What do you have? You have a functor \( R : \mathcal{G}^{op} \to \mathcal{S}_B \) (locally representable?)

What precisely is \( R \)? \( R \) consists of sheaves \( R_x \), \( \forall x \in \mathcal{G}_0 \), and maps \( \mu_{xy} : R_x \times (\mathcal{G}_{xy}) \to R_y \) satisfying id and assoc. conditions.

Locally representable? First do for a point.

Fix a pt be \( B \), then \( R_x \) becomes a set \( \forall x \), \( R_x \cdot \mu_{xy} \) define a fun \( \mathcal{G}_{xy} \to \) sets. Rep. means. \( \exists x \in \mathcal{G}_0 \), and \( \exists \in R_x \) such that \( \forall Y \)

\[
\text{Hom}(Y, X) \xrightarrow{f} R_y
\]

\[
\mu^{-1} \quad f^* f^* \quad f^* f^* \quad f^* f^*
\]

I think the good way to proceed is to form \( \mathcal{G}/R \) and to worry about this having a \( \mathcal{G}_0 \) final object locally.

Review notation. \( R : \mathcal{G}^{op} \to \mathcal{S}_B \). Picture of \( \mathcal{G}_0 \):

\[
\mathcal{G}_2 \quad \xrightarrow{f} \quad \mathcal{G}_1 \quad \xrightarrow{f} \quad \mathcal{G}_0
\]
The nerve of a category \( C \) consists of \( C_0, A_2, \text{id}, s, t, \).

What you need is a notation that fits well with modules.

\[
X, X, X, X, X, X, X, X
\]

You want a composition notation, product, which

\[
R \times C \times L
\]

\[
A s = \frac{1}{(x_0, x_1)} \quad b'(x_0, x_1)
\]

Notation \( b'(x_0) \)

\[
\begin{array}{c}
0 \leq a \leq a \times_0 a \\
a \rightarrow a \times a \xrightarrow{p_{11}} a \rightarrow a \\
\end{array}
\]

you write down sets. A set \( O \) of objects, a set \( A \) of arrows. Each arrow has source and target. You have to decide conventions about composable arrows. Try \( A \times_0 A = \{(f, g) \in A \times A \mid s(f) = t(g)\} = \{x \leftarrow y\} \)

When you have \( A \rightarrow O \)

\[
\begin{array}{c}
d_0 (x \leftarrow x) = x \\
d_1 (x' \leftarrow x) = x' \\
\end{array}
\]

\[
d_0 = s \\
d_1 = t
\]
You want a notation that will enable transition from categories to rings.

$R$ right module over $A$, $L$ left module

$$R \otimes A \otimes A \otimes L \xrightarrow{d_2} R \otimes A \otimes L \xrightarrow{d_1} R \otimes L$$

and the faces $d_i$ replace $\otimes$ by

$$d_2 (r \otimes a \otimes a' \otimes l) = d_0 (r \otimes a \otimes a' \otimes l) = r \otimes a \otimes a'$$

$$d_1 d_0 (---) = d_1 (r \otimes a \otimes a' \otimes l) = r \otimes a' \otimes l$$

What's different involves the objects (set of)

There seems to be something interesting here. When you pass from a category $C$ to its arrow ring $Z[C]$ the partially defined composition in the category is extended by zero.

$$G \xrightarrow{h} G \xrightarrow{f^*} Sh_B \quad f^* h : G^b \rightarrow Sh_B$$

$x \mapsto h^x \xrightarrow{f^*} f^*(h^x)$

$R$ is $G^b$-sheaf (q.m. of $G^b$-set)

What does a $G^b$-set look like?

$$R \subseteq R \times A \subseteq R \times A \times A$$

nerve of $G/R$

$$0 \subseteq A \subseteq A \times A$$

nerve of $G$
Given an ordered pair \((X, Y)\) of Objects, you have a set \(C(X, Y)\) of arrows, you have to say the direction of the arrows — usually \(X\) is the source and \(Y\) is the target. This is relevant for composition.

\[
C(X, Y) \times C(Y, Z) \rightarrow C(X, Z)
\]

\[
f \quad g \\
gf
\]

\[
C^{\text{op}}(X, Y) = C(Y, X)
\]

\[
C^{\text{op}}(X, Y) \times C^{\text{op}}(Y, Z) \rightarrow C^{\text{op}}(X, Z)
\]

\[
f \quad g \\
gf
\]

\[
C(Y, X) \times C(Z, Y) \rightarrow C(Z, X)
\]

\[
f \quad g \\
gf
\]

\[
C^{\text{op}}(Y, X) \times C^{\text{op}}(Z, Y) \rightarrow C^{\text{op}}(Z, X)
\]

\[
f^t \quad g^t \\
f^tg^t \quad f^t g^t
\]
The problem then arising from the notation used for the composition. It won’t make any difference for a groupoid because $G$ and $G^o$ are_iso. categories.

decide on simplest notation. What happens in a category is that given a triple of objects $(X, Y, Z)$ and maps $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, there is a composite map from $X$ to $Z$. The problem is whether to denote the composition of $f$ first followed by $g$ as $gf$ or $fg$. Functions lead to $g(f(x)) = (gf)(x)$. The way to proceed might be
Where to start? Suppose given \( R : G^{op} \to \text{Sh}_B \) is a \( G^{op} \)-sheaf.

- You form \( \mathcal{A}/R \) which is a category (groupoid).
- What is \( R \)? \( R \) consists of a family \( R(X), X \in \mathcal{A} \) of sheaves.

Outline:

- \( \text{Nerve}(C) \)
  - \( C \) small cat, nerve of \( C \)
  - \( C \) cons. of \( C^{op}, C^\circ \) two sets and maps
    \[
    C^{op} \leftrightarrow C^\circ \leftrightarrow C^\circ \times C^\circ
    \]

Yesterday you learned that the composition of arrows can be written in two ways:

- Left, corresponding to functions: \( (fg)(x) = f(g(x)) \)
- Right, left ops.: \( x(gf) = (xg)f \)

This means that

- the nerve will depend on your choice.
- Use left operators:

\[
C^\circ \leftrightarrow C^\circ \times C^\circ \leftrightarrow C^\circ \times C^\circ \ldots
\]
$R : \mathfrak{g}^{ob} \to \text{Sh}_B$ means family of sheaves, $R(X), X \in \mathfrak{g}^{ob}$, and family of maps
\[
\text{source } X \quad \text{target } Y
\]
\[
(\underline{\text{?}}X \mapsto Y) \mapsto \text{Hom}_B (R(Y), R(X))
\]
Assemble the $R(X)$ into
\[
R^{ob} = \bigsqcup_{X \in \mathfrak{g}^{ob}} R(X)
\]
p : $R^{ob} \to \mathfrak{g}^{ob}$
and the $f^*$ into
\[
R^{an} = \bigsqcup (Y \mapsto X) \in \mathfrak{g}^{an}
\]

Spend next ½ hour on maths
\[
A = \text{universal alg. gen. by the components } P_{ij}, i, j \leq n,
\]
of a proj in a $M_2$-graded alg.

Define $\Delta : A \to \Omega M_2 \otimes A$
\[
\Delta \langle P_{ij} \rangle = e_{ij} \otimes P_{ij}
\]
Define $\Delta : A \rightarrow \text{CM}_A \otimes A$

$\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$

$\Delta(p_{ik}) = e_{ik} \otimes p_{ik}$

$\Delta\left(\sum_j p_{ij} p_{jk}\right) = \sum_j \left(e_{ij} \otimes p_{ij}\right)\left(e_{jk} \otimes p_{jk}\right)$

$= \sum_j e_{ik} \otimes p_{ij} p_{jk} = e_{ik} \otimes p_{ik}$

So you have this alg, $A$, which is $M_A$ graded.

The next point. Recall in the case of group $\Gamma$

$\Delta : A \rightarrow \text{C} \Gamma \otimes A$

$\Delta(p_s) = s \otimes p_s$

This you got. You what you did. You made $P$ act internally somehow. Suppose you take an $A$-module $V$, i.e., you have operators $p_{ij} \in \mathcal{L}(V)$ satisfying the relations

One thing you have is $\sum_{ij} e_{ij} \otimes p_{ij} = (p_{ij})$

Think: You are so slow. What you are trying to do is to use the universal properties to construct a retraction. So have this $A$ module $V$, you form $\text{CM}_2 \otimes V$ but it has
Then apply \( \mathbf{p} = (-1)_{ij} \) to \( M_2 \times V = (-1) \)

to get something interesting.

Review: \( A = \text{alg. gen. by components } p_{ij} \)
of a proj in a \( M_2 \) graded algebra.

\[ \Delta: A \rightarrow \text{CM}_n \otimes A \]

\[ p_{ij} \otimes e_{ij} = e_{ij} \otimes p_{ij} \]

\[ \sum_j \text{tr}(e_{ij} \otimes p_{ij}) (e_{jk} \otimes p_{jk}) \]

\[ = \sum_j e_{ij} e_{jk} \otimes p_{ij} p_{jk} = e_{ik} \otimes p_{ik} \]

\[ \text{CM}_n \otimes A = M_n A \]

has a canonical projective, namely
\[ \sum e_{ij} p_{ij} = \begin{pmatrix} p_{11} & p_{1n} \\ p_{n1} & p_{nn} \end{pmatrix} \]

Now how can you use this? The first thing that occurs to me is that \( M_n A \) acts on \( \text{CM}_n \otimes A \)
with commuting with \( A \) action.

Column vectors:

\[ Q: \ A \ 	ext{unital?} \]

\[ \tilde{A} = \text{alg. gen. by } p_{ij} + \text{above rels.} \]

\[ \Delta: \tilde{A} \rightarrow \text{CM}_n \otimes \tilde{A} \]

What happens?

\[ \text{CM}_n \otimes \tilde{A} = M_n C \otimes M_n A \]
A \xrightarrow{\Delta} M_n A \xrightarrow{(M_n \tilde{A})} \text{unital algebra} \quad 550

\pi_j \mapsto e_j \otimes p_j = e_j \otimes p_j

You have a proj \( p = \sum e_j \otimes p_j \). Let's true using Greek letters for the maps \( (\gamma_j) \) in the groupoid. \( p = \sum e_j \otimes p_j \in M_n A \). You can split \( M_n \tilde{A} \) using \( p \).

What is interesting is what it means to adjoin an identity to the \( M_n \)-graded algebra \( A \).

