June 21, 02

interpretation

$$\begin{tikzcd} C^0 \arrow[r, "\delta"] & C^1 \arrow[d, "Z^{-1}"] \\ C_0 & C_1 \arrow[l, "\partial"'] \end{tikzcd}$$

an elt of C^0 gives the input voltage E_x at each node x

an elt of C^1 gives a voltage drop E_σ for each (oriented) edge σ

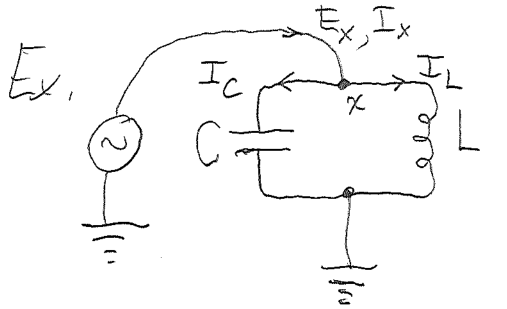
an elt of C_1 gives a current I_σ for each edge σ

――――― C_0 gives the output current I_x for each node x.

$$Z_\sigma = \frac{E_\sigma}{I_\sigma} = \text{the impedance of the edge } \sigma = \begin{cases} L_\sigma s & \text{ind} \\ \dfrac{1}{C_\sigma s} & \text{cap} \end{cases}$$

Then $\partial Z^{-1}\delta : C^0 \longrightarrow C_0$ assigns to any input voltage function the induced output current function.

Q: Is $\partial Z^{-1}\delta$ an isomorphism? good case?



$$I_x = I_C + I_L \qquad \frac{E_C}{I_C} = \frac{1}{Cs}$$

$$E_x = E_C = E_L \qquad \frac{E_L}{I_L} = Ls$$

$$I_x = CsE_x + \frac{1}{Ls}E_x$$

$$I_x = \left(Cs + \frac{1}{Ls}\right)E_x \qquad \text{becomes singular form } s^2 + \frac{1}{LC} = 0$$

You would like to understand again your study of the situation where the input voltage fn and output current function is supported on a given subset S of the nodes.

C^0 = functions on Γ_0 is replaced by C^0_S = fns on S

Idea? Impose $E_x = 0$, $x \in S$ get $C^0_S \subset C_S$.

The point is that E_x for $x \in S$ is given, while E_x for $x \notin S$ is somehow determined by $I_x = 0$, $x \notin S$

Next, leave the graph business, and clear up the

June 10, 2002. Go back to rubber band example, why a rubber band contracts when heated.
To understand Brownian motion, Einstein's derivation of Avogadro's number, equivalently Boltzmann's constant, using Brownian motion. What you would like is an exact model

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ΨDO, spin^c structure

If $U \subset \mathbb{R}^n$, symbol of order $m \in \mathbb{Z}$ = smooth $p: U \times \mathbb{R}^n \to \mathbb{C}$
$\forall K \subset U$, multi-indices α, β : $\exists C_{K\alpha\beta} > 0$

$$\left| \partial_x^\alpha \partial_\xi^\beta p(x,\xi) \right| \leq C_{K\alpha\beta} (1+|\xi|)^{m-|\beta|}$$

Restrict to classical symbols

$$p - \sum_{k\geq 0}^{N-1} \underbrace{p_{m-k}}_{\text{homog of deg } m-k \text{ in } \xi} \in S^{m-N}(U\times\mathbb{R}^n)$$

June 20, 02. LC circuit = finite graph whose edges have either inductance or capacitance type. Choose orientation for each edge, consider $\mathbb{C}$-valued chains, dim 0 & 1, also cochains.

$$C^0 \xrightarrow{\delta} C^1$$

C^0 consists of voltage functions on the nodes

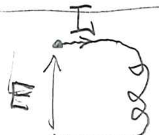
$$C_0 \xleftarrow{\partial} C_1$$

C^1 consists of voltage drops for the edges.

equations?  I_{xy}, $E_x - E_y$ related how?

$$Q = CE \qquad C\dot{E} = \dot{Q} = I.$$
charge

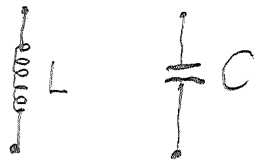
$$I_{xy} = C_{xy}(\dot{E}_x - \dot{E}_y) \qquad E_{xy} = L\dot{I}_{xy}$$



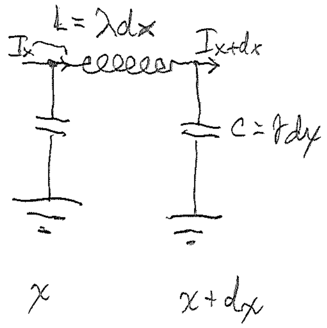
$E = L\dot{I}$ take L.T. $\hat{E} = Ls\hat{I}$ $Z = Ls$ ind

$Q = CE$ $\hat{I} = Cs\hat{E}$ $= \frac{1}{Cs}$ cap
$I = C\dot{E}$

quadratic form picture. (Idea: You might have a symplectic situation where E_x, I_x are dual variables. Symplectic reduction?)

General linear picture: Start with C^1 which is a polarized Hilb space $H = H_{-1} \oplus H_1$. To be specific $H = H_L \oplus H_C$. You seem to want an energy around. Simple cases. 

You should recall the energy somehow. Go back to transmission line?



$$E_x - E_{x+dx} = \lambda dx\, \dot{I}_x$$

$$I_x - I_{x+dx} = \gamma dx\, \dot{E}_x$$

$$-\partial_x E = \lambda \partial_t I \quad , \quad -\partial_x I = \gamma \partial_t E$$

$$\partial_x^2 E + \lambda \partial_{tx}^2 I = 0 \quad , \quad -\partial_{tx}^2 I = \gamma \partial_t^2 E$$

$$\partial_x^2 E - \lambda\gamma \partial_t^2 E = 0$$

Suppose $\lambda = \gamma = 1$.

$$\boxed{\begin{aligned} \partial_x E + \partial_t I &= 0 \\ \partial_t E + \partial_x I &= 0 \end{aligned}} \qquad \begin{aligned} (\partial_x - \partial_t)(E - I) &= 0 \\ (\partial_x + \partial_t)(E + I) &= 0 \end{aligned}$$

Energy should be $\int EI\,dx$.

$$\partial_t \int EI\,dx = \int \big((-\partial_x I) I + E(-\partial_x E)\big)\,dx = -\tfrac{1}{2}\int \partial_x(I^2 + E^2)\,dx$$

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time dep. e^{st}

$$(\partial_x - s)(E - I) = 0 \qquad E - I \quad e^{sx} \quad \text{outgo}$$

$$(\partial_x + s)(E + I) = 0 \qquad E + I \quad e^{-sx} \quad \text{income}$$

Consider $H = H_+ \oplus H_-$ equipped with the family $s\|\xi_+\|^2 + s^{-1}\|\xi_-\|^2$ of pos. hermitian forms. $s \in \mathbb{R}_{>0}$.

This is the abstract version of C'. δC° subspace of H. Call it V.

$$\begin{tikzcd} V \arrow[r, "j"] & H_+ \oplus H_- \arrow[d, "s \oplus s^{-1}"] \\ V & H_+ \oplus H_- \arrow[l, "j^*"'] \end{tikzcd}$$

Probably you want to use the spectrum idea for the Grassmannian

$j = (j_+, j_-)$ $\qquad$ $j^*(s \oplus s^{-1}) j = s j_+^* j_+ + s^{-1} j_-^* j_-$

Move on to the induced hermitian form on the "cokernel of j"

$$\begin{tikzcd} V \arrow[r, tail, "i"] \arrow[d, "i^*Ai"'] & H \arrow[r, two heads, "j"] \arrow[d, "A"] & W \arrow[d, dashed, "A''"] \\ V^* & H^* \arrow[l, two heads, "i^*"'] & W^* \arrow[l, tail, "j^*"'] \end{tikzcd}$$

assume i^*Ai invertible, then you get an induced map $W \to W$

Given $w \in W$ choose h s.t. $j(h) = w$. Form $(i^*Ai)^{-1} i^*Ah \in V$ and $A(h - i(i^*Ai)^{-1} i^*Ah)$. Operator you want is

$$A' = A - Ai(i^*Ai)^{-1} i^*A : H \longrightarrow H^*$$

$$A'i = 0, \qquad i^*A' = 0$$

so A' induces a map $W \longrightarrow W^*$.

$\exists! A'' : W \longrightarrow W^*$ st $j^*A''j = A'$.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & H^+ \oplus H^- \arrow[d, "?"] \\ {} \arrow[r, hook] & {} \end{tikzcd}$$

5 Abstract LC circuit consists of $H = H_+ \oplus H_-$, the complex Hilbert space of 1-cochains split into L and C parts. You should link the inner product to energy (or power).

$$\begin{tikzcd} C^0 \arrow[r, "\delta"] \arrow[d, "\partial Z^{-1}\delta"'] & C^1 \arrow[d, "Z^{-1}"] \\ C_0 & C_1 \arrow[l, "\partial"] \end{tikzcd}$$

C^1 = space of $\mathbb{C}$-valued fns on edges

C_1 = space of $\mathbb{C}$-valued 1-chains... also = functions on edges.

$$Z_\sigma = \begin{cases} Ls \\ \frac{1}{Cs} \end{cases} \qquad E \in C^1$$

$$\left(Z^{-1}E\right)_\sigma = Z_\sigma^{-1} E_\sigma = I_\sigma$$

Power: pairing between C^1 and C_1, $\sum_\sigma E_\sigma I_\sigma = \sum_\sigma Z_\sigma I_\sigma^2$

what about conjugation.

