Review. You are making a calculation carefully so as to handle the left-right choices. You start with a representation of $\Gamma$ on $H$ with op. $h_i > 0$ such that $(h_s = s h_1 s^{-1})_{s \in \Gamma}$ is a partition of unity: $\Sigma h_s = 1$.

Put $V_s = h_s^{1/2} H = sV_1$. Canonical maps

$$
\begin{align*}
\mathcal{H} & \xrightarrow{\chi} \bigoplus_{s \in \Gamma} V_s^2 \\
\xi & \mapsto (s \mapsto h_s^{1/2} \xi) \\
(s \mapsto \eta_s) & \mapsto \sum_s h_s^{1/2} \eta_s
\end{align*}
$$

This much could be done for an arb. partition of unity on a Hilbert space. Now bring in the group actions of $\Gamma$ on $\bigoplus V_s$. Here there is this system of simplicity which is simply transitive under $\Gamma$. So you get an isom.

$$
\begin{align*}
\bigoplus_{s \in \Gamma} V_s^2 & \xrightarrow{\theta} \bigoplus_{s \in \Gamma} V_s \\
(s \mapsto \xi_s) & \mapsto (s \mapsto s \xi_s)
\end{align*}
$$

What is $t$ on $(s \mapsto \eta_s)$, it should be $(ts \mapsto t \eta_s)$ equiv. replacing $s$ by $t^{-1}s$: $s \mapsto t^{-1}s \eta_s$.

$$
\begin{align*}
\bigoplus_{s \in \Gamma} V_s & \xrightarrow{\theta} \bigoplus_{s \in \Gamma} V_s \\
(s \mapsto \xi_s) & \mapsto (s \mapsto s \xi_t^{-1}s)
\end{align*}
$$
\[ H \xrightarrow{\alpha} \bigoplus_s V_s \xrightarrow{\Theta^{-1}} \bigoplus_s V_1 \]

\[ \xi \xrightarrow{\Theta} (\alpha' \xi) = h_1^{1/2} \xi \xrightarrow{\Theta' \xi} \Theta(\alpha' \xi) = h_1^{1/2} h_s^{1/2} \xi = h_1^{1/2} s^{-1} \xi \]

So replace \( \alpha \) by \( \alpha' = \Theta^{-1} \alpha \)

\[ h_1^{1/2} s^{-1} \xi \]

Formulas: On \( \bigoplus_s V_1 \) the action of \( t \in \Gamma \) is

\[ (t \eta)_s = \eta t^{-1} s \]

where \( \eta : \Gamma \rightarrow V_1 \)

Check \( (t_1 t_2 \eta)_s = (t_2 \eta) t_1^{-1} s = \eta t_2^{-1} t_1^{-1} s = \eta (t_2 t_1)^{-1} s = (t_1 t_2 \eta)_s \)

\[ H \xrightarrow{\alpha'} \bigoplus_s V_1 \]

\[ \alpha'(\xi) = h_1^{1/2} s^{-1} \xi \]

Check \( (t \alpha' \xi)_s = (\alpha' \xi) t^{-1} s = h_1^{1/2} s^{-1} t \xi \)

\[ (\alpha' \eta, \xi) = (\eta, \alpha' \xi) = \sum_s (\eta_s, h_1^{1/2} s^{-1} \xi) \]

\[ = \sum_s (sh_1^{1/2} \eta_s, \xi) \]

\[ \alpha' \eta = \sum_s h_1^{1/2} s \eta_s \]

Check \( \alpha' \alpha' \xi = \sum_s h_1^{1/2} h_1^{1/2} s^{-1} \xi = \sum_s h_s^2 \xi = \xi \)
\[ x'^* (t \eta) = \sum_s \text{sh}^{\frac{1}{2}} (t \eta)_s = \sum_s \text{sh}^{\frac{1}{2}} \eta t^{-s} = \sum_t \text{sh}^{\frac{1}{2}} \eta_t s \]

What remains is to do the descent. I am not being clear, but I mean to do the GNS \[ t \text{sh}^{\frac{1}{2}} \eta \]

business: H with \( \Gamma \) action and equivariant partition of unity can be reconstructed from \( \chi \) and the function \( s \mapsto \text{sh}^{\frac{1}{2}} s \) from \( \Gamma \) to \( L (V) \).

Why: \( p = \alpha' \alpha'^* : \bigoplus_{s \in \Gamma} V_s \) is a projection \( \Gamma \)-equivariant whose image is \( H \).

Look at \( \bigoplus_{s \in \Gamma} V_s = L^2 (\Gamma, V) \) \( \eta = (\eta_s \mapsto \eta_s) \)

where \((t \eta)_s = \eta t^{-s} \). OKAY can you write this as \( V \otimes \mathbb{C}[\Gamma] \)? Let left = right straight.

Stick to the Hill. Space picture. The key situation is a rep \( H \) of \( \Gamma \) such that there exists a closed subspace \( j : V \hookrightarrow H \) such that \( H = \bigoplus_{s \in \Gamma} s j V \).

Thus \( H \) is completion of \( C[\Gamma] \otimes V = \bigoplus_{s \in \Gamma} s j V \). Description elements of \( H \) as functions equiv. to functions on \( \Gamma \) to \( V \) as follows:\[ \sum_s s j \eta_s \in \bigoplus_{s \in \Gamma} s j V \]

If you describe elements of \( H \) as \( \sum_{s \in \Gamma} s j \eta_s \), what is action of \( t \in \Gamma \): \( t (\sum_{s \in \Gamma} s j \eta_s) = \sum_{s \in \Gamma} ts j \eta_s = \sum_{s \in \Gamma} s j \eta_t^{-s} \)

so you get \((t \eta)_s = \eta t^{-s}\) action of \( t \in \Gamma \) on \( L^2 (\Gamma, V) \).

Note that you are using the left regular representation consistent with \( \Gamma \) left acting on \( C[\Gamma] \otimes V \).

Next to understand projection...
Return to replacing $V$ by $h_{1/2}V$

\[ H \xrightarrow{\alpha} \bigoplus_{s \in I} s V \]

\[ \xi \xrightarrow{\tau} h_{1/2} \xi \]

\[ \frac{h_1^2 H}{h_1^2} = s \frac{h_1^2 H}{h_1^2} = s V \]

\[ H \xrightarrow{\alpha} \bigoplus_{s \in I} s V \]

\[ \xi \xrightarrow{\tau} \sum_{s} h_{1/2}^s \xi = \sum_{s} s h_{1/2}^s \xi \]

\[ \lambda^{\prime}(\xi) = \sum_{s \in I} s \left( h_{1/2}^s \xi \right) \in \bigoplus_{s \in I} s V \]

At the moment you have $\Gamma$ acting on $H$, $h_1 > 0$,

\[ h = s h_1 s^{-1}, \quad \sum h_s = 1, \quad V = V = h_1 H \]

\[ H \xrightarrow{\alpha} \bigoplus_{s \in I} s V \xrightarrow{\alpha^*} H \]

\[ h_{1/2}^s \xi = s h_{1/2}^s \xi \]

\[ \left( \sum_{s} s \eta_s \right) \left( \sum_{s} s h_{1/2}^s \xi \right) = \sum_{s} \left( s \eta_s h_{1/2}^s \xi \right) \]

\[ \left( t \eta \right) = \eta t^{-1} \xi \]

\[ t \sum_{s} s \eta_s = \sum_{s} ts \eta_s = \sum_{s} \eta_{t^{-1} s} \]

\[ t \sum_{s} s \eta_s = \sum_{s} ts \eta_s = \sum_{s} \eta_{t^{-1} s} \]
So now you have the $\Gamma$-module picture pretty clear. Next find the data needed to reconstruct $H$ from $V$. Recall $V = h_1^2 H = h_1 H$.

