June 21, 02

\[ C^0 \xrightarrow{\mathcal{E}} C^1 \quad \text{an isomorphism of } C^0 \text{ gives the input} \]
\[ E_x^0 \quad \text{voltage at each mode } \]
\[ E_x^0 \quad \text{an isomorphism } C^1 \text{ gives a voltage} \]
\[ \text{drop } E_x^1 \text{ for each (oriented) edge } \]
\[ \text{of } C^1 \]
\[ \text{gives a current for each edge } \]
\[ \text{the output current } I_{x^1} \text{ for each mode } x^1 \]
\[ Z_0 = \frac{E_0}{I_0} = \text{the impedance of the edge } \]
\[ \text{the induced output current function.} \]

Then \[ \mathcal{Z}^{-1} : C^0 \rightarrow C^0 \] assigns to any input

Q: Is \( \mathcal{Z}^{-1} \) an isomorphism? good case?

\[ E_x, I_x \]
\[ I_x = I_c + I_L \]
\[ E_x = E_c = E_L \]
\[ \frac{E_x}{I_c} = \frac{1}{L_s} \]
\[ \frac{E_L}{I_L} = L_s \]
\[ I_x = C_s E_x + \frac{1}{L_s} E_L \]
\[ I_x = (C_s + \frac{1}{L_s}) E_x \]
\[ \text{becomes singular form } s^2 + \frac{1}{L_s} = 0 \]

You would like to understand again your study

of the situation where the input voltage for and output

current function is supported on a given subset of the nodes.

\[ C^0 \text{ functions on } \Gamma \]

is replaced by \( C^0 = \text{free on } S \)

Idea? Impose \( E_x = 0, \quad x \in S \) get \( C^0 = C_s \)

The point is that \( E_x \) for \( x \in S \) is given, while \( E_x \) for \( x \notin S \) is somehow determined by \( I_x = 0, \quad x \notin S \)

Next, leave the graph business, and clean up the
June 10, 2002. So back to rubber band example, why a rubber band contracts when heated.

To understand Brownian motion, Einstein's derivation of Arrhenius number, equivalently Boltzmann's constant, using Brownian motion, what you would like is an exact model.

\[ \nabla D, \text{ spin}^\circ \text{ structure} \]

if \( U \subset \mathbb{R}^n \), symbol of order \( m \in \mathbb{Z} \) smooth \( p: U \times \mathbb{R}^n \to \mathbb{C} \)

\( \forall k < U \), \( \alpha, \beta \) \( \alpha \circ \beta : \mathcal{E} \overset{\leq 0}{\longrightarrow} \)

\[ \sum_{n=0}^{m} \nabla \cdot p(x, \xi) \leq C_k \alpha \beta \left( 1 + \| \xi \|^2 \right)^{m-k} \]

Restrict to classical symbols

\[ p = \sum_{k=0}^{m-N} p_{m-k} \in \mathcal{S}^{m-N}(U \times \mathbb{R}^n) \]

June 20, 2002. LC circuit = directed graph whose edges have either inductance or capacitance type. Choose orientation for each edge, consider \( C \)-valued chains, \( \dim C = 1 \), also cochains.

\[ C^0 \overset{\delta}{\longrightarrow} C^1 \]

\[ C^0 \overset{\partial}{\longleftarrow} C^1 \]

\( C^0 \) consists of voltage functions on the nodes

\( C^1 \) consists of voltage drops for the edges.

Equations?

\[ \begin{pmatrix} I_{xy} \end{pmatrix} \quad \begin{pmatrix} E_x - E_y \end{pmatrix} \]

\[ \begin{pmatrix} I_{xy} \end{pmatrix} = C_{xy} \left( \begin{pmatrix} E_x - E_y \end{pmatrix} \right) \]

\[ E_{xy} = L \begin{pmatrix} I_{xy} \end{pmatrix} \]

Charge

\[ Q = CE \quad C \dot{E} = \dot{Q} = I \]

\[ I_{xy} = C_{xy} \left( \dot{E}_x - \dot{E}_y \right) \]

\[ E_{xy} = L \dot{I}_{xy} \]

\[ E = L \dot{I} \quad \text{take L.T.} \quad \hat{E} = L \hat{s} \hat{I} \]

\[ Z = L \hat{s} \quad \text{ind} \]

\[ \hat{I} = C \dot{s} \hat{E} \quad \dot{s} = \frac{1}{C} \quad \text{cap} \]
quadratic form picture. (Idea: You might have a symplectic situation where \( E, I_x \) are dual variables. Symplectic reduction?)

General linear picture: Start with \( C^1 \) which is a polarized Hilb space \( H = H_{-1} \oplus H_1 \). To be specific \( H = H_L \oplus H_C \). You seem to want an energy around. Simple cases. \( \frac{1}{L} \frac{I}{C} \). You should recall the energy somehow. Go back to transmission line?

\[
\begin{align*}
E_x - E_{x+dx} &= \lambda dx \frac{d}{dx} I_x \\
I_x - I_{x+dx} &= \gamma dx \frac{d}{dx} E_x
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 E}{\partial x} + \lambda \frac{\partial^2 I}{\partial x} &= 0 \\
\frac{\partial^2 E}{\partial t} - \gamma \frac{\partial^2 E}{\partial x} &= 0
\end{align*}
\]

Suppose \( \lambda = \gamma = 1 \). \( \left\{ \begin{array}{l}
\frac{\partial}{\partial x} E + \frac{\partial}{\partial t} I = 0 \\
\frac{\partial}{\partial t} E + \frac{\partial}{\partial x} I = 0
\end{array} \right. \) \( \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) (E - I) = 0 \) \( \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \right) (E + I) = 0 \)

Energy should be \( \int EI \, dx \).

\[
\frac{1}{2} \int EI \, dx = \int \left( (-\frac{\partial}{\partial x} I + E (-\frac{\partial}{\partial x} E) \right) \, dx = \frac{1}{2} \int \frac{\partial}{\partial x} \left( I^2 + E^2 \right) \, dx
\]

Time dep. \( e^{xt} \)

\[
\begin{align*}
\left( \frac{\partial}{\partial x} - s \right) (E - I) &= 0 \\
\left( \frac{\partial}{\partial x} + s \right) (E + I) &= 0
\end{align*}
\]

\( E - I \) \( e^{sx} \) outgo \( E + I \) \( e^{-sx} \) income
Consider $H = H_+ \oplus H_-$ equipped with the family $s/\|x_+\|^2 + s^{-1}/\|x_-\|^2$ of pseudo-hermitian forms, $s \in \mathbb{R}_{>0}$.

This is the abstract version of $C^1$, $SC$ subspace of $H$.

Call it $V$. $V \rightarrow H_+ \oplus H_-$ probably you want $s \circ s^{-1}$ to use the spectrum idea for the Grassmannian

$V \leq \leftarrow H_+ \oplus H_-$

$j = (j^+ + j^-)$

$j^* (s \circ s^{-1}) j = s j^+ j^+ + s^{-1} j^- j^-$

Move on to the induced hermitian form on the "cokernel of $j$"

\[ V \xrightarrow{i} H \xrightarrow{j} W \]

\[ i^* A i \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ V^* \xrightarrow{i^*} H^* \xrightarrow{j^*} W^* \]

Assume $i^* A i$ invertible, then you get an induced map $W \rightarrow W$.

Then $w \in W$ choose $h \in V$, $j(h) = w$. Form $(i^* A i)^{-1} i^* A h \in V$ and $A(h - i(i^* A i)^{-1} i^* A h)$. Operator you want is

$A' = A - A i (i^* A i)^{-1} i^* A : H \rightarrow H^*$

$A' i = 0$, $i^* A' = 0$

So $A'$ induces a map $W \rightarrow W^*$.

$\exists! A^* : W \rightarrow W^*$ s.t. $j^* A^* j = A'$.
Abstract: LC circuit consists of \( H = H_+ \oplus H_- \), the complex Hilbert space of 1-cochains split into \( L \) and \( C \) parts. You should link the inner product to energy (or power).

\[
\begin{align*}
&\mathbb{C}^0 \xrightarrow{S} \mathbb{C}^1 \\
&\mathbb{R}^{-1} \\
&\mathbb{C}_0 \xleftarrow{\partial} \mathbb{C}_1
\end{align*}
\]

\( C' = \) space of \( C \)-valued functions on edges
\( C_i = \) space of \( C \)-valued 1-cochains... also = functions on edges.

\[
Z_\sigma = \begin{cases} 
L_s, & \text{if } E \in C' \\
\frac{1}{C_s}, & \text{if } (Z'^{-1}E)_\sigma = Z'^{-1}_\sigma E_\sigma = I_\sigma
\end{cases}
\]

Power: pairing between \( C' \) and \( C_i \), \( \sum_{\sigma} E_\sigma I_\sigma = \sum_{\sigma} Z'^{-1}_\sigma I_\sigma^2 \)

what about conjugation.