You need a better picture, try coupling a 1-port to a transmission line

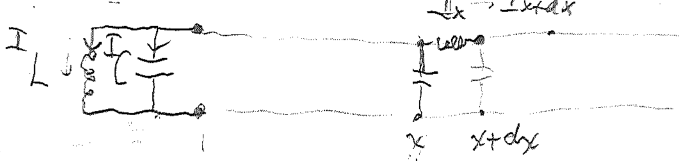
$$\partial_t E + \partial_x I = 0 \qquad (\partial_t + \partial_x)(E+I) = 0$$

$$\partial_t I + \partial_x E = 0 \qquad (\partial_t - \partial_x)(E-I) = 0$$

L.T. $\quad (\partial_x + s)(E+I) = 0 \qquad E+I = e^{-sx}$

$\quad (\partial_x - s)(E-I) = 0 \qquad E-I = e^{sx}$

June 23, 02. Work out transmission line coupled to an LC 1-port, say



$$E_x - E_{x+dx} = dx\, \dot{I}_x \qquad \partial_x E + \partial_t I = 0$$

$$I_x - I_{x+dx} = dx\, \dot{E}_x \qquad \partial_x I + \partial_t E = 0$$

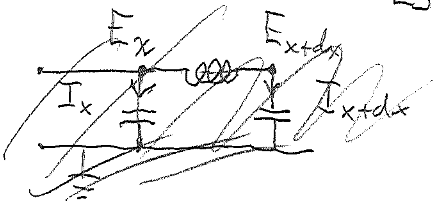
$$(\partial_x + \partial_t)(E+I) = 0 \qquad I_L + I_C + I_0 = 0 \qquad E_L = L\dot{I}_L$$

$$(\partial_x - \partial_t)(E-I) = 0 \qquad E_L = E_C = E_0 \qquad I_C = C\dot{E}_C$$

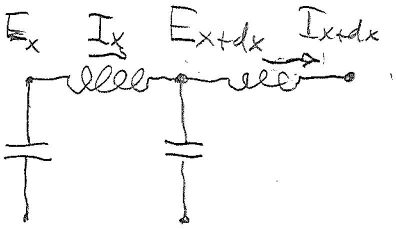
do LT. $\quad I_0 = I_L - I_C = -E_0\left(\frac{1}{Ls} + Cs\right)$

$$-\frac{E_0}{I_0} = \frac{1}{\frac{1}{Ls} + Cs} = \frac{Ls}{1 + LCs^2}$$

again



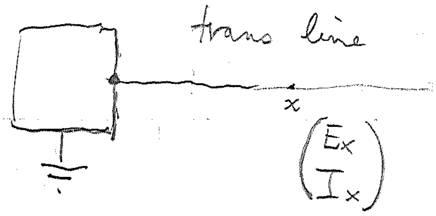
$\cancel{E_x - E_{x+dx} = dx}\ I$



$$E_x - E_{x+dx} = dx\, \dot I_x$$
$$I_x - I_{x+dx} = dx\, \dot E_{x+dx}$$
$$\partial_x E + \partial_t I = 0$$
$$\partial_x I + \partial_t E = 0$$

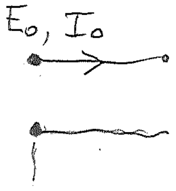
$$(\partial_x + \partial_t)(E + I) = 0$$
$$(\partial_x - \partial_t)(E - I) = 0$$

$$E - I = Ae^{sx}$$
$$E + I = Be^{-sx}$$



1-port with impedance $Z(s) = \frac{-E_0}{}$

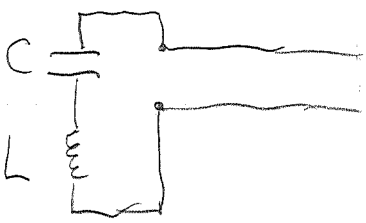
You need more detail.



At $x = 0$ let E_0 be the voltage relative to ground and let I_0 be the current into the transmission line. Then E_0 is the voltage at the 1-port and $-I_0$ is the current into the 1-port, so

$$\frac{E_0}{I_0} = -Z \qquad Z(s) \text{ impedance of 1-port}$$

e.g.



$$-\frac{E_0}{I_0} = Ls + \frac{1}{Cs}$$

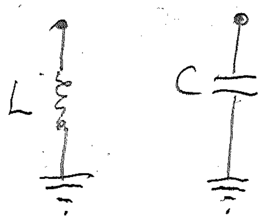
$$E_0 - I_0 = A$$
$$E_0 + I_0 = B$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} E_0 \\ I_0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -Z \\ 1 \end{pmatrix}$$

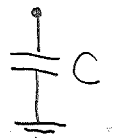
$$\frac{A}{B} = \frac{Z+1}{Z-1}$$

7 June 24, 02 You want to understand properly the ~~general~~ abstract setting for a LC circuit with ~~a set of~~ external nodes. Call this an LC port. To learn again your thm that an abstract LC port is a subquotient of a polarized Hilb space. Start with simple cases, namely:



You probably want to assume the graph is connected, and provided with a ground node.

Q: Is there a space of states for the LC circuit? Could this be a polarized Hilbert space? Consider the inductance. A state for L can any pair (E, I) of real numbers. A history of states is a pair $(E(t), I(t))$ of real functions of time satisfying $E = L\partial_t I$. Can you fit this into some Lagrangian or Hamiltonian formalism? Particle moving with constant velocity. In the C case.



states $(E, I) \in \mathbb{R}^2$, equation of motion: $I = C\partial_t E$.

For a general LC circuit you start with a state space consisting of pairs $(E, I)_\sigma$ for each oriented edge. Then you impose constraints. This should involve something like symplectic quotient, where you restrict the edge voltage function to be conservative, and kill currents with boundary $= 0$.

Your problem is how to make progress. Let's begin with an LC circuit which is connected and has a ground node specified, also suppose the edges are oriented. Ignore the nodes - you get a state space, consisting of pairs $(E, I)_\sigma$ for each edge σ.

Somehow you need to get Kirchoff equations

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{C}^0 & \xrightarrow{\ \delta\ } & C^1 & \longrightarrow & H^1 & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle Z^{-1}} & & & & \\
0 & \longleftarrow & \tilde{C}_0 & \xleftarrow{\ \partial\ } & C_1 & \longleftarrow & H_1 & \longleftarrow & 0
\end{array}$$

main point here is that $\partial Z^{-1}\delta : \tilde{C}^0 \xrightarrow{\sim} \tilde{C}_0$ invertible.

$$\begin{tikzcd} 0 \arrow[r] & \tilde{C}^0 \arrow[r, tail, "\delta"] & C^1 \arrow[r] \arrow[d, "Z^{-1}"] & H^1 \arrow[r] & 0 \\ 0 & \tilde{C}_0 \arrow[l] & C_1 \arrow[l, two heads, "\partial"'] & H_1 \arrow[l] & 0 \arrow[l, tail] \end{tikzcd}$$

Points: $\left(\begin{array}{ll} \tilde{C}_0, \tilde{C}^0 & \text{dual pair} \\ C_1, C^1 & \text{———} \end{array} \right.$

$Z : C_1 \longrightarrow C^1$ is equivalent to the diagonal quadratic form on C_1 with values $Z_\sigma = \begin{cases} Ls & \text{ind} \\ (Cs)^{-1} & \text{cap} \end{cases}$ assume s real

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June 25, 02.

$$V \rightarrow \begin{matrix} H_+ \\ \oplus \\ H_- \end{matrix} \longrightarrow H/V \qquad H = \begin{matrix} H_+ \\ \oplus \\ H_- \end{matrix} \text{ is a polarized Hilb space f.d.}$$

V is a subspace, H/V is the corresp subspace. On H you consider the pos hermitian form $s\|\xi_+\|^2 + s^{-1}\|\xi_-\|^2$, say $s>0$, which induces pos. herm. forms on V and H/V. To work out the formulas. On H you have F, ε and the situation splits according to eigenvalues of $g = F\varepsilon$.

Exact sequence $V \rightarrowtail H \twoheadrightarrow W$ together with a hermitian form on H. Assume the form rest to V is non deg. Then you get an induced form on W

Do the real case: $V \rightarrowtail H \twoheadrightarrow W$, A = quadratic form on H, $A|V$ non degenerate. Given $\xi + V \in W$ where $\xi \in H$. Look for stationary values of $A(\xi+v)$ for $v \in V$. $0 = \delta(A(\xi+v)) = 2A(\xi+v, \delta v) = \langle \delta v | A(\xi+v) \rangle$

$$\delta \langle \xi+v | A(\xi+v) \rangle = \langle \delta v | A(\xi+v) \rangle + \langle \xi+v | A\,\delta v \rangle$$

$$= 2\langle \delta v | A(\xi+v) \rangle = 0 . \quad \text{Means } \xi+v \perp V$$

for the quadratic form A

$$0 \longrightarrow V \rightarrowtail H \twoheadrightarrow W \longrightarrow 0$$ short exact seq.

$a(\xi) = \langle \xi | A\xi \rangle$ quadratic function on H

$A : H \longrightarrow H^*$ symmetric map.

Given coset $\xi_0 + V = \{ \xi_0 + v \mid v \in V \}$

$$a(\xi_0 + v) = \langle \xi_0 + v | A\xi_0 + Av \rangle$$

apply inf. variation δv

$$\delta a(\xi_0 + v) = \langle \delta v | A(\xi_0 + v) \rangle + \langle \xi_0 + v | A\delta v \rangle$$

$$= \langle \delta v | A(\xi_0 + v) \rangle + \langle A(\xi_0 + v) | \delta v \rangle$$

$$= 2 \langle \delta v | A(\xi_0 + v) \rangle = 2 \langle \xi_0 + v | A\delta v \rangle$$

$\xi_0 + v$ is a stationary point for $a(\xi)$ $\xi \in \xi_0 + V$

iff $\langle \xi_0 + v | AV \rangle = 0$.

$\xi = \xi_0 + v$

$$\begin{tikzcd} V \arrow[r, tail, "j"] & H \arrow[r, two heads] \arrow[d, "A"', "A\xi"] & W \\ V^* & H^* \arrow[l, two heads, "j^*"'] & W^* \arrow[l, tail] \end{tikzcd}$$

$$V \overset{\iota}{\hookrightarrow} H \xrightarrow{j} W$$

$A : H \longrightarrow H^*$ symm.

