An interesting point: If B is Γ-graded alg one can form $\mathbb{C}[\Gamma]\otimes B$ and $B\otimes\mathbb{C}[\Gamma]$ which should both be Γ-graded algebras ??? Wait.

Let $A = \bigoplus_{\Gamma} A_s$, $B = \bigoplus_{\Gamma} B_s$ be Γ-graded algebras:

$A_s A_t \subset A_{st}$, $B_s B_t \subset B_{st}$. We know that $A\otimes B$ is a Γ-graded vector space

$$A\otimes B = \bigoplus_{s,t\in\Gamma\times\Gamma} A_s\otimes B_t = \bigoplus_{u}\bigoplus_{u=st} A_s\otimes B_t$$

$$(A\otimes B)_s = \bigoplus_{t} A_t\otimes B_{t^{-1}s}$$

$$A\otimes B = \bigoplus_{t,u\in\Gamma\times\Gamma} A_t\otimes B_u = \bigoplus_{s\in\Gamma}\underbrace{\left(\bigoplus_{s=tu} A_t\otimes B_u\right)}_{(A\otimes B)_s}$$

What happens when you multiply

$$\begin{array}{ccc} (A_t\otimes B_u) & (A_{t'}\otimes B_{u'}) & \subset A_{tt'}\otimes B_{uu'} \\ \cap & \cap & \\ (A\otimes B)_{tu} & (A\otimes B)_{t'u'} & \end{array}$$

You get a problem since $tut'u' \neq tt'uu'$ in general.

Maybe this is the origin of your problems. Recall ~~[illegible]~~ what you want. Mainly to take a Γ-graded P_Φ-module M and to construct a Γ-invariant projection p on $\mathbb{C}[\Gamma]\otimes M$. You would like $p(\mathbb{C}[\Gamma]\otimes M)$ to be Γ-graded.

Basic obj might be $p(\mathbb{C}[\Gamma]\otimes \underbrace{P_\Phi}_{A})$

what do you think is true? A ~~[illegible]~~ firm $B \simeq C_\Phi \rtimes \Gamma$ module E is the same as a Γ-module equipped with an operator h_1 not respecting the Γ action such that $h_1 t h_1 = 0$ $t \notin \Phi$ and $\sum s h_1 s^{-1} = 1$. Factoring $h_1 : E \xrightarrow{\alpha_1 = h_1} h_1 E \xrightarrow{\beta_1} E$ you get $M = h_1 E$ and ~~[illegible]~~

$$E \xrightarrow{\quad\alpha\quad} \mathbb{C}[\Gamma] \otimes M \xrightarrow{\quad\beta\quad} E$$

$$\xi \longmapsto \alpha(\xi) = \sum_s s \otimes \alpha_1 s^{-1} \xi$$

$$\sum_{s} s \otimes f_s \longmapsto \beta\Big(\sum_s s \otimes f_s\Big) = \sum_s s \beta_1 f_s$$

$$\beta(\alpha(\xi)) = \beta\Big(\sum_s s \otimes \alpha_1 s^{-1} \xi\Big) = \sum_s s \beta_1 \alpha_1 s^{-1} \xi = \xi$$

$$\alpha\beta\Big(\sum_t t \otimes f_t\Big) = \alpha\Big(\sum_t t \beta_1 f_t\Big) = \sum_s s \otimes \alpha_1 s^{-1} \sum_t t \beta_1 f_t$$

$$= \sum_s s \otimes \sum_t (\alpha_1 s^{-1} t \beta_1) f_t$$

You want to know ~~the~~ whether M a Γ-graded P_Φ-module $\Longrightarrow$ E a Γ-graded Γ-module. One might hope in this case that E is a sum of copies of $\mathbb{C}[\Gamma]$.

Look carefully at ~~a sum a Γ module~~ a vector space with both Γ action and Γ grading.. ~~[illegible]~~ Suppose M ~~[illegible]~~ is a ~~Γ graded~~ vector space ~~with~~ ~~[illegible]~~ equipped with both Γ-grading $M = \bigoplus_\Gamma M_s$ and Γ action. ~~Then you have a~~ The a

Let M be a v.s. equipped with Γ-grading: $M = \bigoplus_{s\in\Gamma} M_s$ and Γ-action: $\mathbb{C}[\Gamma]\otimes M \longrightarrow M$.

~~Note that~~ Use standard grading on $\mathbb{C}[\Gamma]$, namely $\mathbb{C}[\Gamma]_s = \mathbb{C}s$. Then $\mathbb{C}[\Gamma]\otimes M$ is Γ graded

$$(\mathbb{C}[\Gamma]\otimes M)_s = \bigoplus_{s=tu} \mathbb{C}t\otimes M_u = \bigoplus_{t\in\Gamma} t\otimes M_{t^{-1}s}$$

$$\text{also} \quad = \bigoplus_{t} st^{-1}\otimes M_t$$

~~Assume~~ Compatibility of Γ action and Γ-grading should mean $\mathbb{C}[\Gamma]\otimes M \longrightarrow M$ $\quad s\otimes m \longmapsto sm$ preserves the Γ-grading i.e. $t M_{t^{-1}s} \subset M_s$

Now consider a ~~right Γ-module~~ vector space with Γ-grading: $W = \bigoplus_{s\in\Gamma} W_s$ and right Γ-action: $W_s t \subset W_{st}$

The point should be that inversion in Γ changes a ~~left~~ right Γ action to a left Γ action, and also changes the Γ-grading

$$\begin{array}{lll} \text{go from} & W = \bigoplus W_s & W_s t \subset W_{st} \\ & \quad\Vert \qquad\quad \Vert & \\ \text{to} & M = \bigoplus M_{s^{-1}} & t^{-1}M_{s^{-1}} \subset M_{t^{-1}s^{-1}} \end{array}$$

which agrees with canonical isom. $\mathbb{C}[\Gamma]\otimes M_1 \simeq M$

$W_1 \otimes \mathbb{C}[\Gamma] \xrightarrow{\sim} W$

~~[illegible]~~ Baaj - Sk

$p_s = h_1^{1/2} s\, h_1^{1/2}$ $\qquad \sum_t p_t p_{t^{-1}s} = \sum_t h_1^{1/2} t h_1 t^{-1} s h_1^{1/2} = h_1^{1/2} s h_1^{1/2} = p_s$

In your case $\quad p_s = \alpha_1 s \beta_1 : M \xrightarrow{\beta_1} E \xrightarrow{\alpha_1} M$

what's important. $p_s \in B = C_\Phi \rtimes \Gamma$

B is a Γ-graded algebra $\quad B_s = C_\Phi s \quad\quad p_s = h_1^{1/2} h_s^{1/2} s$

$p_s = \alpha_1 s \beta_1$

Review. Yesterday I discovered that the tensor product of Γ-graded algebras is <u>not</u> defined.

$$A \otimes B = \bigoplus_{t,u} A_t \otimes B_u = \bigoplus_s \bigoplus_{s=tu} A_t \otimes B_u$$

$$(A_t \otimes B_u)(A_{t'} \otimes B_{u'}) \subseteq A_{tt'} \otimes B_{uu'}$$

$$tu \cdot t'u' \qquad tt'uu'$$

Thus if $A = P_\Phi$, then the alg $\mathbb{C}[\Gamma] \otimes A$ is not ~~naturally~~ obviously Γ graded. But $\mathbb{C}[\Gamma] \otimes A = A \rtimes \Gamma$ where Γ acts trivially on A. No here A is ungraded

Go back to $B = C_\Phi \rtimes \Gamma$

Construction $\quad p_s = h_1^{1/2} s h_1^{1/2}$ or $\alpha_1 s \beta_1$ $\quad$ Thus $p_s = \alpha_1 s \beta_1$ is a projection in the $\hat{\Gamma}$-alg B. $\quad \sum_t \alpha_1 t \underbrace{\beta_1 \alpha_1}_{id} t^{-1} s \beta_1 = \alpha_1 s \beta_1$

get $\quad P_\Phi \longrightarrow B = C_\Phi \rtimes \Gamma$

$\quad p_s \longmapsto \alpha_1 s \beta_1$

actually what are α_1, β_1? In B you have h_1, ~~and the way you calculate~~ there should be suitable choices for α_1, β_1. You want $\beta_1 \alpha_1 = h_1$ for a typical firm B-module E you expect

$$E \xrightarrow{h_1} E$$

$$E \xrightarrow{\alpha_1 = h_1} M \xleftarrow{\beta_1 = inc} E$$

Try $\alpha_1 = h_1$

$\beta_1 = \sum_{s \in K} h_s$

Anyway $B = C_{\Phi} \rtimes \Gamma$ is naturally a Γ-graded algebra and it has this $h_1 \in C_{\Phi}$ such that $\sum s h_1 s^{-1} = 1$ in the appropriate sense. In your Morita equiv. you make a firm B-modules E corresp to $M = h_1 E$, you factor $h_1 : E \longrightarrow E$ into

$$\begin{tikzcd} E \arrow[rr, "h_1"] \arrow[dr, "\alpha_1 = h_1"'] & & E \\ & h_1 E \arrow[ur, "\beta_1 = \mathrm{in}"'] & \end{tikzcd}$$

Something interesting happens because of units.

$h_s h_t = 0$ for $s^{-1}t \notin \Phi$, $s \notin t\Phi^{-1}$

$$\underbrace{\Big(\sum_{s \in K} h_s\Big)}_{\beta_1} \underbrace{h_1}_{\alpha_1} = h_1$$

$K \supset \Phi^{-1}$!

Anyway you get in this way a Γ-graded projection

$$p_s = \alpha_1 s \beta_1 = h_1 s \sum_{t \in K} h_t \quad \text{in } C_{\Phi} \rtimes \Gamma$$

~~What does~~ Is there something ~~else~~ interesting here because of the local unit for h_1?

Review the ~~...~~ arguments

You still think there is a Morita equivalence between $B = C_{\Phi} \rtimes \Gamma$ and $A = P_{\Phi}$. Both algebras have natural Γ-gradings. There is also a Γ-graded alg homom. $A \longrightarrow B = C_{\Phi} \rtimes \Gamma$ which depends on a choice of left unit for h_1. Other facts: This Γ-gr alg homom. is given by a p in B, namely

$$p = \sum_s h_1 s h_K \qquad p^2 = \sum_{s,t} h_1 s \cancel{h_K} h_1 t h_K = \sum_u h_1 u h_K = p$$

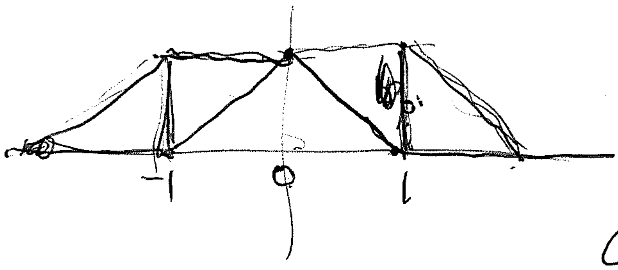
So p gives rise to $A \longrightarrow B$ by the univ. property of A. Better $p \in B = C_\Phi \rtimes \Gamma$ gives rise to a Γ-graded alg map

$$\begin{array}{ccc} A & \longrightarrow & B \\ p_s & \longmapsto & h_1 s h_K \end{array}$$

Suppose you examine the example $\Gamma = \mathbb{Z}$, $\Phi = \{u^{-1}, 1, u\}$ except replace C_Φ by $C_c(\mathbb{R})$.

$C_c(\mathbb{R}) \rtimes \mathbb{Z}$ and you know ~~about this~~ this is Morita equivalent to $C(\mathbb{R}/\mathbb{Z})$

Certainly you have interesting $C_\Phi \rtimes \Gamma$ modules such as $C_c(\mathbb{R})$. What's the corresponding P_Φ-module? Look at h_0 acting on $C_c(\mathbb{R})$



$$\begin{tikzcd} C_c(\mathbb{R}) \arrow[r, two heads] \arrow[d] & h_0 C_c(\mathbb{R}) \arrow[r, hook] \arrow[d, equal] & C_c(\mathbb{R}) \arrow[d] \\ C([-1,1]) \arrow[r] & h_0 C([-1,1]) \arrow[r, hook] & C_c([-1,1]) \end{tikzcd}$$

So $M \simeq C([-1,1])$ on which should be 3 operators p_{-1} p_0 p_1.

Have $\alpha_0 = h_0$ and $\beta_0 = k_0 = \underbrace{h_{-1} + h_0 + h_1}$ = identity on $C([-1,1])$.

$p_0 = h_0$. $\qquad p_1 = h_0 u$

$$M = h_0 C_c(\mathbb{R}) \simeq C([-1,1])$$

$B = C_\Phi \rtimes \Gamma$. Things do not seem to be any clearer. What are you trying to do?

Repeat. Take $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$. $B = C_\Phi \rtimes \Gamma$ acts on C_Φ which is a noncomm. version of $C_c(\mathbb{R})$. In any case C_Φ is generated by the partition of unity $\{h_n\}$ with only nearest neighbor overlap. $C_c(\mathbb{R})$ is naturally a form B-module, and it should correspond to a P_Φ-module given by $h_0 C_c(\mathbb{R})$ with operators p_{-1}, p_0, p_1, where the rest results from Γ-action. So basically you have this space $M = h_0 C_c(\mathbb{R}) \xleftarrow[h_0]{\sim} C([-1,1])$

$$\cancel{C_c(\mathbb{R}) \xrightarrow{\text{res}} C([-1,1])}$$

Try replacing $h_0 C_c(\mathbb{R})$ with $C([-1,1])$

Perhaps you want to consider ~~$C(\mathbb{R}/\mathbb{Z})$~~ $C(\mathbb{R}/\mathbb{Z})$, looks more like $\mathbb{R}/2\mathbb{Z}$?