Recall that \( \Delta \) is compatible with the \( M_n \) grading on \( A \) and the \( M_n \) grading on \( M_n A = M_n \mathcal{C} \otimes A \) where \( A \) has \( \mathcal{C} \)-graded grading \( \triangleleft \).

\[ \Gamma_+ \times \Gamma_+ \rightarrow \Gamma_+ \]

So over your \( \Gamma \times \Gamma_+ \) is a semigroup with absorbing element \( 0 \).

\[ \Gamma_+ \times \Gamma_+ \xrightarrow{\mu \times 1} \Gamma_+ \times \Gamma_+ \xrightarrow{1 \times \mu} \Gamma_+ \times \Gamma_+ \xrightarrow{\mu} \Gamma_+ \]

\[ \pi \xrightarrow{\times} \pi \]

\[ \Gamma_+ \times \Gamma_+ \xrightarrow{\mu \times 1} \Gamma_+ \times \Gamma_+ \xrightarrow{1 \times \mu} \Gamma_+ \times \Gamma_+ \xrightarrow{\mu} \Gamma_+ \]

\[ \Gamma_+ \times \Gamma_+ \xrightarrow{\mu \times 1} \Gamma_+ \times \Gamma_+ \xrightarrow{1 \times \mu} \Gamma_+ \times \Gamma_+ \xrightarrow{\mu} \Gamma_+ \]

\[ \Gamma_+ \times \Gamma_+ \xrightarrow{\mu \times 1} \Gamma_+ \times \Gamma_+ \xrightarrow{1 \times \mu} \Gamma_+ \times \Gamma_+ \xrightarrow{\mu} \Gamma_+ \]

\[ \Gamma_+ \times \Gamma_+ \xrightarrow{\mu \times 1} \Gamma_+ \times \Gamma_+ \xrightarrow{1 \times \mu} \Gamma_+ \times \Gamma_+ \xrightarrow{\mu} \Gamma_+ \]
So go back to $\Gamma = M_2$. 

$$CM_2 = M_2C$$

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$\Delta : A \rightarrow \Gamma \otimes A \xrightarrow{\Delta \otimes 1} \Gamma \otimes \Gamma \otimes A \xrightarrow{1 \otimes \Delta} \Gamma \otimes \Gamma \otimes A \otimes \Gamma$$

$$\Delta(a_x) = x \otimes a_x$$

$$\Delta(a_x a_y) = (x \otimes a_x) \cdot (y \otimes a_y) = x \otimes (y \otimes a_y)$$

$$\Delta(a_x a_y) = (x \otimes a_x) \cdot (y \otimes a_y) = x \otimes (y \otimes a_y)$$

$$A_{a_x a_y} \subset \begin{cases} 0 & x y = 0 \\ A_{a_x a_y} & x y \neq 0 \end{cases}$$

Question: Is $\Gamma \otimes A$ a $\Gamma$-graded alg?

Review: If $\Gamma$ is a semi-group with absorbing then $\Gamma \otimes A = \Gamma / \Gamma \{ \ast \}$ is a bialgebra with

co-product $\Delta s = s \otimes s$, product $s \otimes t = st$.

$$\Gamma \otimes \Gamma = \Gamma / \Gamma \{ \ast \} \otimes \Gamma / \Gamma \{ \ast \} = \Gamma / \Gamma \{ \ast \} \otimes \Gamma / \Gamma \{ \ast \} = \Gamma$$

Point is that if $st \in \Gamma$

then $\mu : \Gamma \otimes \Gamma \rightarrow \Gamma$ given by $\mu(st) = \begin{cases} 0 & st \neq \ast \\ 1 & st = \ast \end{cases}$

Observe you can adjoin an identity to any semi-group to make it a monoid.

Now you understand the $\Gamma$-grading on $\Gamma \otimes A$, every elt of $A$ has degree $1$. 
Look briefly at adjoining identity to a $\Gamma$-graded alg. $A$

$$A \xrightarrow{\Delta} \text{CG} \otimes A \xrightarrow{\Delta \otimes 1} \text{CG} \otimes \text{CG} \otimes A$$

$$A = \bigoplus_{s \in \Gamma} A_s$$

$$\Delta(a_s) = s \otimes a_s \in \text{CG} \otimes A$$

$$\Delta(a_s a_t) = st \otimes a_s a_t \implies a_s a_t = 0 \text{ when } st = 0$$

$$a_s a_t \in A_{st} \text{ when } st \neq 0.$$

Back to $\Gamma = M_2$. $\Gamma$ is a $A$ is $M_2$-graded i.e. a Morita context

$$\begin{pmatrix} A_1 & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad A \xrightarrow{\Delta} M^2 \otimes A$$

$$a_s \xrightarrow{\Delta} e_s \otimes a_s$$

Now you want to understand a unital Morita context. This should mean that the $\Delta$ Mor. cont. $A = (A_1, A_{12})$ is a unital ring

Let $L = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix}$

$$\sum_{ij} \varepsilon_{ij} a_{jk} = a_{ik}$$

$$\begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{nq_{11}} & 0 \\ \varepsilon_{nq_{21}} & 0 \end{pmatrix}$$
\[ \sum_{j} \varepsilon_{ij} a_{jk} = a_{ik} \]

\[
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} \\
\varepsilon_{21} & \varepsilon_{22}
\end{pmatrix}
\begin{pmatrix}
a_{11} & 0 \\
0 & 0
\end{pmatrix}
= 
\begin{pmatrix}
\varepsilon_{11} q_{11} & 0 \\
\varepsilon_{21} q_{11} & 0
\end{pmatrix}
\]

\( \varepsilon_{11} a_{11} = a_{11} \)
\( \varepsilon_{21} a_{11} = 0 \)
\( \varepsilon_{12} q_{21} = 0 \)
\( \varepsilon_{22} q_{21} = a_{21} \)

\[
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} \\
\varepsilon_{21} & \varepsilon_{22}
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
= 
\begin{pmatrix}
\varepsilon_{11} a_{11} + \varepsilon_{12} a_{12} \\
\varepsilon_{21} a_{11} + \varepsilon_{22} a_{12}
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

\( 0 = \varepsilon_{12} A_{21} = \varepsilon_{12} A_{22} \)
\( 0 = \varepsilon_{21} A_{11} = \varepsilon_{21} A_{12} \)

\[
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix}
\varepsilon_{11} & \varepsilon_{12} \\
\varepsilon_{21} & \varepsilon_{22}
\end{pmatrix}
= 
\begin{pmatrix}
a_{11} \varepsilon_{11} + a_{12} \varepsilon_{12} \\
a_{21} \varepsilon_{11} + a_{22} \varepsilon_{12}
\end{pmatrix}
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\]

\[
\varepsilon_{12} A_{21} = \varepsilon_{21} A_{22}
\]

\( \varepsilon_{11} q_{11} = a_{11} \)
To understand a unital Morita context, consider a graded ring $A = (A_{ij})_{i,j}$ when is such a graded ring unital?

Use multipliers. The matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is a $\Gamma$-graded algebra $A \xrightarrow{A} \text{CG} \otimes A$

$a \otimes 1 \mapsto s \otimes a$

Can you see multipliers on such an? For example suppose $\Gamma$ is a group.

$A_{st}A_{t} = A_{st}$

Consider $\Gamma = M_2$ and $\text{CG} = M_2 \mathcal{C}$ arrow ring of the groupoid $M_2 \mathcal{C}$.

So it is probably important to emphasize the special case of a groupoid. What structure does the arrow ring of a groupoid have?

So look at $\text{CG}$ the arrow ring. Note that

Let $\Gamma$ be the set of arrows in a cat. Point. For each object $x$ you get an identity $1_x$ and $\sum 1_x$ should be a local left + right unit.

$f \in \text{Ar}(Y,Z)$, $g \in \text{Ar}(Y,X)$

$fg \in \text{Ar}(X,Z)$ (reduced)

Then it should be clear that left $\text{CG}$ modules are the same as covariant functors from the category to $\text{Mod}(\mathcal{C})$ and right ones are contrav. funs.
The arrow ring for $M_n$ is $M_n \otimes A$. The category picture gives the sum $\sum_{i=1}^n e_{ii}$. What does adjoining an identity mean? Take a semigroup.

What you have at this point an understanding of the arrow ring for a category, grouped. Where next?

In the group case you have

$$A \xrightarrow{\Delta} \otimes A$$

$$\Delta(a) = a \otimes a \quad \text{for } a \in A$$

$$A \rightarrow \otimes A$$

$$a \mapsto a \otimes a$$

$$p = \sum_{\alpha} p_{\alpha} \xrightarrow{\Delta} \sum_{\alpha} \alpha \otimes p_{\alpha}$$

$$p^2 = \sum_{\beta \otimes \alpha} p_{\beta} p_{\alpha} \rightarrow \sum_{\beta \otimes \alpha} \otimes \alpha \otimes p_{\beta} p_{\alpha}$$

$$p_{\alpha} = \sum_{\alpha = \beta \otimes \gamma} p_{\beta} p_{\gamma}$$

In the end you have the following problem:

$p \in M_n \otimes A \quad p \in \otimes A \otimes A$ acts on $\otimes A \otimes V$

$p = \sum e_{\alpha} \otimes p_{\alpha} \quad p = \sum s \otimes p_s$

There is something which involves $s^{-1}$ which commutes left right, might try the same thing for $p = \sum e_{\alpha} \otimes p_{\alpha}$ on $M_n \otimes V$
maybe \( \sum_{\alpha \in M_2} \omega^\alpha \) internally on \( M_2 \otimes V \) 556

This commutes with left \( M_2 \) \( \oplus \) operations.

This looks like it ought to work.

So start with \( \Gamma \) a groupoid, let \( A \) be the \( \Gamma \)-graded algebra (means \( A = \bigoplus A_n \)) \( \) where \( \alpha \) ranges over the arrows of the groupoid and product is like \( \otimes \) the arrow ring \( \mathbb{C} \Gamma \) i.e.

\[
A_\alpha A_\beta = \begin{cases} 0 & \text{if } \alpha \beta = 0 \\ A_\alpha A_\beta & \text{if } \alpha \beta \neq 0 \end{cases}
\]

you want

No at this point you don’t care about the grading.

You’ll want an \( A \)-module \( V \), which means operators \( \alpha \cdot \) on \( V \) \( \) \( \alpha \cdot = \sum \beta P_{\alpha \beta} \)

Then you want a proj op \( \mathcal{P} \)

\[
\begin{array}{ccc}
W & \xrightarrow{\mathcal{C} \Gamma \otimes V} & W \\
\end{array}
\]

\[V\]

Let’s see if you can guess the algebraic picture of a \( \mathcal{S} \)-torsor. Take \( \mathcal{S} = M_2 \) first.

