Finite-duration Impulse Response

Sampling theorem. Signal $x_a(t)$ all occurring frequencies $\in [-B,B]$

F_s sampling rate $> 2B$, then

$$x_a(t) = \sum x_a\left(\frac{n}{F_s}\right) \frac{\sin 2\pi B(t - n/F_s)}{2\pi B(t - n/F_s)}$$

apparently $F_s = 2B$ is allowed, called Nyquist rate

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a\left(\frac{n}{2B}\right) \frac{\sin(2\pi Bt - \pi n)}{2\pi Bt - \pi n}$$

take $B = \frac{1}{2}$

$$x_a(t) = \sum_{n=-\infty}^{\infty} x_a(n) \frac{\sin \pi(t-n)}{\pi(t-n)}$$

provided $x_a(t)$ has only frequencies $\in [-\frac{1}{2}, \frac{1}{2}]$.

Obvious thing to try is $x_a(t) = e^{i\omega t}$ with $|\omega| < \frac{1}{2}$, possibly smoothed a bit, i.e. $\int_{-\frac{1}{2}}^{1/2} d\omega\, e^{i\omega t}$

$$e^{i\omega t} \stackrel{?}{=} \sum_{n=-\infty}^{\infty} e^{i\omega n} \frac{\sin \pi(t-n)}{\pi(t-n)}$$

Go back to Poisson summation. $f(t)$

$f \in \mathcal{S}(\mathbb{R})$ translation $(Tf)(t) = f(t-1)$

mult. $f(t) \xmapsto{U} e^{2\pi i t} f(t)$

$$f(t) \longmapsto F(t,\omega) = \sum e^{2\pi i \omega n} f(t-n)$$

Special case

$$e^{i\omega t} \overset{?}{=} \sum_{n\in\mathbb{Z}} e^{i\omega n} \frac{\sin \pi(t-n)}{\pi(t-n)} \qquad -\tfrac{1}{2} < \omega < \tfrac{1}{2}$$

To prove in the form

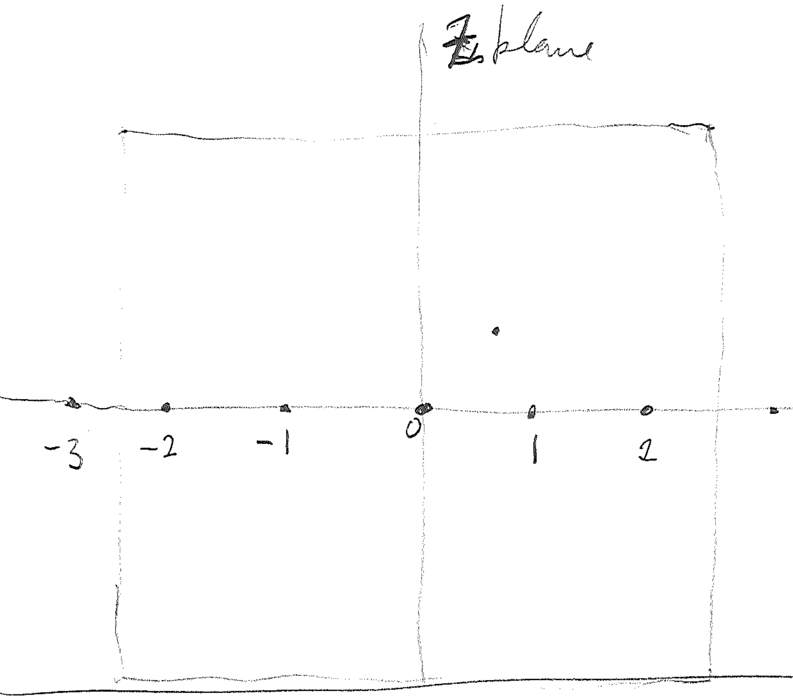
$$\frac{e^{st}}{\sin(\pi t)} \overset{?}{=} \sum_{n\in\mathbb{Z}} \frac{1}{t-n} \frac{e^{sn}}{\pi\cos(\pi n)} = \sum_{n\in\mathbb{Z}} \frac{1}{t-n} \frac{(e^s)^n(-1)^n}{\pi}$$

$$f(z) = \frac{2i\, e^{sz}}{e^{i\pi z} - e^{-i\pi z}}$$

simple poles at $t = n \in \mathbb{Z}$

residue $\dfrac{2i\, e^{sn}}{i\pi e^{i\pi n} + i\pi e^{-i\pi n}} = \dfrac{2i e^{sn}}{2i\pi (-1)^n}$

$$= \frac{1}{\pi} e^{sn}(-1)^n$$



$s = i\omega, \quad -\pi < \omega < \pi$

$$\frac{e^{sz}}{e^{i\pi z} - e^{-i\pi z}}$$

if $z \in$ UHP $\quad \dfrac{e^{-\omega y}}{e^{-\pi y} + \text{small}}$

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try again

$$\sum_{\text{res}} \frac{e^{sz}}{\sin(z)} \frac{1}{z-t}\, dz = \frac{e^{st}}{\sin t} + \sum_n \frac{e^{s\pi n}}{\cos(\pi n)} \frac{1}{\pi n - t}$$

if this is zero, then $\dfrac{e^{st}}{\sin t} = \sum_n e^{sn} \dfrac{(-1)^n}{t - n\pi}$

need $\displaystyle\int \frac{e^{sz}}{\sin z} \frac{1}{z-t}\, dz$ to go to zero.

no problem with $=$ horizontal sides

The problem seems to be to represent $e^{i\omega t}$ in terms of ?

$$f(t) = e^{i\omega t} \qquad \omega \text{ small real}$$

you want to reconstruct $f(t)$ from the sequence $f(n) = e^{i\omega n}$

$$e^{i\omega t} = \sum_{n\in\mathbb{Z}} e^{i\omega n}\, g(\omega, t-n)$$

Different viewpoint: Look at interpolation with polys.

$a_1, \dots, a_n$ dist.

$$f(x) = \sum_{i=1}^{n} \frac{\prod_{j\neq i}(x-a_j)}{\prod_{j\neq i}(a_i-a_j)} f(a_i)$$

residue

$$\frac{f(x)}{\prod_j (x-a_j)} = \sum_i \frac{f(a_i)}{\prod_{j\neq i}(a_i-a_j)} \frac{1}{x-a_i}$$

Next try your

$$\cancel{\frac{\pi f(x)}{\sin(\pi x)} = \sum_{n\in\mathbb{Z}} \frac{f(n)}{x-n}} \qquad \frac{d}{dx}\frac{\sin(\pi x)}{\pi} = \cos\pi x$$

$$\cancel{\frac{f(t)}{\frac{\sin(\pi x)}{\pi}}} \qquad f(t)\frac{\pi\cos(\pi t)}{\sin(\pi t)}$$

$$\frac{\pi\cos(\pi t)}{\sin(\pi t)} = \frac{d}{dt}\log(\sin\pi t)$$

$$f(t)\frac{\pi\cos(\pi t)}{\sin(\pi t)} = \sum_{n\in\mathbb{Z}} \frac{f(n)}{t-n}$$

$$\sin(\pi(t-n))$$
$$= \sin(\pi t)\cos(-\pi n) = (-1)^n \sin(\pi t)$$
$$\cancel{+\cos(\pi t)\sin(-\pi n)}$$

Interpolation formula

Start with Poisson, given $f(x) \in \mathcal{S}(\mathbb{R})$ form

~~$F(x,y) = \sum$~~ periodic fn. $\sum_{n\in\mathbb{Z}} f(x+n)$

more weight by a character $n \mapsto e^{2\pi i n y}$

$$F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i n y} f(x+n) \qquad F(x,y+1) = F(x,y)$$

$$F(x+1,y) = \sum_n e^{2\pi i(n-1)y} f(x+n) = e^{-2\pi i y} F(x,y)$$

Sampling. You have

F.T. of  FIR finite duration impulse response

Situation to understand first is a pure frequency signal $f(t) = e^{-i\omega t}$ $-\infty < t < \infty$, ω real, use the sampling $f \mapsto (f(n), n\in\mathbb{Z})$. The question is that yo

Signal $f(t)$, $t\in\mathbb{R}$, finite frequency band with assumption

$$f(t) = \int_{-B}^{+B} e^{i\omega t}\, \hat{f}(\omega)\, \frac{d\omega}{2\pi} \qquad B = \tfrac{1}{2}\ ?$$

so $f(t)$ is an entire function of t with certain growth properties

$$|f(t)| \leq \int_{-B}^{B} \underbrace{e^{Re(i\omega t)}}_{e^{-\omega(Im\,t)}}\, M\, \frac{d\omega}{2\pi}$$

Finite band width $\Longrightarrow$ analytic fn of t. Then ask about $f(t) \longmapsto (f(n))_{n\in\mathbb{Z}}$. so $\sin \pi t$ has simple zeros at $t\in\mathbb{Z}$

$$\frac{f(t)}{\sin(\pi t)} = \sum_n \frac{f(n)}{\pi \cancel{\cos(\pi n)}} \frac{(-1)^n}{t-n}$$

$\sin \pi(t-n)$

$= \sin(\pi t)\cos(\pi n)$

$- \cancel{\cos(\pi t)} \cancel{\sin(\pi n)}$

$= \sin(\pi t)(-1)^n$

$$f(t) = \sum_n \frac{f(n)}{\pi \cancel{\cos(\pi n)}} \frac{\sin(\pi t) \overset{(-1)^n}{\cancel{\cos(\pi n)}}}{t-n}$$

$$\boxed{f(t) = \sum_n f(n) \frac{\sin \pi(t-n)}{\pi(t-n)}}$$

$$\frac{\sin \pi t}{\pi t} \text{ at } t = n \text{ is } \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

Binary Fixed-Point + Floating-Point Reps.

Mult of 2 DFT's and Circular Convolution

Zero padding to get linear convolution

$$\underset{\text{output}}{y} = \underset{\substack{\text{impulse} \\ \text{response}}}{h} * \underset{\text{input}}{x}$$

$x(n)$ supp $\in [0, L-1]$

$h(n)$ —— $[0, M-1]$

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$$

$y(n)$ supp $\in [0, L+M-2]$

a DFT of size $N \geq L+M-1$ is needed to represent $\{y(n)\}$ in the frequency domain

Discrete signals & continuous ones: $x(n)$ vs. $x(t)$
$(x(n),\ n\in\mathbb{Z})$ versus $(x(t),\ t\in\mathbb{R})$. Next stage should be discrete signals of finite duration $(x(n),\ 0\le n<N)$. LTI linear time independent filter:

$$y(t) = \int h(t-t')\,x(t')\,dt'$$

$$Y(\omega) = H(\omega)\,X(\omega) \qquad \text{frequency version}$$

FIR Finite duration Impulse Response

Discrete LTI filter

$$y(n) = \sum_k h(k)\,\underbrace{x(n-k)}_{k\text{ step delay}}$$

Linear convolution, circular convolution

Linear filtering based on DFT.

$$y(n) = \sum_k h(k)\,x(n-k)$$

Assume x has duration L: $\mathrm{Supp}(x(n)) \subset \{0,\dots,L-1\}$

h —— M: — $y(n)$ — $\{0,\dots,M-1\}$.

Then y —— $L+M-1$. Why

$$X(z) = \sum_{n=0}^{L-1} x(n)\,z^n \qquad H(z) = \sum_{k=0}^{M-1} h(k)\,z^k$$

$$Y(z) = H(z)\,X(z) = \sum_{k=0}^{L-1} h(k)z^k \sum_{m=0}^{M-1} x(m)\,z^m$$

$$= \left(\sum_{k=0}^{L-1}\sum_{m=0}^{M-1} h(k)\,x(m)\right) z^{k+m}$$

$$= \sum_n \left(\sum_{\substack{0\le k<L\\ 0\le m<M\\ n=k+m}} h(k)\,x(m)\right) z^n$$

largest n occurring is $L-1+M-1 = (L+M-1)-1$

Link between trapezoidal fmla for numerical int and the Cayley transform $s = c\,\frac{z-1}{z+1}$

$$H(s) = \frac{b}{s+a} \qquad \text{analog}\overset{\text{linear}}{\ } \text{filter IR function}$$

for the DE $\frac{d}{dt}y(t) + a\,y(t) = b\,x(t)$

Use

$$y(t) = \int_{t_0}^{t} y'(t)\,dt + y(t_0)$$

$$y(nT) \doteq \frac{T}{2}\Big[\overbrace{y'(nT)}^{-ay(nT)+bx(nT)} + y'(nT-T)\Big] + y(nT-T)$$

Let $y(n) = y(nT)$, $x(n) = x(nT)$, get difference eqn

$$\left(1+\frac{aT}{2}\right)y(n) - \left(1-\frac{aT}{2}\right)y(n-1) = \frac{bT}{2}\big(x(n)+x(n-1)\big)$$

$$\left(1+\frac{aT}{2}\right)Y(z) - \left(1-\frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}\left(1+z^{-1}\right)X(z)$$

$$\Rightarrow \quad H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$$

Claim the above works ~~fff~~ in the case of a general first order (vector) system: $y' \cancel{+} ay \cancel{=} bx$ where a, b are $\overset{\text{constant}}{\text{matrices}}$. You write as an integral equation, then approx.

$$y(\cancel{t}) = y(t_0) + \int_{t_0}^{t} y'\,dt \sim y(t_0) + (t-t_0)\,\frac{y'(t_0) + y'(t)}{2}$$

then put $y' = -ay + bx$.

DSP Digital Signal Processing.
Book by Proakis + Manolakis? 2 Greeks.

You hope this might yield insight into the relations between discrete + continuous phenomena. Analog is the engineering term for continuous. It's better to say that an analog signal is a function $x(t)$ on the real time line, and a discrete signal is a function $x(n)$ on $\mathbb{Z}$. A basic tool is the Fourier transform $x(t) \longmapsto X(\omega)$, resp. the z transform $x(n) \longmapsto X(z)$ or Fourier-Laurent series.

STATE EQNS

$x(t)$ = state vector

$$\dot{x}(t) = Ax(t) + B\overset{\text{input}}{u(t)}$$
$$\underset{\text{output}}{y(t)} = Cx(t) + Du(t)$$

$$(s-A)X = BU$$
$$Y = CX + DU$$
$$Y = \left(C(s-A)^{-1}B + D\right)U$$

Dec 2, 01. Go over interpolation

polynomial interpolation. given $a_1, \ldots, a_n$ distinct $\in \mathbb{C}$, you want poly $f(t)$ with given $f(a_i)$, there is a unique f of least degree $n-1$:

$$* \qquad f(t) = \sum_{i=1}^{n} f(a_i) \frac{\prod_{j \neq i} (t-a_j)}{\prod_{j\neq i} (a_i - a_j)} \qquad = \begin{cases} 0 & t = a_j \quad j \neq i \\ 1 & t = a_i \end{cases}$$

all partial fractions fmla.

$$\frac{f(t)}{\prod_j (t-a_j)} = \sum_i \frac{1}{t-a_i} \frac{f(a_i)}{\prod_{j \neq i}(a_i - a_j)}$$

In Digital Signal Processing you encounter

Analog to Digital Converter, symbol A/D given by

Sampling: $f(t) \longmapsto (f(n), n \in \mathbb{Z})$ more generally $f(nT)$

Digital to Analog Converter, symbol D/A given by

Interpolation. The analog of $*$ is

$$** \qquad f(t) = \sum_{n \in \mathbb{Z}} f(n) \frac{\sin \pi (t-n)}{\pi (t-n)} \qquad = \begin{cases} 0 & \text{if } t \in \mathbb{Z},\ t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

or

$$\frac{f(t)}{\sin \pi t} = \sum_n \frac{1}{t-n} \frac{f(n)(-1)^n}{\pi}$$

Sampling Thm: If $f(t) = \int e^{i\omega t} \hat{f}(\omega) \frac{d\omega}{2\pi}$ where $\operatorname{Supp} \hat{f}(\omega) \subset (-\frac{1}{2}, \frac{1}{2})$, then the interpolation formula $**$ holds.

The assumption on f should imply that $\frac{f(t)}{\sin \pi t}$ decays sufficiently as $\operatorname{Im}(t) \to \pm\infty$, so that one can use Cauchy's residue formula.

$$\frac{e^{i\omega t}}{\sin(t)} = \frac{1}{2\pi i}\oint_{\substack{\text{small circle}\\ \text{around } t}} \frac{1}{z-t}\,\frac{e^{i\omega z}}{\sin(z)}\,dz$$

if ~~you~~ ~~can~~ push the small circle to infinity and ~~the~~ you have convergence, then you get

$$\frac{e^{i\omega t}}{\sin(t)} = \sum_n \operatorname{Res}\left(\frac{1}{z-t}\,\frac{e^{i\omega z}}{\sin z}\,dz,\quad z=n\pi\right)$$

$$= \sum_n \frac{-1}{n\pi-t}\,\frac{e^{i\omega n\pi}}{\cos n\pi} = \sum_n \frac{1}{t-n\pi}\left(e^{i\omega\pi}\right)^n(-1)^n$$

So there ~~should~~ might be a "symmetric" way to sum this series, which is not abs. convergent. Put $e^{i\theta} = -e^{i\omega\pi}$

$$\frac{e^{in\theta}}{t-n\pi} + \frac{e^{-in\theta}}{t+n\pi} = \frac{(t+n\pi)e^{in\theta} + (t-n\pi)(e^{-in\theta})}{t^2-n^2\pi^2}$$

$$= \underbrace{\frac{n\pi\left(e^{in\theta}-e^{-in\theta}\right)}{t^2-n^2\pi^2}}_{\substack{\text{hopeless unless}\\ \theta=\omega\pi \text{ is real}}} + \underbrace{\frac{t\left(e^{in\theta}+e^{-in\theta}\right)}{t^2-n^2\pi^2}}_{\text{abs. conv.}}$$

How about Weil reciprocity? Look at two rational functions $f(z)$, $g(z)$, whose divisors are disjoint. View f as a finite degree map from the Riemann sphere to itself. Vague idea is that you have a push forward by f of values of g?

Look at the interpolation formula for ideas. Given a "divisor", better a finite set $\{a_1, \dots, a_n\}$ of distinct points, equiv. a poly $\underbrace{\prod_{i=1}^n (z-a_i)}_{p(z)}$, with distinct roots, you have the interpolation formula

$$\boxed{\frac{f(z)}{\prod_j (z-a_j)} = \sum_i \frac{1}{z-a_i} \frac{f(a_i)}{\prod_{j\neq i}(a_i-a_j)}}$$

$$\boxed{f(z) = \sum_i f(a_i) \frac{\prod_{j\neq i}(z-a_j)}{\prod_{j\neq i}(a_i-a_j)}}$$

What goes on here, and how should it generalize to an (ctble) infinite set $\{a_i\}$? You want no accumulation point in $\mathbb{C}$, i.e. $a_i \to +\infty$. You expect similar formula to hold. You should introduce $p(z) = \prod_j (z-a_j)$

then $$\frac{p'(z)}{p(z)} = \sum_j \frac{1}{z-a_j}$$ and it should follow that

$$\frac{p'(a_i)}{\prod_{j\neq i}(a_i-a_j)} = 1. \qquad p'(z) = \sum_j \frac{p(z)}{z-a_j}$$

$$p'(z) = \frac{d}{dz}\Big((z-a_1)\cdots(z-a_n)\Big) = \sum_j (z-a_1)\cdots(z-a_{j-1})(z-a_{j+1})\cdots(z-a_n)$$

$$p'(a_i) = (a_i-a_1)\cdots(a_i-a_{i-1})(a_i-a_{i+1})\cdots(a_i-a_n)$$

$$p(z) = \prod_j (z-a_j) \qquad \cancel{p'(a_i) = \sum_j \frac{p(a_i)}{a_i-a_j}}$$

$$p'(a_i) = \prod_{j\neq i}(a_i-a_j)$$

$$p'(a_i) = \lim_{z\to a_i} \frac{p(z)-\overbrace{p(a_i)}^{0}}{z-a_i} = \lim_{z\to a_i} \prod_{j\neq i}(z-a_j) = \prod_{j\neq i}(a_i-a_j)$$

simpler still $$p(z) = (z-a_i)\prod_{j\neq i}(z-a_j) \qquad p'(a_i) = \prod_{j\neq i}(a_i-a_j)$$

You are trying to understand an interpolation formula that arises from the poly $p(z)$ whose zeroes are the points $a_1, \dots, a_n$. Interpolation seems to be very close to RR.

D divisor, $0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}[+D] \longrightarrow ?$

$e^{2\pi i z} - 1$, $e^{\pi i z} - e^{-\pi i z}$ vanish exactly when $z \in \mathbb{Z}$

$$p(z) = \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad p(z) = 0 \Leftrightarrow z \in \pi\mathbb{Z}$$

$$p'(z) = \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$p(z) = (\text{formally}) \prod_{n \in \mathbb{Z}} (z - \pi n)$$

$$\frac{p'(z)}{p(z)} = \sum_n \frac{1}{z - \pi n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z^2 - \pi^2 n^2}$$

$$p(z) = z \prod_1^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right)$$

$$\boxed{\frac{f(z)}{p(z)} = \sum_i \frac{1}{z - a_i} \frac{f(a_i)}{p'(a_i)}}$$

$$p(z) = e^{2\pi i z} - 1$$

$$p'(z) = 2\pi i e^{2\pi i z}$$

$$\frac{p'(z)}{p(z)} = \frac{2\pi i e^{2\pi i z}}{e^{2\pi i z} - 1}$$

$$\frac{f(z)}{\sin(\pi z)} = \sum_n \frac{1}{z - n} \frac{f(n)}{\pi (-1)^n}$$

$$f(z) = \sum_n f(n) \frac{\sin(\pi(z-n))}{\pi(z-n)}$$

$$f(z) = e^{i\omega z} \qquad \text{could } \omega \in \mathbb{R}\ ?$$

Look at ℓ^2 because
$$\left(n \mapsto \underbrace{\frac{\sin \pi(z-n)}{\pi(z-n)}}_{\frac{\sin(\pi z)}{\pi} \frac{(-1)^n}{z-n}}\right) \in \ell^2.$$

So for each z you get an l^2 sequence

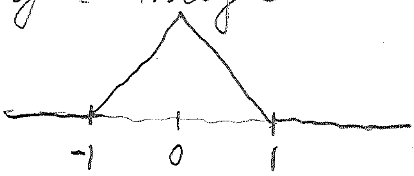
$$n \longmapsto \frac{\sin(\pi z)}{\pi} \frac{(-1)^n}{z-n}$$

Ideas to explore. Interpolation means extending a function $f(n)$ on $\mathbb{Z}$ to a function $f(t)$ on $\mathbb{R}$ or $\mathbb{C}$ in some nice way. You have an explicit way to do this:

$$f(z) = \sum_n f(n) \frac{\sin \pi(z-n)}{\pi(z-n)}$$

at least when this series makes sense, e.g. $f(n)$ has finite support.

First examine $\frac{\sin \pi z}{\pi z}$ which is the interpolation above in the case of $\delta_0(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$. You might look for other functions $\varphi(z)$ satisf e.g.



here z real. $\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$ is an entire analytic fn

Consider the interpolation

$$f(z) = \sum_n f(n)\, \varphi(z-n)$$

require $\varphi(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$

If φ has bdd support then the sum is finite

$$\varphi(z) \neq 0 \implies z \in K$$
$$\varphi(z-n) \neq 0 \implies z - n \in K \implies -n \in -z + K$$

The important case ~~is~~ should be $f(z) = e^{i\omega z}$, because sampling: $f(z) \longmapsto (f(n), n \in \mathbb{Z})$ followed by interpolation is again f only for finite bandwith signals, less than Nyquist rate.