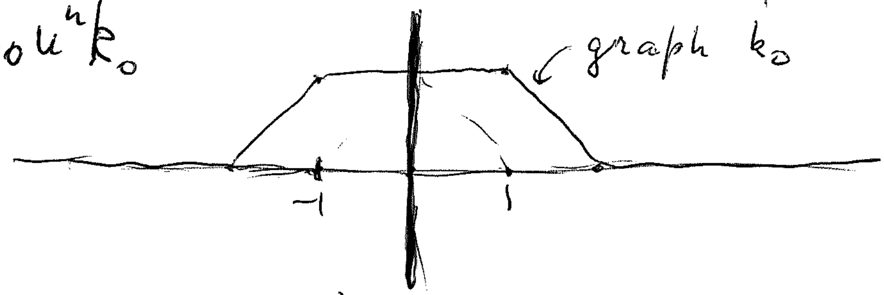
$$M = h_0 C_c(\mathbb{R}) = \{ h_0 g \mid g \in C([-1,1]) \}.$$

$$C_c(\mathbb{R}) \xrightarrow{h_0 \cdot} M \xrightarrow[k_0]{\text{inc.}} C_c(\mathbb{R})$$

$k_0 = h_{-1} + h_0 + h_1$
$= 1$ on $[-1,1]$.

You got 3 operators on M, from

$p_n = h_0 u^n k_0$



start with $g \in C([-1,1])$ mult by h_0 WAIT

$$\cancel{C_c(\mathbb{R}) \xrightarrow{\text{inc.}} M \xrightarrow{\alpha} C_c(\mathbb{R})}$$

$$\begin{tikzcd} h_0 C_c(\mathbb{R}) = M \arrow[r, hook, "\cdot k_0"', "\text{inclusion}"] \arrow[d, "\wr"] & C_c(\mathbb{R}) \arrow[r, two heads, "h_0 \cdot"] \arrow[loop above, "u^n"] & M \\ C([-1,1]) \arrow[r, "\cdot h_0"] & C_c(\mathbb{R}) \arrow[loop above, "u"] & \end{tikzcd}$$

In general $p_s = h_1 s k_1$ so for $\Gamma = \mathbb{Z} = \{u^n \mid n \in \mathbb{Z}\}$ you have $p_n = h_0 u^n k_0$. $\exists$ choice of models for M: either $h_0 C_c(\mathbb{R}) = h_0 C([-1,1])$ or $C([-1,1])$?

Back to the main problem - the Γ-graded alg. map $P_\Phi \longrightarrow C_\Phi \rtimes \Gamma$

Review $C_\Phi =$ alg gens. $h_s, s \in \Gamma$, rels $h_s h_t = 0 \quad s^{-1}t \notin \Phi$.

$$\sum_{s \in t\Phi^{-1}} h_s h_t = h_t = \sum_{s \in t\Phi} h_t h_s$$

action of Γ on C_Φ: $u h_s u^{-1} = h_{us}$. $B = C_\Phi \rtimes \Gamma$ is Γ-graded. Projections in B. Let K finite $< \Gamma$

If $K \supset \Phi^{-1}$, then $\sum_{s \in \Phi^{-1}} h_s h_1 = h_1 \qquad h_s h_1 \neq 0 \Rightarrow s^{-1} \in \Phi$

$$(1 - h_{\Phi^{-1}}) h_1 = 0 \qquad h_1 (1 - h_\Phi) = 0$$

So you have $(1 - h_K) h_1 = 0$ for $K \supset \Phi^{-1}$

$h_1 (1 - h_L) = 0$ — $L \supset \Phi$.

But now put $p_s = h_1 s h_K$ or $p_s = h_L s h_1$

$$\sum_t p_t p_{t^{-1}s} = \sum_t h_1 t h_K h_1 t^{-1} s h_K = h_1 s h_K = p_s$$

or $$= \sum_t h_L t h_1 h_L t^{-1} s h_1 = \sum_t h_L s s^{-1} t h_1 t^{-1} s h_1$$

$$= h_L s h_1 = p_s$$

~~[illegible]~~ New idea – namely there is <u>no canonical</u> ~~[illegible]~~ <u>projection</u> in $B = C_\Phi \rtimes \Gamma$. This is also true for the comm. case. This is perhaps a reason for your confusion. Maybe look at the

$\mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ situation again.

Consider $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ which operates on $C_c(\mathbb{R})$. I think you know ~~that~~ $C_c(\mathbb{R})$ is a fin. gen. proj. left $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ module.

$$
\begin{tikzcd}
Y \times_X Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d] \\
Y \arrow[r] & X
\end{tikzcd}
\qquad
\begin{matrix}
B = P \otimes_A Q & Q \\
P & A
\end{matrix}
$$

6.1	6263
7.2	7340
5.6	5708
6.2	6373

$$C_c(\mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R}) \xleftarrow{\sim} C_c(\mathbb{R}) \otimes_{C(\mathbb{R}/\mathbb{Z})} C_c(\mathbb{R})$$

~~[illegible]~~ $C_c(-)$ is a contravariant fun for proper maps and covariant for etale maps. ~~You need to understand the~~

Yesterday I learned that the projection in ~~[illegible]~~ the Γ-graded algebra $B = C_\Phi \rtimes \Gamma$ is not canonical, there's a choice involved. Review this now. ~~[illegible]~~

You approach this from your Morita equivalence, namely, given a firm B-module N you ~~want the~~ make the factorization $N \xrightarrow{\alpha_1 = h_1} h_1 N \xhookrightarrow{\beta_1 = \mathrm{inc}} N$. Now there seems to be a way to realize this factorization using operators in B, because B has local units.

Review formulas. $h_s h_t \neq 0 \iff h_1 s^{-1} t h_1 \neq 0 \Rightarrow s^{-1} t \in \Phi$
$\iff t^{-1} s \in \Phi^{-1} \iff s \in t\Phi^{-1}$

$$K \text{ fin } \supset \Phi^{-1} \Rightarrow \underbrace{\Big(\sum_{s\in K} h_s\Big)}_{h_K} h_1 = h_1 .$$

$$K \text{ fin. } \supset \Phi \Rightarrow h_1 \Big(\sum_{s\in K} h_s\Big) = h_1$$

Once you choose K, then ~~[illegible]~~ you get a projection in B, namely

$$\sum_s h_1 s h_K \qquad \text{for } K \supset \Phi^{-1}$$
$$\sum_s h_K s h_1 \qquad \text{for } K \supset \Phi$$

Check $\displaystyle\sum_t p(t)p(t^{-1}s) = \sum_t h_1 t \underbrace{(h_K h_1)}_{=h_1} t^{-1} s h_K = h_1 s h_K = p(s)$

Clearly things ~~are not~~ involve choice of the local unit for h_1.

In any case you do get a projection in the Γ graded algebra B.

What's important I think is that the factorization $h_1 = \beta_1 \alpha_1$: $E \xrightarrow{\alpha_1 = h_1} h_1 E \xhookrightarrow{\beta_1 = inc} E$ has been carried out with operators from C_Φ. Check this.

$$h_1 = h_K h_1$$

$$0 \to \mathrm{Ker}(h_1) \to E \xrightarrow{h_1} E \to \mathrm{Cok}(h_1) \to 0$$

$$h_1 = h_K h_1$$

put $\alpha_1 = h_1 \qquad \beta_1 = h_K$

Then $\beta_1 \alpha_1 = h_K h_1 = h_1$

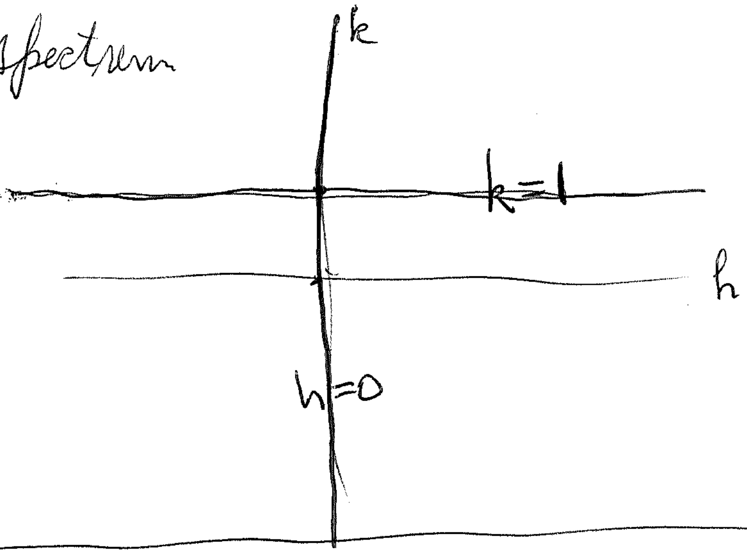
Assume $h_1 h_K = h_K h_1 = h_1$

So you need to understand alg gens h, k rel $hk = kh = h$.

Commutative alg. Look at unital alg.

$(1-k)h = 0$

spectrum



$$\begin{tikzcd} & E \arrow[d, "k_1"] \\ E \arrow[ur, "h_1"] \arrow[r, "h_1"'] & E \end{tikzcd}$$

Start again. ~~You want to understand representations~~
Consider the ring Λ gens: h, k rels $kh = h$.
Can you describe Λ-modules. A Λ-module is a v.s. V equipped with operators h, k ~~[illegible] satisfying~~ such that k is the identity on the image of h.

Calculate the ring. $kh = h$

$h\mathbb{C}[h]$

Λ is a ring with local left unit k, AND

To calculate the ring with gens h, k subj to reln $kh = h$
It should have basis $h^m k^n$ $m, n \geq 0$. Check: these are the words not containing kh. Let A be the unital ring generators + relations as above. Define obvious map $\mathbb{C}[h] \otimes \mathbb{C}[k] \longrightarrow A$:

$$\hat{h}(h^m k^n) = h^{m+1} k^n$$

$$\hat{k}(h^m k^n) = \begin{cases} k^{n+1} & \text{if } m = 0 \\ h^m k^n & \text{if } m \geq 1. \end{cases}$$

clearly $\hat{k}\hat{h}(h^m k^n)$
$= \hat{k}(h^{m+1}k^n) = h^{m+1}k^n$
$= \hat{h}(h^m k^n)$.

so LHS becomes an A-module

Try calculations in $C_c(\mathbb{R}) \rtimes \mathbb{Z}$

$$\begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd} \qquad \begin{tikzcd} B = C_c(X \times_Y X) & C_c(X) = Q \arrow[l, "pr_2^*"'] \\ P = C_c(X) \arrow[u, "pr_1^*"] & C(Y) = A \arrow[l] \arrow[u] \end{tikzcd}$$

$P \otimes_A Q \xrightarrow{\sim} B$. There's the pairing $Q \otimes P \longrightarrow A$, $q \otimes p \mapsto \pi_*(qp)$. P, Q are unitary modules over the unital alg A, π_* surjective, so have M.eq ~~$P \otimes A$~~ P fin. gen. proj. B module Q fin gen proj B^{op} module. **How to see this?**

To show the identity map on P is nuclear.

$$\begin{tikzcd} Q \otimes_B P \arrow[r, hook] \arrow[d] & \mathrm{Hom}_B(P, B) \otimes_B P \arrow[d] \\ A \arrow[r] & \mathrm{Hom}_B(P, P) \end{tikzcd}$$

This square should commute, should show that whenever $\sum \langle q_i, p_i \rangle = 1$ ~~then ...~~ you get an expression showing that the identity map of P is nuclear.

So all you need is to express $1 \in A$ as $\sum \langle q_i, p_i \rangle$ and you take $\beta_1 \alpha_1 = h_1$ $\quad \pi_*(h_1) = 1$.

Yesterday you learned something from the geometric case - non uniqueness. There's still the problem with P_Φ not ~~being~~ unital. So what can you do?

You have $B = C_\Phi \rtimes \Gamma$

~~[scribbled]~~ Try to mimic the geometric case.
You have $X \times \Gamma = X \times_Y X$ and our square becomes

$$\begin{tikzcd} X \times \Gamma \arrow[r, "\mu"] \arrow[d, "pr_1"'] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Now you have a Morita equivalence of same sort between $C_\Phi \rtimes \Gamma$ and P_Φ.

Example $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$.
You have the Morita equivalence between
$B = C_c(\mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R}) \simeq C_c(\mathbb{R}) \rtimes \mathbb{Z}$ and $A = C(\mathbb{R}/\mathbb{Z})$.

This is our basic example which is to be refined.
~~[scribbled out]~~ Picture of $C_c(\mathbb{R})$ as Γ-module with the operator mult. by h_0, whose norm is the identity. Yesterday I looked at ~~factory~~ factoring h_0 into $h_0 k_0$ where $k_0 = h_{-1} + h_0 + h_1$

$$\begin{tikzcd}[column sep=small] C_c(\mathbb{R}) \arrow[dr, "res"'] & & C_c(\mathbb{R}) \arrow[dr, "res"] & & C_c(\mathbb{R}) \\ & C([-2,2]) \arrow[ur, "k_0"] \arrow[rr, "res"'] & & C([-1,1]) \arrow[ur, "h_0"'] & \end{tikzcd}$$

You are missing the proper point of view. So what do you mean?

There should be something simple waiting to be found.

$$\begin{tikzcd} C_c(X \times_Y X) & C_c(X) \arrow[l] \\ C_c(X) \arrow[u] & C(Y) \arrow[l] \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} X \times \Gamma = X \times_Y X \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

$\underbrace{C_c(X \times_Y X)}_{k \text{ in}}$ appears as a space of kernels, namely k in becomes the operator ~~$f \mapsto (pr_1)_* k(pr_2^* f)$~~

$f \mapsto pr_{2*}(k \cdot pr_2^* f)$. k is a function on $X \times_Y X = X \times \Gamma$. Have section $X \xrightarrow{\Delta} X \times_Y X$ which leads to a disjoint union of copies of X indexed by Γ. How do you get any further?