A \( \mathcal{S} \)-torsor over \( B \) is a functor \( \mathcal{G}^{op} \rightarrow \mathcal{S} \) \( \mathcal{B} \) which is locally representable. When \( \mathcal{S} = M_2 \), a functor \( R \) without the last condition is equivalent to two sheaves \( F_1, F_2 \) over \( B \) and an isom. between them. Representable means \( F_1, F_2 \) are final sheaves. Linearize to see what happens. \( R \) should become a \( \mathcal{D}[M_2^{op}]- \)sheaf over \( B \)
Look at $M_2$, groupoid, $\mathcal{C}M_2 = M_2^C$. What was the last idea yesterday. You tried Grothendieck's version of an $M_2$ torus, and it doesn't lead anywhere. Now to linearizing. Before: $M_2^{op} \to \text{Ab}_B$ which is an ordered pair $F_1, F_2$ of sheaves on $B$ and an arrow $F_1 \to F_2$. You can control the situation via stabiales. When you linearize you replace sheaves of sets by sheaves of $(M_2^{op})^{op}$ modules. Morita equivalence still holds. Another point is you want continuous functions.

You've reached the following situation: There seems to be a version of "assembly" for $\Gamma = M_2$, which

- Groupoid $M_2$, object 1,2 unique map for each ordered pair of objects.
- $\mathcal{C}M_2 = M_2^C$, arrow alg of the groupoid $M_2$ functors $M_2 \to \text{vector spaces}$ are left $M_2^C$-modules.
- $\mathcal{C}[C] = \text{arrow ring of } C$, basis given by the set of arrows in $C$, $\mathcal{C}$. So it should be true that a left reduced $\mathcal{C}[C]$ module is a covariant functor.

Your idea now is to treat $\mathcal{C}_G$ in analogy with $\mathcal{C}_\Gamma$, so life goes on.
Let $G$ be the groupoid $M_n$, i.e., $n$ distinct objects say $1, 2, \ldots, n$ and $\text{Ar} = Ob \times Ob, f \mapsto (\text{target}(f), \text{source}(f))$.

You want to start with a retract of a free $C^*$ module

$$W \xrightarrow{\alpha} C^* \otimes V \xrightarrow{\beta} W \quad \beta \alpha = 1$$

In our situation $C^* = M_n C = C^n \otimes (C^n)^*$

Now use the Morita equivalent of $M_n C$ with $C$.

above retract equivalent to

$$C^n \otimes W \xrightarrow{C^*} (C^n)^* \otimes V \xrightarrow{C^n \otimes V} C^n \otimes W$$

So if you start with $V$ then the possible $W$'s are retracts of $\otimes \text{End}(V)$ to a projection of an $\otimes \text{End}(V)$, i.e., $p \in M_n C \otimes \text{End}(V)$.

Take $n=2$. Want proj.

Repeat basic object is a retract of the free $M_n C$-module generated by $V$.

$$W \xrightarrow{\alpha} M_n C \otimes V \xrightarrow{\beta} W \quad \beta \alpha = 1.$$
so what is a retract of \( V^\otimes n \otimes \theta = C^n \otimes V \)?

It's equivalent to a projector of \( p = p^2 \) in \( \mathbb{C} \otimes \text{End}(C^n \otimes V) = M_n \otimes \text{End}(V) \)

i.e. to \( p = \sum e_{ij} \otimes p_{ij} \) where the \( p_{ij} \) satisfy

\[
 p = (\sum_i e_{ij} \otimes p_{ij}) (\sum_k e_{kl} \otimes p_{kl}) \\
 = \sum_{i,j;k,l} e_{ij} \otimes p_{ij} p_{kl} = \sum_{i,j;k,l} e_{ik} \otimes p_{ij} p_{jl}
\]

i.e. \( p_{ik} = \sum_j p_{ij} p_{jl} \).

So one has the following equivalence:

A \( \mathbb{C} \)-module structure on \( V \)
\( W \) is a \( \mathbb{C} \)-module retract of \( C^n \otimes V \)

\[
 M_n \otimes C \quad \xrightarrow{\text{Wf}} \quad M_n \otimes V
\]

Remaining step

\[
 W \xrightarrow{\alpha} M_n \otimes V \xrightarrow{\beta} W
\]

Here \( \alpha \) is induced by \( \delta \)
\( \beta \) is induced by \( \lambda \)
Review. Retract of a free $M_n(C)$ module

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

$\alpha f = \rho = \rho^2$, $\rho \in \Lambda \otimes \text{End}(V)$

\[ T^* \otimes W \rightarrow T^* \otimes V \rightarrow T^* \otimes W \]

You are trying to set up an equivalence between between $A$ module $V$ and $B$ modules $W$. But you don't know what $B$ is.

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

a $C$ linear

because $\Lambda$ is a unital ring, $\beta$ equivalent to $\Lambda \ i : V \rightarrow W$

Dually $\alpha$ should be equivalent to $j : W \rightarrow V$

$C$ linear, but a choice has to be made

$$\text{Hom}_C(\Lambda, V) = \text{Hom}_C(\Lambda, \Lambda) \otimes V$$

$$\text{Hom}_\Lambda(\Lambda \otimes V, \text{Hom}_C(\Lambda, V)) = \text{Hom}(\Lambda \otimes \Lambda \otimes V, V)$$

$$= \text{Hom}_C(\Lambda \otimes V, V)$$

so you need an $\alpha$ with

$$\Lambda \otimes V \rightarrow \text{Hom}_C(\Lambda, V) = \text{Hom}_C(\Lambda, C) \otimes V$$

$$\text{Hom}_\Lambda(\Lambda \otimes V, \text{Hom}_C(\Lambda, V)) = \text{Hom}(\Lambda \otimes \Lambda \otimes V, V)$$

$= \text{Hom}_C(\Lambda \otimes V, V)$ so you need a linear functional on $\Lambda$. Trace
Basic object is retract of a free $\Lambda$-module $W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$ with $\Lambda = M_n \mathbb{C}$

same as projection in $\text{End}_\Lambda(\Lambda \otimes V) = \Lambda \otimes \text{End}(V)$

same as an $A$-module structure on $V$. Problem to find any $B$ operating on any such retract $W$ such that you have a Morita equivalence between $A$ and $B$. You propose to use the identity $1 \in \Lambda$ and the trace $\text{tr} : \Lambda \to \mathbb{C}$ to define

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

$$\text{tr} \otimes 1 \downarrow \quad \varepsilon = 1 \otimes - \quad \uparrow$$

But doesn't look right since $\text{tr}(1) = 2$.

You want an "equivariant" splitting of $\Lambda \otimes V$.

Partition of $\Lambda$.

In the case of $\Lambda = M_n \mathbb{C}$ you have the partition $\sum_{i=1}^n e_{ii}$.

So what are you trying to say? Take

$$\sum \beta e_{ii} \chi$$

so on $W$, besides the $\Lambda$-operation adding up to $1$ on $W$,

you have operator $h_i = \beta e_{ii} \chi$ for $i = 1, \ldots, n$. 

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

On here have \( e_{ij} \)

with relations \( e_{ij} e_{ke} = \begin{cases} 0 & j \neq k \\ e_{ik} & j = k \end{cases} \)

Translates to

\[
\begin{align*}
\beta_{ij} &= \beta e_{ij} \\
\beta_{ij} \cdot \beta_{kl} &= \beta e_{ij} \beta e_{kl} \\
\beta_{ij} \cdot \beta_{kl} &= \beta e_{ij} \cdot \beta e_{kl} \\
\beta_{ij} \cdot \beta_{kl} &= \begin{cases} 0 & j \neq k \\
\beta e_{ij} & j = k \end{cases}
\end{align*}
\]

\[
\Lambda = M_{\mathbb{C}} \text{ basis } e_{ij}
\]

You want the actual projection \( T_{ii} : \Lambda \to \mathbb{C} e_{ii} \)

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

\( \Lambda = M_{\mathbb{C}} \) \( \mathbb{A} \)-graded

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

Retract of the free \( \mathbb{A} \)-mod gen. by \( \mathbb{A} \).

Problem: find natural operators as any such \( W \). \( \Lambda \) itself operates on the left.

Perhaps you have made the same mistake as before, namely, thinking that \( C \otimes \text{End}(V) \) is the endo alg of the free \( \Gamma \)-module \( C \otimes V \) (say \( \Gamma \) finite).
Yes. \( \Lambda \otimes V \quad \Lambda = M_2 \mathbb{C} \)

\[
\operatorname{End}(\Lambda \otimes V) = \Lambda^p \otimes \operatorname{End}(V)
\]

Namely

\[
(s \otimes \Phi)(t \otimes \sigma) = ts \otimes \Phi \sigma
\]

\[
(s_1 \otimes \Phi_1)[(s_2 \otimes \Phi_2)(t \otimes \sigma)] = (s_1 \otimes \Phi_1)[ts_2 \otimes \Phi_2 \sigma]
\]

\[
= ts_2 s_1 \otimes \Phi_1 \Phi_2 \sigma
\]

\[
(s_1 \otimes \Phi_1)(s_2 \otimes \Phi_2) = s_2 s_1 \otimes \Phi_1 \Phi_2
\]

Therefore

\[
T^*_{\Lambda} W \rightarrow T^* \otimes V \rightarrow T^* \otimes W
\]

\[
p = \sum_{ij} p_{ij} \quad \longrightarrow \sum_{ij} \epsilon_{ij} \otimes p_{ij}
\]

"Internally" on \( \Lambda \otimes V \).

\[
p(\lambda \otimes \sigma) = \sum_{ij} \lambda \epsilon_{ij} \otimes p_{ij} \sigma
\]

\[
p(p(\lambda \otimes \sigma)) = \sum_{kl} \sum_{ij} \lambda \epsilon_{ij} \epsilon_{ik} \otimes p_{kl} p_{ij} \sigma
\]

\[
= \sum_{k \neq j} \lambda \epsilon_{jk} \otimes \sum_{l} p_{kl} p_{lj} \sigma
\]

\[
= \sum_{k \neq j} \lambda \epsilon_{jk} \otimes p_{kj} \sigma = p(\lambda \otimes \sigma)
\]

Note: there's some resemblance between this
and the way a $\Gamma$ action and $\Gamma$ grading combine to yield a crossproduct.

So now you understand $p$ on $T^*V$.

Still a way to go.

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \quad \beta \alpha = 1.$$ 

You have $\Lambda$ acting on the left.

$$T^* \Lambda W \quad T^* \otimes V \quad T^* \otimes \Lambda W$$

Basically want

Repeat. $\Lambda$ arrow of a groupoid

$$W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$$

guess that there should be an operator on $W$, an $h$ operator. $\beta \alpha = 1_W$

$$T \otimes T^* \otimes V$$

$$M_a t \otimes V \rightarrow W$$

$$\uparrow$$

$$V$$

maybe the idea is that there are two projections

$C$ levi-civita on $\Lambda \otimes V$
Latest idea: \( \Lambda = \text{arrow alg of the groupoid} \)

\( \Lambda \) is a left \( \Lambda \)-module (also a right module, but you have used this when applying \( p \) to \( \Lambda \otimes V \)).

\( \Lambda \) is also graded with \( \mathbb{N} \). This gives a partition of \( \Lambda \).

