Joachim did understand the case $M$. He says that the alg $A$ generated by the components of an $M$ graded projection is Morita equivalent to functions on a simplex. Look at $A$ generated by $p_{ij}$ satisfying:

$$
\begin{align*}
\sum_j p_{ij} p_{jk} &= \delta_{ik}, \\
p_{ij} p_{kj} &= 0, \\
p_{ij} p_{kk} &= 0.
\end{align*}
$$

$$
\Delta : A \to C(M) \otimes A
$$

$$
p_{ij} \mapsto e_{ij} \otimes p_{ij}
$$

You want to find $B$ which might be a cross product:

$$
\Gamma \text{ a set, let } \Gamma_+ = \Gamma \cup \{0\}
$$

a generalization of the group ring $C\Gamma$ assoc to a group. Suppose given $\mu : \Gamma_+ \times \Gamma_+ \to \Gamma_+$ assoc. s.t. $0$ is absorbing. $\Gamma$ semi group ring $C\Gamma$ is a Hopf alg. $\Delta(1) = 1 \otimes s$

$$
\Delta \in \mathcal{C}(0) \otimes \mathcal{C}(0) \otimes \mathcal{C}(0)
$$

$$
\Delta : C[\mathcal{C}(0)] \to C[\mathcal{C}(0)] \otimes C[\mathcal{C}(0)]
$$

$\Delta : \mathcal{C}(\mathcal{C}(0)) \to \mathcal{C}(\mathcal{C}(0)) \otimes \mathcal{C}(\mathcal{C}(0))$

$\mathcal{C}(\mathcal{C}(0))$, $\mathcal{C}(\mathcal{C}(0)) \otimes \mathcal{C}(\mathcal{C}(0))$ is quotient ring $\mathcal{C}(\mathcal{C}(0))/\mathcal{C}(0)$.
\[ \Gamma_+ \times \Gamma_+ \xrightarrow{\times} \Gamma_+ \] semigroup with 0 abs.

semigroup ring \( \mathcal{C} \Gamma_+ \) in which \( \mathcal{C}[0] \) is ideal.

so get ring structure on \( \mathcal{C} \Gamma = \mathcal{C} \Gamma_+/\mathcal{C}[0] \) s.t.
\[
[st][t] = [st] \quad \text{if} \quad st \in \Gamma
\]
\[
= 0 \quad \text{if} \quad \text{not}
\]

\( \mathcal{C} \Gamma \) coalg structure \( \Delta \gamma = \gamma \otimes \gamma \), get \( \mathcal{C} \Gamma \) bialgebra.

\( \Gamma \)-graded alg \( A = \bigoplus_{\gamma \in \Gamma} A_{\gamma} \) s.t. \( A_{\gamma} A_{\delta} \subset \{ \begin{array}{ll}
A_{\gamma \delta} & \text{if} \quad \gamma \delta = 0 \\
0 & \text{otherwise}
\end{array} \}
\]

\[ A \xrightarrow{\Delta} \mathcal{C} \Gamma \otimes A \]
\[ a_\gamma \mapsto \gamma \otimes a_\gamma \]

Now you want to understand what you missed. This time you have to start with the \( \mathcal{C} \Gamma \) which is harder. So maybe you ought you try special cases:

\( \Gamma \)-groupoid, \( \Gamma \)-semigroup

 simpleset semigroup is \( \Gamma_+ \) which is a gp.

semigroup structure on \( \Gamma_+ = \) two points \( 0, x \)

\( 0, x \), \( x0 = 0x = 0 \), \( x^2 = 0 \) \( \Gamma \)-graded alg.

\[ A = A_0 \oplus A_x \otimes 0 \text{ mult.} \]

\[ \Gamma_+ = \{ x, 0 \} \]

\[ A = A_x \]

\[ A^2 = 0 \]

\[ \Gamma_+ = \{ x^0, x^2, 0 \} \]

\[ A = A_x \oplus A_{x^2} \]

\[ A^2 = 0 \]
So it looks like you want to restrict attention to groupoids. In other words, what is assembly for a groupoid? You want the basic idea. You need to go back to Serre's idea - trying to construct a vector bundle on the classifying space $BG$. You have some background from Haefliger theory as to what the classifying space of a groupoid should be. But you probably don't understand sufficiently the group case.

Look at group case. Principal bundle $X \to Y$ with fibre $\Gamma$. Open covering of $Y$ over which the bundle becomes trivial.

Groupoid case? You should know how to do this. Semi-simplicially there is a classifying map to the nerve of the groupoid. In the case of $M$ where the objects are elements of $F$

Model: Given $Y$, an open covering of $Y$, you have the nerve of the covering, which is a simplicial set.

$$
Y \leftarrow U \rightarrow U \times U \rightarrow U \times U \times U
$$

$$
F \rightarrow F \times F \rightarrow F \times F \times F
$$

This leads to a simplicial set.

Lots of ideas to be reviewed.
Sheaf version. The simplest situation where things work well is the case of a topological groupoid where the source and target arrows are étale. This is the groupoid generalization of a discrete group. Ex: Action of a discrete group on a top space $F$. Nerve of groupoid is

\[ E \subseteq E \times G \subseteq E \times G \times G \ldots \]

Point is that there is a nice topos picture of what should be a principal bundle for such a groupoid. It's a sheaf over the base $B$ acted on by the groupoid.

The stalk at any point of $B$ is going to be a functor on the ?

Special case of $\Gamma$ disc. acting on $F$ space.

Top. group class. space $E \Gamma \times F$

\[
\begin{array}{ccc}
E & \longrightarrow & E \Gamma \times F \\
\downarrow & & \downarrow \\
B & \longrightarrow & E \Gamma \times F
\end{array}
\]

So the first
The first thing to do is to analyze carefully the case of the etale groupoid arising from a \( \Gamma \) disc acting on a space \( F \). The classical space for this groupoid should be \( E \Gamma \times \Gamma F \). Over a base space \( B \) the principal bundles associated to this groupoid are obtained by pullback:

\[
\begin{array}{ccc}
E & \longrightarrow & E \Gamma \times F \\
\downarrow & & \downarrow \\
B & \longrightarrow & E \Gamma \times \Gamma F
\end{array}
\]

Equivalently, a principal bundle over \( B \) assoc. to the groupoid is the same as a principal \( \Gamma \)-bundle \( E \) over \( B \) equipped with equivariant maps:

\[
E \longrightarrow F.
\]

\( \Gamma = \{ e, o^2 \} \quad q^2 = o_0 = g o = o^5 = o^2 \)

Then a \( \Gamma \)-graded \( A \) is just \( A_q \) such that \( A^2_q = 0 \).

\( \Gamma = \{ e, o^2 \} \quad e^2 = e, e o = 0 e = 0 \)

\( A = A_o \quad \text{s.t.} \quad A_e a_e < A_e \quad \)

Program for today: Understand \( \bigcirc \) assembly for a groupoid, e.g., first you have to go over past ideas, Haefliger, etale groupoid.
Find examples. First take disc group $\Gamma$ acting on a space $F$. This gives a nerve on etale groupoid. $F$ is the space of objects, $F \times \Gamma$ is the space of arrows (right action notation).

$$
\begin{align*}
\text{source} (\bar{s}, s) &= \bar{s} \\
\text{target} (\bar{s}, s) &= s,
\end{align*}
$$

the nerve of this groupoid is

$$
\Rightarrow \quad F \xleftarrow{\bar{1}} F \times \Gamma \xrightarrow{\bar{1}} F \times \Gamma \xleftarrow{\bar{1}} F
$$

$$
\Rightarrow \quad \frac{s}{\bar{s}} \xleftarrow{\bar{1}} (\bar{s}, s)
$$

$$
\text{Nerve } (F/\Gamma) = F \times \Gamma \text{ for the Atiyah semi-simplified } E\Gamma \to B\Gamma.
$$

And there is the sheaf picture for torsors associated to an etale groupoid. Actually, this idea should precede. What is a torsor over $B$ assoc. to $(F/\Gamma)$? Answer it is a $\Gamma$-torsor $E \to B$ and an equiv. map from $E$ to $F$, should be same as the fibre bundle over $B$ with $\Gamma$-action. START AGAIN with the first example, namely the etale groupoid $\Gamma$ given by $\Gamma$ disc acting on space $F$. You have nerve

$$
\exists \quad F \times \Gamma \xrightarrow{\bar{1}} F \times \Gamma \to F
$$
whose realization is $F \times \Gamma \text{ E}_\Gamma$, the fibre bundle associated.
A map $B \to F \times \Gamma \overset{E \Gamma \times F}{\to} \Gamma$ should yield first of all a map $B \to B \Gamma$ i.e. a principal $\Gamma \to E \to B$ over $B$ and an equivariant map $E \to F$. In fact, any $B \to F$ should be equivalent to such a pair by pull-back.

\[
\begin{array}{c}
F \times E \Gamma \to E \Gamma \\
\downarrow \text{cart.} \quad \downarrow \\
B \quad F \Gamma \quad B \Gamma
\end{array}
\]

not yet clear.