To prove

$$e^{i\omega z} = \sum_n e^{i\omega n} \frac{\overbrace{\sin \pi(z-n)}^{\sin(\pi z)(-1)^n}}{\pi(z-n)}$$

ω real
z

or

$$\frac{e^{i\omega z}}{\sin(\pi z)} = \sum_{n\in\mathbb{Z}} \frac{1}{z-n} \frac{e^{i\omega n}}{\pi(-1)^n}$$

where $|\omega|$ is small enough (guess $< \pi$ or $\frac{\pi}{2}$

Question: Can you calculate the sum when ω is (rational) π ?

growth of
$$\frac{e^{i\omega z}}{\sin(\pi z)} = 2i \frac{e^{i\omega z}}{e^{i\pi z} - e^{-i\pi z}}$$

$$= 2i \frac{e^{i\omega x - \omega y}}{-e^{i\pi x - \pi y} + e^{-i\pi x + \pi y}}$$

$z = x + iy$
$i\pi z = i\pi x - \pi y$

$$\left|\frac{e^{i\omega z}}{e^{i\pi z} - e^{-i\pi z}}\right| \leq \frac{e^{-\omega y}}{-e^{-\pi y} + e^{\pi y}} = \frac{1}{1-e^{2\pi y}} e^{-(\omega+\pi)y}$$

$y > 0$

$$\frac{e^{i\omega z}}{\sin(\pi z)} = \frac{2i\, e^{i\omega z}}{e^{i\pi z} - e^{-i\pi z}} = \frac{2i\, e^{i\omega x - \omega y}}{e^{i\pi x - \pi y} - e^{-i\pi x + \pi y}}$$

$$\left|-e^{i\pi x - \pi y} + e^{-i\pi x + \pi y}\right| = e^{+\pi y}\left| e^{-i\pi x} - e^{i\pi x - 2\pi y}\right|$$

$$\geq e^{\pi y}\left(1 - e^{-2\pi y}\right)$$

since
$\|\alpha - \beta\| \geq \|\alpha\| - \|\beta\|$, because $\|\alpha - \beta\| + \|\beta\| \geq \|\alpha\|$

$$\left|\frac{e^{i\omega z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \frac{e^{-\omega y}|e^{i\pi z}|}{|e^{2i\pi z}-1|}$$

$$= \frac{e^{-\omega y}e^{-\pi y}}{|e^{-2\pi y}-1|} = \frac{e^{-(\omega+\pi)y}}{1-e^{-2\pi y}} \sim e^{-(\omega+\pi)y} \text{ as } y\to+\infty$$

decay if $\omega+\pi>0$

$$\left|\frac{e^{i\omega z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \frac{e^{-\omega y}|e^{-i\pi z}|}{|1-e^{-2i\pi z}|} = \frac{e^{-\omega y}e^{\pi y}}{|1-e^{2\pi y}|}$$

$$\underset{y\to-\infty}{\sim} e^{(\pi-\omega)y} \qquad \text{decay if } \pi-\omega>0$$

$$-\pi<\omega<\pi$$

Repeat this calc. $\left|\frac{e^{i\omega z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \left|\frac{e^{i(\omega+\pi)z}}{e^{2i\pi z}-1}\right|$

$$\le \frac{e^{-(\omega+\pi)y}}{1-e^{-2\pi y}} \underset{\text{as } y\to+\infty}{\sim} e^{-(\omega+\pi)y}$$

need $\omega+\pi>0$ or $-\pi<\omega$

$$\left|\frac{e^{i\omega z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \left|\frac{e^{i(\omega-\pi)z}}{1-e^{-2i\pi z}}\right| \le \frac{e^{-(\omega-\pi)y}}{1-e^{2\pi y}} \underset{y\to-\infty}{\sim} e^{-(\omega-\pi)y}$$

need $\omega-\pi<0$ or $\omega<\pi$

suppose $i\omega$ not purely imag. $\alpha=i\omega=a+bi$

$$\left|\frac{e^{(\overset{a+bi}{\overset{\shortparallel}{\alpha}}+i\pi)(x+iy)}}{e^{2\pi iz}-1}\right| \le \frac{e^{ax-(b+\pi)y}}{1-e^{-2\pi y}}$$

decays as $y\to+\infty$ $\quad b+\pi>0$

$$\left|\frac{e^{\alpha z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \left|\frac{e^{(\alpha+i\pi)z}}{e^{2i\pi z}-1}\right| \leqslant \frac{\left|e^{(a+bi+\pi i)(x+iy)}\right|}{1-e^{-2\pi y}}$$

$$= \frac{e^{ax-(b+\pi)y}}{1-e^{-2\pi y}} \underset{y\to+\infty}{\sim} e^{ax-(b+\pi)y}$$

decays for
$b+\pi>0$
$-\pi<b$

$$\left|\frac{e^{\overset{a+bi}{(\alpha}-i\pi)(x+iy)}}{1-e^{-2\pi i z}}\right| \leqslant \frac{e^{ax-(b-\pi)y}}{1-e^{2\pi y}} \underset{\text{as } y\to-\infty}{\sim} e^{ax-(b-\pi)y}$$

decays for
$b-\pi<0$
$b<\pi$

So $-\pi < \mathrm{Im}(\alpha) < \pi$ implies that $\dfrac{e^{\alpha z}}{e^{i\pi z}-e^{-i\pi z}}$ decays in the vertical directions. $\updownarrow$

You want to show that

$$\frac{e^{\alpha z}}{\sin(\pi z)} = \sum_n \frac{1}{z-n}\,\frac{e^{\alpha n}}{\pi(-1)^n}$$

The right side will not converge if $Re(\alpha)\neq 0$. So you have to assume $\alpha = i\omega$ with ω real.

$$\int_{-1}^{1} e^{i\omega t}\,d\omega = \left[\frac{e^{i\omega t}}{it}\right]_{\omega=-1}^{\omega=1} = \frac{e^{it}-e^{-it}}{it} = 2\,\frac{\sin(t)}{t}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega z}\,d\omega = \left[\frac{e^{i\omega z}}{2\pi i z}\right]_{\omega=-\pi}^{\omega=\pi} = \frac{e^{i\pi z}-e^{-i\pi z}}{2\pi i z}$$

$$= \frac{\sin \pi z}{\pi z}$$

Review: Interpolation for polys.

$$p(z) = \prod_j (z-a_j) \quad \text{deg } n \text{ dist roots} \qquad f \text{ poly of deg} < n$$

$$\frac{f(z)}{p(z)} = \sum_i \frac{1}{z-a_i} \frac{f(a_i)}{p'(a_i)} \qquad \text{partial fraction exp.}$$

$$f(z) = \sum_i f(a_i) \underbrace{\frac{p(z)}{z-a_i} \frac{1}{p'(a_i)}}_{\dfrac{\prod_{j\neq i}(z-a_j)}{\prod_{j\neq i}(a_i-a_j)}}$$

entire case interpolating from $\mathbb{Z}$ to $\mathbb{C}$.

$$\frac{f(z)}{\sin(\pi z)} = \sum_n \frac{1}{z-n} \frac{f(n)}{\pi(-1)^n} + \text{entire}$$

problem with the convergence of this sum. Mittag-Leffler method to get convergence.

You are interested in $f(z) = e^{i\omega z}$ $\quad \omega$ real

$$\left| \frac{e^{i\omega z}}{e^{i\pi z} - e^{-i\pi z}} \right| = \left| \frac{e^{i(\omega-\pi)z}}{1-e^{-2i\pi z}} \right| \underset{y>0}{\leq} \frac{e^{(\omega-\pi)y}}{1-e^{2\pi y}} \underset{y\to+\infty}{\sim} e^{(\omega-\pi)y}$$

decays for $\omega < \pi$

next you want the case $\quad f(z) = \int_{-\pi}^{\pi} \varphi(\omega) e^{i\omega z} \frac{d\omega}{2\pi}$

$$f(n) = \int_{-\pi}^{\pi} \varphi(\omega) e^{i\omega n} \frac{d\omega}{2\pi}$$

$$in\, f(n) = \int_{-\pi}^{\pi} \underbrace{\varphi(\omega) \frac{d}{d\omega} \overbrace{e^{i\omega n}}^{in e^{i\omega n}}}_{\frac{d}{d\omega}\left(\varphi(\omega)e^{i\omega n}\right) - \varphi' e^{i\omega n}} \frac{d\omega}{2\pi} = -\int_{-\pi}^{\pi} \varphi'(\omega) e^{i\omega n} \frac{d\omega}{2\pi}$$

$$\frac{f(z)}{\sin(\pi z)} = \sum_n \frac{1}{z-n} \frac{f(n)}{\pi(-1)^n} \qquad \text{want this to be true}$$

for $f(z) = \frac{1}{2a}\int_{-a}^{a} e^{i\omega z}\, d\omega = \frac{1}{2a}\left[\frac{e^{i\omega z}}{iz}\right]_{-a}^{a}$

$$= \frac{e^{iaz} - e^{-iaz}}{2i\, az} = \frac{\sin(az)}{az}$$

For this $f(z)$ the series $\sum_n \frac{1}{z-n}\frac{f(n)}{\pi(-1)^n}$ is abs. conv.

You want this series to equal $f(z)$.

$$\frac{\sin az}{az} \overset{?}{=} \sum_n \frac{\sin(\pi z)}{z-n} \frac{\sin(an)}{an} \frac{(-1)^n}{\pi}$$

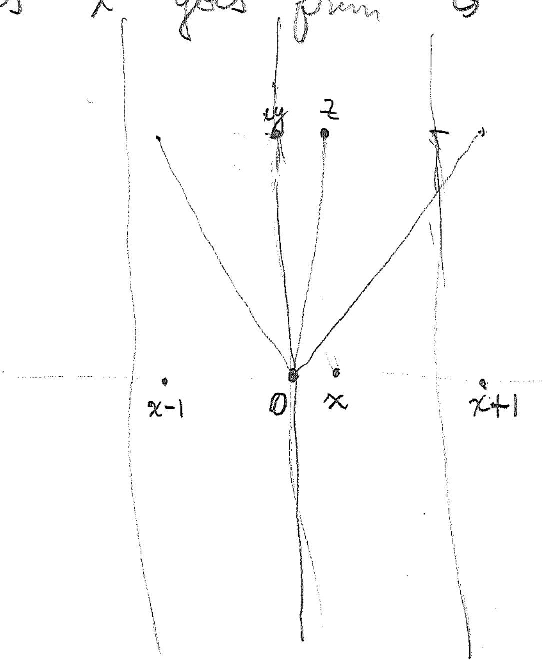
$$\frac{\sin az}{az} \overset{?}{=} \sum_n \frac{\sin(an)}{an} \frac{\sin(\pi(z-n))}{\pi(z-n)} \qquad \text{for } -\pi < a < \pi$$

need to estimate the size of $\sum_{n\in\mathbb{Z}} \frac{1}{z-n}\frac{f(n)}{\pi(-1)^n}$

Cauchy-Schwartz.

$$\sum_n \frac{1}{|z-n|^2} = \sum_n \frac{1}{(x-n)^2+y^2} \qquad \text{periodic in } x$$

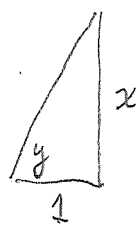
as x goes from 0 to 1. ~~Then~~ Approx sum by



$$\int_{-\infty}^{\infty} \frac{1}{x^2+y^2}\, dx = F(y)$$

$y = \arctan x$

$\tan y = x$



$\sec y = \sqrt{1+x^2}$

$$(\sec^2 y)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{(\sec y)^2} = \frac{1}{1+x^2}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+a^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{a^2x^2+a^2}\,a\,dx = \frac{1}{a}\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$

$$= \frac{1}{a}\arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{a}$$

$$\therefore \sum_n \frac{1}{|z-n|^2} \underset{y\to\infty}{\sim} \frac{1}{y}$$

Cauchy Schwarz should give

$$\left| \sum_n \frac{1}{z-n}\frac{f(n)}{\pi(-1)^n} \right| \le \frac{\text{const}}{\sqrt{|y|}} \cdot \left(\sum_n |f(n)|^2\right)^{1/2}$$

Euler-Maclaurin summ. fmla

$$\sum_n \frac{1}{z-n} = \pi\frac{\cos(\pi z)}{\sin(\pi z)}$$

Go over DSP

$x(n)$ input

$y(n)$ output

$$y(n) = \sum_k h(k)\,x(n-k)$$

unit impulse input: $x(n) = \delta_0(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0. \end{cases}$

response is $y(n) = \sum_k h(k)\,\delta_0(n-k) = h(n).$

FIR Finite Impulse Response h fin. supp.

typical difference eqn.

$$z^{-n} y(n) - \sum_{k\ge 1} \overset{z^{-k}}{a(k)}\, \overset{z^{-n+k}}{y(n-k)} = \sum_{\ell \ge 0} b(\ell)\,x(n-\ell)$$

$\sum_{n\in\mathbb{Z}} z^{-n} y(n)$

$$Y - AY = BX \qquad \Big| \qquad Y = (1-A)^{-1}BX$$

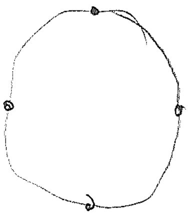
Try to understand aliasing. You are studying DSP, time is discrete, $t = nT$, reminds you of difference equation technique $(D_\omega f)(x) = f(x+\omega) - f(x)$ and related factorials, see Milne-Thompson book.

Need appropriate philosophy, or viewpoint. You want electrical eng. viewpoint for discrete signals.

~~understand~~ Given $h_0 + z^{-1}h_1 + z^{-2}h_2$ polynomial, it gives rise to a mult. op $X \mapsto HX$.

$$h = \sum_{k=0}^{m-1} h_k u^k \quad \text{poly of degree} < m.$$

$h \in F_m$, $\quad x \in F_n \qquad hx \in F_{m+n}$



F_m

$$F_n \xrightarrow{h} F_{n+d}$$

---

linear conv $\qquad y(n) = \sum_{k=0}^{L-1} h(k)\,x(n-k)$

$H(z) = h_0 + h_1 z^{-1}$ $\qquad X(z) = \sum_{n=0}^{N} x_n z^{-n}$

degree 1 $\qquad$ degree N

$Y(z) = H(z)X(z)$ has degree $N+1$. $\qquad$ W

$y_0 = h_0 x_0$ $\qquad h_0 \quad h_1$

$y_1 = h_0 x_1 + h_1 x_0$

$\boxed{L}\boxed{O \qquad N}$

$y_2 = h_0 x_2 + h_1 x_1$

$y_N = h_0 x_N + h_1 x_{N-1}$

$y_{N+1} = \qquad h_1 x_N$

Review: Poly int. $p(z) = \prod_i (z - a_i)$ dist roots.

$$\frac{f(z)}{p(z)} = \sum_i \frac{1}{z-a_i} \frac{f(a_i)}{p'(a_i)} \quad \text{if } \deg f < \deg p$$

$$f(z) = \sum_i \prod_{j \neq i} (z - a_j) \frac{f(a_i)}{p'(a_i)}$$

$$\cancel{\Rightarrow f(a_k) = \prod_{j \neq i} (a_i - a_j) \frac{f(a_i)}{p'(a_i)}} \Rightarrow \quad p'(a_i) = \prod_{j \neq i} (a_i - a_j)$$

simplest is to take $f = p'$

$$p'(z) = \sum_i \prod_{j \neq i} (z - a_j)$$

next entire fns., interpolate from $\mathbb{Z}$ to $\mathbb{C}$

$$p(z) = \sin(\pi z)$$

Yes.

$$\frac{f(z)}{p(z)} \overset{?}{=} \sum_n \frac{1}{z-n} \frac{f(n)}{p'(n)}$$

$$\frac{f(z)}{\sin(\pi z)} \overset{?}{=} \sum_n \frac{1}{z-n} \frac{f(n)}{\pi (-1)^n}$$

strategy: to make the sum converge, so that the difference is entire, then show this difference is bdd.

$n \longmapsto \frac{1}{z-n}$ is an ℓ^2 sequence

$$\sum_n \left| \frac{1}{z-n} \right|^2 = \sum_n \frac{1}{(x-n)^2 + y^2} \text{ "}\sim\text{" } \int_{-\infty}^{\infty} \frac{1}{x^2+y^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{(uy)^2 + y^2} y \, du = \frac{1}{y} \int_{-\infty}^{\infty} \frac{1}{u^2+1} du = \frac{1}{y} \pi$$

Interpolation formula

$$\frac{f(z)}{\sin(\pi z)} = \sum_n \frac{1}{z-n} \frac{f(n)}{\pi(-1)^n}$$

Q: For what $f(z)$ and $(f_n)_{n\in\mathbb{Z}}$ does this hold?

When true? have finite bandwidth condition.

What results might be true?

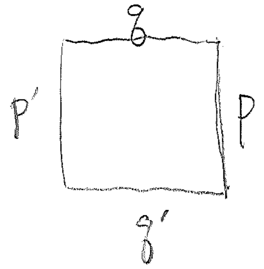
$$f(z)\frac{\pi\cos(\pi z)}{\sin(\pi z)} = \sum_n \frac{f(n)}{z-n}$$

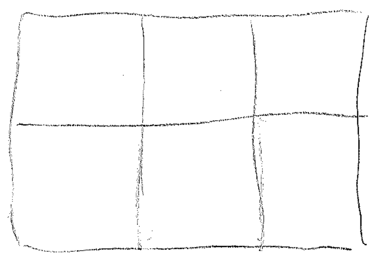
---

review grid spaces

rectangular grid

grid space has 1 generator each edge, 2 relations each sq.





$$\begin{pmatrix} p \\ g \end{pmatrix} = \frac{1}{k}\underbrace{\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}}_{\in SU(1,1)}\begin{pmatrix} p' \\ g' \end{pmatrix}$$

formulas motivated by Szegö

$$k^2 = 1 - |h|^2$$

Let $\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{g} \in SU(1,1)$

$$g^*\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$g^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & +b \\ +c & a \end{pmatrix} \qquad \begin{array}{l} \bar{a} = d \\ \bar{b} = c. \end{array}$$

$$g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}. \qquad |a|^2 - |b|^2 = 1.$$

Szegö: $L^2(S^1, d\mu)$ $\quad \int d\mu = 1.$

$$F_n = \mathbb{C} + \mathbb{C}z + \cdots + \mathbb{C}z^n$$

~~Let V be a 2 dim $\mathbb{C}$ v.s., let p, q, p', q' be linear functionals on V such that~~

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$$

$d \neq 0$

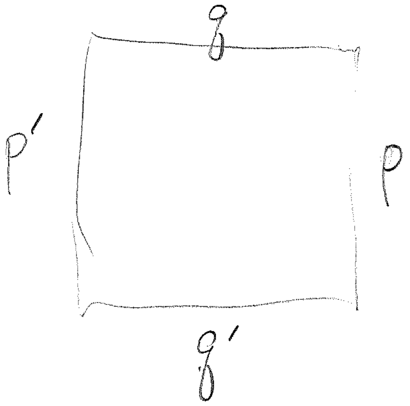
$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}$$

$q = cp' + dq'$

$q' = \frac{1}{d}(q - cp')$

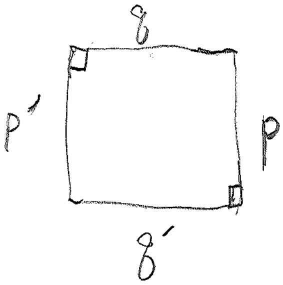
$p = ap' + \frac{b}{d}(q - cp')$

$= \left(a - \frac{bc}{d}\right) p' + \frac{b}{d} q$



You want $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$

$\Longleftrightarrow \frac{1}{d} \begin{pmatrix} \Delta & b \\ -c & 1 \end{pmatrix} \in U(2)$

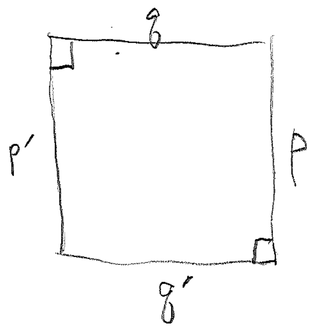


$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}$$

Still confused.

$p = (>0) p' \mod q'$

Go back to



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \underline{a > 0}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} & \\ \gamma & \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix} \qquad \gamma > 0 \ ?$$

$$\begin{array}{ccc} zF_{n-1} & \overset{q_n}{\subset} & F_n \\ zp_{n-1}\ \cup & & \cup\ p_n \\ zF_{n-2} & \underset{q_{n-1}}{\subset} & F_{n-1} \end{array}$$

Look inside $F_n \cap (zF_{n-2})^\perp$ which is 2 dim'l $\quad \| \quad F_n / zF_{n-2}$

Inside F_n / zF_{n-2} are two lines

$$\left|\begin{array}{ll} zF_{n-1}/zF_{n-2} & \text{spanned by } z^n \\ F_{n-1}/zF_{n-2} & \text{———— } 1 \end{array}\right.$$

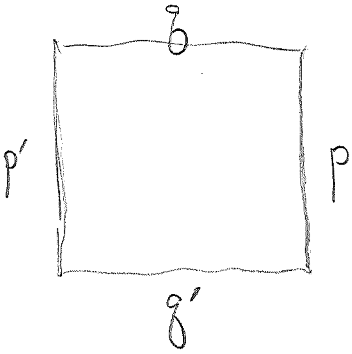
~~You project $\perp$ to zF_{n-2} and normalize to get unit vectors $p_n \in \mathbb{R}_+ z^n + F_{n-1}$~~

Properties $\quad p_n \in (\mathbb{R}_+ z^n + F_{n-1}) \cap F_{n-1}^\perp$, $\|p_n\| = 1$

$q_n \in (\mathbb{R}_+ + zF_{n-1}) \cap (zF_{n-1})^\perp$, $\|q_n\| = 1$.

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix} \qquad a > 0$$

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{\Delta}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix} \qquad q_{n-1} = -\frac{c}{d} zp_{n-1} + \frac{1}{d} q_n$$



$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{array}{l} a > 0 \\ d > 0 \end{array}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\Delta}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}}_{\in U(2)} \begin{pmatrix} p' \\ q \end{pmatrix}$$

$$\frac{\overset{ad-bc}{\Delta} + bc}{d^2} = \frac{a}{d} > 0$$

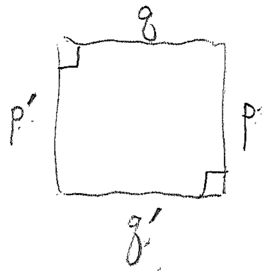
$$\therefore \frac{a}{d} = 1$$

so in $SU(2)$ with

$|h|<1$

$k=\sqrt{1-|h|^2}$

Summarize.



$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar h & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

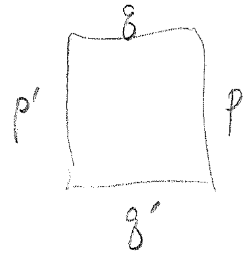
arg. V 2 diml Hilbert space

p, q' and p', q two orth bases

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

---

try again by matrix: V complex v.s. of dim 2 two bases related

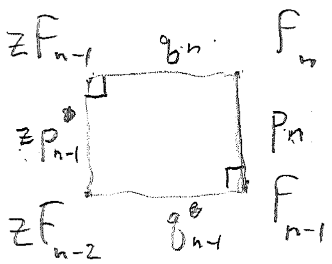


$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix}$$

$$\begin{pmatrix} p \\ q' \end{pmatrix} = \frac{1}{d}\begin{pmatrix} \Delta & b \\ -c & 1 \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

---

do the orth poly situation



$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}}_{\in U(2)}\begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$$

$\alpha > 0$

$\delta > 0$

$\alpha^2 + |\beta|^2 = 1$

$\delta^2 + |\beta|^2 = 1$

$\therefore\ \alpha = \delta$

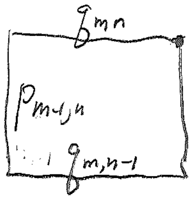
$\alpha\beta + \bar\gamma\delta = 0$ $\quad \gamma = -\bar\beta$

$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar h & k \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$$

$\Delta = k^2 + |h|^2 = 1$

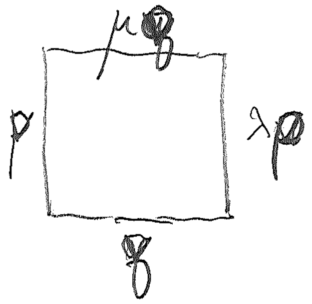
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ -\bar h & 1 \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$$

Constant h grid space has $\mathbb{Z}\oplus\mathbb{Z}$ symmetry.



$$\begin{pmatrix} p_{mn} \\ q_{mn} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p_{m-1,n} \\ q_{m,n-1} \end{pmatrix}$$

difference equations



$$\cancel{\begin{pmatrix} \mu u \\ \lambda v \end{pmatrix}} \qquad \begin{pmatrix} \lambda p \\ \mu q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix}$$

$$k\lambda p = p + hq \qquad (k\lambda - 1)p = hq$$

$$k\mu q = \bar{h}p + q \qquad (k\mu - 1)q = \bar{h}p$$

$$(k\lambda - 1)(k\mu - 1) = |h|^2 = 1 - k^2$$

$$\mu = \frac{1}{k}\left(\frac{1-k^2}{k\lambda - 1} + 1\right) = \frac{1}{k}\left(\frac{\cancel{1} - k^2 + k\lambda \cancel{- 1}}{k\lambda - 1}\right)$$

$$\mu = \frac{\lambda - k}{k\lambda - 1} = \begin{pmatrix} 1 & -k \\ k & -1 \end{pmatrix}(\lambda)$$

---

grid space E def'd by gens. + rels is a $\Gamma = \mathbb{Z}\oplus\mathbb{Z}$ module

$$(\mathbb{C}\Gamma)^2 s \xrightarrow{\quad} (\mathbb{C}\Gamma)p \oplus (\mathbb{C}\Gamma)q \longrightarrow E \longrightarrow 0$$

Γ-map

$\therefore E$ Γ-module 2 gen p, q

2 rels. $\begin{cases} (k\lambda - 1)p = hq \\ (k\mu - 1)q = \bar{h}p \end{cases}$

one gen. is enough ($h \neq 0$). Finally take

Fourier transform. $Y(z) = H(z)X(z)$

$$X(z) = \sum_{0 \le m < M} x_m z^{-m} \qquad H(z) = \sum_{0 \le \ell < L} h_\ell z^{-\ell}$$

$\downarrow$ to understand the DFT

$$X\left(e^{ik\frac{2\pi}{M}}\right) = \sum_{k \in \mathbb{Z}/M\mathbb{Z}} \sum_{m \in \mathbb{Z}/M\mathbb{Z}} x_m e^{-\frac{2\pi i km}{M}}$$

$$X\left(e^{ik\frac{2\pi}{M}}\right) = \sum_{0 < m < M} x_m \left(e^{-\frac{2\pi i}{M} k}\right)^m$$

$$Y = \sum_{0 \le n < L+M-1} y_n z^{-n} = \left(\sum_{0 \le \ell < L} h_\ell z^{-\ell}\right)\left(\sum_{0 \le m < M} x_m z^{-m}\right)$$

method. somehow use FT on $L+M-L$ to avoid aliasing

Idea you are missing: dropping the first L sites

$L = 2 \quad M = 5$

$y_0 = h_0 x_0$

$y_1 = h_0 x_1 + h_1 x_0$

$y_2 = h_0 x_2 + h_1 x_1$

$y_3 = h_0 x_3 + h_1 x_2$

$y_4 = h_0 x_4 + h_1 x_3$

$y_5 \qquad h_1 x_4$

input L

imp. resp M $\qquad N = L + M - 1$

$x_1(n) = M-1$ 0's $x(0) \quad x(L-1)$

$x_2(n) =$ last $M-1$ data pts from $x_1(n)$ + L new data pts.

$$Y(z) = H(z)\,X(z)$$

$$\left(\sum_0^{M-1} h_m z^{-m}\right)\left(\sum_0^{L-1} x_\ell z^{-\ell}\right)$$

The aim is to use the DFT of a given size to process large strings.

