Topic: $\Gamma$-graded vector spaces and algebras

Background: You have already encountered these when $\Gamma$ is a group (more generally a monoid, a semi group even). A $\Gamma$-graded algebra is an algebra $B$ equipped with a splitting into subspaces $B = \bigoplus B_s$ indexed by $\Gamma$ such that $B_s B_t \subset B_{st}$. Example: If $A$ is a $\Gamma$ algebra, the cross product $B = \Gamma \times A = \bigoplus s A = \bigoplus As$ is a $\Gamma$-graded algebra.

Example: The algebra $A = P_{\Gamma, F}$ is defined by generators $p(s)$ for $s \in \Gamma$ subject to the relations $p(s) = 0$ for $s \not\in F$, $p(s) = \sum_{t} p(t)p(t^{-1}s)$. These are homogeneous (both generators + relations) if $p(s)$ is assigned the degree $s$, so $A$ should be a $\Gamma$-graded algebra with the $\Gamma$-grading specified in this way. One way to show this would be to define the $\Gamma$-grading on the free algebra with the generators $p(s)$ - this uses the tensor product operation on $\Gamma$-graded vector spaces defined by

$$(V \otimes W)_s = \bigoplus_t V_t \otimes W_{t^{-1}s}.$$  

Then you check that any element in the ideal generated by the relations is a sum of homogeneous elements lying in this ideal. Hence the ideal is a $\Gamma$-graded subspace of the free algebra, and
b. the quotient $A$ inherits a $Γ$-grading making it a $Γ$-graded alg.

Here is a clearer way to show that $A$ is a $Γ$-graded algebra. The tensor product algebra $CΓ ⊗ A$ is a $Γ$-graded algebra with $(CΓ ⊗ A)_s = s ⊗ A$. Using the generators and relations defining $A$ one has a unique $γ$-homomorphism

$$Δ : A → CΓ ⊗ A \quad Δp(s) = s ⊗ p(s)$$

Let $A_s$ be the subspace of $A$ spanned by monomials $p(s_1)p(s_2)\cdots p(s_n)$ in the generators of total degree $s_1, s_2, \ldots, s_n$ equal to $s$. Clearly $Δ(a_s) = s ⊗ a_s$ for any $a_s ∈ A_s$.

Also $A = \sum_{s ∈ Γ} A_s$. This must be a direct sum since if we have $\sum q_s = 0$ with $q_s ∈ A_s$, then applying $Δ$ yields $\sum s ⊗ q_s = 0$ in $CΓ ⊗ A$, which is possible only if $q_s = 0$ for all $s$. Thus we have a $Γ$-grading on $A$ making $A$ a $Γ$-graded algebra.

Problem: A Morita context is a ring equipped with a grading $A = \bigoplus_{s ∈ Γ} A_s$ such that if we write an element $a$ of $A$ as a $2 \times 2$ matrix:

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} ∈ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A$$

the multiplication in $A$ is given by matrix multiplication.
that is

\[
\begin{pmatrix}
a_{11}' & a_{12}' \\
a_{21}' & a_{22}'
\end{pmatrix}
\begin{pmatrix}
a_{11}'' & a_{12}'' \\
a_{21}'' & a_{22}''
\end{pmatrix}
= 
\begin{pmatrix}
a_{11}' a_{11}'' + a_{12}' a_{21}'' & a_{11}' a_{12}'' + a_{12}' a_{22}'' \\
a_{21}' a_{11}'' + a_{22}' a_{21}'' & a_{21}' a_{12}'' + a_{22}' a_{22}''
\end{pmatrix}
\]

Here we have a grading indexed by the set \( \Gamma \) of ordered pairs \((i,j)\) with \( i, j = 1 \) or \( 2 \). The product \( A_{ij} A_{ik} \) is zero when \( j \neq k \) and contained in \( A_{il} \) when \( j = k \). Thus \( \Gamma \) is the set of arrows in the groupoid having objects in \{1, 2\} and with exactly one map from one object to another.

The analog of \( C \Gamma \) in this situation is the path algebra of the groupoid, which is \( M_2(C) \).

Our aim now is to find a generalization of these two types of \( C \Gamma \) discussed above.

The coalgebra \( C \Gamma \). If \( \Gamma \) is any set, then \( C \Gamma \) equipped with the co-product

\[
\Delta: C \Gamma \rightarrow C \Gamma \otimes C \Gamma \quad \Delta = \sigma \otimes \sigma
\]

is a coassociative, commutative, and counital coalgebra. The counit is \( \eta: C \Gamma \rightarrow C \), \( \eta \circ \Delta = 1 \).