Given a $\Gamma$-space $F$, there is $F_\Gamma = E \Gamma \times F$, the total space of the fibre bundle

\[
\begin{array}{c}
F \to F_\Gamma \to B \Gamma
\end{array}
\]

over $B \Gamma$ assoc. to the $\Gamma$-space $F$. A map $B \to F_\Gamma$ yields a map $B \to B \Gamma$, whence a principle $\Gamma$-bundle $E = B \times_{B \Gamma} E \Gamma$.

\[
\begin{array}{c}
E \Gamma \times F \to F_\Gamma \\
\downarrow \\
E \Gamma \to B \Gamma
\end{array}
\]
It seems that a map from \( B \) to \( F \) still occurs.

Boil mixing diagram. Camp from \( B \) to \( B' \) by pullback of principal \( F \)-bundle over \( B \).

Start again: \( F \) space acted on by \( \Gamma \) disc.
Simpler version.

\[ E \xrightarrow{f} E \Gamma \times F \]

A map \( f : B \to F \Gamma \) yields a principal \( \Gamma \)-bundle \( E \to B \) by pull-back and \( \Gamma \)-map \( f : E \to E \Gamma \times F \) covering \( f \).

Maybe you should work \( \ast \) in the category of principal \( \Gamma \)-bundles, i.e., free \( \Gamma \)-spaces and equivariant maps. Given such an \( E \), then \( B = E/\Gamma \) and \( E \) is up to homotopy a map \( E \to E \Gamma \).

Still confusing!!!

Program: To understand what \( E \Gamma \times F \) classifies. First answer is a pair consisting of a principal \( \Gamma \)-bundle \( E \to B \) and a \( \Gamma \)-map \( E \to \Gamma \). Then you must adjust for homotopy.

Review. You are trying to recall classifying spaces for etale groupoids, such as Haefliger’s classifying spaces. Do you think there is a good classifying topos, that is a good analogous to the category of \( \Gamma \)-sets for a discrete...
group $\Gamma$. How to understand? The idea I think is that given an etale groupoid, you have sheaves (of sets) on the space $\mathcal{O}$ of objects and it makes sense to ask for the arrows to act on such sheaves. Think of etale spaces:

$\Gamma \xrightarrow{\text{source}} \Gamma \leftarrow \text{target}$

Have sheaf $\mathcal{F}$ over $\Gamma$, together with an $s \ast \mathcal{F} \cong \mathcal{F}$. So you need to handle a groupoid. You want to handle $\Gamma$. You still lack control of the simplest case. Anyway, list what you know.

The easiest example to understand is the etale topological groupoid arising from a discrete group $\Gamma$ acting on a space $\mathcal{F}$. The nicest case is when $\mathcal{F} \to \mathcal{F}/\Gamma$ is a principal $\Gamma$-bundle. In this case, you can form the Mischenko line bundle, associated fibre bundle over $\mathcal{F}/\Gamma$ with fibre $\mathcal{C}_\Gamma$.

Missing Point to work on. Review the Grothendieck topos picture to understand what should be a torsor for a simple groupoid such as $\mathbb{M}_2$. In fact $\mathbb{M}_2$ can be described as $\mathbb{Z}/2$ acting on itself by translation. Guess that a torsor for a $\mathbb{Z}/2 \times (\mathcal{F}, \Gamma)$ over a space $\mathcal{B}$ should be a principal $\Gamma$-bundle.
together with a map $E \to F$. When $F = \mathbb{Z}/2$, this means what.

So consider $\Gamma = \mathbb{Z}/2$ acting by translation on $E$. $F = \mathbb{Z}/2$.

The classifying space for $(F, \Gamma)$ should be $E \times F$. A map $B \to E \times F$ should yield a principal $\Gamma$-bundle $E$ over $B$ together with an equivariant map $E \to F$. When $F = \Gamma$ an equivariant map $E \to \Gamma$ should trivialize.

Another idea is the cross product algebra.

If $\mathcal{G}$ is a discrete group, then you get a topos $\mathcal{G}$ consisting of $\mathcal{G}$ sets. Suppose $\mathcal{G}$ is a groupoid, then you should have a topos consisting of functors from $\mathcal{G}$ to sets. What can you say about such functors?

Suppose you take a groupoid.

Pairing between right and left $\mathcal{G}$ sets.

You want both viewpoints. $\mathcal{G}$ groupoid, objects and maps. $\mathcal{G}$-sets $\mathcal{G}$-set = functors from $\mathcal{G}$ to sets.

If $\mathcal{C}$ is a small cat, then...
Today you want to understand classifying space for a groupoid. Let’s begin with Grothendieck’s approach. He treats the case of a group $G$, by associating to $G$ the topos of $G$-sets, i.e., functors from the category $(pt, G)$ to sets. A basic result is that for any space $B$ (eventually any topos), then a map of toposi from $\mathbf{B} = \text{sheaves of sets on } B$ to $\{G\text{-sets}\}$ is equivalent to a $G$-torsor over $B$. Why? In $\{G\text{-sets}\}$ one has the object $G$ given by the set $G$ with $G$ acting by right multiplication, so a map $f : \mathbf{B} \to \{G\text{-sets}\}$ should yield $f^*G$, which should be a $G$-torsor over $B$.

**Important Point.**

B space, let’s understand why a principal $G$-torsor over $B$ yields a morphism of toposi $f^* \rightarrow \{G\text{-sets}\}$.

This should be easy, namely a $G$-torsor over $B$ is an etale space $\pi : E \rightarrow B$ together with a right $G$ action on $E$ such that locally one has an isomorphism $E \cong B \times G$. If $S$ is a $G$-set one can twist $E \times G S$ to get a sheaf over $B$ functorial in $S$. This gives $f^*$ which is clearly
Right cont. (rops are lies) and resp. fin. lies. What is $f^*$?

\[ \text{Hom}(f^*S, F) \]

\[ = \text{Hom}_{\text{sh}}(E \times^G S, F) \text{ restrict to } U \subset B \]

\[ = \text{Hom}_{\text{sh}}(U \times S, F) = \]

Better might be \[ \text{Hom}_{\text{sh}}(E \times^G S, F) = \text{Hom}_{\text{sh}}(E \times S, F) \]

\[ \text{Hom}_{G}(S, \text{Hom}_{\text{sh}}(E, F)) \]

\[ \text{Hom}_{G}(E \times S, F) = \text{Hom}_{G}(E, F) \]

\[ \text{Hom}_{G}(E \times^G S, F) = \text{Hom}_{G}(E, F) \]

\[ f^*(F) = \text{Hom}_{\text{sh}/B}(E, F) \]

Something here reminds you of EG $\times^G S$

A map $\text{sh}/B \rightarrow \{G\text{-sets}\}$ is equivalent to the $G$-torsor $E$ over $B$ given by $E = f^*(G)$

and then \[ f^*(S) = f^*(G_x \times^G S) = E \times^G S \]

\[ \text{Hom}_{G}(S, f^*(F)) = \text{Hom}_{\text{sh}/B}(f^*S, F) = \text{Hom}_{\text{sh}/B}(E \times^G S, F) \]

\[ = \text{Hom}_{G}(S, \text{Hom}_{\text{sh}/B}(E, F)) \]
Now you want to go to groupoids, say discrete, $G$ groupoid $\{\text{G-set}\} = \text{Functors from } G \text{ to Set}_{\text{\skele}}$. This is a topos. $G$ decomposes into connected components (inside which any two objects are isomorphic), and a connected groupoid is equivalent to a group once a basepoint is chosen. Restrict to $G$ connected. You want to know what a map of toposes from $\mathcal{Sh}/B$ to $\{\text{G-set}\}$ looks like.

$G$ groupoid $\{\text{G-set}\} = \text{Fun} \text{ at } (G, \text{ sets}),$ 
this is a topos, $\{\text{G-set}\}$ topos coproduct (disjoint union) of $\{\text{G-set}\}$ where $G_x$ are the components of $G,$ $\{\text{G-set}\}$ equiv to $G\text{-sets}$ where $x \in \text{Ob } G_x,$ and $G_x = \text{Aut}_G(x).$ You have. In the case of $G$ a G-torsor over a space $B$ is an etale space $E \to B$ with a right action of $G$ on $E$ which is over $B,$ which is free. You might view $E$ as an object with free $G$-action in the $G$-tangent to any topos. $G$-torsor $E$ is an object with right $G$-action such that $E \times G \to E \times E$ is surjective.

Philosophy here is that toposes are the good generalization of the category Sets, so that whatever you do in Sets should carry over to any topos.
It should now be possible to define $G$-torsor, but first you need to understand $G$-torsor in sets. This should be a functor from $G$ to sets of some sort.