Repeat: \[ \begin{align*}
W & \xrightarrow{\alpha} \Lambda \otimes V \\
& \xrightarrow{\beta} W
\end{align*} \]

You want to find the \( \mathbb{R} \)-algebra \( B \) which operates on all \( W \)'s arising from \( A \)-modules \( V \).

\( B \) contains left mul by els of \( \Lambda \) and also operators \( \bullet h_s = \beta e_s \otimes \), where \( e_s \) is the projection of \( \Lambda \) into \( C_0 \), arising from the \( \frac{\lambda}{s} \) grading of \( \Lambda \). Then \( \sum_{s \in S} h_s = 1 \) on \( W \).

\( B \) should be generated by \( \Lambda \) and the \( h_s \), ideally a kind of cross product. Is there a cross-product algebra using a groupoid?

\( \mathbb{R} \times D \) should a \( \mathbb{R} \) graded alg.

Can you define what it means for \( \mathbb{R} \) to act on \( D \)? \( D \) may not be a ring.

Repeat: Start with an \( A \)-module structure on \( V \), where a projection

\[ p(\Lambda \otimes v) = \sum_{s \in S} \lambda_s \otimes p_s, v \]

on the \( \otimes \) free \( \Lambda \)-module \( \Lambda \otimes V \), hence a
\( \Lambda \)-module retract

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \quad \beta \alpha = 1. \]

The problem now is to define the right appropriate alg \( B \) which acts on any \( W \), and which leads to Morita equiv.

Examples of operators to go in \( B \), left mult by \( \Lambda \).

\[ h_s \ast = \beta e_s \alpha \quad \text{where } e_s \text{ is proj onto } s \otimes V \text{ defined by the } \mathbb{Z} \text{ grading. This family of } h_s \text{ satisfies } \sum_{s \in \mathbb{Z}} h_s = 1 \text{ on } W, \text{ but }\]

Cuntz has a partition indexed by the objects of the groupoid.

So go back to first idea, using \( 1 \in \Lambda \) and the trace \( tr : \Lambda \rightarrow \mathbb{C} \)

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

\[ \Lambda \xrightarrow{\beta} T^* \otimes V \xrightarrow{\beta^*} W^\# \quad \text{where } W^\# = T^* \otimes W \]

You know

\[ T^* \otimes V \xrightarrow{\beta^*} W^\# \rightarrow T^* \otimes V \]

\[ p(\lambda \otimes v) = \sum \lambda e_j \otimes \beta e_j v \]
\[
p(\lambda \otimes \sigma) = \sum_{i,j} A_{e_i \otimes p_j} e_{i,j} \\
p(p(\lambda \otimes \sigma)) = \sum_{k,l,e_i,j} A_{e_k \otimes p_l e_i p_j} e_{i,j} \\
= \sum_{k,l} A_{e_k} \otimes \sum_{p_l e_i} p_l e_i p_j e_{i,j} \\
\]

Review

\[W \xrightarrow{\lambda \otimes \beta} W \quad \beta \alpha = 1_w\]

List all the operators on \(W\) you get and any relations between them.

\[h_{i,j} = \beta_{\Pi_{i,j}} \quad \text{\(TP_{i,j}\) projects onto \(Ae_{i,j}\)}\]

How to handle \(M_n\), \(A = M_n(C)\), basic object is a retract of a free \(A\)-module

\[W \xrightarrow{\lambda \otimes V} \lambda \otimes V \xrightarrow{\beta} W \quad \beta \alpha = 1 \quad \alpha \beta = 1_p\]

\(p\) is equivalent to an \(A\)-module structure on \(V\).

Now want to find the ring \(B\) whose modules are such \(W\). Idea: You have \(A\) left mult on \(W\).

Other operators are \(\pi_{i,j}: \lambda \otimes V \rightarrow Ce_{i,j} \otimes V\)

\[\pi_{i,j}(\lambda) = e_{i,i} \lambda e_{i,j}\]

\[\sum \pi_{i,j} = \text{id in } \lambda \quad \alpha \lambda = 1 \quad \lambda \beta = 1_p\]
\[ W \xrightarrow{\alpha} A \otimes V \xrightarrow{\beta} W \]

Partition of unity on \( \Omega \Lambda \) is \( \sum \pi_{ij} = \text{id} \)

where \( \pi_{ij}(x) = e_{ij} \otimes e_{ij} \)

\( \pi_{ij}(e_{kl}) = e_{ij} e_{kl} e_{ij} \)

\( \pi_{ij}(x \otimes y) = e_{ij} x \otimes y \)

These ops all map to \( \text{id} \).

\[ \beta \omega = \sum_i \beta_i \pi_{ij} \omega = \sum_i e_{ij} \beta_i \]

Let \( \omega = \sum_i e_{ij} \otimes v(ij) \). Then

\[ \pi_{ij}(\omega) = e_{ij} \otimes v(ij) \]

\( \pi_{ij}(x \otimes y) = e_{ij} \left( \sum_{kl} e_{kl} \otimes v_{kl} \right) e_{ij} \)

\[ \pi_{ij}(x \otimes y) = e_{ij} e_{kl} \otimes v(ij) \]

Let \( \xi \in \Lambda \otimes V \), then \( \xi = \sum_i e_{ij} \otimes v(ij) \)

and

\[ e_{kl} \xi e_{ll} = \sum_i e_{kk} e_{ij} e_{kl} \otimes v(ij) \]

\[ = e_{kl} \otimes v(kl) \]

\[ \alpha \omega = \sum_{k, l} e_{kk} \alpha(\omega) e_{ll} = \sum_l \alpha \]
\[ W \xrightarrow{\alpha \beta} \Lambda \otimes V \xrightarrow{\rho} W \]

\[ p(\alpha \otimes \nu) = \sum \lambda e_{ij} \otimes p_{ij} \nu \]

Typical element of \( \Lambda \otimes V \) has the form

\[ \sum e_{ij} \otimes \nu(ij) \]

\[ p \left( \sum e_{ij} \otimes \nu(ij) \right) = \sum \sum e_{ij} e_{kl} \otimes p_{kl} \nu(ij) \]

Try instead

\[ p \left( \sum e_{ij} \otimes \nu(ij) \right) = \sum \sum e_{ij} e_{kl} \otimes p_{kl} \nu(ij) \]

\[ = \sum e_{jk} \otimes p_{ki} \nu(ij) \]

\[ p \left( \sum e_{ji} \otimes \nu(ij) \right) = \sum e_{ji} e_{lk} \otimes p_{kl} \nu(ij) \]

\[ = \sum e_{jk} \otimes \sum p_{ki} \nu(ij) \]

\[ = \sum e_{kj} \otimes \sum p_{ji} \nu(ik) \]

\[ p \]
\[ p \left( \sum_{i,j} e_{ij} \otimes u(c_{ij}) \right) = \sum_{i,j,k,l} e_{ij} e_{kl} \otimes \nu(i) \nu(j) \]

\[ = \sum_{k,l} e_{kl} \otimes \sum_{i} \nu(i) \]

\[ p \left( \sum_{s} s^{-1} \otimes v(s) \right) = \sum_{s} \sum_{t} s^{-1} t^{-1} \otimes \nu(t) \otimes v(s) \]

\[ = \sum_{u} u^{-1} \otimes \sum_{u=t} \nu(t) \otimes v(s) \]

\[ s^{-1} t^{-1} \otimes v(s) \]

\[ (p \nu)(u) = \sum_{t} p_{t} t^{-1} \otimes v(t^{-1} \otimes u) \]

\[ = \sum_{s} p_{s} s^{-1} t^{-1} \otimes v(s) \]

\[ \Lambda \text{ groupoid ring, arrow ring,} \]

\[ W \xleftarrow{\otimes} \Lambda \otimes V \xrightarrow{\otimes} W \]

\[ p \left( \sum_{s} s \otimes f(s) \right) = \sum_{s,t,u} s \otimes t^{-1} \otimes p_{t} f(s) \]

\[ = \sum \]

\[ p \left( \sum_{t} t \otimes f(t) \right) = \sum_{t,u} t u^{-1} \otimes p_{u} f(t) \]

\[ = \sum_{s} s \otimes \sum_{t} p(s^{-1} t) f(t) \]

s = tu^{-1}

su = t

u = s^{-1}t
Discuss situation

\[ W \xrightarrow{x} V \xrightarrow{p} W \]

May think of \( W \) as a cov. functor from the groupoid to vector spaces.

\[ W = \bigoplus_{x \in \text{Ob}} 1_x \circ W \]

\[ \Lambda = \bigoplus_{x \in \text{Ob}} 1_x \Lambda \]

\( \Lambda \) spanned by the arrows \( x \xleftarrow{\gamma} y \)

What do you know about a category?

\( \text{Ar}(X, Y) \)

Representable functors.

\( \Lambda = \bigoplus_{x} \Lambda_{x} \)

\( \Lambda \) as a left \( \Lambda \)-module splits into left ideals, corresponding to the representable functors.

So far you begin with the category of \( A \)-modules \( V \), i.e. vector space tog. with ops \( p(s) \) for each \( s \in \Gamma \) satisfying the idempotence condition \( p(u) = \sum_{u = st} p(s)p(t) \). Put another way?

\[ p = \sum_{s \in \Gamma} s \otimes p(s) \in \Lambda \otimes \text{End}(V) \]

\[ p = \sum_{s, t \in \Gamma} s \otimes p(s)p(t) = \sum_{u} u \otimes \sum_{u = st} p(s)p(t) \]
Review: You are trying to extend from a group to a groupoid using essentially the same formulas. You begin with an $A$-module structure $V$ that is a vector space with operators $p(s) \in \text{End}(V)$, set

$$p = \sum_{s \in \text{set}} s \otimes p(s) \in \Lambda \otimes \text{End}(V)$$

such that $p^2 = \sum_{s,t} s \otimes t \otimes p(s)p(t) = \sum_u u \otimes \sum_{s \in \text{set}} p(s)p(t)$

where $\sum_{u=s} p(s)p(t) = p(u)$.

Given such a family of operators $p(s)$ define $p$ on $\Lambda \otimes V$ by

$$p\left(\sum_t t \otimes f(t)\right) = \sum_{u,t} tu^{-1} \otimes p(u)f(t)$$

Better might be

$$p\left(\sum_t t \otimes f(t)\right) = \sum_t \left(\sum_u tu^{-1} \otimes p(u)f(t)\right)$$

$$\sum_{u,t} tu^{-1} \otimes p(u)f(t) = \sum_s s \otimes p(s^{-1}t)f(t)$$

$t$ fixed then we have

$$\{ u \mid tu^{-1} \text{ defined} \} \quad \{ s \mid s^{-1}t \text{ defined} \}$$

$$z \to y \xrightarrow{u} t \overset{s^{-1}}{\to} u$$
Repeat: $p\left(\sum_{t} t \circ f(t)\right) = \sum_{t} \left(\sum_{u} tu^{-1} \circ p(u)f(t)\right)$

In $\sum_{u} tu^{-1} \circ p(u)f(t)$ think $t$ is fixed and $u$ runs over all arrows with same source as $t$.