M is the length of the h sequence. A long string X will be subdivided into length L segments. Use DFT of size $N = M-1+L$

$$(\overbrace{0,\dots,0}^{M-1}, \overbrace{x_0,\dots,x_{L-1}}^{L}) = x_0 z^{-M+1} + \dots + x_{L-1} z^{\overbrace{-M-L+1}^{-N}}$$

multiply by $H(z) = \sum_{m=0}^{M-1} h_m z^{-m}$

$$(h_0 + h_1 z^{-1})(0 + x_0 z^{-1}$$

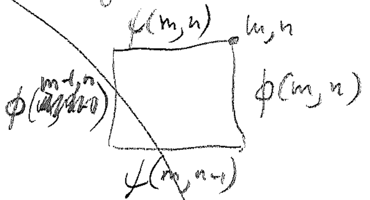
$M = 2 \qquad L = 3 \qquad N = M+L-1 = 4$

$$(x_0, x_1, x_2) \longmapsto (0, x_0, x_1, x_2)$$

$$\longmapsto (0, h_0 x_0, h_1 x_0 + h_0 x_1, h_1 x_1 + h_0 x_2, h_1 x_2$$

---

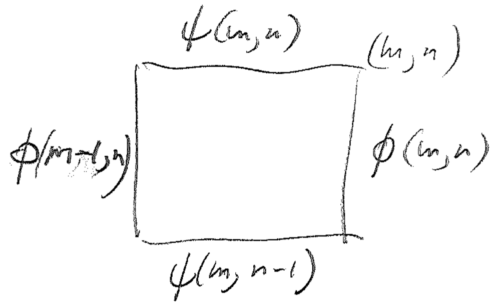
field eqns.



$$\begin{pmatrix}\phi(m,n)\\ \psi(m,n)\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}\phi(m-1,n)\\ \psi(m,n-1)\end{pmatrix}$$

$$k\phi(m,n) - \phi(m-1,n) = h\psi(m,n-1)$$

$$k\psi(m,n) - \bar{h}\psi(m,n-1) =$$



$$\begin{pmatrix}\phi(m,n)\\ \psi(m,n)\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}\phi(m-1,n)\\ \psi(m,n-1)\end{pmatrix}$$

$$k\phi(m,n) - \phi(m-1,n) = h\psi(m,n-1)$$
$$k\psi(m,n) - \psi(m,n-1) = \bar h\phi(m-1,n)$$

$$k\,\phi(x,n) - \phi(x-\varepsilon,n) = h\varepsilon\,\psi(x,n-1)$$
$$k\cdot\psi(x,n) - \psi(x,n-1) = \bar h\;\phi(x-\varepsilon,n)$$

$$k = \sqrt{1-\bar h h\varepsilon} \simeq 1 - \tfrac{1}{2}\bar h h\varepsilon$$

equations in lim. $\varepsilon \to 0$.

$$\phi(x,n) - \phi(x-\varepsilon,n) - \overbrace{\tfrac{1}{2}\bar h h}^{a}\varepsilon\,\phi(x,n) = h\varepsilon\,\psi(x,n-1)$$

$$\boxed{\begin{aligned}(\partial_x - a)\,\phi(x,n) &= h\,\psi(x,n-1)\\ \psi(x,n) - \psi(x,n-1) &= \bar h\,\phi(x,n)\end{aligned}}$$

$$\phi(x,n) = e^{sx} z^n A$$
$$\psi(x,n) = e^{sx} z^n B$$

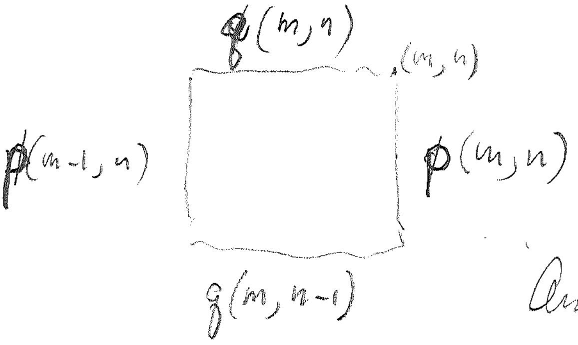
$$(s-a)A = h z^{-1} B \qquad A = \frac{h z^{-1}}{s-a} B$$
$$B - z^{-1}B = \bar h A \qquad B = \frac{\bar h}{1-z^{-1}} A$$

$\cancel{(s-a)}\cancel{(1-z^{-1})}$

$$\frac{h\bar h z^{-1}}{(s-a)(1-z^{-1})} = 1$$

$$2a = h\bar h = (s-a)(z-1)$$

$$z-1 = \frac{2a}{s-a}$$
$$z = 1 + \frac{2a}{s-a} = \frac{s+a}{s-a}$$



$$\begin{pmatrix} p(m,n) \\ q(m,n) \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \tilde h & 1 \end{pmatrix}\begin{pmatrix} p(m-1,n) \\ q(m,n-1) \end{pmatrix}$$

Aim: continuous limit horizontally

$$k\, p(m,n) - p(m-1,n) = h\, q(m,n-1)$$
$$k\, q(m,n) - q(m,n-1) = \tilde h\, p(m-1,n)$$

Replace $m, m-1$ by $x, x-\varepsilon$

$$k_\varepsilon\, p(x,n) - p(x-\varepsilon,n) = h_\varepsilon\, q(x,n-1)$$
$$k_\varepsilon\, q(x,n) - q(x,n-1) = \tilde h_\varepsilon\, p(x-\varepsilon,n)$$

$h_\varepsilon = c\varepsilon$

$\tilde h_\varepsilon = \tilde c$

$$k_\varepsilon = \sqrt{1-\tilde c c\,\varepsilon} = 1 - \underbrace{\tfrac{1}{2}\tilde c c}_{a}\,\varepsilon + O(\varepsilon^2)$$

$$\frac{(k - a\varepsilon)\,p(x,n) - p(x-\varepsilon,n)}{\varepsilon} = c\, q(x,n-1)$$

$\downarrow$

$$\boxed{\begin{aligned} (\partial_x - a)\,p(x,n) &= c\, q(x,n-1) \\ q(x,n) - q(x,n-1) &= \tilde c\, p(x,n) \end{aligned}}$$

$2a = \tilde c c$

look for exp. solns.
$$\begin{pmatrix} p(x,n) \\ q(x,n) \end{pmatrix} = e^{sx} z^n \begin{pmatrix} p \\ q \end{pmatrix}$$

$$(s-a)p = c z^{-1} q$$
$$q - z^{-1} q = \tilde c\, p$$

$$\cancel{p = \frac{cz^{-1}}{s-a}q \qquad q = \frac{\tilde c z^{-1}}{1-z^{-1}}p = \frac{\tilde c}{z-1}p}$$

$$\cancel{\frac{cz^{-1}}{s-a}\frac{\tilde c}{z-1} = z}$$

$$(z-1)q = z\tilde c p = \frac{z\tilde c c z^{-1} q}{s-a} = \frac{2a}{s-a}q$$
$$z = 1 + \frac{2a}{s-a} = \frac{s+a}{s-a}$$

probably you did the continuous limit on the analytic function level.

$$(k_\varepsilon \lambda^\varepsilon - 1)p = \varepsilon h q = \varepsilon c q$$

$$(k_\varepsilon \mu - 1) q = \tilde{h} p = \tilde{c} p \qquad\qquad k_\varepsilon = \sqrt{1 - \varepsilon \tilde{c} c} = 1 - \underbrace{\tfrac{1}{2}\tilde{c}c}_{a}\,\varepsilon$$

$$\lambda^\varepsilon = e^{\varepsilon s} \qquad \frac{k_\varepsilon \lambda^\varepsilon - 1}{\varepsilon}\, p = c q \qquad (\mu - 1) q = \tilde{c} p$$

$$(-a + s) p = c q \qquad\qquad \begin{pmatrix} 1 & \varepsilon c \\ \tilde{c} & 1 \end{pmatrix}$$

$$\begin{pmatrix} p(m,n) \\ q(m,n) \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \tilde{h} & 1 \end{pmatrix} \begin{pmatrix} p(m-1,n) \\ q(m,n-1) \end{pmatrix}$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \tilde{h} & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$kp = \lambda^{-\varepsilon} p + h \mu^{-1} q$$

$$kq = \tilde{h} \lambda^{-\varepsilon} p + \mu^{-1} q$$

better might be

$$\begin{tikzcd} \cdot \arrow[r, dash, "\mu q"] \arrow[d, dash, "p"'] & \cdot \arrow[d, dash, "\lambda p"] \\ \cdot \arrow[r, dash, "q"'] & \cdot \end{tikzcd}$$

$$\begin{pmatrix} \lambda p \\ \mu q \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \tilde{h} & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

$$\begin{pmatrix} \lambda^\varepsilon p \\ \mu q \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & c\varepsilon \\ \tilde{c} & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} \qquad \begin{aligned} (k\lambda^\varepsilon - 1) p &= \varepsilon c q \\ (k\mu - 1) q &= \tilde{c} p \end{aligned}$$

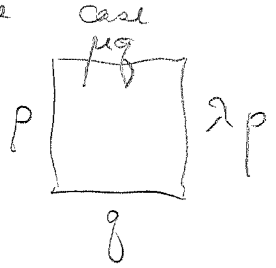
So now you the grid equation.

$$(-a+\partial_x)\,p(x,n) = c\, q(x,n-1)$$

$$q(x,n) - q(x,n-1) = \tilde c\, p(x,n) \qquad 2a = \tilde c c \quad ?$$

~~basis~~ generators for grid space should consist of $\lambda^x z^n \begin{pmatrix} p \\ q \end{pmatrix}$

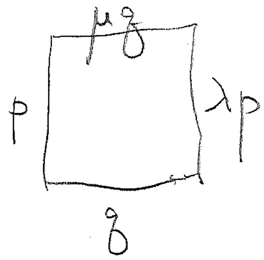
discrete case



$$k\begin{pmatrix} \lambda p \\ \mu q \end{pmatrix} = \begin{pmatrix} 1 & h \\ \tilde h & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} \qquad k = \sqrt{1-\tilde h h}$$

$$(k\lambda - 1)p = hq$$
$$(k\mu - 1)q = \tilde h p$$
$$(k\lambda^\varepsilon - 1)p = \varepsilon h q \rightsquigarrow$$
$$(k\mu - 1)q = \tilde h p$$

$$(k\lambda - 1)p = hq$$
$$(k\mu - 1)q = \tilde h p$$



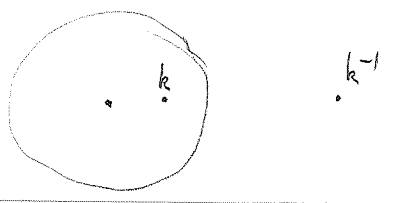
$$k\mu \cancel{-1} = \frac{1-k^2}{k\lambda-1} + 1$$

$$\cancel{k}\mu = \frac{-k^{\cancel{2}} + \cancel{k}\lambda}{k\lambda - 1}$$

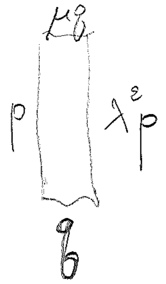
$$p = \frac{h}{k\lambda - 1}\, q \rightsquigarrow \frac{h}{kz-1}$$

completion of grid space for pos. def. herm. form.

$$\lambda = z \qquad L^2(S^1)$$
$$\mu = \frac{z-k}{kz-1}$$
$$q = 1 \qquad p = \frac{h}{kz-1} \qquad \text{what else.}$$

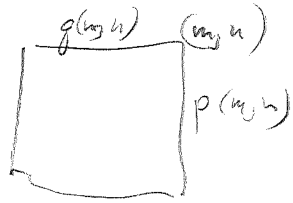


$$\lambda^x \mu^n \begin{pmatrix} p \\ q \end{pmatrix}$$



$$(k\lambda^\varepsilon - 1)p = \varepsilon h q$$
$$(k\mu - 1)q = \tilde{h} p$$

$$\begin{pmatrix} p(m,n) \\ q(m,n) \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \tilde{h} & 1 \end{pmatrix}\begin{pmatrix} p(m-1,n) \\ q(m,n-1) \end{pmatrix}$$



$$k\,p(x,n) - p(x-\varepsilon,n) = \varepsilon h\, q(x,n-1)$$
$$k\,q(x,n) - q(x,n-1) = \tilde{h}\, p(x-\varepsilon,n)$$

$$k = \sqrt{1-\varepsilon h\tilde{h}}$$
$$= 1 - \tfrac{1}{2}\varepsilon h\tilde{h}$$
$$a = \tfrac{1}{2}h\tilde{h}$$

$$-\cancel{\varepsilon} a\, p(x,n) + \frac{p(x,n)-p(x-\varepsilon,n)}{\varepsilon} = \cancel{\varepsilon} h\, q(x,n-1)$$

$$\boxed{\begin{aligned} (\partial_x - a)\,p(x,n) &= h\, q(x,n-1) \\ q(x,n) - q(x,n-1) &= \tilde{h}\, p(x,n) \end{aligned}}$$

$$\begin{pmatrix} p \\ q \end{pmatrix}(x,n) = e^{sx} z^n \begin{pmatrix} A \\ B \end{pmatrix}$$

$$(s-a)A = h z^{-1} B \qquad A = \frac{hz^{-1}}{s-a}B$$

$$B - z^{-1}B = \tilde{h}A \qquad B = \frac{\tilde{h}}{1-z^{-1}}A$$

$$\frac{h \cancel{z^{-1}} \tilde{h}}{(s-a)(\underbrace{\cancel{1-z^{-1}}}_{z-1})} = 1 \qquad z = \frac{2a}{s-a} + 1 = \frac{s+a}{s-a}$$

~~The idea is going to be clear.~~ Need a functional repn. Hilb. space to be $L^2(\mathbb{R})$. λ^x = mult by e^{xs}. What is ~~going~~ happening. λ^n = mult by z^n is replaced ~~replaced~~ by λ^x = mult. by e^{ikx} $\quad s = e^{ik}$

Attempt universal description of grid space. It should be a representation of $\mathbb{R} \times \mathbb{Z}$ horizontal + vertical translations.

---

digress back to DFT, how it can be used to filter a signal $(x_n,\ n \geq 0)$. Filter $(h_m,\ 0 \leq m \leq M-1)$. Decide to split x into blocks of length L. Follow instructions given $(x_0, \dots, x_{L-1})$ you shift to get $(\underbrace{0, \dots, 0}_{M-1}, x_0, \dots, x_{L-1})$, which has length $M-1+L = N$. Next convolve with $(h_0, \dots, h_{M-1}, \underbrace{0, \dots, 0}_{L-1})$, use circular convolution to get $(y_0, \dots, y_{N-1})$. ~~and~~ split this into $(y_0, \dots, y_{M-1})$ and $(y_M, \dots, y_{N-1})$

$N = L + M - 1$

L	$L+M-1$
0	$M-1$

Then for the next step you use the sequence

$y_0, \dots, y_{M-1}, x_M, \dots, x_{M+L-1}$?

Start again. $H = (h_0, \dots, h_{M-1})$ $\quad X^0 = (\underbrace{0, \dots, 0}_{M-1}, \underbrace{x_0, \dots, x_{L-1}}_{L})$

Circularly convolve. First linearly convolve.

$$H * X^0 = Y^0 = (\underbrace{0, \dots, 0}_{M-1}, y_0, \dots, y_{L-1}, y_L, \dots, y_{N-1})$$

$$\left(\sum_0^{M-1} h_m u^m\right)\left(\sum_0^{L-1} x_\ell u^{M-1+\ell}\right).$$

$M = 2 \qquad L = 3. \qquad X = x_0 + x_1 u + x_2 u^2 \qquad h_0 = 0,\ h_1 = 1$

$H = u$

$$X^0 = 0 + x_0 u + x_1 u^2 + x_2 u^3$$
$$H = 0 + u + 0 + 0$$

length 4

$M-1+L = 1+3 = 4.$

$$HX^0 = x_2 + 0u + x_0 u^2 + x_1 u^3$$

circular convolution

$H = h_0 + h_1 u \quad , \quad M = 2$

linear convolution $\quad Y = HX$

Fix a block size L so that a block consists of $X = (x_0, \dots, x_{L-1})$. Form the linear convolution $Y = H * X$ which has ~~a~~ size $L+1$. ~~Ms.~~ Focus on $x_0, \dots, x_{L-1}$

$M = 2 \qquad L = 3 \qquad H = u$

$$x_0 \; x_1 \; x_2 \xmapsto{\text{incl}} (0 \; x_0 \; x_1 \; x_2)$$

$$\downarrow u$$

$$(x_2 \; 0 \; x_0 \; x_1)$$

$$X^0$$

$$[x_0 \; x_1 \; x_2 \,|\, x_3 \; x_4 \; x_5] \mapsto [0 \; x_0 \; x_1 \; x_2 \,|\, x_3 \; x_4 \; x_5]$$

$\downarrow$ circ. conv $\qquad \downarrow$ id

$$[x_2 \; 0 \; x_0 \; x_1 \,|\, x_3 \; x_4 \; x_5]$$

$$[0 \; x_0 \; x_1 \; x_2] \qquad [x_2 \; x_3 \; x_4 \; x_5 \,|\, x_6$$

$$\downarrow$$

$$x_5 \; x_2 \; x_3 \; x_4 \,|\, x_6$$

$$[0 \; x_0 \; x_1 \; x_2]$$

Convolution operator on sequences

$$(x_n) \longmapsto (h*x)_n = \sum_{m=0}^{M-1} h_m x_{n-m}$$

Apparently there is a method of obtaining this operator by ~~splitting~~ partitioning the integers into equal segments of length L (accounting periods?). Instead of scalar functions on $\mathbb{Z}$ you have L-diml vectors on $L\mathbb{Z}$.

There's a standard 1-1 corresp between $\mathbb{Z}$ and $L\mathbb{Z} \times \mathbb{Z}/L\mathbb{Z}$

Bring into play Poisson summation? You w

$$f \in \mathcal{S}(\mathbb{R}) \qquad \sum_{n\in\mathbb{Z}} f(x+n)\, e^{2\pi i y n} = F(x,y)$$

then

$$F(x, y+1) = F(x,y)$$

$$F(x+1, y) = \sum_{n\in\mathbb{Z}} f(x \cancel{+1} + n)\, e^{2\pi i y(n-1)}$$

$$= e^{-2\pi i y} F(x,y)$$

Consider $\mathbb{C}[\mathbb{Z}]$ as $L\mathbb{Z}$-module, so $\mathbb{C}[\mathbb{Z}]$

$$\mathbb{C}[\mathbb{Z}] = \bigoplus_{i=0}^{L-1} \mathbb{C}[i + L\mathbb{Z}].$$

Suppose instead of $\mathbb{Z}$ you had $L\mathbb{Z} \times \mathbb{Z}/L\mathbb{Z}$ group. Then $\mathbb{C}[L\mathbb{Z} \times \mathbb{Z}/L\mathbb{Z}]$ is the tensor product rep $\mathbb{C}[L\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/L\mathbb{Z}]$. On each L block you get the translation group $\mathbb{Z}/L\mathbb{Z}$ and its characters.

$M=2,\ L=3,\ N = M+L-1 = 4$

$$\begin{array}{ccccc} h_0x_{-1} & h_0x_0 & h_0x_1 & h_0x_2 & \\ & + & + & + & \\ & h_1x_{-1} & h_1x_0 & h_1x_1 & h_1x_2 \end{array}$$

Yesterday idea: Partition time into blocks, replace $\mathbb{Z}$ by $L\mathbb{Z} \times \mathbb{Z}/L\mathbb{Z}$ seemed a good idea. But you don't see why you need blocks of size $M-1+L$. Maybe you need to calculate an example carefully. Take $M=2$ $L=3$ so that $N = M-1+L = 1+3 = 4$. Focus on the 0th block of date (length L)

$$x_{-1} \, \overbrace{\boxed{x_0, x_1, x_2} \, x_3} \qquad \text{two overlapping blocks of length 4.}$$

You work with size 4. ~~Note~~ $h_0 = h_1 = 1$

$$\begin{array}{cccccc} x_{-1} & x_0 & x_1 & x_2 & x_3 & \\ & x_{-1} & x_0 & x_1 & x_2 & x_3 \\ \hline & y_0 & y_1 & y_2 & & \end{array}$$

$$\begin{array}{ccccc} 0 & x_0 & x_1 & x_2 & \\ & 0 & x_0 & x_1 & x_2 \end{array}$$

$M=2$ $L=3$ $N = M+L-1 = 4$

$[0 \;\; x_0 \;\; x_1 \;\; x_2] \mapsto 0u^0 + x_0 u + x_1 u^2 + x_2 u^3$

$[h_0 \;\; h_1] \mapsto h_0 + h_1 u$

$y_0 + y_1 u + y_2 u^2 + y_3 u^3 + y_4 u^4$

$\downarrow$

restrict to 4th roots of unity.

$y_4 + y_1 u + y_2 u^2 + y_3 u^3$

Take $M = 2$ and L large.

Organizing the situation. Objects: discrete signal (a sequence $(x_n)_{n\in\mathbb{Z}}$), filter (translation invariant, or time independent, operator on signals), e.g. convolution operator $(x_n) \longmapsto \left((h_k) * (x_n)\right)$ $\quad y_n = \sum_m h_m x_{n-m}$ to simplify say $h = (h_0, \dots, h_{m-1}, 0, \dots)$. ~~Thus you~~ ~~Laurent the~~ z-transform picture multiplication by a Laurent poly on formal Laurent series $\sum_{n\in\mathbb{Z}} x_n u^n$. Simplify suppose $h(u) = \sum_{m=0}^{M-1} h_m u^m$ is a Laurent poly in $u = z^{-1}$.