Let us try to get from the geometric Morita equivalence (maybe faithful flat descent idea) to the one we want. How?

Other idea. If C_Φ corresp to P_Φ and $C_c(\mathbb{R})$

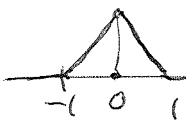
So back to Hilbert space picture for insight. You want a $*$ repn. of $C_\Phi \rtimes \Gamma$, which should amount to a Hilb. space $\mathcal{H}$, unitary op u, and hermitian h_0 such that $h_0 u^n h_0 = 0$ $|n| > 1$ and $\sum u^n h_0 u^{-n} = 1$ on $\mathcal{H}$. Let's examine the structure of $\mathcal{H}$ carefully. Let $V_n = \overline{h_n \mathcal{H}} = \overline{u^n h_0 \mathcal{H}}$. You know that $V_m \perp V_n$ when $|m-n| > 1$.

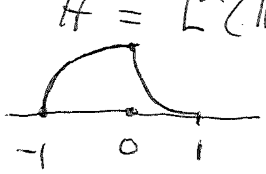
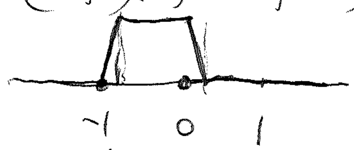
V_{-1} V_0 V_1. Think GNS, get dense subspace of $\mathcal{H}$ from $\mathbb{C}[z, z^{-1}] \otimes V_0$. I think you want to introduce $h_0^{1/2}$ which gives ?

Hilbert space $\mathcal{H}$ with unitary u and pos. op $h_0 \geq 0$ such that

$$\begin{cases} h_0 u^n h_0 = 0 & |n| > 1. \\ \sum \underbrace{u^n h_0 u^{-n}}_{h_n} = 1 & \text{sup in sense of pos. ops.} \end{cases}$$

$V_n = \overline{h_n \mathcal{H}} = u^n V_0$ $\qquad V_n \perp V_{n'}$ for $|n - n'| > 1$.

Example: $\mathcal{H} = L^2(\mathbb{R}, dx)$, $(uf)(x) = f(x-1)$, $h_0(x)$: 

other , maybe , you see that the overlap $h_0 h_1$ can be made fairly small.

Because of the condition $\sum h_n = 1$ its natural to introduce $h_n^{1/2} = u^n h_0^{1/2} u^{-n}$, and then you get your basic isometric embedding of $\mathcal{H}$.

$$\mathcal{H} \xrightarrow{\alpha = (h_n^{1/2})} \bigoplus_{n \in \mathbb{Z}} V_n \xrightarrow{\beta = \alpha^* = (h_n^{1/2})} \mathcal{H} \qquad \alpha^* \alpha = 1$$

proj op is $h_n^{1/2} h_m^{1/2} = u^n h_0^{1/2} u^{-n+m} h_0^{1/2} u^{-m} = 0$ for $|n-m| > 1$

$$\mathcal{H} \xrightarrow{(h_0^{1/2} u^{-n})} \bigoplus_{\mathbb{Z}} V_0 \xrightarrow{(u^n h_0^{1/2})} \mathcal{H}$$

and you end up with $h_0^{1/2} u^{-n+m} h_0^{1/2}$ — this is the pos. type function on $\mathbb{Z}$ with values in $\mathcal{L}(V_0)$

~~In this situation you are on~~

Review GNS $\quad \rho : A \longrightarrow B \quad$ linear $\quad \rho(1) = 1$.

cal of $\quad N \underset{i}{\overset{j}{\rightleftarrows}} M \qquad$ N a B-mod, M an A-mod

$\rho(a) n = j a i n \qquad \therefore \; ji = 1_N$.

In this case N can be recovered from M.

ring of operators on M is $A \oplus A\otimes B\otimes A$

$$m \longmapsto (a' \mapsto j a' m)$$

$$A\otimes N \longrightarrow M \longrightarrow \mathrm{Hom}(A,N)$$

$$a\otimes n \longmapsto ain \longmapsto (a' \mapsto \underbrace{j a' a i n}_{\rho(a'a)})$$

$A = \mathbb{C}[\Gamma]$, $M =$ Hilb. space $\mathcal{H}$,

$$h_0 = ij : \mathcal{H} \xrightarrow{j} V \xrightarrow{i} \mathcal{H}$$

$$\rho(a) = j a i$$

the formulas seem the same except for the condition $ji = 1$.

So what happens???? ~~Nothing formulas simple~~

Try to understand. Previous GNS: A C^*-alg, $\mathcal{H}$ $*$rep. of A, V closed subspace of $\mathcal{H}$ $\ni$ $\overline{AV} = \mathcal{H}$. Then have $B = \mathcal{L}(V)$, $\rho : A \to B$ defd by $\rho(a) = jai$ where $V \underset{j=i^*}{\overset{i}{\rightleftarrows}} \mathcal{H}$. Then ρ is a completely positive map from which you can reconstruct the repn $\mathcal{H}$ of A. Idea of proof. Since AV is dense in $\mathcal{H}$, $\mathcal{H}$ is the completion wrt a herm. scalar prod on $A\otimes V$

$$\langle a' i v' \mid a i v\rangle = \langle v' \mid \underbrace{i^* a'^* a\, i}_{\rho(a'^* a)}\, v\rangle$$

This is confusing, probably not sufficiently general. Begin with A acting on $\mathcal{H}$, and First understand GNS, the original GNS in the nonunital case. This goes back to measures, states.

Reasoning: Take $C(\mathbb{R})$ continuous functions vanishing at ∞. Positive linear functional on $C(\mathbb{R})$? This should be a pos. measure on $\mathbb{R} \cup \{\infty\}$ such that $\{\infty\}$ is a null set. Total mass finite

Let's analyze the possibilities.

Let A be a non-unital C^*-algebra, and $A \longrightarrow \mathcal{L}(H)$ a $*$ representation. Assume there is a vector $\xi \in H$ such that $A\xi$ is dense in H. You need to find the correct question, so assume A commutative.

Possible questions? Let A be C^*-algebra, let H be a $*$-rep. of A, and let $\xi \in H$ with $\xi \neq 0$. $\overline{A_+\xi} \overset{\text{def}}{=}$ the cyclic subrep of H generated by ξ. It is the smallest closed subspace of H closed under A and containing ξ. The question is whether $\overline{A\xi} = \overline{A_+\xi}$, i.e. whether $\xi \in \overline{A\xi}$.

Look at some simple examples. $A_+ = C(S^1)$ = the C^*-alg generated by a unitary operator, let A = subalg of $f(z)$ vanishing at $z = -1$ say. We know a cyclic rep of A_+ is described by a measure on the circle: the Hilbert space is $L^2(S^1, d\mu)$, the unitary operator is $\underline{z\cdot}$, and ξ is a constant function. What is $\overline{A\xi}$? This should be the closure of the space of continuous functions on S^1 which vanish at $z=-1$, closure inside $L^2(S^1, d\mu)$. If the point $z = -1$ has > 0 mass, then ~~there is all~~ there is a line in the Hilbert space killed by A etc. if $z=-1$ has mass $= 0$, then you probably can approximate the identity

A C^*-algebra, Def: A state on A is a linear functional f which is positive: $f(x^*) = \overline{f(x)}$, $f(x^*x) \geq 0$. ~~hermitian~~ inner product $(x, y) = f(x^*y)$, ~~mult is~~ left mult by A on itself gives $*$ representation:

$$f(x^*ay) = f((a^*x)^*y).$$

When $1 \in A$ this representation has an obvious ~~unit ve~~ cyclic vector given by 1 in A. Note $\|1\|^2 = f(1^*1) = f(1)$ so $f(1) \geq 0$,

Repeat the preceding. Given $f : A \to \mathbb{C}$ be a bdd. linear fnl such that $f(x^*) = f(x)^*$ and $f(x^*x) \geq 0$ $\forall x \in A$, then $(x|y) = f(x^*y)$ is a herm. scalar product on A, so you can complete to get $A \longrightarrow H$ where A is a Hilb. space. One has $(a^*x|y) = f((a^*x)^*y) = f(x^*ay)$
$= (x, ay)$

You need $\|ax\| \leq C\|x\|$ for $a\cdot$ to be defined on the Hilbert space. So $(ax|ay) = f(x^*a^*ay)$

$$\leq C\,\|x^*a^*ay\| \leq C'\|x^*\|\|y\| = C'\|x\|\|y\|$$

It seems OK.

When A unital, then $1 \in A$ is an obvious cyclic vector in H, since $a \cdot 1 = a$.

Note: A $*$-rep of A on H, i.e. a $*$-hom. $A \longrightarrow \mathcal{L}(H)$ is the same as a unital $*$-hom $A_+ \longrightarrow \mathcal{L}(H)$. So unital

Start again. Your idea was to begin with GNS, specifically, to find what nonunital ~~[illegible]~~ modifications are necessary. Look first at ~~the relation [illegible]~~ Hilbert space representations of a C^*-alg. A. The original GNS theorem identifies cyclic* reps. of a unital C^* alg A with positive functionals on A, i.e. $\varphi : A \to \mathbb{C}$ bdd respecting $*$ such that $\varphi(x^*x) \geq 0 \quad \forall x \in A$. Given a ~~[illegible]~~ $*$-rep $A \longrightarrow \mathcal{L}(H)$ ~~and [illegible]~~ with cyclic vector ξ, then $\varphi(x) = \xi^* x \xi$ is the corresp. pos. fun on A. Given φ let $H =$ completion of A ~~with~~ resp to $(x, y) = \varphi(x^*y)$, ~~[illegible]~~ the A action on H induced by extending left mult. on A to the completion, $\xi =$ element of H corresp. to $1 \in A$.

Simple example. $A = C(S^1)$. A rep. of A on H i.e. $*$-homom. $C(S^1) \longrightarrow \mathcal{L}(H)$ ~~[illegible]~~ preserving 1 is equivalent to a unitary op u on X. If this rep. ~~[illegible]~~ is cyclic with cyclic vector ξ, then $\varphi(x) = \xi^* x \xi$ is a pos. lin. fnl. on the group $\mathbb{Z}$, which is equiv. to a measure on S^1. ~~[illegible]~~ You can go between pos. fnls on A, pos. def functions on $\mathbb{Z}$, measures $d\mu$ on S^1, cyclic representations of $A = C(S^1)$, $\quad H = L^2(S^1, d\mu)$, $\xi = 1$.

Now the problem is to ~~[illegible]~~ understand the changes when A is nonunital. The key change seems to be that there is no obvious ~~vector~~ generating vector in the completion: $H_\varphi =$ completion of A ~~with~~ wrt $(x, y) = \varphi(x^*y)$. Now you expect the multiplier algebra ~~to~~ $M(A)$ to ~~[illegible]~~ enter in some ~~[illegible]~~ way. In the commutative case $M(A)$ is the Stone-Čech compactification.

Simplest example is $A = C(\mathbb{Z})$ = functions on $\mathbb{Z}$ vanishing at ∞ with limit $= 0$ as $|n| \to \infty$. A is nonunital. What is a $*$ homom. from A to $\mathcal{L}(H)$? Inside A is $C_c(\mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} e_n$ $\quad e_n = e_n^* = e_n^2$, $e_m e_n = 0$ $(m \neq n)$

It looks simple to take $A = \mathbb{C}(\mathbb{N})$. A $*$ homo from $C_c(\mathbb{N})$ to $\mathcal{L}(H)$ is equivalent to an orthog family $\{H_n, n \geq 0\}$ of closed subspaces. So you have $H = \bigoplus_{n \geq 0}^{(2)} H_n \oplus H'$ orthogonal direct sum.

~~You now understand~~

There's a thm. that any $*$ homo between C^*-algs is continuous. But it is clear that the hom. $C_c(\mathbb{N}) \to \mathcal{L}(H)$ extends by cont. to $C(\mathbb{N}) \to \mathcal{L}(H)$. Note

$C(\mathbb{N})$ = continuous functions on the compact space $\mathbb{N} \cup \{\infty\}$ vanishing at ∞.

Next look at GNS for $A = C(\mathbb{N})$. Take a positive linear functional on A. Restricting to $C_c(\mathbb{N})$ you get a sequence ~~measure~~ $\mu_n \geq 0$, $n \in \mathbb{N}$. You want this to extend to $C(\mathbb{N})$, then you need $\sum \mu_n < \infty$. For if this sum is infinite you can choose N_1 so that

$$\sum_{0}^{N_1} \mu_n \geq 1, \quad \sum_{N_1+1}^{N_2} \mu_n \geq 2, \quad \text{etc.}$$

~~$1 + \frac{1}{2} + \frac{1}{3} + \cdots$~~ then you get a func vanishing at ∞ which is not integrable.

what do we know, where are we?

$A = C(\mathbb{N})$ $\qquad C_c(\mathbb{N}) = \bigoplus_{n\geq 0} \mathbb{C} e_n$ $\qquad$ ann. idempotent.

a $*$-homom. from A to $\mathcal{L}(H)$ is the same as a sequence $(H_n)_{n\in\mathbb{N}}$ of orthog closed subspaces of H:

$$H = \bigoplus_{n\geq 0}^{(2)} H_n \oplus H'$$

~~[illegible]~~ Objects

φ positive functional on A.

cyclic ~~[illegible]~~ Hilbert space representation of A.

A positive fnl on $C_c(\mathbb{N})$ is a measure $d\mu = (\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{N}$. $d\mu$ extends to $C(\mathbb{N})$ iff $\sum \mu_n < \infty$

Suppose H is a H.S. rep of A, so $H = \bigoplus_{n\geq 2}^{(2)} H_n \oplus H'$ and $\xi = \sum_{n\geq 0} \xi_n + \xi'$ is a vector in H. Then $\varphi(x) = \xi^* x \xi$ is a pos. fnl on A.