Ex: If \( \Gamma \) is a point, then \( C \Gamma = C \) with \( \Delta(1) = 1 \otimes 1 \in C \otimes C \).

By a point of a coalgebra \( C \Gamma \) we mean a coalgebra morphism \( \phi: C \rightarrow C \Gamma \), equivalently an element \( \phi \in C \) satisfying \( \Delta \phi = \phi \otimes \phi \). If \( C \) has counit \( \eta: C \rightarrow C \), then the point \( \phi \) will be called unital action of \( \phi \) respects counits: \( \eta \circ \phi = \text{id} \).
equivalently, \( f(1) = 1 \). Note that the identity and zero are the only coaly morphisms from \( C \) to \( C \). Thus a point \( \phi \) is unital iff it is zero.

Calculation of the points of \( C \Gamma \). Let \( \delta = \sum \lambda_s \delta_s \) in \( C \Gamma \) satisfy \( \Delta(\delta) = \delta \otimes \delta \), that is

\[
\sum \lambda_s \delta_s \delta_s = \sum \lambda_s \lambda_t \delta_s \delta_t.
\]

Then \( \lambda_s \lambda_t = 0 \) for \( s \neq t \) and \( \lambda_s^2 = \lambda_s \). Thus either all \( \lambda_s = 0 \) and we have the zero point, or there is exactly one \( \lambda_s = 1 \) and the rest are zero.

Therefore

\[
\text{Points } (C \Gamma) = \Gamma \circ \{0, \delta\}
\]

\[
\text{Unital Points } (C \Gamma) = \Gamma
\]

Clearly one has functions \( C \rightarrow \text{Points}(C) \) from coalgebras to sets with basepoint, and \( C \rightarrow \text{Unital Points}(C) \) from counital coalgebras to sets. In the unital case one can recover \( \Gamma \) from \( C \Gamma \) as \( \Gamma = \text{Unital Points}(C \Gamma) \). Moreover one has an equivalence between the category of sets and the category of counital coalgebras which are set-like, that is, spanned by the points.

In the nonunital case one gets an equivalence of categories between sets with basepoint and set-like coalgebras as follows. Note first that there is a 1-1 correspondence (essentially) between sets and sets with basepoint given by \( \Gamma \rightarrow \Gamma_+ = \Gamma \circ \{0, \delta\} \). However there are more morphisms in the category of sets with basepoint. A map \( \Gamma_+ \rightarrow \Gamma_+ \)
Amounts to a partially defined map from $\Gamma$ to $\Gamma'$, that is, a map from a subset of $\Gamma$, the domain $D(f)$, to $\Gamma'$.

Next observe that the coalgebra $C\Gamma$ can be expressed

$$C\Gamma = C\Gamma_+/C\Gamma_0$$

showing that $\Gamma_+ \mapsto C\Gamma$ is a well-defined functor from sets with basepoint to set-like coalgebras. This functor has the quasi-inverse $C \mapsto \text{Points}(C)$, yielding the desired equivalence of categories.

Since $C\Gamma \otimes C\Gamma' = C[\Gamma \times \Gamma']$, one has

$$\text{Points}(C\Gamma \otimes C\Gamma') = (\Gamma \times \Gamma')_+ = \Gamma_+ \wedge \Gamma'_+.$$ 

In other words, the tensor product operation in set-like coalgebras corresponds to the smash product in sets with basepoint.

Perhaps you should be more careful, namely, check that $C\Gamma \otimes C\Gamma' \rightarrow C[\Gamma \times \Gamma']$, $s \otimes t \mapsto (s,t)$ is an isomorphism of coalgebras. The tensor product coalgebra has coproduct

$$\Delta: C\Gamma \otimes C\Gamma' \rightarrow C\Gamma \otimes C\Gamma' \otimes C\Gamma \otimes C\Gamma'$$

s \otimes t \quad s \otimes t \otimes s \otimes t

which corresponds dually to $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$ for algebra tensor product. (x) under $C\Gamma \otimes C\Gamma' \rightarrow C[\Gamma \times \Gamma']$

becomes $C[\Gamma \times \Gamma'] \rightarrow C[\Gamma \times \Gamma'] \otimes C[\Gamma \times \Gamma']$

$(s,t) \mapsto (s,t) \otimes (s,t)$

so it's clear.
Continue to identify sets and pointed sets via \( \Gamma \rightarrow \Gamma_+ = \Gamma \cup \{0\} \). Recall the equivalence of categories between pointed sets and set-like coalgebras given by \( \Gamma_+ \rightarrow C\Gamma \), \( C \rightarrow \) points of \( C \).