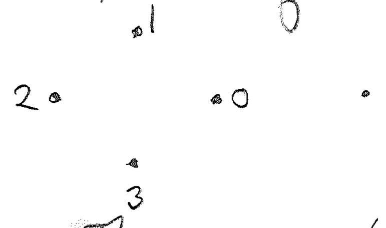
Other objects $\to$ evaluation at roots of unity

Link between subdividing $\mathbb{Z}$ into $L\mathbb{Z} \times \{0, \dots, L-1\}$, breaking time into intervals of length L.

$$| \; \dots, x_{-1} \,|\, x_0, x_1, \dots, x_{L-1} \,|\, x_L \; \dots$$

~~Process of restricting Finite FT~~ DFT

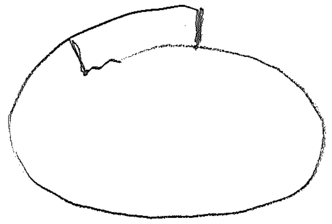
How to proceed? Fix L and look at signals of length L. Think of these as functions on the discrete circle



This is a quotient $\mathbb{Z}/L\mathbb{Z}$ of $\mathbb{Z}$. The convolution operator ~~depends~~ descends. ~~One of the things~~ You want to understand

The aim now is to ~~combine~~ use subdividing + enlarging so as to compute the filtering process

Begin with time divided into blocks of length L. Do the Fourier transform in each ~~b~~ block



Problem concerns crossing the ends of ~~the~~ a block. You want the finite conv. op of mult by $h(u) = \sum_{m=0}^{M-1} h_m u^m$ to be described from a signal

$$|x_{-L}, \ldots, x_{-1} | x_0, \ldots, x_{L-1} | x_L, \ldots, x_{2L-1}|$$

Enough to handle case where $x_n = 0 \quad n < 0$
$\qquad\qquad\qquad\qquad\qquad\qquad = 0 \quad n \geq L$

$h(u) = 1 - au$

$$x_0 \qquad \begin{matrix} x_1 u \\ -ax_0 u \end{matrix} \qquad \begin{matrix} x_{L-1} u^{L-1} \\ -ax_{L-2} u^{L-1} \end{matrix} \qquad -ax_{L-1} u^L$$

$$y_0 \qquad y_1 \qquad\qquad\qquad\qquad y_L$$

At ~~what~~ this point you need data $(y_0, \ldots, y_L)$ of length $L+1$.

$M = 2 \qquad H(z) = h_0 + h_1 u$, divide time into segments of length L. Any signal is the sum of its segments; operation is linear. $X = \sum_{n \in \mathbb{Z}} X^n[n]$

$HX = \sum_n (HX^n)[n]$. You need only $X^0 \longmapsto HX^0$ together with addition. If X^0 has length L:

$X^0(z) = x_0 + x_1 u + \cdots + x_{L-1} u^{L-1}$, then HX^0 has length L
$y^0 = HX^0 = y_0 + y_1 u + \cdots + y_L u^L$.

My method: ~~blocks~~ segment $x_{g,r}$ $n = gL + r$ where $0 \le r \le L-1$. Take segment $(x^g_r, 0 \le r < L)$ form $\sum_{r=0}^{L-1} x^g_r u^r$ mult by $\sum_{m=0}^{1} h_m u^m = h_0 + h_1 u$
(linear convolution

to get $(h_0 + h_1 u) \sum_{r=0}^{L-1} x^g_r u^r = \sum_{r=0}^{L} y^g_r u^r$

you want to compute this via DFT of size $L+1$. What does this mean? You evaluate polys in the variable u on μ_{L+1}. Send a poly $p = \sum_{r=0}^{L} c_r u^r$ into the function $p(\zeta)$, $\zeta^{L+1} = 1$. This linear map is bijective with IDFT for inverse. Thus you get the desired output (linear convolution) for segments of length L. But the output $(y^g_r, 0 \le r \le L)$ has length $L+1$, there's an extra entry y^g_L which must be added to y^{g+1}_0

Start again write signal $(x_n, n \in \mathbb{Z})$ as a chain of blocks $X^g, g \in \mathbb{Z}$, of length L.

$X^g =$

Divide time into segments of length L via $n = qL + r$, $0 \le r < L$. Our signal $x = (x_n, n \in \mathbb{Z})$ is the direct sum of the segments $x^q = (x^q_r = x_{qL+r},\ 0 \le r < L)$. Suppose the IR (h_0, h_1) has length 2. The output $y = h * x$ is the sum of the terms $y^q = h * x^q$ which have length $L+1$.

$$\sum_{r=0}^{L} y^q_r u^r = (h_0 + h_1 u) \sum_{r=0}^{L-1} x^q_r u^r$$

$L = 3$

$$(h_0 + h_1 u)(x^q_0 + x^q_1 u + x^q_2 u^2) = (y^q_0 + y^q_1 u + y^q_2 u^2 + y^q_3 u^3)$$

the $(y^q_r, 0 \le r \le L)$ are computed by the DFT + IDFT

$$| x^0_0 \quad x^0_1 \quad x^0_2 | \quad x^1_0 \quad x^1_1 \quad x^1_2 | \quad x^2_0 \quad x^2_1 \quad x^2_2 |$$

$$y^0_0 \quad y^0_1 \quad y^0_2 \quad y^0_3$$

$$y^1_0 \quad y^1_1 \quad y^1_2 \quad y^1_3$$

$$y^2_0 \quad y^2_1 \quad y^2_2 \,|\, y^2_3$$

This describes your process. Next comes the better one, where it seems the addition is handled via aliasing, the addition of y^0_3 to y^1_0. Start with (x^0_0, x^0_1, x^0_2), your segment. send this not to $x^0_0 + x^0_1 u + x^0_2 u^2 + 0u^3$, but to $x^0_0 u + x^0_1 u^2 + x^0_2 u^3$, mult by $h_0 + h_1 u$ to get

$$y^0_0 u + y^0_1 u^2 + y^0_2 u^3 + y^0_3 u^4$$

restrict u to 4th roots of 1

$$y^0_3 + y^0_0 u + y^0_1 u^2 + y^0_2 u^3$$

Object: Convolution operator $x \mapsto y = h * x$ on discrete time signals $x = (x_n, n \in \mathbb{Z})$, $h = (h_0, h_1, 0, 0 \cdots)$; $y_n = (h * x)_n = \sum_{m=0}^{1} h_m x_{n-m}$. You want to understand a scheme for computing this operator, which involves splitting the signal x into segments of length L: $x = (\ldots, x^0, x^1, \ldots)$, where

$$x^g = (x^g_0, \ldots, x^g_{L-1}), \qquad x^g_r = x_{gL+r},$$

and then involves the DFT of size $L+1$ to handle these signals of length L. The output $y^g = h * x^g$ has length $L+1$, so you need roots of 1 of this size at least to distinguish the different segments of this size, i.e. to avoid aliasing.

$\boxed{\text{Overlap-Add}}$

Take $L = 3$. Your picture involves taking a segment $x = (x_0, x_1, x_2)$, extending by 0 to other sites, taking z transf.

$$\sum_n x_n z^{-n} = \sum_n x_n u^{-n} = x_0 + x_1 u + x_2 u^2 + 0 u^3$$

~~restricting~~ to μ_4 (getting the ~~which means~~ DFT of length 4), then mult by $h_0 + h_1 u + 0u^2 + 0u^3$ rest. to μ_4, next the IDFT of length 4 to get the output

$$(y^0_0, y^0_1, y^0_2, y^0_3) = (h_0, h_1) * (x^0_0, x^0_1, x^0_2, \overset{0}{\overset{\shortparallel}{x_3}})$$

Then handle the next segment (x^1_0, x^1_1, x^1_2) in the same way to get $(y^1_0, y^1_1, y^1_2, y^1_3)$, then the answer is

$$y^0_0, y^0_1, y^0_2, y^0_3 + y^1_0, y^1_1, y^1_2, y^1_3$$

The book has a sly way to avoid the sum

$M=2 \qquad N = M-1+L = L+1. \qquad L=3$

$$(x_0, x_1, x_2)$$

$$(0, x_0, x_1, x_2)$$

$$\Big\downarrow \text{DFT}$$

$$(x_0u + x_1u^2 + x_2u^3)_{u\in\mu_4}$$

$$\Big\downarrow$$

$$\Big((h_0+h_1u)(x_0u + x_1u^2 + x_2u^3),\ u\in\mu_4 \Big)$$

$$\|$$

$$(y_0u + y_1u^2 + y_2u^3 + y_3u^4,\ u\in\mu_4)$$

$$\|$$

$$(y_0u + y_1u^2 + y_2u^3 + y_3,\ u\in\mu_4)$$

The point you missed. Use input data blocks of size $N = L+1$. ~~Instead~~ of a splitting of the input into a sum of disjoint blocks of length L, you use a covering by blocks of the larger size $N = M-1+L$. Focus on what you need to compute the output y_j at time j. $\boxed{\text{Method called Overlap-Save}}$

Go over carefully what's involved. You have the input signal $(x_n, n\in\mathbb{Z})$ and the output signal $(y_n, n\in\mathbb{Z})$ related by linear convolution $y = h * x$, $y_n = \sum_m h_m x_{n-m}$ where $h = (h_0, \ldots, h_{M-1})$ has length M. You want to compute this linear convolution operator by working with blocks ~~of~~ data of length N and using the DFT & IDFT

of size N. Suppose $N = M-1+L$ with $L>0$.
What do we need to know about the input x in order to calculate y_n? Since $y_n = \sum_0^{M-1} h_m x_{n-m}$ it suffices to give $(x_n, x_{n-1}, \dots, x_{n-M+1})$, better order: $(x_{n-M+1}, x_{n-M+2}, \dots, x_{n-1}, x_n)$ which is the data block of size M ending with time n.
If you want $(y_0, y_1, \dots, y_{L-1})$ you need

$$(\underbrace{x_{-M+1}, x_{-M+2}, \dots, x_{-1}}_{M-1}, \underbrace{x_0, x_1, \dots, x_{L-1}}_{L}).$$

It seems that you have factored the linear convolution operator $x \longmapsto h * x$ as follows.

Overlap-Save should be dual (adjoint?) to Overlap-Add. Consider the latter, where signal space $C(\mathbb{Z})$ is written $\mathbb{C}(L\mathbb{Z} \times (\mathbb{Z}/L\mathbb{Z})) \simeq \mathbb{C}(L\mathbb{Z}, C(\mathbb{Z}/L\mathbb{Z}))$, equivariantly for $L\mathbb{Z}$ translation.

Maybe it's better to look at finite support signals, the space $C_c(\mathbb{Z}) = \mathbb{C}[\mathbb{Z}]$, which is $\mathbb{C}[L\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/L\mathbb{Z}]$ as $L\mathbb{Z}$ module. Then $h_0 : \mathbb{C}[\mathbb{Z}] \longrightarrow \mathbb{C}[\mathbb{Z}]$ becomes

$$\mathbb{C}[L\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/L\mathbb{Z}] \longrightarrow \mathbb{C}[\mathbb{Z}]$$

an $L\mathbb{Z}$-module map, which is determined by its restriction to $\mathbb{C}[\mathbb{Z}/L\mathbb{Z}] = \{(x_0, \dots, x_{L-1})\}$ signals supported in $[0, \dots, L-1]$.
Then $h_0 : \mathbb{C}[\mathbb{Z}/L\mathbb{Z}] \longrightarrow \mathbb{C}[\mathbb{Z}]$ factors

$$\begin{tikzcd} \mathbb{C}[\mathbb{Z}/L\mathbb{Z}] \arrow[rr] \arrow[dr, "\alpha"'] & & \mathbb{C}[\mathbb{Z}] \\ & \mathbb{C}[\mathbb{Z}/N\mathbb{Z}] \arrow[ur, "\beta"'] & \end{tikzcd}$$

subspace with supp in $[0, \dots, M-1+L]$

then $h_0 : \mathbb{C}[\mathbb{Z}] \to \mathbb{C}[\mathbb{Z}]$ becomes the composition of $L\mathbb{Z}$-mod maps

$$\mathbb{C}[L\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/L\mathbb{Z}] \xrightarrow{1 \otimes \alpha} \mathbb{C}[L\mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/N\mathbb{Z}] \xrightarrow[\text{OVERLAP ADD}]{\tilde{\beta}} \mathbb{C}[\mathbb{Z}]$$

Next look at signals of arb. supp, the space $C(\mathbb{Z})$, and identify the target of the conv. of h_0 with:

$$C(\mathbb{Z}) = \underbrace{C(L\mathbb{Z}, C(\mathbb{Z}/L\mathbb{Z}))}$$

coinduced $L\mathbb{Z}$ module from $\mathbb{C}(\mathbb{Z}/L\mathbb{Z})$.

Then

$$\begin{array}{ccc} C(\mathbb{Z}) & \xrightarrow{h_0} & C(\mathbb{Z}) \\ {\scriptstyle \text{rest to}}\downarrow & & \downarrow{\scriptstyle \text{rest to } \{0,\dots,L-1\}} \\ C(\mathbb{Z}/N\mathbb{Z}) & \longrightarrow & C(\mathbb{Z}/L\mathbb{Z}) \end{array}$$

---

back to grid space continuous horizontally, discrete vertically. You want to make the link with the trapezoidal fmla for integration.

D.E. is $\left(\frac{d}{dt} - a\right) y = x$

Introduce the L.T. variable s. $(s-a)\hat{y} = \hat{x}$

The trap. rule with $x = 0$, consider this..

$$\int_0^n \left(\frac{dy}{dt} - ay\right) dt = 0$$

$$y(n) - y(0) = a \int_0^n y\, dt \qquad \Bigg| \qquad \frac{e^{na} - 1}{a} = \int_0^n e^{at}\, dt$$

The trapezoidal ~~rule~~ approx is

$$\int_0^n f(t)\, dt = \frac{1}{2} f(0) + \sum_{i=1}^{n-1} f(i) + \frac{1}{2} f(n)$$

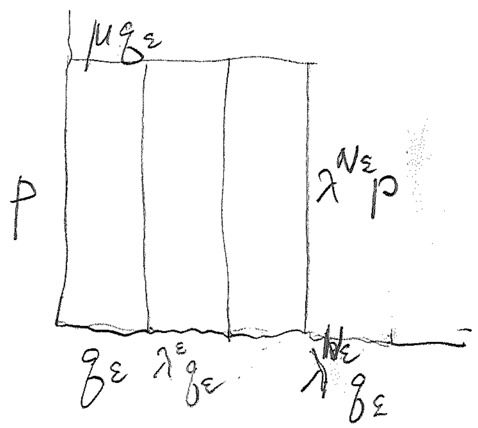
$$\frac{1}{2} + \sum_1^{n-1} a^i + \frac{1}{2} a^n$$

$$= \frac{1}{2} + \frac{a - a^n}{1-a} + \frac{1}{2} a^n = \frac{1 - a + 2a - 2a^n + a^n(1-a)}{2(1-a)} = \frac{1 + a - a^n - a^{n+1}}{2(1-a)}$$

$$\frac{1}{2} + \sum_{i=1}^{n-1} a^i + \frac{1}{2}a^n = \sum_{i=0}^{n-1} a^i - \frac{1}{2} + \frac{1}{2}a^n$$

$$= \frac{1-a^n}{1-a} - \frac{1}{2}(1-a^n) = (1-a^n)\left(\frac{1}{1-a} - \frac{1}{2}\right)$$

$$= (1-a^n)\left(\frac{2-1+a}{(1-a)2}\right) = \frac{(1-a^n)}{1-a}\,\frac{1+a}{2}$$



$$(k_\varepsilon \lambda^\varepsilon - 1)p = h g_\varepsilon$$

$$(k_\varepsilon \mu - 1) g_\varepsilon = \tilde{h} p$$

$\|g_\varepsilon\| = \frac{1}{\sqrt{\varepsilon}}$

?

Poisson interpolation. $f \in \mathcal{S}(\mathbb{R})$

$$F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i y n} f(x+n)$$

$$F(x,y) = F(x,y+1)$$

$$F(x+1,y) = e^{-2\pi i y} F(x,y)$$

better might be

$$f(x) \longmapsto \sum_{n\in\mathbb{Z}} f(x+n) \quad \text{period 1 in } x$$

$$\sum_{k\in\mathbb{Z}} a_k e^{2\pi i k x} = \sum_{n\in\mathbb{Z}} f(x+n)$$

$$\int_{-\infty}^{\infty} e^{-2\pi i k u} f(u)\,du = \hat{f}(k)$$

$$a_k = \int_0^1 dx\, e^{-2\pi i k x} \sum_{n\in\mathbb{Z}} f(x+n) \qquad u = x+n$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 dx\, e^{-2\pi i k x} f(x+n) = \sum_{n\in\mathbb{Z}} \int_n^{n+1} e^{-2\pi i k u} f(u)\,du$$

difference equation vs. diff'l equ.

$$\frac{dy}{dt} = ay$$

$$y(n) = \int_{n-1}^{n} \frac{dy}{dt}\,dt + y(n-1)$$

$$= a\int_{n-1}^{n} y\,dt + y(n-1)$$

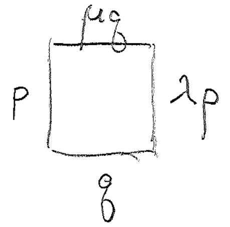
$$\sim a\,\frac{y(n-1)+y(n)}{2} + y(n-1)$$

$$\left(1-\frac{a}{2}\right)y(n) = \left(1+\frac{a}{2}\right)y(n-1)$$

$$y(n) = \left(\frac{1+\frac{a}{2}}{1-\frac{a}{2}}\right)^n y(0)$$

Review grid space | vertically disc, horiz. cont.

Begin with



$$(k\lambda-1)p = hq$$
$$(k\mu-1)q = \bar{h}p$$

isomorphism

$$\mathbb{C}[\lambda][\lambda^{-1},(\lambda-k)^{-1},(k\lambda-1)^{-1}] \xrightarrow{\sim} \text{grid space} \hookrightarrow L^2(S^1)$$

$$\lambda = \text{mult by } z$$
$$\mu = \frac{z-k}{kz-1}$$
$$q \longmapsto 1$$
$$p \longmapsto \frac{h}{k\lambda-1}$$

The idea was to consider the way the Fourier transform on $\mathbb{R}$ is obtained from Fourier series on $\mathbb{R}/L\mathbb{Z}$ as $L \to \infty$.

$$f(x) = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}} f_k\, e^{ikx} \quad\Big|\quad f_k = \frac{1}{L}\int_{-L/2}^{L/2} e^{-ikx} f(x)\,dx$$

$\longleftarrow$

Hormander's notation.

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} \hat{f}(k)\, dk$$

Discrete version of Poisson summation, for $L\mathbb{Z} \subset \mathbb{Z}$

$f(n), n \in \mathbb{Z} \longmapsto g = \sum_{k\in\mathbb{Z}} f(n+kL)$ periodic of period L, can be expanded in characters of $\mathbb{Z}/L\mathbb{Z}$ i.e. Lth roots of 1, should get $\hat{g}$ on μ_L is $\hat{f}(z)$ restr. to μ_L.

Vague idea - similarity between finite bandwith condition determining the signal from its sampling and the degree condition on h which prevents aliasing

---

interpolation: polynomials $p(z) = \prod_i (z - a_i)$ a_i distinct

$$\frac{f(z)}{p(z)} = \sum_i \frac{1}{z-a_i} \frac{f(a_i)}{p'(a_i)} \qquad \deg f < \deg p$$

$$f(z) = \sum_i \frac{p(z)}{z-a_i} \frac{f(a_i)}{p'(a_i)}$$

$$\frac{p(z)}{z-a_i} = \prod_{j\neq i} (z - a_j)$$

$$\left.\right\} \quad f(a_i) = \prod_{j\neq i}(a_i - a_j)\frac{f(a_i)}{p'(a_i)}$$

$$\therefore \quad p'(a_i) = \prod_{j\neq i}(a_i - a_j)$$

Take $p(z) = \sin(\pi z)$. You want an entire version B_4

$$\frac{f(z)}{\sin \pi z} = \sum_n \frac{1}{z-n} \frac{f(n)}{\pi (-1)^n}$$ if so then

$$f(z) = \sum_n f(n) \frac{\sin \pi(z-n)}{\pi (z-n)}$$

growth condition on $f(z)$ analogous to $\deg f(z) < \deg p(z)$

the important case is $f(z) = e^{sz}$ s purely imag

You want a good theory about this interpolation formula

Given values a_n $n \in \mathbb{Z}$, there should be an entire $f(z)$ with $f(n) = a_n$ for all n. Mittag-Leffler theorem gives a meromorphic function with appropriate principal parts.

There's the finite bandwidth stuff

$$f(z) = \int e^{sz} \underbrace{\hat{f}(s)}\, ds$$

distribution or hyperfn compact support in $i\mathbb{R}$

For what entire $f(z)$ is it true that

$$\frac{f(z)}{\sin(\pi z)} = \sum_{n \in \mathbb{Z}} \frac{1}{z-n} \frac{f(n)}{\pi (-1)^n} \quad ?$$

$$f(z) \frac{\pi \cos(\pi z)}{\sin(\pi z)} \stackrel{?}{=} \sum_{n \in \mathbb{Z}} \frac{f(n)}{z-n}$$

$$f(z)\, d \log p(z) = \sum_n \frac{f(n)}{z-n}$$

$$\oint_C f(z)\, d \log p(z) = 2\pi i \sum_{n \in C} f(n)$$

Suppose $f(-z) = f(z)$, Then

$$\sum_n \frac{f(n)}{z-n} = \sum_n \frac{f(-n)}{z+n} = \sum_n \frac{+f(n)}{z+n} = \sum_n \frac{f(n)}{(-z)-n}$$

and if $f(-z) = -f(z)$, then

$$\sum_n \frac{f(n)}{z-n} = \sum_n \frac{f(-n)}{z+n} = \sum_n \frac{-f(n)}{z+n} = \sum \frac{f(n)}{(-z)-n}$$

Let

$$g(z) = \sum_{n\in\mathbb{Z}} \frac{f(n)}{z-n} = \sum_n \frac{f(-n)}{z+n}$$ assuming convergence

So $f(-n) = f(n) \Rightarrow g(z) = \sum_{n\in\mathbb{Z}} f(n)\left(\frac{1}{z-n} + \frac{1}{z+n}\right) = 2z\sum_n \frac{f(n)}{z^2-n^2}$

in particular $g(-z) = -g(z)$.

---

repeat:

$$f(z)\frac{\pi\cos(\pi z)}{\sin(\pi z)} = \sum_n \frac{f(n)}{z-n}$$

$$f(z)\frac{\pi}{\sin(\pi z)} = \sum_n \frac{f(n)(-1)^n}{z-n}$$

$$\frac{f(z)\pi}{\sin(\pi z)}\left(\frac{1 \pm \cos(\pi z)}{2}\right) = \sum_n \frac{f(n)}{z-n}\left(\frac{1\pm(-1)^n}{2}\right)$$

$$g(z) = \sum_{n\in\mathbb{Z}} \frac{f(n)}{z-n} = \sum_n \frac{f(-n)}{z+n}$$

f odd $\Rightarrow$ g even
f even $\Rightarrow$ g odd

$$g(-z) = \sum_n \frac{f(n)}{-z-n} = \sum_n \frac{-f(n)}{z+n}$$

$$\frac{g(z)-g(-z)}{2} = \sum_n \frac{f(n)}{2}\left\{\frac{1}{z-n} + \frac{1}{z+n}\right\} = \sum_n f(n)\frac{z}{z^2-n^2}$$

$$\frac{g(z)+g(-z)}{2} = \sum_n \frac{f(n)}{2}\left\{\frac{1}{z-n} - \frac{1}{z+n}\right\} = \sum_n f(n)\frac{n}{z^2-n^2}$$

You need a viewpoint toward interpolation, i.e. a digital/analog converter. You give values $f(n)$.
Focus upon whether the interpolation formula

1) $$f(z) = \sum_n f(n) \frac{\sin \pi(z-n)}{\pi(z-n)} = \sum_n \frac{\sin(\pi z)}{z-n} \frac{f(n)}{\pi(-1)^n}$$

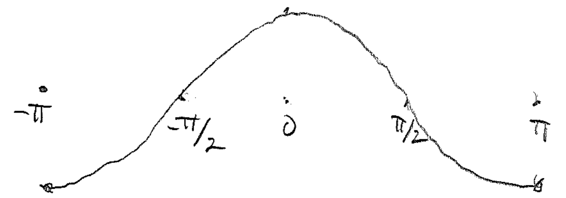
holds for $f(z) = e^{sz}$ s purely imaginary. There's another interpolation formula

2) $$f(z) \frac{\pi \cos(\pi z)}{\sin(\pi z)} = \sum_n \frac{f(n)}{z-n}$$

Can you deduce one from the other? Take $f(n) = \delta_0(n)$.

First formula 1) says $$f(z) = \frac{\sin(\pi z)}{\pi z}$$

2) says $$f(z) = \frac{\sin(\pi z)}{\pi \cos(\pi z)} \frac{1}{z}$$ extra poles where $z \in \frac{1}{2} + \mathbb{Z}$



It looks like you want to stick to 1).