$q(\xi) = \langle \xi | A\xi \rangle$

Consider a coset $j^{-1}w$ of V in H

let ξ be stationary for $q|_{j^{-1}w}$

$$\delta q(\xi) = q(\xi + \delta v) - q(\xi) = \langle \xi + \delta v | A\xi + A\delta v \rangle - \langle \xi | A\xi \rangle$$

$$= \langle \xi | A\delta v \rangle + \langle \delta v | A\xi \rangle$$

So $j^*A\xi = 0$ is the condition which says that ξ is a stationary point for $q(\xi) = \langle \xi | A\xi \rangle$ when this form is restricted to the coset $\xi + iV = j^{-1}j\xi$

But this condition $j^*A\xi = 0$ doesn't require A to be symmetric, so it's probably not a good idea to use the quadratic form $\langle \xi | A\xi \rangle$.

Review the basic decomposition

$$\begin{tikzcd} V \arrow[r, hook, "i"] & H \arrow[r, two heads, "j"] \arrow[d, "A"] & W \\ V^* & H^* \arrow[l, two heads, "i^*"'] & W^* \arrow[l, hook', "j^*"'] \end{tikzcd}$$

Assume $i^*Ai : V \to V^*$ is an isomorphism

Then there should be canonical complements for $i(V) \subset V$, $j^*W^* \subset H^*$ with nice properties wrt A. Why? For $\xi \in H$ the map

$$\xi \mapsto i(i^*Ai)^{-1}i^*A\xi$$

is a projection with image iV since

$$\left(i(i^*Ai)^{-1}i^*A\right)i = i$$

Then $\xi \longmapsto \xi - i(i^*Ai)^{-1}i^*A\xi$ is a projection on H whose image is a complement to iV, hence this image is mapped isom. to W via j.

$$\left(Ai(i^*Ai)^{-1}i^*\right)Ai = Ai$$

so $Ai(i^*Ai)^{-1}i^*$ is a projection with image AiV, which is mapped isom. to V^* via i^*.

Lift $\xi \longmapsto \xi - i(i^*Ai)^{-1}i^*A\xi \xmapsto{i^*A} 0$, so $\xi \longmapsto A(\xi - i(i^*Ai)^{-1}i^*A\xi)$ is killed by i^* on the left and i on the right; get A induced map $W \longrightarrow W^*$.

Look at $\iota : V \longrightarrow H_+ \oplus H_-$, $\quad \iota^*\iota = 1_V$

Replace $H_\pm$ by $\overline{e_\pm \iota V}$ so that $\iota_\pm : V \to H_\pm$ is onto

$$1_V = \iota^*\iota = \iota_+^*\iota_+ + \iota_-^*\iota_-$$

self adjoint ρ such that $0 \le \rho \le 1$

$$\rho = \sum_\lambda \lambda \pi_\lambda \quad \text{on} \quad V = \bigoplus V_\lambda$$

Special cases where $\iota_+^*\iota_+ = \lambda$, $\iota_-^*\iota_- = 1-\lambda$ and $0 < \lambda < 1$. Then

$$\iota : V \xrightarrow{\begin{pmatrix}\lambda^{1/2} \\ (1-\lambda)^{1/2}\end{pmatrix}} \begin{matrix} V \\ \oplus \\ V \end{matrix}$$

Instead of λ you might use $\lambda^{1/2} = \cos\theta$, $(1-\lambda)^{1/2} = \sin\theta$

$$H = \begin{pmatrix}\mathbb{C} \\ \mathbb{C}\end{pmatrix} \quad \text{and} \quad V = \mathbb{C}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}, \quad V^\perp = \mathbb{C}\begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}$$

So $\begin{pmatrix}\iota_+ \\ \iota_-\end{pmatrix} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$ $\quad$ Try $\begin{pmatrix}1 & -t \\ t & 1\end{pmatrix}\Big(1+$

$$\frac{1+X}{\sqrt{1-X^2}} \qquad X^2 = \begin{pmatrix}0 & -t \\ t & 0\end{pmatrix}\begin{pmatrix}0 & -t \\ t & 0\end{pmatrix} = \begin{pmatrix}\cancel{0}-t^2 & 0\ \cancel{2t} \\ 0\ \cancel{\quad} & -t^2\end{pmatrix}$$

$$\sqrt{1-X^2} = \begin{pmatrix}(1+t^2)^{1/2} & 0 \\ 0 & (1+t^2)^{1/2}\end{pmatrix}$$

$$\frac{1+X}{\sqrt{1-X^2}} = \begin{pmatrix}1 & -t \\ t & 1\end{pmatrix}\begin{pmatrix}(1+t^2)^{-1/2} & 0 \\ 0 & (1+t^2)^{-1/2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{1+t^2}} & \frac{-t}{\sqrt{1+t^2}} \\ \frac{t}{\sqrt{1+t^2}} & \frac{1}{\sqrt{1+t^2}}\end{pmatrix}$$

$$\frac{1+X}{1-X} = \frac{1}{1+t^2}\begin{pmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{pmatrix} = \begin{pmatrix} \frac{1-t^2}{1+t^2} & \frac{-2t}{1+t^2} \\ \frac{2t}{1+t^2} & \frac{1-t^2}{1+t^2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix}\begin{pmatrix} 1 & -t \\ t & 1 \end{pmatrix} = \begin{pmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{pmatrix}$$

So what are you trying to get

June 26, 02 $\quad i : V \longrightarrow \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$, $i^* i = 1_V$, on H you have pos. herm. form $\quad i^*(se_+ + s^{-1}e_-)i = s i_+^* i_+ + s^{-1} i_-^* i_-$,

$= sp + s^{-1}(1-p)$ where $p = i_+^* i_+ \in \mathcal{L}(V)$ satisfies $0 \leq p = p^* \leq 1$. The point you've been missing: you have to include $W = V^\perp$ and look at ~~the~~ ~~smallest~~ joint eigenspace decomposition for the pair of involutions F, ε.

So what next. $\quad H_+, H_-$ dim 1.

$V \subset \begin{pmatrix} H_+ \\ H_- \end{pmatrix}$

a

Recall what you relearned

$$\begin{array}{ccccc} V & \overset{i}{\rightarrowtail} & H & \overset{j}{\twoheadrightarrow} & W \\ & & \downarrow{\scriptstyle A} & & \\ V^* & \overset{i^*}{\twoheadleftarrow} & H^* & \overset{j^*}{\leftarrowtail} & W^* \end{array}$$

If i^*Ai is invertible, then there is a push forward form $A_W : W \longrightarrow W^*$ given by

$$j^* A_W(j\xi) = A\xi - Ai(i^*Ai)^{-1} i^*A\xi$$

kills iV

killed by i^* $\therefore$ descends to W^*

Also when A invertible one has $A_W = (jA^{-1}j^*)^{-1}$ because

$$jA^{-1}j^* A_W(j\xi) = j\xi \quad \text{as } ji = 0$$

?

$$j^* A_W(jA^{-1}j^* w') = j^* w'$$

13 Try again

$$\begin{tikzcd} V \arrow[r, hook, "\iota"] \arrow[d] & H \arrow[r, two heads, "f"] \arrow[d, "A"] & W \\ V^* & H^* \arrow[l, two heads, "\iota^*"] & W^* \arrow[l, hook', "f^*"] \end{tikzcd}$$

Assume $A_V = \iota^* A \iota$, the restriction of the bilinear form A from H to V, is invertible

Then $\iota(\iota^* A \iota)^{-1}\iota_* A$ is a projection on V with image ιV.

Also $A\iota(\iota^* A \iota)^{-1}\iota_*$ is a projection on H^* with image $A\iota V$

too hard

Next project: harmonic oscillators. Question: Can you make a theory of ports out of harmonic oscillators. forced harmonic oscillator?

June 25, 02. Observe that the transmission line equation is the Dirac equation with zero mass in 2 diml space time:

$$(\partial_t + \partial_x)(E + I) = 0 \qquad E+I = \text{fn}(x-t) \quad \text{right-moving}$$

$$(\partial_t - \partial_x)(E - I) = 0 \qquad E-I = \text{fn}(x+t) \quad \text{left-moving}$$

Can you make a link with your picture of LC circuits?

Bosonization of fermions might provide an LC picture for a harmonic oscillator type of.

June 30, 02 Explain viewpoint for an LC circuit.

First there is a geometric object: the graph of the circuit. Each edge σ of the graph has a 2 diml complex(?) vector space of states (E_σ, I_σ), where E_σ is the voltage drop and I_σ is the current along the edge. (Actually E_σ is a 1-cochain and I_σ is a 1-chain, so you need the edge to be oriented for E_σ, I_σ to be numbers). The power of the state (E_σ, I_σ) is $E_\sigma I_\sigma$. Thus the space of states of all the edges is $C^1 \oplus C_1$, and the power form is the usual pairing between 1-cochains and 1-chains.

July 1, 02 Philosophy for LC circuits. You are given a graph with inductance and capacitance edges. This is a geometric object to which you want to assign a state space. Analogy with a mechanical system with constraints, e.g. a particle moving on a submanifold of configuration space. In this case you start with phase space (cotangent bundle) of config space, then obtain the phase space of the constrained particle (cotangent bdl of submanifold) by "symplectic reduction".

For the LC circuit the ~~state~~ space is $C^1 \oplus C_1$, where an elt of C^1 (resp C_1) is a family of voltage drops (resp currents) for the edges of the graph. These chains and cochains become functions on the edges when the edges are oriented. Note that C^1 and C_1 are naturally dual by the power pairing $E \cdot I = \sum_\sigma E_\sigma I_\sigma$. You can think of $C^1 \oplus C_1$ as the cotangent bundle of the configuration space C^1.

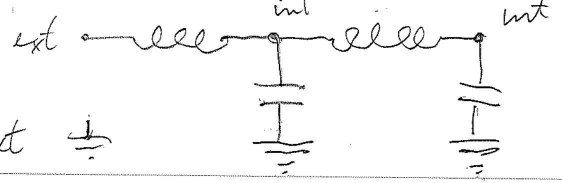
Next you constrain the configurations $E \in C^1$ to be conservative, i.e. to come from a "potential" function on the nodes. To simplify suppose the graph is connected and equipped with a basepoint (the ground). Thus ~~you~~ restrict via $C^0 \overset{\delta}{\hookrightarrow} C^1$, where $C^0 =$ 0-cochains vanishing at $*$. On the state space level you get "symplectic reduction" from $C^1 \oplus C_1$ to $C^0 \oplus C_0$:

$$C^0 \hookrightarrow C^1$$

$C_0 = \{0$ cochains$\}$, view the current going into the node

$$C_0 \twoheadleftarrow C_1$$

Next ~~suppose~~ the nodes divided into internal + external nodes, e.g.



the ground should be external

July 3, 02. Let C^0_{ext} be the voltage functions on the external nodes with the ground value $= 0$, let C_0^{ext} contain "currents" going into the external nodes, with the current going into $*$ adjusted so that the sum of the external currents is 0. This is consistent with the potential being zero at the ground. Then you should have another symplectic reduction from $C^0 \oplus C_0$ to $C^0_{ext} \oplus C_0^{ext}$.