You need a better picture. Try coupling a 1-port to a transmission line.

\[
\begin{align*}
&\partial_t E + \partial_x I = 0 \\
&\partial_t I + \partial_x E = 0
\end{align*}
\]

L.T.
\[
\begin{align*}
&(\partial_x + s)(E + I) = 0 \\
&(\partial_x - s)(E - I) = 0
\end{align*}
\]

\[
\begin{align*}
E + I &= e^{-sx} \\
E - I &= e^{sx}
\end{align*}
\]

June 23, 02: Work out transmission line coupled to an LC 1-port, say

\[
\begin{align*}
E_x - E_{x+dx} &= dx \dot{I}_x \\
I_x - I_{x+dx} &= dx \dot{E}_x
\end{align*}
\]

\[
\begin{align*}
(\partial_x + \partial_t)(E + I) &= 0 \\
(\partial_x - \partial_t)(E - I) &= 0
\end{align*}
\]

\[
\begin{align*}
I_L + I + I_0 &= 0 \\
E_L &= E_C = E_0 \\
I_C &= C \dot{E}_C
\end{align*}
\]
\[ I_0 = -I_L - I_C = -E_0 \left( \frac{1}{L_s} + C_s \right) \]

\[ \frac{-E_0}{I_0} = \frac{1}{L_s + C_s} = \frac{L_s}{1 + L C s^2} \]

\[ E_x - E_{x+dx} = dx \frac{d}{dx} I_x \]

\[ I_x - I_{x+dx} = d\frac{d}{dx} E_{x+dx} \]

\[ \partial_x E + \partial_y I = 0 \]

\[ \partial_x I + \partial_y E = 0 \]

\[ E - I = A e^{s x} \]

\[ E + I = B e^{-s x} \]

You need more detail.

At \( x = 0 \) let \( E_0 \) be the voltage relative to ground and let \( I_0 \) be the current into the transmission line. Then \( E_0 \) is the voltage at the 1-port and \( -I_0 \) is the current into the 1-port, so

\[ \frac{E_0}{I_0} = -Z \]

\( Z \) is the impedance of 1-port.

\[ \frac{-E_0}{I_0} = L_s + \frac{1}{C_s} \]

\[ E_0 - I_0 = A \]

\[ E_0 + I_0 = B \]

\[ \frac{A}{B} = \frac{Z + 1}{Z - 1} \]

\[ \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} \]
June 24, 02

You want to understand properly the abstract setting for a LC circuit with external modes. Call this an LC port. To begin again your theme that an LC port is a subagreement of a polarized Hilbert space. Start with simple cases, namely:

$$L \quad C \quad \frac{1}{L} \quad \frac{1}{C}$$

You probably want to assume the graph is connected, and provided with a ground mode.

Q: Are there a space of states for the LC circuit? Could this be a polarized Hilbert space? Consider the inductance. A state for $\frac{1}{L}$ can any pair $(E, I)$ of real numbers. A history of states is a pair $(E(t), I(t))$ of real functions of time satisfying

$$E = L \frac{d}{dt} I.$$ 

Can you fit this into some Lagrangian or Hamiltonian formalism?

Particle moving with constant velocity, in the C case.

$$\frac{1}{C} \quad \text{states} \quad (E, I) \in \mathbb{R}^2, \quad \text{equation of motion:} \quad \frac{d}{dt} I = C \frac{d}{dt} E.$$ 

For a general LC circuit you start with a state space consisting of pairs $(E, I)$ for each edge. Then you impose constraints. This should involve something like symplectic quotient, where you restrict the voltage function to be conservative, and kill currents with boundary $= 0$.

Your problem is how to make progress. Let's begin with an LC circuit which is connected and has a ground mode specified, also suppose the edges are oriented. Ignore the modes—you get a state space, consisting of pairs $(E, I)$ for each edge $e$.

Somehow you need to get Kirchhoff equations

$$0 \rightarrow \tilde{C}_o \rightarrow \delta \rightarrow C' \rightarrow H' \rightarrow 0$$

$$\downarrow L^{-1}$$

$$0 \rightarrow \tilde{C}_o \rightarrow \delta \rightarrow C_1 \leftarrow H_1 \leftarrow 0$$

The main point here is that $\exists Z^{-1}: \tilde{C}_o \rightarrow \tilde{C}_o$ invertible.
\[0 \to \tilde{C}_0 \xrightarrow{g} C_1 \to H' \to 0 \]
\[0 \to \tilde{C}_0 \to \mathcal{C}_1 \to H \to 0\]

Points: \( (\tilde{C}_0, \tilde{C}_1) \) dual pair
\( (\mathcal{C}_1, C_1) \)

\[Z : \mathcal{C}_1 \to C_1 \] is equivalent to the quadratic form on \( C_1 \) with values
\[Zx = \begin{cases} Ls & \text{if } x \in \mathcal{C}_1 \text{ is diagonal} \\ (Cs)^{-1} & \text{assume } s \neq 0 \end{cases} \]

---

June 25, 02

\[V \to H_+ \otimes H_- \to H/V \]

\[H = H_+ \oplus H_- \text{ is a polarization} \]

\( V \) is a subspace, \( H/V \) is the corresuden subspace. On \( H \)
you consider the semi-hermitian form \( s \| \xi \|^2 + s^{-1} \| \eta \|^2 \), say \( s > 0 \)
which induces semi-hermitian forms on \( V \) and \( H/V \). To
work out the formulas, on \( H \) you have \( F, E \) and
the situation splits according to eigenvalues \( J = FE \).

Exact sequences \( V \to H \to W \) together with
a hermitian form on \( H \). Assume the form rest to \( V \)
is non-deg. Then you get an induced form on \( W \)

Do the real case: \( V \to H \to W \), \( A \) = quadratic
form on \( H \), \( A|_V \) non-degenerate. Given \( \xi^0 + V \in W \)
where \( \xi \in H \). Look for stationary values \( \xi \) of \( A(\xi + \eta) \)
for \( \eta \in V \), \( \eta = 8(A(\xi + \eta)) = 2A(\xi + \eta, \delta \eta) = \langle \delta \eta | A(\xi + \eta) \rangle\)
\[\langle \xi + \eta | A(\xi + \eta) \rangle = \langle \delta \eta | A(\xi + \eta) \rangle + \langle \xi + \eta | A \delta \eta \rangle\]
\[= 2 \langle \delta \eta | A(\xi + \eta) \rangle = 0 \]
Means: \( \xi + \eta \perp V \)
for the quadratic form \( A \).
\[ 0 \rightarrow V \xrightarrow{\delta} H \xrightarrow{A} W \rightarrow 0 \quad \text{short exact seq} \]

\[ a(\xi) = \langle \xi | A \xi \rangle \quad \text{quadratic function on } H \]

\[ A : H \rightarrow H^* \quad \text{symmetric map} \]

Given coset \( \xi_0 + V = \{ \xi_0 + \sigma \mid \sigma \in V \} \)

\[ a(\xi_0 + \sigma) = \langle \xi_0 + \sigma | A \xi_0 + A \sigma \rangle \]

Apply variation \( \delta \sigma \)

\[ \delta a(\xi_0 + \sigma) = \langle \delta \sigma | A(\xi_0 + \sigma) \rangle + \langle \xi_0 + \sigma | A \delta \sigma \rangle \]

\[ = \langle \delta \sigma | A(\xi_0 + \sigma) \rangle + \langle A(\xi_0 + \sigma) | \delta \sigma \rangle \]

\[ = 2 \langle \delta \sigma | A(\xi_0 + \sigma) \rangle = 2 \langle \xi_0 + \sigma | A \delta \sigma \rangle \]

\( \xi_0 + \sigma \) is a stationary point for \( a(\xi) \) if \( \xi \in \xi_0 + V \)

iff \( \langle \xi_0 + \sigma | AV \rangle = 0 \)

\[ \xi = \xi_0 + \sigma \]

Consider a coset \( q^{1\omega} \) of \( V \) in \( H \)

\[ g(\xi) = \langle \xi | A \xi \rangle \]

let \( \xi \) be stationary for \( q^{1\omega} \)

\[ \delta g(\xi) = g(\xi + \delta \omega) - g(\xi) = \langle \xi + \delta \omega | A \xi + A \delta \omega \rangle - \langle \xi | A \xi \rangle \]

\[ = \langle \delta \omega | A \xi + A \delta \omega \rangle + \langle \xi | A \delta \omega \rangle \]
So \( \mathbf{j}^* \mathbf{A}^\# = 0 \) is the condition which says that \( \mathcal{E} \) is a stationary point for \( \mathcal{S}(\mathcal{E}) = \langle \mathcal{E} | \mathbf{A}^\# \mathcal{E} \rangle \) when this form is restricted to the coset \( \mathcal{E} + \mathcal{i} \mathcal{V} = \mathcal{j}^{-1} \mathcal{j} \mathcal{E} \).

But this condition \( \mathbf{j}^* \mathbf{A}^\# = 0 \) doesn't require \( \mathcal{A} \) to be symmetric, so it's probably not a good idea to use the quadratic form \( \langle \mathcal{E} | \mathbf{A}^\# \mathcal{E} \rangle \).

Review the basic decomposition

\[
\begin{array}{ccc}
\mathbf{V} & \overset{i}{\rightarrow} & \mathbf{H} \\
\downarrow & \searrow & \downarrow \mathbf{A} \\
\mathbf{V}^* & \overset{i^*}{\leftarrow} & \mathbf{H}^* \\
\end{array}
\]

Assume \( i^* \mathbf{A}^i : \mathbf{V} \rightarrow \mathbf{V}^* \) is an isomorphism.