$= \left(\sum_{t}\right) \sum_{s} s \circ p(s^{-1}t)f(t)$. So you have the formula

$$(p\circ f)(s) = \sum_{t} p(s^{-1}t)f(t)$$

Sum takes place over all $t$ with same target as $s$.

You want to factor $p$ appropriately.

$\alpha \beta \left(\sum_{t} t \circ f(t)\right) = \sum_{s} s \circ j^{-1} \sum_{t} ti f(t)$

$= \sum_{s} s \circ \sum_{t} \left(p(s^{-1}ti)f(t)\right)$

Is there some way to factor $p(s^{-1}ti)$ into $j^{-1} \circ t \circ i$, say according to intermediate object

[diagram]

and ask for solutions of $u = s^{-1}t$

$$\{s, t\} | u = s^{-1}t\} = \Pi_{Y} \text{Ar}(Y, Z) \times \text{Ar}(Y, X)$$

for each $Y$, the piece should be an orbit under the isomorphism $\text{gp} \text{Ar}(Y, Y)$
Now go back to $M_n$

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \longrightarrow \Lambda \otimes V \]

What you have is $p$

\[ (p(f))(s) = \sum_{t} p(s^{-1}t) f(t) \]

Repeat earlier idea. You have a functor from $A$-modules $V$ to vector spaces $W$ and you want to find an algebra $B$ operating naturally on this functor so that the functor is a Morita equivalence.

Go back to earlier idea that the projection operators $\pi_s$ on $V$ as a vector space yield to "compressed" operators on $W.$

\[ h_s = \beta \pi_s \alpha. \]

$W = p(\Lambda \otimes A) \otimes_A V$

\[ p(\Lambda \otimes A) \xrightarrow{\alpha} \Lambda \otimes A \xrightarrow{\beta} p(\Lambda \otimes A) \]

These are all $A^{op}$-maps.

Can you recover $V$ from $W$?

In any case, it is clear that you have a partition of unity $W: \sum_{s \in I} h_s = 1.$
\[ \text{\textnormal{\Gamma groupoid}, \; \Lambda = \text{CG}, \; \text{does } \Gamma \text{ act on the alg } \text{CG} \text{ allowing us to form a cross product alg } \text{CG} \otimes \text{CG} \text{ as in the group case? thereby getting a Monta equiv. in the group case the Monta equivalence arises from left mult by } \Gamma \text{ on the } \Gamma\text{-graded vects space } \mathcal{CG}. \text{ Better to say that a } \Gamma \text{-module } M \text{ with } \Gamma \text{-grading induced by } \Gamma \text{ such that } \Gamma \text{ respects } M = \bigoplus_{s \in \Gamma} M_s \text{ such that } \gamma M_s \subset M_s \text{ is canonically sim to } \text{CG} \otimes M_1. \]

Is there a concept statement in the groupoid case?

Consider then \[ M \text{ a } \Gamma\text{-graded module, where } \Gamma \text{ is a groupoid } \; M = \bigoplus_{s \in \Gamma} M_s \; \Gamma \text{ finite set only identity maps.} \]

\[ \text{CG groupoid alg, suppose finitely many objects so that CG is unital. A unital CG-mod is the same as a functor from } \Gamma \text{ to Vect.} \]

Repeat. If \[ V \text{ is A-module, get } p \text{ on } A \otimes V \text{ given by } p(\sum_t t \otimes f(t)) = \sum_t \sum_s t a^s \otimes p(\kappa) f(t) \]

\[ = \sum_s s \otimes \sum_t p(s^{-1} t) f(t) \]
You ultimately want to express \( p(u) \) as a sum over factorizations of \( u \) as \( s^{-1}t \).

\[
(pf)(s) = \sum_{t} p(s^{-1}t) f(t).
\]

This is to be written as a sum over possible intermediate objects occurring in a factorization of \( u \).

What would you like to happen? Note that \( s^{-1} \) should be viewed, or written as \( s^* \).

At some point you should explore this. You might want to show that the groupoid ring should be a \( * \) algebra in an obvious way.

What would you like to happen.

\[
p(u) = \sum_{s\in G} q(s)^* q(t)
\]

Suppose you have the \( M_n \) case. Then

\[
(pf)(s) = \sum_{t} p(s^*t) f(t)
\]

\[
= \sum_{s^*t = s_i^*t_i} p(s_i^*) p(t) f(t)
\]
On $\Lambda \otimes V$ you have the operator $\pi_s$ onto $CS = \Lambda^\vee V \in \Gamma$.

$$\sum \pi_s = 1, \quad h_s = \beta \pi_s \alpha, \quad \sum h_s = 1$$

So the $h_s$ operator on $W$ as well as the $s_i$.

Look at $t h_s u = \beta(t \pi_s u)\alpha$.

$s = iy$ then $\pi_s(\lambda) = e^{\lambda x} \pi_s$.

Repeat: $W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W$

On $\Lambda \otimes V$ have projections $\pi_s : \Lambda \otimes V \rightarrow \mathbb{R} \otimes V$.

Also have left mult $s_i$ operators $\otimes \Lambda \otimes V$.

So the $\Gamma$ action and the $\Gamma'$-grading.

How are they related? Take the $M_n$ case.

Two kinds of operators, namely left mult by $s$ on $\Lambda \otimes V$ and projection operator $\pi_f : \Lambda \otimes V \rightarrow \text{image } t \otimes V$.

Can suppose $V = \mathbb{C}$, $\Lambda = \mathbb{C} \Gamma$ where $\Gamma$ is a groupoid trivial isotropy groups. Let $x,y,z$ be objects s.t. $u$ arrows.

Question: Assuming $\text{Ob}$ finite, what is the alg. generated by the $s_i$, $\alpha$ and $\pi_f$?
\[ M_n(\mathbb{C}) \] has basis \( e_{ij} \) for left mult by \( \lambda \).

\[ \Pi_s \] proj onto \( \mathbb{C}s \).

You are working in the adjoint rep, not the standard rep.

\[
\begin{pmatrix}
1 \\
0
\end{pmatrix}
\]

\[ \Pi_{ij}(\lambda) = e_{ii} \lambda e_{jj} \]

find the relations

\[ \Pi_{ij} \]

\[ \begin{align*}
M_n(\mathbb{C}) & . \\
(e_{ij} \Pi_{kl})(\lambda)(e_{kl}) = e_{ij} e_{kk} \lambda e_{ll} = \begin{cases} \\
0 & \text{if } j \neq k \\
e_{ij} \lambda e_{ll} & \text{if } j = k
\end{cases}
\end{align*} \]

\[ \begin{align*}
(e_{ij} \Pi_{kl})(\lambda) = e_{ij} \Pi_{kl}(\lambda) = e_{ij} e_{kk} \lambda e_{ll} = \begin{cases} \\
0 & \text{if } j \neq k \\
e_{ij} \lambda e_{ll} & \text{if } j = k
\end{cases}
\end{align*} \]

\[ \Pi_{kl} (e_{ij}) = \begin{cases} \\
0 & \text{if } k \neq i \\
e_{ij} \lambda e_{ll} & \text{if } k = i
\end{cases} \]

\[ e_{kk} e_{ij} \lambda e_{ll} = \begin{cases} \\
e_{ij} \lambda e_{ll} & \text{if } k = i \\
0 & \text{if } k \neq i
\end{cases} \]
\[ \Pi_{kl}(\lambda) = e_{kk} \lambda e_{ll} \]

\[
(e_{ij} \circ \Pi_{ke})(\lambda) = e_{ij} e_{kk} \lambda e_{ll} = \begin{cases} e_{ik} \Pi_{ke}(\lambda) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}
\]

\[
(\Pi_{kl} \circ e_{ij})(\lambda) = e_{kk} e_{ij} \lambda e_{ll} = \begin{cases} e_{kj} \lambda e_{ll} & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}
\]

\[
\Pi_{kl}(e_{ij}) = e_{kk} \ e_{ij} \lambda e_{ll} = \begin{cases} 0 & \text{if } k \neq i \\ e_{ij} \lambda e_{ll} & \text{if } k = i \\ e_{kk} e_{ij} \lambda e_{ll} & \text{if } k = i \end{cases}
\]

Repeat this calculation. You are looking at operators on \( \Lambda = C \Gamma \) in particular \( \Pi_s \) proj ops assoc. to the grading and opp. \( e_s \) shift mult by \( s \in \Gamma \).

\[
\Pi_{kl}(\sum \lambda_i e_{ij}) = \lambda_i \phi_{kl}
\]

\[
\Pi_{kl}(\lambda) = e_{kk} \lambda e_{ll}.
\]

\[
(\Pi_{ke} \circ e_{ij})(\lambda) = e_{kk} e_{ij} \lambda e_{ll} = \begin{cases} 0 & \text{if } k \neq i \\ e_{ij} \lambda e_{ll} & \text{if } k = i \\ e_{ij} e_{jj} \lambda e_{ll} = (e_{ij} \Pi_{je})(\lambda) \end{cases}
\]
Repeat the calculation
\[ \Pi_k^L (\lambda) = e_{kk} \lambda e_{ll} \]

\[ (e_{ij} \cdot \Pi_k^L) (\lambda) = e_{ij} e_{kk} \lambda e_{ll} = \begin{cases} 0 & j \neq k \\ e_{ik} \Pi_k^L (\lambda) & j = k \end{cases} \]

\[ e_{ij} \Pi_k^L = \begin{cases} 0 & j \neq k \\ e_{ij} \end{cases} \]

\[ (e_{ij} \cdot \Pi_k^L) (\lambda) = e_{ij} e_{kk} \lambda e_{ll} = \begin{cases} 0 & j \neq k \\ e_{ik} \Pi_k^L & j = k \end{cases} \]

\[ (\Pi_k^L \cdot e_{ij} ) (\lambda) = e_{kk} (e_{ij} \lambda) e_{ll} = \begin{cases} 0 & k \neq i \\ e_{ij} \lambda e_{ll} \end{cases} \]

\[ (e_{ij} \cdot \Pi_k^L) (\lambda) = e_{ij} e_{dj} \lambda e_{ll} \]

\[ \Pi_k^L (e_{ij} \lambda) = e_{kk} e_{ij} \lambda e_{ll} = \begin{cases} 0 & k \neq i \\ e_{ij} (\Pi_k^L \lambda) \end{cases} \]

So it looks like there is a kind of normal form. Note that the $L$ doesn't change. This is the effect of right multiply by $e_{ll}$.
Repeat again. \( \Gamma = M_n \quad \Lambda = M_n \mathbb{C} \)

\[ \pi_{kl} \text{ grading projection into } \mathbb{C} e_{kl} \]

\[ \pi_{kl}(\lambda) = e_{kk} \lambda e_{ll} \quad \text{in } M_n \mathbb{C} \]

\[ (\pi_{kl} \circ \delta_{ij})(\lambda) = e_{kk}(\delta_{ij} \lambda) e_{ll} = \sum_{d} \delta_{ki} e_{ij} \delta_{dl} \]

\[ \pi_{kl} \cdot e_{ij} = \delta_{ki} \; e_{ij} \; \pi_{jl} \]

So look at the operators \( \beta \pi_{kl} \alpha = h_{kl} \)

Try \( \pi_{l}(\lambda) = \lambda e_{ll} \)

\[ (\pi_{l} \circ e_{ij})(\lambda) = \pi_{l}(e_{ij} \lambda) = e_{ij} \lambda e_{ll} = e_{ij} \pi_{l}(\lambda) \]

\[ \beta \pi_{kl} \alpha \; e_{ij} = \delta_{ki} \; e_{ij} \; \beta \pi_{jl} \alpha \]

\[ h_{kl} \; e_{ij} = \delta_{ki} \; e_{ij} \; h_{jl} \]

Repeat. \( \Gamma = M_n \quad \Lambda = M_n \mathbb{C} \).