Review the philosophy for LC circuits. You are given a connected graph together with a distinguished node (the ground) and a distinguished subset of nodes containing the ground called external nodes. This is the geometry which you linearize to obtain three configuration spaces as follows:

$$\begin{tikzcd} C^0 \arrow[r, hook, "\delta"] \arrow[d] & C^1 \\ C^0_{ext} & \end{tikzcd}$$

C^1 = space of 1-cochains on the graph

C^0 = space of 0-cochains vanishing at $*$

C^0_{ext} = space of 0-cochains on the external nodes vanishing at $*$

Element of C^1 is a family of voltage drops on the edges

——— C^0 —— potential function on the nodes, which $= 0$ on $*$

——— C^0_{ext} ——— external ———.

Associated to these three configuration spaces are phase spaces obtained by taking the direct sum with the dual space:

$$C^0_{ext} \oplus C_0^{ext} \qquad C^0 \oplus C_0 \qquad C^1 \oplus C_1$$

Each one is a "symplectic reduction" of the following one.

So far you have kinematics only. Dynamics arises when you allow the phase space states to be time dependent and require for each edge σ that

$$E_\sigma = L_\sigma \partial_t I_\sigma \quad \text{for an inductive edge of inductance } L_\sigma$$

$$I_\sigma = C_\sigma \partial_t E_\sigma \quad \text{—— a capacitive edge of capacitance } C_\sigma.$$

You now need to understand how this time flow, which makes sense on $C^1 \oplus C_1$, induces a time flow on $C^0_{ext} \oplus C_0^{ext}$.

July 4, 02. Your aim is to link partial unitary operators: $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ $\quad a^*a = b^*b = 1$ with abstract LC circuits: subquotient

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] & H \arrow[d] \\ V/W \arrow[r, hook] & H/W \end{tikzcd}$$

of a polarized Hilbert space $H = H_+ \oplus H_-$

Let's review partial unitaries. Example. Suppose given a Hilb space H equipped with a unitary operator u, and a closed subspace $Y \subset H$. Put $X = Y \cap u^{-1}Y$, $a =$ inclusion $X \hookrightarrow Y$, $b = u|_X : X \longrightarrow uX = uY \cap Y \subset Y$.

Eigenvector equation: You have

$$Y = X \oplus V_+ = V_- \oplus uX \qquad \begin{aligned} V_+ &= \operatorname{Ker} a^* \\ V_- &= \operatorname{Ker} b^* \end{aligned}$$

so
$$H = X \oplus V_+ \oplus Y^\perp = V_- \oplus uX \oplus Y^\perp.$$

Let $\xi \in H$ satisfy $\quad u\xi - \lambda\xi \perp uX$. One has

$$\xi = x_1 + v_+ + p = v_- + ux_0 + p \qquad x_1, x_0 \in X,\ v_\pm \in V_\pm$$

$$u\xi - \lambda\xi = ux_1 + uv_+ + up - \lambda v_- - \lambda ux_0 - \lambda p.$$

Now project onto uX: $\ u\xi - \lambda\xi \mapsto 0$, $uv_+ \mapsto 0$, $up \mapsto 0$, $\lambda v_- \mapsto 0$, $\lambda p \mapsto 0$

$$\Longrightarrow \quad ux_1 = u(\lambda x_0) \Longrightarrow x_1 = \lambda x_0.$$

17 Thus get $\boxed{\lambda x_0 + v_+ = v_- + u x_0}$ in Y

Rewrite as $\boxed{(\lambda a - b)x = -v_+ + v_-}$

Review how this determines a scattering operator $S(\lambda) : V_- \longrightarrow V_+$. $a^*(\lambda a - b)x = a^* v_- \implies$

$$x = (\lambda - a^*b)^{-1} a^* v_- = a^*(\lambda - ba^*)^{-1} v_-$$

$$\begin{aligned} v_+ &= v_- - (\lambda a - b)a^*(\lambda - ba^*)^{-1} v_- \\ &= \left[(\lambda - \cancel{ba^*}) - \lambda aa^* + \cancel{ba^*}\right](\lambda - ba^*)^{-1} v_- \\ &= (1 - aa^*)\frac{\lambda}{\lambda - ba^*} v_- . \end{aligned}$$

Next you need a diagram to interpret this

<u>July 5, 02:</u>

$$\begin{tikzcd} & V_+ \arrow[d, tail] \arrow[dr] & \\ X \arrow[r, "a\lambda - b"] \arrow[dr, "\lambda - a^*b"'] & Y \arrow[r] \arrow[d, two heads, "a^*"] & L_\lambda \\ & X & \end{tikzcd}$$

$(\lambda - a^*b)^{-1} \ \exists \implies V_+ \xrightarrow{\sim} L_\lambda$

so you get

$$Y \twoheadrightarrow L_\lambda \xleftarrow{\sim} V_+$$

which assigns to each y a function $y(\lambda) \in V_+$ such

$$y - y(\lambda) \in (a\lambda - b)X$$

Check: Assume $v_+ \in V_+$ sat $(a\lambda - b)x = -v_+ + y$

Then $(\lambda - a^*b)x = a^*y \implies x = a^*(\lambda - ba^*)^{-1} y$

$$\begin{aligned} \Rightarrow \quad y - (a\lambda - b)x &= y - (\lambda aa^* - ba^*)(\lambda - ba^*)^{-1} y \\ &= \left[(\lambda - \cancel{ba^*}) - (\lambda aa^* - \cancel{ba^*})\right](\lambda - ba^*)^{-1} y \\ &= \underbrace{(1 - aa^*)\frac{\lambda}{\lambda - ba^*} y}_{y(\lambda)} \in V_+ \end{aligned}$$

Let T be symplectic v.s., W an isotropic subspace, $W^\perp = \{\xi \in T \mid \omega(\xi, W) = 0\}$, then $W^\perp/W$ is the symplectic quotient corresponding to W. Let L be a Lagrangian subspace of T. Then

$$(L+W) \cap W^\perp = L \cap W^\perp + W$$

by the modular law. Also $L = L^\perp$ implies

$$\left((L+W) \cap W^\perp\right)^\perp = (L+W)^\perp + W = L \cap W^\perp + W$$

Put $\bar{L} = (L+W) \cap W^\perp = L \cap W^\perp + W$. Then $W \subset \bar{L} \subset W^\perp$ and $\bar{L}^\perp = \bar{L}$ which should mean that $\bar{L}/W$ is a Lagrangian subspace of $W^\perp/W$.

Examples. Let $T = \begin{pmatrix} D \\ D^* \end{pmatrix}$ with hyperbolic symplectic form

$$\begin{pmatrix} d_1 \\ \delta_1 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_2 \\ \delta_2 \end{pmatrix} = (d_1 \ \ \delta_1) \begin{pmatrix} \delta_2 \\ -d_2 \end{pmatrix} = (d_1, \delta_2) - (d_2, \delta_1).$$

Here the pairing between $d \in D$, $\delta \in D^*$ is written symmetrically: $(d, \delta) = (\delta, d)$.

Now let $T = \begin{pmatrix} D \\ D^* \end{pmatrix}$, let $L = \begin{pmatrix} 1 \\ Q \end{pmatrix} D \subset \begin{pmatrix} D \\ D^* \end{pmatrix}$, where $Q : D \longrightarrow D^*$ is symmetric. L is isotropic in T:

$$\begin{pmatrix} d_1 \\ Qd_1 \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} d_2 \\ Qd_2 \end{pmatrix} = (d_1, Qd_2) - (Qd_1, d_2), \text{ as } (Qd_1, d_2) = (d_1, Q^* d_2)$$
$$= (d_1, Qd_2).$$

(1) Let $V \subset D$; then $V^* = D/V^\circ$, $V^\circ = \{\lambda \in D \mid \lambda(V) = 0\}$

Take $W = \begin{pmatrix} 0 \\ V^\circ \end{pmatrix} \subset \begin{pmatrix} D \\ D^* \end{pmatrix}$, then $W^\perp = \begin{pmatrix} V \\ D^* \end{pmatrix}$ and

$$W^\perp/W = \begin{pmatrix} V \\ D^*/V^\circ \end{pmatrix} = \begin{pmatrix} V \\ V^* \end{pmatrix}. \quad \text{Let } L = \begin{pmatrix} 1 \\ Q \end{pmatrix} D$$

 Then $L \cap W^\perp = \begin{pmatrix} 1 \\ Q \end{pmatrix} D \cap \begin{pmatrix} V \\ D^* \end{pmatrix} = \left\{ \begin{pmatrix} \xi \\ Q\xi \end{pmatrix} \middle| \xi \in V \right\}$

and $\dfrac{L \cap W^\perp + W}{W} = \operatorname{Img}\left\{ \begin{pmatrix} 1 \\ Q \end{pmatrix} : V \longrightarrow \begin{pmatrix} V \\ V^* \end{pmatrix} \right\}$. Note

$$\begin{tikzcd} V \arrow[r, tail] & D \arrow[r] \arrow[d, "Q"] & D/V \\ V^* & D^* \arrow[l, two heads] & V^0 \arrow[l, tail] \end{tikzcd}$$

Thus $L \longmapsto \dfrac{L \cap W^\perp + W}{W}$ sends $\begin{pmatrix} 1 \\ Q \end{pmatrix} D$ to the graph of the composite:

$$\begin{tikzcd} V \arrow[r] & D \arrow[d] \\ V^* & D^* \arrow[l] \end{tikzcd}$$

This is pull-back of Q on D to Q restricted to V.