$= \sum_{n\in\mathbb{N}} x_n \|\xi_n\|^2$

$\therefore$ in this example pos. fnls on A are the same as measures.

Given finite measure on $\mathbb{N}$: $d\mu = (\mu_n)_{n\in\mathbb{N}}$ ~~[illegible]~~ $\sum \mu_n < \infty$, what is the GNS representation corresp. to $d\mu$. $\quad L^2(\mathbb{N}, d\mu)$

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review: program. GNS for nonunital C^*-algebras

example $\quad A = C(\mathbb{N}) = \{$ sequences $(x_n)_{n\in\mathbb{N}} \mid x_n \to 0$ as $n\to\infty \}$

$A =$ sup norm completion of $C_c(\mathbb{N}) = \bigoplus_{n\in\mathbb{N}} \mathbb{C} e_n$

~~a [illegible] space [illegible]~~ Suppose $A \longrightarrow \mathcal{L}(H)$ $*$-alg hom. cont.

~~then get $H = \bigoplus e$~~ $\quad e_n H$ orthog closed subsp of H

H

Review: Program is GNS for nonunital C^*-alg

Example: $A = C(\mathbb{N}) \simeq \{ f : \mathbb{N} \to \mathbb{C} \mid \lim_{n\to\infty} f(n) = 0 \}$.

$A_+ = C(\mathbb{N}_+)$. $\quad A \longrightarrow \mathcal{L}(H)$ $\quad$ *alg-homom.

$\{e_n\} \in A$ yields orth closed subs. $e_n H = H_n$ when

$$H = \bigoplus_{n \geq 0}^{(2)} H_n \oplus H'$$

$$A = C(\mathbb{N}) \simeq \{ f : \mathbb{N} \longrightarrow \mathbb{C} \mid \lim_{n\to\infty} f(n) = 0 \}.$$

$A_+ = C(\mathbb{N}_+)$ unital C^* alg.

A * rep of A on a Hilb. space H should be a family $H_n = e_n H$ of orthogonal closed subspaces

$$\therefore \quad H = \bigoplus_{n \geq 0}^{(2)} H_n \oplus H'$$

~~[illegible]~~ It should be true a * rep of A on H is the same as a * rep of A_+ on H.

Now look at cyclic representations, same for A, A_+ hence corresp to finite measures on $\mathbb{N}_+$. If ξ has components ξ_n, ξ' then $\|\xi\|^2 = \sum_n \|\xi_n\|^2 \oplus \|\xi'\|^2$

cyclic rep $\quad L^2(\mathbb{N}_+, d\mu)$

You should have an equiv. between cyclic reps of A, cyclic unital reps of A_+, finite measures on $\mathbb{N}_+$ = collection $\mu_n, \mu' \geq 0$ $\quad \sum_{n \geq 0} \mu_n + \mu' < \infty$.

You are interested in representations $\ni H' = 0$.

There should be some link between

measures with zero mass for $\{\infty\}$

cyclic representations of A such that $A\xi$ is dense in the Hilbert space.

cyclic rep such that $\xi \in \overline{A\xi}$.

This seems clear.

Next look at $A = C(S^1 - \{-1\})$ $\qquad A_+ = C(S^1)$

A rep. of A_+ on a Hilb. space H is equivalent to a unitary op on H

Problem: You seem to have messed up the duality. So return to your example and be careful. $A = C(\mathbb{N})$. A rep of A on H is a splitting $H = \bigoplus_{n\geq 0} H_n \oplus H'$.

cyclic representation $\quad \xi = \sum \xi_n + \xi'$

$$\|\xi\|^2 = \sum_n \underbrace{\|\xi_n\|^2}_{\mu_n} + \underbrace{\|\xi'\|^2}_{\mu'}$$

$$\xi^* x^* x \xi = \sum_n |x_n|^2 \mu_n + |x_\infty|^2 \mu' = \int |x|^2 d\mu$$

Usual GNS for a unital C^*-algebra gives an equivalence between 1) (H, ξ) up to isomorphism, where H is a unital Hilbert space representation of A and ξ is a cyclic vector ($\overline{A\xi} = H$)

2) positive functionals φ on A, (this means $\varphi : A \to \mathbb{C}$ respects $*$ and $\varphi(x^*x) \geq 0$ for all $x \in A$.)

Given (H, ξ) let $\varphi(x) = (\xi | x\xi)$ so that

$$\varphi(x^*) = (\xi | x^*\xi) = (x\xi | \xi) = \overline{(\xi | x\xi)} = \overline{\varphi(x)}$$

$$\varphi(x^*x) = (\xi | x^*x\xi) = \|x\xi\|^2.$$

Given φ you get hermitian ≥ 0 form on A $(x|y) = \varphi(x^*y)$, so you get a Hilbert space H by completing A wrt $\|x\|^2 = \varphi(x^*x)$. $\therefore H = \bar{A}$

$x \longmapsto ax$ induces a on H

since $\|ax\|_H^2 = \varphi(x^*a^*ax) \leq C\|x^*a^*ax\|_A \leq C\|x\|^2\|a\|^2$

Get $*$ rep since $(ax | y) = \varphi(x^*a^*y) = (x | a^*y)$ + completing $\therefore$ You have a H.S. rep of A. $\xi = 1$ is a cyclic vector since $a.\xi = a.1 = a$, so $A\xi = A \subset \bar{A} = H$ dense.

$(\xi, a\xi) = \varphi(1^*a1) = \varphi(a)$.

Next discuss A non unital. You have equivalence between unital H reps of A^+ equipped with cyclic vector ξ (up to isom) and pos. fnls. φ on A^+

Look at $\overline{A\xi} \subset \overline{A^+\xi} = H$. Since $A\xi + \mathbb{C}\xi = A^+\xi$ either $\overline{A\xi} = H$ (equiv. $\xi \in \overline{A\xi}$) or H splits into the orthogonal sum of $\overline{A\xi}$ and a line $\perp$ $\overline{A\xi}$.

What would you like to say happens?

If $\xi \notin \overline{A\xi}$, then $\overline{A^+\xi} = \mathbb{C}\xi_0 + A\xi_1$. Want

$\xi = \underset{\perp \text{ to } A\xi}{\xi_0} + \underset{\in \overline{A\xi}}{\xi_1}$ $\qquad$ ξ_1 is a cyclic vector for $\overline{A\xi}$

Do the example carefully. $A = C_0(\mathbb{N})$.
A rep of A on H is the same as a sequence $H_n = e_n H$ of orthog closed subspaces. A positive functional φ on A is a ~~sequence finite~~ measure on $\mathbb{N}$ of finite mass: $\mu_n = \varphi(e_n) \geq 0$, $\sum \mu_n < \infty$.

Given φ you can make A into a pre Hilbert space $(x|y) = \varphi(x^* y) = \sum \bar{x}_n y_n \mu_n$, form completion $A \longrightarrow \bar{A} = H$, and A acts by mult. So you are getting $L^2(\mathbb{N}, d\mu)$ = completion of $\mathbb{C}_0(\mathbb{N})$. Now what? You could try to find a cyclic vector yielding φ. This ~~is~~ the function 1 (equiv. unital $*$ rep of A^+)

Start again. $A = C_0(\mathbb{N})$. A $*$-repn of A on a Hilb. space H is a sequence $H_n = e_n H$ of orth closed subspaces, i.e. an orth splitting $H = \overset{(2)}{\bigoplus_{n \geq 0}} H_n \oplus H'$
Positive functional on A $\varphi(e_n) = \mu_n \geq 0$ $\sum \mu_n < \infty$. Then completion of A wrt $\varphi(x^* x) = \sum |x_n|^2 \mu_n$ is $L^2(\mathbb{N}, d\mu)$

$$A = C_0(\mathbb{N}) \longrightarrow L^2(\mathbb{N}, d\mu) \qquad \text{inj if all } \mu_n > 0$$

Look at the functional $\varphi(x) = \sum_{n \geq 0} x_n \mu_n$ and represent it as (ξ, x). ξ = image of const. fn. 1 in H.

Outline: Given $A \longrightarrow \mathcal{L}(H)$ $\xi \in H$ get $\varphi(x) = \xi^* x \xi$ a pos. ~~functional~~ on A^+, hence a positive functional on A by restriction. Given φ a pos. fnl on A^+ you introduce $(x|y) = \varphi(x^* y)$ on A^+, complete to obtain H and $A \longrightarrow \mathcal{L}(H)$. Also $\exists \xi \ni \varphi(y) = (\xi | y)$ note 1 has this prop.

Picture Given $A \xrightarrow{* \text{hom} + \text{cont}} \mathcal{L}(H)$ and $\xi \in H$ get bdd pos fnl $\varphi(x) = \xi^* x \xi$ on A^+, hence a bdd pos fnl on A.

Given φ bdd pos. fnl on A, get herm. inner prod $(x|y) = \varphi(x^*y)$ on A, hence get a Hilbert space H_φ by completing A, and action of A on H_φ induced by left mult on A.

You need to extend φ to H_φ, so you need $|\varphi(x)| \leq C\, \varphi(x^*x)^{1/2}$.

In the unital case you have $\varphi(1^* x) = (\xi|x)$

~~Unital case.~~ $A \xrightarrow{\theta} \mathcal{L}(H)$ $*$ homom. $\xi \in H$

$\varphi(x) = \xi^* \theta(x) \xi$ bdd lin. fnl. $|\varphi(x)| \leq \underbrace{\|\theta x\|}_{\leq \|x\|} \|\xi\|^2$

$(y|x) = \varphi(y^*x)$ hermitian form on A. positive

$= \xi^* \theta(y)^* \theta(x) \xi = (\theta(y)\xi)^* \theta(x)\xi$

$x \longmapsto \theta(x)\xi$ from A to H induces an isom. $\bar{A} \xrightarrow{\sim} \overline{\theta(A)\xi}$.

~~the direction: given φ a pos. lin. fnl on A~~

Assume now that A is unital and θ respects 1. Then $\overline{\theta(A)\xi}$ is the cyclic subrep of H generated by ξ. ~~It contains ξ~~ It's a closed subspace, closed under the operators from A and it contains $\xi = \theta(1)\xi$. ~~To say this better~~

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unital cyclic repn of A $\rightsquigarrow$ bdd pos fnl φ

$\theta : A \to \mathcal{L}(H)$, $\xi \in H$

$\overline{\theta(A)\xi} = H$

$|\varphi(x)| \leq \|\varphi\| \cdot \|x\|$

$\varphi(x^*) = \overline{\varphi(x)}$

$\varphi(x^*x) \geq 0$.

trying to understand GNS thm for nonunital A. Let A be a C^*-alg not nec. unital. Let us define a <u>cyclic representation of A</u> to be a triple (H, θ, ξ) where H is a Hilbert space, $\theta : A \longrightarrow \mathcal{L}(H)$ is a (continuous) $*$ homom., and $\xi \in H$, such that $\xi \in \overline{\theta(A)\xi}$.

Given a cyclic repn. (H, θ, ξ) let $\varphi(x) = (\xi, \theta(x)\xi)$ for $x \in A$. Properties of φ: linear functional $\varphi : A \longrightarrow \mathbb{C}$ <u>respecting $*$</u>: $\varphi(x^*) = (\xi, \theta(x^*)\xi) = (\xi, \theta(x)^*\xi)$ $= (\theta(x)\xi, \xi) = \overline{\varphi(x)}$, satisfying <u>positivity condition</u>

$$\varphi(x^*x) = (\xi, \theta(x^*x)\xi) = (\xi, \theta(x)^*\theta(x)\xi) = \|\theta(x)\|^2 \geq 0 \quad \forall x \in A.$$

<u>Continuous</u> $\quad |\varphi(x)| = |(\xi, \theta(x)\xi)| \leq \|\xi\| \, \|\theta(x)\| \, \|\xi\|$

$$\leq C\|x\|, \quad C = \|\xi\|^2 \|\theta\|.$$

Thus a cyclic rep of A determines a cont. pos. functional φ on A. <u>Claim</u> you get a bijection between iso classes of cyclic repns of A and cont pos. fnls, in this way.

To prove you construct a map going the other way. Suppose given φ a cont. pos. fnl on A. Define a pairing on A by $(y, x) = \varphi(y^*x)$. This is a <u>hermitian bilinear form</u> – evidently $\mathbb{C}$ linear in x, $\mathbb{C}$ anti linear in y, and $\overline{(x,y)} = \overline{\varphi(x^*y)} = \varphi((x^*y)^*) = \varphi(y^*x) = (y,x)$. Also the pos. cond. $(x,x) = \varphi(x^*x) \geq 0$ holds by assumption, so A ~~equipped with this form is a pre Hilbert space. One~~ we can complete A with respect to the "L^2" norm $\|x\|_2 = \varphi(x^*x)^{1/2}$ to obtain a Hilbert space H. H comes equipped with a canonical linear map $A \xrightarrow{L} H$

such that $(y, x) = (\rho y, \rho x)$, the latter being the inner product in H. $\operatorname{Ker}\rho$ = subspace of $x \in A \ni \|x\|_2 = 0$. and $\operatorname{Im}\rho$ is dense in H.

Thus you have constructed a Hilbert space H. Next need action θ of A on H. So you want to know that $y \longmapsto xy$ on Y is bounded for $\| \ \|_2$

$$\|xy\|_2^2 = \varphi((xy)^* xy) = \varphi(y^* x^* x y) \quad ?$$

What's the relation for $\|x\|$ in C^*-alg. sense with $\|x\|_2 = \varphi(x^*x)^{1/2}$. Basic C^* axiom $\|T^*T\| = \|T\|^2$. You should look at the comm. examples.

