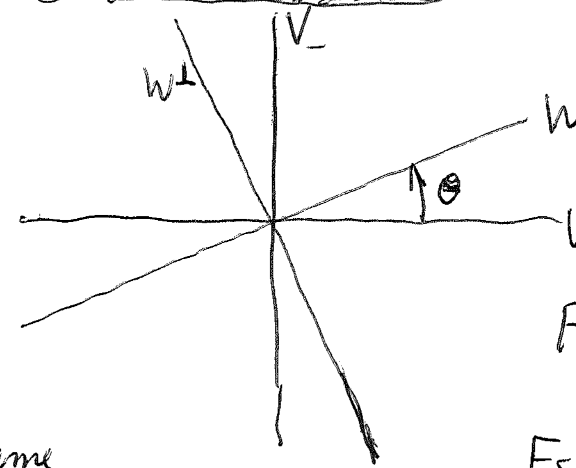
$$\frac{1}{2}\begin{pmatrix}1&1\\-i&i\end{pmatrix}\begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix}\begin{pmatrix}1&i\\1&-i\end{pmatrix}=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}$$

Main question. Given an orthogonal repn of F, ε on a Euclidean space V such that $\frac{1}{2}(g+g^{-1})$ is a scalar operator λI where $\lambda \in (-1,1)$, can you construct a canonical isom between V and the 2 dim'l irred rep V_λ belonging to λ $\otimes$ vector space.

I think the answer is YES because the endo ring of V_λ over $\mathbb{R}$ is $\mathbb{R}$.

[redacted]

$\sigma = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ on $\mathbb{R}^2$



$$F\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}=\begin{pmatrix}\cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{pmatrix}\varepsilon$$

$$=\varepsilon\begin{pmatrix}\cos\theta&\sin\theta\\ -\sin\theta&\cos\theta\end{pmatrix}$$

$F\varepsilon = \begin{pmatrix}\cos 2\theta&-\sin 2\theta\\ \sin 2\theta&\cos 2\theta\end{pmatrix}$, $\quad F=\begin{pmatrix}\cos 2\theta&\sin 2\theta\\ \sin 2\theta&-\cos 2\theta\end{pmatrix}$

Assume $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\frac{F\varepsilon+\varepsilon F}{2} = \cos(2\theta)$$

10 Classify irred orthogonal representations V of the infinite dihedral group F,ε. In the group algebra one has ~~the~~ element $\frac{g+g^{-1}}{2} = \frac{F\varepsilon+\varepsilon F}{2}$ which is ~~symmetric~~ self adjoint. Decompose V into eigenspaces for $\frac{g+g^{-1}}{2}$. Let V_κ be the eigenspace with $\frac{g+g^{-1}}{2} = \kappa$. $V_\kappa \neq 0 \Rightarrow \kappa \in [-1,1]$. Look at a κ in $(-1,1)$.

Here's the way to proceed. Take the complex case, that is, unitary reps of $F,\varepsilon = \langle g\rangle \rtimes \langle \varepsilon\rangle$. s.t. $\frac{g+g^{-1}}{2} = \lambda$ where $\lambda \in (-1,1)$. Form the quotient ring of $\mathbb{C}[g,g^{-1}]$ by the relation $g^2-2\lambda g+1=0$, you get an algebra $\simeq \mathbb{C}\times\mathbb{C}$. Yes. define $\mathbb{C}\langle g\rangle \longrightarrow \mathbb{C}\times\mathbb{C}$ using the two roots. Then adjoin ε, tensoring with $\mathbb{C}[\varepsilon]$ which is 2-diml. Get an alg of dim 4 over $\mathbb{C}$ quotient of $\mathbb{C}[\langle F,\varepsilon\rangle]$. $\longrightarrow \boxed{\mathbb{C}[\langle F,\varepsilon\rangle]\big/\left(\frac{g+g^{-1}}{2}=\lambda\right)}$

Real case $\mathbb{R}[\langle F,\varepsilon\rangle] \twoheadrightarrow \mathbb{R}[\langle F,\varepsilon\rangle]\big/\left(\frac{F\varepsilon+\varepsilon F}{2}=\lambda\right)$

$$= \mathbb{R}[g,g^{-1}] \twoheadrightarrow \mathbb{R}[g,g^{-1}]\big/\tfrac{g+g^{-1}}{2}=\lambda \xrightarrow{\sim} \mathbb{C}$$

Take semi-direct prod. $\mathbb{C}\rtimes\varepsilon$ dim 4 over $\mathbb{R}$.

$$\begin{pmatrix}1&0\\0&1\end{pmatrix} \underbrace{\begin{pmatrix}0&1\\-1&0\end{pmatrix}}_{-i} \underbrace{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}_{i} \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{\varepsilon} \simeq M_4\mathbb{R}$$

Now what you need is a standard form for these F,ε modules satisfying $\frac{1}{2}(F\varepsilon+\varepsilon F)=\lambda$. If S_λ is irreducible then ~~then~~ since Endo ring of S_λ is $\mathbb{R}$, one should have

$$V \xleftarrow{\sim} S_\lambda \otimes_{\mathbb{R}} \mathrm{Hom}_{F,\varepsilon}(S_\lambda, V)$$

α' You want to link this standard picture based on S_λ to what arises from the C.T.

Let $\lambda \in (-1,1)$ $\qquad \lambda \pm i\sqrt{1-\lambda^2} = \cos\theta \pm i\sin\theta$

$$R_\lambda = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbb{R}\underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{g} \subset M_2(\mathbb{R})$$

$$\frac{g+g^{-1}}{2} = \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

You want to see that $\mathbb{R}[g,g^{-1}]/\left(\frac{g+g^{-1}}{2} - \lambda\right)$ is ~~so~~ isom. to $\mathbb{C}$ via the map $g \longmapsto \lambda + i\sqrt{1-\lambda^2}$

$$R_\lambda = \underbrace{\mathbb{R}[g,g^{-1}]/\left(\frac{g+g^{-1}}{2} - \lambda\right)}_{\dim 2} \xrightarrow{\sim} \mathbb{C}$$

$g \longmapsto \lambda + i\sqrt{1-\lambda^2}$

$g^{-1} \longmapsto \lambda - i\sqrt{1-\lambda^2}$

$\frac{g+g^{-1}}{2} \longmapsto \lambda$

$\varepsilon \longleftrightarrow$ conj. in $\mathbb{C}$

$R_\lambda \rtimes \varepsilon \xrightarrow{\sim} \mathbb{C} \rtimes \varepsilon$ $\qquad a+bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$(a+bi)\varepsilon \longmapsto \begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ?$$

$$\mathbb{C} \otimes_{\mathbb{R}} (\mathbb{R} + \mathbb{R}\varepsilon)$$

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idea that since -1 is not an eigenvalue of g (when $\lambda \in (-1,1)$), the C.T. theory yields a choice for $g^{1/2} = \dfrac{1+X}{(1-X^2)^{1/2}}$