(2) $T = \begin{pmatrix} D \\ D^* \end{pmatrix}$, $L = \begin{pmatrix} 1 \\ Q \end{pmatrix} D$ as above, but $W = \begin{pmatrix} V \\ 0 \end{pmatrix}$

$W^\perp = \begin{pmatrix} D \\ V^0 \end{pmatrix}$, $W^\perp / W = \begin{pmatrix} D/V \\ V^0 \end{pmatrix}$. Then

$$L \cap W^\perp = \begin{pmatrix} 1 \\ Q \end{pmatrix} D \cap \begin{pmatrix} D \\ V^0 \end{pmatrix} = \left\{ \begin{pmatrix} \xi \\ Q\xi \end{pmatrix} \middle| Q\xi \in V^0 \right\} \simeq D \times_{D^*} V^0$$

$$\operatorname{Im}\left\{ L \cap W^\perp \longrightarrow W^\perp / W \right\} = \left\{ \begin{pmatrix} \xi \bmod V \\ Q\xi \end{pmatrix} \middle| \begin{array}{l} \xi \in D \\ Q\xi \in V^0 \end{array} \right\}$$

$$\begin{tikzcd} V \arrow[r, tail] & D \arrow[r, two heads, "\xi"] \arrow[d, "Q"'] & D/V \\ V^* & D^* \arrow[l, two heads] & V^0 \arrow[l, tail] \end{tikzcd}$$

This Lagrangian subspace (call it Γ') of $\begin{pmatrix} D/V \\ V^0 \end{pmatrix} = \begin{pmatrix} D/V \\ (D/V)^* \end{pmatrix}$ is the correspondence $\Gamma' \subset D/V \times V^0$ arising from $\xi \in D$ which satisfy $Q\xi \in V^0$. This condition should mean that ξ is stationary for Q restricted to $\xi + V$.

 July 8, 02

Given T symplectic, W isotropic in T, $W^\perp/W$ the associated symplectic quotient, you would like to understand better the map

$$* \qquad SG(T) \longrightarrow SG(W^\perp/W)$$
$$L \longmapsto (L\cap W^\perp + W)/W = (L+W)\cap W^\perp/W$$

Here SG stands for symplectic Grassmannian: the set of Lagrangian subspaces.

You have discussed examples in the hyperbolic case $T = \begin{pmatrix} D \\ D^* \end{pmatrix}$, where $L = \begin{pmatrix} 1 \\ Q \end{pmatrix} D$ is the graph of a symmetric $Q: D \to D^*$, and W has the form $\begin{pmatrix} V_1 \\ V_2^0 \end{pmatrix}$, with $V_1 \subset V_2$, in other words where W is stable under ε. You found that the map $*$ above amounts to induced quadratic forms on a sub-quotient space. However, the induced quadratic form on a quotient space seems to be defined only under suitable nondegeneracy conditions, whereas $*$ is defined in general.

Let's study a simple case: T^4, W a line in T, L an arbitrary Lagrangian subspace of T. Two cases: 1) $L \supset W$, whence $L \subset W^\perp$ and the "symplectic reduction" of L is L/W. 2) $L \cap W = 0$, whence $L + W^\perp = T$. Lattice picture:

$$\begin{tikzcd}[column sep=small]
 & & T \arrow[dl, dash] \arrow[dr, dash] & & \\
 & L+W \arrow[dl, dash] \arrow[dr, dash] & & W^\perp \arrow[dl, dash] & \\
L \arrow[dr, dash] & & L\cap W^\perp + W \arrow[dl, dash] \arrow[dr, dash] & & \\
 & L\cap W^\perp \arrow[dr, dash] & & W \arrow[dl, dash] & \\
 & & 0 & &
\end{tikzcd}$$

Fix the symplectic red. of L in $SG(W^\perp/W)$, i.e. fix the Lagrangian subspace $L\cap W^\perp + W = (L+W)\cap W^\perp$ between W and $W^\perp$. You want to describe the possible L. L determines a line $L\cap W^\perp \neq W$ in $L\cap W^\perp + W$. Applying $\perp$ you get $L+W$. It seems that $L/L\cap W^\perp$ can be any Lagrangian subspace (line) in the symplectic ~~quotient~~ $L+W/L\cap W^\perp$

Simpler example: $\dim(T) = 2$, W line in T, L another line. The symplectic reduction of L to $W^\perp/W = 0$ is $L \cap W^\perp + W/W = (L+W) \cap W^\perp/W = 0$. The possible L's are <u>any</u> line in T.

Important diagram: Assume $L \cap W = 0$

equiv: $L + W^\perp = T$

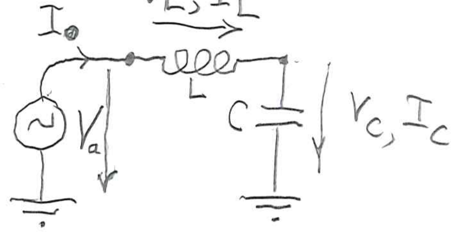
$$\begin{tikzcd}
 & & T \arrow[dl, dash, "n-r"'] \arrow[dr, dash, "r"] & & \\
 & L+W \arrow[dl, dash, "r"'] \arrow[dr, dash, "r"] & & W^\perp \arrow[dl, dash, "n-r"] & \\
L \arrow[dr, dash, "r"'] & & (L+W)\cap W^\perp = (L \cap W^\perp)+W \arrow[dl, dash, "r"] \arrow[dr, dash, "n-r"] & & \\
 & L \cap W^\perp \arrow[dr, dash, "n-r"'] & & W \arrow[dl, dash, "r"] & \\
 & & 0 & &
\end{tikzcd}$$

IDEAS to explore: Complex structure J on T and J-linear subspaces; this is the linear version of an almost complex structures on a symplectic manifold and J-holomorphic curves.

Krein space picture for the Riemann sphere with the action of $SU(1,1)$. If $T = \mathbb{C}^2$ with hermitian form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and Y is a Krein space then $T \otimes Y$ should be a Krein space, and $\mathcal{O}(-1) \otimes Y \hookrightarrow \mathcal{O} \otimes T \otimes Y$ should be a family of Lagrangian subspaces.

July 10, 02

Discuss LC circuit:



V_a = applied voltage to the LC series circuit

I_0 = output current. These 6 variables satisfy equations of motion (time-evolution)

$$V_L = L\partial_t I_L, \quad I_C = C\partial_t V_C$$

and the circuit equations

$$I_0 = I_L = I_C$$
$$V_a = V_L + V_C$$

V_a is a given function of time, say of compact support. There are 5 equations for 5 unknowns.

You would like to obtain these equations from a variational principle. You want a functional of the 5 functions V_L, I_L, V_C, I_C, I_0 of t such that the stationary values are the solutions of the equations.

Use the Laplace Transform to replace time t by the frequency variable s, and the operator ∂_t by multiplication by s.

July 13, 02

Partial unitary operator: $X \underset{b}{\overset{a}{\rightrightarrows}} Y$ $\quad a^*a = b^*b = 1_X$, where X, Y are Hilbert spaces. Form the Krein space $\binom{Y}{Y}$ with the hermitian form $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \|y_1\|^2 - \|y_2\|^2$

Then X, a, b can be identified with the isotropic subspace $W = \binom{a}{b} X \subset \binom{Y}{Y}$ of the Krein space. Consider

$$W^\perp = \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, x^* (a^* \; b^*) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \quad \forall x \right\}$$

$$= \left\{ \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \,\middle|\, a^* y_1 = b^* y_2 \right\}$$

One has
$$y_1 = a(a^*y_1) + (1-a^*a)y_1 \in aX + V_+$$
$$y_2 = b(b^*y_2) + (1-b^*b)y_2 \in bX + V_-$$
which yields a splitting
$$W^\perp = \left(\begin{pmatrix} a \\ b \end{pmatrix} X\right)^\perp = \begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} V_+ \\ V_- \end{pmatrix}$$
Next look at $\begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \subset \begin{pmatrix} Y \\ Y \end{pmatrix}$ which is isotropic iff $|\lambda| = 1$. This is the analogue of the family of Lagrangian subspaces in the LC circuit case giving the dynamics.

Analogues of eigenvectors. 1) Bound states $=$ $x \in X$, $x \neq 0$ such that $\lambda a x = bx$ for some $\lambda \in \mathbb{C}$; then $|\lambda| = 1$.

2) When
$$0 \neq \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \cap W^\perp = \begin{pmatrix} 1 \\ \lambda \end{pmatrix} Y \cap \left(\begin{pmatrix} a \\ b \end{pmatrix} X \oplus \begin{pmatrix} V_+ \\ V_- \end{pmatrix}\right).$$
Then one has nontrivial solutions of
$$\begin{cases} y = ax + v_+ \\ \lambda y = bx + v_- \end{cases} \quad \text{or} \quad (\lambda a - b)x = -\lambda v_+ + v_-$$

July 16, 02

Consider a partial unitary $X \underset{b}{\overset{a}{\rightrightarrows}} Y$, $Y = aX \oplus V_+ = V_- \oplus bX$ where $V_\pm$ is 1-diml, let $\xi_\pm$ be a unit vector in $V_\pm$. Let $c_h = ba^* + \xi_- h \xi_+^*$ for $h \in \mathbb{C}$. Then c_h extends the partial unitary ba^{-1} by the operator $V_+ \to V_-$ sending ξ_+ to $\xi_- h$. Note c_h is a contraction for $|h| \le 1$ and it's unitary for $|h| = 1$.