TFAE: (1) A product \( \mu: C\Gamma \otimes C\Gamma \rightarrow C\Gamma \) which respects the coalgebra structures.

(2) A binary operation \( \Gamma_+ \times \Gamma_+ \rightarrow \Gamma_+ \) such that 0 is absorbing: \( 0 \times \gamma = 0 = \gamma \times 0 \).

(3) A pointed set map \( \Gamma_+ \wedge \Gamma_+ \rightarrow \Gamma_+ \)

**Proof.** \( \Gamma_+ \) is the subset of points in \( C\Gamma \). Because \( \mu \) is a coalgebra map it restricts to a binary operation on \( \Gamma_+ \). In effect given points \( \eta, \eta' \) in \( C \), then

\[
\eta \otimes \eta' \mapsto \mu(\eta \otimes \eta')
\]

\( C\Gamma \otimes C\Gamma \xrightarrow{\mu} C\Gamma \)

\[
\Delta \downarrow \quad \Delta
\]

\[
(C\Gamma \otimes C\Gamma) \otimes (C\Gamma \otimes C\Gamma) \xrightarrow{\mu \otimes \mu} C\Gamma \otimes C\Gamma
\]

\[
\eta \otimes \eta \mapsto \mu(\eta \otimes \eta) \otimes \mu(\eta \otimes \eta)
\]

Thus (1) yields (2). Next the absorbing property of 0 means that the \( \mu \) operation on \( \Gamma_+ \) descends to the smash product \( \Gamma_+ \times \Gamma_+ / (\Gamma_+ \triangleright \Gamma_+) = \Gamma_+ \wedge \Gamma_+ \), so (2) yields (3). Finally \( \Gamma_+ \wedge \Gamma_+ = (\Gamma \times \Gamma)_+ = \Gamma \) Points of \( C\Gamma \otimes C\Gamma = C[\Gamma \times \Gamma] \), so that a pointed set map \( \Gamma_+ \wedge \Gamma_+ \rightarrow \Gamma_+ \) is equivalent to a coalg map \( C\Gamma \otimes C\Gamma \rightarrow C\Gamma \).
Next associativity

TFAE: (1) \( \mu : C\Gamma \otimes C\Gamma \to C\Gamma \) is associative
(2) the induced product \( \Gamma_+ \times \Gamma_+ \to \Gamma_+ \) is associative
(3) the map \( \overline{\mu} : \Gamma_+ \wedge \Gamma_+ \to \Gamma_+ \) satisifies
\[ \overline{\mu}(\overline{\mu} \wedge 1) = \overline{\mu}(1 \wedge \overline{\mu}) \]
from \( \Gamma_+ \wedge \Gamma_+ \to \Gamma_+ \)

Proof. (1) \( \Rightarrow \) (2) because you are restricting the product in \( C\Gamma \) to the subset \( \Gamma_+ \). (2) \( \Rightarrow \) (3) because the product on \( \Gamma_+ \times \Gamma_+ \) descends to \( \Gamma_+ \wedge \Gamma_+ \). In other words, the two maps \( \Gamma_+ \times \Gamma_+ \times \Gamma_+ \to \Gamma_+ \) giving associativity descend to \( \overline{\mu}(\overline{\mu} \wedge 1) \) and \( \overline{\mu}(1 \wedge \overline{\mu}) \). Finally (3) \( \Rightarrow \) (1) by the equivalence between coalgebras \( C\Gamma \) and ftd sets \( \Gamma_+ \).

At this point one has described bialgebras with set-like coalgebra structure in terms of

semi groups \( \Gamma_+ = \Gamma \cup \{0\} \) with absorbing basepoint 0. Note that any subset of a
ring closed under product and containing zero yields
such a \( \Gamma_+ \), and that \( C\Gamma \) is the largest ring
generated by \( \Gamma_+ \). \( C\Gamma \) is an obvious generalization
of the group ring of a group.

Next discuss \( \Gamma \)-graded vector spaces and algebras.