β'

complex case: unitary rep of F, ε on V such that $\frac{1}{2}(g+g^{-1}) = \lambda$ $\quad \lambda \in (-1,1)$ assume.

g satisfies $\quad g^2 - 2\lambda g + 1 = 0 \quad$ eigenvalues $\lambda \pm i\sqrt{1-\lambda^2} = e^{\pm i\theta}$

~~V splits into eigenspaces for~~

$$V = V_{e^{i\theta}} \oplus V_{e^{-i\theta}}$$

~~You have to go over~~

$$V = \begin{pmatrix} V_{e^{i\theta}} \\ V_{e^{-i\theta}} \end{pmatrix} = \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} V_{e^{i\theta}} \qquad g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$V = \mathbb{C}^2 \qquad g = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

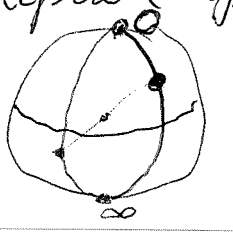
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ +i \end{pmatrix} \qquad \frac{1}{2}\begin{pmatrix} 1 & +1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & \pm i \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} e^{i\theta} & -e^{i\theta} \\ e^{-i\theta} & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\frac{-e^{i\theta} + e^{-i\theta}}{2} = -i\sin\theta \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & +i \\ 1 & -i \end{pmatrix}$$

Riemann sphere $\quad \mathbb{P}^1\mathbb{C} \quad$ An involution on $\mathbb{C}^2$ corresponds to (is the same as) an ordered pair of antipodal points. $\quad z \mapsto -\bar{z}^{-1} \qquad \frac{1}{t} \mapsto -\bar{t}$



So you see there is a phase defined involving the longitudes.

γ' Next you want ~~to~~ to look at the other picture of a unitary rep V of F, ε such that $(g+g^{-1})/2 = \lambda \in (-1,1)$. Namely, eigenspaces

$$V_\xi = \mathrm{Ker}(g-\xi), \quad V_{\bar\xi} = \mathrm{Ker}(g-\bar\xi) \qquad \xi = \lambda + i\sqrt{1-\lambda^2} = e^{i\theta}$$

$$V = \begin{pmatrix} V_\xi \\ V_{\bar\xi} \end{pmatrix} \qquad g = \begin{pmatrix} \xi & 0 \\ 0 & \bar\xi \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & \xi \\ \bar\xi & 0 \end{pmatrix}$$

[redacted] Next you want to convert ε to the standard form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (corresponding to electric / magnetic)

[redacted]

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

You can further conjugate by $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$, which fixes the standard $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

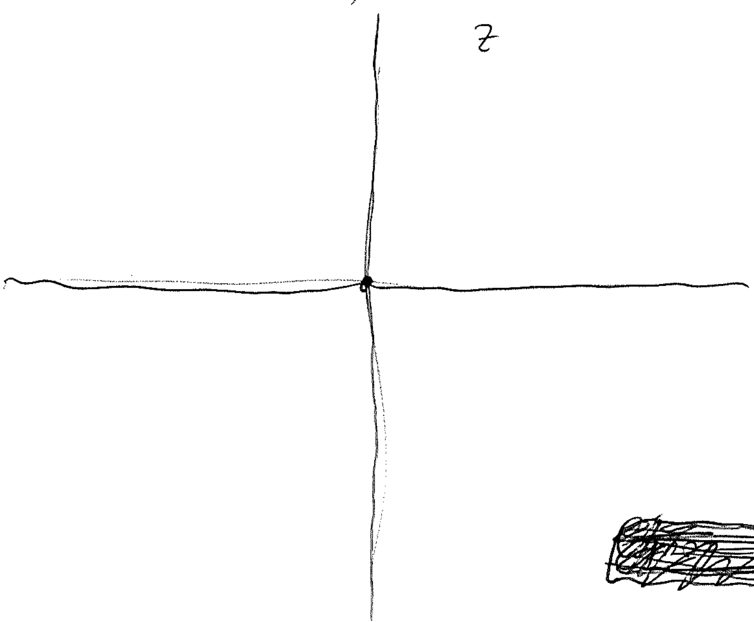
For the model $\quad g = \dfrac{1+X}{1-X} \qquad X = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}$

Reimann sphere $\qquad\qquad e^{2i\theta}z$

$$\varepsilon z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} z = \frac{z}{-1} = -z$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} z = \frac{1}{z} \qquad \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}(z) = \frac{e^{i\theta}z + 0}{0 + e^{-i\theta}}$$



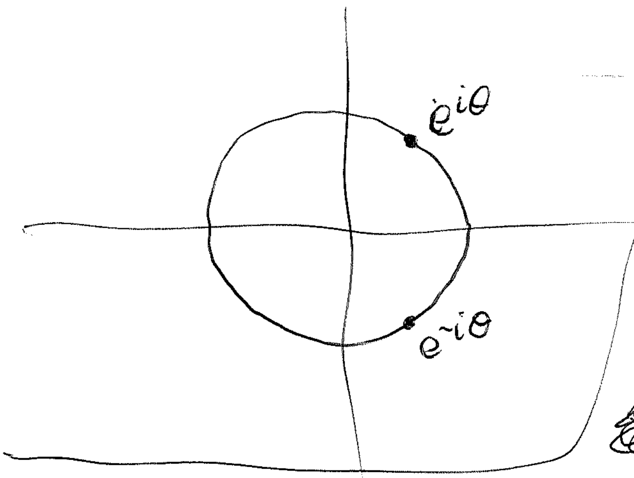
any involution fixes 2 pts.

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$$

[redacted]

δ' $\begin{pmatrix}0 & e^{i\theta}\\ e^{-i\theta} & 0\end{pmatrix}(z) = \dfrac{0z+e^{i\theta}}{e^{-i\theta}z+0} = e^{2i\theta}z^{-1}$

fixpts are $e^{2i\theta} = z^2 \Longrightarrow z = e^{\pm i\theta}$



$gz = e^{2i\theta}z$

$\begin{pmatrix}0&1\\1&0\end{pmatrix} z = z^{-1}$

Review the problem. Given a unitary repn of F, ε on V such that

$\frac{1}{2}(g+g^{-1}) = \lambda \in (-1,1)$.