Try the following viewpoint. You have the groupoid $G$ and the topos $\text{Fun}(G, \text{Sets})$ of covariant functors. This topos should be the classifying topos for $G$-torsors. So you should analyze what is a map from toposi from any $T$ to $\text{Fun}(G, \text{Sets})$. In particular what is a map from sets to $\text{Fun}(G, \text{Sets})$. More generally you can take a small category $C$ and the topos $\text{Fun}(C, \text{Sets})$. What is a map of toposi $\text{Sets} \xleftarrow{f_*} \text{Fun}(C, \text{Sets})$? The important condition is that $f_*$ respect finite colim's.

**Example.** If $C$ and $D$

$$f: C \to D$$

is a functor, then it induces

$$\text{Fun}(C, \text{Sets}) \xleftarrow{f_*} \text{Fun}(D, \text{Sets})$$

Given $f: C \to D$ you have always

$$\text{Fun}(C, \text{Sets}) \xleftarrow{f_*} \text{Fun}(D, \text{Sets})$$

set $f_*F = Ff$
A category, *topos* \( \text{Fun}(C, \text{sets}) \) is a topos. Let \( u : C \to C' \) be a functor.

\[
\begin{align*}
\text{Fun}(C', \text{sets}) \xrightarrow{u^*} \text{Fun}(C, \text{sets}) & \\
\uparrow_{u_*} & \\
\text{Fun}(C', \text{sets}) & \xleftarrow{u_*^*} \text{Fun}(C, \text{sets})
\end{align*}
\]

\( (u^*F)(X) = F(uX) \)

\( (u_*F')(X) = \lim_{(X',uX' \to X)} F'(X') \)

\[
\forall \phi \in \text{Hom}(\lim_{X',uX' \to X} F'(X'), F(X)) \Rightarrow \exists \psi \in \text{Hom}(F'(X'), F(X))
\]

\( F \) consists of maps \( F'(X') \to F(X) \) \( \forall X', uX' \to X \)

Anyway, do this later. The idea is that the pair \((u^*, u_*)\) constitutes a map of toposi when \( u_* \) is exact, which should be equivalent to \( \forall X, \) the category of \((X', uX' \to X)\) is filtering.

Look at the case \( C' = \text{pt} \). So \( \exists X' \).

Start again. \( C \) small cat, \( \text{Fun}(C, \text{sets}) \) is a topos. Look at morphisms of toposi

\[
\begin{align*}
\text{sets} & \to \text{Fun}(C, \text{sets}) \\
f_* \text{sets} & \to \text{Fun}(C, \text{sets})
\end{align*}
\]

Geometrically this is a map from a point to a space, actually it is a pair of adjoint functors \((f^*, f_*)\) such that \( f_* \) resp. \( f^* \) functor. Example: \( X \) obj of \( C \) and you have \( f^*(F) = F(X) \)
\[ \text{sets} \leftrightarrow \text{Fun}(\mathcal{C}, \text{sets}) \]

\[ f^*(F) = F(X) \]

Given \( F \in \text{Fun}(\mathcal{C}, \text{sets}) = \hat{\mathcal{C}} \) and \( G \in \hat{\mathcal{D}} \), let

\[ (u^*G)(C) = G(u(C)) \]

Given \( F \xrightarrow{\Theta} u^*G = Gu \) for all \( C \in \mathcal{C} \). Then given \( D \xrightarrow{} u(C) \) we have \( F(C) \xrightarrow{} G(u(C)) \) for all \( C \in \mathcal{C} \).

Thus given \( u : C \rightarrow D \) there is the intermediate category consisting of \( (C, D, D \rightarrow u(C)) \) cofibred over \( C \), fibred over \( D \).

Left and right fibres, try to recall the notation \( u : C \rightarrow \mathcal{D} \).
Given \( u: C \rightarrow D \) and \( \mathcal{D} = \text{Set} \), then you have the categories whose objects are \((C, uC \rightarrow D)\) and \((C, D \rightarrow uC)\).

\[ u/D \quad \quad \quad \quad \quad \quad \quad D \setminus u \]

so what happens is you factor the functor.

\[ C \rightarrow (C, u, D) \rightarrow D \]

Over \( D \) you have the cofibred cat of \((C, uC \rightarrow D)\).

\[ C \rightarrow (C, D, uC \rightarrow D) \rightarrow D \]

\[ C \rightarrow (C, uC, id_{uC}) \]

\[ \text{C cat, get topos } \hat{\mathcal{C}} = \text{Fun}(C, \text{hito}) \]

\[ C \rightarrow D \text{ fun, get adjoint functors } \hat{D} \leftrightarrow \hat{\mathcal{C}} \]

get map of toposi \( \hat{C} \leftrightarrow \hat{D} \) with \( u^* = f^* \) \( u_* = f_* \)

When \( f \) respects finite limits, you also have a map of toposi \( \hat{D} \leftrightarrow \hat{\mathcal{C}} \) with \( v^* = f^* \) \( v_* = f_* \)

Important to remember that the functor to factors inverse image functor is the important one, and it determines the other.
Recall that a topos \( \mathcal{E} \) is a map from \( \mathcal{T} \) to \( \mathcal{T}' \) can be defined as a functor \( \mathcal{T} \xrightarrow{\ast} \mathcal{T}' \) respecting all limits.

The first condition implies (using \( \mathcal{T} \) as a set of generators, a site, for a topos) the existence of the adjoint functor \( \ast^\triangleright \)

\[ \text{Ex. } \mathcal{T} = \text{sets, } \text{then } \mathcal{T}' = \hat{\mathcal{C}} \]

as functors \( \text{pt} \xrightarrow{f} \hat{\mathcal{C}} \) given by the object \( f(\text{pt}) = X \). Special case of

\[ \mathcal{B} \xrightarrow{f} \hat{\mathcal{C}} \]

So an object \( X \) of \( \mathcal{C} \) gives \( \mathcal{h}_X : \text{pt} \rightarrow \hat{\mathcal{C}} \)

whence map of topos \( \hat{\mathcal{C}} \) \( \xleftarrow{\mathcal{h}_X} \mathcal{C} \). Do you find you get other "points" \( \mathcal{h}_X : \mathcal{C} \rightarrow \hat{\mathcal{C}} \) by taking filtered limit (other ind or pro object, depending on whether \( \mathcal{C} \) is cov or contrav).

\[ \mathcal{C} \xrightarrow{\mathcal{h}_X} \hat{\mathcal{C}} \]

\[ X \xrightarrow{\mathcal{h}_X} \mathcal{h}_X = (X \mapsto \text{Hom}_\mathcal{C}(X, Y)) \]

Thus you get \( \text{Pro}(\mathcal{C}) \xrightarrow{\mathcal{C}} \hat{\mathcal{C}} \)

This is fully faithful, as image consists the left exact fun (help for finite)

\[ \text{Pro}(\mathcal{C}) = \text{col of points in } \hat{\mathcal{C}} \]
To now should be the time to understand what is a $G$ torser over a space $B$.

It should be a map of topoi $\text{Ab}_B \leftarrow \text{Fun}(G, \text{sets})$.

For each point of $B$, you get a prorepresentable functor. But for a groupoid, prorepresentable = representable. Thus for a groupoid you want

$L^*_f \rightarrow \text{Fun}(G, \text{sets})$

$x \rightarrow (x' \mapsto h^x(x') = \text{hom}(x, x'))$

What do you learn? What is the want to understand torsors in an arbitrary topos $\mathcal{T}$?

By def'n, a $G$ torser in a topos $\mathcal{T}$ is a topos map $\mathcal{T} \leftarrow \text{Fun}(G, \text{sets})$.

What does this mean? For $\mathcal{T} = \text{sets}$.

Assume $G$ conn. $\text{Fun}(G, \text{sets}) = \{G\text{-sets}\}$

Canonical map $f^x(S) \leftarrow f^*G \times G S$ which is an iso.

Next ask when its left exact — must amount to $G$ acting freely on $f^*(G)$.

So what next?
Here seems to be the idea. You want to describe a topos map

\[ \text{Sh}_B \leftarrow \text{Fun}(\mathcal{G}_p, \text{Sets}) = \hat{\mathcal{G}_p} \]

\[ \uparrow \text{(Yoneda)} \]

\[ \mathcal{G}_p B \]

The composition is a functor from \( \mathcal{G}_p B \) to \( \text{Sh}_B \), so first of all you have an object in \( \text{Fun}(\mathcal{G}_p, \text{Sh}_B) \), which means a family of sheaves \( E^X \in \text{Sh}_B \) associated to each \( X \in \mathcal{G}_p \) and \( \text{sheaf maps} \ g : E^X \to E^{X'} \) in \( \text{Sh}_B \) associated to each map \( g : X \to X' \) in \( \mathcal{G}_p \), all this amounts to a \( \mathcal{G}_p \)-sheaf over \( B \), the generalization of a \( \mathcal{G}_p \)-set from sets to \( \text{Sh}_B \). So far you have described a \( \mathcal{G}_p \)-sheaf over \( B \), but next you want the action to be free, which means for each point of \( \mathcal{G}_p \) that that the \( \mathcal{G}_p \)-set is representable.