The point is any projector $p = p^* = p^2$ on $\bigoplus_{s \in \Gamma} s V$ commuting with $\Gamma$ determines $H = \text{Im}(p)$ which is a unitary repn. of $\Gamma$ and an operator $h_1$ which is $H \xrightarrow{p} \bigoplus_{s \in \Gamma} s V \xrightarrow{h_1} V \xrightarrow{(a)} \bigoplus_{s \in \Gamma} s V \xrightarrow{x^*} H$.

$h_i = x_i i^* x_i \geq 0$, should satisfy $\sum_{s} s h_i s^{-1} = 1$ on $H$ since $\sum_{s} s x_i x_i^* s^{-1} = 1$ on $\bigoplus_{s \in \Gamma} s V$. It might not be true that $H \xrightarrow{h_1} V$ is surjective.

Problem. You have $p = p^* = p^2$ on $\bigoplus_{s \in \Gamma} s V$.

$p$ commutes with left $\Gamma$ multiplication

$p = x x^* : \bigoplus_{s \in \Gamma} s V \rightarrow \bigoplus_{s \in \Gamma} s V$

$p$ commutes with left mult by $\Gamma$

You know that $\sum_{s} s g s = x 1$ on $\bigoplus_{s \in \Gamma} s V$.

So that? What is an equivalent map from $\bigoplus_{s \in \Gamma} s V$ to itself? Same as a linear map $V \rightarrow \bigoplus_{s \in \Gamma} s V$. 


\[
\text{Hom}_\Gamma (C[\Gamma] \otimes V, C[\Gamma] \otimes V) = \text{Hom}(V, C[\Gamma] \otimes V)
\]

\[
c[\Gamma] \otimes V = \bigoplus_s sV \quad \text{typical element is } \Sigma s \eta_s \quad \text{with } \eta : \Gamma \to V. \quad t \sum s \eta_s = \sum t s \eta_s = \sum s \eta_t \cdot s.
\]

You want to understand how an element of \(C[\Gamma] \otimes V\) looks. It amounts to a linear map \(V \to C[\Gamma] \otimes V\), thus it has the form of a function on \(\Gamma\) with values in \(\text{Hom}(V, V)\), call this function \(s \mapsto t_s\). No.

You want to understand operators

\[
T = \bigoplus_s sV \longrightarrow \bigoplus_t tW
\]

which can \(\Gamma\)-module maps. Assume \(\dim(V) < \infty\).

\(T\) is equivalent to a linear map \(V \longrightarrow \bigoplus_t tW\)

\[
\sum t \eta_t : v \mapsto \sum t \eta_t v \in \bigoplus_t tW
\]

Then \(s v \mapsto \sum s t \eta_t v = \sum s t s_t v\).

Maybe you should look at \(T : \bigoplus_s sV \longrightarrow \bigoplus tW\)

does focus on \(t \rotateleft{ss'} T \rotateleft{ss'} : sV \longrightarrow tW\).

\[
\begin{array}{cc}
V & \xrightarrow{s} sV \\
\bigoplus_s sV & \xrightarrow{T} \bigoplus_t tW
\end{array}
\]

These given \(V \longrightarrow \bigoplus_t tW\) \(\sum t \eta_t \quad s \in \text{Hom}(V, W)\)

Conclude that \(\text{Hom}_\Gamma (\bigoplus_s sV, \bigoplus_t tW) = \text{Hom}(V, \bigoplus_t tW)\)

\[
\left\{ s \in \bigoplus t \eta_t \mid s \in \text{Hom}(V, W) \right\}
\]

\(T(sV) = s \otimes \sum t \eta_t v \quad T\).
So what is going on?

\[ sV = \frac{h_s}{3} \vec{H} = \frac{h_s}{3} \vec{H} \]

\[ \sum_{s} h_{\frac{1}{2}}^s \sum_{t} \eta_t \]

\[ \sum_{s} h_{\frac{1}{2}}^s \sum_{t} th_{\frac{1}{2}}^t \eta_t = \sum_{s} s h_{\frac{1}{2}}^s \sum_{t} th_{\frac{1}{2}}^t \eta_t \]

\[ \text{Try again.} \]

\[ H \xrightarrow{\alpha} \bigoplus sV \xrightarrow{\alpha^*} H \]

\[ \frac{h_{\frac{1}{2}}^s}{3} = s h_{\frac{1}{2}}^s \eta^s \]

\[ \alpha(\xi) = \sum_{s} h_{\frac{1}{2}}^s \xi^s \]

\[ \alpha(\xi, \sum_{t} \eta_t) = \sum_{s} h_{\frac{1}{2}}^s \xi^s \quad \left( \text{Try again.} \right) \]

\[ \alpha^*(\xi, \sum_{s} h_{\frac{1}{2}}^s \eta^s) = \sum_{t} h_{\frac{1}{2}}^t \sum_{s} h_{\frac{1}{2}}^s \eta^s \]
\( V = h_{1/2}^1 H \quad \Rightarrow \quad s V = h_{1/2}^s H \)

\[ \| A \|_2 = \sum_s (h_s \| \xi \|)^2 = \| \xi \|_2^2 \]

\[ H \xrightarrow{\alpha} \bigoplus_s s V \xrightarrow{\alpha^*} H \]

\[ j_1 s^{-1} = s \downarrow \Gamma_s = s \xi \]

\[ \alpha^* (\sum_s \eta_s) = \sum_s s h_{1/2}^s \eta_s \]

\[ \alpha^* \xi_1 = h_{1/2}^{1/2} \eta_1 \]

Remaining point: \( p = \alpha \alpha^* \) is a projector on \( \bigoplus_s s V \)

which is \( \Gamma \)-graded. This should mean \( p \) is a projector in \( \Gamma \)-graded algebra.

\[ \bigoplus_s s V \xrightarrow{\alpha \alpha^*} \bigoplus \xi t V \]

\[ j_1 \alpha \alpha^* \xi_1 s = j_1 t^{-1} \alpha \alpha^* \xi_1 = j_1 t^{-1} \alpha \xi_1 t^{-1} \alpha \]

So here you have \( V \xrightarrow{\alpha \xi_1 = h_{1/2}^{1/2}} H \) and \( H \xrightarrow{j_1} V \)

\[ j_1 p s = h_{1/2}^1 t^{-1} h_{1/2}^s \] group elt. compressed to \( V \)

Summary: From \( H, \Gamma, h_1 \), you get \( V = h_{1/2}^{1/2} H \) and \( p = (p t) \)

\( p t = h_{1/2}^1 t^{-1} h_{1/2}^{1/2} \in \mathcal{L}(V) \). Conversely GNS allows you to reverse this.
Review: \( H, \Gamma, h_6 = s h_1 s^{-1} \varphi, \Sigma h_s = 1 \).

\[
V = h_1^{1/2} H
\]

\( s \cdot V = h_1^{1/2} H \)

\( \alpha(\xi) = \bigoplus_s h_5^{1/2} \xi \cdot \bigoplus_s V \)

\( (a^*(\bigoplus_s q_s), \xi) = (\bigoplus_s q_s, \bigoplus_s h_5^{1/2} \xi) = \sum_s (q_s, h_1^{1/2} s^{-1} \xi) = \sum_s (h_1^{1/2} q_s, s \xi) \)

\( a^*(\bigoplus_s q_s) = \sum_s h_1^{1/2} q_s = \sum_s h_5^{1/2} s q_s \)

\( t(\bigoplus_s q_s) = \bigoplus_s q_s \cdot h_5^{-1} \)

\[
H \xrightarrow{\alpha} \bigoplus_s s V \xrightarrow{a^*} H
\]

\( j_s \xi = \frac{1}{j_s} \xi, j_s = \sigma_s \quad \forall s \)

\( j_s \alpha(\xi) = h_1^{1/2} \xi \quad j_s (\bigoplus_s q_s) = j_s \alpha(\bigoplus_s q_s) = h_1^{1/2} j_s (\bigoplus_s q_s) = h_1^{1/2} \bigoplus_s q_s \)

\[
\alpha^* \xi = h_1^{1/2} \xi
\]

Check

\( \alpha^* \alpha = \alpha^* (\sum_s j_s) \alpha = \sum_s (\alpha^* j_s \alpha) s^{-1} = \text{id}_H \)

\( \alpha^*: \bigoplus_t s V \rightarrow \bigoplus_t s V \)

\( j_s \alpha^* \xi = j_s \alpha \xi^{-1} \quad \alpha^* \xi_1 = h_1^{1/2} \xi_1 \)

is the compression of the group element \( s^{-1} t \) to an operator in \( V \).