You want an isomorphism of representations between V and standard irred repn U_λ $\otimes$ vectorspace.

You have two methods to construct

$V \simeq U_\lambda \otimes\ ?$

What is you aim? You start with a unitary rep on F, ε on a Hilbert space V such that $\frac{1}{2}(g+g^{-1}) = \lambda \in (-1,1)$. You have ~~a standard~~ an irreducible 2 diml repn U_λ with these properties. You want to construct a canonical isomorphism $V = U_\lambda \otimes \mathrm{Hom}_{F,\varepsilon}(U_\lambda, V)$ respecting herm. scalar product

ε' At this point I am beginning to believe that the vague idea about coherent phases is naive. Explain, describe idea. You consider a unitary (f.d.) repn of F, ε such that $g = F\varepsilon$ sat $\frac{1}{2}(g+g^{-1}) = \lambda \in (-1,1)$. You ~~seek~~ a canonical picture of this rep. (IDEA: Every automorphism of a simple (maybe semi simple) f.d. C^* algebra is inner. So such an autom. determines an invertible element modulo a central invertible. In the real case this should be ± 1, in the $\mathbb{C}$ case S^1.)

If you have any hope of utilizing the phases ignored (or suppressed) by taking the polar decomposition of the classical motion operator, then you need somehow to get the correct (i.e Bose-Einstein) statistics. There might be possibilities of utilizing algebraic K-theory ideas.

Basic Idea: $A_\lambda = C^*$-algebra gen. by $F, \varepsilon \ni \frac{1}{2}(g+g^{-1}) = \lambda$ where λ assumed in $(-1,1)$. A_λ should be isomorphic to $M_2(\mathbb{C})$ as C^* alg. You have a $*$-rep $A_\lambda \longrightarrow M_2(\mathbb{C})$ s.t.

$$g \longmapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \varepsilon \longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\cos\theta = \lambda$. This $*$-rep should be a C^*-alg isom. You should know that A_λ has a unique irreducible $*$-repn V_λ up to isomorphism. The isom should be unique up to an autom. of V_λ, and such an autom is mult by a scalar in S^1.

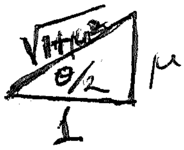
A arbitrary (say f.d.) $*$-repn. of A_λ should be isomorphic to $V_\lambda \otimes E$, where E is a f.d. Hilb space. If you restrict to reps of mult. n, the arbitrariness in the choice of E should amount to the space U_n.

§′ Real case $A_{\lambda,\mathbb{R}}$ = real C^*-alg with same gens. and relations. You should get an irred *repn. on $\mathbb{R}^2$ given by $g \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\varepsilon \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The centralizer of this repn., call it $V_{\lambda,\mathbb{R}}$, should be $\mathbb{R}$, so the autos respecting the $\| \ \|$ should be ± 1. Then an arbitrary *repn of $A_{\lambda,\mathbb{R}}$ should have the form $V_{\lambda,\mathbb{R}} \otimes E$, E Euclidean space. If E has dim n, then the arbitrariness in the choice of E should amount to O_n.

How does this compare to the C.T. picture. In the complex case you have $g = \frac{1+X}{1-X}$ where $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$ where $T: V_+ \to V_-$ satisfies $TT^* = T^*T = \mu^2$

$$\frac{g+g^{-1}}{2} = \frac{1+X^2}{1-X^2} = \frac{1-\mu^2}{1+\mu^2} = \cos\theta$$



$$\cos\left(\frac{\theta}{2}\right)^2 = \frac{1}{1+\mu^2}$$

so $\mu = \left|\tan\left(\frac{\theta}{2}\right)\right|$

$$\sin\left(\frac{\theta}{2}\right)^2 = \frac{\mu^2}{1+\mu^2}$$

What do you want? You seek a standard picture, canonical form, for an orthogonal repn of F, ε such that $\frac{1}{2}(g+g^{-1}) = \lambda \in (-1,1)$.

From the C.T. viewpoint you get

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\mu}{1+\mu^2}$$

$$g = \begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}^2 \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+TT^* \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1-T^*T & -2T^* \\ 2T & 1-TT^* \end{pmatrix} \begin{pmatrix} 1+T^*T & 0 \\ 0 & 1+T^*T \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \cos\theta & [-\sin\theta]u^* \\ (\sin\theta)u & \cos\theta \end{pmatrix}$$

$u: V_+ \xrightarrow{\sim} V_-$ orthogonal

η' Look at harmonic oscillator from the general symplectic viewpoint. What you've been looking at is harmonic oscillators from LC circuits. Review the mechanics situation $L = T - U = \frac{1}{2}\dot{x}^t m \dot{x} - \frac{1}{2} x^t k x$;

$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\quad H = p^t \dot{x} - L(x, \dot{x})$ but with $\dot{x}$ put $= m^{-1}p$

then $H = p^t m^{-1} p - \frac{1}{2}(m^{-1}p)^t m (m^{-1}p) + \frac{1}{2} x^t k x$

$$= \frac{1}{2} p^t m^{-1} p + \frac{1}{2} x^t k x$$

$$\dot{x} = \frac{\partial H}{\partial p} = m^{-1} p \qquad \dot{p} = -\frac{\partial H}{\partial x} = -kx$$

$$\begin{pmatrix} x \\ p \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & m^{-1} \\ -k & 0 \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}$$

This matrix should be skew-symmetric wrt H.

It's clear that you have a polarized Euclidean space corresponding to the two energy types: kinetic + potential. Things are mysterious from the Legendre transform to Hamilton's equations.

There should be a good viewpoint, probably using path integrals. Hamilton's principle $\delta \int L\,dt = 0$ which yields $p = \frac{\partial L}{\partial \dot{q}}$ and the contact tranformation between $t = a$ and b: $\left[p\,\delta q \right]_a^b = 0$

How do you handle LC networks? An easier question might be to take the symplectic situation and ask if it fits into Hamilton's equations. Recall the basic idea that $\Gamma_s = \begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix} \begin{pmatrix} \{V_C^d\} \\ \{I_L^d\} \end{pmatrix} \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ should be transversal to the Kirchhoff ~~conditions~~ space.

8'

$$\begin{pmatrix} x \\ p \end{pmatrix}^{\circ} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\text{symplectic form}} \underbrace{\begin{pmatrix} k & 0 \\ 0 & m^{-1} \end{pmatrix}}_{\text{Hamiltonian quad form}} \begin{pmatrix} x \\ p \end{pmatrix}$$

tomorrow (after lecture preparation!) you need to go over again the flow X arising from ω and H.