Prop. Equivalence between \( \Gamma \)-comodule \( V \) for the
c coalgebra \( C\Gamma \), where \( \Gamma \) is a set, and a grading
of \( V = \bigoplus_{\sigma \in \Gamma} V_\sigma \) with respect
to \( \Gamma_+ \). The comodule \( V \) is coriental \( \Leftrightarrow V_0 = 0 \),
so that \( V \) is graded with \( \Gamma \).
Proof. Given the co-product \( \Delta_V \) which is co-associative:

\[
\begin{align*}
V \xrightarrow{\Delta_V} C \otimes V & \xrightarrow{\Delta \otimes 1} C \otimes C \otimes V \\
& \xrightarrow{1 \otimes \Delta_V} C \otimes C \otimes C \otimes V
\end{align*}
\]

\( \Delta_V \) has the form \( \Delta_V \sigma = \sum_{s \in \Gamma} s \otimes e_s(\sigma) \) where the \( e_s \in \text{End}(V) \) satisfy the finiteness condition \( V \oslash e_s(\sigma) = 0 \) for all \( s \). Then equality of \( (\Delta \otimes 1) \Delta_V \sigma = \sum_{s} s \otimes s \otimes e_s(\sigma) \) and \( (1 \otimes \Delta_V) \Delta_V \sigma = \sum_{s \neq t} s \otimes t \otimes e_se_t(\sigma) \) for all \( \sigma \) is equivalent to \( e_s e_t = 0 \) for \( s \neq t \) and \( e_s^2 = e_s \). The \( e_s \) are annihilating projections on \( V \) such that \( \sum e_s \) is defined by the finiteness condition and it is a projection. Then we have the splitting

\[
V = \bigoplus_{s \in \Gamma} e_s V \oplus (1 - \sum e_s) V
\]

which yields the grading with \( V_s = e_s V \), \( V_0 = (1 - \sum e_s) V \).

Also \( \sum e_s = (\eta \otimes 1) \Delta_V \), so that \( \Delta_V \) is counital coproduct \( \Leftrightarrow \sum e_s = 1 \).

Next let \( C \Gamma \) be the bialgebra arising from a semi-group \( \Gamma \) with absorbing basepoint \( 0 \). Define a \( \Gamma \) graded algebra \( A \) to be an algebra equipped with a \( \Gamma \)-grading

\[
A = \bigoplus_{s \in \Gamma} A_s \quad \text{s.t.} \quad A_s A_t \subseteq A_{st} \quad \text{if} \quad st \in \Gamma \\
= 0 \quad \text{if} \quad st = 0.
\]

The \( \Gamma \)-grading is equivalent to a comodule structure on \( A \) for the coalgebra \( C \Gamma \), i.e. a coproduct.
\[ \Delta : A \longrightarrow C \otimes A, \quad \Delta a = \sum_{s \in I} s \otimes a_s, \quad \sum s = 1 \]

In other words \( \Delta(a_s) = s \otimes a_s \) for \( a_s \in A_s \).

The compatibility condition between grading and product can be expressed as saying that \( \Delta \) is an algebra homomorphism.

In effect, \( \Delta(a_s a_t) = (s \otimes a_s)(t \otimes a_t) = st \otimes a_{st} \) which implies that \( a_{st} \in A_{st} \) for \( st \neq 0 \), and \( a_{st} = 0 \) if \( st = 0 \) (since \( \Delta \) is injective because of the counit \( \eta \)).
Review the multiplier algebra $\text{Mult}(A)$ for an algebra $A$. A multiplier on $A$ is defined to be a pair of operators on $A$

$$\mu = (a \mapsto \mu a, \ a \mapsto a\mu)$$

satisfying

$$\mu(a_1 a_2) = (\mu a_1) a_2$$

$$a_1 (\mu a_2) = (a_1 \mu) a_2$$

$$(a_1 a_2) \mu = a_1 (a_2 \mu)$$

The product $\mu \nu$ of two multipliers is defined by

$$\nu \mu a = \mu(\nu a) \quad a(\nu \mu) = (a \nu) \mu$$

and it makes $\text{Mult}(A)$ into a subalgebra:

$$\text{Mult}(A) = \left\{ \mu \in \text{Hom}_{\text{op}}(A, A) \times \text{Hom}(A, A)^{\text{op}} \left| (\alpha_1 \mu) a_2 = a_1 (\mu a_2) \right\} \right\}$$

left multipliers \right\} \text{ right multipliers}$$

More generally if $(X, Y, \langle y, x \rangle)$ is a dual pair over $A$, one can define its multiplier algebra to be

$$\text{Mult} (X, Y, \langle y, x \rangle) = \left\{ \mu \in \text{Hom}_{\text{op}}(X, X) \times \text{Hom}(Y, Y)^{\text{op}} \left| \langle y, \mu x \rangle = \langle y, \mu x \rangle \right\} \right\}$$