This should simplify. Again you want \( \mathcal{G}_p B \to \text{Sh}_B \), i.e., \( X \mapsto E^X \), \( (X \to X') \mapsto (E^X \to E^{X'}) \).

Now you want to recover your past understanding, motivated by Haene's stuff.
A groupoid has a nerve which is an set.

First example. \( G \) acting on \( F \).

\[
G \times G \times F \cong G \times F \Rightarrow F
\]

Let's try to construct the topos map. \( F \in \text{Fun}(\mathcal{G}, \text{Sets}) \) \( E \in \text{Fun}(\mathcal{G}^{\text{op}}, \text{Sets}) \). There will be some sort of

\[
\text{Cohom} \left\{ \quad \begin{array}{c}
E \times \mathcal{G} \times F \\
\xrightarrow{\theta} \quad E \\
\end{array} \right. \xrightarrow{\otimes} E \otimes F \right\} \cong E \otimes F
\]

\[
\bigoplus \ E_x \times \mathcal{A}_y \times F
\]

\[\begin{array}{c}
\bigoplus \\
x \leftrightarrow y
\end{array} \]

\[
\bigoplus \ E_x \times F_y \Rightarrow \bigoplus \ E_x \times F_x \Rightarrow E \otimes F
\]

\[\mathcal{G}^{\text{op}} \hookrightarrow \hat{\mathcal{G}} = \text{Fun}(\mathcal{G}, \text{Sets}) \xrightarrow{f^*} \text{Sh}_B\]

You get a functor \( \mathcal{G}^{\text{op}} \to \text{Sh}_B \) with a freeness property namely

\[
E \times \text{Hom}_B(X, X') \Rightarrow \text{Hom}_B(E, E') \text{ via } \phi
\]

\[E \times \mathcal{A} \to \]
A groupoid, \( \mathcal{G} \), is \( \text{Fun}(\mathcal{G}, \text{Set}) \).

Take \( \mathcal{G} \) = groupoid id_0 \rightrightarrows \text{id}_2 

i.e. \( \text{Hom}_\mathcal{G}(x, y) = \text{pt} \) for \( x, y \in \{1, 2\} \).

A functor \( \phi \) from \( \mathcal{G} \) to any cat \( \mathcal{C} \) consists of two objects \( \phi(1), \phi(2) \) and an isomorphism \( \phi(1) \sim \phi(2) \). An \( M_2 \) set consists of two sets and an isomorphism between them. There are two representable \( M_2 \) sets objects but they are ism, so the two rep functors are isom. There is one represent.

Look at \( 1/2 \) acting on \( 2/2 \) by translation. In general, look at \( G \) acting on a set \( S \). The answer is \( EG \times G S \).

A map from \( B \) to \( EG \times G S \) is essentially equivalent to a \( G \) torsor \( E \) over \( B \) together with an equivariant map \( E \to S \).

\[
\begin{array}{ccc}
EG & \to & EG \times G S \\
\downarrow & & \downarrow \\
BG & \to & EG \times G S
\end{array}
\]

Start again: you want disc \( G \) acting on \( F \) to give examples of \( \mathcal{G} \) etale groupoids.
A groupoid \((\Gamma, F)\) is top cat so has classifying space by Graeme's theory; geom. real of nerve.

\[
F \cong \Gamma \times F \cong \Gamma \times \Gamma \times F
\]

\[
\Gamma \cong \Gamma \times \Gamma
\]

yields then \(E\Gamma \times \Gamma F\) for the classifying space.

What does this classifying space classify?

\[
E \rightarrow E\Gamma \times \Gamma F \rightarrow F
\]

\[
\Gamma \text{-} \text{torsor over } B \quad \text{together with a } \Gamma \text{-} \text{map } E \rightarrow F.
\]

Problem: Assembly maps. If \(F\) is a point.

Go over it carefully - themes: Look at \(R \xrightarrow{m} R/Z\).

Recall the link with locally compact spaces.

What did you learn? \(R \xrightarrow{m} R/Z\), \(R\) is the total space of the principal bundle. You found that \(\Gamma(R/Z, \pi^* \Theta) = C_c(R)\). Ideas occurring:

\[
\begin{array}{ccc}
R \times Z & \rightarrow & R \\
\downarrow & & \downarrow \text{cent} \\
R & \rightarrow & R/Z
\end{array}
\]

\[
C_c(R) \times \mathbb{Z} \text{ operates on } C_c(R)
\]

You remember getting a finger frig module over the torus as well as over the cross product. mult. alg?
Given a principal $\Gamma$-bundle $E \to B$, in other words a locally trivial family of $\Gamma$-torsors parametrized by points of $B$. You want to move from this fibre bundle to a kind of vector bundle, why? Motivation from N.C. At this point maybe review N.C. Closed (available) manifold with fund. gp. $\pi_1$. Too bad, the NC affirms that the numbers obtained by pairing the $L$ classes with cohomology from $B\pi$ are linear invariants.

See if you can try to find a assembly in the case of the groupoid $(\Gamma, F)$ with classifying space $E\Gamma \times F$. First review the case $\Gamma = F = \text{pt}$. Given $\pi : \Gamma \to F$, then there is an assoc. fibre bundle with fibre $\Gamma \times F$ considered as free $\Gamma \times F$ module with one generator. So over $B$ you have a locally trivial fibre bundle with fibre $\Gamma$ the $\Gamma$ module $\Gamma \times F$. Now you wish to apply the same thin argument that this fibre bundle (when $B$ has a finite partition of it over which the bundle is trivial) of free $\Gamma$ module is a retract of a locally trivial free $\Gamma$ module bundle.

A partition of unity involves cent. fun. on $B$ ask first what a retract of $B \times \Gamma \to B$ should look like. It should be given by a
an idempotent operator on the trivial $\Gamma$-module bundle $\mathcal{B} \times \mathcal{C} \rightarrow \mathcal{B}$, that is, an idempotent section $\rho \in C(\mathcal{B}, M_n(\mathcal{C} \Gamma))$. Serre then gives an argument: you start with a geometric situation, namely, a principal $\Gamma$-bundle $\pi: \mathcal{E} \rightarrow \mathcal{B}$. You form the associated fibre bundle $L$ with fibre the $\Gamma$-module $\mathcal{C} \Gamma$. There's a problem with topology on $\mathcal{C} \Gamma$ -- maybe this is where the fine topology enters.

What exactly do you have locally on $\mathcal{B}$? You have isomorphisms $L \xrightarrow{\sim} \mathcal{B} \times \mathcal{C} \Gamma$ which are related by left multiplication via a group element. So any topology on $\mathcal{C} \Gamma$ preserved by left multiplication should be okay. Next you want continuous sections of $\mathcal{E}$, really a suitable space of sections of $L$ over $\mathcal{B}$ such that $C(\mathcal{E})$ is a $C(\mathcal{B})$ module. It seems that algebraically there is only one candidate, relative to a trivial $L \xrightarrow{\sim} \mathcal{B} \times \mathcal{C} \Gamma$ and a section $s(\mathcal{b}) = \sum \gamma f(\gamma) \mathcal{C}$ sum finite over compact subsets, coeffs are cont. fun. on $\mathcal{B}$.

Repeat: you begin with principal $\Gamma$-bundle $\pi: \mathcal{E} \rightarrow \mathcal{B}$ with $\mathcal{B}$ compact. Look at $C_c(\mathcal{E})$. This should be a module over $C_c(\mathcal{B}) \otimes \mathcal{C} \Gamma$. 
So next look at $\Gamma$ operating on $F$.

A torsor for $(\Gamma, F)$ should be a $\Gamma$-torsor $E/B$ together with $\Gamma$-map $E \to F$.

\[
\begin{array}{ccc}
E & \rightarrow & E\Gamma \times F \\
\downarrow & & \downarrow \\
B & \rightarrow & E\Gamma \times F
\end{array}
\]

and

\[
\begin{array}{ccc}
E & \phi & \rightarrow & F \\
\downarrow & & & \downarrow \\
B & & & 
\end{array}
\]

Your idea is to involve $C(F)$ or $C_0(F)$.
Really you should work only w.

The obvious idea is that cross product of $\Gamma$ and $C_\infty(F)$ should be relevant, should assemble should yield

In the case you get a bimodule $\text{ for } C(B)$ and $C_\Gamma$. You actually construct a retract of the trivial bundle with base $B$ and fibre $\text{ End}_B \text{ of the } \Gamma^\text{op} \text{ module } C_\Gamma$.