The problem is to start with H, i.e. treat phase space as a Euclidean space equipped with a skew-symmetric non singular transformation X. The phase of X gives a complex structure. ??

~~It should be clear~~ 3 objects on phase space Ω:

1) symplectic form
2) Hamiltonian H (pos. def.)
3) invertible operator X

Roughly two of these (with appropriate compatibility) $\Longrightarrow$ third

Recall

$$\underbrace{\begin{pmatrix} s & T^* \\ -T & s \end{pmatrix}}_{s-X} \begin{pmatrix} \hat{V}_C \\ \hat{I}_L \end{pmatrix} = \begin{pmatrix} V_C(0) \\ I_L(0) \end{pmatrix}$$

IVP for LC network.

$s - X$ where $X = \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$

Therefore [redacted] an LC network seems to yield a harmonic oscillator of the phase space, kinetic + potential energy types. X is skew-symmetric invertible on $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$. Polar decomposition for X and $T: V_+ \to V_-$ should be equivalent?

i Start with a real vector space V equipped with a symplectic bilinear form and a positive def symmetric form, whence V is a Euclidean space equipped with an invertible skew ~~symmetric~~ operator X. Then apply spectral theory to X. Just what is spectral theory of X? Say dim $V=2$. You pick an orth basis for V, the matrix for X should be $\begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}$, $0 \neq a \in \mathbb{R}$. So frequency is $|a|$ and the ~~[illegible]~~ direction of rotation is given by the sign of a

Vague Idea: Extract a square root of a negative quantity. There's an ambiguity in sign which you might treat as a random variable.

~~[illegible]~~ Suppose that your oscillator has a single frequency, that is, $X^2 = -\omega^2$, (note $s^2 = -\omega^2$!)

$O(2n)/U(n)$ = space of complex structures on the Euclidean space $\mathbb{R}^{2n}$. These are the possible square roots of $-I$.

Recall your old old ideas about buildings using finite dim unitary groups + self adjoint A, $0 \leq A \leq 1$. There should be some connection with less old ~~[illegible]~~ work on the moment map. (Atiyah, Kirwan, Guillemin-Sternberg)

~~[illegible]~~

Complex analog, consider f.d Hilbert space and a non-deg skew-hermitian operator X. Then $X^*X = -X^2$ so $|X| = (-X^2)^{1/2}$ and $J = X|X|^{-1}$ satisfies $J^2 = X^2|X|^{-2} = X^2(|X|^2)^{-1} = X^2(-X^2)^{-1} = -1$. Then you can identify J with the involution F given by $iF = J$. In other words, because i is already present, an operator J satisfying $J^2 = -1$ is equivalent to the F ~~[illegible]~~ corresponding to the eigenspaces of J. Except you have to pick either i or $-i$ for $F = +1$.

K′ So in the complex case the space of skew-hermitian X on $\mathbb{C}^n$ satisfying $X^2 = -\omega^2$, $\omega \neq 0$ is the space of involutions $\{F \mid F = F^*, F^2 = 1\}$. This is the total Grassmannian $\coprod_{0 \le p \le n} Gr_p(\mathbb{C}^n) \rightsquigarrow U(n)/U(p) \times U(n-p)$

Idea not to forget, about an LC network with "external" nodes & the corresponding response function. You probably also want a systematic treatment of a forced harmonic oscillator.

There is probably a quotient symplectic picture.

$$
\begin{array}{ccc}
\bar{C}^0 & \longrightarrow & C^1_C \oplus C^1_L \\
& & \Big\downarrow {\scriptstyle \Gamma_S \oplus \Gamma_S^{\perp}} \\
& & C_{1C} \oplus C_{1L} \rightsquigarrow H_1
\end{array}
$$

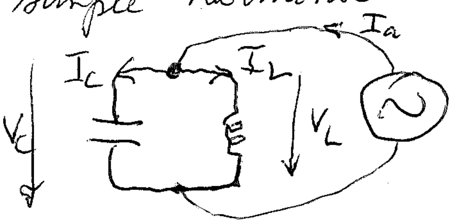
What's important is that Γ_S is transversal to $\bar{C}^0 \times H_1$.

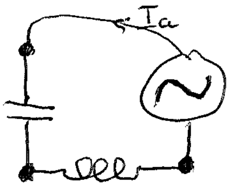
When you label 2 nodes in the graph "external" you fix the voltage drop between these nodes, which means that you restrict the node voltage function to lie in a codim 1 coset (affine hyperplane in $\bar{C}^0$). So you decrease $\bar{C}^0$. The voltage drop across the external nodes should be viewed as an applied voltage source which causes a node current to flow between the external nodes.

In effect you are changing the network by adding an edge joining the two external nodes and putting the applied voltage source in this edge. You're dealing with the Thevenin equivalent circuit. ?

λ' You are trying to understand the inhomogenous equation, i.e. the motion of an LC network with an applied voltage source. Recall that you have [redacted] first order ODE describing the motion in which the 2e dynamical variables (V_σ, I_σ) ~~Vedges~~ [redacted] reduce to the e dominant variables V_C, I_L modulo the Kirchhoff constraints. This should mean that once the forcing terms are expressed in terms of V_C, I_C then the solution of the perturbed motion is clear.

What might be a good question? Look at simple harmonic oscillator. 1 node other than the ground.





$$V_a = V_C + V_L = \left(Cs + \frac{1}{Ls}\right) I_a$$

Problem: When you connect an applied voltage source does this change the homogeneous situation? That is, the dominant variables.

$$Z = \frac{V_a}{I_a} = \frac{1}{Cs + \frac{1}{Ls}} = \frac{Ls}{LCs^2 + 1}$$

$$V_C = V_L = V_a$$

$$I_a = I_C + I_L = CsV_C + \frac{1}{Ls}V_L = \left(Cs + \frac{1}{Ls}\right) V_a$$

You are bothered by a memory of pulling back a quadratic form to a subspace, and also pushing it down to a quotient space. You spent a lot of time on this idea without getting the free motion of the LC network.

μ' Quadratic extension of $\mathbb{Q}$. $\mathbb{Q}[\sqrt{a}] = \mathbb{Q} + \mathbb{Q}\sqrt{a}$
$= \{\alpha + \beta\sqrt{\alpha} \mid \alpha, \beta \in \mathbb{Q}\}$.

~~$a \notin \mathbb{Q}$ $\sqrt{a}$~~

quadratic equation over $\mathbb{Q}$.

$$x^2 + bx + c = 0 \qquad x = -b \pm \sqrt{b^2 - c}$$

Let $a \in \mathbb{Q}$ assume $\nexists\, b \in \mathbb{Q}$ s.t. $b^2 = a$.