Then there should be canonical complements for \( i(\mathcal{V}) \subset \mathcal{V}, j^* \mathcal{W}^* \subset \mathcal{H}^* \).

With nice properties with \( \mathcal{A} \). Why? For \( \mathcal{E}, \mathcal{F} \in \mathcal{H} \),

the map \( \mathcal{E} \mapsto i((i^* \mathbf{A}^i)^{-1} i^* \mathbf{A}^i) \mathcal{E} \)

is a projection with image \( i \mathcal{V} \), since

\[
(i(i^* \mathbf{A}^i)^{-1} i^* \mathbf{A}^i)i = i
\]

Then \( \mathcal{E} \mapsto \mathcal{E} - i(i^* \mathbf{A}^i)^{-1} i^* \mathbf{A}^i \mathcal{E} \) is a projection on \( \mathcal{H} \) whose image is a complement to \( i \mathcal{V} \), hence this image is isom. to \( \mathcal{W} \) via \( j^* \).

\[
(A i (i^* \mathbf{A}^i)^{-1} i^*) \mathbf{A} i = \mathbf{A} i
\]

so \( A i (i^* \mathbf{A}^i)^{-1} i^* \) is a projection with image \( A i \mathcal{W} \), which is mapped isom. to \( \mathcal{V}^* \) via \( i^* \).

Lift \( \mathcal{E} \mapsto \mathcal{E} - i(i^* \mathbf{A}^i)^{-1} i^* \mathbf{A}^i \) \( \mathbf{A} \to \mathbf{0} \), so

\[
\mathcal{E} \mapsto A(\mathcal{E} - i(i^* \mathbf{A}^i)^{-1} i^* \mathbf{A}^i \mathcal{E}) \text{ is killed by } i^* \text{ over the left,}
\]

and \( \mathcal{E} \) on the right; get \( A \) induced map \( \mathcal{W} \rightarrow \mathcal{W}^* \).

Look at $x: V \rightarrow H_+ \oplus H_-$, $x^* x = 1_V$

replacing $H_+$ by $\mathbb{C} i V$ so that $x: V \rightarrow H_+$ is onto

$v_i = x^* x_i = \left( x_i^* x_i + \lambda^* x_i \right)$

self-adjoint $\lambda$ such that $0 < \lambda < 1$

$\lambda = \sum_{\lambda} \lambda^x x_\lambda$ in $V = \bigoplus V_\lambda$

special case, where $x^* x = \lambda$, $x^* x = 1 - \lambda$ and $0 < \lambda < 1$. Then

\[ i: V \xrightarrow{\{x^x, x_{x_{\lambda}}\}} V \bigoplus V \]

instead of $\lambda$ you might use $\lambda' = \cos \theta$

$(1 - \lambda)' = \sin \theta$

\[ H = \begin{pmatrix} C \\ C \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}, \quad V^+ = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix} \]

so \( (x_i^*) = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \)

Try \( \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \)

\[ \frac{1 + x}{\sqrt{1 - x^2}} \]

\[ x^2 = \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix} = \begin{pmatrix} t^2 & 0 \\ 0 & t^2 \end{pmatrix} \]

\[ \sqrt{1 - x^2} = \begin{pmatrix} (1 + t^{1/2}) & 0 \\ 0 & (1 - t^{1/2}) \end{pmatrix} \]

\[ \frac{1 + x}{\sqrt{1 - x^2}} \begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} \begin{pmatrix} (1 + t^{1/2}) & 0 \\ 0 & (1 - t^{1/2}) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1 + t^2}} & \frac{-t}{\sqrt{1 + t^2}} \\ \frac{t}{\sqrt{1 + t^2}} & \frac{1}{\sqrt{1 + t^2}} \end{pmatrix} \]
\[
\frac{1 + X}{1 - X} = \frac{1}{1 + t^2} \begin{pmatrix} 1 - t^2 & -2t \\ 2t & 1 - t^2 \end{pmatrix} = \frac{1}{1 + t^2} \begin{pmatrix} 1 - t^2 & -2t \\ 2t & 1 - t^2 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 - t \\ t \end{pmatrix} \begin{pmatrix} 1 - t \\ t \end{pmatrix} = \begin{pmatrix} 1 - t^2 & -2t \\ 2t & 1 - t^2 \end{pmatrix}
\]

So what are you trying to get

June 20, 02
\[i : V \rightarrow \frac{H^+}{H^-}, \quad \kappa^* = 1_V, \quad \text{on } H \text{ you have po-s. herm. form } \mu^* (s^*_+ s^*_-) V = s^*_+ s^*_+ + s^*_+ s^*_- \kappa^* \kappa_- \]

\[= s \rho + s^* (1 - \rho) \quad \text{where } \rho = s^*_+ s^*_+ \quad \text{e. } \mathcal{L}(V) \text{ satisfies}
0 \leq \rho = \rho^* \leq 1 \]

The point you've been missing: you have to include \( W = V^* \) and look at the eigenspace decomposition for the pair of involutions \( F, \xi \).

So what next \( H^+, H^- \quad \text{dim } 1 \)

\[V \subset \begin{pmatrix} H^+ \\ H^- \end{pmatrix} \]

Recall what you rehearsed

\[V \xrightarrow{i} H \xrightarrow{\xi} W \]

If \( i^* A_i \) is invertible, then there is a push-forward formal process

\[A_W : W \rightarrow W^* \]

\[V^* \leftarrow i^* \leftarrow H^* \leftarrow i^* \leftarrow W^* \]

Given by

\[j^* A_W(j^* y) = A^*_j \cdot -A_i (i^* A_i)^{-1} i^* A^*_j \]

Kills \( i_V \)

\[\text{killed by } i^* \quad \text{descends to } W^* \]

Also when \( A \) invertible one has \( A_W = (j_* A_i j^*)^{-1} \) because

\[j_* A_W (j^* y^i \omega^j) = y^i \omega^j \quad \text{as } y_i = 0 \]

\[j^* A_W(j^* \omega \omega^i) = j^* \omega^i \]
Try again

\[ V \xrightarrow{H} W \]

Assume \( A_V = i^*A_\Lambda \), the restriction of the bilinear form \( A \) from \( H \) to \( V \), is invertible.

Then \( A(i^*A_\Lambda)^{-1}i^*A \) is a projection on \( V \) with range \( iV \).

Also \( i^*(i^*A_\Lambda)^{-1}i^*A \) is a projection on \( H^* \) with range \( AiV \).

too hard

Next project: harmonic oscillators. Question: Can you make a theory of forces out of harmonic oscillators? forced harmonic oscillator?

June 25, 02. Observe that the transmission line equation is the d'alembert equation with pro mass in 2 diml space time:

\[(\partial_t + \partial_x)(E + i) = 0 \]

\[(\partial_t - \partial_x)(E - i) = 0 \]

\(E + i = f_n(x - t) \) right-moving
\(E - i = f_n(x + t) \) left-moving

Can you make a link with your picture of LC circuits?

Bozonic quantization of fermionic might provide an LC picture for a harmonic oscillator type of.

June 30, 02. Explain viewpoint for an LC circuit.

First there is a geometric object: the graph of the circuit. Each edge of the graph has a 2 diml complex vector space of states \((E_\sigma, I_\sigma)\), where \(E_\sigma\) is the voltage drop and \(I_\sigma\) is the current along the edges. (Actually \(E_\sigma\) is a 1-cochain and \(I_\sigma\) is a 1-chain, so you need the edge to be oriented for \(E_\sigma, I_\sigma\) to be numbers). The power of the state \((E_\sigma, I_\sigma)\) is \(E_\sigma I_\sigma\). Thus the space of states of all the edges is \(C^1 \oplus C_1\), and the power form is the usual pairing between 1-cochains and 1-chains.
Philosophy for LC circuits. You are given a graph with inductance and capacitance edges. This is a geometric object to which you want to assign a state space. Analogy with a mechanical system with constraints, e.g., a particle moving on a submanifold of configuration space. In this case you start with phase space (cotangent bundle) of simplex space, then obtain the phase space of the constrained particle (cotangent bundle of submanifolds) by "symplectic reduction".

For the LC circuit the state space is $C^1 \oplus C_1$, where an elt of $C^1$ is a family of voltage drops for the edges of the graph. These chains and cochains become functions on the edges when the edges are oriented. Note that $C^1$ and $C_1$ are naturally dual by the pairing $E \cdot I = \sum \epsilon I_\epsilon$. You can think of $C^1 \oplus C_1$ as the cotangent bundle of the configuration space $C^1$.

Next you constrain the configurations $E \in C^1$ to be conservative, i.e., to come from a "potential" function on the modes. To simplify suppose the graph is connected and equipped with a basepoint (the ground). Thus you restrict via $C^0 \xrightarrow{\delta} C^1$, where $C^0 = 0$-cochains vanishing at $x$. On the state space level you get "symplectic reduction" from $C^1 \oplus C_1$ to $C^0 \oplus C_0$:

$$
C^0 \longrightarrow C^1 \\
C_0 \longleftarrow C_1
$$

$C_0 = \{0 \text{ cochains}\}$, view the current going into the node

Next suppose the modes divided into internal + external modes, e.g.,

- external
- internal

... the ground should be external
July 3, 02. Let $C^s_0$ be the voltage functions on the external modes with the ground value = 0. Let $C^s_0$ contain "currents" going into the external modes, with the current going into $\mathcal{X}$ adjusted so that the sum of the external currents is 0. This is consistent with the potential being zero at the ground. Then you should have another symplectic reduction from $C^s_0 \oplus C_0$ to $C^s_0 \oplus C^s_0$.