Given \( H, \Gamma, h_1 = \sum h_s = 1 \) get \( V = h_1^{1/2} H \)

and \( H \xrightarrow{\alpha = h_1^{1/2}} V \xrightarrow{\alpha^* \xi_1 = h_1^{1/2}} H \).
Summary: Given $H, \Gamma, h_1, \sum h_s = 1$, you get
\[ V = h_1^{1/2} H \] and maps
\[ H \xrightarrow{\rho \times \alpha} V \xrightarrow{\alpha^* \rho} H \]
such that $P_s = \rho_s \times \alpha^* \nu_1 = h_1^{1/2} s h_1^{1/2} \in L(V)$ has a complete positivity property.

\[ P_s P_t = h_1^{1/2} s h_1^{1/2} P_t = h_1^{1/2} s h_1^{1/2} s h_1^{1/2} = P_t \]

\[ \sum_s P_s P_{s^{-1}} \]

Go backwards. Suppose given $V$ a Hilbert space and a family of operators $P_s \in L(V)$ satisfying $P_s^* = P_s^{-1}$.

Then you define a
\[ p = \bigoplus_{s \in \Gamma} s P_s \]

\[ V = \bigoplus_{s \in \Gamma} s P_s \]

\[ p = \sum_s P_s \sum_t t \overline{f}_t \overline{f} = \sum_s \sum_t \overline{f}_t \overline{f} \overline{f}_s^{-1} P_s \sum_t t \overline{f}_t \overline{f} \overline{f}_s^{-1} \]

\[ = \sum_{s, t} s \overline{f}_t \left( \overline{g}_1 s^{-1} P_s \overline{f}_1 \right) \overline{f} t^{-1} \]

So you need to know something.

You have to get the data of the Morita equivariant control
\[ (H, \Gamma, h_1) \longrightarrow (V, P_s) \]

Conversely given $V, P_s$
\[ \int P_u = \sum_{s, t} P_s P_{s^{-1}} P_t. \]

\[ P_{s^{-1}} = P_{\alpha^* \rho} \]
You construct \( p \) on \( \bigoplus sV \), \( p = \lambda \) for some \( \lambda \).

\[
p^{1} = \sum_{s} s \gamma_{s} p^{1} = \sum_{s} s \gamma_{s} \frac{1}{e_{s}^{1} \times e_{s}^{1} \alpha^{s} \lambda^{s}}
\]

\[
p^{1} = \sum_{s} \frac{s \gamma_{s} p^{1}}{e_{s}^{1} \times e_{s}^{1} \alpha^{s} \lambda^{s}}
\]

Again:

\[
H \xrightarrow{\alpha} \bigoplus sV \xrightarrow{\alpha^{*}} H \xrightarrow{\alpha} \bigoplus sV
\]

\[
d_{s} p_{s} t = f_{1} s^{-1} \alpha^{*} t \gamma_{s} = (f_{1} \alpha) s^{-1} t \alpha^{*} \gamma_{s}
\]

\[
p_{s} t = \sum_{s} s \gamma_{s} p_{s} t = \sum_{s} s \gamma_{s} (f_{1} \alpha) s^{-1} t \alpha^{*} \gamma_{s}
\]

The problem? What should happen?

Given \( V, \Gamma \) you get \( \bigoplus sV \) unitary rep. of \( \Gamma \)

\[
t(\bigoplus s \eta_{s}) = \bigoplus s \eta_{s} \gamma_{s}
\]

This is forced because you want \( \Gamma \) to left act and \( \alpha \) to yield a \( \Gamma \)-grading. Next you can consider any projector \( p \) on \( \bigoplus sV \) commuting with \( \Gamma \).

The image of \( p \) will give an \( H \) with \( \bigoplus sV \) unitary \( \Gamma \)-action. Moreover, you have

\[
\text{Hom}_{\Gamma}(\bigoplus sV, \bigoplus sV) = \text{Hom}(V, \bigoplus sV)
\]

Hence an \( T : V \rightarrow \bigoplus sV \) it extends to a \( \Gamma \)-map \( \bigoplus sV \rightarrow \bigoplus sV \)

\[
d_{s} T_{s} t = f_{1} (s^{-1} t) T_{s} t
\]

\[
T = \sum_{s, t} d_{s} T_{s} t f_{s, t}
\]
Something should be very simple

Let \( T_s = f_s T_1 : V \rightarrow \bigoplus_s V \rightarrow \bigoplus_s V \rightarrow V \rightarrow \bigoplus_s V \rightarrow \bigoplus_s V \rightarrow V \)

\[ = f_s T_1 \]

Then,

\[ T t = \bigoplus_s f_s T_1 T t = \bigoplus_s f_s T_1 T t \]

What's going on? It should be simple to describe. You want to describe \( T : \bigoplus_s V \rightarrow \bigoplus_s V \) commuting with \( \Gamma \). For \( dim V < \infty \) you know \( \Gamma \)-linear \( T \) is equiv to \( T_1 : V \rightarrow \bigoplus_s V \). So \( T \) splits into components. \( T = \sum_s T_s \). \( T_s \) unique \( \Gamma \)-linear ops \( \rightarrow T s = f_s T_1 \)

Here the problem. Starting point (algebraic version)

is a \( \Gamma \)-module \( M \) containing a subspace \( V \) such that the canon. map \( \bigoplus_s V \rightarrow M \) is an isom. Equivalently, the vector space \( M \) is given a grading \( M = \bigoplus_m M \) indexed by the set \( \Gamma \) which is compatible with the left \( \Gamma \)-action on \( M \). \( t M = \bigoplus_m t M \).

Now consider operators \( \Gamma \)-linear on such a "free" \( \Gamma \)-module. A \( \Gamma \)-linear operator \( T : M = \bigoplus \rightarrow M = \bigoplus \rightarrow M \) is the same as a linear map \( V \rightarrow M = \bigoplus \rightarrow M \). So it splits \( T = \sum_s T_s \), where \( T_s V \subset s \). Say \( T \) is homogenaus of degree \( t \). When \( TV \subset t V \).