Back to **harmonic oscillators**. For some reason you thought that an applied voltage at some node produced a response function.

Explain: Begin with a connected LC network. You want to apply a voltage source between some nodes to the ground. Thevenin idea should explain what's happening.

The real puzzle is why you were able to get a response function at an external node without understand the free oscillations. [redacted] Actually this is not surprising, because it's easy to [redacted] find the steady state motion arising a forcing term with a **fixed** frequency ω not one of the normal modes.

What's puzzling to me is how to connect, link your result [about a subquotient of a polarized **Euclidean** space being equivalent to a type of response function] to your [redacted] picture of the free motion.

$$\begin{tikzcd}
C^0_{ext} \arrow[d, dashed, "\text{push forward}"'] & \bar{C}^0 \arrow[l, two heads] \arrow[r, hook] \arrow[d, dashed, "\text{res}"] & C^1 \arrow[d, "\Gamma"] \\
C_{0,ext} \arrow[r, hook] & \bar{C}_0 & C^1 \arrow[l, two heads]
\end{tikzcd}$$

What's wrong here is that there is nothing about the other side, the loop current space H_1 which together with $\bar{C}^0$ is the Kirchhoff constraint space.

<u>Problem</u>: How do the external nodes affect the network? You apply a voltage at the

V' node which fixes the potential at this node, and you get a current passing through the node. You need to analyze restricting voltage function and also allowing a node current. What does this do to the space of loop currents?

So what should be the approach to an applied voltage? Ideas. There should be a dynamical response not confined to $\bar{C}^0$ (the node voltage space). Something involving node currents

You expect that declaring a node to be external might change the graph by adding an extra edge joining the external node to ground.

Ideas: Thevenin, look at resistance network response

$$\begin{tikzcd}
\bar{C}^0 \arrow[r] & C^1 \arrow[r, two heads] \arrow[d, "\Gamma"] & H^1 \\
\bar{C}_0 & C_1 \arrow[l, two heads] \arrow[r, hook] & H_1
\end{tikzcd}$$

Γ positive def symm. bilinear form.

no dynamics at all.

Suppose you make one of the nodes ($\neq$ ground) external and apply a voltage source. Two changes: the potential at the external node is fixed, there is a possible node current. It **looks like** H_1 increases and $\bar{C}^0$ decreases.

Restricting the node potential to a hyperplane should be an inhomogeneous constraint, which is expressed via a map $C^1 \longrightarrow H^1 \times \mathbb{R}$. Maybe this is the same as adding an edge joining the ground to the external node.

9'

Go back to the IVP for a closed LC network.

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, "\binom{1}{T}"] & C^1_C \oplus C^1_L \arrow[r, "(-T\ 1)"] \arrow[d, "s"'] & H^1 \\
\bar{C}_0 & C_{1C} \oplus C_{1L} \arrow[l, "(1\ T^*)"'] \arrow[u, "s^*"', shift right=3] & H_1 \arrow[l, "\binom{-T^*}{1}"']
\end{tikzcd}
\qquad (-T\ 1)\begin{pmatrix}\hat{V}_C \\ \hat{V}_L\end{pmatrix} = 0$$

$$\dot{V}_C = I_C \qquad \dot{I}_L = V_L \qquad \hat{\dot{f}}(s) = -f(0) + s\hat{f}(s)$$

$$\mathcal{L}\dot{V}_C = -V_C(0) + s\mathcal{L}V_C$$

$$\hat{V}_L = \hat{\dot{I}}_L = -I_L(0) + s\hat{I}_L \qquad T\hat{V}_C = \hat{V}_L$$

$$\hat{I}_C = \hat{\dot{V}}_C = -V_C(0) + s\hat{V}_C \qquad \hat{I}_C = -T^*\hat{I}_L$$

$$s\hat{V}_C + T^*\hat{I}_L = V_C(0)$$

$$-T\hat{V}_C + s\hat{I}_L = I_L(0)$$

$$\left\{ s - \begin{pmatrix} & -T^* \\ T & \end{pmatrix} \right\} \begin{pmatrix}\hat{V}_C \\ \hat{I}_L\end{pmatrix} = \begin{pmatrix} V_C(0) \\ I_L(0)\end{pmatrix}$$

There should be no problem about the forced oscillator here. This is a first-order linear ordinary DE constant coeff. The [redacted] forcing term can be any [redacted] function of time with values in the space of dominant variables.

This [redacted] means that you can put a varying voltage source

0′ in series with a capacit~~ance~~or for each capacitance edge, and a varying current source in series with ~~the~~ inductor for each inductance edge.

This seems to be the most you can say about a given connected LC network.

Next you want to apply a voltage to a node (Really: from the ground to the node). This means you add an edge to the graph, where the edge contains the voltage source.

The interesting point is that the edge has neither L nor C type. If the

Recall the idea that in a resistance network it is useful to assume each edge is a pure e.m.f. in series with an internal resistance. (This is part of the Thevenin thm.)

DeRh ex.
$\bar{C}^0 \xhookrightarrow{S} C^1$

Review R networks.

$$\begin{tikzcd}
\bar{C}^0 \arrow[r, hook] \arrow[d, bend right, "\cong"'] & C^1 \arrow[d, "R^{-1}"] & \\
C_1/H_1 & C_1 \arrow[l, two heads] & H_1 \arrow[l, hook']
\end{tikzcd}$$

The problem: How to handle no resistence of the attached edge. No problem because if your voltage source is a battery, $\exists$ internal resistance.

You have to handle the case where the resistance of the new edge is 0, maybe also the case $R = \infty$.

The point should be that even though the Ohm's Law correspondence Γ

π' between C' and C_1 is singular, it might be nonsingular modulo the Kirchhoff space, i.e. Γ is a complement for $\bar{C}^0 \oplus H_1$ in $C' \oplus C_1$.

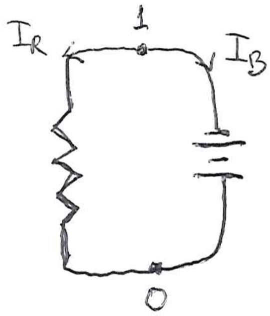
Let σ denote the added edge, let $\bar{C}^0, C', H', \bar{C}_0, C_1, H_1$ pertain to the original graph. Then the augmented graph has extra variables V_σ, I_σ. Its picture

$$\bar{C}^0 \hookrightarrow C' \oplus \{V_\sigma\} \twoheadrightarrow H' \oplus \{V_\sigma\}$$

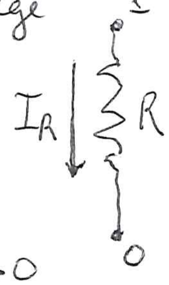
$$\bar{C}_0 \twoheadleftarrow C_1 \oplus \{I_\sigma\} \hookleftarrow H_1 \oplus \{I_\sigma\}$$

It's probably better to begin with a conn. resistance network and remove an edge such that the new graph is connected.