Let $\mu = X = \text{Mult}(A)$ be the special case with the $A^{\text{op}}$-module $A$, the $A$ module, $Y = A$ and the pairing $\langle y, x \rangle = yx$.

Let $A$ be an ideal in the algebra $R$. Then each $r \in R$ yields a multiplier

$$\mu_r = (a \mapsto ra, \ a \mapsto ar)$$

whence one has \text{alg homomorphisms } \mu : R \rightarrow \text{Mult}(A). \text{ Restricting}
to $A$ (in other words, taking $R=A$)

one gets a canonical algebra map

$$A \xrightarrow{\phi} \text{Mult}(A),$$

with the following properties:

1) $\ker \phi = \{ a \in A \mid Aa = aA = 0 \}.$

2) $\mu \phi a = \phi \mu a$ and $\phi a \mu = \phi \mu a.$

hence $A/\ker \phi = \phi A$ is an ideal in $\text{Mult}(A)$.

Check 2). $(\mu \phi a) a' = \mu(\phi a) = \mu(\phi a') = (\mu a) a' = \phi \mu a a' = (\mu a') \phi a = (\mu a') a = a'(\mu a) = a' \phi \mu a \mu a = \phi \mu a$, and similarly for the other order.

Next look at semi-direct products for algebras which are analogous to such products for groups, where to form $Q \times K$ one needs a homomorphism from $Q$ to $\text{Aut}(K)$. For algebras the analogy is an alg map $R \xrightarrow{\phi} \text{Mult}(A)$ and the product on $R \times A$ is defined by $(r+a)(r'+a') = rr' + (\phi r a' + a \phi r') + aa'$.

There is a slight problem with associativity as follows. It's enough to consider $R = \text{Mult}(A)$. There are 8 associativities to check: $a_1, a_2, a_3$; three involving one $\mu$: $\mu a_1 a_2, a_1 \mu a_2, a_2 \mu$ $\mu$ OK by defn. of multiplier; three involving one $a$: $\mu a a, \mu a a, \mu a a \mu$, where the first and third are $\mu a$, defn. of product of multipliers; one involving three multipliers which is $\mu a a$.

So there is a problem with $(\mu a) a = \mu(a a')$, and
There are two ways to proceed. If $A = A^2$, then
is assumed, then OK because
\[
(\mu(a_1 a_2)\nu) = (\mu(a_1) a_2)\nu = (\mu a_1)(a_2 \nu) \\
\mu(a_1 a_2)\nu = \mu(a_1(\nu a_2)) = (\mu a_1)(\nu a_2)
\]
Thus no problem with $\text{Mult}(A) \times A$ when $A = A^2$.

On the other hand, applying $\phi$ takes $\mu, a, \nu$ into $\mu, \phi_0, \nu$ which satisfies associativity as $\text{Mult}(A)$ is a ring.

Recall that if \( e^2 = e \) in a ring \( B \), then one has a Morita context:

\[
\begin{pmatrix}
eBe & eB \\
eBe & B
\end{pmatrix} \leq M_{2B}
\]

which is associated to the dual pair over \( B \) given by \( eB, Be \) and the pairing \( \langle b_1 e, e b_2 \rangle = b_1 e b_2 \). (Note: \( eB \otimes B e = eBe \)).

This Morita context yields a Morita equivalence between the unital ring \( eBe \) and the ideal \( BeB \) which is idempotent. One has a canonical surjective ring morphism \( Be \otimes eB \rightarrow BeB \) whose kernel is killed by \( B \) (hence by \( BeB \)) on both left and right.