Let \( U \) be homogenaus of degree \( u \). \( U \) is \( \Gamma \)-linear and \( UV \subset u V \). Then \( T(U(V)) \subset T(u V) = u TV \subset u t V \). So the degree of \( TU \) is backwards: \( ut \).
$\text{Hom}_R(C[R]\otimes V, C[R]\otimes V) = \text{Hom}(V, C[R]\otimes V)$
$q : V \rightarrow C[R]\otimes V \quad q = \bigoplus_{s \in L(V)} \sum s \otimes q_s \otimes \psi_s \in L(V)$

Let $\hat{q} : C[R]\otimes V \rightarrow C[R]\otimes V$ be the $R$ linear extension

$\hat{q}(t \otimes \eta_t) = \sum t \otimes q_s \otimes \psi_s \otimes \eta_t = \sum t \otimes q_s \otimes \psi_s \otimes \eta_t$

Thus if $q = \bigoplus_{s \in L(V)} q_s \in \bigoplus_{s \in L(V)} C[R]\otimes V$

then $\hat{q}(\bigoplus_{t \in T} t \otimes \eta_t) = \bigoplus_{t \in T} \sum t \otimes q_s \otimes \psi_s \otimes \eta_t = \bigoplus_{t \in T} \sum t \otimes q_s \otimes \psi_s \otimes \eta_t$

Note wrong order

Another notation maybe better for the analysis.

$C[R]\otimes V = \{ \eta : R \rightarrow V \mid \text{finite support} \}$

left action of $R$ is $(t \eta)(s) = \eta(ts)$

End$(C[R]\otimes V) = C[R] \otimes \text{End}(V)$ at least to dim $V < \infty$

$(\sum_s s \otimes q_s)(\sum_t t \otimes \eta_t) = \sum_{s,t} ts \otimes q_s \otimes \eta_t$

composition is $(\sum_s s \otimes q_s)(\sum_t t \otimes \psi_t) = \sum_{s,t} ts \otimes q_s \otimes \psi_t = \sum_{s,t} t \otimes q_s \otimes \psi_t$
Let $\rho : \text{End}_F(C\Gamma \otimes V) \to \text{End}_F(C\Gamma \otimes V)$, assume $\rho^2 = \rho$.

Then $\rho = \sum s \otimes p_s \in C[\Gamma] \otimes L(V)$, assume $\rho^2 = \rho$.

$p^2 = \sum_{s,t} ts \otimes p_s p_t = \sum_{s \in \Gamma} u_0 \otimes \sum_{t \in S} p_s p_t$

So a $\Gamma$-projection in $C[\Gamma] \otimes V$ is the same as a function $s \mapsto p_s \in L(V)$ satisfying

$p_u = \sum_{ts = u} p_s p_t$

So back to the Hilb. space notation

\[ H, \Gamma, h_1, h_5 = sh_5^{-1} \quad V = \frac{h_{1/2}}{h_1} \leftarrow H \]

\[ H \xrightarrow{\alpha^*} \bigoplus S V \xrightarrow{\alpha} H \]

\[(\alpha^* \alpha)(s) = \sum_s h_{1/2} h_{5/2} \frac{s_5}{s} = \sum_s h_6 \frac{s}{s} \frac{s}{s} = s .\]

This since $\alpha^* \alpha = \text{id}_H$, $\rho = \alpha^* \alpha$ is a projector commuting with $\Gamma$ so it should have the form $\sum_{s \in S} s \otimes p_s \otimes \sum_{t \in \Gamma} u_0 \otimes \sum_{s \in S} p_s p_t$

\[(\sum_{s \in S} s \otimes p_s)(1 \otimes u) = \sum_{s \in S} s \otimes p_s u\]

\[p(1 \otimes u) = \alpha^* \alpha(1 \otimes u) = \alpha h_1^{1/2} u = \bigoplus_s h_5^{1/2} h_6^{1/2} u \]

\[= \sum_s h_5^{1/2} s^{-1} h_6^{1/2} u\]

\[p_s = h_5^{1/2} s^{-1} h_6^{1/2} . \text{ Check this} \]

\[\sum_{ts = u} p_s p_t = \sum_{ts = u} h_{1/2} (s \otimes h_5^{-1} h_6^{1/2} t^{-1} h_1^{1/2} = \sum h_{1/2} u^{-1} h_6 h_{1/2}^{1/2} u = p_u \]
You seem to be missing the good notation for a cross product. Once this is straightened out things should go smoothly.

Method: use inverses. \( H \) a Hilbert space with \( \Gamma \) action. \( \mathcal{V} \) subspace \( s \mathcal{V} \) are orthogonal and sum dense in \( H \). Then \( \bigoplus \mathcal{sV} \sim H \). However, you want to assign \( \deg(sV) = s^{-1} \). So you write \( s^{-1}_V \) to mean \( s^{-1} \)?

What was the point yesterday? Homogeneous components of an operator on a \( \mathbb{F} \)-graded space. Consider

\[
\mathcal{C}[[s]] \otimes \mathcal{V} = \bigoplus_{s \in \mathcal{F}} s \mathcal{V}
\]

\( \Gamma \) action \( t(s \mathcal{V}) = ts \mathcal{V} \)

\( \Gamma \) grading \( (\mathcal{C}[[s]] \otimes \mathcal{V}) = s \mathcal{V} \)

\[
\text{Hom}_\mathbb{F}(\mathcal{R} \otimes \mathcal{V}, \mathcal{R} \otimes \mathcal{V}) = \text{Hom}(\mathcal{V}, \mathcal{R} \otimes \mathcal{V})
\]

Start with a "free" \( \Gamma \) representation module \( M = \mathcal{R} \otimes \mathcal{V} = \bigoplus_{s \in \mathcal{F}} s \mathcal{V} \). An operator \( \phi \) in \( M \) commuting with \( \Gamma \)-action is the same as a linear map \( \phi: \mathcal{V} \to M \) via \( \phi(t \mathcal{v}) = t \phi(\mathcal{v}) \).

Among the \( \phi \in \text{Hom}(\mathcal{V}, \mathcal{R} \otimes \mathcal{V}) \) are those of the form

\[
\sum \phi_t \otimes \phi_t \in \mathcal{R} \otimes \text{Hom}(\mathcal{V}, \mathcal{V})
\]

Where the sum is finite.
One has
\[
\left( \sum_t t \otimes \phi_t \right) \left( \sum_s s \otimes \psi_s \right) = \sum_{t,s} t s \otimes \psi_t \phi_s
\]
and the composition of \( \Delta \) of operators assoc. to \( \psi, \phi \) is given by the same formula.
\[
\left( \sum_t t \otimes \phi_t \right) \left( \sum_s s \otimes \phi_s \right) = \sum_{t,s} t s \otimes \psi_t \phi_s
\]

You want to focus on the grading, what grading?

Given a graded vector space \( M = R \otimes V \) has a natural grading indexed by the set \( \Gamma \), namely \( M = \bigoplus_{s \in \Gamma} M_s \) where \( M_s = s \otimes V \) of \( R \otimes V \).

An operator \( T \) on \( M \) is said to be homogeneous of degree \( t \in \Gamma \) when \( TM_s \subseteq M_t s \) \( \forall s \). If \( T_i \) has deg \( t_i \) for \( i = 1, 2 \), then \( T_1 T_2 M_s \subseteq T_1 M_{t_1 s} \subseteq M_{t_1 t_2 s} \) so \( T_1 T_2 \) has deg \( t_1 t_2 \).

The general framework should be comodules over the coalgebra \( C[\Gamma] \), \( \Delta s = s \otimes s \) for a grading over the set \( \Gamma \). Perhaps a tensor product defined for comodules over a Hopf alg.

\[
M \otimes N = \bigoplus_{s,t} M_s \otimes N_t = \bigoplus_{s,t} \left( \bigoplus M_s N_t \right)
\]

This tensor product is appropriate for \( R_s M_t \subseteq M_{s t} \), \( R_s R_t = R_{s t} \), \( W_s R_t \subseteq W_{s t} \) for a \( \Gamma \)-graded alg \( R \).