Orient edges by ordering the nodes



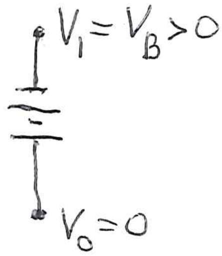
R edge



The positive direction is from 1 to 0. So

$$V_R = V_1 - V_0, \quad V_0 = 0$$

B edge + direction



I_B has wrong sign $I_B = -I_R$

$TV_C = V_L \qquad I_C + T^* I_L = 0$

$$\bar{C}^0 \xrightarrow{\binom{1}{T}} C' \xrightarrow{(-T\ 1)} H'$$

$$s\hat{V}_C - \underbrace{\hat{I}_C}_{-T^*\hat{I}_L} = V_C(0)$$

$$\underbrace{-\hat{V}_L}_{-TV_C} + s\hat{I}_L = I_L(0)$$

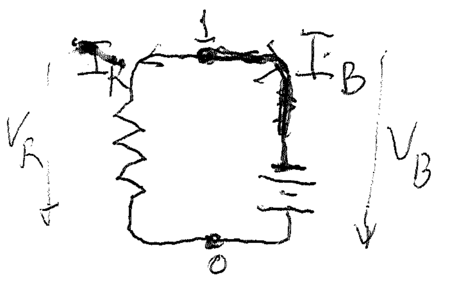
$$\bar{C}_0 \xleftarrow{(1\ T^*)} C_1 \xleftarrow{\binom{-T^*}{1}} H_1$$

$$\begin{pmatrix} s & +T^* \\ -T & s \end{pmatrix}$$

§′ external edge joining different nodes 0 resistance pure emf sources. What equations do you need to solve for the response?

First suppose the external edge has a (small) resistance. The equations to solve should be inhomogeneous, the associate homogeneous equations are Ohm's Law for the edges (e equations) together with Kirchhoff's **1st + 2nd** Laws (e equations). The inhomogeneous term should arise from the applied emf on the external edge.

What form should **the "forcing" term** take? Recall that the full collection of $2e$ state variables is redundant because of the e Kirchhoff **constraints**. So you probably have to select e state variables which are independent of the e Kirchhoff constraints, like choosing the dominant variables in the LC case. Back to



$$\begin{tikzcd}
\{V_1\} \arrow[r, "\begin{pmatrix}1\\1\end{pmatrix}"] & \left\{\begin{pmatrix}V_R\\V_B\end{pmatrix}=\begin{pmatrix}V_1\\V_1\end{pmatrix}\right\} \arrow[r, "\begin{pmatrix}1 & -1\end{pmatrix}"] \arrow[d, "\begin{pmatrix}R^{-1} & 0\\0 & 0\end{pmatrix}"'] & \{\ \} \\
\{I_1\} & \left\{\begin{pmatrix}I_R\\I_B\end{pmatrix}\right\} \arrow[l, "\begin{pmatrix}1 & 1\end{pmatrix}"'] & \{I_\lambda\} \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"']
\end{tikzcd}$$

??

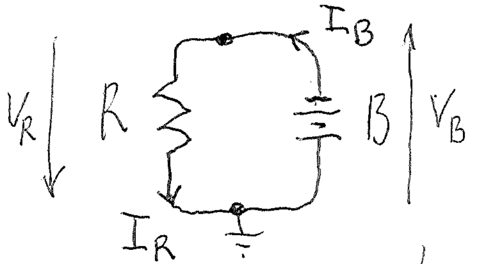
$V_R = V_B = V_1$

$I_\lambda = I_R = -I_B$

$V_R = R I_R$

V_B fixed $= V_1$

Let's try again to understand the 2 edge loop circuit. There are 4 variables



V_R, I_R, V_B, I_B. The arrows on the R and B indicate the orientation chosen for these edges. I_R and I_B are expected to be positive. V_R is a positive voltage drop, the beginning at the top node is higher than the ending at the bottom node. But V_B is negative because it start with the potential zero at the ground, and it ends with the higher potential at the top node.

You have the three equations

$$\underbrace{V_R + V_B = 0, \quad I_R = I_B}_{\text{Kirchhoff}}, \quad \overbrace{V_R = R I_R}^{\text{Ohm}}$$

There does not seem to be a 4th equation linking I_B, V_B.

But the three equations yield the relation

$$-V_B = +V_R = +RI_R = +RI_B$$

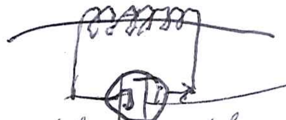
between the voltage drop and current in a battery edge.

The situation is unsatisfactory. You hoped to get a clear picture of the effect of applied voltage and applied current sources, by exploiting Thevenin theory. In this theory one treats the applied source by adding a new edge to the graph.

Idea: Voltage source is attached to nodes; it involves $\tilde{C}^0$, really a linear functional on $\tilde{C}^0$, an element of $\tilde{C}_0$. So maybe an applied current source

σ' should be associated to a linear functional on the space H_1, i.e. an element of H'. You would then be linking external sources to the dual of the Kirchhoff space.

Here's how to produce a current source in an edge, namely, put the wire through a solenoid



A.C. Voltage source

run current thru the solenoid, you get a magnetic field induced in the wire which should induce a alternating current in the wire. Maybe simpler would be to run A.C. generator with permanent magnet field core.

The central problem is to understand forcing terms, inhomogeneous terms in an LC network. There is a straightforward answer when you start with the IVP for the network.

But what happens when you are not given the forcing term in terms the dominant variables?

Ideas look at s.h.o., see if you can handle non dominant forcing terms.

Elec. Eng State Variables $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, somehow this yields a $C(s-A)^{-1}B + (D?)$. Something similar occurs when you have a space of 1-particle states for a quantum field, the Hamiltonian is A, interacting - being scattered by heavy particle

---

$|A| + |B| \geq |A+B|$

I'

So what to do next? Maybe look at the degenerate frequencies. Start with polarized Euclidean space and subspace. Everything gets split into into s.h.o.'s and certain degenerate cases. Exactly what should be related to $g = \pm 1$, so the $\breve{B}, C$ wings should perhaps be linked to L and C. Question about $g = \pm 1$ and frequencies $0, \infty$?