Look at the convergence of $\sum_n \frac{f(n)}{z-n}(-1)^n$. For each $z \notin \mathbb{Z}$ the sequence $\frac{(-1)^n}{z-n} \in l^2$, so can use Cauchy-Schwartz

$$\left| \sum_n \frac{f(n)(-1)^n}{z-n} \right|^2 \leq \sum_n |f(n)|^2 \sum_n \left| \frac{1}{z-n} \right|^2$$

something similar should hold for Hölder estimates:

$$\sum_n \left| \frac{1}{z-n} \right|^2 = \sum_n \frac{1}{(x-n)^2+y^2} \sim \int_{-\infty}^{\infty} \frac{1}{x^2+y^2}\, dx = \int_{-\infty}^{\infty} \frac{1}{|y|^2(u^2+1)} |y|\, du$$
$$= \frac{1}{|y|} \arctan(u) \Big|_{-\infty}^{\infty} = \frac{\pi}{|y|}$$

$$\frac{\sin \pi z}{\pi z} = \int_{-\pi}^{\pi} e^{i\omega z} \frac{d\omega}{2\pi} \quad \left(= \frac{e^{i\omega z}}{iz} \frac{1}{2\pi} \Big]_{-\pi}^{\pi} \right)$$

$$\frac{e^{i\pi z} - e^{-i\pi z}}{2\pi i z}$$

$$\int_{-a}^{a} e^{i\omega z} \frac{d\omega}{2a} = \left[\frac{e^{i\omega z}}{2a\, i\omega} \right]_{-a}^{a} = \frac{e^{i\omega a} - e^{-i\omega a}}{2i\ \omega a} = \frac{\sin \omega a}{\omega a}$$

So is it true that

$$\frac{\sin \omega z}{\omega z} = \sum_n \frac{\sin \omega n}{\omega n} \frac{\sin \pi(z-n)}{\pi(z-n)}$$

$$= \sum_n \frac{\sin \omega n}{\omega n} \frac{\sin \pi z}{\pi(z-n)} (-1)^n$$

$$e^{i\omega t} \stackrel{?}{=} \sum_n \frac{\sin(\pi t)}{t-n} \frac{(-1)^n}{\pi} e^{i\omega n}$$

$$e^{i\omega t} \stackrel{?}{=} \sum_n \boxed{\frac{\sin \pi(t-n)}{\pi(t-n)}} e^{i\omega n}$$

$$\frac{\sin \pi(t-n)}{\pi(t-n)} = \int_{-\pi}^{\pi} e^{i\omega'(t-n)} \frac{d\omega'}{2\pi}$$

$$e^{i\omega t} \stackrel{?}{=} \sum_n \int_{-\pi}^{\pi} e^{i\omega' t} e^{-i(\omega'-\omega)n} \frac{d\omega'}{2\pi}$$

idea is that $\quad \frac{1}{2\pi} \sum_n e^{i(\omega-\omega')n} = \sum_{k\in\mathbb{Z}} \delta(\omega - \omega' - 2k\pi)$

$$e^{i\omega t} \overset{?}{=} \sum_{n\in\mathbb{Z}} \frac{\sin\pi(t-n)}{\pi(t-n)}\, e^{i\omega n}$$

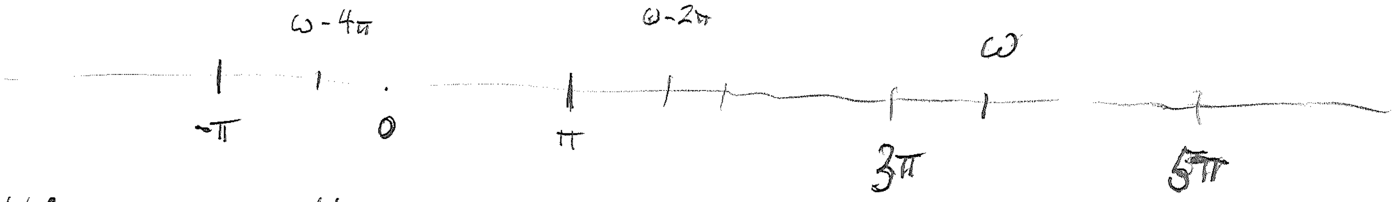
$$= \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\omega'(t-n)}\, \frac{d\omega'}{2\pi}\, e^{i\omega n}$$

$$= \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} \frac{d\omega'}{2\pi}\, e^{i\omega' t}\, e^{i(\omega-\omega')n}$$

$$= \int_{-\pi}^{\pi} \frac{d\omega'}{2\pi}\, e^{i\omega' t} \underbrace{\sum_n e^{i(\omega-\omega')n}}_{2\pi\sum_m \delta(\omega-\omega'-2\pi m)}$$

$$= \sum_m \int_{-\pi}^{\pi} \frac{d\omega'}{\cancel{2\pi}}\, e^{i\omega' t}\, \delta(\omega - 2\pi m - \omega')$$

But now ω' sat. $-\pi < \omega' < \pi$ (ignore bdry)
and there's exactly one ω' in this range with
$\omega - \omega' \in 2\pi\mathbb{Z}$.



Something is wrong!! ω is arbitrary $\in\mathbb{R}$ in the above calculation. But ω' is definitely in $(-\pi, \pi)$

Note $e^{i\omega n} = e^{i(\omega+2\pi m)n}$

so the right side $\displaystyle\sum_n \frac{\sin\pi(t-n)}{\pi(t-n)} \left(e^{i\omega}\right)^n$

depends only on $\omega + 2\pi\mathbb{Z}$

$$f(t) = e^{-t^2/2}$$ entire fn. of the [illegible] t.

$\sum_{n\in\mathbb{Z}} f(t+n)$ should converge to yield a periodic holom. function of period 1, equivalently a Laurent series in the variable $z = e^{2\pi i t}$ converging on $\mathbb{C}^\times = \mathbb{C} - \{0\}$

You should end up with the usual Jacobi Θ fns.

do something rigorous. Assume $-\pi < \omega < \pi$. Can you prove that

$$e^{i\omega t} = \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\xi(t-n)} e^{i\omega n} \frac{d\xi}{2\pi} \quad ?$$

Try to get a smoothed version, namely, replace $e^{i\omega t}$ by $\hat{\varphi}(t) = \int_{-\pi}^{\pi} e^{i\omega t} \varphi(\omega)\, d\omega$ where $\varphi \in C_c^\infty((-\pi,\pi))$.

$$\hat{\varphi}(t) \overset{?}{=} \sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{i\xi(t-n)} \hat{\varphi}(n) \frac{d\xi}{2\pi}$$

$$= \int_{-\pi}^{\pi} e^{i\xi t} \underbrace{\left(\sum_{n\in\mathbb{Z}} e^{-i\xi n} \hat{\varphi}(n) \right)} \frac{d\xi}{2\pi}$$

$$\sum_{n\in\mathbb{Z}} \int_{-\pi}^{\pi} e^{-i\xi n} \underbrace{\int e^{i\omega n} \varphi(\omega)\, d\omega}_{\text{seq. of Fourier coeff. for } \varphi \text{ extended periodically period } 2\pi} \frac{d\xi}{2\pi}$$

so this should be $\varphi(\xi)$ extended periodically to $\mathbb{R}$

Before proceeding further, let's review Poisson summation and the line bundle on the 2-torus

$f(x) \in \mathcal{S}(\mathbb{R})$ yields a period 1 function $g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$, which can be expanded in a F.S.

$$g(x) = \sum_{k\in\mathbb{Z}} c_k\, e^{2\pi i k x},$$

$$c_k = \int_0^1 e^{-2\pi i k x}\, g(x)\, dx$$

$$= \int_0^1 e^{-2\pi i k x} \sum_n f(x+n)\, dx$$

$$= \sum_n \int_0^1 e^{-2\pi i k x} f(x+n)\, dx \qquad x' = x+n$$

$$= \sum_n \int_n^{n+1} \underbrace{e^{-2\pi i k (x'-n)}}_{e^{-2\pi i k x'}} f(x')\, dx'$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i k x'} f(x')\, dx' = \hat{f}(k)$$

Next ~~points~~ twist by a character

$$F(x,y) = \sum e^{2\pi i n y} f(x+n) \qquad e^{2\pi i x y}$$

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{k\in\mathbb{Z}} \hat{f}(k)\, e^{2\pi i k x}$$

Poisson summation formula for $\mathbb{Z} \subset \mathbb{R}$

$f(x) \in \mathcal{S}(\mathbb{R})$.

$$\cancel{\hat{f}(2\pi\xi) = \int e^{-\frac{2\pi}{\lambda}\xi x} f(x)\,dx}$$

$$\cancel{f(x) = (2\pi)^{-1}\int e^{2\pi i \xi x}\hat{f}(\xi)\,d\xi}$$

$$\hat{f}(k) = \int e^{-2\pi i k x} f(x)\,dx = \int e^{-i\xi x} f(x)\,dx \qquad \xi = 2\pi k$$

$$f(x) = \int e^{i\xi x}\hat{f}(k)\frac{d\xi}{2\pi} = \int e^{2\pi i k x}\hat{f}(k)\,dk$$

Starting with $f(x) \in \mathcal{S}(\mathbb{R})$ there are two kinds of $\mathbb{Z}$ sums. Start with $f \mapsto \sum f(x+n)$ $\quad \pi_! f$

$$\pi \quad \begin{array}{c}\mathbb{R}\\ \downarrow\\ \mathbb{R}/\mathbb{Z}\end{array} \qquad \pi_! : \mathcal{S}(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R}/\mathbb{Z})$$

Maybe look at analog of F.T. where instead of the inner product $\langle e^{2\pi i\xi x}, f\rangle = \int e^{2\pi i \xi x} f(x)\,dx$ you use

$$\pi_!\left(e^{-2\pi i\xi x} f(x)\right) = \underbrace{\sum_n e^{-2\pi i\xi(x+n)} f(x+n)}_{F(x,\xi)}.$$

$$F(x,\xi) = e^{-2\pi i\xi x}\sum_n e^{-2\pi i\xi n} f(x+n)$$

$$G(x,\xi) = e^{2\pi i\xi x}F(x,\xi) = \sum_n e^{-2\pi i\xi n} f(x+n)$$

$$G(x,\xi+1) = G(x,\xi) \qquad G(x+1,\xi) = \sum_n e^{-2\pi i\xi n} f(x+1+n)$$

$$= \sum_n e^{-2\pi i\xi(n-1)} f(x+n)$$

$$= e^{2\pi i\xi} G(x,\xi)$$

$$F(x,\xi) = e^{-2\pi i\xi x} G(x,\xi)$$

$$F(x+1,\xi) = e^{-2\pi i\xi(x+1)}\, e^{2\pi i\xi} G(x,\xi)\ \cancel{G(x+1,\xi)}$$

$$= e^{-2\pi i\xi x} G(x,\xi) = F(x,\xi)$$

$$F(x,\xi+1) = e^{-2\pi i\xi} F(x,\xi).$$

The interesting point seems to be that you can derive the inversion formula for the F.T. from the Fourier series inversion formula (?)

Begin with sections of the Poincaré line bundle over $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Where should you start? Take the principal bundle $\mathbb{Z} \longrightarrow \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ and form the associated fibre bundle with fibre the group ring $\mathbb{C}[\mathbb{Z}]$. Identify $\mathbb{C}[\mathbb{Z}]$ with the ring of Laurent polys $\mathbb{C}[z,z^{-1}]$. You may want to enlarge $\mathbb{C}[\mathbb{Z}]$ to the ring of Laurent series $\sum_{n\in\mathbb{Z}} c_n z^n$ converging for $z \neq 0,\infty$.

You would also like to be able to extend $\mathbb{Z} \to \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ to the principal $\mathbb{Z}$-bundle $\mathbb{Z} \to \mathbb{C} \longrightarrow \mathbb{C}^\times$. But this probably doesn't work.

$$f(x) \longmapsto \sum_n e^{2\pi i ny} f(x+n) \qquad \text{periodic in } y$$

$$\overset{\text{alt}}{\longmapsto} \sum_n e^{2\pi i (x+n)y} f(x+n) \qquad \text{periodic in } x$$

$$F(x,y) = \sum_n e^{2\pi i ny} f(x+n) \qquad \boxed{F(x,y+1) = F(x,y)}$$

$$e^{2\pi i xy} F(x,y) = \sum_n e^{2\pi i(\cancel{x}+n)} f(x+n)$$

$$e^{2\pi i(x+1)y} F(x+1,y) = e^{2\pi i xy} F(x,y)$$

$$e^{2\pi i y} F(x+1,y) = F(x,y) \qquad \boxed{F(x+1,y) = e^{-2\pi i y} F(x,y)}$$

$$\boxed{\begin{aligned} F(x,y) &= \sum_n e^{2\pi i n y} f(x+n) \\ F(x,y+1) &= F(x,y) \\ F(x+1,y) &= e^{-2\pi i y} F(x,y) \end{aligned}}$$

$$f(x+n) = \int_0^1 e^{-2\pi i n y} F(x,y)\,dy$$

$$= \int_0^1 F(x+n,y)\,dy$$

$$\boxed{f(x) = \int_0^1 F(x,y)\,dy}$$

Since $F(x,y+1) = F(x,y)$ can expand in F.S.

$$F(x,y) = \sum \phi_n(x) e^{2\pi i n y}$$

$$F(x+1,y) = \sum_n \phi_n(x+1)\, e^{2\pi i n y}$$

$$\|$$

$$e^{-2\pi i y} F(x,y) = \sum_n \phi_n(x)\, e^{2\pi i (n-1) y} = \sum_n \phi_{n+1}(x)\, e^{2\pi i n y}$$

$$\therefore \quad \phi_{n+1}(x) = \phi_n(x+1) = \phi_{n-1}(x+2) = \cdots = \phi_0(x+n+1)$$

so $\phi_n(x) = f(x+n)$ where $f(x) = \phi_0(x) = \int_0^1 F(x,y)\,dy$

Other idea

$$e^{2\pi i x y} F(x,y) = \sum_n e^{2\pi i (x+n) y} f(x+n)$$

$$\int_0^1 e^{2\pi i xy} F(x,y)\,dx = \int_0^1 \sum_n e^{2\pi i (x+n)y} f(x+n)\,dx$$

$$= \sum_n \int_n^{n+1} e^{2\pi i (\cancel{x+n})y} f(\cancel{x+x})\,dx = \underbrace{\int_{-\infty}^{\infty} e^{2\pi i xy} f(x)\,dx}_{\hat{f}(y)}$$

Repeat: Start with $F(x,y)$ C^∞-section of line bdl.

$$F(x,y+1) = F(x,y)$$
$$F(x+1,y) = e^{-2\pi i y} F(x,y)$$

ex. $F(x,y) = \sum_n e^{2\pi i n y} f(x+n)$, $f \in \mathcal{S}(\mathbb{R})$

$$F(x,y) = \sum_n e^{2\pi i n y} f_n(x) \qquad f_n(x) = \int_0^1 e^{-2\pi i n y} F(x,y)\,dy$$

Yes. $\qquad f_{n-1}(x+1) = \int_0^1 e^{-2\pi i (n-1) y} F(x+1,y)\,dy$

$\therefore$ $f_n(x) = f_0(x+n)$ where $f_0(x) = \int_0^1 F(x,y)\,dy$

$$e^{2\pi i xy} F(x,y) = e^{2\pi i xy} e^{2\pi i y} F(x+1,y) = e^{2\pi i (x+1)y} F(x+1,y)$$

periodic in x. Also clear from

$$e^{2\pi i xy} F(x,y) = \sum_n e^{2\pi i (x+n)y} f(x+n)$$

Use same arg

$$e^{2\pi i xy} F(x,y) = \sum_k e^{2\pi i k x} g_k(y)$$

$$g_k(y) = \int_0^1 e^{2\pi i \cancel{x}(y-k)} F(x,y)\,dx = g_{k+1}(y+1)$$

$$\therefore\ g_k(y) = g_0(y+k) \qquad g_0(y) = \int_0^1 e^{2\pi i xy} F(x,y)\,dx$$

$$F(x,y) = \sum_n e^{2\pi i n y} f(x+n) \qquad F(x,y+1) = F(x,y)$$

$$F(x+1,y) = \sum_n e^{2\pi i \cancel{n}^{(n-1)} y} f(x\cancel{+1}+n) = e^{-2\pi i y} F(x,y)$$

Conversely given $F(x,y)$ smooth with these properties

expand

$$\underset{\parallel}{F(x,y)} = \sum_n e^{2\pi i n y} f_n(x)$$

then

$$e^{2\pi i y} F(x+1,y) = \sum_n e^{2\pi i \cancel{n}^{n+1} y} f_n(x+1)$$

$$= \sum_n e^{2\pi i n y} f_{n-1}(x+1)$$

$$\therefore \quad f_n(x) = f_{n-1}(x+1) = \cdots = f_0(x+n)$$

where $f_0(x) = \int_0^1 F(x,y)\,dy$

$$f_n(x) = \int_0^1 e^{-2\pi i n y} F(x,y)\,dy$$

$$= \int_0^1 e^{-2\pi i (n+1) y} \underbrace{e^{+2\pi i y} F(x,y)}_{F(x-1,y)} dy = f_{n+1}(x-1)$$

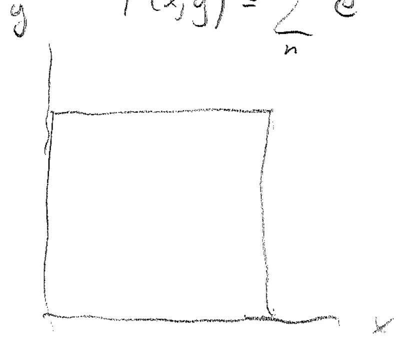
$$f_n(x) = f_{n-1}(x+1) = \cdots = f_0(x+n)$$

Poisson Summ. Formula $\qquad \mathbb{R}^2/\mathbb{Z}^2$

$$F(x,y) = \sum_n e^{2\pi i n y} f(x+n)$$

$$F(x,y+1) = F(x,y)$$

$$F(x+1,y) = e^{-2\pi i y} F(x,y)$$



$f(x)$

There should be a good idea here. Interpolation from $\mathbb{Z}$ to $\mathbb{R}$ should generalize from $\mathbb{Q}$ to $\mathbb{A}$. You should understand interpolating from $\mathbb{Z}$ to $\mathbb{Z}\frac{1}{N}$. Question: You have seen how to obtain the F.T. on $\mathbb{R}$ from the extension $\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ and two applications of F.S. The question is whether you can apply this idea in passing from the discrete to the half continuous grid space.

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interpolation: $p(z) = \prod_i (z-a_i)$ $\quad a_i$ dist.

deg $f(z) <$ degree $p(z)$. Then

$$\frac{f(z)}{p(z)} = \sum_i \frac{1}{z-a_i}\frac{f(a_i)}{p'(a_i)} \qquad \text{partial fractions}$$

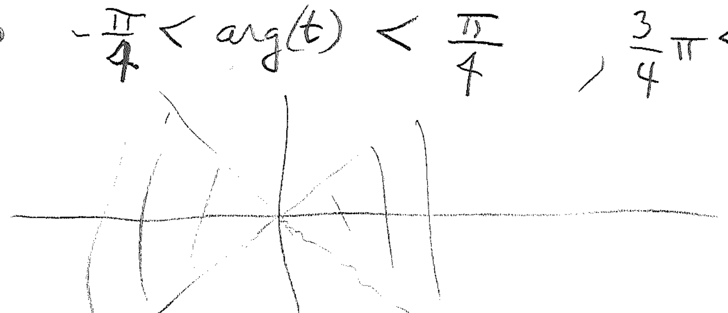
If deg f. arb. then $\quad f(z) = qp + r \quad$ division alg

$$\frac{f(z)}{p(z)} = q(z) + \sum_i \frac{1}{z-a_i}\frac{f(a_i)}{p'(a_i)}$$

because $\deg r < \deg p$ and $f(a_i) = r(a_i)$

Question: Is there a holomorphic Poincare line bundle over $\mathbb{C}^\times \times \mathbb{C}^\times$? Probably not, but you should understand why. It might be better to understand the interpolation situation, what's special about $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{C}$, the kind of entire functions corresponding to finite bandwith signals.

Let's start with $f(t) = e^{-\frac{t^2}{2}}$ $\quad t \in \mathbb{C}$ which decays faster than exponentially as $t \to \infty$ along the rays $-\frac{\pi}{4} < \arg(t) < \frac{\pi}{4}$, $\frac{3}{4}\pi < \arg(t) < \frac{5}{4}\pi$



You should be able to form

$$F(t,u) = \sum_n u^n f(t+n)$$

which should be holomorphic on $\mathbb{C} \times \mathbb{C}^\times$

$$e^{-(t+n)^2/2} = e^{-\frac{t^2}{2} - nt - \frac{n^2}{2}}$$

Maybe go back to the notation.

$$F(t,\omega) = \sum_n e^{2\pi i n\omega} f(t-n)$$

$$F(t,\omega+1) = F(t,\omega)$$

$$F(t+1,\omega) = \sum_n e^{2\pi i n\omega} f(t+1-n)$$

$$= \sum_m e^{2\pi i(m+1)\omega} f(t - \overbrace{(n-1)}^{m})$$

$$= e^{2\pi i\omega} F(t,\omega)$$

So it seems there are entire holomorphic functions of two complex variables $F(t,\omega)$ satisfying $F(t,\omega+1) = F(t,\omega)$ and $F(t+1,\omega) = e^{2\pi i t} F(t,\omega)$. Since F is periodic in ω, such an F should have a Laurent series expansion

$$F(t,\omega) = \sum_n e^{2\pi i n\omega} f_n(t)$$

where

$$f_n(t) = \int_0^1 e^{-2\pi i n\omega} \underbrace{F(t,\omega)}\, d\omega$$

$$= \int_0 e^{-2\pi i n\omega}\, e^{2\pi i\omega} F(t-1,\omega)\, d\omega = f_{n-1}(t-1)$$

$$\therefore \quad f_n(t) = f_{n-1}(t-1) = f_{n-2}(t-2) = \cdots = f_0(t-n)$$

Similarly you can introduce the factor $e^{2\pi i t\omega}$ to shift the periodicity

$$G(t,\omega) = e^{-2\pi i t\omega} F(t,\omega)$$

$$G(t-1,\omega) = e^{-2\pi i \omega t} \underbrace{e^{2\pi i\omega} F(t-1,\omega)}_{F(t,\omega)} = G(t,\omega)$$

Review the interpolation formula (sampling theorem)
What you know how to do. Generalize polyn. interpolation to the case $p(t) = \sin(\pi t)$ which is entire having simple zeroes at $t \in \mathbb{Z}$, no other zeroes. Let $f(t)$ be entire, then $\frac{f(t)}{\sin(\pi t)}$ is meromorphic $\sim \frac{1}{t-n} \frac{f(n)}{\pi\cos(\pi n)} = \frac{1}{t-n}\frac{(-1)^n}{\pi} f(n)$

so

$$* \quad \frac{\pi f(t)}{\sin(\pi t)} \underset{\text{formally}}{=} \sum_n \frac{f(n)(-1)^n}{t-n} + \text{entire}$$

Basic assumption: $f(t)$ is a finite bandwith signal

F.T. Review what you have learned.

$$\int_{-a}^{a} e^{i\omega t} \frac{d\omega}{2a} = \left[\frac{e^{i\omega t}}{it}\frac{1}{2a}\right]_{\omega=-a}^{\omega=a} = \frac{e^{iat}-e^{-iat}}{2iat}$$

$$= \frac{\sin(at)}{at}$$

Write $*$ above in the form.

$$f(t) = \underbrace{\sum_n \frac{\sin\pi(t-n)}{\pi(t-n)} f(n)}_{\text{this is the interpolation}} = \sum_n \int_{-\pi}^{\pi} e^{i\omega(t-n)} f(n) \frac{d\omega}{2\pi}$$

$$= \int_{-\pi}^{\pi} e^{i\omega t} \underbrace{\sum_n e^{-i\omega n} f(n)}_{\text{the Fourier series with Fourier coeffs } f(n)} \frac{d\omega}{2\pi}$$

But now use Poisson summ. fmla. Suppose

$$f(t) = \int_{-\infty}^{\infty} e^{i\omega' t} \hat{f}(\omega') \frac{d\omega'}{2\pi}$$

$$\sum_n e^{-i\omega n} f(n) = \int_{-\infty}^{\infty} \underbrace{\sum_n e^{-i\omega n} e^{i\omega' n}}_{\sum_k \delta(\omega' - \omega - 2\pi k)} \hat{f}(\omega') \frac{d\omega'}{2\pi}$$

$$= \sum_{k\in\mathbb{Z}} f(\omega + 2\pi k)$$

This seems correct but there's a lot to get straight.