Keep things simple. \( \Gamma \) set, then you have notion of \( \Gamma \)-graded space \( M = \bigoplus_{s \in \Gamma} M_s \), same as a \( \bigoplus \) of comodule for \( C[\Gamma], \Delta s = s \otimes s \)
If \( \Gamma \) is a group, you have notion of \( \Gamma \)-graded v.s. \( M = \bigoplus_{s \in \Gamma} M_s \). Given two \( \Gamma \)-graded v.s. \( M = \bigoplus_{s \in \Gamma} M_s \), \( N = \bigoplus_{s \in \Gamma} N_s \), then \( M \otimes N = \bigoplus_{s,t \in \Gamma} M_s \otimes N_t \) is \( \Gamma \times \Gamma \)-graded. Now if \( \Gamma \) is a group you can push forward \( \Gamma \times \Gamma \rightarrow \Gamma \) to get a \( \Gamma \)-graded v.s.

\[
M \otimes N = \bigoplus_{s,t \in \Gamma} (M_s \otimes N_t)
\]

This is a kind of convolution type tensor product:

\[
(M \otimes N)_s = \bigoplus_{s,t} M_s \otimes N_t = \bigoplus_{s,t \in s \otimes t} M_s \otimes N_t
\]

So \( \Gamma \)-graded vector spaces form \( \otimes \)-category allowing one to define \( \Gamma \)-graded algebras \( R \) and \( \Gamma \)-graded \( R \)-modules:

\[
R_s \otimes R_t < R_{st}, \quad R_s M_t < M_{st}
\]

also left modules. Note that \( \Gamma \)-graded vector spaces are the same as comodules for the comm. coalg. \( \mathcal{O}(\mathbb{Z}) \), \( \Delta s = s \otimes s \).

You feel that the important question concerns the behavior of operators on a \( \Gamma \)-graded module. Specifically look at the group ring \( R = \bigoplus_{s \in \Gamma} \mathbb{C}s \) which is both a \( \Gamma \)-graded left (resp. right) \( R \)-module.

\[
R_s R_t < R_{st}, \quad R_t R_s < R_{ts}
\]

and these two actions of \( \Gamma \) commute.

Look at things as follows. Let \( M = \bigoplus_{s \in \Gamma} M_s \) be \( \Gamma \)-graded, let \( \Gamma \) act on \( M_s \), \( t : M \rightarrow M \) \( tM_s < M_t \). Then \( t : M_1 \rightarrow M_5 \) so \( M = \bigoplus_{s \in \Gamma} s M_s \).
Back to grading. Starting point is a \( \Gamma \)-module \( M \) equipped with a subspace \( V \) whose translates \( sV \) for \( s \in \Gamma \) are independent from \( M \):

\[
\bigoplus_{s \in \Gamma} sV \xrightarrow{\sim} M, \quad C[\Gamma] \otimes V \xrightarrow{\sim} M.
\]

In this way \( M \) acquires a grading indexed by the set \( \Gamma \). Alt: A comodule with \( C[\Gamma] \), \( \Delta s = s \otimes s \)

Maybe a better starting point would be to consider a graded \( \Gamma \)-module \( X \) with respect to a set \( \Gamma \). Given \( M = \bigoplus_{x \in X} M_x \), \( N = \bigoplus_{y \in Y} N_y \), you have

a tensor product \( \bigoplus_{x \in X, y \in Y} M_x \otimes N_y \). Also given \( f : X \to Y \), \( M = \bigoplus_{x \in X} M_x \)

get \( f_!(M) = \bigoplus_{y \in Y} \left( \bigoplus_{x \in f^{-1}(y)} M_x \right) \) pushforward of the system \( M_x \) under \( f : X \to Y \).

Then given \( M = \bigoplus_{s \in \Gamma} M_s \), \( N = \bigoplus_{t \in \Gamma} N_t \), you can form \( M \otimes N = \bigoplus_{s \in \Gamma, t \in \Gamma} M_s \otimes N_t \) and push forward with \( (st)^{-1} \cdot s \)

\( (st)^{-1} \cdot s \) to get a \( \Gamma \)-graded \( \bigoplus_{s \in \Gamma} M_s \otimes N_t \)

This construction should correspond to \( \otimes \) of comodules under the Hopf algebra \( C[\Gamma] \).

Look at the group ring \( C[\Gamma] = \bigoplus_{s \in \Gamma} \mathbb{C} s \)
two actions of \( \Gamma \): \( t \cdot s = ts \) \( st^{-1} \) which commute.

How do you clarify things. Start with operators on a graded vector space. Fix look at \( \mathbb{Z} \) grading. \( V = \bigoplus_{n \in \mathbb{Z}} V_n \). The operators on a graded vector space should be a graded ring,
at least under appropriate finiteness. What about use the obvious grading: \( C[\Gamma] = \bigoplus_{\sigma \in \Gamma} C_\sigma \). Put \( \mathcal{R} = C[\Gamma], \quad \mathcal{R}_\sigma = C_\sigma \).

Start again with a Hilbert space rep'n \( H \) of \( \Gamma \) and a closed \( V \subset H \) such that \( \bigoplus_{\sigma \in \Gamma} H_\sigma = \bigoplus_{\sigma \in \Gamma} sV \). Consider \( \Gamma \)-invariant operator \( T \) on \( H \).

\[
\text{Hom}_{\mathcal{R}}(\mathcal{R} \otimes V, \mathcal{R} \otimes V) = \text{Hom}(V, \mathcal{R} \otimes V) \\
\cup \\
\mathcal{R} \otimes \text{Hom}(V, V).
\]

Point is that with this way \((r \otimes \varphi)(r' \otimes \psi) = (r' r \otimes \psi \varphi)\). So as a ring one has \( \mathcal{R} \otimes \text{End}(V) \). \( \text{Hom}_{\mathcal{R}}(\mathcal{R}, \mathcal{R}) = \mathcal{R} \otimes \text{End}(V) \). For \( \mathcal{R} = C[\Gamma] \) and \( r \) is sum of \( \sigma \) in \( \Gamma \) and \( s \) homogeneous of \( r \).

Maybe you want simply to use the ring \( \mathcal{R} \otimes \text{End}(V) \)?

Let's begin again with \( \bigoplus_{\sigma \in \Gamma} V_\sigma \subset H \)

What's important? Start again with Answer:
- a grading on an algebra \( A \). This means a splitting \( A = \bigoplus A_\mu \) of \( A \) as vector space such that \( A_\mu A_\nu \subset A_{\mu \nu} \) for some \( \mu \times \nu = \mu \nu \). Then the set of indices has some sort of product, which should be sorted out. There's a problem with \( A_1 = 0 \), which should be sorted out.

Monita contexts.
Question: Is there a way to interpret a Morita context as a graded algebra? A Morita context \((A, \mathcal{B})\) is a ring with a splitting into 4 abelian subgroups such that 8 of the possible 16 products are zero. Perhaps you have a grading not a category with two objects and id\(\mathcal{C}\) for arrows.

Question: What is a unital Morita context?

This should be exactly the case of a unital ring \(R\) equipped with idempotent \(e : R = (eRe, eRe^t)\)

Claim: \(R = (e + e^t)R(e + e^t)\)

To start with the notion of grading, i.e., vector space equipped with a splitting indexed by a set \(S : V = \bigoplus_{s \in S} V_s\).