Idea: Eliashberg rigidity thm: a C^0 limit of C^∞ symplectic transformations is a C^∞ limit. Better: a sequence of C^∞ symplectic transformations which converges in the C^0-topology in fact converges in the C^∞-topology. You might be able to use the C^∞ limit to prove decay.

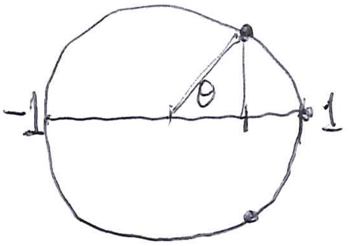
Return to the Grassmannian $\boxed{SL_2(\mathbb{R}) \text{ [illegible]} \to \text{closed UHP}}$

You have polarized Euclidean space $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ and a subspace $W \subset V$, whence $F = \begin{cases} +1 \text{ on } W \\ -1 \text{ on } W^\perp \end{cases}$ $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on and orthog

Then decompose according to spectrum of $\frac{1}{2}(g + g^{-1})$, $g = F\varepsilon$

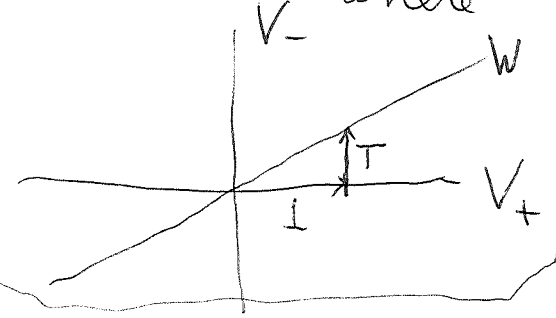
You should split V, F, ε into s.h.o.'s for eigenvalues $\cos\theta \in (-1, 1)$. Next you need to understand the cases $g = +1$ whence $F = \varepsilon$

$g = -1$ — $F = -\varepsilon$



$|A - B| \le \big||A| - |B|\big|$?

u′ Back to $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ $W \subset V$. You've been looking at the case where $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$ with $T : V_+ \to V_-$.



~~occurs~~ when $W \to V_+$ is

This case the projection an isomorphism.

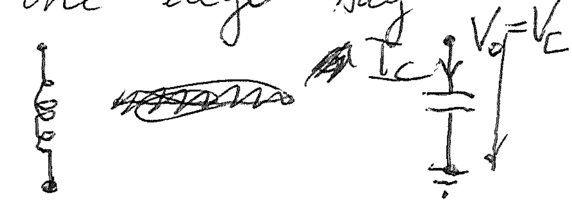
In general it has a kernel + cokernel — kernel $= W \cap V_-$, cokernel $W^\perp \cap V_+$. $W \cap V_-$ is where $F = +1$, $\varepsilon = -1$ and $W^\perp \cap V_+$ is where $F = -1$ and $\varepsilon = +1$. These comprise the -1 eigenspace for g. Remove these, reduce to case where $W \xrightarrow{\sim} V_+$ whence $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$, then look at proj. $W \to V_-$. Kernel is $W \cap V_+$, $F=+1, \varepsilon=+1$) cokernel $V_- \cap W^\perp$, $F = -1$, $\varepsilon = -1$, get the $+1$ eigenspace for g.

Free ??

This is the case where $H^1 = 0 = H_1$

$$\begin{array}{ccccc} \bar{C}^0 & \hookrightarrow & \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} & \twoheadrightarrow & H^1 \\ & & s \downarrow \uparrow s & & \\ \bar{C}_0 & \twoheadleftarrow & \begin{pmatrix} C_{1,C} \\ C_{1,L} \end{pmatrix} & \hookleftarrow & H_1 \end{array}$$

Simplest case is one edge say



Take the C case first

constraints $I_C = 0$

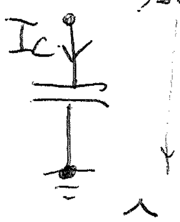
$$\{V_0\} = \bar{C}^0 \xrightarrow{\sim} C^1_C = \{V_C\}. \quad V_C = V_0$$
$$Cs \downarrow$$
$$I_0 = I_C = CsV_0 \quad \bar{C}_0 \xleftarrow{\sim} C_{1,C},\ I_C = CsV_0$$

IVP: $C\dot{V}_C = I_C$

$Cs\hat{V}_C - V_C(0) = \hat{I}_C$

$Cs\hat{V}_C - \hat{I}_C = CV_C(0)$

ϕ' Go over the 4 tried degenerate cases: 1st $\bar{C}^0 \xrightarrow{\sim} C^1_C$

circuit 

~~[illegible]~~ Constraint Kirchhoff I

$\therefore I_C = 0$. $\quad C\dot{V}_C = I_C$

L.T. $\quad Cs\hat{V}_C = -CV_C(0) + \hat{I}_C$

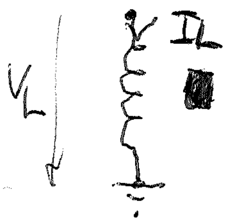
$$\int_0^\infty e^{-st}\dot{F}(t)\,dt = \int_0^\infty \left((e^{-st}F)^{\cdot} + se^{-st}F\right)dt$$

$$\boxed{\hat{\dot{F}} = -F(0) + s\hat{F}}$$

$$Cs\hat{V}_C = -CV_C(0)$$

$$\hat{V}_C = \frac{1}{s}V_C(0)$$

$$V_c(t) = H(t)V_c(0).$$



$I_L = 0$. Kirchhoff I ~~[illegible]~~

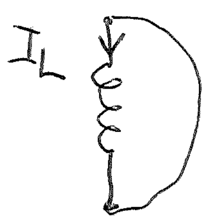
$L\dot{I}_L = V_L$?? $\quad \widehat{L\dot{I}_L} = Ls\hat{I}_L - LI_L(0)$