Let's go over things again A is a C^* algebra φ linear functional on A which is positive, which means $\varphi(x^*) = \varphi(x)^*$, $\varphi(x^*x) \geqslant 0$ $\forall x \in A$. Then $(y, x) = \varphi(y^* x)$ is positive $(\geqslant 0)$ herm. form on A.

H_φ = completion of A wrt pseudonorm $\|x\|_\varphi = \varphi(x^*x)^{1/2}$. canonical map $\varepsilon : A \longrightarrow H_\varphi$, $x \longmapsto \hat{x}$, $(y, x) = (\hat{y}, \hat{x})$

$\operatorname{Ker}(\varepsilon) = \{x \in A \mid \varphi(x^*x) = 0\}$, $A/\operatorname{Ker}(\varepsilon) = \varepsilon(A)$ dense in H_φ

[Properties of completion: H_φ is a Hilbert space, there is a canonical linear map $\varepsilon : A \longrightarrow H_\varphi$ such that $(y, x) = (\varepsilon y, \varepsilon x)$, ~~Ker(ε) = Ker~~ and $\varepsilon(A)$ is dense in H_φ.
$\operatorname{Ker}\varepsilon = \{x \in A \mid \varphi(x^*x) = 0\}$]

Now you need to define A-action, ~~$\forall a \in A$~~ to show $\forall a$ that $x \longmapsto ax$ on A induces a bdd operator $\theta(a)$ on H_φ. The obvious method would be to seek an estimate $\|ax\|_\varphi \leq C\|x\|_\varphi$, i.e. $\varphi(x^*a^*ax) \leq C\,\varphi(x^*x)$. But there might be some way to use the form $\varphi(y^* a x)$.

The other point you need is for $\varphi : A \longrightarrow \mathbb{C}$ to descend to a bdd linear functional on H_φ, in which case $\exists ! \ \xi \in H_\varphi$ such that $\varphi(x) = (\xi, \varepsilon x)$. At this point you will have (H_φ, θ, ξ), Hilb. space, $*$action of A, vector. It remains to show $\varphi(x) = (\xi, \theta(x)\xi)$.

Proof in unital case (A unital, $\theta(1) = 1$).

$$\varphi(x) = (1, x) = (\varepsilon 1, \varepsilon x) \qquad \therefore \ \xi = \varepsilon 1$$

$$\theta(a)\,\varepsilon(x) = \varepsilon(ax) \qquad \therefore \ \theta(a)\,\varepsilon 1 = \varepsilon(a).$$

$$\theta(a)\,\xi = \varepsilon(a).$$

Again. You start with φ on A, get inner product $(y, x) = \varphi(y^* x)$, complete to form H_φ with $\varepsilon : A \longrightarrow H_\varphi$ respecting inner product and dense image.

$$\varepsilon x = 0 \iff 0 = (\varepsilon x, \varepsilon x) = (x, x) = \varphi(x^* x)$$

~~Ask what it means~~

Example. $A = C_0(\mathbb{N})$. Let φ be a pos. fnl on A, get $\varphi(e_n) = \mu_n \geq 0 \quad \forall n$. This sequence is equivalent to the rest. of φ to $C_c(\mathbb{N})$. ~~Because φ is positive you know that~~ Look at functions $f \geq 0$ in A, then $f = (f^{1/2})^* f^{1/2}$ so $\varphi(f) \geq 0$.

φ pos. fnl on A $\rightsquigarrow \mu_n = \varphi(e_n) \geq 0$. ~~monotone limits~~

Take $f_n = \sum_{j \leq n} e_j$, $\quad f_n \uparrow 1$

harmonic divergence argument shows $\sum \mu_n < \infty$.

$A = C_0(\mathbb{R}) = C_0(S^1 - \{1\})$ aug ideal of the group algebra $\mathbb{C}[\mathbb{Z}]$

rep. of A are ~~unitary~~ operators.

---

positivity says φ increasing on the ~~...~~ cone of pos. herm. ops.

$$\varphi((ax)^*(ax)) = \varphi(x^*a^*ax) \leq \varphi(x^* \|a^*a\| x) \leq \|a^*a\| \, \varphi(x^*x).$$

$$\|ax\|_\varphi \leq \|a\| \, \|x\|_\varphi$$

~~...~~ Remains to show φ is represented by a $\xi \in H_\varphi$. ?

Suppose A comm. $A = C_0(X)$ X locally comp.

$$\varphi(f) = \int_X f \, d\mu \qquad \varphi(f^*f) = \int_X |f|^2 d\mu$$

Need ~~...~~ that $\varphi(f) < \infty$ for all $f \geq 0$ in A

$\Longrightarrow \int 1 \, d\mu < \infty$. The same harmonic divergence argument should work.

Back to Γ. What have you learned? When dealing with non unital A you define a cyclic vector ξ to be such that $\xi \in \overline{A\xi}$. ~~...~~ life is going on. How should GNS look algebraically.

A $\quad$ M an A-module $\quad N \underset{j}{\overset{i}{\rightleftarrows}} M$

Go back to $C = C_\Phi$ generators h_s $s \in \Gamma$

relations $h_s h_t = 0$ for $s^{-1}t \notin \Phi$ $\qquad s^{-1}t \in \Phi$

$$h_t = \sum_{s \in t\Phi} h_s h_t = \sum_{s \in t\Phi} h_t h_s \qquad t^{-1}s \in \Phi$$

This is a ring with local units. In fact if K finite in Γ, then $h_K = \sum_{s\in K} h_s$ has the property $h_K x = x$ and $x h_K = x$ for K large enough. ~~Next you have Morita equiv~~

$$0 \longrightarrow C \longrightarrow \tilde{C} \longrightarrow \mathbb{C} \longrightarrow 0$$

$$0 \longrightarrow C \rtimes \Gamma \longrightarrow \tilde{C} \rtimes \Gamma \longrightarrow \mathbb{C}[\Gamma] \longrightarrow 0$$

Let E be a firm $C \rtimes \Gamma$ module – this is the same as a $\tilde{C} \rtimes \Gamma$-module such that

$$E = \sum_{s\in\Gamma} h_s E = \sum_{s\in\Gamma} s h_1 E$$

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \qquad h_1 = \beta_1 \alpha_1$$

W

$$\mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V$$

$$\xi \longmapsto \sum_s s \otimes \alpha_1(s^{-1}\xi)$$

$$\Big\{ \sum_{s\in\Gamma} s \otimes f(s) \Big\} \longmapsto \sum_s s \beta_1 f(s)$$

Supp f finite

$$\beta\alpha\xi = \sum_s s \beta_1 \alpha_1 s^{-1} \xi = \sum_s h_s \xi = \xi$$

$$(\alpha\beta f)(s) = \alpha_1 s^{-1} \sum_t t \beta_1 f(t) = \sum_{t\in\Gamma} \alpha_1 (s^{-1}t) \beta_1 f(t)$$

---

basic proj from Hilb. space viewpoint is $\sum h_1^{1/2} s h_1^{1/2}$

$= \sum h_1^{1/2} h_s^{1/2} s$ $\quad$ $\alpha_1 = h_1^{1/2}$ $\quad$ $\beta_1 = h_1^{1/2}$

Question: Suppose $\beta_1\alpha_1 : E \longrightarrow V \longrightarrow E$ is a given factorization of h_1. Then

$p_s = \alpha_1 s \beta_1$ is a Γ-graded projection in $\underbrace{\mathbb{C} \rtimes \Gamma}_{B}$

$$\sum_t p_{st^{-1}} p_t = \sum_t{}' \alpha_1 st^{-1}\beta_1\alpha_1 t \beta_1 = \alpha_1 s \beta_1 = p_s.$$

id

$$E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V$$

idemp.

$$(\alpha\beta f)_s = \sum_t (\alpha_1 s^{-1} t \beta_1) f(t)$$

~~It looks as if a kernel $k(s^{-1}t)$~~

Consider an operator T on $\mathbb{C}[\Gamma]$ which is left invariant $\quad (Tf)(s) = \sum_t{}' T(s,t) f(t)$

$\{s \mid T(s,t) \neq 0\}$ finite $\forall t$. Left means $L_u T = T L_u$

$$(TL_u f)(s) = \sum_t T(s,t) f(u^{-1}t) = \sum_t T(s,ut) f(t)$$

$$(L_u Tf)(s) = (Tf)(u^{-1}s) = \sum_t T(u^{-1}s, t) f(t)$$

$\therefore \quad T(s,ut) = T(u^{-1}s, t)$

$T(us, ut) = T(s,t)$

$T(t^{-1}s, 1) = T(us,ut) = T(1, s^{-1}t)$

Left invariant op is $(Tf)(s) = \sum_t T(t^{-1}s) f(t)$

or $(Tf)(s^{-1}) = \sum_t \underbrace{T(t^{-1}s^{-1})}_{T(s^{-1}} f(t^{-1})$

Look at

$$\begin{tikzcd} E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[rr, bend right, "p"'] & E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V \end{tikzcd}$$

$$p = \alpha\beta \qquad (pf)(s) = \sum \underbrace{\left(\alpha_1(s^{-1}t)\beta_1\right)}_{p_{s^{-1}t}} f(t)$$

$$\sum_{s=tu} p_t p_u = \sum_{s=tu} \alpha_1 t \beta_1 \alpha_1 u \beta_1 = \sum_t \alpha_1 t h_1 t^{-1} s \beta = p_s$$

---

nicest multd by ~~$(Tf)(s) = \sum$~~

$$(f\varphi)(s) = \sum_t f(t)\varphi(t^{-1}s)$$

$$(f\varphi)(us) = \sum_t f(t)\varphi(t^{-1}us)$$

$$= \sum_t f(ut)\varphi(t^{-1}\cancel{u^{-1}u}s) =$$

$$\boxed{(f\varphi)L_u = f(\varphi L_u)}$$

---

left-invariant kernel $\varphi(t^{-1}s)$ or $\varphi(s^{-1}t)$

So if you want to write things in convolution form you need to put these on the right

$$(f\varphi)(s) = \sum_t f(t)\varphi(t^{-1}s)$$

$E = C_\Phi$ is a left $C_\Phi \rtimes \Gamma = B$ module (ungraded). $h_1 : E \longrightarrow E$

Question: Is E a fin gen proj B-module. You do get a map

$C_\Phi \rtimes \Gamma$

$X \times \Gamma$

$\shortparallel$

geom. case:
$$\begin{tikzcd} C_c(X \times_Y X) \arrow[r] \arrow[d, "pr_{1,*}"'] & C_c(X) \arrow[d, "\pi_*"] \\ C_c(X) \arrow[r] & C(Y) \end{tikzcd} \qquad \begin{tikzcd} X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\ X \arrow[r] & Y \end{tikzcd}$$

$\pi_*(h_1) = 1$. ~~[illegible]~~

~~[illegible]~~ You have this ~~[illegible]~~ isom.

$$C_c(X) \underset{C(Y)}{\otimes} C_c(X) \xrightarrow{\sim} C_c(X \times_Y X)$$

$$p \otimes q \longmapsto pr_1^*(p) \otimes pr_2^*(q)$$

$\langle q, p \rangle = \pi_*(qp)$

$P \otimes_A Q \xrightarrow{\sim} B \qquad Q \otimes P \longrightarrow A$

How do you see that P, Q are f.g. proj B-mods.

You choose $\pi_*(\alpha_1 \beta_1) = \sum s(\alpha_1 \beta_1) s^{-1} = 1$.

~~[illegible]~~ You want to find
$$\begin{tikzcd} B \arrow[d, shift left, "\pi_*"] \\ C \arrow[u, hook, shift left] \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d, shift left, "\pi_*"] \\ A \arrow[u, hook, shift left] \end{tikzcd}$$

~~Question. Is the situation~~

$C = C_\Phi$ generators h_s, $s \in \Gamma$ $h_s h_t = 0$ $s^{-1}t \notin \Phi$

$$h_t = \sum_{s \in t\Phi^{-1}} h_s h_t = \sum_{s \text{ such that } t^{-1}s \in \Phi,\ s \in t\Phi} h_t h_s$$

local left + right units $h_F = \sum_{s \in F} h_s$

30,000.
60,000

I need to understand ~~something~~ ~~about~~

C is a firm module over $B = C \rtimes \Gamma$. What happens in the case of $C_c(\mathbb{R}) \rtimes \mathbb{Z} = C_c(\mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R})$? This a space of kernels giving rise to operators on $C_c(\mathbb{R})$

Back to C_Φ: gens h_s $s \in \Gamma$

$h_s h_t = 0$ $s^{-1}t \notin \Phi$

$h_t = \sum_s h_s h_t = \sum_s h_t h_s$ (finite sums)

$B = C_\Phi \rtimes \Gamma$, Regard C as a B-module in the obvious way. In the geom. case

$$C_c(\mathbb{R}) \rtimes \mathbb{Z} = C_c(\mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R}) \quad \text{kernels.}$$

$B = C_c(\mathbb{R} \times_{\mathbb{R}/\mathbb{Z}} \mathbb{R})$ $\quad P = C_c(\mathbb{R})$

$Q = C_c(\mathbb{R})$ $\quad A = C(\mathbb{R}/\mathbb{Z})$

$$P \otimes_A Q \xrightarrow{\sim} B$$

You should know that because A is unital etc, that Q, P are f.g. proj B-mods. In fact the pairing $Q \times P \to A$ is $\langle qp \rangle = \pi_*(qp)$

Now arrange $qp = h_1 =$ 

Idea: $\mathrm{Hom}_\Gamma(\mathbb{C}[\Gamma]\otimes V, \mathbb{C}[\Gamma]\otimes W) = \mathrm{Hom}(V, \mathbb{C}[\Gamma]\otimes W)$

$$\uparrow$$

$$\mathbb{C}[\Gamma]\otimes \mathrm{Hom}(V,W)$$

$$(s\otimes\theta)(t\otimes v) = ts^{-1}\otimes\theta v$$

$$(s\otimes\theta)\big((s'\otimes\theta')(t\otimes v)\big) = ts'^{-1}s^{-1}\otimes\theta\theta' v = t(ss')^{-1}\otimes\theta\theta' v$$

$u = ts^{-1}$
$us = t$

$$\Big(\sum_s s\otimes\theta(s)\Big)\Big(\sum_t t\otimes v(t)\Big) = \sum_{s,t} \overbrace{ts^{-1}}^{u}\otimes\theta(s)v(t)$$

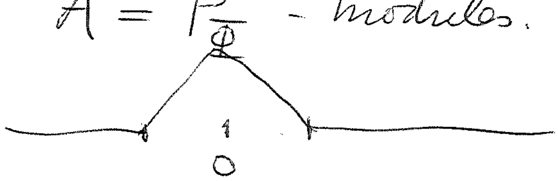
$$= \sum_u u\otimes\sum_s \theta(s)v(us)$$

$$= \sum_u u\otimes\sum_s \theta(u^{-1}s)v(s)$$

$$(\theta v)(s) = \sum_t \theta(s^{-1}t)v(t)$$

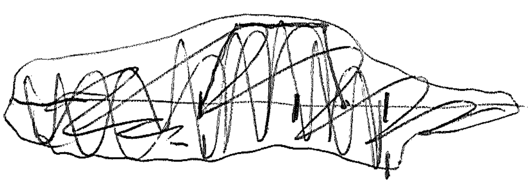
notation awkward.