To define graded algebra what you need is to assign to an ordered pair of elements of \(S\) a third element \(s \ast s'\), partially defined operation \(s \ast s' \rightarrow S\), want

\[
A_s A_{s'} \subseteq \begin{cases} A_{s \ast s'}, & \text{if defined} \\ 0, & \text{otherwise} \end{cases}
\]

Let skip this & focus on \(S = \text{group } \Gamma\) graded algebra \(A\) is one with \(s\)-f.p.

\[
A = \bigoplus_{s \in \Gamma} A_s \quad A_s A_t \subseteq A_{s \ast t}
\]

have notion of graded left module right

\[
M_t A_s \subseteq N_{ts}
\]

bimodule \(\Gamma\)-graded is ok.

\[
\begin{align*}
A M_t & \subseteq M_{ts} \\
M_t B_s & \subseteq M_{ts}
\end{align*}
\]
Suppose $V = \bigoplus_{s \in \Gamma} V_s$ is $\Gamma$-graded.

To get past the obstruction,

Let $\Gamma$ be a group. Notion of $\Gamma$-graded vector space $V = \bigoplus_{s \in \Gamma} V_s$, tensor product of these:

$$(V \otimes W)_s = \bigoplus_{t+u=s} V_t \otimes W_u,$$

$\Gamma$-graded algebra: $A = \bigoplus_{s \in \Gamma} A_s$, $A_s \cdot A_t \subseteq A_{st}$, left and right $\Gamma$-graded modules over a $\Gamma$-graded alg.

Question: Given a $\Gamma$-graded vector space $V$, is there a natural $\Gamma$-graded alg of endomorphisms? More generally you want $\text{Hom}(V, V)$ $\Gamma$-graded v.s.

Universal:

$$\text{Hom}_{\text{mod}}(U, \text{Hom}(V, W)) = \text{Hom}_{\Gamma}(U \otimes V, W)$$

It looks as if there are two $\text{Hom}$'s correspond. to choose $U \otimes V$ and $V \otimes U$.

Take $U = Cu$ $\text{deg}(u) = u$. Then degree a part of $\text{Hom}(V, W)$, denoted $\text{Hom}^u(V, W)$, should be

$$\text{Hom}_{\Gamma}((Cu \otimes V, W)) = Cu \otimes \bigoplus_{s} V_s = \bigoplus_{s} uV_s$$

$$\sum_{u} V = \sum_{u} \otimes V = \bigoplus_{s} V_s \otimes u^{-1}$$

$$(\sum_{u} \otimes V)_s = \bigoplus_{s=ut} \sum_{u} \otimes V_t$$
Review: \( \Gamma \) group, consider \( \Gamma \)-graded vector space.

\[ V = \bigoplus_{s \in \Gamma} V_s \]

Tensor prod.

\[ (V \otimes W)_s = \bigoplus_{t \in \Gamma} V_t \otimes W_t \]

They are the same as comodules over \( \mathbb{C}[\Gamma] \), \( A_s = s A_s \).

So you have a tensor category.

Definition of \( \Gamma \)-graded alg and \( \Gamma \)-graded left and right modules. \( A = \bigoplus A_s \), \( M = \bigoplus M_s \), \( A_s M_t \subseteq M_{st} \) resp \( N_s A_t \subseteq N_{st} \).

You have this tensor product operation on \( \mathbb{C} \)-modules, assoc. but not commutative. Question about internal \( \text{Hom} \):

\[ \text{Hom}_\mathbb{C}(U, \text{Hom}(V, W)) = \text{Hom}(U \otimes V, W) \]

Take \( U = \sum u \mathbb{C} \), \( \sum u \mathbb{C} \) = \{ \mathbb{C} \} \text{ if } s \neq u \text{,} \mathbb{C} \text{ if } s = u \).

Assume \( \text{Hom}(V, W) \) satisfies formula above get

\[ \text{Hom}_\mathbb{C}(V, W)_u = \text{Hom}(\sum u \mathbb{C}, \text{Hom}(V, W)) = \text{Hom}(\sum u \mathbb{C} \otimes V, W) \]

\[ (\sum u \mathbb{C} \otimes V)_s = \bigoplus_{t \in \Gamma} (\sum u \mathbb{C})_t \otimes V_{st} \]

\[ = \bigoplus_{u \neq t} V_{st} = V_{u^{-1}s} \]

\[ \therefore \text{Hom}_\mathbb{C}(V, W)_u = \bigoplus_{s \in \Gamma} \text{Hom}(V_{u^{-1}s}, W_s) \]
Click to there a way to compose homogeneous operators: 

\[ V^\text{deg a} \xrightarrow{\Sigma^a} W^\text{deg b} \xrightarrow{\Sigma^b} X \]

\[ \Sigma^a \longrightarrow \text{Hom}(V^a, W) \]

\[ \Sigma^b \longrightarrow \text{Hom}(W^b, X) \]

i.e., \[ \Sigma^a \otimes V \rightarrow W, \quad \Sigma^b \otimes W \rightarrow X \]

\[ \Sigma^b \otimes \Sigma^a \otimes V \rightarrow \Sigma^b \otimes W \rightarrow X \]

\[ (\Sigma^b \otimes \Sigma^a)^s = \bigoplus_{b_tu} (\Sigma^b \otimes \Sigma^a)^s = \begin{cases} \Sigma & s = ba \\ 0 & s \neq ba \end{cases} \]

\[ \{ t(u) | tu = s \} \]

\[ (\Sigma^b \otimes \Sigma^a)^s = \bigoplus_{\{ t(u) | tu = s \} \epsilon(tu)} (\Sigma^b)^t \otimes (\Sigma^a)^u = \begin{cases} \Sigma & t = b \text{ and } u = a \\ 0 & \text{not} \end{cases} \]

\[ \{ \Sigma & \text{if } ba = s = \Sigma \]

\[ 0 & \text{otherwise} \]

Alternative: Given \[ V^a \rightarrow W^b \rightarrow X \]

and \[ W^b \rightarrow X \]

\[ V^a \rightarrow W^b \rightarrow X \]

and \[ (ba)^s \]

Alternative: Here use \[ \text{Hom}^b(\nu, W) \] defined by

\[ \text{Hom}^b(\nu, \text{Hom}(V, W)) = \text{Hom}^b(\nu \otimes U, W) \]
\[ \text{Hom}'(V, W)_a = \text{Hom}(V \otimes \Sigma^a, W) \text{ where} \]

\[ (V \otimes \Sigma^a)_s = \bigoplus_{t \in u_s} V \otimes \Sigma^a_t = V_{s^{-1}} \]

\[ V \otimes \Sigma^a \rightarrow W \quad W \otimes \Sigma^b \rightarrow X \]

\[ V \otimes \Sigma^a \otimes \Sigma^b \rightarrow W \otimes \Sigma^b \rightarrow X \]

\[ \sum_{ab} \]

\[ V_{s^{-1}} \rightarrow W_s \quad W_{t^{-1}} \rightarrow X_t \]

\[ V_{s^{-1}} \rightarrow W_{t^{-1}} \rightarrow X_t \]

\[ t(ab)^{-1} \]

I'm confused. If \( V \) is \( \Gamma \)-graded, then there are apparently two \( \Gamma \)-graded algebras, \( \text{Hom}(V, V) \) and \( \text{Hom}'(V, V) \). Degree 0 elements of the former are maps \( V_{a^t} \rightarrow V_s \) vs. and of the latter are map \( V_{s^{-1}} \rightarrow V_s \) vs.

Maybe what's useful to remember is that there are two ways to shift indexing - left \& right translation. This gives two types of homogeneous operators namely

\[ V_{a^{-t} s} \rightarrow V_s \text{ vs. comp. } V_{b^{-t} s} \rightarrow X_s \]

\[ V_{a^{-t} s} \rightarrow V_s \text{ vs. comp. } V_{b^{-t} s} \rightarrow X_s \]

\[ \text{deg}(a) \quad \text{deg}(b) \]

\[ \text{deg}(b) \quad \text{deg}(a) \]

\[ \text{deg}(b) \quad \text{deg}(a) \]

\[ \text{deg}(b) \quad \text{deg}(a) \]
You should now go back to a free $\Gamma$-module $M = \bigoplus s V$. You need to describe $\sigma$-$\Gamma$-invariant projections on $M$.