Review the philosophy for LC circuits. You are given a connected graph together with a distinguished mode (the ground) and a distinguished subset of modes containing the ground called external modes. This is the geometry which you linearize to obtain three configuration spaces as follows:

\[
\begin{array}{ccc}
C^0 & \xrightarrow{\phi} & C^1 \\
\downarrow & & \downarrow \\
C^0 & & C^s_0
\end{array}
\]

- $C^1$ = space of 1-chains in the graph
- $C^0$ = space of 0-chains vanishing at $\mathcal{X}$
- $C^s_0$ = space of 0-chains on the external modes vanishing at $\mathcal{X}$

Element of $C^1$ is a family of voltage drops on the edges

$C^0$ potential function on the modes, which = 0 on $\mathcal{X}$

$C^s_0$ potential function on the external modes

Associated to these configuration spaces are Jordan spaces obtained by taking the direct sum with the dual space:

$C^0 \oplus C^s_0 \quad C^0 \oplus C_0 \quad C^1 \oplus C_1$

Each one is a "symplectic reduction" of the following one.

So far you have kinematics only. Dynamics arises when you allow the phase space states to be
16 time dependent and require for each edge $v$ that
\[ E_v = L_v \mathcal{A}_v I_v \] for an inductive edge of inductance $L_v$.

\[ I_v = C_v \mathcal{A}_v E_v \] a capacitive edge of capacitance $C_v$.

You now need to understand how this time flow, which makes sense on $C^1 + C_1$, induces a time flow on $C^\text{out} + C^\text{out}$.

July 4, 02. Your aim is to link partial unitary operators: $X \xrightarrow{\alpha} Y, \alpha \circ \alpha = b \circ b = 1$ with abstract LC circuits: subquotient \[ V \twoheadrightarrow H \] of a polarized Hilbert space \[ V/W \twoheadrightarrow H/W \] \[ H = H_+ \oplus H_- \]

Let's review partial unitaries. Examples. Suppose given a Hilbert space $H$ equipped with a unitary operator $u$, and a closed subspace $Y \subset H$. Put $X = Y \cap u^{-1} Y$, $a = \text{inclusion } X \hookrightarrow Y$, $b = u|_X : X \longrightarrow uX = uY \cap Y \subset Y$.

Eigenvector equation: You have \[ Y = X \oplus V_+ = V_- \oplus uX \]
\[ V_+ = \text{Ker } a^* \]
\[ V_- = \text{Ker } b^* \]

Thus \[ H = X \oplus V_+ \oplus V_- = V_- \oplus uX \oplus V_- \].

Let $\xi \in H$ satisfy $u \xi - \lambda \xi \perp uX$. One has
\[ \xi = x_1 + u_1 + p = u_2 + u\xi_0 + p \]
\[ u \xi - \lambda \xi = u(\xi_1 + u_1 + p) - \lambda(\xi_0 + p) \]

Now project onto $uX$.

\[ \Rightarrow uX_1 = u(\lambda \xi_0) \Rightarrow x_1 = \lambda \xi_0. \]
Thus get \( \lambda x_0 + u_+ = u_- + u_0 \) in \( Y \)

Rewrite as \( (\lambda a - b) x = -u_+ + u_- \)

Review how this determines a scattering operator

\[ S(\lambda) : V_- \rightarrow V_+ \quad \Rightarrow \quad \alpha^*(\lambda a - b) x = \alpha^* u_- \]

\[ x = (\lambda - a^* b)^{-1} \alpha^* u_- = \alpha^* (\lambda - b a^*)^{-1} u_- \]

\[ u_+ = u_- - (\lambda a - b) \alpha^* (\lambda - b a^*)^{-1} u_- \]

\[ = [(\lambda^2 - b a^*) - \lambda a a^* + b a^*] (\lambda - b a^*)^{-1} u_- \]

\[ = \left( 1 - a a^* \right) \frac{\lambda}{\lambda - b a^*} u_- \]

Next you need a diagram to interpret this -

\[ \begin{array}{c}
X \\
\downarrow \alpha_a - b \\
Y \\
\downarrow \alpha^* \\
\lambda - a^* b \\
\downarrow \alpha \\
X
\end{array} \]

\[ \xrightarrow{(\lambda - a^* b)^{-1} \alpha^*} \quad \mathbb{E} \quad \Rightarrow \quad V_+ \xrightarrow{\lambda} V_+ \]

so you get

\[ y \rightarrow \lambda y \leftarrow V_+ \]

which assigns to each \( y \) a function \( y(\lambda) \in V_+ \) such

\[ y - y(\lambda) \in (\alpha_a - b) X \]

Check: Assume \( u_+ \in V_+ \) sat \( (\lambda a - b) x = -u_+ + y \)

Then \( \lambda a - b x = a^* y \Rightarrow x = \alpha^* (\lambda - b a^*)^{-1} y \)

\[ \Rightarrow y - (\lambda a - b) x = y - (a a^* - b a^*) (\lambda - b a^*)^{-1} y \]

\[ = \left[ (\lambda - b a^*) - (a a^* - b a^*) \right] (\lambda - b a^*)^{-1} y \]

\[ = \left( 1 - a a^* \right) \frac{\lambda}{\lambda - b a^*} y \in V_+ \]

\[ y(\lambda) \]
Let $T$ be symplectic v.s., $W$ an isotropic subspace, $W^\perp = \{ \xi \in T : \omega(\xi, W) = 0 \}$, then $W^+/W$ is the symplectic quotient corresponding to $W$. Let $L$ be a Lagrangian subspace of $T$. Then
\[(L+W) \cap W^\perp = L \cap W^\perp + W\]
by the modular law. Also $L = L^\perp$ implies
\[(L+W) \cap W^\perp)^\perp = (L+W)^\perp + W = L \cap W^\perp + W\]
Put $\bar{L} = (L+W) \cap W^\perp = L \cap W^\perp + W$. Then $W \subset \bar{L} \subset W^\perp$ and $\bar{L}^\perp = \bar{L}$ which should mean that $\bar{L}/W$ is a Lagrangian subspace of $W^+/W$.

**Example.** Let $T = \begin{pmatrix} D & D^* \end{pmatrix}$ with hyperbolic symplectic form
\[
\begin{pmatrix} d_1 & d_2 \\ s_1 & s_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_1 & d_2 \\ s_1 & s_2 \end{pmatrix} = (d_1, s_1) (\delta_1, \delta_2) = (d_1, s_2) = (d_2, s_1).
\]
Here the pairing between $d \in D$, $s \in D^*$ is written symmetrically. $(d, s) = (s, d)$.

Now let $T = \begin{pmatrix} D \\ D^* \end{pmatrix}$, let $L = \begin{pmatrix} 1 \end{pmatrix} D \subset \begin{pmatrix} D \\ D^* \end{pmatrix}$, where $Q : D \to D^*$ is symmetric. $L$ is isotropic in $T$:
\[
(d_1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} d_2) = (d_1, Qd_2) = (Qd_1, d_2), \quad \text{as } (Qd_1, d_2) = (d_1, Qd_2).
\]
(4) Let $V \subset D$; then $V^* = D/V^0$, $V^0 = \{ \lambda \in D : \lambda(V) = 0 \}$.
Take $W = \begin{pmatrix} 0 \\ V^0 \end{pmatrix} \subset \begin{pmatrix} D \\ D^* \end{pmatrix}$, then $W^+ = \begin{pmatrix} V \\ D^* \end{pmatrix}$ and
\[W^+/W = \begin{pmatrix} V \\ D^* \end{pmatrix} = \begin{pmatrix} V \\ V^0 \end{pmatrix}.
\] Let $L = \begin{pmatrix} 1 \end{pmatrix} D$.
Then \( L \cap W^\perp = (\frac{1}{q})D \cap (\frac{V}{D^*}) = \{ (\frac{q}{\xi}) \mid \xi \in V^0 \} \)

and \( \frac{L \cap W^\perp + W}{W} = \text{Im} \{ (\frac{1}{q}) : V \rightarrow (\frac{V}{V^0}) \} \). Note

\[
\begin{array}{ccc}
V & \rightarrow & D \\
\downarrow \quad \frac{q}{\xi} & & \downarrow D \\
V^* & \leftarrow & D^* \\
\end{array}
\]

Thus \( L \rightarrow \frac{L \cap W^\perp + W}{W} \) sends \((\frac{1}{q})D\) to the graph of \( V \rightarrow D \). This is pull-back of \( q \) and \( D \)

the composite:

\[
\begin{array}{ccc}
V^* & \leftarrow & D^* \\
\downarrow \quad \frac{q}{\xi} & & \downarrow D \\
V & \rightarrow & D \\
\end{array}
\]

This is pull-back of \( q \) and \( D \)

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V & \rightarrow & D \\
\end{array}
\]

This Lagrangean submanifold of \((\frac{D}{V}) = (\frac{D}{V^0})\) is the correspondence \( F' \subset D/V \times V^0 \) arising from \( \xi \in D \)

which satisfy \( Q \xi \in V^0 \). This condition should mean

that \( \xi \) is stationary for \( Q \) restricted to \( \xi + V \).
July 8, 02

Given $T$ symplectic, $W$ isotropic in $T$, $W/W$ the associated symplectic quotient, you would like to understand better the map

$$
\star \quad \text{SG}(T) \longrightarrow \text{SG}(W^+/W)
\quad L \quad \longmapsto \quad (L \cap W^+ + W)/W = (L+W) \cap W^+/W
$$

Here $\text{SG}$ stands for symplectic Grassmannian, the set of Lagrangian subspaces.