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

Each object determines a pro. Right mult on \( \Lambda \) by the identity maps of the groupoid yields...
Look then at $\Lambda \otimes V = M_n V$

Define $\Pi_k$ in $\Lambda \otimes V$ by

Start again $V$ is a $A$-module ie vector space with $p(s)$ set $\Gamma$.

$$W \xrightarrow{\sigma} \Lambda \otimes V \xrightarrow{\beta} W$$

If the case of a group alg you have projections $e_s$ of $\Lambda \otimes V$ onto $s \otimes V$.

Last night tried to review topos idea

$C \overset{\text{C}^\wedge}{\longrightarrow} \text{Fun} (C, \text{sets})$

There a topos map $\mathcal{F} \xrightarrow{f^*} \mathcal{C}^\wedge$ is given

by $f^* : C^\wedge \rightarrow \mathcal{F}$ if $f^*$ retent, left exact

$\mathcal{F} = \text{sets}$ (the pt topos), then $f^*$ retent

means $f^*$ given by "twisting" with $R \in (C^\text{op})^\wedge$

$$f^*(L) = R \times L = \lim_{X \in C/R} h_X \times L(x)$$

$\text{C}^\text{op} \xrightarrow{\text{C}^\wedge} \text{Yoneda}$
In the case of a groupoid

IDEA: Groth has all these nice category ideas which should be linearized

In the case of a groupoid the pre-representable is the same as representable.

\[ G^{op} \rightarrow \text{Fun}(G, \text{sets}) \xrightarrow{f^*} \text{Sh}_B \]

\[ C^{op} \xrightarrow{\text{Pro}(C)^{op}} \rightarrow \text{Fun}(C, \text{sets}) \]

\[ X \times X^* \quad \text{height} \; x^*(y) = \text{height}(x^*y) \]

So now take

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

\[ \Lambda = \text{arrow ring of groupoid} \]

You want a partition of \( \Lambda \otimes V \), really of \( \Lambda \).

What you want is to see if there is a relation, link between the category situation:

\[ G^{op} \rightarrow \text{Fun}(G, \text{sets}) \]

and the assembly stuff you are studying.

Let's begin:

\[ \Lambda = C[A_r] = \bigoplus C[A_r(y, x)] \quad \forall, x \in Ob \]

\[ y \xleftarrow{f} x \]

\( \Lambda \) is the arrow ring, reduced \( \Lambda \)-modules same as cov funs, etc. \( R \otimes \Lambda \). Yoneda? You want a category inside of \( \Lambda \)-modules.
\[ \Lambda = \text{arrow ring of } \Gamma \]

\[ \{ \text{red } \Lambda \text{-modules} \} = \mathcal{F} \text{un}\left( \Lambda \Gamma, \Lambda e \right) \]

You want Yoneda. For each \( g : X \rightarrow Y \), you want an arrow \( \Gamma \rightarrow \Lambda e \), i.e., a left \( \Lambda \)-module \( \Lambda e_X = \bigoplus_Y \mathbb{C}[g(X,Y)] = \mathbb{C}[h^X] \)

\[ \text{Hom} \left( \mathbb{C}[h^X], M \right) = L(X) \]

\[ \Lambda = \bigoplus_X \mathbb{C}[h^X] \]

\[ \Lambda = \mathbb{C}[\Gamma] = \bigoplus_X \mathbb{C}[g(X,Y)] = \bigoplus_X \mathbb{C}[h^X] \]

So you have this splitting of \( \bigoplus \Lambda \) as a left \( \Lambda \)-module.

\[ W \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} W \]

Aim towards reconstructing \( V \) from \( W \).

\[ \Lambda e_X = \sum \Lambda e_{yx} \]

It seems that you have some type of inducing taking place.
First see about the $M_{aq}$ situation

$$T^* \otimes W \xrightarrow{\text{dx}} T^* \otimes V \xrightarrow{\text{V}} T^* \otimes W$$

The point to make is that for each object $X$, there seems to be a map $W$.

Try to understand in the $M_{aq}$ situation how $V$ might be recovered, assuming it is reduced. $V$ is reduced when $V = \sum p(s) V$ and $\cap \text{ Ker}(p(s) \text{ on } V) = 0$

$$\overline{W} \xrightarrow{\text{C}^n \otimes V} \overline{W}$$

It seems time for better details. Begin with the $A$ for $M_{a}$.

$p = \left(\begin{array}{cc} p_{11} & p_{12} \\ p_{21} & p_{22} \end{array}\right) \in M_2 \otimes \text{End}(V)$

$p^2 = p$ \quad $\text{End}(C^\infty V)$

$p = \sum e_{ij} \otimes p_{ij}$

Typical element of $C^\infty V$ is $\sum e_k \otimes f_k = f$

$$p \sum_k e_k \otimes f_k = \sum \frac{\partial g_{ij}}{\partial e_k} e_k \otimes p_{ij} f_k = \sum e_i \otimes p_{ij} f_i$$

$$= \sum e_i \otimes \sum p_{ij} f_i$$

$$(p\psi)^{(i)} = \sum p^{(i)} f_{(j)}$$
To given \( p = p^2 \) on \( C^n \otimes V = V^{\oplus n} \).

\((p_{ij}) \in M_n(\text{End}(V)), \sum_j p_{ij} p_{jk} = p_{ik}\)

What does it mean for \( V \) to be reduced.

\[ (\beta_1, \beta_2) \quad \implies \quad (\alpha_1, \alpha_2) \quad \implies (\beta_1, \beta_2) \quad \text{idem} \]

\[ p_{ij} = \alpha_i \beta_j \]

\[ \sum_j \alpha_i \beta_j = V \]

\[ \sum_j p_{ij} p_{jk} = \sum_j \alpha_i \beta_j \alpha_j \beta k = \alpha_i \beta k = p_{ik} \]

\[ W = \beta_1 V + \beta_2 V \]

\[ \alpha_1 W = \alpha_1 \beta_1 V + \alpha_1 \beta_2 V \]

\[ \alpha_2 W = \alpha_2 \beta_1 V + \alpha_2 \beta_2 V \]

\[ \therefore V = \sum p_{ij} V \iff V = \alpha_1 W + \alpha_2 W \]

\[ \bigcap \ker(p_{ij} \text{ on } V) = 0 \iff 0 = \bigcap \ker(\beta_j \text{ on } V) \]

\[ 0 = \alpha_i \beta_j V \quad i = 1, 2 \iff (\beta_j \cdot 0 = 0) \quad j = 1, 2 \]
So what do we learn? An $A$-module structure on $V$ consists of $p_{ij} \in \text{End}(V)$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ such that $\sum_{j=1}^{n} p_{ij} p_{jk} = p_{ik}$.

whence you have a retract

$W \leftarrow \mathbb{C}^n \oplus \mathbb{C}^n \rightarrow W$

with $p_{ij} = a_i \beta_j$. $V$ is reduced $A$-module iff

$V = \sum_{i=1}^{n} a_i W$ and $\bigcap_{j=1}^{n} \ker(\beta_j : V \to W) = \{0\}$

Assuming $V$ is reduced, you should be able to recover $V$ from the retract $W$.

$W \leftarrow \mathbb{C}^n \oplus V \rightarrow W$

$\alpha_1 W + \alpha_2 W = V$

$p_{ij} = a_i \beta_j$. Assume $V = a_1 W + a_2 W$, let $v \in V$.

write $v = a_1 w_1 + a_2 w_2$, write $w_1 = \beta_1 v_1 + \beta_2 v_2$, $w_2 = \beta_1 v'_1 + \beta_2 v'_2$.

then $v = a_1 \beta_1 v_1 + a_1 \beta_2 v_2 + a_2 \beta_1 v'_1 + a_2 \beta_2 v'_2 \in \sum_{i,j} p_{ij} \mathbb{C}$.
To understand $M_2$ completely. A base generator $p_{ij}$ $i, j = 1, 2$ subject to relations $\sum_{j} p_{ij} p_{jk} = p_{ik}$, i.e. $(\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix})$ is idempotent.

Let $V$ be an $A$-module. Then $V \oplus \frac{p_{ij}}{p_{21}} V$ is idempotent.

\[
p \left( \sum_k e_k \otimes f(k) \right) = \sum_{k, j} e_{ij} e_k \otimes p_{ij} f(k)
\]

\[
= \sum_{i, j} e_{ij} \otimes p_{ij} f(j) = \sum_i e_i \otimes \sum_j p_{ij} f(j)
\]

\[
W \left( \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \right) V = \sum \beta_i \alpha_i = 1_W
\]

$P = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} (\beta_1, \beta_2) = \begin{pmatrix} \alpha_i \beta_j \end{pmatrix}$

What seems to happen is that by introducing $W$ the image of $P$ on $V$, the retract of $V \oplus V$ corresponds to $P$, you actually get a factorization of $P$ into $(\cdot_1 \cdot \cdot \cdot)$. \[
(\cdot_1 \cdot \cdot \cdot) (\beta_1 \cdot \cdot \cdot \beta_n)
\]
 Mayer Victor's, the simplest partition situation

\[ (\beta_1, \beta_2) \quad V_1 \quad V_2 \]

It appears your mistake was trying to use \( \Lambda \otimes V \) instead of allowing \( V \) to depend on the source object. You want a free \( \Lambda \) module to be a direct sum of representable functors.

\[ \bigoplus \Lambda e \otimes V \cdot [k^x] \times \]

Let's work this out in the simplest case \( M_2 \):

two objects, \( V_1, V_2 \).