Start with a $\Gamma$-module $M$, whether left or right is irrelevant via $sm = ms^{-1}$. Assume given a subspace $V$ of $M$ such that $\bigoplus s M \rightarrow M$ is an isomorphism. This means that $M$ is the free $\Gamma$-module gen. by the v.s. $V$.

Yesterday what did you learn? You looked at the tensor category of $\Gamma$-modules ($= \Gamma$-graded modules)

\[(V \otimes W)_s = \bigoplus_{t \in \Gamma} V_t \otimes W_{s^{-1}t} \quad \text{for left and right} \quad \text{translations}, \]

yielding notions of homogeneous maps of degree $a$: \[V_{a^{-1}s} \rightarrow W_s \quad V_{sa^{-1}} \rightarrow W_s \]

Next go back to a free $\Gamma$-module $M = \bigoplus s V$, the same via $sm = ms^{-1}$, so there are two obvious ways to grade a free $\Gamma$-module. First define the free $\Gamma$-module gen by v.s. $V$ to be $C[\Gamma] \otimes V$.

Let $M$ be a free $\Gamma$-module, more precisely $M \cong C[\Gamma] \otimes V$, so if subspace $V$ of $M$ such that $M = \bigoplus s V$. Here you use left action, but give $\sigma \in \Gamma$ and $M = \bigoplus V s^{-a}$ for the right. What's the point? The point is that there are two gradings you didn't say this right.
Let $M$ be a $\Gamma$ module. You can view $\Gamma$ as operating on the left or on the right via $sm = ms^{-1}$.

Let $M$ be free, i.e. a subspace $V$ s.t. $\bigoplus_{s \in \Gamma} V_s = M$. Equivalently, $\bigoplus_{s \in \Gamma} V_s = M$. Thus, you have two $\Gamma$-gradings of $M$ which are related by inverse since $sV = Vs^{-1}$.

You are looking at idempotent operators $s$ on a free $\Gamma$-module, which commute with the $\Gamma$-action.

Recall. Equivalence between left + right $\Gamma$-modules via $sm = ms^{-1}$.

Notion of free $\Gamma$-module generated by $V$: $M = \bigoplus_{s \in \Gamma} V_s$. Up to isomorphism, this gives a $\Gamma$-module with subspace $V$ s.t. $\bigoplus_{s \in \Gamma} V_s \cong M$.

This gives a $\Gamma$-grading with $M = sV$, making $M$ a $\Gamma$-graded vector space.

Have $\Gamma M_s = M_{s^2}$, so $M$ is a left $\Gamma$-graded module.

Summary:

$\Gamma$-modules = $\Gamma$-graded vector spaces

$$V = \bigoplus_{s \in \Gamma} V_s$$

form a cat with $\Gamma$-morphisms = linear maps preserving grading.

Tensor product $$(V \otimes W)_s = \bigoplus_{t \in \Gamma} V_t \otimes W_s$$

forms a cat in general when $\Gamma$-commutative.

Tensoring with Ca $(\alpha F)$ leads to shift or susp. ops.
Then to two kinds of maps of degree $a$:

\[ V_a \to W, \forall s \text{ or } V_{sa^{-1}} \to W, \forall s. \]

**Question:** What is the degree of a composition of homog. maps?

**Left shift:**

\[
\begin{array}{ccc}
V_{b^{-1}a^{-1}s} & \xrightarrow{\beta} & W_{a^{-1}s} \\
\downarrow & & \downarrow \alpha \\
V_{(ab)^{-1}s} & \xrightarrow{} & X_s \\
\end{array}
\]

\[
\text{so } \quad \text{Hom}(W, X)_a \times \text{Hom}(V, W)_b \to \text{Hom}(V, X)_{ab}
\]

\[
\begin{array}{ccc}
\alpha & \times & \beta \\
\end{array}
\]

**Right shift**

\[
\begin{array}{ccc}
V_{sa^{-1}(ba)^{-1}} & \xrightarrow{\beta} & W_{sa^{-1}} \\
\downarrow & & \downarrow \alpha \\
V_{(ab)^{-1}s} & \xrightarrow{} & X
\end{array}
\]

\[
\text{Hom}(W, X)_a \times \text{Hom}(V, W)_b \to \text{Hom}(V, X)_{ba}
\]

\[
\begin{array}{ccc}
\alpha & \times & \beta \\
\end{array}
\]

Rewrite = Kasparov composition

\[
\text{Hom}(V, W)_b \times \text{Hom}(W, X)_a \to \text{Hom}(V, X)_{ba}
\]

You are interested ultimately in \textbf{left} or \textbf{right} idempotent operators on a free $\Gamma$-module. \textbf{Left} or \textbf{right} does for the $\Gamma$-action does not make any difference, nor does the order of composition matter.

\[
\text{free } \Gamma\text{-module: } M = C[\Gamma] \otimes V
\]

\[
\text{Hom}(C[\Gamma] \otimes V, C[\Gamma] \otimes V) = \text{Hom}(V, C[\Gamma] \otimes V)
\]

\[
\cong C[\Gamma] \otimes V \otimes V^* 
\]
Look at the group ring \( C[\Gamma] \)

First consider a free \( \Gamma \)-module?  

Assume you understand \( \Gamma \)-graded, i.e. \( \hat{\Gamma} \)-modules. Now consider a vector space \( V \) equipped with \( \Gamma \)-action.

The idea is to replace the?

**Generalization.** The shifting \( V_{\alpha} \)'s generalizes to
\[
V = \bigoplus_{\alpha \in \Delta} V_{\alpha}
\]
where \( \Delta \) is a \( \Gamma \)-torsor.

So far you have looked at vector spaces with \( \Gamma \)-grading. Next look at \( \Gamma \)-modules and compatibility.

Start with splitting
\[
V = \bigoplus_{k \in K} V_k
\]
giving \( \Gamma \)-set \( K \), say \( \Gamma \) operates on \( V \) permuting the \( V_k \)'s. \( \Gamma \) acts on the set \( K \). \( K \) is a \( \Gamma \)-set so can be split into orbits, each orbit is described by a representation of stabilizers.

Mackey's imprimitivity theory. Interesting case for you is where \( K \) is a \( \Gamma \)-torsor.

Free module: where \( K = \Gamma \), i.e. \( K \) is a \( \Gamma \)-torsor with basepoint chosen.

Consider a free \( \Gamma \)-module \( M = \bigoplus_{s \in \Gamma} sV \). Then
\( M \) has a \( \Gamma \)-grading with \( M_s = sV \) such that
\( tM_s \subset M_{ts} \), whence \( M \) is a graded \( C[\Gamma] \)-module.
You are interested in operators on the $\Gamma$-module $M = C[\Gamma] \otimes V$, which commute with $\Gamma$-action:

$$\text{Hom}_\Gamma (C[\Gamma] \otimes V, C[\Gamma] \otimes V) = \text{Hom}_\Gamma (V, C[\Gamma] \otimes V)$$

given $\Theta : V \to C[\Gamma] \otimes V$ one has

$$\Theta (v) = \sum s \otimes \Theta_s v$$

for unique $\Theta_s \in \text{End}(V)$. Assume $\{s | \Theta_s \neq 0\}$ finite.

$$\Theta = \sum s \Theta_s \in C[\Gamma] \otimes \text{End}(V).$$

Important is how $\Theta$ extends uniquely to a $\Gamma$-module endo $\tilde{\Theta}$ of $C[\Gamma] \otimes V$, namely

$$\tilde{\Theta}(t \otimes v) = t \Theta (v) = \sum ts \Theta_s v$$

so $\sum ts \Theta_s \in C[\Gamma] \otimes \text{End}(V)$ becomes the op

$$t \mapsto t \sum s \Theta_s v$$

You still haven't focussed properly. You persist using left $\Gamma$-action. Since left and right $\Gamma$-actions are equivalent this should be okay, but the $\Gamma$-grading should be changed.