You have discussed examples in the hyperbolic case $T = (D^x, D)$, where $L = \left( \begin{smallmatrix} 0 \\ D \end{smallmatrix} \right)$ is the graph of a symmetric $Q : D \rightarrow D^*$, and $W$ has the form $(V_1, V_2)$, with $V_1 \subseteq V_2$, in other words where $W$ is stable under $\varepsilon$. You found that the map $\star$ above amounts to induced quadratic forms on a sub-quotient space. However, the induced quadratic form on a quotient space seems to be defined only under suitable non-degeneracy conditions, whereas $\star$ is defined in general.

Let's study a simple case: $T^4$, $W$ a line in $T$, $L$ an arbitrary Lagrangian subspace of $T$. Two cases: 1) $L \subseteq W$, whence $L \subseteq W^+$ and the "symplectic reduction" of $L$ is $L/W$. 2) $L \cap W = \emptyset$, whence $L + W^+ = T$. Lattice picture:

1. Fix the symplectic red of $L$ in $\text{SG}(W^+/W)$, i.e., fix the Lagrangian subspace $L \cap W^+ + W = (L+W) \cap W^+$ between $W$ and $W^+$. You want to describe the possible $L$. $L$ determines a line $L \cap W^+ \neq W$ in $L \cap W^+ + W$. Applying $L$, you get $L + W$. It seems that $L/\cap W^+$ can be any Lagrangian subspace (line) in the symplectic quotient $L + W/\cap W^+$. 
Simpler example: $\dim (T) = 2$, $W$ line in $T$, $L$ another line. The symplectic reduction of $L$ to $W^\perp/W = 0$ is $L \cap W^\perp + W/W = (L + W) \cap W^\perp/W = 0$. The possible $L$s are any line in $T$.

Important diagram: Assume $L \cap W = 0$  

\[ T \xrightarrow{n-r} L^+ + W \xrightarrow{n} W^\perp \]  

$\left( L^+ + W \right) \cap W^\perp = (L \cap W^\perp) + W$

IDEAS to explore: Complex structure $J$ on $T$ and $J$-linear subspaces; this is the linear version of an almost complex structure on a symplectic manifold and $J$-holomorphic curves.

Krein space picture for the Riemann sphere with the action of $SU(1,1)$. If $T = \mathbb{C}^2$ with hermitian form $(\cdot,\cdot)$ and $Y$ is a Krein space then $T \otimes Y$ should be a Krein space, and $\mathcal{O}(T \otimes Y) \rightarrow \mathcal{O} \otimes T \otimes Y$ should be a family of Lagrangian subspaces.
July 10, 02

Discuss LC circuit:

- \( V_a = \) applied voltage to the LC series circuit
- \( I_o = \) output current
- \( V_c, I_c \) are 6 variables

Satisfy equations of motion (time-evolution):

\[
V_L = L \frac{d}{dt} I_L, \quad I_c = C \frac{d}{dt} V_c
\]

and the circuit equations:

\[
I_o = I_L = I_c
V_a = V_L + V_c
\]

\( V_a \) is a given function of time, say of compact support. There are 5 equations for 5 unknowns. You would like to obtain these equations from a variational principle. You want a functional of the 5 functions \( V_L, I_L, V_c, I_c, I_o \) of \( t \) such that the stationary values are the solutions of the equations.

Use the Laplace Transform to replace time \( t \) by the frequency variable \( s \), and the operator \( \frac{d}{dt} \) by multiplication by \( s \).

July 13, 02

Partial unitary operator: \( X \frac{a}{b} Y \quad a^* a = b^* b = 1 \), where \( X, Y \) are Hilbert spaces. Form the Krein space \( (Y) \) with the hermitian form \( (y_1)^* (0 \ 1) (y_2) = \|y_1\|^2 - \|y_2\|^2 \).

Then \( X, a, b \) can be identified with the isotropic subspace \( W = (a | X \in (Y)) \) of the Krein space. Consider:

\[
W^\perp = \{ (y_1, y_2) \mid a^* y_1 = b^* y_2 \}
\]

\[= \{(y_1, y_2) \mid a^* y_1 = b^* y_2 \}\]
One has
\[ y_1 = a(a^*y_1) + (-a^*a)y_1 \in aX + V_+ \]
\[ y_2 = b(b^*y_2) + (-b^*b)y_2 \in bX + V_- \]
which yields a splitting
\[ W^\perp = (b^*)^\perp = (b^*)X \oplus (y_2^\perp) \]
Next look at \((1)\) \(Y \subset (Y)\) which is isotropic iff \(|A| = 1\). This is the analogue of the family of Lagrangian subspaces in the LC circuit case giving the dynamics.
Analogues of eigenvectors. 1) Bound states : \(x \in X, x \neq 0\)
such that \(\lambda aX = bX\) for some \(\lambda \in \mathbb{C}\); then \(|\lambda| = 1\).
2) When \(0 \neq (\lambda) Y \cap W^\perp = (\lambda) Y \cap (b^*)X \oplus (y_2^\perp)\).
There are no non-trivial solutions of
\[
\begin{cases}
  y_1 = ax + v_+ \\
  y_2 = bx + v_-
\end{cases}
\]
or \((\lambda a - b)x = -\lambda v_+ + v_-\)

---

**July 16, 02**

Consider a partial unitary \(X \xrightarrow{\alpha} Y, Y = aX \oplus V_+ = V_+ \oplus bX \)
where \(V_+\) is 1-dim. Let \(\xi_n\) be a unit vector in \(V_+\). Let \(c_0 = \alpha a^* + \xi_n^* \) for \(he \mathbb{C}\). Then \(c_0\) extends the partial unitary \(a^*\) by the operator \(V_+ \to V_+\) sending \(\xi_+\) to \(\xi_n\). Note \(c_0\) is a contraction for \(|h| < 1\) and its unitary for \(|h| = 1\).

Treat \(c_h\) as a perturbation of \(c_0 = \alpha a^*\) by \(\Delta = \xi_n^*\)

\[
\frac{1}{\xi + z - c_0} = \frac{1}{\xi + z - c_0} + \frac{1}{\xi + z - c_0} \frac{1}{\Delta} \frac{1}{\xi + z - c_0} + \frac{1}{\xi + z - c_0} \left( \frac{1}{\xi + z - c_0} \right)^2 + \cdots 
\]

\[
\xi + z - c_0 = \frac{\xi}{\xi + z - c_0} + \left( \frac{\xi}{\xi + z - c_0} - h \right) \frac{\xi}{\xi + z - c_0} + \left( \frac{\xi}{\xi + z - c_0} - h \right)^2 \frac{\xi}{\xi + z - c_0} 
\]

\[
\frac{1}{\xi + z - c_0} = \left( \frac{\xi}{\xi + z - c_0} \right) + \frac{\xi}{\xi + z - c_0} 
\]
Put $S_h = \frac{\xi^*_+}{1 - z - c_h}\xi_-$ and recall that $S_0 = \frac{\xi^*_+}{1 - z - \text{ba}^*}\xi_-$ is essentially the scattering operator $(\text{I} - \text{aa}^*)\frac{1}{z - \text{ba}^*} : V_+ \to V_+$. So $S_0(z)$ should be analytic outside and of $|z| = 1$ on the unit circle. From the previous page one should have

$$S_h = \frac{1}{1 - S_0 h} S_0 \quad \text{at least for } |h| < 1.$$  

\[ \text{July 18, 02} \]

Consider a partial unitary $Y = aX \oplus V_+ = V_- \oplus bX$ with $Y$ finite dimensional, $X$ of codim 1. Let $\xi_\pm \in V_\pm$ be unit vectors, spanning $V_\pm$ resp.

$$c_h = \frac{\text{ba}^*}{\xi^*_+ h} \xi^*_+ = c_0 + \Delta$$

$$(z - c_h)^{-1} = (z - c_0)^{-1}(1 - \Delta(z - c_0)^{-1})^{-1} = (1 - (z - c_0)\Delta)^{-1}(z - c_0)^{-1}$$

$$\xi^*_+(z - c_h)^{-1} = \xi^*_+(1 - (z - c_0)\xi^*_+ h \xi^*_+)^{-1}(z - c_0)^{-1}$$

$$= (1 - \xi^*_+(z - c_0)\xi^*_+ h \xi^*_+) \xi^*_+(z - c_0)^{-1}$$

Then $S_h \triangleleft \xi^*_+(z - c_h)^{-1}\xi_- = (1 - S_0 h)^{-1}S_0$. Take $h = 1$, so $c_1 = \frac{\text{ba}^*}{\xi^*_+ \xi^*_+} = \text{the unitary operator u on Y sending } aX \text{ to bX and } \xi^*_+ \text{ to } \xi^-$. One has

$$S_1 = \frac{\xi^*_+}{z - u} \xi^*_+ + \frac{1}{2} + \frac{u}{2 - z - u} = \frac{1}{2} \frac{z + u}{z - u}$$

$$\xi^*_+ \left( \frac{1}{2} \frac{z + u}{z - u} \xi^*_+ \right) = \frac{1}{2} + S_1 = \frac{1}{2} + S_0 = \frac{1}{2} \frac{1 + S_0}{1 - S_0}$$