First digest to understand the ring \( \Lambda \), which should be slightly different from what you expected.

\[ M_2 \mathcal{C} = \Lambda \quad \text{this is the arrow ring of} \quad \text{the groupoid} \ M_2. \]

Now you consider \( \Lambda = T \otimes T^* \).

You are after a retract a "free" \( \Lambda \)-module

\[ W \leftarrow (\Lambda e_{11} \otimes V_1) \quad \Lambda e_{22} \otimes V_2 \]

What is new is the meaning of free \( \Lambda \) module.

So you do get \( T \otimes (V_1, V_2) \) for your free module.
So by $M$, eq. back to

$$W^\# \leftarrow \left( \beta_1, \beta_2 \right) V_1 \bigoplus \left( \alpha_1, \alpha_2 \right) V_2 \leftarrow W^\#$$

Start again. You have the notion of free $\Lambda$-module for $\Lambda = CM_2 = M_2 \mathbb{C}$, which leads to retracts of the form

$$W \leftarrow \mathbb{C} \otimes \left( V_1 \bigoplus V_2 \right) \leftarrow W$$

Then by $M$, eq. to retracts

$$W^\# \leftarrow \left( \beta_1, \beta_2 \right) V_1 \bigoplus \left( \alpha_1, \alpha_2 \right) V_2 \leftarrow W^\#$$

You propose now to study the latter to understand $A$.

You need to look at arb. projections on

$$V_1 \bigoplus V_2$$

$$p = p^2 \text{ in } \text{End}(V_1 \bigoplus V_2)$$

Naturally an $M_2$ graded alg.

Your $p_\sigma = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \in \begin{pmatrix} \text{End}(V_1) & \text{Hom}(V_1, V_2) \\ \text{Hom}(V_2, V_1) & \text{End}(V_2) \end{pmatrix}$

so as before you get $p = (\alpha_1)(\beta_1, \beta_2)$
I'm confused. You have the notion of free module where you are given \( V_x \) for each object \( X \). Thus it should be clear that the modules are graded.

Back to \( M_2 \). The notion of free module which involves representable functors. Looks good because of topos background. If \( G \) is a groupoid then a topos map \( \mathcal{L}_B \xleftarrow{f^*} \mathcal{G}^\wedge \) is described by a functor \( R : \mathcal{G}^{op} \to \mathcal{L}_B \), i.e. a right \( \mathcal{G} \) sheaf over \( B \), whose stalks are representable. In other words \( R \) is a sheaf over \( B \) with right \( \mathcal{G} \) action, which means you are given \( R \xrightarrow{\text{res}} \text{Ob}(B) \) and \( R \times \text{Ob}(B) \to R \) making a contravariant functor.

A groupoid, \( \mathcal{G}^\wedge = \text{Fun}(\mathcal{G}, \text{sets}) = \{ \mathcal{G}\text{-sets} \} \) is a topos. A topos map from \( \mathcal{L}_B \) to \( \mathcal{G}^\wedge \) is given by a functor \( \mathcal{L}_B \xleftarrow{f^*} \mathcal{G}^\wedge \). \( f^* \) right adjoint. \( f^* \) right adjoint implies \( f^L : \text{Fun}(\mathcal{L}_B, \text{sets}) \to \text{Fun}(\mathcal{G}^\wedge, \text{sets}) \)

\( R \) is the \( \mathcal{G}^{op} \)-sheaf over \( B \) given by

\[
\begin{align*}
\mathcal{G}^{op} \xrightarrow{\text{Yoneda}} \mathcal{G}^\wedge \xrightarrow{f^*} \mathcal{L}_B
\end{align*}
\]

Finally, \( f^* \) left adjoint means that \( \mathcal{G}^{op}/R \) (this should be the coproduct of the \( \mathcal{G}^{op} \) action on \( B \)) has a final object locally over \( B \).
Repeat: A groupoid, $Y^\uparrow = \text{Fun}(Y, \text{sets})$

topos map $sh_B \to Y^\uparrow$ given by $f^*: Y^\uparrow \to sh_B$

constant + left exact. $f^*$ constant implies $f^*$ has the form $f^*(L) = R \times g L$ where

$R$ is the $Y^{op}$-sheaf $\xrightarrow{\text{const.}} Y^{op} \to Y^\uparrow \xrightarrow{f^*} sh_B$

$f^*$ left exact means the functor $f^*$ at each $b \in B$ the functor given by $R_b$ with $Y^{op}$ acting is representable, better to say the $Y^{op}$-set given by $R_b$ is representable.

Where to start. $\Lambda = \text{arrow ring of } Y$. Yesterday you learned that there might be a new notion of free $\Lambda$-module, namely $\bigoplus_{x} \Lambda_{E_{x}} \otimes V_{x}$, a direct sum of representable functors $\Lambda_{E_{x}} = C[h^{x}] = \bigoplus_{y} C[\text{ar}(y, x)]$. For a connected groupoid the functors $C[h^{x}] = \Lambda_{E_{x}}$ are all isomorphic, so it is not really a new notion. Only in so far that the old version of free module with one generator $\Lambda$ is replaced by $\Lambda_{E_{x}}$ which is smaller. Now you need
Repeat situation. Look at $S = M_2$.

An $M_2$ graded ring is a Monot matrix context.

Let's review what you know. Consider $A \otimes V$.

Because $\Lambda$ is graded over $M_2$?

Non comm. Mayer-Vietoris.

First understand groupoid consisting of 2 ets.

Only the identity maps allowed.

\[
W \leftarrow (\beta_1, \beta_2) \oplus W \leftarrow (\alpha_1', \alpha_2') \oplus W
\]

\[
\beta_1 \alpha_1 + \beta_2 \alpha_2 = 1
\]

\[
W \leftarrow (\beta_1, \beta_2) \oplus W
\]

\[
V_1 \oplus V_2 = \Lambda = C e_{11} \oplus C e_{22}
\]

$p = p^2$ in a $\Gamma$ graded ring.

\[
\Gamma = \{ e_1, e_2 \}
\]

\[
e_1^2 = e_1, \quad e_2^2 = e_2 \]

\[
e_1 e_2, e_2 e_1, \text{ undefined}
\]

\[
A = A_1 \oplus A_2
\]

\[
A \rightarrow C \Gamma \otimes A
\]

\[
a_1 \rightarrow e_1 \otimes a_1
\]

\[
a_2 \rightarrow e_2 \otimes a_2
\]

\[
A A_1 \subset A_1, \quad A_2 A_1 \subset A_2
\]

\[
A_1 A_2 = A_2 A_1 = 0
\]

\[
p_1 = p_1, \quad p_2 = p_2
\]

\[
p_1 p_2 = p_2 p_1 = 0
\]
Next $\Gamma = \mathbb{M}_2$. $A = \mathbb{C} \bigoplus A_{ij}$

$$A_{ij} \otimes A_{kl} = \begin{cases} 0 & j \neq k \\ A_{il} & j = k \end{cases}$$

**IDEA** that you should be careful about $e_x^2 = e_x$

$M_2$-graded alg = Morita context.
Interested in $p = p^2$ in an $M_2$-graded alg.
get an $M_2$ graded ring $A$

$$\Delta : A \rightarrow M_2 \otimes A$$

$$\Delta : a_{ij} \rightarrow e_{ij} \otimes a_{ij}$$

But the units $e_{ii}, e_{jj}$ are multiphers on $A$ so that reduced modules naturally split

**Argument**: A universal alg gen. by components $p_{ij}$ of a projection in an $M_n$-graded alg. Then $A$ is naturally $M_n$ graded also idempotent. The diagonal units $e_{ii}$ can be adjoined to $A$

$$e_{hh}(a_{ij} b_{mn}) = (a_{kj})_{ij} (b_{mn}) = a_{kj} b_{jn}$$
Let $A$ be graded wrt a groupoid $\Gamma$. Can you show that the units $e_x$ for $x$ any object. $A$ is a graded ring if $A$ is graded with $\Gamma$ a groupoid. For each object $x$ of $\Gamma$ define a multiplier $\mu_x$ of $A$ by:

$$\mu_x(z \triangleright y) = \begin{cases} 0 & x=2 \\ (x \triangleright y) & x \neq 2 \end{cases}$$

$$(z \triangleright y) . \mu_x = \begin{cases} 0 & y \neq x \\ (z \triangleright x) & y = x \end{cases}$$

$$\mu_x(z \triangleright y)(y \triangleright g \cdot u) = \mu_x$$

$$(\mu_x a_f) a_g = \mu_x(a_f a_g)$$

$$a_f (\mu_x) a_g = a_f (\mu_x a_g)$$

If $f \neq 0$ then $(f) = x$

Source$(f) = x$, Target$(f) = g$

You have a $\Gamma$-graded ring $A : A_f A_g \subseteq \begin{cases} 0 & f \text{ not defined} \\ A_f & \text{other} \end{cases}$

$x$ object let $e_x$ be the operator on $A$ defined by $e_x a_f = \begin{cases} a_f & \text{if} \ \text{Target}(f) = x \\ 0 & \text{if not} \end{cases}$
Maybe take care of the cases by using the graded:

\[
\begin{align*}
A & \xrightarrow{\Delta} \mathcal{C} \otimes A \\
& \xrightarrow{\Delta} f \otimes a_f
\end{align*}
\]

what elements of \( \mathcal{C} \otimes A \) yield multipliers on \( A \):

\[
(\otimes 1)(f \otimes a_f) = \begin{cases} 
0 & \text{if } X \neq \text{ils} \\
\sum f \circ a_f & \text{if } X = \text{ils}
\end{cases}
\]

\[
(f \otimes a_f)(\otimes 1) = \begin{cases} 
\sum f \circ a_f & \text{if } X = \text{ils} \\
0 & \text{if } X \neq \text{ils}
\end{cases}
\]

So it seems clear that you can adjoin units belonging to objects. Back to \( M_2 \). Now your \( A \) is \( M_2 \)-graded and idempotent. Idempotent implies so a red. \( A \)-module \( V \) splits into \( e_{11} V \oplus e_{22} V \).

At this point this means is

Repeat: Given the groupoid \( M_2 \), you have the notion of a \( M_2 \)-graded algebra (= Mor. context), and can form \( A \) the unital alg gen. by the components of a proj in a \( M_2 \)-graded alg. \( A \) is \( M_2 \)-graded and we can adjoin \( e_{11}, e_{22} \) to \( A \) to get a untied Morita context. So next consider an \( A \)-module \( V \):

\[
W \leftarrow \, \otimes \, M_2 C \otimes V \leftarrow \alpha W
\]
$A$ = univ. alg gen by components $p_{ij}$ of a proj in a $M_2$-graded alg.

$A$ is idempotent, $M_2$-graded, and can be enlarged to a unital $M_2$-graded alg.