We now generalize this construction to any element \( h \) of \( B \). Consider the dual pair over \( B \) given by the right ideal \( hB \), the left ideal \( Bh \), and the pairing \( b_1 h * h b_2 = b_1 h b_2 \) which is well-defined since \( b_1 h = 0 \) or \( h b_2 = 0 \), \( \Rightarrow b_1 h b_2 = 0 \). This yields the Morita context

\[
\begin{pmatrix}
hB @ B h & hB \\
hB @ B h & B
\end{pmatrix}
\]

where the product in the ring \( hB @ B h \) is

\[
(hb_1 @ b_2 h) \ast (hb_3 @ b_4 h) = hb_1 @ b_2 h b_3 b_4 h
\]

Define the *-product on \( hBh \) by

\[
hb h * h b' h = h b b' h
\]
Then the canonical map \( h_b \otimes b_2 h \mapsto h_b b_2 h \) from \( hB \otimes B h \) to \( hB h \) respects \( \times \) product, showing that \( \times \) product on \( hB h \) is associative.

Similarly,

\[
(h_b \otimes b_2 h) \times h_b = h_b b_2 h_b \\
\quad b_0 h \times (h_b \otimes b_2 h) = b_0 h b_2 h.
\]

The actions of \( hB \otimes B h \) on \( hB \) and \( B h \) respectively descend to actions of \( hB h \) given by \( \times \) product:

\[
h_b h \times h_b = h_b h_b h_b \\
b_0 h \times h_b h = b_0 h b h
\]

(These statements are not accurate unless \( B = B^2 \), which is the case when \( B h B = B \). Thus it would have been better to proceed as follows.)

Consider the \( M_2 \)-graded abelian group \((hB h, hB, B, B)\) and define the \( \times \) product on it, using the formulas which hold when \( h^2 \).

More precisely, there are 8 products associated to this Morita context, 4 of which lead to expressions containing \( h^2 \); these give the \( \times \) products

\[
h_b h \times h_b' h = h_b h b h' h \\
h_b h \times h_b' = h_b h b h'
\]

\[
b_0 h \times h_b' h = b_0 h b h' \\
b_0 h \times h_b h' = b_0 h b h'
\]
Here's a way to understand better the Macauley context \((hB, hB)\). This context is essentially determined by the dual pair over \(B\) given by the \(B\)-module \(hB\), the \(B\) module \(X = Bh\), and the pairing \(bh \times hb = b'h'b\). So \((hB, Bh, \langle b'h'b \rangle = b'h'b)\) is a quotient of the dual pair \((B, B, \langle b'h'b \rangle = b'h'b)\). You have eliminated from the latter Macauley context the obvious degeneracies arising from the annihilators \(hB\) and \(B_h\).

Relation of \((hBh, hB)\) to \((jBj, jB)\)

where the latter Mac. context is supposed to correspond to quadruples \((V, W, j:W \rightarrow V \rightarrow V\langle j \rangle W)\). Now

\[ Bi = B/\{ b \mid bj = 0 \}, \text{ but } bi = 0 \Leftrightarrow bj = 0 \text{ when } j \text{ is surjective; so } Bi = B/B_h. \text{ Similarly } jB = B/\{ b \mid jb = 0 \} \text{ and } jb = 0 \Leftrightarrow gb = 0 \text{ when } s \text{ is injective; so } jB = B/B_h. \]

Conclude that the dual pairs

\[
\begin{cases}
(B, B, \langle b'h'b \rangle = b'h'b) \\
(B_hB, B/B_h \text{ same}) \\
(jB, Bc, \langle b'c, gb \rangle = b'h'b)
\end{cases}
\]

are essentially equivalent.
Let's consider the Morita context (here $h \in B$)

\[
\begin{pmatrix}
A & Y \\
X & B
\end{pmatrix} =
\begin{pmatrix}
hB & hB \\
Bh & B
\end{pmatrix}
\]

the product is the $\ast$-product, i.e. as if $h^2 = h$.

Then this context is strictly idempotent assuming $BhB = B$.

$B = BhB \subseteq BB$, $XY = Bh \ast hB = BhB = B$

$YX = hBBh = hB0h = A$, $A^2 = hBhBh = hBh = A$

$YB = hB^2 = hB = Y$, $AY = XY = YB = Y$

$BX = B^2h = Bh = X$, $XA = XYX = BX = X$.

Let's describe the Morita equivalence associated to this Morita context. We use the reduced module picture, i.e. $M_n(A)$ is the category of $A$-modules $V$ such that $AV = V$ and $V = 0$. You know that the functor of the equivalence from $M_n(B)$ to $M_n(A)$ is given by

\[
W \mapsto \text{Im} \left\{ Y \otimes W \rightarrow \text{Hom}_B(X, W) \right\}
\]

\[
y \otimes \omega \mapsto (x \mapsto (xy) \omega)
\]

(This should be true quite generally, certainly for a strictly idempotent Morita context.)