Next we generalize to include $F$.
Next to include $F$. Given
\[ E \xrightarrow{\phi} F \]
\[ \phi(\cdot \phi) = \phi' \phi(\cdot) \]
You expect to have $B$ a bimodule on the right. Take a $b \in B$, get
\[ \Gamma \times E \xrightarrow{\cdot b} F \]
\[ \Gamma \text{ disc acting on space } F, \text{ get groupoid whose} \]
torsors over a space $B$ are $\Gamma$-torsors $E \rightarrow B$ to $\Gamma$-
\[ \text{a } \Gamma \text{-map } E \xrightarrow{\phi} F \]. What you absolutely need
What you should look at maybe is fibres of the
\[ E \times F \]
\[ E \times \Gamma F \]
\[ E \times F \]
\[ E \times F \]
Start again: You are given space $B$, a $\Gamma$-torsor
\[ E/B \]
and a $\Gamma$-map $\phi : E \rightarrow F$. You can form $E \times F$ the
\[ \text{assoc. fibre bundle with} \]
fibre $F$.
You don't seem to get somewhere. Go back
to the groupoid $G = M_2$ two objects objects $\{ 1, 2 \} = \mathcal{O}$
map $E \times \mathcal{O}$. You have
\[ \text{topos.} \]
Let's go over again what you did for the groupoid $\mathcal{M}_2$. You started with Grothendieck topos viewpoint. If $G$ is a groupoid then $\text{Fun}(\mathcal{G}, \text{sets})$ (\(\mathcal{G}\text{-sets}\)) is a topos and a topos map
\[
\text{Sh}_B \xleftarrow{f^*} \text{Fun}(\mathcal{G}, \text{sets})
\]
should be equivalent to a \(G\)-torsor in \(\text{Sh}_B\). 

Picture to keep in mind: \((G\text{-sets}) = \text{Fun}(\mathcal{G}, \text{sets})\)

A representable functor
\[
\text{Sets} \xleftarrow{f^*} \text{G-sets}
\]
should have the form $f^*(G) \times G S \xleftarrow{} S$, but $f^*(G)$ is a $G$-torsor iff $G$ action has one orbit (trivial isot. gps). So a topos map \(\text{Sets} \xleftarrow{f^*} \text{G-sets} \Rightarrow f^*(G)\) is a $G$-torsor.

Another ingredient is that you have the Yoneda embedding
\[
\mathcal{C} \to \text{Fun}(\mathcal{C}, \text{sets}) = \hat{\mathcal{C}}
\]
\[
x \mapsto h^x = (y \mapsto \text{Hom}_\mathcal{C}(x, y))
\]
The points in $\hat{\mathcal{G}}$ should be the representable functors. Yes.

Let's work out exactly how a $G$-torsor looks.

You want also to link with Graeme classifying space.
\[ \mathbb{B} \leftarrow f^* \quad \text{Fun}(\mathbb{B}, \text{sets}) \]

A topos map as above should be roughly the same as assigning to each point of \( \mathbb{B} \) a point in \( \text{Fun}(\mathbb{B}, \text{sets}) \), i.e., an object of \( \text{sets} \).

You will need some examples. This is hard but not

Try to guess what the structure should be.

First do \( \text{sets} \leftarrow f^* \quad \text{Fun}(\mathbb{A}, \text{sets}) \).

Let \( f^* \) be a right adjoint from \( \mathbb{B} \) to \( \text{sets} \).

Let \( \mathcal{C} \) be a small category, let you have basic pairing
\[
\mathcal{C}^\text{op} \times \mathcal{C} \rightarrow \text{sets} \quad (X, Y) \mapsto \text{Hom}(X, Y)
\]

This should be an analog of dual.

Let \( \hat{\mathcal{C}} = \text{Fun}(\mathcal{C}, \text{sets}) \), and let
\[
\mathcal{G} : \hat{\mathcal{C}} \rightarrow \text{sets}, \quad \text{be a functor.} \quad \text{You have}
\]
\[
\begin{array}{cccc}
\mathcal{C}^\text{op} & \xrightarrow{h} & \hat{\mathcal{C}} & \xrightarrow{\mathcal{G}} & \text{sets} \\
X & \mapsto & (h_X : Y \mapsto \text{Hom}(X, Y)) & \mapsto & \mathcal{G}(h_X)
\end{array}
\]

This gives a \( \mathcal{C}^\text{op} \text{-set.} \) Need now a tensor product operation between the \( \mathcal{C}^\text{op} \text{-set} \mathcal{G}(h) \) and \( \mathcal{C} \text{-set} \quad \_ F \quad \mathcal{G}(h_X), \quad F(Y) \)
Start again. In a small category, you have left and right $C$-sets. If there a kind of tensor product? $R \otimes_L R$.

It should be constructed from $R(X) \times L(Y)$, disjoint union modulo equiv. relation.

Obvious guess is $R(X) \times L(X) \rightarrow R(X) \times L(Y)$.

What is the basic idea? Any $R$ in $\mathbb{C}ob$ is a colim of representable functors.

$\prod_{X \rightarrow Y \rightarrow Z} R(Y) \times L(X)$

$\prod_{X \rightarrow Y} R(Y) \times L(X)$

$\prod_{X \rightarrow Y} R(X) \times L(X)$

$\prod_{X \rightarrow Y} R(Y) \times L(X)$

$\prod_{X \rightarrow Y} h^X \rightarrow \prod_{X \rightarrow Y} h^X \rightarrow R$

$\text{Hom}_{\mathbb{C}ob} \left( R, T \right) = \frac{\prod_{X \rightarrow Y \in R(Y)} (X, \eta \in R(X))}{\prod_{X \rightarrow Y}}$

To give $\phi : R \rightarrow T$ you give $\forall X, \phi_X : R(X) \rightarrow T(X)$ such that $\forall X \rightarrow Y$ you have the commutes $\phi^\ast \eta = u^\ast \phi_Y \eta$.

\[
\begin{align*}
\text{for } \eta \in R(Y) \quad u^\ast \phi_Y \eta &= \phi^\ast \eta
\end{align*}
\]
Start again. $R: C^{op} \rightarrow \text{sets}$, $L: C \rightarrow \text{sets}$.

Presentation of $R$ by $h$?

$$
\begin{array}{c}
\text{?} \\
\xymatrix{\text{?} \ar[r]^-{h^X} & \ar@{-->}[r]^X & R \\
\ar[r]_-{X, \xi} & \ar@{-->}[r]^X & R(X) \times h^X \\
x & \ar[r]_-{\text{?}} & \ar[r]^-{\text{?}} & R }
\end{array}
$$

$R: C^{op} \rightarrow \text{sets}$. You want presentation of $C$ via representable functors $h_x(-) = \text{Hom}(-, X)$.

You could form $C/\text{R}$ whose ogy are $(X, \xi)$ with $X$ in $C$ and $\xi \in R(X)$, whose maps are $(X, \xi) \rightarrow (Y, \eta)$ are $\alpha: X \rightarrow Y$ s.t. $\xi = \alpha^* \eta$.

You're forgotten so much.

$C \rightarrow \text{Fun}(C^{op}, \text{sets}) \leftarrow$.

$X \rightarrow h_x$. So given $R \in$ you can talk about $C/\text{R}$ i.e. $\text{Obj}(C/\text{R}) = \text{pairs } (X, \xi) \in X \times \text{Ob} C \Rightarrow \xi \in R(X)$.

$$
\begin{array}{c}
\xymatrix{\text{?} \ar[r]^-{\text{?}} & \ar@{-->}[r]^X & \ar@{-->}[r]^X & R(Y) \\
\ar[r]_-{X} & \ar[r]_-{\text{?}} & \ar[r]^-{\text{?}} & \text{?} \\
x \ar[r]_-{\text{?}} & \ar[r]^-{\text{?}} & \ar[r]^-{\text{?}} & \text{?} \times R(X) }
\end{array}
$$
R : C^o \rightarrow \text{sets}. Claim that

R is the ind. limit of the functors from C/R to C^o sending (X, \xi) to \( h_x \)
equipped with the map \( h_x \rightarrow R \) correspond to \( \xi \).

\[
\lim_{\leftarrow} \left( (X, \xi) \mapsto h_x \right) \quad \xrightarrow{\sim} \quad R
\]

\[
\text{Hom}(\cdot, T) = \lim_{\leftarrow} \left( (X, \xi) \mapsto T(X) \right)
\]

\[
\text{Hom}(R, T)
\]
given \( \phi : R \rightarrow T \) and \( X, \xi \in R(X) \)

\( \xi \in R(X) \rightarrow T(X) \)

An element of \( \lim_{\leftarrow} \left( (X, \xi) \mapsto T(X) \right) \)
is \( \forall X, \xi \in \text{Ob} \ C, \xi \in R(X) \) an element \( \alpha(X, \xi) \in T(X) \)
such that \( \forall \alpha \in \text{Hom}(Y, X) \) one has

\( u^* \alpha(X, \xi) = \alpha(Y, u^* \xi) \)

Define \( \alpha_X : R(X) \rightarrow T(X) \) by \( \alpha_x(\xi) = \alpha(X, \xi) \)

Then given \( u : Y \rightarrow X \)

\( \alpha_y : R(Y) \rightarrow T(Y) \)

\( u^* \alpha_y(\xi) = \alpha_y u^* \xi \)
Let now \( L \in \hat{C} \).

Recall what you want to do. You are trying to find the analogue of the fact that a right continuous functor \( \text{Mod}(R) \xrightarrow{F} \text{Ab} \) is given by \( F(M) = F(R) \otimes_R M \).