This is essentially the Pick function (analytic on D with $> 0$ imaginary part) corresponding to the measure on $|z| = 1$ associated to the unitary $u$ and the vector $\xi^*_+$. It is essentially the C.T. of $S_0$, which is a finite degree map from $|z| = 1$ to itself.
Variational approach to \( \det (z-c_h) \):

\[
S \log \det (z-c_h - \delta h \xi^*_+ ) = \text{tr} \left( \frac{1}{z-c_h} (-\delta h \xi^*_+) \right)
\]

\[
= -\delta h \frac{1}{z-c_h} \xi^*_+ \xi^- = -S_h \delta h = -\frac{S_0}{1-S_0 h} S_h
\]

\[
= S \log (1-S_0 h). \quad \text{Thus}
\]

\[
\frac{\det (z-c_h)}{1-S_0 h} = \det (z-c_0)
\]

Simpler might be

\[
\frac{\det (z-c_h)}{\det (z-c_0)} = \det \left( 1 - (z-c_0)^{-1} \xi^*_+ \xi^- \right)
\]

\[
= \det \left( 1 - \xi^*_+ (z-c_0)^{-1} \xi^- \right)
\]

\[
= 1 - S_0 h
\]

Aim to link LC circuits + partial unitaries. Consider a rank 1 partial unitary \((V_1 \text{ are done})\) with \(V\) for drinks. Recall the Schur-Szego recursion formulas

\[
\begin{pmatrix} p_n \\ \tilde{p}_n \end{pmatrix} = \frac{1}{k_n} \begin{pmatrix} 1 & h_n \\ \frac{1}{h_n} & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ \tilde{p}_{n-1} \end{pmatrix}
\]

\[
k_n = \sqrt{1-h_n^2}
\]

Assume \(h_n\) real and note

\[
\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1+h & 1+h \\ 1-h & h-1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+h & 0 \\ 0 & 1-h \end{pmatrix}
\]

\[
\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} z & 1 \\ z & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}
\]

This conjugation transforms

\[
\frac{1}{k} \begin{pmatrix} h & 1 \\ 1 & h \end{pmatrix} \Rightarrow \frac{1}{k} \begin{pmatrix} 1+h & 0 \\ 0 & 1-h \end{pmatrix}, \quad \begin{pmatrix} z^{1/2} & 0 \\ 0 & \frac{z}{2} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{z^{1/2}}{2} & \frac{z^{1/2}+z^{-1/2}}{2} \\ \frac{z^{1/2}-z^{-1/2}}{2} & \frac{z^{1/2}+z^{-1/2}}{2} \end{pmatrix}
\]
Thus this conjugation transforms a real Schur-Szego recursion to one involving
\[
\begin{pmatrix}
\frac{1}{k} & 0 \\
0 & \frac{1}{k}
\end{pmatrix}
\]
real positive and
\[
\begin{pmatrix}
\cos(\theta/2) & i \sin(\theta/2) \\
i \sin(\theta/2) & \cos(\theta/2)
\end{pmatrix}
\]
where \( z = e^{i \theta} \).

The kernel is
\[
\begin{pmatrix}
1 & i \tan(\theta/2) \\
i \tan(\theta/2) & 1
\end{pmatrix}
\]
up to a scalar.

Put \( s = i \tan(\theta/2) = \frac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{-1/2}} = \frac{z - 1}{z + 1} \). Then
\[
z = \begin{pmatrix}
1 & i \\
i & 1
\end{pmatrix}(s) = \frac{1 + s}{1 - s}, \quad s \in i \mathbb{R}
\]
and \( z \in \mathbb{H} \).

July 20, 02

\[
\begin{align*}
I & \rightarrow C \\
V & \rightarrow \text{charge of capaci}t	ext{al} \\
C & \text{power going into capaci}t	ext{al}
\end{align*}
\]

\( Q = CV \Rightarrow I = CV \)

Electrical Energy stored in \( C \) at time \( t \)
\[
\int_{-\infty}^{t} V I \, dt = \left[ \frac{1}{2} CV^2 \right]_{-\infty}^{t} = \frac{1}{2} CV(t)^2
\]

\[
\begin{align*}
I & \rightarrow L \\
V & \rightarrow \text{Faraday Induction:} \\
L & \text{power going into}
\end{align*}
\]

\( V = LI \)

\( VI = LII = \frac{1}{2} \left( \frac{d}{dt} LI^2 \right) \)

Magnetic Energy stored in \( L \) at time \( t \)
\[
\int_{-\infty}^{t} V I \, dt = \left[ \frac{1}{2} LI^2 \right]_{-\infty}^{t} = \frac{1}{2} LI(t)^2
\]

Aug 9, 02

Begin with the algebraic picture of BNS. Consider a unital algebra $A$, a unital (left) $A$-module $M$, and a $C$-linear retract

$$N \xrightarrow{i} M \xrightarrow{j} N,$$

of $M$. Define

$$\varphi: A \rightarrow \mathcal{L}(N), \quad \varphi(a) = i \circ a \circ j.$$ 

$\varphi$ is linear and satisfies $\varphi(1) = 1$. Our aim is to reconstruct as far as possible the $A$-module $M$ from $(N, \varphi)$. We introduce $A$-module maps

$$A \otimes N \xrightarrow{\tilde{\iota}} M \xrightarrow{\tilde{\varphi}} \text{Hom}(A, N)$$

$$\tilde{\iota}(a \otimes n) = a \cdot n \quad \tilde{\varphi}(m) = \{a \mapsto \varphi(a) m\}$$

$$\tilde{\varphi}(a \otimes n) = \{a \mapsto \varphi(a \circ n)\}$$

$\tilde{\iota}$ is the unique $A$-module map extending $i$ coextending $j$ in the sense that $\tilde{\varphi}m$ evaluated at $a = 1$ is $j \circ m$. Here $A \in A$ acts on $\text{Hom}(A, N)$ by right mult on $A$.

Thus one has an $A$-module factorization: $\tilde{\varphi} = \tilde{\iota} \tilde{\varphi}$ of $\varphi$ through $M$. Conversely given any such factorization

$$\varphi = \beta \alpha$$

$$A \otimes N \xrightarrow{\alpha} M' \xrightarrow{\gamma} \text{Hom}(A, N)$$

one has $\alpha = \tilde{\iota}$, $\beta = \tilde{\varphi}$ and $\gamma' m = \text{eval}_1 \beta \alpha (1 \otimes n) = \text{eval}_1 \tilde{\varphi}(1 \otimes n) = \varphi(1) n = n$.

There's a smallest factorization, namely when $M$ is the image of $\varphi$. 
Example: $A = C(S)$, $S$ finite, is a $A$-module, so a linear retract has the form

$$W \leftarrow \bigoplus_{s \in S} V_s \leftarrow W$$

$$\beta = (\ldots, \beta_s, \ldots) \quad \alpha = (\ldots, \alpha_s, \ldots) \quad \beta \alpha = \sum_s \beta_s \alpha_s = 1_w$$

$\varphi(a) = \beta \alpha$ amounts to family $h_s = \beta_s \alpha_s = \beta_s \alpha_s$ in $L(W)$, which is a partition of 1. $V_s$ can be shrunk to $\alpha_s W$ and then to $\alpha_s W/Ker f_s$ without affecting $h_s$. Thus the minimal case is when $V_s = \text{Im}(f_s)$, $f_s$.

Hilbert picture of GNS. $A$ is a $C^*$-algebra, the $A$-module is a *-rep of $A$ on a Hilbert space $V$, and the retract has the form

$$W \leftarrow \mathbb{C}^* V \leftarrow i_* W,$$

where $i$ is an isometry as $i^* i = 1_W$. In this situation $\varphi : A \rightarrow L(W)$ is a completely positive function yielding a positive hermitian inner product on $A \otimes W$, and $V$ is the corresponding completion.

$\varphi$ completely positive should mean that the hermitian form on $A \otimes W$ defined for every $\xi = \sum_i \omega_i \otimes \omega_i$ by

$$\langle \xi | \eta \rangle = \sum_i \langle \omega_i | \varphi(a_i^* a_j) \omega_j \rangle$$

is $\geq 0$.

In other words, a finite sequence $(a_i)$ in $A$, the block matrix $\varphi(a_i^* a_j)$ with blocks in $L(W)$ is $\geq 0$.