Treat c_h as a perturbation of $c_0 = ba^*$ by $\Delta = \xi_- h \xi_+^*$
$$\frac{1}{z - c_h} = \frac{1}{z - c_0} + \frac{1}{z - c_0}\Delta\frac{1}{z-c_0} + \frac{1}{z-c_0}\left(\Delta\frac{1}{z-c_0}\right)^2 + \cdots$$
$$\xi_+^*\frac{1}{z - c_h} = \xi_+^*\frac{1}{z-c_0} + \left(\xi_+^*\frac{1}{z-c_0}\xi_- h\right)\xi_+^*\frac{1}{z-c_0} + \left(\xi_+^*\frac{1}{z-c_0}\xi_- h\right)^2 \xi_+^*\frac{1}{z-c_0}$$
$$= \frac{1}{1 - \left(\xi_+^*\frac{1}{z-c_0}\xi_-\right)h}\ \xi_+^*\frac{1}{z-c_0}$$

Put $S_h = \xi_+^* \frac{1}{z-c_h} \xi_-$ and recall that

$S_0 = \xi_+^* \frac{1}{z-ba^*} \xi_-$ is essentially the scattering operator $(1-aa^*)\frac{1}{z-ba^*} : V_- \to V_+$. So $S_0(z)$ should be analytic outside and of $|\ |=1$ on the unit circle. From the previous page one should have

$$S_h = \frac{1}{1-S_0 h} S_0 \qquad \text{at least for } |h|<1.$$

July 18, 02

Consider a partial unitary $Y = aX \oplus V_+ = V_- \oplus bX$ with Y finite dimensional, X of codim 1. Let $\xi_\pm \in V_\pm$ be unit vectors spanning $V_\pm$ resp.

$$c_h = ba^* + \xi_- h \xi_+^* = c_0 + \Delta$$

$$(z-c_h)^{-1} = (z-c_0)^{-1}\left(1-\Delta(z-c_0)^{-1}\right)^{-1} = \left(1-(z-c_0)^{-1}\Delta\right)^{-1}(z-c_0)^{-1}$$

$$\xi_+^*(z-c_h)^{-1} = \xi_+^*\left(1-(z-c_0)^{-1}\xi_- h\, \xi_+^*\right)^{-1}(z-c_0)^{-1}$$

$$= \left(1-\xi_+^*(z-c_0)^{-1}\xi_- h\right)^{-1}\xi_+^*(z-c_0)^{-1}$$

Then $S_h \overset{\text{def}}{=} \xi_+^*(z-c_h)^{-1}\xi_- = (1-S_0 h)^{-1} S_0$. Take $h=1$, so $c_1 = ba^* + \xi_-\xi_+^*$ is the unitary operator u on Y sending ax to bx and ξ_+ to ξ_-. One has

$$S_1 = \xi_+^* \frac{1}{z-u} u\, \xi_+ \qquad \frac{1}{2} + \frac{u}{z-u} = \frac{1}{2}\frac{z+u}{z-u}$$

$$\underbrace{\xi_+^* \frac{1}{2}\frac{z+u}{z-u}\xi_+} = \frac{1}{2} + S_1 = \frac{1}{2} + \frac{S_0}{1-S_0} = \frac{1}{2}\frac{1+S_0}{1-S_0}$$

this is essentially the Pick function (analytic on D with ≥ 0 imaginary part) corresponding to the measure on $|z|=1$ associated to the unitary u and the vector ξ_+. is essentially the C.T. of S_0 which is a finite degree map from $|z|=1$ to itself.

Variational approach to $\det(z - c_h)$:

$$\delta \log\det(z - c_0 - \xi_- h\, \xi_+^*) = \operatorname{tr}\left(\frac{1}{z-c_h}(-\xi_- \delta h\, \xi_+^*)\right)$$

$$= -\xi_+^* \frac{1}{z-c_h}\xi_- \delta h = -S_h\, \delta h = \frac{-S_0}{1-S_0 h}\delta h$$

$$= \delta \log(1 - S_0 h). \quad \text{Thus}$$

$$\boxed{\frac{\det(z-c_h)}{1-S_0 h} = \det(z-c_0)}$$

Simpler might be

$$\frac{\det(z-c_h)}{\det(z-c_0)} = \det\left(1 - (z-c_0)^{-1}\xi_- h\, \xi_+^*\right)$$

$$= \det\left(1 - \xi_+^*(z-c_0)^{-1}\xi_- h\right)$$

$$= 1 - S_0 h$$

Aim to link LC circuits & partial unitaries. Consider a rank 1 partial unitary ($V_\pm$ are dim 1) with Y fin dim. Recall the Schur-Szegö recursion formulas

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \begin{array}{l} |h_n| < 1 \\ k_n = \sqrt{1-|h_n|^2} \end{array}$$

Assume h_n real and note

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+h & 1+h \\ 1-h & h-1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1+h & 0 \\ 0 & 1-h \end{pmatrix}$$

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} z & 1 \\ z & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{z+1}{2} & \frac{z-1}{2} \\ \frac{z-1}{2} & \frac{z+1}{2} \end{pmatrix}$$

This conjugation transforms

$$\frac{1}{k}\begin{pmatrix} 1 & h \\ h & 1 \end{pmatrix} \rightsquigarrow \frac{1}{k}\begin{pmatrix} 1+h & 0 \\ 0 & 1-h \end{pmatrix}, \qquad \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{z^{1/2}+z^{-1/2}}{2} & \frac{z^{1/2}-z^{-1/2}}{2} \\ \frac{z^{1/2}-z^{-1/2}}{2} & \frac{z^{1/2}+z^{-1/2}}{2} \end{pmatrix}$$

Thus this conjugation transforms a real Schur - Szegö recursion to one involving

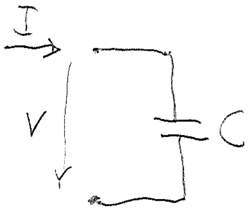
$\begin{pmatrix} \frac{1+h}{k} & 0 \\ 0 & \frac{1-h}{k} \end{pmatrix}$ real positive diagonal and $\begin{pmatrix} \cos(\theta/2) & \iota\sin(\theta/2) \\ \iota\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$ where $z = e^{i\theta}$

The latter is $\begin{pmatrix} 1 & \iota\tan(\theta/2) \\ \iota\tan(\theta/2) & 1 \end{pmatrix}$ up to a scalar

Put $s = \iota\tan(\theta/2) = \dfrac{z^{1/2} - z^{-1/2}}{z^{1/2} + z^{1/2}} = \dfrac{z-1}{z+1}$. Then

$z = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}(s) = \dfrac{1+s}{1-s}$, so we have the C.T. between $s \in iR$ and $z \in \mathbb{T}$.

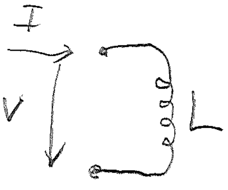
July 20, 02



charge of capacitor $Q = CV \implies I = C\dot{V}$

power going into Capac. $VI = CV\dot{V} = \partial_t\left(\tfrac{1}{2}CV^2\right)$

Electrical Energy stored in C at time t $\displaystyle\int_{-\infty}^{t} VI\,dt = \left[\tfrac{1}{2}CV^2\right]_{-\infty}^{t} = \tfrac{1}{2}CV(t)^2$



Faraday Induction: $V = L\dot{I}$

power going into L $VI = LI\dot{I} = \partial_t\left(\tfrac{1}{2}LI^2\right)$

Magnetic Energy stored in L at time t $\displaystyle\int_{-\infty}^{t} VI\,dt = \left[\tfrac{1}{2}LI^2\right]_{-\infty}^{t} = \tfrac{1}{2}LI(t)^2$

<u>Aug 9, 02</u> Begin with the algebraic picture of GNS. Consider a unital algebra A, a unital (left) A-module M, and a $\mathbb{C}$-linear retract

$$N \xrightarrow{i} M \xrightarrow{j} N, \quad ji = 1_N$$

of M. Define

$$\varphi : A \longrightarrow \mathcal{L}(N), \quad \varphi(a) = jai.$$

φ is linear and satisfies $\varphi(1) = 1$. Our aim is to reconstruct as far as possible the A-module M from (N, φ). We introduce A-module maps

$$\begin{tikzcd} A \otimes N \arrow[r, "\tilde{\imath}"] \arrow[rr, bend right, "\tilde{\varphi}"'] & M \arrow[r, "\tilde{\jmath}"] & \mathrm{Hom}(A, N) \end{tikzcd}$$

$$\tilde{\imath}(a' \otimes n) = a' i n, \quad \tilde{\jmath}(m) = \{a \mapsto jam\}$$

$$\tilde{\varphi}(a' \otimes n) = \{a \mapsto \varphi(aa')n\}$$

$\tilde{\imath}$ is the unique A-module map extending i

$\tilde{\jmath}$ ______________________ coextending j in the sense that $\tilde{\jmath}m$ evaluated at $a = 1$ is jm. Here $a \in A$ acts on $\mathrm{Hom}(A, N)$ by right mult on A.

Thus one has an A-module factorization: $\tilde{\varphi} = \tilde{\jmath}\tilde{\imath}$ of $\tilde{\varphi}$ through M. Conversely given any such factorization $\tilde{\varphi} = \beta\alpha$

$$\begin{tikzcd} A \otimes N \arrow[r, "\alpha"] & M' \arrow[r, "\beta"] \arrow[dr, "j'"] & \mathrm{Hom}(A, N) \arrow[d, "\mathrm{eval}_1"] \\ N \arrow[u, "1 \otimes -"] \arrow[ur, "i'"'] & & N \end{tikzcd}$$

one has $\alpha = \tilde{\imath}'$, $\beta = \tilde{\jmath}'$ and $j' i' n = \mathrm{eval}_1\, \beta\alpha(1 \otimes n) = \mathrm{eval}_1\, \tilde{\varphi}(1 \otimes n) = \varphi(1) n = n$.

There's a smallest factorization, namely when M is the image of $\tilde{\varphi}$.

Example: $A = C(S)$, S finite, an A-module is a vector space V with splitting indexed by S, so a linear retract has the form

$$W \xleftarrow{\beta = (\cdots \beta_s \cdots)} \bigoplus_s V_s \xleftarrow{\alpha = \begin{pmatrix} \vdots \\ \alpha_s \\ \vdots \end{pmatrix}} W \qquad \beta\alpha = \sum_s \beta_s \alpha_s = 1_W$$

$\varphi(a) = \beta a \alpha$ amounts to family $h_s = \beta e_s \alpha = \beta_s \alpha_s$ in $\mathcal{L}(W)$, which is a partition of 1. V_s can be shrunk to $\alpha_s W$ and then to $\alpha_s W / \mathrm{Ker}\, \beta_s$ without affecting h_s. Thus the minimal case is when $V_s = \mathrm{Img}(h_s)$, $\forall s$.