You want to describe all \( \text{rep.} \) funs. \( F: \text{Fun}(C, \text{sets}) \rightarrow \text{sets} \). Discuss examples. \( C = \text{group} G \), \( F: G\text{-sets} \rightarrow \text{sets} \).

\[
F(\theta) \times^G S \rightarrow F(S)
\]
defined \( (\tilde{\xi}, \lambda) = F(g \rightarrow g\theta)(\tilde{\xi}) \)

Here use \( S = \text{Hom}_G(G, S) \),

\( C \) small cat. Construct tensor product, a set \( R \times^C L \) where \( R \in C^{\text{op}}\text{-set} \), \( L \in C \text{ set} \).

Idea is any \( R, L \) can be expressed as end lim of rep. funs.

\[
R = \varinjlim \frac{h_x}{C/R} \quad L = \varinjlim \frac{h_Y}{C^{\text{op}}/L}
\]

\[
R \times^C L = \left( \varinjlim \frac{h_x}{C/R} \right) \times \left( \varinjlim \frac{h_Y}{C^{\text{op}}/L} \right) = \varinjlim \text{Hom}_C(Y, X) \quad (xy) \in \left( \frac{C/R}{C^{\text{op}}/L} \right)
\]
\[ R \times \mathcal{L} = \lim_{(x,y) \in \mathcal{C} \times \mathcal{C}^\text{op} / (R \times \mathcal{L})} \hom_{\mathcal{C}}(y, x) \]

\[ \lim_{(x, \xi) \in R(x)} \{ (x, \xi) \mapsto h_x \} = R \]

\[ \lim_{X} \]

Do it as follows. \(
\mathcal{C}/R \) cat with \( \text{Ob} = (X, \xi) \)
\( X \in \text{ObC}, \ \xi \in R(X) \) equiv \( \xi : h_x \rightarrow R \) in \( \mathcal{C}^\text{op}-\text{cats} \). Claim
\( \forall \mathcal{Z} \in \text{ObC} \)
\[ \lim_{(X, \xi) \in \mathcal{C}/R} \hom_{\mathcal{C}}(\mathcal{Z}, X) \xrightarrow{u} R(\mathcal{Z}) \]

\[ u^* \eta \]

\[ X, \xi \xrightarrow{\eta} Y \]

\[ H_{\mathcal{C}}(\mathcal{Z}, X) \xrightarrow{V} \hom_{\mathcal{C}}(\mathcal{Z}^x, Y) \]

\[ R(2) \xrightarrow{u^* \eta} \]

\[ u \mapsto v \eta = v \xi \mapsto (v \xi)^* \eta = u^* v^* \xi \eta = u^* \xi \]
Start again, C small cat. have

\[ C^{op}\text{-sets} \cong \text{Fun}(C^{op}, \text{sets}) \cong h^\sim C \]

\[ C\text{-sets} \cong \text{Fun}(C, \text{sets}) \cong h^\sim C^{op} \]

Let \( R \) in \( C^{op} \), \( L \) in \( C \)

A basic idea is that the representable functors are generators. Thus \( \text{Fun}(C^{op}, \text{sets}) \) has gen \( h^Y = \text{Hom}(\_, Y) \) for \( Y \in \text{Ob} C \). Similarly \( C \) has gen \( h^Y = \text{Hom}(Y, \_) \) as generators.

To define \( R \times^C L \). This is a set defined by universal "bilinearity property".

\[ \phi \in \text{Hom}_{\text{sets}}(R \times^C L, T) \]

should be a family of maps \( \phi_X : R(X) \times L(X) \to T \) for \( X \in \text{Ob} C \)

such that for \( f : X \to Y \) one has

\[ f \times 1_L : R(X) \times L(X) \to R(Y) \times L(Y) \]  
\[ \phi_X \]

\[ = (f \circ \_Y)^* : R(Y) \times L(Y) \to T \]

so \( T \) should be the quotient of \( \coprod_X R(X) \times L(X) \) gen. by the relns.

\[ (f \circ \_Y)^* \cdot \text{Id}_X = (f^*_Y, f^*_X) \]

What does this mean? Suppose

\[ R = h_A \quad L = h_B \]
\[
\text{Let } \phi : R \to \text{Hom}_\mathcal{C}(L, T) \text{ be a map in } \mathcal{C}.
\]

\[
\phi_x : R(X) \to \text{Hom}(L(X), T) \quad \text{contrav. in } X.
\]

\[
\phi_y : R(Y) \to \text{Hom}(L(Y), T)
\]

\[
\begin{array}{c}
R(X) \times L(X) \\
\downarrow \phi_x
\end{array}
\begin{array}{c}
\begin{array}{c}
\exists (f^* \times 1) \\
\phi_y (1 \times f^*) \\
\downarrow \phi_y
\end{array}
\end{array}
\begin{array}{c}
T
\end{array}
\]

So this seems to work. Thus must get

\[
\rightarrow \times \mathcal{L}
\]

\[
R \times \mathcal{E} \rightarrow \mathbb{B}
\]

\[
\text{Hom}(h_A \times \mathcal{L}, T) = \text{Hom}(h_A, \text{Hom}_\mathcal{C}(L, T))
\]

\[
= \triangleright \text{Hom}_\mathcal{C}(L(A), T)
\]

So progress is being made. But what does it all mean? What would be Greem's viewpoint?

Go back to groupoid \( M_2 \)

Go back over the ideas
Summarize events. From Cuntz you learned that for \( \Gamma = M_n \), there is an analogue of the universal algebra by the components of a projection in a \( \Gamma \)-graded algebra, and \( \mathbf{A} \) is a non-commutative \( n \)-simplex. You tried to study the question for a general \( \Gamma \) (i.e. \( \Gamma \in \mathbb{Z} \)) has assoc. mult. with 0 absorbing but it seems you want \( \Gamma \) to be a groupoid. Also you know assembly exists for groupoids.

Then arises the question of the classifying space for a groupoid \( G \). Your idea: to use Grothendieck topos picture (at least when \( G \) is étale e.g. a space with discrete \( \mathbb{G}_m \) acting.

You've now understood Grothendieck classifying for at least a discrete groupoid \( G \), but there's nothing simplicial about it.

Take \( G = M_2 \), \( \text{Ob} \) has 2 elts, \( A_r = \text{Ob} \times \text{Ob} \) with source + target given by projections. What does a \( G \)-torsor look like over a space \( B \)? A \( G \)-torsor is equivalent to a topos map

\[
\mathcal{S}h_B \xrightarrow{f^*} \hat{G} = \text{Fun}(G, \text{sets})
\]

\[\xrightarrow{\bigcup} \text{Yoneda embedding of } \hat{G} \]

So your \( G \)-torsor should amount to a \( \text{centra fun from } G \) to \( \mathcal{S}h_B \). \( \forall \psi \in G \) get
How should you picture a torsor for the discrete groupoid $G$ over the space $B$?

It should be a $G\text{-fop}$-sheaf, i.e., a functor from $G\text{-fop}$ into $\text{Sh}_B$. So this means that you will have sheaves $E_x$ over $B$, where $x$ runs over $\text{Ob} G$, and for each $x \in G$ you are given $g^*: E_y \to E_x$.

**Torsor**

You are still stuck on the assembly stuff for a groupoid. Maybe because you are not paying enough attention to localization. Take a discrete groupoid $G$ and define in sheaf terms what a $G$-torsor over a space $B$ is. The answer should be it is a functor $G_{\text{fop}} \to \text{Sh}_B$ which is locally representable in a suitable suitable sense.

Suppose $B = \text{point}$. What is a functor $G_{\text{fop}} \to \text{sets}$?

\[ \forall x \in \text{Ob} \quad \underset{x}{{\text{E}_x}} \leftrightarrow \underset{x \to y}{{\text{E}_y}} \leftrightarrow \underset{x \to y \to z}{{\text{E}_z}} \]

\[ \forall x \in \text{Ob} \quad \underset{x}{{\text{Ob}}} \leftrightarrow \underset{x \to y}{{\text{Ar}}} \leftrightarrow \underset{x \to y \to z}{{\text{Ar} \times \text{Ar}}} \]
The problem is to understand assembly for an etale groupoid. First case is for a discrete groupoid $\mathcal{G}$. It consists of a set of objects and a set of arrows, denoted by $\mathcal{G}_0$, $\mathcal{G}_1$ say. Together with arrows

$$\mathcal{G}_0 \xleftarrow{s} \mathcal{G}_1 \xrightarrow{t} \mathcal{G}_1 \sqcap (s,p) \mathcal{G}_1$$

Maybe you should straighten this out so that you can deal with left and right $\mathcal{G}$-sets.

left $\mathcal{G}$-set is $\mathcal{G}_1 \times (s,p) \mathcal{G}_0 \xrightarrow{(s,p)} \mathcal{G}_1 \times \mathcal{G}_0$ is a set $F$ over $\mathcal{G}_0$.

which is associative, and any cat leads to a arrow ring.