In the case $A = C(S)$ above, since $A \otimes W = \bigoplus \varepsilon_s \otimes W_s$, it suffices to consider the sequence $(\varepsilon_s)$ in $A$. Then

$$\varphi(\varepsilon_s^* \varepsilon_t) = \begin{cases} 0 & t \neq s \\ \varphi(\varepsilon_s) = h_s & t = s \end{cases}$$

so for $\xi = \sum \varepsilon_s \otimes w_s \in A \otimes W$

one has

$$\langle \xi | \xi \rangle = \sum \langle w_s | h_s w_s \rangle = \sum_s \| h_s^\frac{1}{2} w_s \|^2.$$
Thus $V$ is the completion of $\bigoplus \mathcal{H}_W$ where

$$V = \bigoplus \mathcal{H}_W$$

with the inner product $\|h_{\mathcal{H}}\|^2 = \sum h_{\mathcal{H}}^2 \mathcal{H}_W^2$. So

$$V = \bigoplus \mathcal{H}_W$$

where $V_\mathcal{H}$ is the completion of $\mathcal{H}_W$ with the inner product $\|h_{\mathcal{H}}\|^2$.

Aug 11, 02: Discuss $A = C(i \pm i^\alpha)$, which is the Grassmannian.

Example:

$$\begin{pmatrix} \alpha_+^* & \alpha_-^* \end{pmatrix} \begin{pmatrix} V_+ \end{pmatrix} \begin{pmatrix} \alpha_+ \end{pmatrix} W = h_\pm = x_\pm^* x_\pm$$

Then $h_+ + h_- = 1_W$, because $\mathcal{H}$ is assumed an isometry. Also, the minimal choice for $V$ is $V_\pm = h^{1/2}_\pm W$ with $x_\pm^* x_\pm = h^{1/2}_\pm$.

Aug 12, 02: The above "Grassmannian situation" depends only on $W$ and the hermitian operator $h_\pm$, so one gets a spectral decomposition of $(\mathcal{H})$ from the spectral decom of $h_\pm$. Suppose $h_\pm = \lambda$ the scalar operator on $W$, so that $0 \leq \lambda \leq 1$ and $h_- = 1 - \lambda$. First consider the generic case $0 < \lambda < 1$. Then $V_\pm = W$ (with scaling of inner product) and $(\mathcal{H})$ is

$$\begin{pmatrix} \lambda^{1/2} & (1-\lambda)^{1/2} \end{pmatrix} W \begin{pmatrix} \lambda^{1/2} & (1-\lambda)^{1/2} \end{pmatrix} = \begin{pmatrix} \alpha_+^* & \alpha_-^* \end{pmatrix} \begin{pmatrix} V_+ \end{pmatrix} \begin{pmatrix} \alpha_+ \end{pmatrix} W$$

Of interest for LC circuits is the hermitian (or real)

form

$$\alpha_+^* (5e_+ + s^{-1} e_-) = \lambda h_+ + s^{-1} h_- = 5\lambda + s^{-1} (1-\lambda)$$

Written in terms of $\omega = h^{1/2}_- h^{1/2}_+$ (the eigenvalue of "$T"$

one has $\omega^2 = \frac{1 - 2 \lambda}{\lambda}$, $\lambda = \frac{1}{1 + \omega^2}$

$$5\lambda + s^{-1} (1-\lambda) = \frac{s + s^{-1} \omega^2}{1 + \omega^2}$$
The other scalar cases are \( \lambda = 1 \) \((\omega = 0)\) where \( \pi : V_+ \to W \) is an isometry \(+ V_- = 0\) and \( \lambda = 0 \) \((\omega = \infty)\) where \( \pi \) is an isometry and \( V_+ = 0 \).

Next discuss the graph viewpoint. Let \( W \) be a closed subspace of \((V_+)\) as above, and consider the map \( \pi : p^+ : W \to V_+ \). \( \text{Ker}\pi = W \cap V_- \); if this is \( \neq 0 \), then \( W \) as a correspondence from \( V_+ \) to \( V_- \) is many-valued, so \( W \) is not the graph of a map. One can shrink \( V_- \) to the orthogonal complement of \( W \cap V_- \) and reduce to the case \( W \cap V_- = 0 \).

Let \( D = \text{Im}\{\pi^+ : W \to V_+\} \). \( \forall x \in D \exists y \in V_- \text{ s.t. } (x,y) \in W \). Moreover \( y \) is unique, so that \( W = (\pi^+)^D \) is the graph of a linear transformation \( T \) with domain \( D \) and values in \( V_- \). Now shrink \( V_+ \) to the closure \( \overline{D} \), and reduce to the case where \( T \) is a densely defined linear transformation from \( V_+ \) to \( V_- \) whose graph \( W \) is closed.

Next bring in von Neumann's theory, which says that \( W = (\pi^+)^T \overline{D} \), where \( T^* \) is the adjoint of \( T \), which is again a closed densely-defined operator that also \( (\pi^+)^T \overline{D} \) is invertible, leading to the C.T. etc. etc.

Philosophy: You get involved with unbed of because of the frequency variable is dual to time \( t \) is unbounded \((\omega = 2\pi \text{ under the L.T.})\).

Problem: What structure does the family of factorizations of a map have? For example, if the map is \( 1 \times \), then factorizations are equivalent to embeddings as retract of other objects.
Glueing two Hilbert spaces via a contraction $c : X \to Y$. You want a Hilbert space $Z$, isometric

\[ x \xrightarrow{\alpha} Z \xrightarrow{b} y \]

such that $c = b \alpha$ and $Z = \frac{c}{a \ x + b \ y}$.

\[ \|a x + b y\|^2 = (x^* y^*) (a^* b)(y^*) = (x^*)(1_{C^*})(y) = \|x^* y\|^2 + \|g\|^2 - \|c^* g\|^2 > 0.\]

There is an orthogonal splitting

\[ Z = aX \oplus \text{ker}(a^*) \quad a x + b y = a(x + c^* y) + (1 - a a^*) b y \]

\[ \|a x + b y\|^2 = \|x + c^* y\|^2 + \|\alpha^* b y\|^2, \quad \text{where} \quad \|\alpha^* b y\|^2 = (y | b^*(1 - a a^*) b y) = (y | (1 - c c^*) y) = \| (1 - c c^*)^{1/2} y \|^2. \]

Thus $Z = aX \oplus (1 - c c^*)^{1/2} X$ and similarly $Z = bX \oplus (1 - c c^*)^{1/2} Y$.

Consider a retract of a free $A$-module $A \otimes V$, $A = C(\{\bar{z}\})$.

\[
\begin{pmatrix}
W_+ \\
W_-
\end{pmatrix} \xrightarrow{\begin{pmatrix}
\alpha_+ \\
\alpha_-
\end{pmatrix}} \begin{pmatrix}
V \\
V
\end{pmatrix} \xleftarrow{\begin{pmatrix}
\beta_+ \\
\beta_-
\end{pmatrix}} \begin{pmatrix}
W_+ \\
W_-
\end{pmatrix}
\]

\[ \beta \alpha = 1_{W}. \]

Since $\beta \alpha = 1_{W}$, one has two projections $p_\pm = \alpha_\pm \beta_\pm$ on $V$. Using the obvious identification between the two copies of $V$ one get an odd operator on $W$:

\[ C = (\begin{pmatrix}
\alpha_+ \\
\alpha_-
\end{pmatrix}) (\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}) (\begin{pmatrix}
\alpha_+ \\
\alpha_-
\end{pmatrix}) = (\begin{pmatrix}
0 & \beta_+ \alpha_-
\\
\beta_\alpha_+ & 0
\end{pmatrix}) \]

Let's now shrink $V$ to obtain the reduced linear space equipped with two projections which is Morita equivalent to $W$, equipped with the odd operator $C$. Shrinking $V$ to $\alpha_+ W_+ + \alpha_- W_-$ one can assume $V \xleftarrow{(\alpha_+ \alpha_-)} (W_+)$.
is surjective. Then passing to the quotient of $V$ by $\ker(\beta_+) \cap \ker(\beta_-)$ one can assume

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{(\beta_+)} V \xrightarrow{(\beta_-)} V$$

is injective.

Then $V$ can be identified with the image of

$$\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} \begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix} = \begin{pmatrix} 1 + \beta_+ \alpha_- \\ \beta_- \alpha_+ \end{pmatrix} = 1 + C$$

Let's introduce dilation language. In the Grass example, the embedding of $W$ as a retract of $V_+$ can be viewed as a dilation of partition $h_+ + h_- = 1_W$ to a partition $e_+ + e_- = 1_V$ which is disjoint in the sense that $e_+ e_- = e_- e_+ = 0$. Any dilation is equivalent to a factorization of $h_+ : W \xleftarrow{\beta_+} V_+ \xrightarrow{\alpha_+} W$.

Consider next a retract of a free $A = C(\mathbb{S}^1)$ module

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{(\beta_+ \circ 0)} V \xrightarrow{(0 \circ \beta_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

We can view this as a dilation of the odd operator

$$C = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & \beta_+ \alpha_- \\ \beta_- \alpha_+ & 0 \end{pmatrix}$$
on $W$

to the operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $V$. Note the factorization

$$1_W + C : \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{(\beta_+ \circ 0)} V \xrightarrow{(0 \circ \beta_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$
Conversely given \((W_+ \leftrightarrow W_-)\) equipped with an odd operator \(C\) we get a equivalence between factorizations of \(1 + C\) through the vector space \(V\) on one hand and embeddings of \(W\) as a retract of \(A \otimes V\). The smallest \(V\) is the image of \(1 + C\).