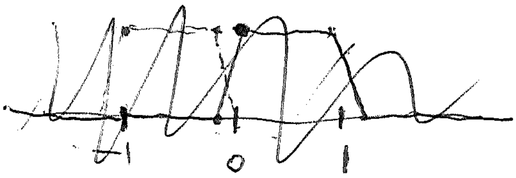
E $C\rtimes\Gamma$-module. Maybe you should examine a bit $A = P_\Phi$ - modules. $\Gamma = \mathbb{Z} = \{u^n \mid n\in\mathbb{Z}\}$.

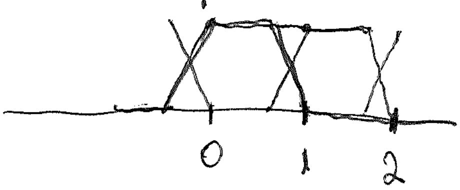
h_0  $h_0(h_{-1}+h_0+h_1) = h_0$

You have to find what to understand, look for

Think about the Hilbert space picture. What about







serious effort to understand ~~diffy~~ difference between $C_c(\mathbb{R})$ and C_Φ in the case of $\mathbb{Z}$, $\{-1,0,1\}$.

$$\begin{tikzcd} X\times\Gamma = X\times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

$$C_c(X)\otimes_{C(Y)} C_c(X) \xrightarrow{\sim} C_c(X)\rtimes\Gamma$$

$$P\otimes_A Q \xrightarrow{\sim} B$$

~~So what are you doin~~ Look at

$$\begin{matrix} \cancel{B} & \cancel{P} & A & Q \\ \cancel{Q} & \cancel{A} & P & B \end{matrix}$$

~~You want to show that~~

B acts on P, you want to show $P = Be$ for some idemp. e.

$$\begin{tikzcd} B \arrow[d, shift left] \\ P \arrow[u, shift left] \end{tikzcd} \qquad P\otimes_A \left(\begin{tikzcd} Q \arrow[d, shift left, "p_0"] \\ A \arrow[u, shift left, "\xi_0"] \end{tikzcd} \right)$$

You need to find $\langle \xi_0, p_0\rangle = 1$ to get projection $e = p_0\xi_0$ in B. Apparently doesn't exist in the noncommutative situation so there should be a substitute. How to proceed?

E firm $B = \underline{C\rtimes\Gamma}$ module e.g. $E = C_c(\mathbb{R})$

Γ, h_1 on E. $\quad E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \qquad h_1 = \beta_1\alpha_1$

$$E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V$$

$$\xi \longmapsto \sum s\otimes \alpha_1 s^{-1}\xi \longmapsto \sum s\beta_1\alpha_1 s^{-1}\xi = \xi$$

$$\sum s\otimes f_s \longmapsto \sum s\beta_1 f_s \longmapsto \sum_t s\otimes(\alpha_1 s^{-1}t\beta_1)f_t$$

on $\mathbb{C}[\Gamma]\otimes V$ you have the projection

$$p\Big(\sum_s s\otimes f_s\Big) \quad \text{or} \quad (pf)_s = \sum_t \big(\alpha_1(s^{-1}t)\beta_1\big) f_t$$

~~You are pleased~~ There is some alg of ops here!

Anyway once you factor h_1: $\underline{h_1 = \beta_1 \alpha_1}$
then you get $p_s = \alpha_1 s \beta_1$

$$\sum_s p_s p_{s^{-1}t} = \alpha_1 \Big(\sum_s s\beta_1\alpha_1 s^{-1}\Big) t \beta_1 = \alpha_1 t \beta_1 = p_t .$$

In your situation you will take $h_K h_1 = h_1$

$B = C_\Phi \rtimes \mathbb{Z}$ $\qquad \Phi = \{u^{-1}, u^0, u^1\}$

How does $C_c(\mathbb{R})$ become a B-module?
use translation action of $\mathbb{Z}$, need to give ~~the~~ action of h_0. ~~The obvious~~

What are you going to do? You study Morita equivalence between $B = C_\Phi \rtimes \Gamma$ and $A = P_\Phi$ in the case $\Gamma = \mathbb{Z}$, $\Phi = \{-1, 0, 1\}$. You should keep the Γ grading clear. B is Γ graded $B = \bigoplus_{s\in\Gamma} Cs$. A is also Γ-graded. You have defined A by gen. p_s, $s \in \Gamma$ subject to relns. $p_s = \sum_t p_t p_{t^{-1}s}$ and $p_s = 0$ for $s \notin \Phi$, $p_s \neq 0 \Rightarrow s \in \Phi$. ~~Since these~~
The generator p_s has degree s: $p_s \in A_s$. The relns. are homog. Therefore you can make a homom.

$$A \longrightarrow \mathbb{C}[\Gamma] \otimes A \qquad \longleftarrow \Gamma\text{-graded with } \deg(A) = 0$$

$$p_s \longmapsto s \otimes p_s$$

So the relations are satisfied. Need the ~~coalgebra setta~~ to check

$$\begin{tikzcd} A \arrow[r, "\Delta_A"] \arrow[dr, "1"'] & \mathbb{C}[\Gamma] \otimes A \arrow[r, shift left, "\Delta_\Gamma \otimes 1"] \arrow[r, shift right, "1 \otimes \Delta_A"'] \arrow[d, "\eta \otimes 1"] & \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \otimes A \\ & A & \end{tikzcd}$$

Γ-graded vector spaces, Γ set $\mathbb{C}[\Gamma]$ counital coalgebra with
$$\Delta s = s \otimes s \quad \forall s \in \Gamma$$
$$\eta(s) = 1 \quad \forall s \in \Gamma.$$

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "e_s"'] & \mathbb{C}[\Gamma] \otimes V \arrow[d, "\delta_s \otimes 1"] \\ & V \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & v \overset{\Delta_V}{\longmapsto} \sum_s s \otimes e_s(v) \arrow[dl, "(1\otimes\Delta_V)"'] \arrow[dr, "\Delta_\Gamma"] & \\ \sum_{s,t} s \otimes t \otimes e_t(e_s v) & & \sum_{s\in\Gamma} s \otimes s \otimes e_s v \end{tikzcd}$$

$$e_t e_s v = \begin{cases} e_s v & s = t \\ 0 & \text{otherwise} \end{cases} \qquad \therefore\ e_t \text{ annih. idempotents}$$

$$(\eta \otimes 1)\Delta v = v$$
$$\sum_{s\in\Gamma} e_s v = v \qquad \forall v.$$

$$V = \bigoplus_{s\in\Gamma} V_s \qquad \mathrm{Mod}(\mathbb{C}) \qquad \mathrm{Mod}(\hat{\Gamma})$$

$$\left\{\begin{matrix}\text{vector}\\ \text{spaces}\end{matrix}\right\} \underset{\varepsilon_*}{\overset{\varepsilon_!}{\leftleftarrows}} \!\!\xrightarrow{\varepsilon^*}\! \left\{\begin{matrix}\Gamma\text{-graded}\\ \text{vector sp.}\end{matrix}\right\} \qquad \varepsilon : \Gamma \longrightarrow pt$$

$$\varepsilon_!(V_s)_{s\in\Gamma} = \bigoplus_{s\in\Gamma} V_s$$
$$\varepsilon^*(W) = (W)_{s\in\Gamma}$$
$$\varepsilon_*(V_s)_{s\in\Gamma} = \prod_{s\in\Gamma} V_s$$

$$\mathrm{Hom}\Big(\bigoplus_{s\in\Gamma} V_s, W\Big) = \prod_{s\in\Gamma} \mathrm{Hom}(V_s, W) = \mathrm{Hom}_{\hat{\Gamma}\text{-mod}}\big((V_s)_{s\in\Gamma}, (W)_{s\in\Gamma}\big)$$

To understand the grading: P_Φ ~~...~~ defined to be the alg gen. by p_s, $s \in \Gamma$ subj to the relations $p_s = \sum_t p_t p_{t^{-1}s}$, $p_s = 0$ for $s \notin \Phi$.

Proof that P_Φ is a Γ-graded algebra. There is a canon. homom.

$$P_\Phi \xrightarrow{\Delta_p} \mathbb{C}[\Gamma] \otimes P_\Phi \quad , \quad p_s \longmapsto s \otimes p_s$$

$$\sum_{t \in \Gamma} (t \otimes p_t)(t^{-1}s \otimes p_{t^{-1}s}) = s \otimes \sum_{t \in \Gamma} p_t p_{t^{-1}s} = s \otimes p_s$$

Moreover what? $\quad (\Delta_\Gamma \otimes 1)\Delta_p \overset{?}{=} (1 \otimes \Delta_p)\Delta_p$

$$p_s \xrightarrow{\Delta_p} s \otimes p_s \underset{1 \otimes \Delta_p}{\overset{\Delta_\Gamma \otimes 1}{\rightrightarrows}} \begin{matrix} s \otimes s \otimes p_s \\ s \otimes s \otimes p_s \end{matrix} \qquad \begin{matrix} (\eta \otimes 1)\Delta_p = 1 \\ (\eta \otimes 1)(s \otimes p_s) = p_s \end{matrix}$$

So P_Φ has natural Γ-grading. Is it clear that P_Φ is a Γ-graded algebra?

Answer is that $\mathbb{C}[\Gamma] \otimes D$ is always a Γ-graded algebra with D in degree 1.

$$\mathrm{Hom}_{\mathbb{C}}\left(\oplus V_s, W\right) = \mathrm{Hom}_{\hat{\Gamma}}\left(\bigoplus_{s \in \Gamma} V_s, \mathbb{C}[\Gamma] \otimes W\right)$$

What about $\otimes$ product for Γ-graded vector spaces

$$(V \otimes W)_s = \bigoplus_{s = tu} V_t \otimes W_u = \bigoplus_t V_t \otimes W_{t^{-1}s}$$

$$V \otimes W \xrightarrow{\Delta_V \otimes \Delta_W} \mathbb{C}[\Gamma] \otimes V \otimes \mathbb{C}[\Gamma] \otimes W$$

$$\mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \otimes V \otimes W \xrightarrow{\mu_* \otimes 1_V \otimes 1_W} \mathbb{C}[\Gamma] \otimes V \otimes W$$

So go over Γ-grading

$\otimes$ product for Γ-graded v.s. $\quad (V\otimes W)_s = \bigoplus_t V_t \otimes W_{t^{-1}s}$

$$V \otimes W = \bigoplus_{t,u} V_t \otimes W_u = \bigoplus_s \bigoplus_t V_t \otimes W_{t^{-1}s}$$

alg. $\quad A = \bigoplus A_s$

$$\begin{tikzcd} A\otimes A \arrow[r] \arrow[d, phantom, "\parallel"] & A \arrow[d, phantom, "\parallel"] \\ \bigoplus_s \left(\bigoplus_t A_t \otimes A_{t^{-1}s}\right) \arrow[r] & \bigoplus_s A_s \end{tikzcd}$$

not very clear it seems

What is a Γ graded algebra A. First a Γ grading $A = \bigoplus_{s\in\Gamma} A_s$, then a product $A\otimes A \longrightarrow A$ assoc. etc. respecting the total Γ grading on $A\otimes A$ where $(A_s \otimes A_t) \subset A_{st}$.

So go back to your $B = C \rtimes \Gamma$. This is naturally a Γ-graded alg so there ~~is a~~ should be a homom. $B \longrightarrow B \otimes \mathbb{C}[\Gamma]$ such that $B_s \longrightarrow B_s \otimes s$. You are confused again

$$B = \bigoplus_{s\in\Gamma} B_s = \bigoplus_{s\in\Gamma} C_s = \bigoplus_{s\in\Gamma} sC$$

What is unclear. What is the significance of the canonical homom. $B \longrightarrow B \overset{\times}{\otimes} \Gamma$

$$\begin{array}{ccc} B & \longrightarrow & B\otimes \mathbb{C}[\Gamma] \\ \cup & & \cup \\ B_s & \longrightarrow & B_s \otimes s \\ \uplus & & \\ b & & b\otimes s \end{array}$$

It is a Γ-graded alg homos. where B is trivially graded on the right side?

What sort of meaning?