Again: Begin with if a discrete groupoid $(\mathcal{G}_0, \mathcal{G}_1, s, t, \text{id}, \circ, \text{inv})$, Category $\text{Fun}(\mathcal{G}, \text{sets})$ of $\mathcal{G}$-sets.

Problem: Describe a torsor over a space $\mathcal{B}$. Possible approaches:

- topos map $\mathcal{B} \xrightarrow{\mathcal{T}} \{\mathcal{G}\text{-sets}\}$.
- If $\mathcal{B}$ a point, then you get a point in $\{\mathcal{G}\text{-sets}\}$ which should be the same as an object of $\mathcal{G}$. The category of points in the topos $\{\mathcal{G}\text{-sets}\}$, namely $\text{Sh}(\mathcal{G}, \text{sets})$ the category of functors $\text{sets} \xleftarrow{-} \mathcal{G}\text{-sets}$ which are right col. and left exact should be the full subcat of representable functors: $\mathcal{G}^\text{op} \xrightarrow{\mathcal{B}} \mathcal{G}$.
You have reached the viewpoint that a $G$-torsor over a space $B$ is some sort of map from $B$ to $\hat{G}^\text{op}$, or from $B$ to the category of representable funs in $\hat{G}$. Let's explore this idea, try to find a precise version. Take $\mathcal{G}$ to be the groupoid given by a group $G$, the category with one object whose self maps are all of $G$.

If $\mathcal{G} = G$, then $\hat{G} = G$-sets, and the Yoneda embedding $G^\text{op} \hookrightarrow \hat{G}$ sends $X$ to the $G$-set given by $G$ operating on itself by left mul.

The right idea of a $G$-torsor over $B$ is a family of $G$-sets.

At some spot here it should become clear that to regard a $G$-torsor as a "map" from $B$ to $\hat{G}^\text{op}$, i.e., a $G$-torsor as a family param. by $b \in B$ of objects of $\hat{G}^\text{op}$, is not going to work.

Maybe shift to a covering viewpoint or give an open covering of $B$, say $B = U \cup V$, over $U$ you give an object of $\hat{G}$ over again $B$ top space $\mathcal{G}$ discrete groupoid you must define a $G$-torsor over $B$. Idea: Such a torsor should provide a topos map

$$f^* \colon \hat{G} \rightarrow \text{Fun}(\mathcal{G}, \text{sets})$$

$f^*$ is right continuous and left exact. Right cont should imply that $f^*$ is given by $f^* G$.
twisting w.r.t a $\hat{A}^\text{op}$-sheaf over $B$, means a functor $\hat{A}^\text{op} \to \text{Sh}_B$. The way this
functor should arise is via the Yoneda embedding

$$\hat{A}^\text{op} \to \hat{A} = \text{Fun}(\hat{A}, \text{sets})$$

$$x \mapsto h^x(y) = \text{Hom}_y(x, y)$$

This is clear because $\hat{A}^\text{op}$ is "dense" in $\hat{A}$.

So your tensor should be

$$\hat{A}^\text{op} \xleftarrow{h} \hat{A} \xrightarrow{f^*} \text{Sh}_B$$

$$\lim \ x \mapsto L$$

out of $(x, \frac{h^x}{h^y}(L))$ since $\frac{h^x}{h^y}(L) 

\Rightarrow f^*(L) \leftarrow \lim_{x, y; h^x = L} f^*(h^x) = (f^* h)^\wedge L$$

Review: Problem: Understand $\hat{A}$ tensor over a space $B$, where $\hat{A}$ is a discrete groupoid.

First def. A topos map: $\hat{A} \xleftarrow{f^*} \hat{B} \xrightarrow{\text{Fun}(\hat{A}, \text{sets})}$

means $f^*$ exact and left exact.

Examine $\hat{A}$. Yoneda $\hat{A}^\text{op} \xleftarrow{h} \hat{A}, x \mapsto h^x$.

Given $f^*: \hat{A} \to \text{Sh}_B$, get $f^* h: \hat{A}^\text{op} \to \text{Sh}_B$.

whence a map from $\hat{A} \to \text{Sh}_B$, $x \mapsto f^* h^x \times \text{Sh}_B$.

and a map of maps $f^* h \times \text{Sh}_B \to f^* \text{Sh}_B$.

In fact

$$h^x \times \text{Sh}_B \to \text{L}(x)$$

$$\text{Hom}_{\text{Sh}_B}(h^x \times \text{Sh}_B, L) = \text{Hom}_{\text{sets}}(\text{L}(x), S) =$$

$$\text{Hom}_{\hat{A}^\text{op}}(h^x, \text{Hom}(L, S)) = \text{Hom}_{\text{sets}}(\text{L}(x), S).$$
Still not clear. You first have to study $R \times^g L$ for $R$ a $G$-set, $L$ a $G$-set.

$$R \times^g L = \text{Coker} \left\{ \frac{R(Y) \times L(X)}{X \sim Y} \right\}$$

$\text{Hom}_{\text{set}}(R \times^g L, S) = \text{Hom}_{G^{\text{op}}}((R, \text{Hom}(L, S)))$

$$= \text{Hom}_{\text{set}}(L, \text{Hom}_{G^{\text{op}}}(R, S))$$

$f^*: \tilde{G} \to \mathcal{S}_B$ given, compose with $G^{\text{op}} \xrightarrow{h} \tilde{G}$

$f^* h : x \mapsto f^*(h^x)$ get from $L^{G^{\text{op}}} \to \mathcal{S}_B$

(generalization of $(L^{G^{\text{op}}})^*$)

$f^* h \times^g L$ twisting of $G$-sheaf $f^* h$ by $G$-set $L$

This should be a $\text{Steat}$ functor of $L$. Should there be a canonical map:

$$f^* h \times^g L \to f^* L$$

suffices to consider $L = h^x$ by using

$$L = \lim_{(X, \xi \in L(X))} h^x$$

In effect

$$\lim_{(X, \xi)} f^*(h^x) \xrightarrow{\text{can}} f^*(L)$$

and

$$\lim_{(X, \xi)} f^* h \times^g X = (f^* h)(X) \xrightarrow{\text{can}} f^*(h^x) \xrightarrow{\text{can}} f^*(L)$$
Repeat: Problem is to understand a $\mathcal{G}$-torsor over a space $B$, starting from the Grothendieck defn of it as a topos map $\mathcal{G} \rightarrow \mathcal{B}$, where topos map means $f^*$ exact and left exact, means resp. finite lim's (suffices to respect ~$\sim$~, kernels).

Now you have seen that $f^*$ exact.

Have Yoneda $h: \mathcal{G} \rightarrow \mathcal{B}$, $X \mapsto h_X(y) = \text{Hom}(y,X)$, so get $\mathcal{G} \rightarrow \mathcal{B}$.

So the functors $(f^* h) \times \mathcal{G}$ (gen. of $\mathcal{G}$-set), namely $f^* h$.

So the functor $\mathcal{G} \rightarrow \mathcal{B}$ is defined, and is a functor $\mathcal{G} \rightarrow \mathcal{B}$.

You have constructed a canonical map of functors $\mathcal{G}$.

Next remains the meaning of $f^*$ left exact.

You have this $\mathcal{G}$-sheaf $\mathcal{E}$ in $\mathcal{B}$ in other words, $\forall X$ a sheaf $\mathcal{E}_X$ and $\forall x \rightarrow y$ a map $\mathcal{E}_x \rightarrow \mathcal{E}_y$, $\xi \mapsto \xi f$ such that $(\xi f)_y = \xi (f_y)$ etc. But left exact.

Now use what you know about groupoids.

$G$ splits into a disj union of amn. groupoids.

This analysis should suffice over any point of $B$. So $f^* L = (f^* h) \times \mathcal{G}$ is left exact to understand when can support $B = \text{pt}$, so that $f^* h$ is a $\mathcal{G}$-set.
The problem is to understand when a Hopf set \( R \) has the property that \( L \mapsto R \times ^{\text{op}} L \) is left exact.

Example 1: \( R = h^X \), then \( h^X \times ^{\text{op}} L = L(X) \) respects arb. limits.

Example 2: \( R = h^X \sqcup h^Y \). Then get

\[ R \times ^{\text{op}} L = L(X) \sqcup L(Y) \]

This is not compatible with products.

\[
\begin{align*}
(L \times L)(X) & \sqcup (L \times L)(Y) \\
(L(X) \sqcup L(Y)) \times (L(X) \sqcup L(Y))
\end{align*}
\]

\[
(R_1 \sqcup R_2) \times ^{\text{op}} (L \times L)
\]

\[ R_1 \times ^{\text{op}} (L \times L) \sqcup R_2 \times ^{\text{op}} (L \times L) \]

\[ R \times ^{\text{op}} (L \times L) \rightarrow (R \times ^{\text{op}} L) \times (R \times ^{\text{op}} L) \]

Getting too hard.

You should first understand a Hopf set \( R \) is a point i.e. \( L \mapsto R \times ^{\text{op}} L \) is left exact. This happens if \( R \) is representable, i.e. of the form \( h_Y \) for some object \( Y \).
Let's take up the next step. What does a $G$-torsor over $B$ really look like?