The map $\otimes$ factors

\[
hB \otimes W \rightarrow hW \hookrightarrow \text{Hom}_B(Bh, W)
\]

\[
hb \otimes \omega \mapsto hb\omega, \quad hw \mapsto (b'h \mapsto b'h \ast hw')
\]

The second map is injective, since $b'hw' = 0$ for all $b' \in B$ implies $hw' \in B^W$ which is zero assuming $W$ reduced.

The first map is surjective as $hBW = hW$ since $BW = W$ for $W$ reduced.
Thus the functor giving the equivalence $M_n(B) \rightarrow M_n(A)$ is $W \mapsto hW$. As a check note that $A(hW) = hBhW = hBhBW = hBW = hW$; also $A(hw) = 0 \iff hBhw = 0 \Rightarrow Bhw = BhBW = 0 \Rightarrow hw \in B W = 0$.

Next look at the inverse functor from $M_n(A)$ to $M_n(B)$ which sends $V$ to $W = \text{Im} \{ BhA \rightarrow \text{Hom}_A (hB, V) \}$. Thus

\[
\begin{array}{ccc}
BhA & \rightarrow & W \\
\downarrow \text{B conil free} & & \downarrow \text{B conil free as} \\
\text{Hom}_A (hB, V) & \rightarrow & \text{Hom}_A (hB, V)
\end{array}
\]

as $B = B^2$.

Thus $W$ is $A$-reduced.

Now assume $B$ satisfies $B = 0$ and $B = 0$. Then $hB$ is $A$-reduced, and $Bh$ is $A$-reduced.

Also since $B(\text{hB}) = Bh$ and $Bh \in B B = 0$, one sees that $A = hBh$ is $A$-reduced, and similarly $A$ is $A^\text{op}$-reduced.
July 18, 2001

Motivation: At the conference nearly one month ago Joachim told me that he could extend the Morita equivalences, which arise in the assembly maps for a group $\Gamma$ (finite support condition), to the case of the groupoid $M_n$ ($\text{Ob} = \{1, \ldots, n\}$, $\text{Ar} = \text{Ob} \times \text{Ob}$). On one side one has the algebra $A$, universally generated by the components of a projection on an $M_n$ graded algebra. On the other side one has the algebra $B$, which is a sort of cross product with the non-commutative n-simplex.

In order to reconstruct Joachim’s result it seems worthwhile to look more generally at assembly for a groupoid $\Gamma$. Geometrically assembly for a group involves constructing a $K$ class starting from a principal bundle for the group. So you want to understand principal bundles (or torsors) for a groupoid.

Let’s approach this problem from Groth’s topos viewpoint, which gives an elegant category picture of classifying toposi for groupoids (without homotopies, partitions of unity, etc.).

Let $C$ be a small category, let $C$-sets be the category $\text{Fun}(C, \text{sets})$ of covariant functors $L$ and $C^{op}$-sets the category of contravariant functors $R$. Because we use left functional notation: $$(fg)(x) = f(g(x))$$
it is convenient to write a chain
of composable arrows in \( C \) with the arrows
pointed to the left, so that the composition
of
\[
2 \xleftarrow{f} 1 \xrightarrow{g} 3 \xrightarrow{h} X.
\]
We write \( \text{Ob} \) for the set of objects, and \( \text{Ar} \)
for the set of arrows:
\[
\text{Ar} = \bigsqcup_{X, Y \in \text{Ob}} \text{Ar}(X, Y), \quad \text{Ar}(X, Y) = \text{Hom}_C(Y, X)
\]
composition:
\[
\text{Ar}(X, Z) \times \text{Ar}(Y, Z) \rightarrow \text{Ar}(X, Z),
\]
\[
\left( X \xleftarrow{f} Y, \quad Y \xrightarrow{g} Z \right) \mapsto \left( X \xleftarrow{f \circ g} Z \right)
\]
A \( C \)-set \( L \) is a set over \( \text{Ob} \):
\[
L = \bigsqcup_{X \in \text{Ob}} L(X)
\]
with a left action by \( \text{Ar} \):
\[
\text{Ar} \times \text{Ob} \rightarrow L, \quad \text{source} \rightarrow \text{Ob}
\]
\[
\bigsqcup_{(Y, X) \in \text{Ob}} \text{Ar}(Y, X) \times L(X) \rightarrow \bigsqcup_{Y \in \text{Ob}} L(Y), \quad \text{satisfying appropriate identity and associativity conditions. Similar}
\]
a \( C \)-set \( R \) is a set over \( \text{Ob} \) with right action by \( \text{Ar} \):
\[
R \times \text{Ob} \rightarrow R
\]
With this notation understood, we can
define the "tensor product" \( R \times \_ L \) to be the set
\[
R \times \_ L = \text{Coker} \left\{ R \times \text{Ob} \rightarrow R \times \text{Ob} \text{Ar} \times \text{Ob} \text{L} \right\}
\]
\[
\left( f, \_ , \_ L \right) \left( f_\lambda, f, \_ \right)
\]
In other words a map $R \times \epsilon L \rightarrow S$ is the same as a family of maps