Why $A$ is $M_2$-graded.

Define $\Delta : A \to M_2 \otimes A$ to be the alg map such that $\Delta(p_{ij}) = e_{ij} \otimes p_{ij}$.

Check relations

$$\sum \Delta(p_{ij}) \Delta(p_{jk}) = \Delta \left( \sum (e_{ij} \otimes p_{ij})(e_{jk} \otimes p_{jk}) \right) = \sum e_{ik} \otimes \sum p_{ij} p_{jk} = e_{ik} \otimes p_{ik} = \Delta(p_{ik})$$

$\Delta : A \to M_2 \otimes A$ is an alg map

Check $\xymatrix{ A \ar[r]^-\Delta & M_2 \otimes A \ar[d]^-\eta & M_2 \otimes M_2 \otimes A \ar[l]_-{\Delta \otimes 1} \ar[d]^-1 \otimes \Delta \ar[r] & M_2 \otimes A \ar[l]^-{1 \otimes \Delta} \ar[u]^-s \otimes s }$

$p_s \mapsto s \otimes p_s \mapsto s \otimes s \otimes p_s$

Next point $A \xrightarrow{\Delta} M_2 \otimes A \subset M_2 \otimes A^*$

Claim that $e_{11} \otimes 1$, $e_{22} \otimes 1$ in $M_2 \otimes A^*$ such that left or right mult by these elts preserves $\Delta A = \bigoplus e_{ij} \otimes A_y \subset M_2 \otimes A$

$$(e_{11} \otimes 1)(e_{ij} \otimes a_{ij}) = \delta_{i1} e_{ij} \otimes a_{ij} = \left\{ \begin{array}l 0 \quad i \neq 1 \\
\end{array} \right.$$
$\delta: A \rightarrow \Lambda \otimes A$

Look at $e_x \otimes 1 \in \Lambda \otimes \Lambda$ contains $\Lambda \otimes A$ as ideal

$$(e_x \otimes 1) \Delta(a_s) = (e_x \otimes 1)(s \otimes a_s) = e_x s \otimes a_s$$

$$= \begin{cases} 
  s \otimes a_s & \text{if } X = \text{target}(s) \\
  0 & \text{if } X \neq \text{target}(s)
\end{cases}$$

$\in \Delta(A_s)$

Therefore you find that $e_i$.

Review $M_2$

$A \rightarrow \Lambda \otimes A$, $A = \oplus A_s$, $\Delta(a_s) = s \otimes a_s$

$\Lambda \otimes \Lambda$

Inside $\Lambda \otimes \Lambda$ you have subalgebras $\Delta A \oplus C(e_x \otimes 1)$, you can adjoin elements of the idempotents belong to objects.

$$(e_x \otimes 1) \otimes (e_x \otimes 1)(s \otimes a_s) = e_x s \otimes a_s$$

So let's see how this works for $M_2$. Let $V$

be an $A$-module, $p_s$

$A$-module $V$ has operator $p_s$
You want to understand clearly the situation. But it should be simpler to treat a connected groupoid $\Gamma$. Assembly arrow ring $\text{CR}_\Gamma$, basis $\frac{1}{1+y^x} a_2(Y, X) = a_2 \text{Ham}_\Gamma(X, Y)$.

Notion of $\Gamma$-graded alg. $A \rightarrow \text{CR} \otimes A$.

$\Gamma$-graded alg. $A$ is alg. with splitting $A = \bigoplus_{s \in \Gamma} A_s$.

\[
\Delta(a_s) = s \otimes a_s, \quad \Delta(a_s a_t) = s t \otimes a_s a_t
\]

A $\Gamma$-graded alg.

$p = p_t$ in $\text{Hom}$ means $p_s = \sum_{s = tu} p_t p_u$.

Define $A_\Gamma$ by gens + rels. $V$ an $A$-module.

$\Lambda \otimes V = \sum t \otimes f(t)$

\[
p\left(\sum_t t \otimes f(t)\right) = \sum_{t, u} \sum_t t u^{-1} \otimes p(u) f(t)
\]

\[
= \sum_{s, t} \sum_t p(s^{'t}) f(t)
\]

What do you want to do? Settle question of $V$ being graded with respect to objects.
\[ \Lambda = \mathcal{G} \Gamma = \bigoplus \mathcal{C}[h^\times] \times \text{basis } (\mathcal{G} \Gamma \chi) \]

\[ \mathcal{R} = M_2 \]

\[ W \xleftarrow{\beta} \Lambda \otimes V \xleftarrow{\alpha} W \]

\[ W \xleftarrow{\times (\beta_1, \beta_2)} V \oplus \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) \xrightarrow{\oplus} W' \]

\[ V \xleftarrow{\times (\beta_1, \beta_2)} W' \oplus V \]

So you should start maybe with

\[ V \oplus \left( \begin{array}{cc} p_{11} & p_{12} \\ p_{21} & p_{22} \end{array} \right) \oplus V \]

and introduce the image

\[ V \xleftarrow{\times (\beta_1, \beta_2)} W' \oplus V \]

\[ V_1 \oplus \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) \xrightarrow{\oplus} W' \oplus V_1 \]

\[ V_2 \xrightarrow{\oplus} W' \oplus V_2 \]
There are two situations

\[ \begin{array}{c}
\circlearrowleft \left( \alpha_i \right) \\
\oplus \\
\circlearrowleft \left( \beta_i \right) \\
\oplus \\
\circlearrowright \left( \beta_i \right) \\
\oplus \\
\circlearrowright \left( \alpha_i \right) \\
\end{array} \]

\[ \begin{array}{c}
V \\
\oplus \\
V_1 \\
\oplus \\
V_2 \\
\end{array} \]

\[ \begin{array}{c}
\oplus \\
\oplus \\
\oplus \\
\oplus \\
\oplus \\
\end{array} \]

\[ \begin{array}{c}
V_1 \\
\oplus \\
V_2 \\
\oplus \\
V_1 \\
\oplus \\
V_2 \\
\end{array} \]

In the graded case

\[ (p_{ij}) \in \begin{pmatrix}
\text{Hom}(V_1, V_1) & \text{Hom}(V_1, V_1) \\
\text{Hom}(V_1, V_2) & \text{Hom}(V_2, V_2)
\end{pmatrix} \]

Start with ungraded case, go through the process of making \( V \) reduced. You find that then \( V \) will split.

In the ungraded case when \( V \) is reduced,

\[ V = \sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V \subset \sum_i \alpha_i W^# \]

\[ \sum_{ij} \alpha_i \beta_j v_{ij} \subset \sum_i \alpha_i \sum_j \beta_j V \]

\[ \sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V \]

\[ \subset \sum_i \alpha_i W^# \]

Robert \[ \sum_{ij} p_{ij} V = \sum_{ij} \alpha_i \beta_j V \]

\[ \subset \sum_i \alpha_i W^# \]
\[
\sum p_{ij} V = \sum_{i,j} \alpha_i \beta_j V = \alpha_1 \beta_1 V + \alpha_1 \beta_2 V + \alpha_2 \beta_1 V + \alpha_2 \beta_2 V = V \\
= \alpha_1 \beta V^+ + \alpha_2 \beta V^- \iff V = \alpha_1 \beta V^+ + \alpha_2 \beta V^-
\]

\[
\bigcap \text{Ker}(p_{ij} \circ V) = \bigcap \text{Ker}(\alpha_i \beta_j \circ V)
\]

\[
\alpha_i \beta_j \circ \sigma = 0 \quad \forall i, j \implies \beta_j \circ \sigma = 0 \quad \forall j.
\]

\[
\therefore \bigcap \text{Ker}(p_{ij} \circ V) = \bigcap \text{Ker}(\beta_j \circ V)
\]

So you seem to understand what a reduced ungraded A module is. Question: What is a reduced graded module?

Assume V reduced A-module

\[
A \otimes_A V \xrightarrow{\text{semi}} \text{Hom}_A(A, V)
\]

Therefore you get \(e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\), \(e_{22} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\)

which splits V into \(V = V_1 \oplus V_2\) gives a \(\mathbb{Z}/2\) grading

Suppose \(A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\) is idempotent. Does this imply that \(A_2 A_t = A_{st} = A\; \forall s, t\)?

A graded and \(A = A^2\), does this imply \(A\) is graded?
Consider a Monza context \((A_{11}, A_{12}) \oplus (A_{21}, A_{22})\)

which is idempotent as a ring. Let \(V\) be a reduced \(A\)-module. You know that you can embed \(A\) as ideal in a unital Monza context.

\[
R = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix} = \begin{pmatrix}
C & 0 \\
0 & C
\end{pmatrix} \oplus \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}
\]

Make it clear you know the \(A\) action on \(V\) extends uniquely to a unital \(R\)-action, hence it should be clear that \(V = \begin{pmatrix}
e_{11}V \\
e_{22}V
\end{pmatrix} = \begin{pmatrix}
V_1 \\
V_2
\end{pmatrix}
\]

with \(A_{ij} V_k = \begin{cases} 0 & j \neq k \\ V_i & j = k \end{cases}\)

\[
\begin{pmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{pmatrix}
\]

is idempotent when \(A_{11}, A_{22}\) idem.

but not comp. idemp: \(A_{21} A_{12} = 0\)

Return to \(A = \langle p_{ij} : \sum_j p_{ij} p_{jk} = 0 \quad i \neq k \rangle\)

\[\begin{align*}
A & \xrightarrow{\Delta} M_{nA} \\
p_{ij} & \xrightarrow{\Delta} e_{ij} \otimes p_{ij}
\end{align*}\]
So have $A$ with its universal $M_2$ graded proj $(\pi_{ij})$. Reduced

you need to formulate things clearly

in order to believe them. Given $A$ gen

by $\pi_{ij}$rels $\begin{cases} \pi_{ij}\pi_{kj} = 0 & j \neq \ell \\ \pi_{ik} = \sum \pi_{ij}\pi_{jh} & \end{cases}$

and a reduced $A$-module $V$. You

know $V$ is graded by object projections. Thus

in $V = \bigoplus V_i$ with $\pi_{ij}V_k \subseteq \{0 \quad \text{if } j \neq k \}$

In $\text{End}(V)$ there are besides the $\pi_{ij}$, the object units $e_{ii}$,

\[
\begin{array}{ccc}
V_1 & \overset{(p_{11} \quad p_{12})}{\longrightarrow} & V_1 \\
\oplus & & \oplus \\
V_2 & \overset{(p_{21} \quad p_{22})}{\longleftarrow} & V_2
\end{array}
\]

\[
\sum_{ij} \pi_{ij}V = \sum_{ij} \pi_{ij}V_j = \begin{pmatrix} p_{11}V_1 + p_{12}V_2 \\ p_{21}V_1 + p_{22}V_2 \end{pmatrix}
\]