Hilbert picture of GNS. A is C^*-algebra, the A-module is a $*$-repn of A on a Hilbert space V, and the retract has the form $W \xleftarrow{\iota^*} V \xleftarrow{\iota} W$, whence ι is an isometry as $\iota^* \iota = 1_W$. In this situation $\varphi : A \longrightarrow \mathcal{L}(W)$ is a completely positive function yielding a positive hermitian inner product on $A \otimes W$, and V is the corresponding completion.

φ completely positive should mean that the hermitian form on $A \otimes W$ defined for any $\xi = \sum a_i \otimes w_i$ by

$$\langle \xi | \xi \rangle = \sum_{ij} \langle w_i | \varphi(a_i^* a_j) w_j \rangle \quad \text{is } \geq 0.$$

In other words, $\forall$ finite sequence (a_i) in A the block matrix $\varphi(a_i^* a_j)$ with blocks in $\mathcal{L}(W)$ is ≥ 0.

In the case $A = C(S)$ above, since $A \otimes W = \bigoplus_s e_s \otimes W$, it suffices to consider the sequence (e_s) in A. Then

$$\varphi(e_s^* e_t) = \begin{cases} 0 & t \neq s \\ \varphi(e_s) = h_s & t = s \end{cases} \quad \text{so for } \xi = \sum_s e_s \otimes w_s \in A \otimes W$$

one has $$\langle \xi | \xi \rangle = \sum_s \langle w_s | h_s w_s \rangle = \sum_s \| h_s^{1/2} w_s \|^2.$$

Thus V is the completion of $\bigoplus_s W$ wrt the inner prod. $\|(w_s)\|^2 = \sum_s \|h_s^{1/2} w_s\|^2$. So $V = \bigoplus_s V_s$ where V_s = completion of W wrt the inner product $\|h_s^{1/2} w\|^2$.

Aug 11, 02 example: Discuss $A = \mathbb{C}(\{\pm 1\})$, which is the Grassmannian of a polarized Hilbert space.

$$(*) \qquad W \xleftarrow{(\alpha_+^* \ \alpha_-^*)} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ \\ \alpha_- \end{pmatrix}} W \qquad h_\pm = \alpha_\pm^* \alpha_\pm$$

Then $h_+ + h_- = 1_W$, because α is assumed an isometry. Also the minimal choice for V is $V_\pm = h_\pm^{1/2} W$ with $\alpha_\pm^* = \alpha_\pm = h_\pm^{1/2}$.

Aug 12, 02: The above "Grassmannian situation" depends only on W and the hermitian operator h_+, so one gets a spectral decomposition of $(*)$ from the spectral decomp of h_+. Suppose h_+ = the scalar operator λ on W, so that $0 \le \lambda \le 1$ and $h_- = 1 - \lambda$. First consider the generic case $0 < \lambda < 1$. Then $V_\pm = W$ (with scaling of inner product) and $(*)$ is

$$W \xleftarrow{(\lambda^{1/2} \ (1-\lambda)^{1/2})} \begin{pmatrix} W \\ W \end{pmatrix} \xleftarrow{\begin{pmatrix} \lambda^{1/2} \\ (1-\lambda)^{1/2} \end{pmatrix}} W$$

Of interest for LC circuits is the hermitian (s real) form

$$\alpha^*(s e_+ + s^{-1} e_-) = s h_+ + s^{-1} h_- = s\lambda + s^{-1}(1-\lambda)$$

Written in terms of $\omega = h_-^{1/2} h_+^{-1/2}$ (the eigenvalue of "T") one has $\omega^2 = \frac{1-\lambda}{\lambda}$, $\lambda = \frac{1}{1+\omega^2}$

$$\boxed{s\lambda + s^{-1}(1-\lambda) = \frac{s + s^{-1}\omega^2}{1+\omega^2}}$$

The other scalar cases are $\lambda = 1$ $(\omega = 0)$ where $\alpha_+ : V_+ \longrightarrow W$ is an isometry + $V_- = 0$ and $\lambda = 0$ $(\omega = \infty)$ where α_- is an isometry and $V_+ = 0$.

Next discuss the graph viewpoint. Let W be a closed subspace of $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ as above, and consider the map $\alpha_+ = pr_1 : W \longrightarrow V_+$. $\operatorname{Ker} \alpha_+ = W \cap V_-$; if this is $\neq 0$, then W as a correspondence from V_+ to V_- is many-valued, so W is not the graph of a transf. One can shrink V_- to the othogonal complement of $W \cap V_-$, and reduce to the case $W \cap V_- = 0$.

Let $D = \operatorname{Img}\{\alpha_+ : W \longrightarrow V_+\}$. $\forall x \in D$ $\exists y \in V_-$ s.t. $\begin{pmatrix} x \\ y \end{pmatrix} \in W$. Moreover y is unique, so that $W = \begin{pmatrix} 1 \\ T \end{pmatrix} D$ is the graph of a linear transformation T with domain D and values in V_-. Now shrink V_+ to the closure $\bar{D}$, and reduce to the case where T is a densely defined linear transf. from V_+ to V_- whose graph W is closed.

Next bring in von Neumann's theory, which says that $W^\perp = \begin{pmatrix} -T^* \\ 1 \end{pmatrix} V_-$, where T^* is the adjoint of T, which is again a closed densely-defined operator, that also $\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}$ is invertible, leading to the C.T. etc. --

Philosophy You get involved with unbdd ops because of the frequency variable s dual to time t is unbounded ($s = \partial_t$ under the L.T.)

Problem: What structure does the family of factorizations of a map have? For example, if the map is 1_X, then factorizations are equivalent to embeddings as retract of other objects.

Aug 13, 02

Glueing two Hilb spaces via a contraction $c: X \longrightarrow Y$. You want a Hilb space Z,

$$\text{isometries}\quad \begin{tikzcd} X \arrow[dr, "a"] & \\ & Z \\ Y \arrow[ur, "b"'] & \end{tikzcd} \qquad \begin{matrix} a^*a = 1_X \\ b^*b = 1_Y \end{matrix} \qquad \text{such that } c = b^*a \text{ and } Z = \overline{aX + bY}.$$

$$\|ax + by\|^2 = (x^* \; y^*)\begin{pmatrix} a^* \\ b^* \end{pmatrix}(a \;\; b)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}^* \begin{pmatrix} 1 & c^* \\ c & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \|x + c^*y\|^2 + \|y\|^2 - \|c^*y\|^2 \geqslant 0.$$

There is an orthogonal splitting

$$Z = aX \oplus \operatorname{Ker}(a^*) \qquad ax + by = a(x + c^*y) + (1 - aa^*)by$$

$\|ax + by\|^2 = \|x + c^*y\|^2 + \|(1 - aa^*)by\|^2$, where $\|(1 - aa^*)by\|^2 = (y \mid b^*(1 - aa^*)by) = (y \mid (1 - cc^*)y) = \|(1 - cc^*)^{1/2}y\|^2$.

Thus $Z = aX \oplus \overline{(1 - cc^*)^{1/2}X}$ and similarly $Z = bX \oplus \overline{(1 - c^*c)^{1/2}Y}$.

Consider a retract of a free A-module $A \otimes V$, $A = C(\{\pm 1\})$.

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \qquad \beta\alpha = 1_W$$

Since $\beta_\pm \alpha_\pm = 1_{W_\pm}$ one has two projections $p_\pm = \alpha_\pm \beta_\pm$ on V. Using the obvious identification between the two copies of V one get an odd operator on W:

$$C = \begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix} = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix}$$

Let's now shrink V to obtain the reduced linear space equipped with two projections which is Morita equivalent to W equipped with the odd operator C. Shrinking V to $\alpha_+W_+ + \alpha_-W_-$ we can assume $V \xleftarrow{(\alpha_+ \;\; \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$

is surjective. Then passing to the quotient of V by $\mathrm{Ker}(\beta_+) \cap \mathrm{Ker}(\beta_-)$ one can assume $\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V$ is injective.

Then V can be identified with the image of

$$\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix} (\alpha_+ \ \alpha_-) = \begin{pmatrix} 1_+ & \beta_+\alpha_- \\ \beta_-\alpha_+ & 1_- \end{pmatrix} = 1 + C$$

Let's introduce dilation language. In the Gross example the embedding

$$W \xleftarrow{\beta} \begin{pmatrix} V_+ \\ V_- \end{pmatrix} \xleftarrow{\alpha} W$$

of W as a retract of $\begin{pmatrix} V_+ \\ V_- \end{pmatrix}$ can be viewed as a dilation of partition $h_+ + h_- = 1_W$ to a partition $e_+ + e_- = 1_V$ which is disjoint in the sense that $e_+e_- = e_-e_+ = 0$. Any dilation is equivalent to a factorization of $h_\pm$: $W \xleftarrow{\beta_\pm} V_\pm \xleftarrow{\alpha_\pm} W$.

Consider next a retract of a free $A = C(\{\pm 1\})$ module

$$\begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ & 0 \\ 0 & \beta_- \end{pmatrix}} \begin{pmatrix} V \\ V \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+ & 0 \\ 0 & \alpha_- \end{pmatrix}} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

We can view this as a dilation of the odd operator

$$C = \beta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \alpha = \begin{pmatrix} 0 & \beta_+\alpha_- \\ \beta_-\alpha_+ & 0 \end{pmatrix} \quad \text{on } W$$

to the operator $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $\begin{pmatrix} V \\ V \end{pmatrix}$. Note the factorization

$$1_W + C : \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \beta_+ \\ \beta_- \end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

Conversely given $\begin{pmatrix} W_+ \\ W_- \end{pmatrix}$ equipped with an odd operator C we get a equivalence between factorizations of $1_W + C$ through the vector space V on one hand, and embeddings of W as a retract of $A \otimes V$. The smallest V is the image of $1_W + C$

August 14, 02. Consider next the Hilbert space versions of the above where the vector spaces are Hilbert spaces and $\beta = \alpha^*$, equivalently α is an isometry $(\alpha^* \alpha = 1)$.