Let's look at the analog of P. Sum. in the case of

$L\mathbb{Z} \longrightarrow \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/L\mathbb{Z}$. Handle like $\mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z}$

Given discrete signal $f(n), n \in \mathbb{Z}$ you can sum its translates $\sum_{m\in\mathbb{Z}} f(n + Lm)$ to get a fn on $\mathbb{Z}/L\mathbb{Z}$, this is $\pi_! f$. Can also combine with character of $L\mathbb{Z}$

It seems better to use $\mathbb{Z} \hookrightarrow \mathbb{Z}\frac{1}{L} \twoheadrightarrow \mathbb{Z}\frac{1}{L}/\mathbb{Z}$ sitting inside of $\mathbb{Z} \hookrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$. Then use formulas

$$F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i n y} f(x+n) \qquad x \in \mathbb{Z}\tfrac{1}{L} \subset \mathbb{R}/\mathbb{Z}$$

period 1 for y

$$e^{2\pi i x y} F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i (x+n) y} f(x+n) \qquad \text{period 1 for } x.$$

---

maybe do finite interpolation, really circular interpolation ~~...~~. so you have f given on L-th roots of unity and you want ~~the~~ to extend to the unit circle

It should involve poly. interpolation

$$z^L - 1 = p(z) = \prod_{j=0}^{L-1} (z - e^{2\pi i j/L}) = \prod_{j=0}^{L-1} (z - \zeta^j) \qquad \zeta = e^{2\pi i/L}$$

But maybe F.T. methods are better

$\dfrac{z^L - 1}{z - 1}$ vanishes at $z = \zeta^j$, $1 \le j \le L-1$
$= 1$ at $z = 1$.

You want to introduce $z^{1/2}$ ~~to~~ to get Dirichlet type kernel

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Focus on the problem. Consider the space of sequences $(x_n)_{n \in \mathbb{Z}}$. Translation $T^\ell : (x_n)_n \longmapsto (x_{n-\ell})_n$ and mult. $M_\zeta : (x_n) \longmapsto (\zeta^n x_n)_{n \in \mathbb{Z}}$ should commute if $\zeta^\ell = 1$. Use z transform $\hat{x}(z) = \sum_{n \in \mathbb{Z}} z^n x_n$

$$\widehat{T^\ell(x_n)} = \widehat{(x_{n-\ell})_n} = \sum_n z^n x_{n-\ell} = z^\ell \sum z^n x^n = z^\ell \hat{x}(z)$$

$$\widehat{M_\zeta(x_n)} = \sum_n z^n \zeta^n x_n = \hat{x}(\zeta z)$$

$$\left(T^\ell M_\zeta T^{-\ell} \hat{x}\right)(z) = z^\ell (T^{-\ell}\hat{x})(\zeta z)$$
$$= z^\ell (\zeta z)^{-\ell} \hat{x}(\zeta z) = \zeta^{-\ell} \hat{x}(\zeta z) = \zeta^{-\ell}(M_\zeta \hat{x})(z)$$

really unclear.

Something confusing. You remember the two commuting autos on $\mathcal{S}(\mathbb{R})$ given by translation and multiplication by a character.

$$T : f(x) \longmapsto f(x-1)$$
$$U : f(x) \longmapsto e^{2\pi i x} f(x).$$

$$f(x) \longmapsto f(x-1) \longmapsto e^{2\pi i x} f(x-1)$$
$$f(x) \longmapsto e^{2\pi i x} f(x) \longmapsto e^{2\pi i (x-1)} f(x-1)$$

OKAY you need some progress. Return to

$$F(x,y) = \sum_n e^{2\pi i n y} f(x+n) \qquad F(x,y+1) = F(x,y)$$

$$e^{2\pi i x y} F(x,y) = \sum_n e^{2\pi i (x+n)} f(x+n) \qquad \text{periodic in } x$$

$$e^{2\pi i (x+1) y} F(x+1,y) = e^{2\pi i x} F(x,y)$$

$$\therefore \boxed{F(x+1,y) = e^{-2\pi i y} F(x,y)}$$

$$f_n(x) = \int_0^1 e^{-2\pi i n y} F(x,y)\, dy = \int_0^1 e^{-2\pi i (n-1) y} F(x+1,y)\, dy$$
$$= f_{n-1}(x+1)$$

$$\therefore f_n(x) = f_0(x+n) \qquad \text{etc.}$$

Your idea now is to restrict x, to $\mathbb{Z}\frac{1}{L}$. Better: In the $\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ case you can recover $S(\mathbb{R})$ using $\mathbb{Z}$ translations and characters on $\mathbb{Z}$. You want to handle $\mathbb{Z} \longrightarrow \mathbb{Z}\frac{1}{L} \longrightarrow \mathbb{Z}\frac{1}{L}/\mathbb{Z}$. You want to replace $\mathbb{R}/\mathbb{Z}$ by $\mathbb{Z}\frac{1}{L}/\mathbb{Z}$, restriction of z transforms to roots of 1. Can you use the same formulas as above?

$$F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i n y} f(x+n) \qquad x \in \mathbb{Z}\tfrac{1}{L}$$

Try again. The basic gadget is $f \mapsto \sum f(x+n)$

First point is to describe discrete signals starting with the splitting of $\mathbb{Z}$ into cosets $x + \mathbb{Z}N$ $\quad x = 0, \dots, N-1$

$$\sum_{k\in\mathbb{Z}} \zeta^k f(x+kN)$$

Discrete signal $(f(n), n \in \mathbb{Z})$ $\qquad \mathbb{Z} = \coprod_{r=0}^{L-1} r + L\mathbb{Z}$

z-transform $\displaystyle\sum_{n\in\mathbb{Z}} z^{-n} f(n) = \sum_{r=0}^{L-1} \sum_{g\in\mathbb{Z}} z^{-r-gL} f(r+gL)$

$n = gd + r$

The aim is to recover discrete signals $(f(n), n\in\mathbb{Z})$ as functions on $\widehat{\mathbb{Z}}d$

$u = e^{2\pi i y}$

$$F(x,y) = \sum_n e^{2\pi i n y} f(x+n) = \sum_n u^n f(x+n)$$

$$f_n(x) = \int_0^1 e^{-2\pi i n y} F(x,y)\, dy \qquad f_0(x+n)$$

$$= \int_0^1 e^{-2\pi i (n-1) y} \underbrace{e^{-2\pi i y} F(x,y)}_{F(x+1,y)}\, dy = f_{n-1}(x+1)$$

$$\mathbb{Z} \hookrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$$

$$\mathbb{Z} \hookrightarrow \mathbb{Z}\tfrac{1}{L} \longrightarrow \mathbb{Z}\tfrac{1}{L}/\mathbb{Z}$$

$\mathbb{Z}\tfrac{1}{L} \ni x$

$x \in \mathbb{Z}\frac{1}{L}$

$y \in \mathbb{R}/\mathbb{Z}$

$$f(x) \longmapsto \sum_n e^{2\pi i n y} f(x+n) = F(x,y)$$

$$F(x, y+1) = F(x,y)$$

$$F(x+1,y) = \sum_n e^{2\pi i n y} f(x+1+n) = \sum_n e^{2\pi i (n-1) y} f(x+n)$$

$$= e^{-2\pi i y} F(x,y)$$

$\mathbb{Z} \hookrightarrow \mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z} \qquad (\pi_! f)(x+\mathbb{Z}) = \sum_{n\in\mathbb{Z}} f(x+n)$

twist with a character, meaning? there are two things you can do: 1) mult. $f(x)$ by the character $e^{2\pi i x y}$, then apply $\pi_!$, yielding $\sum_{n\in\mathbb{Z}} e^{2\pi i (x+n) y} f(x+n)$.

2) restrict $x \mapsto e^{2\pi i x y}$ char on $\mathbb{R}$ to subgp $\mathbb{N}$ use twisted translation: $\sum_n e^{2\pi i n y} f(x+n)$.

You want someway to understand. Maybe assembly idea yields something God-given. Meaning: You have the principal bundle

$$\begin{tikzcd} \mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathbb{R} \arrow[r] & \mathbb{R}/\mathbb{Z} \end{tikzcd}$$

Return to

$$f(t) = \sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n) \quad + \ ?$$

$$\int_{-a}^{a} e^{i\xi t} \frac{d\xi}{2a} = \left[\frac{e^{i\xi t}}{i t\, 2a} \right]_{-a}^{a} = \frac{e^{iat} - e^{-iat}}{2iat} = \frac{\sin(at)}{at}$$

$$\cancel{\int_{-\frac{1}{2}}^{1/2} e^{2\pi i \xi t} \frac{2\pi\, d\xi}{1}} \qquad \int_{-\pi}^{\pi} e^{i\xi t} \frac{d\xi}{2\pi} = \frac{\sin(\pi t)}{\pi t}$$

$$\int_{-\frac{1}{2}}^{1/2} e^{2\pi i \xi t} d\xi = \left[\frac{e^{2\pi i \xi t}}{2\pi i t} \right]_{-1/2}^{1/2} = \frac{e^{\pi i t} - e^{-\pi i t}}{2 i \pi t} = \frac{\sin \pi t}{\pi t}.$$

$$\boxed{\int_{-\frac{1}{2}}^{1/2} e^{2\pi i \xi t} d\xi = \frac{\sin \pi t}{\pi t}}$$

$$\sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n)$$

$$= \sum_n \int_{-1/2}^{1/2} e^{2\pi i \xi (t-n)} d\xi \int e^{2\pi i w n} \phi(w)\, dw$$

$$= \int_{-1/2}^{1/2} d\xi\, e^{2\pi i \xi t} \sum_n e^{-2\pi i \xi n} \int e^{2\pi i w n} \phi(w)\, dw$$

$$S = \sum_n \frac{\sin \pi(t-n)}{t-n} f(n) = \sum_n \int_{-1/2}^{1/2} e^{2\pi i \xi (t-n)} d\xi \; f(n)$$

assume $f(t) = \int e^{2\pi i \omega t} \phi(\omega)\, d\omega$ $\qquad \phi$ compact support

$$S = \sum_n \int_{-1/2}^{1/2} e^{2\pi i \xi t} e^{-2\pi i \xi n} f(n)\, d\xi$$

PS should be used somewhere.

Idea should be to use $\sum_n e^{-2\pi i \xi n} e^{2\pi i \omega n} =$

sum of δ functions $\sum_{k \in \mathbb{Z}} \delta(\xi - \omega - k)$

You take $\phi(\omega)$ into $f(t)$ then restrict to $t \in \mathbb{Z}$

You're missing the link with the line bundle on the 2 torus. Review.

$$F(x,y) = \sum_n e^{2\pi i n y} f(x+n)$$

$$e^{2\pi i x y} F(x,y) = \sum_n e^{2\pi i (x+n) y} f(x+n)$$

Simplest form of P.S. says.

$$\sum_n g(x+n) = \sum_m e^{2\pi i m x} h(m)$$

$$h(m) = \int_0^1 e^{-2\pi i m x} \sum_n g(x+n)\, dx$$

$$= \sum_n \int_n^{n+1} e^{-2\pi i m x} g(x)\, dx = \hat{g}(m)$$

$$g(x) = \int e^{2\pi i x y} h(y)\, dy \qquad h = \hat{g}$$

$$\sum_n g(x+n) = \int e^{2\pi i x y} \underbrace{\sum e^{2\pi i n y}}_{\sum_{m \in \mathbb{Z}} \delta(y-m)} h(y)\, dy = \sum_k e^{2\pi i x m} h(m)$$

$$\int_{-1/2}^{1/2} e^{2\pi i \xi t}\, d\xi = \frac{\sin \pi t}{\pi t}, \qquad \sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n)$$

$$= \sum_n \int_{-1/2}^{1/2} d\xi\; e^{2\pi i \xi(t-n)} \int e^{2\pi i \omega n} \hat{f}(\omega)\, d\omega \qquad \boxed{f(t) = \int e^{2\pi i \omega t} \hat{f}(\omega)\, d\omega}$$

$$\therefore \sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n) = \int_{-1/2}^{1/2} d\xi\, e^{2\pi i \xi t} \boxed{\sum_n e^{-2\pi i \xi n} \int e^{2\pi i \omega n} \hat{f}(\omega)\, d\omega}$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$$

$$\sum_n e^{-2\pi i \xi n} f(n) \leftarrow \text{doesn't help.}$$

Repeat. $$\int_{-1/2}^{1/2} e^{2\pi i \xi t}\, d\xi = \frac{e^{2\pi i \xi t}}{2\pi i t}\Bigg]_{\xi = -\frac{1}{2}}^{\xi = \frac{1}{2}} = \frac{e^{\pi i t} - e^{-\pi i t}}{2\pi i t} = \frac{\sin(\pi t)}{\pi t}$$

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i \omega t}\, \underbrace{\hat{f}(\omega)}_{\phi(\omega)}\, d\omega$$

$$\sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n) = \sum_n \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \xi(t-n)} d\xi \int_{-\infty} e^{2\pi i \omega n} \phi(\omega)\, d\omega$$

$$= \sum_n \int_{-1/2}^{1/2} d\xi\; e^{2\pi i \xi t} \int_{-\infty}^{\infty} d\omega\; e^{-2\pi i \xi n} \qquad ??$$

$$\sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n) = \sum_n \int_{-1/2}^{1/2} e^{2\pi i \xi t}\, e^{-2\pi i \xi n}\, d\xi \int_{-\infty}^{\infty} e^{2\pi i \omega n} \phi(\omega)\, d\omega$$

$$= \int_{-1/2}^{1/2} e^{2\pi i \xi t}\, d\xi \int_{-\infty}^{\infty} d\omega \sum_n e^{-2\pi i \xi n + 2\pi i \omega n}\, \phi(\omega)$$

try again. You probably should do the ω integration last.

Look at frequency picture. The ~~[illegible]~~ convolution operator

$$f(t) \longmapsto f(n) \longmapsto \sum_n G(t-n) f(n) \quad \text{becomes a}$$

mult. op. namely

$$\hat{f}(\omega) \longmapsto \sum_k \hat{f}(\omega + k) \longmapsto \chi_{(-1/2,\, 1/2)}(\omega) \sum_k \hat{f}(\omega + k)$$

which is the identity when $\operatorname{Supp} \hat{f}(\omega)$ is contained in $(-\frac{1}{2}, \frac{1}{2})$.

Look at discrete signal case. You hope to be able to bring in higher order principal parts. Question not answered yet: Is there some link between interpolation for $\mathbb{Z} \subset \mathbb{C}$ and the line bundle over T^2?

Look at discrete signal case. Interpolate from $\mathbb{Z}$ to $\mathbb{Z}\frac{1}{L} \subset \mathbb{R}$. z transform $\mathbb{C}[z, z^{-1}] \subset \mathbb{C}[z^{1/L}, z^{-1/L}]$. Shift to $\mathbb{C}[z^L, z^{-L}] \subset \mathbb{C}[z, z^{-1}]$. Given discrete signal $f(n), n \in \mathbb{Z}$ with z-transf $\hat{f}(z) = \sum_n f(n) z^{-n}$ you can restrict $f(n)$ to $\mathbb{Z}L$ and interpolate. Meaning? Various candidates, extend $f(mL), m \in \mathbb{Z}$ by 0 to the other cosets $\mathbb{Z}L + r$

$$\mathbb{Z}L \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/\mathbb{Z}L$$

$$(z^L - 1) \rightarrowtail \mathbb{C}[z, z^{-1}] \twoheadrightarrow \mathbb{C}[\mu_L]$$

Aim to understand interpolation from a function on $\mathbb{Z}L$ (a sampled disc signal) to a function on $\mathbb{Z}$. This is the opposite (adjoint) of restriction from $\mathbb{Z}$ to $L\mathbb{Z}$. Let's follow the $\mathbb{Z} \subset \mathbb{C}$ case. Shift to $\mathbb{Z} \subset \mathbb{Z}L^{-1}$. Interp. fmla $\sum_{n \in \mathbb{Z}} K(t-n) f(n)$ $t \in \mathbb{Z}\frac{1}{L}$.

What should K be? $K(t)$ is a function on $\mathbb{Z}\frac{1}{L}$ equal to 1 for $t=0$ and $=0$ for $t \in \mathbb{Z} - \{0\}$. Some convenient choice for $t \in \mathbb{Z}\frac{1}{L} - \mathbb{Z}$. Idea: Use transitivity

$$\mathbb{Z} \overset{a}{\subset} \mathbb{Z}\tfrac{1}{L} \overset{b}{\subset} \mathbb{C}$$

Define $a_! f = b^*(ba)_! f = \underbrace{b^* b_!}_{1} a_! f$

$$\mathbb{Z} \longrightarrow \mathbb{C}$$

$$f(n) \longmapsto \sum_n \frac{\sin \pi(t-n)}{\pi(t-n)} f(n)$$

$$\hat{f}(z) \longmapsto \chi_{(-\frac{1}{2},\frac{1}{2})}(\omega)\, \hat{f}(e^{2\pi i \omega}) \overset{?}{=} \chi_{(-\frac{1}{2},\frac{1}{2})}(\omega) \sum_k \hat{f}(\omega+k)$$

compare p 807

There's a mistake, contradiction

$$G(t) = \frac{\sin \pi t}{\pi t} = \int_{-1/2}^{1/2} d\xi\, e^{2\pi i \xi t} = \widehat{\chi_{(-\frac{1}{2},\frac{1}{2})}}(-t)$$

$$\int dt\, e^{-2\pi i \omega t} \sum_n G(t-n) f(n) = \sum_n \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} \int_{-1/2}^{1/2} d\xi\, e^{2\pi i \xi (t-n)} f(n)$$

$$= \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} \int_{-\infty}^{\infty} d\xi\, \chi_{(-\frac{1}{2},\frac{1}{2})}(\xi)\, e^{2\pi i \xi t} \underbrace{\sum_n e^{-2\pi i \xi n} f(n)}_{\hat{f}(e^{2\pi i \xi})}$$

$$= \int_{-\infty}^{\infty} d\xi\, \chi_{(-\frac{1}{2},\frac{1}{2})}(\xi) \qquad ?$$

$f(n)$ disc signal $\qquad \hat{f}(z) = \sum_n z^{-n} f(n) \qquad z = e^{2\pi i \omega}$

$$G(t) = \frac{\sin \pi t}{\pi t} = \int_{-1/2}^{1/2} d\xi\, e^{2\pi i \xi t} = \int_{-\infty}^{\infty} d\xi\, e^{2\pi i \xi t} \chi_{(-\frac{1}{2},\frac{1}{2})}(\xi)$$

$$\sum_n G(t-n) f(n) = \sum_n \int_{-\infty}^{\infty} d\xi\, e^{2\pi i \xi (t-n)} \chi_{(-\frac{1}{2},\frac{1}{2})}(\xi) \int d\omega\, e^{2\pi i \omega n} \hat{f}(e^{2\pi i \omega})$$

$$= \int_{-\infty}^{\infty} d\xi\, e^{2\pi i \xi t} \chi_I(\xi) \int_{-\infty} d\omega \sum_n e^{2\pi i (\omega - \xi) n} \hat{f}(e^{2\pi i \omega})$$

$f = (f(n), n \in \mathbb{Z})$ discrete signal

$$F(t) = \sum_{n\in\mathbb{Z}} G(t-n) f(n) \,, \qquad G(t) = \int_{-1/2}^{1/2} d\xi \; e^{2\pi i \xi t} = \frac{\sin(\pi t)}{\pi t}$$

Idea is to find the frequency repr. for $F(t)$, frequency picture.

$$\hat{F}(\omega) = \int_{-\infty}^{\infty} dt \; e^{-2\pi i \omega t} F(t) =$$

$$\int_{-\infty}^{\infty} dt \; e^{-2\pi i \omega t} \sum_n \int_{-\infty}^{\infty} d\xi \, \chi_I(\xi) e^{2\pi i \xi (t-n)} f(n)$$

$$\hat{f}(z) = \sum_n z^{-n} f(n) \quad \text{where } z = e^{2\pi i \xi}$$

$$\int_{-\infty}^{\infty} dt \; e^{-2\pi i \omega t} \int_{-\infty} d\xi \; \chi_I(\xi) e^{2\pi i \xi t} \sum_n e^{-2\pi i \xi n} f(n)$$

$$\iint dt\, d\xi \;\; \chi_I(\xi) \; e^{2\pi i \xi (t-\omega)}$$. Still unclear. You want $\hat{F}(\omega)$.

Focus on the problem. You have a continuous output from a discrete input

$$\int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} \sum_{n\in\mathbb{Z}} G(t-n) f(n) = \sum_{n\in\mathbb{Z}} \underbrace{\int_{-\infty}^{\infty} dt \;\; e^{-2\pi i \omega t} G(t-n)}\, f(n)$$

$$\underbrace{\hat{G}(\omega)}_{\chi_{(-\frac{1}{2},\frac{1}{2})}(\omega)} \underbrace{\sum_n e^{-2\pi i \omega n} f(n)}_{\hat{f}(e^{2\pi i \omega})} = \sum_n \int_{-\infty}^{\infty} dt \;\; e^{-2\pi i \omega (t+n)} G(t) f(n)$$

Conclusion: the convolution operator

$$\underset{\text{disc. signals}}{(f(n), n\in\mathbb{Z})} \longmapsto \underset{\text{cont. signals}}{F(t) = \sum_n G(t-n) f(n)}$$

on the F.T. side becomes

$$\hat{f}(z) \longmapsto \hat{G}(\omega) \hat{f}(z) \qquad z = e^{2\pi i \omega}$$

So what have you learned? Consider a convolution operator from sequences $(x(n), n \in \mathbb{Z})$ to functions $(y(t), t \in \mathbb{R})$:

$$y(t) = \sum_n G(t-n)\, x(n).$$

You want the corresponding frequency picture. Corresp to the sequence $x(n)$ is its z transform $\sum_n z^{-n} x(n) = \hat{x}(z)$ and corresp to the function $y(t)$ is its Fourier transform

$$\hat{y}(\omega) = \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} y(t) = \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} \sum_n G(t-n)\, x(n)$$

$$= \sum_n \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} G(t-n)\, x(n)$$

$$= \sum_n \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega (t+n)} G(t)\, x(n) = \hat{G}(\omega) \sum_n e^{-2\pi i \omega n} x(n)$$

$$\therefore \quad \boxed{\hat{y}(\omega) = \hat{G}(\omega)\, \hat{x}(z) \qquad z = e^{2\pi i \omega}}$$

the interpolation given by the sampling thm. is the case when

$$G(t) = \int_{-1/2}^{1/2} e^{2\pi i \omega t}\, d\omega = \left[\frac{e^{2\pi i \omega t}}{2\pi i t} \right]_{\omega=-1/2}^{\omega=1/2} = \frac{e^{i\pi t} - e^{-i\pi t}}{2\pi i t} = \frac{\sin(\pi t)}{\pi t}$$

$$\hat{G}(\omega) = \chi_{(-\frac{1}{2},\frac{1}{2})}(\omega)$$

Thus $\hat{y}(\omega) = \chi_{(-\frac{1}{2},\frac{1}{2})}(\omega)\, \hat{x}(e^{2\pi i \omega})$. The good case is where the discrete signal $x(n)$ has z transform vanishing at $z = -1$, or $\omega = \pm\frac{1}{2}$, whence $\hat{y}(\omega)$ is $\hat{x}(e^{2\pi i \omega})$ extended by zero outside of $-\pi \le \omega < \pi$.

Observe the similarity with Overlap-Add or Save

Is there a discrete loop group theory?

time domain, frequency domain,

by using the fast F.T. algorithm fewer computations are required to compute the output sequence in the frequency domain than in the time domain.

finite energy signals have continuous spectra

$$\sum_{n\in\mathbb{Z}} |x(n)|^2 < \infty \implies \hat{x}(z) = \sum_{n\in\mathbb{Z}} z^{-n} x(n) \in L^2(S^1)$$

motivate DFT using sampling in the frequency domain

$$\cancel{X(\omega) = \sum_n e^{-2\pi i \omega n} x(n)}$$

$$\cancel{X\left(\frac{k}{N}\right) = \sum_n e^{-2\pi i kn/N} x(n)} \qquad \zeta = e^{2\pi i/N}$$

$$X(\omega) = \sum_m e^{-2\pi i \omega m} x(m) \qquad \omega = \frac{k}{N} \qquad k = 0, \dots, N-1$$

$$e^{-2\pi i \frac{k}{N} m}$$ split $\{m \in \mathbb{Z}\}$ into cosets $r + N\mathbb{Z}$

on such a coset the phases are the same.

$$\frac{k}{N}(qN + r) = kq + \frac{r}{N}$$

$$X\left(\frac{k}{N}\right) = \sum_{q\in\mathbb{Z}} \sum_{r=0}^{N-1} e^{-2\pi i \frac{r}{N}} x(r + qN)$$

$$= \sum_{r=0}^{N-1} e^{-2\pi i \frac{r}{N}} \underbrace{\sum_{q\in\mathbb{Z}} x(r + qN)}_{\text{sum over coset.}}$$

Note that for {interpolation / sampling} one seems to use $\mathbb{Z} \subset \mathbb{C}$ rather than any coset $t + \mathbb{Z}$.