$\bar{C}^0 \xrightarrow{\sim} C^1_L \qquad \{V^0\} \longrightarrow \{V_L\}$

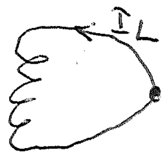
$\bar{C}_0 \xleftarrow{\sim} C_{1,L} \qquad \{I_L\}$

apparently this ~~[illegible]~~ circuit has no motion

$0 = \bar{C}^0 \longleftrightarrow C^1_L \xrightarrow{\sim} H^1$

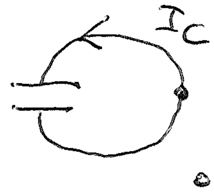


$V_L = 0$



$0 \qquad C_{1,L} \xleftarrow{\sim} H_1$

$0 = \bar{C}^0 \longleftrightarrow C^1_C \xrightarrow{\sim} H^1$



$V_C = 0$

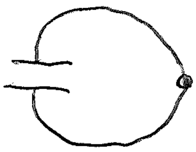
$C\dot{V}_C = I_C$

$C_{1,C} \xleftrightarrow{\sim} H_1$

no motion at all

X' You've examined four cases, and found 2 with no motion. These are

$\bar{C}^0 \simeq C^1_L$ $\varepsilon = -1,\ F = 1$

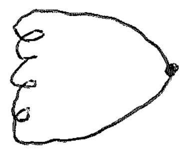


$C^1_C \simeq H^1$ $\varepsilon = +1,\ F = -1$

These gives the $g = -1$ eigenspace

Other two which have motion

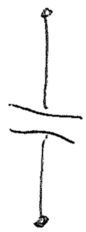
$\bar{C}^0 \xrightarrow{\sim} C^1_C$ $\varepsilon = +1,\ F = +1$



$C_{1,L} \xrightarrow{\sim} H_1$ $\varepsilon = -1,\ F = -1$

These gives the $g = +1$ eigenspace.

Tree



$$\begin{tikzcd} \bar{C}^0 \arrow[r, shift left] & C^1 \arrow[l, shift left] \arrow[d, leftrightarrow, "\Gamma_s"] \\ \bar{C}_0 \arrow[r, shift left] & C_1 \arrow[l, shift left] \end{tikzcd}$$

What happens to the IVP? Is it true that Γ_s is a comp. to the Kirchhoff space C^1.

$$\begin{tikzcd} \bar{C}^0 \arrow[r, "\sim"] & C^1_c \arrow[d, "s"] \\ \bar{C}_0 & C_{1,C} \arrow[l, "\sim"'] \end{tikzcd}$$

$C^1_c = \{V_c\}$

$C_{1,c} = \{I_c\}$

~~[illegible]~~ $\dot{V}_C = I_C$ $\hat{I}_C = \hat{\dot{V}}_C = s\hat{V}_C - V_c(0)$

$\Gamma_s = \left\{ \begin{pmatrix} \hat{V}_C \\ \hat{I}_C \end{pmatrix} \middle| \ s\hat{V}_C = \hat{I}_C \right\}$ $\bar{C}^0 = \left\{ \begin{pmatrix} V_C \\ 0 \end{pmatrix} \right\}$.

ψ'

$$\bar{C}^0 \Rightarrow C^1_L \qquad C^1_L = \{V_L\}$$

$$\bar{C}_0 \Leftarrow C_{1,L} \qquad C_{1,L} = \{I_L\}$$

$$\dot{I}_L = V_L \qquad \hat{V}_L = \hat{\dot{I}}_L = s\hat{I}_L - I_L(0)$$

$$\bar{C}^0 = \begin{pmatrix} C^1_C \\ C^1_L \end{pmatrix} \qquad \begin{matrix} \{V_C\} \\ \{V_L\} \end{matrix}$$

$$C^1_C \xrightarrow{s} C_{1,C} \qquad C_{1,L} \xrightarrow{s} C^1_L$$

$$\bar{C}_0 = \begin{pmatrix} C_{1,C} \\ C_{1,L} \end{pmatrix} \qquad \begin{matrix} \{I_C\} \\ \{I_L\} \end{matrix}$$

You probably need to keep track of $s = 0, \infty$. Consider a tree with L and C type edges. You want the Impedance correspondance $\Gamma_s \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$ to be complementary to the Kirchhoff space $\begin{pmatrix} C^1 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} C^1 \\ C_1 \end{pmatrix} = \begin{pmatrix} C^1_C \\ C_{1,C} \end{pmatrix} \oplus \begin{pmatrix} C^1_L \\ C_{1,L} \end{pmatrix}$$

$$\qquad\qquad \parallel \qquad\qquad\qquad \parallel$$

$$\Gamma_s = \begin{pmatrix} 1 \\ s \end{pmatrix} C^1_C \oplus \begin{pmatrix} s \\ 1 \end{pmatrix} C_{1,L}$$

when is $\Gamma_s \cap \begin{pmatrix} C^1 \\ 0 \end{pmatrix} \neq 0$. If $s = \infty$, this intersection is C^1_L. If $s = 0$, this intersection is C^1_C.

ω' ζ function of a graph $\stackrel{?}{=}$ characteristic polynomial of some correspondence? Kronecker modules. Something related to fixpts of iterates. There are interesting examples related to geodesic flows.

General $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$. You want the projection $W \longrightarrow V_+$ to be an isom. so that $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$ $\quad T: V_+ \longrightarrow V_-$

So look at the kernel of $W \longrightarrow V_+$ i.e. $W \cap V_-$, where $F = 1$ and $\varepsilon = -1$, $\therefore g = -1$. Split off the kernel, look at image $W \hookrightarrow V_+$, orthog space is $W^\perp \cap V_+$, where $F = -1, \varepsilon = +1$ so $g = -1$.

Now you want to find a picture. First make T invertible. Consider the projection $W \longrightarrow V_-$ remove the kernel $W^\perp \cap V_+$, and remove the "cokernel" $V_- \cap W$. These two spaces have $F = -1, \varepsilon = +1$ and $F = +1, \varepsilon = -1$ resp., hence comprise all of $g = +1$ eigenspace.

You would like a picture consisting of the "core" where W projects bijectively to both V_+ and V_- (i.e. $T: V_+ \to V_-$ is invertible), and 4 "wings". Where to start? The case $W = \begin{pmatrix} 1 \\ T \end{pmatrix} V_+$ where T is not invertible. [redacted]

Let's start again, choose V_+, V_- with pos. def. scalar products, put $V = \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ and let $W \subset \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$. Let's digress a little back to W a linear retract of $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ pol. fnc.

$$W \xleftarrow{(\alpha_+^* \ \alpha_-^*)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W \qquad \underbrace{\alpha_+^* \alpha_+}_{h_+} + \underbrace{\alpha_-^* \alpha_-}_{h_-} = 1_W$$

$$h_+ = \frac{1}{1+\omega^2} \qquad h_- = \frac{\omega^2}{1+\omega^2} \qquad \frac{s + \omega^2 s^{-1}}{1+\omega^2} = \frac{s^2+\omega^2}{s(1+\omega^2)}$$

$$\begin{pmatrix} s & T^* \\ -T & s \end{pmatrix} = s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -T^* \\ T & 0 \end{pmatrix}$$