There should be adjoint functors

~~[illegible]~~ $\{\text{algs}\}$: $\{\hat{\Gamma}\text{ algs}\}$?

$$\{\mathbb{C}\text{-modules}\} \underset{\varepsilon_*}{\overset{\varepsilon_!}{\leftarrow}} \xrightarrow{\varepsilon^*} \{\Gamma\text{ graded } \mathbb{C}\text{-modules}\}$$

$\varepsilon : \Gamma \to pt.$

$$\underbrace{\varepsilon_!((V_s)_{s\in\Gamma}) = \bigoplus_{s\in\Gamma} V_s} \qquad \varepsilon_*((V_s)_{s\in\Gamma}) = \prod_{s\in\Gamma} V_s$$

$$\mathrm{Hom}_{\mathbb{C}}(\ \ , W) = \mathrm{Hom}_{\hat{\Gamma}}((V_s)_{s\in\Gamma}, \varepsilon^* W)$$

$$(\varepsilon^* W)_s = W$$

If ~~$V \cong \bigoplus_{s\in\Gamma} V_s$, then~~ $(V_s)_{s\in\Gamma}$ is a $\hat{\Gamma}$-module

$\exists$.

$$\mathrm{Hom}(\varepsilon_! V, W) = \mathrm{Hom}(V, \varepsilon^* W) \qquad 1 \to GF$$

canon, ~~[illegible]~~ $(V_s)_{s\in\Gamma} \longrightarrow (\bigoplus_{\Gamma} V_s)_{s\in\Gamma}$

$$\mathrm{Hom}_{alg}\Big(\bigoplus_{s\in\Gamma} A_s, B\Big) = \mathrm{Hom}_{\hat{\Gamma}\text{-alg}}\big((A_s)_{s\in\Gamma}, (B)_{s\in\Gamma}\big)$$

$$= \mathrm{Hom}_{\hat{\Gamma}\text{-alg}}\Big(\bigoplus_{s\in\Gamma} A_s, \mathbb{C}[\Gamma]\otimes B\Big)$$

So there is always a canonical alg hom.

$$\bigoplus_{s\in\Gamma} A_s \longrightarrow \mathbb{C}[\Gamma] \otimes \bigoplus_s A_s$$

$$A_s \longrightarrow s \otimes A_s$$

So you're still puzzled by this construction.

Dually consider G-set X ~~[illegible]~~

$\{G\text{-sets}\}$ $\{\text{sets}\}$ $f : G \to pt$?

$$\{G\text{-sets}\} \underset{f_*}{\overset{f_!}{\leftarrow}} \xleftarrow{f^*} \{\text{sets}\}$$

canon. $G \times X \longrightarrow X$

$\{G\text{-sets}\} \xleftarrow{f^*} \{\text{sets}\} \qquad G\times X \longleftarrow X$

$$\mathrm{Hom}_G(\underbrace{G\times X}_{f^*X}, E) = \mathrm{Hom}_{\text{sets}}(X, \underbrace{E}_{\text{forget } G\text{-action}})$$

$$\{G \text{ sets}\} \underset{\underset{\text{forget } G\text{-action}}{\phi}}{\overset{\phi_!}{\rightleftarrows}} \{\text{sets}\}$$

$$\mathrm{Hom}_{\text{sets}}(X, \phi E) = \mathrm{Hom}_{G\text{-sets}}(\overbrace{G\times X}^{\phi_!(X)}, E)$$

On sets you have $T(X) = (\phi\phi_!)(X) = G\times X$.

an alg for the triple T is a ~~set X together with~~ pair (X,μ), X set $\mu: T(X) = G\times X \longrightarrow X$.

$\quad R \ L$

$T = \phi\phi_!$

$id \longrightarrow T$

$T^2 \longrightarrow T$

Conditions $\mu: T(X) \longrightarrow X$

$$\begin{tikzcd} TT(X) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(X) \arrow[d, "\mu"] \\ T(X) \arrow[r, "\mu"] & X \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r, "\varepsilon"] \arrow[dr, "id"'] & TX \arrow[d, "\mu"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} x & (1,x) \\ X \arrow[r] & G\times X \arrow[d] \\ & X \end{tikzcd}$$

$$\begin{tikzcd} G\times G\times X \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & G\times X \arrow[d, "\mu"] \\ G\times X \arrow[r, "\mu"] & X \end{tikzcd} \qquad \begin{tikzcd} (g_1, g_2, x) \arrow[r, mapsto] \arrow[d, mapsto] & (g_1, g_2 x) \arrow[d, mapsto] \\ (g_1 g_2, x) \arrow[r, mapsto] & (g_1 g_2)x \quad g_1(g_2 x) \end{tikzcd}$$

g_1

$B = C \rtimes \Gamma$ $\qquad$ Γ-graded alg

C gen $h_s,\ s\in\Gamma$ $\qquad$ $h_s h_t \underset{s^{-1}t\in\Phi}{=} 0$ $\quad s^{-1}t \notin \Phi$

$$h_t = \sum_{s\in t\Phi^{-1}} h_s h_t = \sum_{s\in t\Phi} h_t h_s \qquad \begin{matrix} t^{-1}s\in\Phi \\ s\in t\Phi \end{matrix}$$

$$C \twoheadrightarrow h_1 C \hookrightarrow C \qquad h_1 = \sum h_t h_1$$

$$h_1 = \Big(\sum_{t\in\Phi^{-1}} h_t\Big) h_1 \qquad t^{-1}1 \in \Phi$$

you have lots of choice for ?

Let's organize. When you factor $h_1 = \beta_1 \alpha_1$ you get a projection the other way.

$B = C \rtimes \Gamma$ $\qquad$ C: gen $h_s,\ s\in\Gamma$
rels $h_s h_t = 0 \quad s^{-1}t \notin \Phi$

$$h_t = \sum_s h_s h_t = \sum_s h_t h_s$$

C is a Γ-algebra: $s * h_t = h_{st}$

$$h_1 = \sum_{s^{-1}\in\Phi} h_s h_1 = \sum_{s\in\Phi} h_1 h_s$$

$$h_1 = \Big(\sum_{s\in K} h_s\Big) h_1 \qquad h_1 = h_1 \sum_{s\in K}{}' h_s$$

What's important, deserves emphasis, is that each time you factor $h_1 = \beta_1\alpha_1$ $\in$? ? you get a projection somewhere.

You need to organize the Morita equivalence with ~~the other~~ what happens to the rings.

Basic rings are $B = \mathbb{C}_{\Phi} \rtimes \Gamma$ and $A = P_{\Phi}$ which are both naturally Γ-graded algebras. What you are looking for is a good picture of the bimodules giving the Morita equivalence. B is inherently non commutative. But A might be commutative when Γ is.

$$p_s = 0 \qquad s \notin \Phi$$

$$p_s = \sum_{t} p_t p_{t^{-1}s} \quad \text{in } A$$

$$\Longrightarrow \quad p_s = \sum_{t} p_{t^{-1}s} p_t \quad \text{in } A^{op}$$

$$\text{so } A \simeq A^{op}$$

$$= \sum_{t} p_{st^{-1}} p_t = \sum_{t^{-1}s} p_{t} p_{t^{-1}s} \qquad \text{OK.}$$

Maybe it's easier to look for $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$

i.e. a dual pair (Q, P) over B; yielding A somehow, so you want a pairing $P \otimes Q \longrightarrow B$. ~~and the relations~~

Question: Suppose given P_A, ${}_AQ$ and generators a_i for A. ~~There is $P \times Q$~~ The question is whether

$$\bigoplus_i P \otimes_{\mathbb{Z}} Q \xrightarrow{(\cdot a_i)\otimes 1 - 1 \otimes (a_i \cdot)} P \otimes_{\mathbb{Z}} Q$$

has cokernel $P \otimes_A Q$.

$$p a_1 \cdots a_k \otimes q .$$

What should P, Q be? P is the B-module corresp to the A-module A (or $\tilde{A}$). So take $\mathbb{C}[\Gamma] \otimes A$ with the canon. proj.

How to handle? ~~understand~~

Go back to projections. $B = C \rtimes \Gamma$.
$h_1 = h_K h_1 = h_1 h_K$ if $K \supseteq \Phi \cup \Phi^{-1}$. Then for each choise of K you get a $p = (p_s)_{s \in \Gamma}$

$p = \sum_{s \in \Gamma}' s \otimes p_s \in \mathbb{C}[\Gamma] \otimes B$, where $p_s = \alpha_1 s \beta_1$

$p_s = h_1 s h_K$. This is a Γ-graded projection in B

Puzzling.

~~There is a basis~~

You seem to have something intrinsic ~~interesting, significant,~~ arising from a choice of factorization $h_1 = \alpha_1 \beta_1 : E \xrightarrow{\beta_1} V \xrightarrow{\alpha_1} E$. ~~Once you make~~ ~~What's important?~~ Any choice yields the same $\overset{\text{round}}{}$ A-module up to isomorphism.

nilpotence?

First case: $h_1^2 = h_1$, then what can you say? ~~to look at what hap~~ Among the factorizations of h_1 is the smallest with β_1 surj α_1 injective. Drop 1's.

$$
\begin{tikzcd}
E \arrow[rr, "\alpha"] \arrow[dr] & & V \arrow[rr, "\beta"] \arrow[dr] & & E \\
& \alpha E \arrow[ur] \arrow[dr] & & \beta V \arrow[ur] & \\
& & hV \arrow[ur] & & \\
& & \beta\alpha V \arrow[u, phantom, "\shortparallel"] & &
\end{tikzcd}
$$

Do we actually get a map from V to

So given
$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rr, bend left, "h"] & V \arrow[r, "\beta"] & E \end{tikzcd}$$
$h = \beta\alpha$ you get

$p(s) = \alpha s \beta \in \mathrm{End}(V)$ is Γ-graded projection

$$\begin{tikzcd} & V_1 \arrow[r, "f"] \arrow[drr, "\beta_1 = \beta_2 f"'] & V_2 \arrow[dr, "\beta_2"] & \\ E \arrow[ur, "\alpha_1"] \arrow[urr, "\alpha_2 = f\alpha_1"'] & & & E \end{tikzcd}$$

So what happens. $p_1(s) = \alpha_1 s \beta_1 = (\alpha_1 s \beta_2) f$

$p_2(s) = \alpha_2 s \beta_2 = f(\alpha_1 s \beta_2)$, $\quad f p_1(s) = f(\alpha_1 s \beta_2) f = p_2(s) f$

Thus f induces a map
$$1\otimes f : \mathbb{C}[\Gamma] \otimes V_1 \longrightarrow \mathbb{C}[\Gamma] \otimes V_2$$
which intertwines p_1 and p_2: $(1\otimes f) p_1 = p_2 (1\otimes f)$

$(1\otimes f)(1-p_1) = (1-p_2)(1\otimes f)$

$$\begin{array}{ccccc} \mathbb{C}[\Gamma]\otimes V_1 & = & \mathrm{Im}(p_1) & \oplus & \mathrm{Im}(1-p_1) \\ \downarrow{\scriptstyle 1\otimes f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ \mathbb{C}[\Gamma]\otimes V_2 & = & \mathrm{Im}(p_2) & \oplus & \mathrm{Im}(1-p_2) \end{array}$$

All this seems too easy. Given E a ${}^{\text{finite}}$ $B = C \rtimes \Gamma$ module. ~~[illegible]~~

Guess that your bimodules are versions of C as in the geometric case.

~~[illegible]~~

You have this algebra C on which ~~acted on [illegible]~~ Γ operates, allowing you to form a Γ-graded algebra $B = C \rtimes \Gamma$. Then your first guess is that the ~~bimodule~~ dual pair over B ?

To guess in this fashion ~~seems~~ bad. You should be able to motivate your constructions

Go back to a firm module E over $B = C \rtimes \Gamma$ (which has local units, left and right, hence E firm means $E = BE = \sum sh_1 E$).

Choose a factorizations of the operator h_1 on E:

$h_1 = \beta_1 \alpha_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$, whence

you have

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rr, bend right, "1"'] & \mathbb{C}[\Gamma]\otimes V \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, "\alpha"] & \mathbb{C}[\Gamma]\otimes V \end{tikzcd}$$

$$(\alpha\beta f)(s) = \sum_t \underbrace{\alpha_1 s^{-1} t \beta_1}_{p(s^{-1}t)} f(t) \qquad \boxed{\begin{array}{l} p_s = \alpha_1 s \beta_1 \\ s \in \Gamma \end{array}}$$

$$\beta\alpha = 1 \implies p^2 = \alpha\underbrace{\beta\alpha}_{1}\beta = \alpha\beta = p.$$

Is the converse true? $\quad p^2 = p \iff \alpha(1-\beta\alpha)\beta = 0$

$p^2 = p \iff \quad \alpha\beta = \alpha\beta\alpha\beta$

Q: does $\alpha\beta = \alpha\beta\alpha\beta \implies \beta\alpha = 1$?

However $\alpha\beta = \alpha\beta\alpha\beta \implies \beta\alpha\beta = \beta\alpha\beta\alpha\beta$

$\implies (\beta\alpha)^2 = (\beta\alpha)^3$

$\implies (\beta\alpha)^2(1-\beta\alpha) = (1-\beta\alpha)(\beta\alpha)^2 = 0$

So E splits into $\mathrm{Ker}\left((\beta\alpha)^2\right)$ and $\mathrm{Ker}(1-\beta\alpha)$ and these are compatible with α, β.

Repeat this

$$E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V$$

$$\beta\alpha = 1 \implies p = \alpha\beta = \alpha\beta\alpha\beta = p^2$$

Converse? Assume $\alpha\beta = \alpha\beta\alpha\beta$

$$\beta\alpha\beta = \beta\alpha\beta\alpha\beta = (\beta\alpha\beta)p$$
$$\alpha\beta\alpha = \alpha\beta\alpha\beta\alpha = p(\alpha\beta\alpha).$$

But $(\alpha\beta)^2 = (\alpha\beta)^3 \quad \therefore \quad (\alpha\beta)^2(1-\alpha\beta) = 0$

$$p^2(1-p) = 0$$

So $\mathbb{C}[\Gamma]\otimes V = \mathrm{Ker}(p^2) \oplus \mathrm{Ker}(1-p)$

$$= \mathrm{Im}(1-p) \oplus \mathrm{Im}(p^2)$$

splits into ~~subspace~~ $+1$ eigenspace for p and 0 gen. eigenspace.