Big problem: to link interpolation with Poisson summation. P.S. $f(t) \in \mathcal{S}$ same as $F(x,y)$ C^∞ sat

$$F(x, y+1) = F(x,y), \quad F(x+1, y) = e^{-2\pi i y} F(x,y)$$

$$f_n(x) = \int_0^1 e^{-2\pi i n y} F(x,y)\,dy = \int_0^1 e^{-2\pi i n y} e^{2\pi i y} F(x+1,y)\,dy = f_{n-1}(x+1)$$

$$= f_{n-2}(x+2) = \cdots = f_0(x+n).$$

$$F(x,y) = \sum_n e^{2\pi i n y} f_0(x+n) = \sum_n e^{2\pi i y} e^{2\pi i n y} f_0(x+1+n)$$

Do the same sort of thing, but restrict $x \in \mathbb{Z}\frac{1}{L}$.
So $F(x,y)$ is defined on $\mathbb{Z}\frac{1}{L} \times \mathbb{R}$, periodic in y, again require $F(x,y) = e^{2\pi i y} F(x+1,y)$, whence smoothness in y will imply $f = f_0$ decays. Then $e^{2\pi i x y} F(x,y)$ is periodic in x. $e^{2\pi i x y} e^{2\pi i y} F(x+1,y) = e^{2\pi i x} F(x,y)$.
Seems OK.

Next can you describe (decaying) functions on $\mathbb{Z}\frac{1}{L}$?

It seems that a decaying functions $f(x)$ on $\mathbb{Z}\frac{1}{L}$ has a transform $\sum_n e^{2\pi i n y} f(x+n) = F(x,y)$ which is smooth + periodic in y and satisfies $e^{2\pi i x y} F(x+1,y) = F(x,y)$

Example: $L = 2$. $x \in \mathbb{Z} \amalg \left(\frac{1}{2} + \mathbb{Z}\right)$

$$\mathbb{Z} \hookrightarrow \mathbb{Z}L^{-1} \twoheadrightarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

$$\mathbb{C}/\mathbb{Z} \twoheadleftarrow \mathbb{C}/\mathbb{Z}L^{-1} \hookleftarrow \mu_L$$

You should be finding the frequency picture corresponding to a time picture. First look at $\mathbb{Z} \subset \mathbb{R}$; this means you are comparing discrete time signals $(x(n), n \in \mathbb{Z})$ with cont. signals signals $(y(t), t \in \mathbb{R})$. Frequency pictures $X(z) = \sum_{n\in\mathbb{Z}} z^{-n} x(n)$

$Y(\omega) = \int e^{-2\pi i \omega t} y(t)\,dt$. Sampling operation $y(t) \mapsto y(n)$

$$\hat{y}(\omega) = \int e^{-2\pi i \omega t} y(t)\,dt \qquad \sum e^{-2\pi i n \omega} y(n)$$

$$\hat{y}(\omega) \longmapsto \int_{-\infty}^{\infty} e^{2\pi i \omega t} \hat{y}(\omega)\,d\omega$$

$$\int_{-\infty}^{\infty} e^{2\pi i \omega n} \hat{y}(\omega)$$

?

$\mathbb{Z} \subset \mathbb{R}$ disc signal $(x(n), n \in \mathbb{Z})$ obtained by sampling in the time domain a cont. signal $(y(t), t \in \mathbb{R})$

$$\hat{X}(z) = \sum_n z^{-n} x(n) \qquad \hat{Y}(\omega) = \int e^{-2\pi i \omega t} y(t)\,dt$$

$$y(t) = \int e^{2\pi i \omega t} \hat{y}(\omega)\,d\omega$$

You want to understand the operation defined in the time domain by restricting the cont. signal $y(t)$ to the integers to get a discrete signal.

$$y(t) = \int e^{2\pi i \omega t} \hat{y}(\omega)\,d\omega \qquad x(n) = \int_0^1 e^{+2\pi i \theta n} \hat{x}(\theta)\,d\theta$$

$$\updownarrow \qquad\qquad \updownarrow$$

$$\hat{y}(\omega) = \int e^{-2\pi i \omega t} y(t)\,dt \qquad \hat{x}(\theta) = \sum_n e^{-2\pi i \theta n} x(n)$$

$$\hat{x}(\theta) = \sum_n e^{-2\pi i \theta n} \underbrace{\int_{-\infty}^{\infty} e^{2\pi i \omega n} \hat{y}(\omega)\,d\omega}_{y(n)}$$

$$= \int_{-\infty}^{\infty} \sum_n e^{-2\pi i (\theta - \omega) n} \hat{y}(\omega)\,d\omega = \sum_{k \in \mathbb{Z}} \hat{y}(\omega + k)$$

---

situation you don't understand. Begin with $x = (x(n), n \in \mathbb{Z})$ form $\hat{x}(\theta) = \sum_n e^{-2\pi i n \theta} x(n)$ and $f(t)$, form $\hat{f}(\omega) = \int e^{-2\pi i \omega t} f(t)\,dt$, Then $f(t) = \int e^{2\pi i \omega t} \hat{f}(\omega)\,d\omega$

$$f(n) = \int e^{2\pi i \omega n} \hat{f}(\omega)\,d\omega \quad \Big| \quad \sum_n e^{-2\pi i \theta n} f(n) = \int \sum_n e^{2\pi i (\omega - \theta) n} \hat{f}(\omega)\,d\omega$$

$$= \sum_m \hat{f}(\theta + m)$$

$$\boxed{\sum_n e^{-2\pi i \theta n} f(n) = \sum_m \hat{f}(\theta + m)}$$

$f(x) \in \mathcal{S}(\mathbb{R})$. $\sum_{n\in\mathbb{Z}} f(x+n)$ has period 1

so $\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x} a_m$

$$a_m = \int_0^1 e^{-2\pi i m x} \sum_n f(x+n)\,dx$$

$$= \sum_n \int_0^1 e^{-2\pi i m x} f(x+n)\,dx = \sum_n \int_n^{n+1} e^{-2\pi i m x} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i m x} f(x)\,dx = \hat{f}(m)$$

$$\boxed{\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x} \hat{f}(m)}$$

$$F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i n y} f(x+n) = \sum_n e^{2\pi i (n+1) y} f(x+1+n) = e^{2\pi i y} F(x+1,y)$$

Put $f_n(x) = \int_0^1 e^{-2\pi i n y} F(x,y)\,dy =$

$\parallel$

$$f_{n-1}(x+1) = \int_0^1 \underbrace{e^{-2\pi i (n-1) y} F(x+1,y)}_{e^{-2\pi i n y}\,\underbrace{e^{2\pi i y} F(x+1,y)}_{F(x,y)}}\,dy$$

Make ~~x~~ periodic wrt x

$G(x+1,y) = G(x,y)$

$G(x,y+1) = e^{2\pi i x} G(x,y)$

$$G(x,y) = e^{2\pi i x y} F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i (x+n) y} f(x+n)$$

$\parallel$

$$\sum e^{2\pi i m x} g_m(y) \qquad g_m(y) = \int_0^1 e^{-2\pi i m x} \cancel{e^{2\pi i x y} F(x,y)}\; G(x,y)\,dx$$

$$g_m(y) = g_0(y-m) \qquad g_m(y+1) = \int_0^1 e^{-2\pi i m x} e^{2\pi i x} G(x,y)\,dx$$

$$G(x,y) = \sum e^{2\pi i m x} g_0(y-m) \qquad = g_{m-1}(y)$$

$$g_0(y) = \int_0^1 e^{2\pi i x y} \sum e^{2\pi i n y} f(x+n)\,dx = \int_0^1 \sum e^{2\pi i (x+n) y} f(x+n)\,dx \;\; \parallel \;\; \int_{-\infty}^{\infty} e^{2\pi i x y} f(x)\,dx$$

Sample in time domain

$$f(t) \longmapsto f(n)$$

$$\hat{f}(\omega) \longmapsto \sum_m \hat{f}(\omega+m)$$

easy direction

Proof: $$\int_0^1 d\omega\, e^{+2\pi i \omega n} \sum_m \hat{f}(\omega+m) = \sum_m \int_0^1 d\omega\, e^{+2\pi i \omega n} \hat{f}(\omega+m)$$

$$= \sum_m \int_m^{m+1} d\omega\, e^{+2\pi i (\omega - \cancel{m}) n} \hat{f}(\omega) = \int_{-\infty}^{\infty} d\omega\, e^{+2\pi i \omega n} \hat{f}(\omega) = f(n)$$

hard direction

$$\sum_n e^{-2\pi i \theta n} f(n) = \sum_n e^{-2\pi i \theta n} \int_{-\infty}^{\infty} d\omega\, e^{2\pi i \omega n} \hat{f}(\omega)$$

$$= \int_{-\infty}^{\infty} d\omega\, \underbrace{\sum_n e^{2\pi i (\omega-\theta) n}}_{\sum_m \delta(\omega-\theta-m)} \hat{f}(\omega) = \sum_m \hat{f}(\theta+m)$$

---

easy direction $$\int_{-1/2}^{1/2} d\theta\, e^{-2\pi i n \theta} \sum_n f(\theta+n) = \sum_n \int_{-1/2}^{1/2} d\theta\, e^{-2\pi i m \theta} f(\overset{\varphi}{\theta+n})$$

$$= \sum_n \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} d\varphi\, e^{-2\pi i m (\varphi - \cancel{n})} f(\varphi) = \int_{-\infty}^{\infty} d\varphi\, e^{-2\pi i m \varphi} f(\varphi) = \hat{f}(m)$$

hard direction - why is it hard?

$$\sum_n e^{-2\pi i \theta n} f(n) = \int_{-\infty}^{\infty} d\omega\, \underbrace{\sum_n e^{2\pi i (\omega-\theta) n}}_{\sum_{m\in\mathbb{Z}} \delta(\omega-\theta-m)} \hat{f}(\omega) = \sum_{m\in\mathbb{Z}} \hat{f}(\theta+m)$$

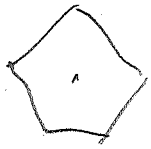
What do I do to organize things?

First try to make sense out of interpolation from $\mathbb{Z}$ to $\mathbb{Z}L^{-1}$. How should you view interpolation from $\mathbb{Z}$ to $\mathbb{R}$. (or $\mathbb{C}$)? some kind of adjoint to the restriction.

space of (fine) signals $(x(t), t \in \mathbb{Z}L^{-1})$

(YOU SHOULD LEARN ABOUT LINKING DIFFERENT SAMPLING RATES)

fine signals $(x(t), t \in \mathbb{Z}\frac{1}{L})$
disc. " $(y(n), n \in \mathbb{Z})$



D space of "disc" signals $(g(n), n\in\mathbb{Z}) \longleftrightarrow \sum_{n\in\mathbb{Z}} z^{-n} g(n)$, $z \in \mathbb{R}/\mathbb{Z}$

F space of "fine" disc. signals $(f(t), t \in \mathbb{Z}L^{-1}) \longleftrightarrow \sum_{t\in\mathbb{Z}L^{-1}} w^{-t} f(t)$, $w \in \mathbb{R}/\mathbb{Z}$

In the frequency domain you have functions on the circle $\mathbb{R}/\mathbb{Z}L^{-1}$ compared with functions on the L-fold covering $\mathbb{R}/\mathbb{Z}$. You have the Galois group of the covering which is $\mathbb{Z}L^{-1}/\mathbb{Z}$ acting by translations. What is your aim?

$K = \mathbb{Z}L^{-1}/\mathbb{Z}$ cycle order L

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \to & \mathbb{Z} & \to & \mathbb{R} & \to & \mathbb{R}/\mathbb{Z} & \to & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \to & \mathbb{Z}L^{-1} & \to & \mathbb{R} & \to & \mathbb{R}/\mathbb{Z}L^{-1} & & \\
 & & \downarrow & & & & & & \\
 & & \mathbb{Z}L^{-1}/\mathbb{Z} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

you have this geometric situation, namely cyclic covering of the circle of degree L.

Apply $\mathcal{S}$

$$\begin{array}{c} \mathcal{S}(\mathbb{R}/\mathbb{Z}) \\ \uparrow \\ \mathcal{S}(\mathbb{R}/\mathbb{Z}L^{-1}) \end{array}$$

inclusion of fixpts for $\mathbb{Z}L^{-1}/\mathbb{Z}$ trans. action on $\mathbb{R}/\mathbb{Z}$

$\mathcal{S}(\mathbb{R}/\mathbb{Z})$ = appropriate Laurent series $\sum_{n\in\mathbb{Z}} z^{-n} g(n)$

$z = e^{2\pi i \omega}$
$w = e^{2\pi i L^{-1}\omega}$

$\mathcal{S}(\mathbb{R}/\mathbb{Z}L^{-1})$ = Laurent series $\sum_{t\in\mathbb{Z}L^{-1}} w^{-t} f(t)$

$$0 \to \mathbb{Z} \longrightarrow \mathbb{Z}\tfrac{1}{L} \longrightarrow \mathbb{Z}\tfrac{1}{L}/\mathbb{Z} \longrightarrow 0$$

$$0 \longleftarrow \mathbb{R}/\mathbb{Z} \xleftarrow{L} \mathbb{R}/\mathbb{Z} \longleftarrow \mu_L \longleftarrow 0$$

$$\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$$

$\{f(n)\} \xleftarrow{\text{res}} \{g(t)\}$

$\sum z^{-n} f(n) = \{f(n)\}$ $\qquad$ $\{g(t)\} = \{\hat{g}(\omega)\}$

$$\boxed{\sum_n e^{-2\pi i \omega n} g(n) = \sum_m \hat{g}(\omega+m)}$$

easy way $\cancel{\int_0^1 d\xi\, e^{-2\pi i m \xi} \sum_m \hat{g}(\omega+m)}$

$$\int_{-1/2}^{1/2} d\omega\, e^{-2\pi i \omega n} \sum_m \hat{g}(\omega+m) = \sum_m \int_{-1/2}^{1/2} d\omega\, e^{-2\pi i \omega n}\, \hat{g}(\omega+m)$$

$$= \sum_m \int_{m-1/2}^{m+\frac{1}{2}} d\omega\, e^{-2\pi i (\omega - m) n}\, \hat{g}(\omega) = \int d\omega\, e^{-2\pi i \omega n}\, \hat{g}(\omega) = g(n)$$

$$\sum_n e^{-2\pi i \omega n} \int d\eta\, e^{2\pi i \eta n}\, \hat{g}(\eta) = \int d\eta \underbrace{\sum_n e^{2\pi i (\eta - \omega) n}}_{\sum_m \delta(\eta - \omega - m)} \hat{g}(\eta) = \sum_m \hat{g}(\omega+m)$$

$$\mathbb{Z} \overset{L}{\hookrightarrow} \mathbb{Z} \longrightarrow\!\!\!\!\to \mathbb{Z}/L\mathbb{Z}$$

apply $\mathrm{Hom}(-, \mathbb{R}/\mathbb{Z})$

$$\begin{tikzcd}
\mathbb{R}/\mathbb{Z}L^{-1} \arrow[d, "L", "\wr"'] & \mathbb{R}/\mathbb{Z} \arrow[l, two heads] \arrow[dl, "L"] & \mathbb{Z}L^{-1}/\mathbb{Z} \arrow[l] \\
\mathbb{R}/\mathbb{Z}
\end{tikzcd}$$

$$\mathbb{Z} \xhookrightarrow{L} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/L\mathbb{Z}$$

$$\mathbb{R}/\mathbb{Z} \xleftarrow{L} \mathbb{R}/\mathbb{Z} \longleftrightarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

Let's try to use sum over group. You should be able

$$\mathbb{Z} \subset \mathbb{R}$$

dual $\quad \mathbb{R}/\mathbb{Z} \twoheadleftarrow \mathbb{R}$

Program for today Friday. Dec 28. Analog of Poisson Summ. for $\mathbb{Z} \subset \mathbb{R}$ replaced by $\mathbb{Z} \subset \mathbb{Z}L^{-1}$.

Remind you: time domain $\left(\begin{array}{l} (g(n),\ n \in \mathbb{Z}) \\ (f(t),\ t \in \mathbb{R}) \end{array} \right.$ $\quad$ disc. signal $\mathcal{S}(\mathbb{Z})$, anal. signal $\mathcal{S}(\mathbb{R})$

restriction sampling $\quad f(t) \longmapsto f(n)$, $\quad \mathcal{S}(\mathbb{R}) \xrightarrow{res} \mathcal{S}(\mathbb{Z})$

frequency domain $\quad \mathcal{S}(\mathbb{Z}) \simeq \mathcal{S}(\mathbb{R}/\mathbb{Z})$ $\quad g(n) = \int_0^1 e^{2\pi i \omega n} \hat{g}(\omega)\, d\omega$

$\mathcal{S}(\mathbb{R}) \simeq \mathcal{S}(\mathbb{R})$ $\quad \hat{g}(\omega) = \sum_n z^{-n} g(n)$, $\quad z = e^{2\pi i \omega}$

$$f(t) \qquad \hat{f}(\omega) = \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} f(t)$$

$$f(t) = \int_{-\infty}^{\infty} d\omega\, e^{2\pi i \omega t} \hat{f}(\omega)$$

So what is restriction?

$$\underset{\text{anal.}}{f(t)} = \int_{-\infty}^{\infty} d\omega\, e^{2\pi i \omega t} \hat{f}(\omega) \longmapsto g(n) = \int_{-\infty}^{\infty} d\omega'\, e^{2\pi i \omega' n} \hat{f}(\omega')$$

dig

$$\text{want } \hat{g}(\omega) = \sum_n e^{-2\pi i \omega n} g(n) = \int_{-\infty}^{\infty} d\omega' \underbrace{\sum_n e^{-2\pi i (\omega - \omega') n}}_{\sum_m \delta(\omega' - \omega - m)} \hat{f}(\omega') = \sum_m \hat{f}(\omega + m)$$

$g(n) = f(t)$ rest to $t \in \mathbb{Z}$

$$\hat{g}(\omega) = \sum_m \hat{f}(\omega + m)$$

What do you learn? You know that restricting from $\mathbb{R}$ to $\mathbb{Z}$ loses information in some way. You see now in the frequency picture that the loss of information is due to aliasing.

Now $\hat{f}(\omega) \longmapsto \sum_m \hat{f}(\omega+m)$ can be made faithful by introducing characters on $\{m \in \mathbb{Z}\}$:

$$\sum_m e^{2\pi i m \xi}\, \hat{f}(\omega+m) = F(\omega,\xi)$$

which satisfies
$$F(\omega,\xi+1) = F(\omega,\xi)$$
$$e^{2\pi i \xi} F(\omega+1,\xi) = F(\omega,\xi)$$

What is the time ~~picture~~ of $\hat{f}(\omega) \longmapsto \sum_m e^{2\pi i \xi m}\hat{f}(\omega+m)$?

$$\int_{-\infty}^{\infty} d\omega\, e^{-2\pi i \omega t} \sum_m e^{2\pi i \xi m}\hat{f}(\omega+m)$$
$$= \sum_m e^{2\pi i(\xi m)} \int_{-\infty}^{\infty} d\omega\, e^{-2\pi i \omega t}\hat{f}(\omega+m)$$
$$= \sum_m e^{2\pi i \xi m} \int_{-\infty}^{\infty} d\omega\, e^{-2\pi i (\omega-m)t}\hat{f}(\omega)$$
$$= \sum_m e^{2\pi i(\xi m + mt)} f(t)$$

$$\sum_m e^{2\pi i \xi m}\hat{f}(\omega+m) = \sum_m e^{2\pi i \xi m}\int_{-\infty}^{\infty} dt\, e^{-2\pi i(\omega+m)t} f(t)$$
$$= \int_{-\infty}^{\infty} dt \sum_m e^{2\pi i(\xi m - tm - \omega t)} f(t)$$
$$= \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t}\sum_n \delta(t-\xi-n) f(t) = \sum_n f(\xi+n)$$

You have some control over restriction for $\mathbb{Z} \subset \mathbb{R}$

ume $\quad \mathbb{Z} \hookrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$

ng $\quad \mathbb{R}/\mathbb{Z} \longleftarrow \mathbb{R} \longleftrightarrow \mathbb{Z}$

Given a cont signal $(f(t), t \in \mathbb{R})$ you can restrict it to $\mathbb{Z}$. Corresp operation in frequency picture takes $(\hat{f}(\omega), \omega \in \mathbb{R})$ to $\sum_{n \in \mathbb{Z}} \hat{f}(\omega + n)$. You can also restrict $f(t)$ to a coset $t + \mathbb{Z}$. What is the frequency picture of such a signal $t+n \longmapsto f(t+n)$? Obvious $\sum e^{-2\pi i (t+n)} f(t+n)$

Keep on trying to make things clear

Go over $\mathbb{Z} \subset \mathbb{R}$ again.

$$\mathbb{R}/\mathbb{Z} \twoheadleftarrow \mathbb{R} \leftarrow \mathbb{Z}$$

Given $\{f(t), t \in \mathbb{R}\}$ its F.T. is $\hat{f}(\omega) = \int_{-\infty}^{\infty} dt\, e^{-2\pi i \omega t} f(t) dt$

Given $\{g(n), n \in \mathbb{Z}\}$ its F.T. is $\hat{g}(\omega) = \sum_n z^{-n} g(n) \qquad z = e^{2\pi i \omega}$

If $g = f|\mathbb{Z}$, then $\hat{g}(\omega) = \sum_m \hat{f}(\omega + m)$

Next look at $\mathbb{Z} \hookrightarrow \mathbb{Z}\frac{1}{L}$ periodic of period L

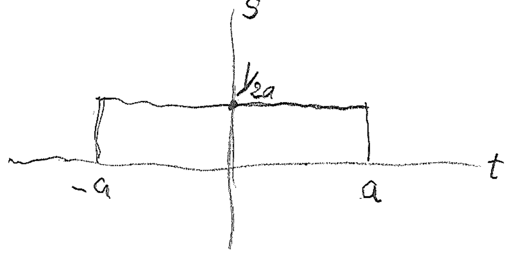
power of $e^{2\pi i \omega \frac{1}{L}} = z^{\frac{1}{L}}$

$$\mathbb{R}/\mathbb{Z} \twoheadleftarrow \mathbb{R}/\mathbb{Z}L$$

Given $\{f(t), t \in \mathbb{Z}\frac{1}{L}\}$ its F.T. is $\hat{f}(\omega) = \sum_t e^{-2\pi i \omega t} f(t)$

Given $\{g(n), n \in \mathbb{Z}\}$ its F.T. is $\hat{g}(e^{2\pi i \omega}) = \sum e^{-2\pi i \omega n} g(n)$

If $g = f|\mathbb{Z}$, then $\hat{g}(e^{2\pi i \omega}) = \sum_{m=0}^{L-1} \hat{f}(\underbrace{e^{2\pi i \frac{\omega}{L} m}}_{z^{\frac{m}{L}}})$



$$\int_{-a}^{a} e^{ist}\frac{dt}{2a} = \frac{e^{ist}}{is2a}\Big]_{-a}^{a} = \frac{e^{ias}-e^{-ias}}{2ias}$$

$$= \frac{\sin(as)}{as}$$

$$\int_{-a}^{a} \frac{e^{i\xi x}}{2a}\,dx = \frac{e^{i\xi x}}{2i\xi a}\Big]_{-a}^{a} = \frac{e^{i\xi a}-e^{-i\xi a}}{2i\xi a} = \frac{\sin(a\xi)}{a\xi}$$

$$\operatorname*{Conv}_{n}\left(\frac{1}{2a_n}\chi_{(-a_n,a_n)}(x)\right) \xmapsto{FT} \prod \frac{\sin(a_n\xi)}{a_n\xi}$$

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \cdots$$

product should converge if $\sum_{n=1}^{\infty} a_n^2 < \infty$.

infinite convolution should be supported in $[-b, b]$ $\quad b = \sum_{n=1}^{\infty} a_n$

$$\mathbb{Z} \hookrightarrow \mathbb{Z}L^{-1} \twoheadrightarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

$$\mathbb{R}/\mathbb{Z} \twoheadleftarrow \mathbb{R}/\mathbb{Z}L \hookleftarrow \mathbb{Z}/\mathbb{Z}L$$

---

better $\quad \mathbb{Z}L \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}/\mathbb{Z}L$

$$\mathbb{R}/\mathbb{Z}L^{-1} \twoheadleftarrow \mathbb{R}/\mathbb{Z} \hookleftarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

sampling or restriction $\quad (f(t), t \in \mathbb{Z}) \longmapsto (f(nL), n \in \mathbb{Z})$

$$\sum_{t\in\mathbb{Z}} e^{-2\pi i\omega t} f(t) \qquad \sum_{n\in\mathbb{Z}} e^{-2\pi i\omega nL} f(nL)$$

$\Vert$

$\hat{f}(\omega)$ $\qquad$ function of $e^{2\pi i\omega L}$

function of $e^{2\pi i\omega}$ $\qquad$ period L^{-1}

period 1

Try again. $\mathbb{Z} \xhookrightarrow{L} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/\mathbb{Z}L$

$\mathbb{R}/\mathbb{Z} \xleftarrow{L} \mathbb{R}/\mathbb{Z} \hookleftarrow \mathbb{Z}L^{-1}/\mathbb{Z}$

Let $f \in \mathcal{S}(\mathbb{Z})$, let $g = fL \in \mathcal{S}(\mathbb{Z})$

$$\hat{f}(\omega) = \sum_{t\in\mathbb{Z}} e^{-2\pi i\omega t} f(t) \qquad \hat{g}(\omega) = \sum_{n\in\mathbb{Z}} e^{-2\pi i\omega n} f(nL)$$

$$\hat{g}(\omega) = \sum_{r=0}^{L-1} \hat{f}\left(\omega + \tfrac{r}{L}\right) \quad ?$$

$$\| $$

$$\sum_{r=0}^{L-1} \sum_{t\in\mathbb{Z}} e^{-2\pi i\left(\omega+\frac{r}{L}\right)t} f(t) = \sum_{t\in\mathbb{Z}} e^{-2\pi i\omega t} \underbrace{\left(\sum_{r=0}^{L-1} e^{-2\pi i\frac{r}{L}t}\right)} f(t)$$

$$\sum_{r=0}^{L-1} \left(e^{-2\pi i t L^{-1}}\right)^r = \begin{cases} L & \text{if } t \in \mathbb{Z}L \\ 0 & \text{otherwise} \end{cases}$$

---

can you make this clearer? can you imitate the line bundle.