So it should be a $G^\text{op}$-sheaf, i.e. contrav. functor from $\mathcal{G}$ to $\mathsf{Sh}_B$ such that each stalk is a representable contrav. functor. You need now to come to grips with open sets.

Look at group case $G = G$. You have over $B$ a sheaf of sets, an étale space with free $G^\text{op}$-action.

Sheaf properties yield local triviality: Pick a point $\xi$ of $E$ over $b \in B$. By étaleness there is a local section $s$ of $E$ over a nbhd $U$ with $s(b) = \xi$ now move this around by $G$.

How do I handle a groupoid $\mathcal{G}$?

Cent. with $G$, $\xymatrix{E \ar[d]_{\pi} \ar[r]^{\pi} & B}$ local homeom.

means $\forall \xi \in E$, $E$ open nbhd $U$ of $\xi$ and an open nbhd $V$ of $\pi(\xi)$, such $\pi$ restricted to $U$ is a homeom. of $U$ with $V$.

Alt. $\forall \xi \in E$ there is an open $V$ containing $\pi(\xi)$ and a continuous $s : V \to E$ $\pi s = \text{id}_V$ and $s(V)$ is open in $E$. 

$$
\begin{array}{ccc}
E & \xymatrix{\ar[r]^{E} & } \\
\downarrow_{\xi} & \downarrow_{\pi} \\
B & \xymatrix{\ar[r]^{\text{open}} & } \\
\end{array} \\
\begin{array}{ccc}
U & \xymatrix{\ar[r]^{\text{cont.}} & E} \\
\downarrow_{\xi} & \downarrow_{\pi} \\
V & \xymatrix{\ar[r] & B} \\
\end{array}
$$
At this point you have a functor 
\[ \mathcal{G} \text{op} \to \mathcal{B}_\mathcal{G} \]. First understand \( \mathcal{B} = \text{pt} \).

\( R : \mathcal{G} \text{op} \to \text{sets} \) contravariant functor call it \( R \)

\( \mathcal{G}_0 = \text{object set} \quad \mathcal{G}_1 = \text{arrow set} \)

\[ \begin{array}{ccc}
R_0 \downarrow & & R_1 \\
\mathcal{G}_0 & \equiv & \mathcal{G}_1
\end{array} \]

\[ \forall X \in \mathcal{G}_0 \quad \text{get} \quad R(X) \]

\[ \forall f : Y \to X \quad \text{get} \quad f^* : R(X) \leftarrow R(Y) \]

\[ \forall X \in \mathcal{G}_0 \quad \text{have set} \quad R(X) \]

\[ \forall x \leftarrow y \in \mathcal{G}_1 \quad \text{have map} \quad R(X) \xrightarrow{f^*} R(Y) \]

\[ \forall x \leftarrow y \leftarrow z \quad \text{have} \quad R(X) \xrightarrow{f^*} R(Y) \xrightarrow{g^*} R(Z) \]

\[ (fg)^* = g^* f^* \]

\[ \begin{array}{ccc}
\prod_{X \in \mathcal{G}_0} R(X) & \leftarrow & \prod_{X \leftarrow Y \in \mathcal{G}_1} R(X) \\
\prod_{X \leftarrow Y \leftarrow Z} R(X) & \leftarrow & \prod_{X \leftarrow Y \leftarrow Z} R(X) \times R(Y) \times R(Z)
\end{array} \]
So you learned the following, that a functor \( C^{\text{op}} \to \text{Sets} \) yields a nerve

\[
\coprod_{x_0} R(x_0) \subseteq \coprod_{x_0 \to x_1} R(x_0) \subseteq \coprod_{x_0 \to x_1 \to x_2} R(x_0)
\]

You have written down the nerve for the category \( \mathbb{C} \) whose objects are \( (x, \xi \in R(x)) \) and whose maps \( (x, \eta) \leftrightarrow (x, \zeta) \) are \( \eta \leftarrow f \to x \), such that \( f^* \eta = \zeta \), seems to get cat \( C/R \)

i.e. objects are \( (x, h_x \xrightarrow{\xi} R) \)

\[
\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
(y, h_y) & \text{mapps} & (x, h_x) \\
\downarrow & \downarrow & \downarrow \\
y & \text{fibre cat over } C \text{ given by } R & R(\xi) \\
\end{array}
\]

\( R(\eta) \)

\( X \xrightarrow{f} Y \)

\( \text{fibre cat over } C \text{ given by } R \)

\( R(\xi) \to R(\eta) \)

\( R(Y) \)

\( \xrightarrow{\xi} \)

\( \text{gives analog of } \)

\( \text{a } \text{g}{\text{op}}\text{-set but in the topos } \text{sh}_B \)

Look at case at each point of \( B \) you get a representable from \( R \) him to \( h_x \) for some \( \text{Ob } X \).
You have a contravariant functor from $\mathcal{G}$ to $\mathbf{B}$, to sets, when $\mathbf{B}$ is replaced by a point.

$\forall X$ there is a set $R(X)$

$\forall \mathbf{A}$, $\mathbf{B}$ have maps $\mathbf{A} \to R(\mathbf{B})$

$\exists Z$, such that $\text{Hom}(X, Z) \to R(X)$

how to keep direction of arrows straight. Think of $R \times \bullet L = \text{Coker} \left\{ \bigsqcup_{X} R(X) \times L(X) \to \bigsqcup_{X, Y} \text{Hom}(X, Y) \times L(Y) \right\}$

$$
\begin{align*}
R(Y) \times L(Y) & \to R(X) \times L(X) \\
\downarrow & \\
R(Y) \times L(Y) & \to R \times \bullet L
\end{align*}
$$

$$(u^*, \lambda) = (\rho, u \cdot \lambda)$$

This shouldn't be important

$$
\frac{\bigsqcup_{X} R(X) \times L(X)}{X} \quad \frac{\bigsqcup_{X, Y} R(X) \times \text{Hom}(X, Y) \times L(Y)}{X, Y}
$$

How do you get the correct notation

$$R \times L \subseteq R \times G \times L \subseteq R \times G \times G \times L$$

$$R \times L \subseteq R \times \mathcal{B_0} \times L \subseteq R \times \mathcal{B_0} \times \mathcal{B_0} \times \mathcal{B_0} \times L$$
You should now be able to finish this off, namely, to handle a \( G \)-torsor over a space \( B \) using an covering. Suppose given a \( G \)-torsor \( R \) over \( B \) i.e. a functor \( \mathcal{Y} \rightarrow B \) such that each stalk, the stalk at each point of \( B \), is a representable functor on \( G \).

\( R \) is a sheaf on \( B \) that is an etale space \( \pi: R \rightarrow B \) over \( B \). It comes with \( R \rightarrow B \times \mathcal{Y} \), which means \( R = \bigcup_{x \in \mathcal{Y}} R_x \)

where each \( R_x \) is an etale space over \( B \). \( R \) is partitioned according to the objects of \( \mathcal{Y} \). Then you have a right action

\[ R \times \mathcal{Y} \xrightarrow{\cdot} R \]

Ask what \( R \) representable means at a point \( b \)

Idea: \( \mathcal{Y}^\circ / R \) is a category, there should be an equivalence between \( R \) representable and \( \mathcal{Y}^\circ / R \) having a final object

\[ (X, \xi : X \rightarrow Y) \]

\[ \Downarrow \xi \]

\[ (\mathcal{Y}, \text{id} : Y \rightarrow Y) \]
Latest idea is that $\mathcal{Q}/R$ has a final object iff $R$ is representable. Here $R \in \text{Fun}(\mathcal{Y}^p, \text{sets})$ and $\mathcal{C}/R$ is the fibred category over $\mathcal{Y}$ with fibre $R(X)$ over $X$:

$\text{Ob}(\mathcal{C}/R) = \{(x, 0) \mid x \in \text{Ob} \mathcal{C}, \ 0 \in R(x)\}$

$\text{Hom}_{\mathcal{C}/R}((x, 0), (x', 0')) = \{f \in \text{Hom}_\mathcal{C}(x, x') \mid f^* 0 = 0'\}$

Suppose $R(x) = h_x(x) = \text{Hom}_\mathcal{C}(x, y)$

$\text{Hom}_{\mathcal{C}/R}((x, 0; x \to y), (x', 0'; x' \to y))$

$= \{f \in \text{Hom}_\mathcal{C}(x, x') \mid f^* 0' = 0\}$

$\text{Hom}_{\mathcal{C}/R}((x, 0; x \to y), (y, \text{id}_y; y \to y))$

$= \{f : x \to y \mid \text{id}_y f = 0\}$

$= \{0 : x \to y\}$

$\mathcal{C}/R$ cat of $\{X, h_x \rightarrow R\}$, maps

$h_x \rightarrow h_x$, cat of $x, h_x \rightarrow h_y$

obtains final elt $(y, h_y \rightarrow h_y)$