What is your aim? You should have this Morita between ~~from~~ $B = C\rtimes\Gamma$ modules E and round $A = P_\Phi$ modules V

P_Φ: alg gen. by $p(s)$, $s\in\Gamma$ subject to relns.

$$p(s) = \sum_{t\in\Gamma} p(t)p(t^{-1}s) \quad , \quad p(s) = 0 \quad s\notin\Phi.$$

Given a P_Φ module V, then you define the operator p on $\mathbb{C}[\Gamma]\otimes V$ by

$$(pf)(s) = \sum_{t\in\Gamma} p(s^{-1}t)f(t)$$

$p(s^{-1}t)\neq 0 \Rightarrow s^{-1}t\in\Phi \Rightarrow s\in t\Phi^{-1}$

then
$$(ppf)(s) = \sum_{t\in\Gamma} p(s^{-1}t)\sum_{u\in\Gamma} p(t^{-1}u)f(u)$$

$$= \sum_{u\in\Gamma}\Big(\underbrace{\sum_{t\in\Gamma} p(s^{-1}t)p(t^{-1}u)}_{\sum_t p(t)p(t^{-1}s^{-1}u) = p(s^{-1}u)}\Big) f(u)$$

You want +1 eigenspace i.e

$$\{f(t) \in \mathbb{C}[\Gamma] \otimes V \mid pf = f\}.$$

You may have learned something here. about the +1 eigenspace being important.

~~What seems to happen is that?~~

What's the problem? If you start with the P_Φ-module V, then you get a projection p on $\mathbb{C}[\Gamma] \otimes V$, an operator comm. with Γ mult and idempotent. The image of p = +1 eigenspace is an exact exact functor of V. It kills nil-modules i.e. modules such that $p(s)V = 0$ all $s \in \Gamma$.

What you want to do understand the converse better

Yesterday some progress was made on the problem following. When you factor $h_1 = \beta_1 \alpha_1$ then the support condition $h_1 s h_1 = 0$ is weaker than $p(s) = \alpha_1 s \beta_1 = 0$.

Hopefully $\alpha_1 s \beta_1$ will be zero on any firm module

Example $\Gamma = \mathbb{Z}/2$. C gen h_+, h_- rels. ?

Suppose $\Phi = \{1\}$, so that C has gen. h_s, $s \in \Gamma$ relations $h_s h_t = 0$ $s \neq t$, $\cancel{h_s = \sum_t h h_{st} = \sum_t h_t h_s}$. $h_s = h_s^2$

So $C = C_c(\Gamma)$ functions under mult. Now factor h_1 into $\beta_1 \alpha_1$, i.e. adjoin to C

Adjoin elements β_1, α_1 to C such that $h_1 = \beta_1\alpha_1$

also $E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$ $\qquad h_s\alpha_1 = 0$, $\beta_1 h_s = 0$

$$\begin{pmatrix} C & C\beta_1 \\ \alpha_1 C & \alpha_1 C \beta_1 \end{pmatrix}$$

Set up the obvious. ~~Start with $B = C \rtimes \Gamma$.~~

Start with your Morita equivalence between

$A = P_\Phi$ and $B = C_\Phi \rtimes \Gamma$.

You have specific functors

~~think of factoring~~ Take $\Phi = \{1\}$ whence $C_\Phi = C_c(\Gamma)$ $\qquad h_1 = \beta_1\alpha_1$

fin supp functors. $\qquad h_1^2 = h_1$. Look at factorizations

in this case, Fin B-modules have form $E = \underset{C_c(\Gamma)\,?}{\mathbb{C}[\Gamma]} \otimes V$, where

$V = h_1 E$. You probably want a general fact.

Note that yesterday you looked at $p = \alpha\beta$.

What did you learn?

$$\begin{tikzcd} E \arrow[r, "\alpha"] \arrow[rr, bend left, "1"] & V \arrow[r, "\beta"] \arrow[rr, bend right, "p"'] & E \arrow[r, "\alpha"] & V \end{tikzcd}$$

$$\beta\alpha = 1 \implies p^2 = \alpha\beta\alpha\beta = \alpha\beta = p$$

Converse $\quad p^2 = p \implies \beta\alpha\beta\alpha\beta\alpha = \beta\alpha\beta\alpha \quad$ i.e. $p^3 = p^2$

You looked at a factorization of an idempotent and found a splitting. The Ring in question was

$$\begin{pmatrix} \mathbb{C}\alpha\beta & \mathbb{C}\alpha \\ \mathbb{C}\beta & \mathbb{C}\beta\alpha \end{pmatrix} \in \mathrm{End}(V \oplus E)$$

$\beta\alpha = 1 \implies$ $\alpha = \alpha\beta\alpha = p\alpha$

$\beta = \beta\alpha\beta = \beta p$

$\alpha\beta = p$

Special $\Phi = \{1\}$. $C_\Phi = C_c(\Gamma)$ $B = C_c(\Gamma) \rtimes \Gamma$

~~so~~ Morita equiv. to $\mathbb{C}$. B has $h_s = s h_1 s^{-1}$ sat. $h_1 s h_1 = 0$ for $s \neq 1$. $h_s^2 = h_s$ $h_s h_t = 0$ $s \neq t$.

~~So take a map~~ Let E be a firm B-module, then $E \cong C_c(\Gamma) \otimes W$, but ~~if~~ suppose you factor h_1 = proj onto W in a stupid way.

$$h_1 = \beta_1 \alpha_1 : E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E$$

30 K

~~[illegible]~~

$$\begin{tikzcd}
E \arrow[d, equal] & & \\
C_c(\Gamma)\otimes W \arrow[r] \arrow[d] & V \arrow[r] & C_c(\Gamma)\otimes W \\
W \arrow[rr, equal] & & W \arrow[u]
\end{tikzcd}$$

so you have

$$\begin{tikzcd}
 & V \arrow[dr] & \\
C_c(\Gamma)\otimes W \arrow[ur] \arrow[dr, two heads] & & C_c(\Gamma)\otimes W \\
 & W \arrow[ur, hook] &
\end{tikzcd}$$

Go over it carefully to ~~u~~ understand.

$$\begin{tikzcd}
E \arrow[r, "\alpha_1"] \arrow[d, two heads] & V \arrow[r, "\beta_1"] \arrow[d, two heads] & E \\
\alpha_1 E \arrow[ur, hook] \arrow[dr, two heads] & \beta_1 V \arrow[ur, hook] & \\
 & \underset{W}{\beta_1 \alpha_1 E} \arrow[u, hook] &
\end{tikzcd}$$

You know that

$$E = W \oplus W^\perp$$

What are you looking for? W should be a summand of V. ~~So you~~ If you split off W, then you get

~~then~~ $C_c(\Gamma)\otimes W \xrightarrow{f} W \xrightarrow{g} C_c(\Gamma - \{1\}) \otimes W$ $\to$ $gf = 0$.

~~Also~~ special case being studied. Γ arb. $\Phi = \{1\}$.

$C_\Phi = C_c(\Gamma)$ = functions fin supp on Γ under mult.

$B = C_\Phi \rtimes \Gamma$ is Meq to $\mathbb{C}$. ~~A~~ firm B-module has form $\mathbb{C}[\Gamma] \otimes W$

But now consider a factorization of h_1

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \qquad h_1 = \beta_1\alpha_1 \text{ idemp.}$$

The good case is when $\alpha_1\beta_1 = 1$ on V. Let's review the analysis of this situation

$$\begin{tikzcd} E \arrow[r, "\alpha"] & V \arrow[r, "\beta"] \arrow[rr, bend left, "p"] & E \arrow[r, "\alpha"] & V \end{tikzcd}$$

$$\beta\alpha = 1 \implies \alpha\beta\alpha\beta = \alpha\beta \quad \text{so } p^2 = p$$

Conversely assume $p^2=p$. ~~Then V~~ Then $(\beta\alpha)^3 = (\beta\alpha)^2$.

You have a splitting

$$E = E_1 \oplus E_0$$
$$V = V_1 \oplus V_0$$

where $E_1 = \mathrm{Ker}(1-\beta\alpha)$, $E_0 = \mathrm{Ker}(\beta\alpha)^2$

$V_1 = \mathrm{Ker}(1-\alpha\beta)$, $V_0 = \mathrm{Ker}(\alpha\beta)$

$$\alpha(1-\beta\alpha) = (1-\alpha\beta)\alpha \qquad \therefore \ \alpha : E_1 \longrightarrow V_1$$
$$\beta(1-\alpha\beta) = (1-\beta\alpha)\beta \qquad \therefore \ \beta : V_1 \longrightarrow E_1$$

inverse isos.

$$\alpha(\beta\alpha)^2 = \underbrace{(\alpha\beta)^2}_{\alpha\beta}\alpha \qquad \alpha : E_0 \longrightarrow V_0$$

$$\beta(\alpha\beta) = \beta(\alpha\beta)^2 = (\beta\alpha)^2\beta \qquad \beta : V_0 \longrightarrow E_0$$

$$\boxed{\begin{array}{ll} \alpha\beta = 0 & \text{on } V_0 \\ (\beta\alpha)^2 = 0 & \text{on } E_0 \end{array}}$$

what does this mean?

simply that $V_0 \xrightarrow{\beta_0} E_0 \xrightarrow{\alpha} V_0$ is a complex

$$\alpha\beta = 0 \implies \beta\alpha\beta\alpha = 0 \quad \text{~~complex~~}.$$

Now arrange this better $\begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$ on $\begin{pmatrix} E \\ V \end{pmatrix}$

The puzzle: interplay between

$$E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \xrightarrow{\alpha_1}$$

$$E \xrightarrow{\alpha} \mathbb{C}[\Gamma]\otimes V \xrightarrow{\beta} E \xrightarrow{\alpha}$$

Repeat $\begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$ on $\begin{pmatrix} E \\ V \end{pmatrix}$. I am reminded of length 1 complexes and homotopy. But in any case you assume $p = \alpha\beta$ is idempotent on V then $(\beta\alpha)^3 = \beta p^2 \alpha = \beta p \alpha = (\beta\alpha)^2$ is idempotent on E. You get then a splitting of $\begin{pmatrix} E \\ V \end{pmatrix}$

$$X^2 = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}\begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} = \begin{pmatrix} \beta\alpha & \\ & \alpha\beta \end{pmatrix}$$

$$X^3 = \begin{pmatrix} 0 & \beta\alpha\beta \\ \alpha\beta\alpha & 0 \end{pmatrix} \qquad X^4 = \begin{pmatrix} (\beta\alpha)^2 & 0 \\ 0 & (\alpha\beta)^2 \end{pmatrix}$$

The important point

You need to understand possible factorizations of an idempotent. Say given $E \xrightarrow{\alpha} V \xrightarrow{\beta} E$ such that $(\beta\alpha)^2 = \beta\alpha$. Put $p = \beta\alpha$ on E. Consider the composition $pE \hookrightarrow E \xrightarrow{\alpha} V \xrightarrow{\beta} E \xrightarrow{p} pE$. This is the identity of pE, which means pE splits off both E and V. It should be true that the complements of pE yield $0 \hookrightarrow E^c \xrightarrow{\alpha} V^c \xrightarrow{\beta} E^c \longrightarrow 0$?

Repeat: $X = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$ on $\begin{pmatrix} E \\ V \end{pmatrix}$ $\qquad E \xrightarrow{\alpha} V \xrightarrow{\beta} E \xrightarrow{\alpha} V$

$X^2 = \begin{pmatrix} \beta\alpha & 0 \\ 0 & \alpha\beta \end{pmatrix}$ the real issue is what you can say when $\beta\alpha$, $\alpha\beta$ are idempotent.

~~Initial~~ question was ~~about~~ to understand factoring a projection p. Thus you have $\alpha\beta = p = p^2$ and you want to work out the structure. The idea is that

$$pV \hookrightarrow V \xrightarrow{\beta} E \xrightarrow{\alpha} V \xrightarrow{p} pV$$

is the identity on pV.

You've switched from $p = \beta\alpha$ on p91 to $p = \alpha\beta$ above.

Go over again. Let $p : M \to M$ be idempotent: $p^2 = p$ so that $M = \underset{\mathrm{Ker}(1-p)}{\underset{\parallel}{pM}} \oplus \underset{\mathrm{Ker}(p)}{\underset{\parallel}{(1-p)M}}$. Suppose that we have maps $M \xrightarrow{\beta} N \xrightarrow{\alpha} M$ such that $p = \alpha\beta$

You are reminded of GNS. $\quad A \xrightarrow{\rho} B \quad M \underset{i}{\overset{j}{\rightleftarrows}} N \quad j\,a\,i(n) = \rho(a)n$

For the moment avoid coincidence of notation.

$$\begin{array}{ccccc} & & m & \longmapsto & (a' \mapsto j(a'm)) \\ A \otimes N & \longrightarrow & M & \longrightarrow & \mathrm{Hom}(A, N) \\ a \otimes n & \longmapsto & a\,i\,n & \longmapsto & (a' \mapsto \underset{\rho(a'a)\,n}{j(a'a\,i\,n)}) \end{array}$$

Something is happening. Go back to

$$E \xrightarrow{\alpha} V \xrightarrow{\beta} E \xrightarrow{\alpha} V$$

You use two things $\quad E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \qquad h_1 = \beta_1\alpha_1$

and $\quad E \xrightarrow{\alpha} \mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E \longrightarrow$