$$\phi_x : R(X) \times L(X) \rightarrow S \quad \forall X \in \text{Ob}$$

such that

$$R(Y) \times \epsilon \text{Ar}(y, x) \times L(X) \rightarrow R(X) \times L(X)$$

$$\downarrow \quad \downarrow \phi_x$$

$$R(Y) \times L(Y) \quad \phi_y \rightarrow S$$

commutes $\forall X, Y \in \text{Ob}$.

It is easy to establish the following bilinearity property

$$\text{Hom}_{\text{sets}}(R \times \epsilon L, S) = \text{Hom}_{\text{sets}}(R, \text{Hom}_{\text{sets}}(L, S))$$

$$= \text{Hom}_{\text{C-sets}}(L, \text{Hom}_{\text{sets}}(R, S))$$

from which it follows that $R \times \epsilon L$ respects arbitrary limits. If you take $R = h_x = \text{Hom}_{\epsilon}(\cdot, X)$, then

$$\text{Hom}_{\text{sets}}(h_x \times \epsilon L, S) = \text{Hom}_{\text{sets}}(L(X), S)$$

by Yoneda's lemma, whence

$$h_x \times \epsilon L = L(X), \quad \text{sim} \quad R \times \epsilon h^x = R(X)$$

Therefore

$$h_x \times \epsilon h^y = \text{Hom}_{\epsilon}(y, x) = \text{Ar}(X, Y)$$

and then using \( \lim_{x/R} h_x = R \) etc., yields the general $R \times \epsilon L$ by right continuity.
The category $C$-sets is a topos. In the Grothendieck theory it is natural to define a $C$-torsor over a space $B$ to be a topos map from $\mathcal{S}_B$, sheaves of sets over $B$, to $C$-sets. Such a map is given by the inverse image functor $f^* : C$-sets $\rightarrow \mathcal{S}_B$ which is required to be right exact and left exact (respect finite proj. lim's).

Consider $B = \text{pt}$. A right continuous functor $F : C$-sets $\rightarrow$ sets has the form (up to a canonical isomorphism) $F(L) = R \times_{C} L$, where $R$ is the $C^{\text{op}}$-set $C^{\text{op}} \xrightarrow{\text{Yoneda}} C$-sets $\xrightarrow{F} \text{sets}$, i.e. $R(X) = F(\mathcal{X})$. When is $F$ left exact? I think this happens iff $C/R$ is filtering, equivalently $R$ is pro-representable. Assuming this, it follows that $\text{Pro} C$ is the category of points in $C$-sets.

For a space $B$ a right exact $F : C$-sets $\rightarrow \mathcal{S}_B$ should have the form $F(L) = R \times_{C} L$, where $R$ is a $C$-sheaf over $B$, i.e. the functor $C^{\text{op}} \xrightarrow{\text{Yoneda}} C$-sets $\xrightarrow{F} \mathcal{S}_B$.

The left exactness of $F$ should be equivalent to each stalk of $R$ being pro-representable.

Next simplify to a groupoid $\Gamma$ where pro-representable functors are representable. A $\Gamma$-torsor over $B$ is a $\Gamma^{\text{op}}$-sheaf such that each...
stalk is representable.

Example: \( \Gamma = \mathbb{M}_2 \). A \( \Gamma \)-set is the same thing as an ordered pair \( (F_1, F_2) \) together with an isomorphism between them. It is representable iff both sets are points. Clearly there is a unique torsor up to canonical isom.