August 14, 02. Consider the Hilbert space versions of the above where the vector spaces are Hilbert spaces and \(\beta = \alpha^*\), equivalently \(\alpha\) is an isometry (\(\alpha^* \alpha = 1\)).

In the Grass case the partition \(h_+ + h_- = 1\) consists of hermitian operators \(> 0\). If \(W \leftarrow \frac{\alpha_+^*}{\alpha_+} \quad V_+ \leftarrow \frac{\alpha_+^*}{h_+} W\) is a minimal factorization of \(h_+\), then it is common room to \(V_+ = \frac{h_+}{h_+^2} W\), \(\alpha_+ = \alpha_+^* = \frac{h_+}{h_+^2}\); similarly for \(h_-\).

In the other case one has the factorization
\[
\left( \begin{array}{cc}
1 & c^* \\
c & 1-
\end{array} \right) = 1 + C : \left( \begin{array}{c}
W_+ \\
W_-
\end{array} \right) \leftrightarrow \left( \begin{array}{c}
\left( \frac{\alpha_+^*}{\alpha_+} \right) \\
\left( \frac{\alpha_-^*}{\alpha_-} \right)
\end{array} \right) \left( \begin{array}{c}
W_+ \\
W_-
\end{array} \right)
\]
where \(c = \alpha_+^* \alpha_+\). This means that the Hilbert space \(V\) contains \(W_+\) and \(W_-\) (isometrically) glued via the contraction \(c\). So the smallest \(V\) is this gluing.
Aug 19, 02. Example of Thévenin's theorem. From Peck's Electrodynamics and Magnetism. Find $I_6$ in the Wheatstone bridge: 187-188

Remove the edge $AB$ and find the equivalent emf $E_0$ in series with the internal resistance $R_0$.

$$E_0 = \frac{E}{R_1 + R_2} - \frac{E}{R_3 + R_4}$$

where $E = \frac{R_1}{R_1 + R_2}$. $R_0$ = resistance between $A$ and $B$ with $E = 0$.

Then

$$E_0 = I_6 (R_6 + R_0)$$

$$I_6 = \frac{\frac{R_1}{R_1 + R_2} - \frac{R_3}{R_3 + R_4}}{\frac{R_1 R_2}{R_1 + R_2} + \frac{R_3 R_4}{R_3 + R_4} + R_6}$$

The bridge balances $(I_6 = 0)$ when $R_1 (R_2 + R_4) - (R_3 + R_2) R_3 = 0$.

i.e. $R_1 R_4 = R_2 R_3$. 

Let's now discuss an LC network which is closed, i.e. there are no applied voltages or current sources. Consider the example:

This graph has two edges denoted \( \sigma_L, \sigma_C \). The arrows indicate the orientation chosen for these edges. \( \sigma_L \) and \( \sigma_C \) form a basis for the space \( C_1 \) of \( \mathbb{C} \)-valued 1-chains. Using the orientations, the boundary operator \( \partial \) is defined by \( \partial \sigma_L = \partial \sigma_C = [A]-[B] \), where E.E. conventions are used (normally the potential at A is higher than at B). Writing a 1-chain in the form \( I_L \sigma_L + I_C \sigma_C \), where \( I_L, I_C \) are members, we have

\[
\partial (I_L \sigma_L + I_C \sigma_C) = (I_L + I_C)([A]-[B]).
\]

Note that \([A]-[B]\) is a generator for the space \( C_0 \) of 0-chains having augmentation = 0, so that the subspace \( \mathcal{H}_1 \) of closed 1-currents, i.e. 1-cocycles, is given by the condition \( I_L + I_C = 0 \).

Consider next the space \( C' \) of 1-cochains, that is, functions \( V = (V_L, V_C) \) on the edges, \( \sigma_L, \sigma_C \). Such a \( V \) yields potential drops for those oriented edges. Let's examine when such a \( V \) is conservative, i.e. is \( \phi \) where \( \phi \in \mathcal{C} \) is a function on the nodes A, B modulo constant functions. Clearly one has \( V_L = \phi(A) - \phi(B) = V_C \). Another way to see this is to use that \( V \) is conservative if the work done in going around a closed path is zero. So you want \( V \) applied to the loop current \( \sigma_L - \sigma_C \) (which generates \( \mathcal{H}_1 \)) to be zero, \( V(\sigma_L - \sigma_C) = V_L - V_C = 0 \).

So we have found the Kirchhoff 1st + 2nd laws, namely \( I_L + I_C = 0 \) and \( V_L = V_C \). Now allow the variables \( I_L, I_C, V_L, V_C \) to be functions of time and introduce the D.E.'s governing \( L \)'s and \( C \)'s:

\[
V_L = L \frac{dI_L}{dt}, \quad I_C = C \frac{dV_C}{dt}.
\]
Then \( V_L = L_s I_L = -L_s I_C = -LC s^2 V_C = -LC s^2 V_L \)

or \( (1 + LC s^2) V_L = 0 \).

Aug 25, 02.

Example of motion of a closed LC network:

\[ \dot{I}_L = L_s V_L, \quad V_C = C^{-1} I_C \]

(dynamics)

\[ I_L + I_C = 0, \quad V_L = V_C \]

(Kirchhoff 1.0.2)

Do you have 4 linear equations in 4 unknowns — is this system well-posed? Write down the relations in the L.T. picture

\[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
-L_s & 0 & 0 & 0 \\
0 & 0 & s & -C^{-1}
\end{pmatrix}
\begin{bmatrix}
V_L \\
I_L \\
V_C \\
I_C
\end{bmatrix}
= 0
\]

\[
\det \lambda = \frac{1}{s} \begin{vmatrix}
0 & 1 & 1 \\
0 & s & 0 \\
0 & 0 & s - C^{-1}
\end{vmatrix}
= \frac{1}{s}(s - C^{-1})(s - L_s) = s^2 + (CL)^{-1}
\]

Next look at the case of a general connected closed LC network. You have two variables: \((V_e, I_e)\) for each of the \(e\) edges of the network; \(n\) \(2e\) unknowns. Kirchhoff 1, 2 namely \(\partial I = 0\); \(C = \text{Ker}(C' \to H')\) amount to \((n - 1) + 2 = e\) conditions. You have one dynamical equation for each edge:

\[
\dot{V}_e = L_s I_e \quad \text{r of L type}
\]

\[
V_e = C^{-1} I_e \quad \text{r of C type}
\]

Thus you have \(2e\) unknowns in \(2e\) equations, and you get a determinant as in the example above.

The good case should be when the determinant is a polynomial of degree \(e\) in \(s\).
Here's a generalization of the LC oscillator case studied above. Consider an LC network with $e$ edges, and let

$$\mathcal{S} = \left( \begin{array}{c} \mathcal{C}_0 \\ \mathcal{H}_1 \end{array} \right) \subset \left( \begin{array}{c} \mathcal{C}_1 \\ \mathcal{C}_1 \end{array} \right)$$

be its state space. $\mathcal{S}$ has dimension $e$. You want to construct a time evolution on $\mathcal{S}$ using the DE's associated to the $e$ edges: 

$$\begin{cases} \dot{L}_0 \mathcal{I}_0 = \mathcal{V}_0 & \text{L-type} \\ \dot{C}_0 \mathcal{V}_0 = \mathcal{I}_0 & \text{C-type} \end{cases}$$

Introduce the splittings:

$$\mathcal{C}_1 = \mathcal{C}_C \oplus \mathcal{C}_L$$

$$\mathcal{C}_1 = \mathcal{C}_C \oplus \mathcal{C}_{1L}$$

where $\mathcal{C}_C$ is the space of voltages drops across the capacitance edges, and $\mathcal{C}_{1C}$ is the space of currents passing through the capacitance edges; similarly for $\mathcal{L}$. Assume that the linear functions (or variables) $\mathcal{I}_0$ for L-type and $\mathcal{V}_0$ for C-type form a coordinate system on $\mathcal{S}$. Put another way, the composition

$$(+) \quad \mathcal{S} = \left( \begin{array}{c} \mathcal{C}_0 \\ \mathcal{H}_1 \end{array} \right) \subset \left( \begin{array}{c} \mathcal{C}_1 \\ \mathcal{C}_1 \end{array} \right) = \left( \begin{array}{c} \mathcal{C}_L \\ \mathcal{C}_{1C} \end{array} \right) \oplus \left( \begin{array}{c} \mathcal{C}_C \\ \mathcal{C}_{1L} \end{array} \right)$$

is an isomorphism.

Let us call dominant the variables $\mathcal{I}_0$ L-type, $\mathcal{V}_0$ C-type. We are assuming the dominant variables form a coordinate system on $\mathcal{S}$. Since $\mathcal{V}_0$ L-type, $\mathcal{I}_0$ C-type are linear functions of the dominant variables, * above is a first order linear DE on the state space $\mathcal{S}$. One has $\mathcal{C}_0 \rightarrow \mathcal{C}_C$ and $\mathcal{H}_1 \rightarrow \mathcal{C}_{1L}$ from (*).