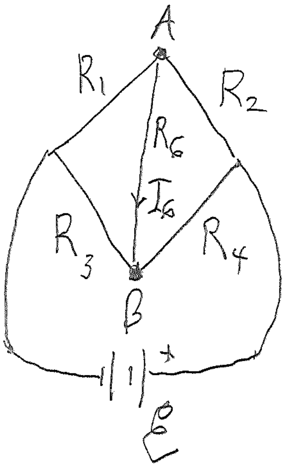
In the Grass case the partition $h_+ + h_- = 1$ consists of hermitian operators ≥ 0. If $W \xleftarrow{\alpha_+^*} V_+ \xleftarrow{\alpha_+} W$ is a minimal factorization of h_+, then it is canon isom. to $V_+ = h_+^{1/2} W$, $\alpha_+ = \alpha_+^* = h_+^{1/2}$; similarly for $-$.

In the other case one has the factorization

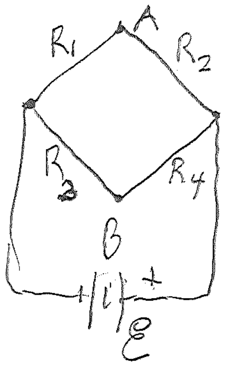
$$\begin{pmatrix} 1 & c^* \\ c & 1_- \end{pmatrix} = 1_W + C : \begin{pmatrix} W_+ \\ W_- \end{pmatrix} \xleftarrow{\begin{pmatrix} \alpha_+^* \\ \alpha_-^* \end{pmatrix}} V \xleftarrow{(\alpha_+ \ \alpha_-)} \begin{pmatrix} W_+ \\ W_- \end{pmatrix}$$

where $c = \alpha_-^* \alpha_+$. This means that the Hilbert space V contains W_+ and W_- (isometrically) glued via the contraction c. So the smallest V is this glueing.

Aug 19, 02. Example of Thevenin's thm. from Peck's Electricity + Magnetism. Find I_6 in the Wheatstone bridge: p187-188



Remove the edge AB and find the equivalent emf $\mathcal{E}_0$ in series with the internal resistance R_0

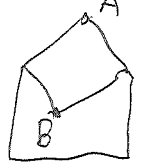


$\mathcal{E}_0 = \underbrace{\text{pot}_A}\ - \text{pot}_B$

$R_1 \underbrace{\dfrac{\mathcal{E}}{R_1+R_2}}_{\text{current thru A.}}$

$$\therefore\ \mathcal{E}_0 = R_1\frac{\mathcal{E}}{R_1+R_2} - R_3\frac{\mathcal{E}}{R_3+R_4}.$$

R_0 = resistance between A and B with $\mathcal{E}=0$.



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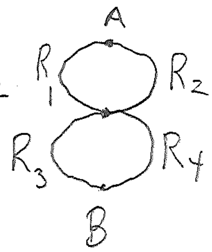
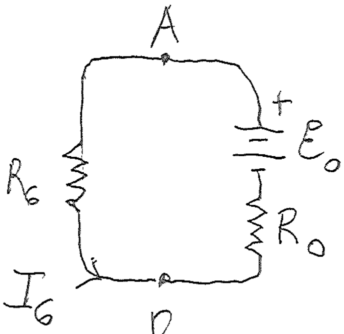


$$R_0 = \frac{R_1R_2}{R_1+R_2} + \frac{R_3R_4}{R_3+R_4}$$

Then

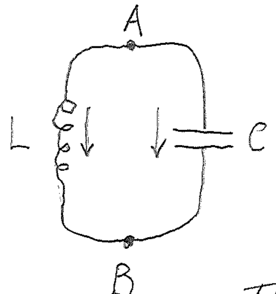


$$\mathcal{E}_0 = I_6(R_6+R_0)$$

$$I_6 = \frac{\dfrac{R_1}{R_1+R_2} - \dfrac{R_3}{R_3+R_4}}{\dfrac{R_1R_2}{R_1+R_2} + \dfrac{R_3R_4}{R_3+R_4} + R_6}\,\mathcal{E}$$

The bridge balances ($I_6 = 0$) when $R_1(\cancel{R_3}+R_4) - (\cancel{R_1}+R_2)R_3 = 0$, i.e. $R_1R_4 = R_2R_3$.

Let's now discuss an LC network which is closed, i.e. there are no applied voltage or current sources. Consider the example:



This graph has two edges denoted σ_L, σ_C. The arrows indicate the orientation chosen for these edges. σ_L and σ_C form a basis for the space C_1 of $\mathbb{C}$-valued 1-chains. Using the orientations the boundary operator ∂ is defined by $\partial\sigma_L = \partial\sigma_C = [A]-[B]$, where E.E. conventions are used (normally the potential at A is higher than at B). Writing a 1-chain in the form $I_L\sigma_L + I_C\sigma_C$, where I_L, I_C are numbers, we have

$$\partial\left(I_L\sigma_L + I_C\sigma_C\right) = \left(I_L + I_C\right)\left([A]-[B]\right).$$

Note that $[A]-[B]$ is a generator for the space $\bar{C}_0$ of 0-chains having augmentation $=0$, so that the subspace H_1 of closed 1-currents, i.e. 1-cocycles, is given by the condition $I_L + I_C = 0$.

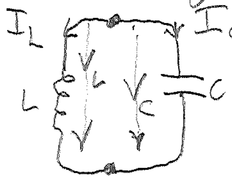
Consider next the space C^1 of 1-cochains, that is, functions $V = (V_L, V_C)$ on the edges σ_L, σ_C. Such a V yields potential drops for those oriented edges. Let's examine when such a V is conservative, i.e. is $\delta\phi$ where $\phi \in \bar{C}^0$ is a function on the nodes A, B modulo constant functions. Clearly one has $V_L = \phi(A) - \phi(B) = V_C$. Another way to see this is to use that V is conservative iff the work done in going around a closed path is zero. So you want V applied to the loop current $\sigma_L - \sigma_C$ (which generates H_1) to be zero. $V(\sigma_L - \sigma_C) = V_L - V_C = 0$.

So we have found the Kirchhoff 1st + 2nd laws namely $I_L + I_C = 0$ and $V_L = V_C$. Now allow the variables I_L, I_C, V_L, V_C to be functions of time and introduce the D.E.'s governings L's and C's:

$$V_L = Ls\, I_L\,, \quad I_C = Cs\, V_C$$

Then $V_L = LsI_L = -LsI_C = -LCs^2 V_C = -LCs^2 V_L$
or $(1+LCs^2)V_L = 0$.

<u>Aug 25, 02.</u> Example of motion of a closed LC network:



4 Variables V_L, I_L, V_C, I_C

$\dot{I}_L = L^{-1}V_L$, $\dot{V}_C = C^{-1}I_C$ (dynamics)

$I_L + I_C = 0$, $V_L = V_C$ (Kirchhoff 1, 2)

So you have 4 linear equations in 4 unknowns - is this system well-posed? Write down the relations in the L.T. picture

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ -L^{-1} & s & 0 & 0 \\ 0 & 0 & s & -C^{-1} \end{pmatrix} \begin{pmatrix} V_L \\ I_L \\ V_C \\ I_C \end{pmatrix} = 0$$

$$\det = \begin{vmatrix} 1 & 0 & 1 \\ s & 0 & 0 \\ 0 & s & -C^{-1} \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 & 1 \\ -L^{-1} & s & 0 \\ 0 & 0 & -C^{-1} \end{vmatrix} = (-s)\begin{vmatrix} 0 & 1 \\ s & -C^{-1} \end{vmatrix} + C^{-1}\begin{vmatrix} 0 & 1 \\ -L^{-1} & s \end{vmatrix}$$
$$= s^2 + (CL)^{-1}$$

Next look at the case of a general connected closed LC network. You have two variables: (V_σ, I_σ) for each of the e edges of the network, $\therefore$ $2e$ unknowns. Kirchhoff 1, 2 namely $\partial I = 0$, $\bar{C}^0 = \mathrm{Ker}(C^1 \to H^1)$ amount to $(v-1)+\ell = e$ conditions. You have one dynamical equation for each edge:

$$\dot{I}_\sigma = L_\sigma^{-1} V_\sigma \qquad \sigma \text{ of } L \text{ type}$$
$$\dot{V}_\sigma = C_\sigma^{-1} I_\sigma \qquad \sigma \text{ of } C \text{ type}$$

Thus you have $2e$ unknowns in $2e$ equations, and you get a determinant as in the example above.

The good case should be when the determinant is a polynomial of degree e in s.

-11-

August 27, 02. Here's a generalization of the LC oscillator case studied above. Consider an LC network with e edges, and let

$$\mathcal{S} = \begin{pmatrix} \overline{C}^0 \\ H_1 \end{pmatrix} \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix}$$ be its state space

$\mathcal{S}$ has dimension e. You want to construct a time evolution on $\mathcal{S}$ using the DE's associated to the e edges:

$$* \quad \begin{cases} L_\sigma \dot{I}_\sigma = V_\sigma & L\text{-type } \sigma \\ C_\sigma \dot{V}_\sigma = I_\sigma & C\text{-type } \sigma \end{cases}$$

Introduce the splittings:

$$C^1 = C^1_C \oplus C^1_L$$
$$C_1 = C_{1C} \oplus C_{1L}$$

where C^1_C is the space of voltages drops across the capacitance edges, and C_{1C} is the space of currents passing thru the capacitance edges; similarly for L.

Assume that the linear functions (or variables) I_σ for L-type σ together with V_σ for C-type σ form a coordinate system on $\mathcal{S}$. Put another way, the composition

$$(+) \qquad \mathcal{S} = \begin{pmatrix} \overline{C}^0 \\ H_1 \end{pmatrix} \subset \begin{pmatrix} C^1 \\ C_1 \end{pmatrix} = \begin{pmatrix} C^1_L \\ C_{1L} \end{pmatrix} \oplus \begin{pmatrix} C^1_C \\ C_{1C} \end{pmatrix} \twoheadrightarrow \begin{pmatrix} C^1_C \\ C_{1L} \end{pmatrix}$$

is an isomorphism.

Let us call dominant the variables I_σ L-type, V_σ C-type. We are assuming the dominant variables form a coord system on $\mathcal{S}$. Since V_σ L-type, I_σ C-type are linear functions of the dominant variables, $*$ above is a first order linear DE on the state space $\mathcal{S}$.

One has $\overline{C}^0 \xrightarrow{\sim} C^1_C$ and $H_1 \xrightarrow{\sim} C_{1L}$ from $(+)$.