Given $(f(t), t \in \mathbb{Z})$ let $g = (f(nL), n \in \mathbb{Z})$

express $\hat{g}(\omega) = \sum_n e^{-2\pi i\omega nL} f(nL)$ in terms of

$\hat{f}(\omega) = \sum_{t\in\mathbb{Z}} e^{-2\pi i\omega t} f(t)$. Note $\hat{g}(\omega + \frac{1}{L}) = \hat{g}(\omega)$, so

you can average $\hat{f}$: $\frac{1}{L}\sum_{r=0}^{L-1} \hat{f}(\omega + rL^{-1}) =$

$$= \sum_{t\in\mathbb{Z}} \frac{1}{L} \sum_{r=0}^{L-1} e^{-2\pi i\omega t} e^{-2\pi i rL^{-1}t} f(t)$$

$$\frac{1}{L}\sum_{r=0}^{L-1} e^{-2\pi i r L^{-1} t} = \begin{cases} 0 & \text{if } e^{-2\pi i L^{-1}t} \neq 1 \\ 1 & \text{if } e^{-2\pi i L^{-1}t} = 1 \end{cases}$$

$$\Updownarrow$$

$$L^{-1}t \in \mathbb{Z}$$

$$t \in \mathbb{Z}L.$$

$$\sum_{\substack{t\in\mathbb{Z}L \\ t=nL}} e^{-2\pi i\omega nL} f(nL)$$

Next, some sort of descent. Two ways to proceed which should be related. On the time side you can include sampling over other cosets $r + \mathbb{Z}L$, and on the frequency side you can multiply by a character before averaging.

Given $f = (f(t), t \in \mathbb{Z})$, let $g(n) = f(r+nL)$.
You want to express $\hat{g}$ in terms of $\hat{f}$.

$$\hat{g}(\omega) = \sum_n e^{-2\pi i \omega n} f(r+nL)$$

$$\hat{f}(\omega) = \sum_{t\in\mathbb{Z}} e^{-2\pi i \omega t} f(t)$$

Problem: Is the usual interpolation adjoint to restriction? Ideally you want appropriate Hilbert space structures such that the adjoint operator respects scalar product. Restriction: $\{f(t), t\in\mathbb{R}\} \mapsto \{f(n), n\in\mathbb{Z}\}$
$f: \mathbb{R}\to\mathbb{C}$ $\qquad f/\mathbb{Z}$
is unbdd, so maybe you expect interpolation to be unbdd.

Look at discrete case:

$$\mathbb{Z} \xhookrightarrow{L} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/L\mathbb{Z}$$

$$\mathbb{R}/\mathbb{Z} \xleftarrow{L} \mathbb{R}/\mathbb{Z} \hookleftarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

sampling operator: $f = (f(t), t\in\mathbb{Z}) \longmapsto g = (g(n) = f(nL))$. What is this in frequency picture.

$$\hat{g}(\omega) = \sum_n e^{-2\pi i \omega n} g(n) = \sum_n e^{-2\pi i \omega n} f(nL)$$

$$f(t) = \int_{-\infty}^{\infty} d\xi\, e^{2\pi i \xi t} \hat{f}(\xi) \qquad = \int_{-\infty}^{\infty} d\xi \sum_n e^{-2\pi i \omega n + 2\pi i \xi L n} \hat{f}(\xi)$$

$$\sum_n e^{2\pi i(-\omega + \xi L)n} \ne \cancel{\sum_{n\in\mathbb{Z}} \delta(\omega - \xi L - n)}$$

$$\sum_n e^{2\pi i(\xi L - \omega)n} = \sum \delta(\xi L - \omega$$

$$\{\xi \mid \xi L - \omega \in \mathbb{Z}\} = \frac{\omega + \mathbb{Z}}{L}$$

$$\hat f(e^{2\pi i\omega}) = \sum_{t\in\mathbb{Z}} e^{-2\pi i\omega t} f(t)$$

$$f(t) = \int_{-1/2}^{1/2} d\omega\, e^{2\pi i\omega t}\hat f(\omega)$$

$$g(n) = f(nL) \qquad \hat g(\xi) = \sum_n e^{-2\pi i\xi n} f(nL)$$

$f(t),\ t\in\mathbb{Z}$ given. $\qquad \hat f(z) = \sum_{t\in\mathbb{Z}} z^{-t} f(t) \qquad z = e^{2\pi i\omega}$

$g(n) = f(nL),\ n\in\mathbb{Z} \qquad \hat g(w) = \sum_n w^{-n} f(nL)$

express $\hat g$ in terms of $\hat f$

$$\overset{g}{\mathbb{Z}} \xrightarrow{L} \overset{f}{\mathbb{Z}}$$

$$\mathbb{R}/\mathbb{Z} \xleftarrow{L} \mathbb{R}/\mathbb{Z} \longleftrightarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

$$\underset{\hat g}{w = z^L} \qquad \underset{\hat f}{z}$$

$$z^L = e^{2\pi i\omega L} = w = e^{2\pi i\xi}$$

to prove

$$\hat g(w) = \sum_{z^L = w} \hat f(z) = \sum$$

$$\hat g(e^{2\pi i\xi}) = \sum_{(e^{2\pi i\omega})^L = e^{2\pi i\xi}} \hat f(e^{2\pi i\xi}) = \sum_{r=0}^{L-1} \hat f\left(e^{2\pi i\left(\frac{\xi+r}{L}\right)}\right)$$

Discrete sampling.

$$\mathbb{Z} \xhookrightarrow{L} \mathbb{Z} \longleftrightarrow\!\!\!\!\rightarrow \mathbb{Z}/L\mathbb{Z}$$

$$\mathbb{R}/\mathbb{Z} \xleftarrow{L} \mathbb{R}/\mathbb{Z} \longleftrightarrow \mathbb{Z}L^{-1}/\mathbb{Z}$$

What's important? You have operations on signals, you need to understand the frequency domain picture of these operations properly. How to go about this?

How to proceed? Study DSP some more.

Look at changing the sampling rate.

Try again: Take discrete signal $(f(n), n\in\mathbb{Z})$, assume square integrables, i.e. in $\ell^2(\mathbb{Z})$ which is isom. to $L^2(S^1)$ via F.T.

$$\hat{f}(z) = \sum_n z^n f(n) \qquad z = e^{i\theta}$$

$$f(n) = \int z^{-n-1}\hat{f}(z)\frac{dz}{2\pi i} = \langle z^n | \hat{f}\rangle$$

$$\langle \hat{f} | z^n \hat{f}\rangle = \int_{S_1} z^n |f|^2 \qquad ?$$

$$\ell^2(\mathbb{Z}) \simeq L^2(S^1) \qquad \text{F.T.}$$

$$(x_n, n\in\mathbb{Z}) \longmapsto \sum_n z^{-n}x_n = X(z) \qquad \int |X|^2 \frac{d\theta}{2\pi} = \sum_n |x_n|^2$$

$$x_n = \langle z^{-n} | X\rangle \longleftarrow X(z)$$

$$\|$$

$$\frac{1}{2\pi i}\int z^{n-1} X(z)\, dz$$

$$X(z) = \sum x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi i}\oint X(z) z^{n-1} dz$$

$$x_1 * x_2 \longrightarrow X_1 X_2$$

DFT conventions.

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-i2\pi n/N} \qquad \underbrace{\left(e^{-2\pi i/N}\right)}_{W_N}{}^{kn}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{i2\pi kn/N}$$

$$\mathbb{Z} \xrightarrow{L} \mathbb{Z} \longrightarrow\!\!\!\!\!\!\rightarrow \mathbb{Z}/\mathbb{Z}L \qquad \text{Given } (f(n), n\in\mathbb{Z})$$

$$S^1 \overset{L}{\twoheadleftarrow} S^1 \hookleftarrow \mu_L$$

want frequency picture of restriction from $\mathbb{Z}$ to $\mathbb{Z}L$. This involves push forward of $\hat{f}$ wrt $L: S^1 \twoheadrightarrow S^1$

So what? Imitate $\mathbb{Z} \hookrightarrow \mathbb{R}$ case

$$F(x,y) = \sum_n e^{2\pi i n y} f(x+n) \qquad F(x,y+1) = F(x,y)$$

$$F(x+1,y) = \sum_n e^{2\pi i (n-1) y} f(x+1+n-1) \qquad F(x,y) = e^{2\pi i y} F(x+1,y)$$

$$= e^{-2\pi i y} F(x,y)$$

Setup $\mathbb{Z} \xhookrightarrow{L} \mathbb{Z}$. $f = (f(t), t \in \mathbb{Z})$

$$F(x,y) = \sum_{m\in\mathbb{Z}} e^{2\pi i m y} f(x+mL) \qquad F(x,y+1) = F(x,y)$$

$$F(x+L,y) = \sum_m e^{2\pi i (m-1) y} f(x+L+(m-1)L) \qquad F(x,y) = e^{2\pi i y} F(x+L,y)$$

$$= e^{-2\pi i y} F(x,y)$$

$$e^{2\pi i \frac{xy}{L}} F(x,y) = e^{2\pi i \frac{xy}{L}} e^{2\pi i y} F(x+L,y)$$

$$= e^{2\pi i \frac{(x+L)y}{L}} F(x+L,y) \qquad ?$$

Notice that $F(x,y) = F(x, y+\mathbb{Z})$ is the Fourier transform of sequence $m \longmapsto f(x+mL)$, essentially the coset of $\mathbb{Z}L$ in $\mathbb{Z}$ containing x.

Look at DSP notation.

Given $(f(n), n\in\mathbb{Z})$, take $\hat{f}(z) = \sum_n z^{-n} f(n)$, then restrict to $z = e^{2\pi i k/N}$ for $k = 0, \ldots, N-1$

$$\hat{f}(z) = \sum_n \underbrace{e^{-2\pi i k n/N} f(n)} \cancel{= \sum_{k=0}^{N-1} \sum_{\ell\in\mathbb{Z}} \cdots f(k)}$$

$$e^{-2\pi i k n/N} \sum_{\ell\in\mathbb{Z}} f(n-\ell N)$$

Review. $\mathbb{Z} \xrightarrow{L} \mathbb{Z} \longrightarrow \mathbb{Z}/\mathbb{Z}L$

$f = (f(n), n \in \mathbb{Z})$ $\qquad F(x,y) = \sum_n e^{2\pi i ny} f(x+nL, y)$

$F(x, y+1) = F(x,y)$

$$e^{2\pi i xy/L} F(x,y) = \sum_n e^{2\pi i (\frac{x}{L} + n) y/L} F(x+nL, y)$$

$$e^{2\pi i xy/L} F(x,y) = \sum_n e^{2\pi i (x+nL)y/L} F(x+nL, y)$$

period L in x

Review $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow \mathbb{Z}/\mathbb{Z}L$.

$f = (f(n), n \in \mathbb{Z})$ $\qquad F(x,y) = \sum_n e^{2\pi i ny} f(x+nL)$

$$e^{2\pi i xy/L} F(x,y) = \sum_n e^{2\pi i (x+nL)y/L} f(x+nL) \qquad \text{has period } L \text{ in } x.$$

$$\|$$

$$e^{2\pi i (x+L)y/L} F(x+L, y) = e^{2\pi i xy/L} e^{2\pi i y} F(x+L, y).$$

$$\begin{cases} F(x, y+1) = F(x,y) \\ F(x+L, y) = e^{2\pi i y} F(x,y) \end{cases} \qquad x \in \mathbb{Z},\ y \in \mathbb{R}/\mathbb{Z}.$$

Next? To recover f from F. This should be easy because $\mathbb{Z}$ splits into cosets $x + \mathbb{Z}L$, $0 \le x < L$, and $F(x,y)$ is the F.T. of the sequence $f(x+nL)$.

So what to do next? You have the Schwartz space $\mathcal{S}(\mathbb{R})$. Go back to

Recall $F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i n y} f(x+n)$

$$e^{2\pi i x y} F(x,y) = \sum_{n\in\mathbb{Z}} e^{2\pi i (x+n) y} f(x+n) \qquad \text{period in } x$$

$$\| $$

$$e^{2\pi i y}\, e^{2\pi i x y} F(x+1,y)$$

$$\boxed{\begin{aligned} F(x,y+1) &= F(x,y) \\ F(x,y) &= e^{2\pi i y} F(x+1,y) \end{aligned}}$$

$$\int_0^1 dx \int_0^1 |F(x,y)|^2 dy = \int_0^1 dx \sum_n |f(x+n)|^2 = \int_{-\infty}^{\infty} dx\, |f(x)|^2$$

Look at interpolation, rather sampling,

$$f(x) \longmapsto (f(n))_{n\in\mathbb{Z}}$$

You know that $f\in L^2 \longmapsto f(a)$ is unbdd.

Evaluation at a point is rep by δ fn not in L^2. Gelfand business.

$\int \delta$ okay —

You have learned something already about ~~your problem~~ interpolation from $\mathbb{Z}$ to $\mathbb{R}$ something involving the dual F.T. picture, namely, restriction from $\mathbb{R}$ to $\mathbb{Z}$ is given by $\mathbb{Z}$-summing, or making $\mathbb{Z}$-periodic:

$$\hat{f}(\xi) \longmapsto \sum_{n\in\mathbb{Z}} \hat{f}(\xi+n)$$

To make things clear you ~~probably~~ should work ~~with~~ in the dual picture. Thus restriction from $\mathbb{R}$ to $\mathbb{Z}$, i.e. sampling is $\hat{f}(\omega) \longmapsto \sum_n \hat{f}(\omega+n)$ from functions on $\mathbb{R}$ to functions on $\mathbb{R}/\mathbb{Z}$. Next examine interpolation. Is there an adjoint???

Next consider a continuous section of the degree 1 line bundle over $\mathbb{T}^2$. This should be a continuous $F(x,y)$ on $\mathbb{R}^2$ satisfying $F(x,y+1) = F(x,y)$ and $F(x,y) = e^{2\pi i y} F(x+1,y)$. Expand $F(x,y)$ which is periodic of period 1 into Fourier series.

$$F(x,y) = \sum_n e^{2\pi i n y} \underbrace{\int_{S^1} e^{-2\pi i n y} F(x,y)\, dy}_{f_n(x).}$$

$$\sum_n e^{2\pi i n y} f_n(x) = e^{2\pi i y} \sum_n e^{2\pi i n y} f_n(x+1)$$

$$= \sum_n e^{2\pi i n y} f_{n-1}(x+1).$$

so $\quad f_n(x) = f_{n-1}(x+1) = \cdots f_0(x+n).$

$$F(x,y) = \sum_n e^{2\pi i n y} f_0(x+n)$$

F assume cont. sat per. conds. Then $F(x,y)$ is a cont. family of cont fns on $S^1 = \mathbb{R}/\mathbb{Z}$, so get

$$f_n(x) = \int_{S^1} e^{-2\pi i n y} F(x,y)\, dy \qquad \text{cont. fn. of } x$$

$$= \int_{S^1} \underbrace{e^{-2\pi i n y} e^{2\pi i y}}_{e^{-2\pi i (n-1) y}} F(x+1,y)\, dy = f_{n-1}(x+1)$$

not clear that $f_n(x) = f_0(x+n)$ decays as $n \to \infty$. Use ℓ^2 norm?

It seems that $F(x,y)$ cont. $\Longrightarrow$ $f(x)$, $\hat{f}(y)$ continuous

$F(x,y)$ continuous on $\mathbb{R}^2$ sat $F(x,y+1) = F(x,y)$
$F(x,y) = e^{2\pi i y} F(x+1,y)$

Form the space of such F. This is a module over $C(\mathbb{T}^2)$. It should be a f.g. proj module of rank 1. F should be the same thing as a path $[0,1] \to LC$ continuous in the sup norm. A loop $g: S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ has Fourier components $\int_{S^1} e^{-2\pi i n y} g(y)\,dy = \hat{g}(n)$ satisfying

$$\sum_n |\hat{g}(n)|^2 = \int_{S^1} |g(y)|^2\,dy \qquad \text{Parseval relation}$$

Go over this. $LC \subset L^2(S^1)$. $L^2(S^1)$ is a $LC = C(S^1)$ module. Inside $L^2(S^1)$ is orth. sequence $e^{2\pi i n y}$ and one knows $\hat{g}(n) = \langle e^{2\pi i n y}, g\rangle$ satisfies $\sum_n |\hat{g}(n)|^2 \leq \|g\|_2^2$ by orthogonality. Also $\|g\|_2 \leq \|g\|_\infty$

Completeness $\Rightarrow$ Parseval.

You want to expand $F(x,y)$ into a Fourier series. First get the coeffs: $\int_{S^1} e^{-2\pi i n y} F(x,y)\,dy = f_n(x)$ cont fn of x.

$$f_n(x) = \int_{S^1} \underbrace{e^{-2\pi i n y}\, e^{2\pi i y}}_{e^{-2\pi i (n-1) y}} F(x+1,y)\,dy = f_{n-1}(x+1) = \cdots = f_0(x+n)$$

where $f_0(x) = \int_{S^1} F(x,y)\,dy$. $f_n(x)$ is the nth Fourier coeff of $F(x,-)$ so

$$\sum_{n\in\mathbb{Z}} |f_n(x)|^2 = \int_{S^1} |F(x,y)|^2\,dy .$$

Use a partition of unity in the x direction to reduce to something on the torus

You have line bundle over $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, whose sections are continuous functions $F(x,y)$ on $\mathbb{R}^2$ sat per. cond.

$$F(x, y+1) = F(x,y)$$
$$F(x,y) = e^{2\pi i y} F(x+1, y)$$

The space of these sections is a module over $C(\mathbb{T}^2)$, the alg of continuous doubly periodic functions

Various ideas: 1) Partition of unity over $\mathbb{T}^2$

2) $SL(2,\mathbb{Z})$ invariant picture.

$$e^{+2\pi i xy} F(x,y) = e^{2\pi i (xy+y)} F(x+1,y)$$

$$G(x,y) = e^{2\pi i xy} F(x,y)$$

$$G(x+1,y) = e^{2\pi i xy} e^{2\pi i y} F(x+1,y) = G(x,y)$$

$$\boxed{G(x,y) = \sum_m e^{2\pi i m x} g_m(y)}$$

$$g_m(y) = \int dx\, e^{-2\pi i m x} G(x,y) = \int dx\, e^{2\pi i (x-m) y} F(x,y)$$

$$\cancel{g_m(y+1) = \int dx\, e^{2\pi i (x-m)(y+1)}}$$

$$G(x,y+1) = e^{2\pi i x(y+1)} F(x, y\cancel{+1}) = e^{2\pi i x} G(x,y)$$

Start again: $F(x,y+1) = F(x,y)$, $F(x+1,y) = e^{2\pi i y} F(x,y)$

$$G(x,y) = e^{2\pi i xy} F(x,y) = e^{2\pi i xy} e^{2\pi i y} F(x+1,y) = G(x+1,y)$$

$$G(x,y+1) = e^{2\pi i x(y+1)} F(x,y\cancel{+1}) = e^{2\pi i x} G(x,y)$$

$$g_m(y) = \oint e^{-2\pi i x m} G(x,y)\, dx = \oint e^{-2\pi i x m - 2\pi i x} G(x,y+1) = g_{m+1}(y+1)$$

$$g_m(y) = g_{m-1}(y-1) = \cdots = g_0(y-m)$$

$$\therefore \quad G(x,y) = \sum_m e^{2\pi i m x} g_0(y-m)$$

$$\cancel{G(x+1,y) = e^{2\pi i x} G(x,y)}$$

$$\cancel{G(x,y+1) = \sum_m e^{2\pi i (m+1) x} g_0(y+1-m)}$$

$$\cancel{G(x+1,y) = \sum_m e^{2\pi i} \quad g_0(y}$$

$$\cancel{G(x+1)} \qquad G(x+1,y) = G(x,y)$$

$$G(x,y+1) = \sum_m e^{2\pi i \overset{m+1}{m} x} g_0(y+\cancel{1}-m) \qquad m \to m+1$$

$$= e^{2\pi i x} G(x,y)$$

$$g_m(y) = \oint e^{-2\pi i m x} G(x,y)\, dx$$

$$g_m(y+1) = \oint e^{-2\pi i m x} e^{2\pi i x} G(x,y)\, dx = g_{m-1}(y)$$

$$g_m(y) = g_{m-1}(y-1) = g_0(y-m).$$

$$g_0(y) = \oint G(x,y)\, dx = \int_{-1/2}^{1/2} e^{2\pi i x y} \sum_n e^{2\pi i n y} f_\phi(x+n)\, dx$$

$$= \sum_n \int_{-1/2}^{1/2} e^{2\pi i (x+n) y} f_\phi(x+n)\, dx$$

$$= \sum_n \int_{n-1/2}^{n+1/2} e^{2\pi i u y} f_\phi(u)\, du = \int_{-\infty}^{\infty} e^{2\pi i u y} f_\phi(u)\, du$$

$$= \hat{f}(y)$$

Make $SL(2,\mathbb{Z})$ act on sections of the degree 1 line bundle over $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Maybe need double cover. You want to consider all translation reps. You need a proper framework, descent? Principal bundle viewpoint $\mathbb{Z}^2 \to \mathbb{R}^2 \longrightarrow \mathbb{T}^2$, so any space F with $\mathbb{Z}^2$ action yields an assoc fibre bundle over $\mathbb{T}^2$, whose sections are cont $\mathbb{Z}^2$ equiv. maps from $\mathbb{R}^2$ to F. What's special here involves the area-volume on $\mathbb{R}^2$ ~~of~~ preserved by $SL(2,\mathbb{R})$. ~~If~~ You start with $C(\mathbb{R})$ with Heisenberg $\mathbb{R}^2$ action, translations + mult by characters.

What is your aim? Each representation of $\mathbb{Z}^2$?

Where to start? Begin with a cont section $F(x,y)$ that is, a cont. fn on $\mathbb{R}^2$ satisfying $F(x,y+1) = F(x,y)$ and $F(x,y) = e^{2\pi i y} F(x+1)$. What's the general pattern?
$e^{-2\pi i y} F(x-1,y) = F(x)$
There should be a formula for $F(x+m, y+n)$, namely

$$F(x+m,y+n) = F(x+m,y) = e^{-2\pi i y} F(x+m-1,y)$$
$$= \cdots = e^{-2\pi i m y} F(x,y).$$

Do things in a straightforward fashion. $\Gamma = SL(2,\mathbb{Z})$ acts on $\mathbb{R}^2$ preserving $\mathbb{Z}^2$. What's important is that $\mathbb{R}^2$ carries non deg symplectic form. $\mathbb{R}^2$ is the real vector space with coords (x, ξ), has hyperbolic symplectic form. You get an action of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$

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oriented Real v.s. with constant volume ω. Lift vol ω to A 1-form $dA = \omega$

You need to get started. Consider