So let $M$ be the free $\Gamma$-module gen by $V$

and right it using standard cross product notation

$$M = \bigoplus M_s$$

where $M_s = \otimes V_s$

Start at the beginning with the Hilbert space situation.
Begin with a Hilbert space $H$, with group $\Gamma$ acting by unitary operators on a closed subspace $V \subset H$ such that $\sum_{s \in \Gamma} sV$ dense in $H$.

Assume $h_s$ is positive operator on $H$ such that $h_sH = V$, let $h_s = s h_s^{-1}$, and assume $\sum_{s \in \Gamma} h_s = 1$ (sum of pos. ops. makes sense).

Let

$$H \xrightarrow{\alpha} \bigoplus_{s \in \Gamma} sV \xrightarrow{\alpha^*} H$$

$$\alpha(\xi) = \bigoplus_{s \in \Gamma} h_s^{1/2} \xi$$

makes sense because $h_s^{1/2} \xi \in s h_s^{-1}H \subset sV$

and $\|\alpha(\xi)\|^2 = \sum_s \|h_s^{1/2} \xi\|^2 = \sum_s (\xi, h_s \xi) = \|\xi\|^2$

So $\alpha$ is an isometry $\implies \alpha^*$ orth proj onto $H$.

If $\sum_{s \in \Gamma} s \eta_s \in \bigoplus_{s \in \Gamma} sV$, then $(\alpha^* \bigoplus_{s \in \Gamma} s \eta_s, \xi) =$

$$(\bigoplus_{s \in \Gamma} s \eta_s, \bigoplus_{s \in \Gamma} h_s^{1/2} \xi) = \sum_s (s \eta_s, s h_s^{1/2} \xi) = \sum_s (s h_s^{1/2} \eta_s, \xi)$$

$$\therefore \quad \alpha^* \left( \bigoplus_{s \in \Gamma} s \eta_s \right) = \sum_s s h_s^{1/2} \eta_s$$
Again: $H$ Hilbert space, $\Gamma$ group acting on $H$ by isometries, $h_s > 0$ on $H$, $h_s = s h_s^{-1}$ for $s \in \Gamma$. Assume $\sum_{s \in \Gamma} h_s = 1$ on $H$ (well-defined since $h_s > 0$).

Put $V = \overline{h_{1/2} H}$, closed subspace of $H$. $s V = \overline{s h_{1/2} H} = h_{1/2} s H$.

Define

$$H \xrightarrow{\alpha} \bigoplus_{s \in \Gamma} s V$$

$$\alpha(\xi) = \bigoplus_{s \in \Gamma} h_s \xi$$

$$\|\alpha(\xi)\|^2 = \sum_{s \in \Gamma} \|h_s \xi\|^2 = \sum_{s \in \Gamma} (h_s \xi, h_s \xi)$$

$\alpha$ isometry, $\alpha^* =$ projection onto $H$.

Calc. $2 \sum_{s \in \Gamma} h_s \xi_s = \sum_{s \in \Gamma} h_{1/2} s h_s \xi_s = \sum_{s \in \Gamma} s h_{1/2} \xi_s$.

So you have

$$\begin{array}{c}
\text{id} \\
H \xrightarrow{\alpha} \bigoplus_{s \in \Gamma} s V \xrightarrow{\alpha^*} H
\end{array}$$

Action of $\Gamma$ on $\bigoplus_{s \in \Gamma} s V$:

$$\alpha(t \xi) = \sum_{s \in \Gamma} h_s t \xi_s$$

$$t \alpha(\xi) = \sum_{s \in \Gamma} h_s \xi_s$$

The notation $\bigoplus_{s \in \Gamma} s \xi_s$ is not so good. You mean the function $$(s \xi_s)_{s \in \Gamma}$$ such that

Start again, but find good notation. $H$, $\Gamma$, $h_s$ as above.

$V = \overline{h_{1/2} H}$, $s V = \overline{h_{1/2} s H} \subset H$. Embedding $\varphi$ takes $s$ to the function $s_1 \mapsto h_{1/2} \xi_s \in s V$. $\varphi(s) = \varphi(s_1)$. $\varphi$ takes $s_1$ to $h_{1/2} \xi_s \in s V$. $\forall s_1 \in s V$. $s \mapsto \varphi(s_1)$. $
\]$
What is \( \bigoplus_{s \in \Gamma} sV \)?

It's the set functions \( \phi \) from \( \Gamma \) to \( V \) such that \( \forall s \in \Gamma, \quad \phi(s) \in sV \), with \( L^2 \) norm. \( \sum_{s \in \Gamma} ||\phi(s)||^2 \)

If you put \( s^* \phi(s) = \eta_s \), then you get \( \{ \eta: \Gamma \to V \} \) with \( L^2 \) norm.

\[
H \xrightarrow{\alpha} L^2(\Gamma; V) \xrightarrow{\alpha^*} H
\]

\[
\xi \mapsto \alpha(\xi)_s = s^{-1} h^{1/2}_s \xi (\xi) \mapsto \sum_s h^{1/2}_s \phi(s)
\]

\[
(\alpha(\xi), \phi) = \sum_s (h^{1/2}_s \xi, \phi(s)) = \sum_s (\xi, s h^{1/2}_s \phi(s))
\]

\( \alpha \) is \( \Gamma \)-equivariant.

How does \( \Gamma \) act on \( L^2(\Gamma; V) \). It has to be \( t \alpha(\phi)(s) = \phi(t^{-1}s) \)

\[
(t \alpha(\xi))(s) = \alpha(\xi)(t^{-1}s) = h^{1/2}_s \xi t^{-1} \xi = \alpha(t^{-1} \xi)(s)
\]

\[
\alpha^*(t \phi) = \sum_s s h^{1/2}_s (t \phi)(s) = \sum_s s h^{1/2}_s \phi(t^{-1}s)
\]

\[
= \sum_s ts h^{1/2}_s \phi(s) = t \alpha^* \phi
\]

Now do \( \alpha^* \) which is an operator on \( L^2(\Gamma; V) \) commuting with \( \Gamma \) action. \( (\theta \phi)(s) = \phi(t^{-1}s) \). Example:

Let \( \Theta: \Gamma \to L(V) \). Let \( \Theta_s \in L(V) \) Can you make \( \Theta \) act on \( \phi: \Gamma \to V \) so as to commute with \( \Theta \) \( \Gamma \) action on \( \phi \).
Go over this: \( \Theta: \Gamma \rightarrow V \), \((\Theta \varphi)(s) = \varphi(\Theta s)\)

\( \Theta: \Gamma \rightarrow \mathcal{L}(V) \). Consider \( \Theta(s) \varphi(t) \). Out of these products you want to construct a \((\Theta \times \varphi): \Gamma \rightarrow V\).

\( L_u \varphi \) you have \( \varphi \in L^2(\Gamma, V) \) and \( \Gamma \)-action \( (u \varphi)(s) = \varphi(u^{-1} s) \). Let \( T \) be an operator on \( L^2(\Gamma, V) \) commuting with this \( \Gamma \)-action. Example: \((R_v \varphi)(s) = \varphi(sv)\). Then \( (L_u R_v \varphi)(s) = (R_v \varphi)(u^{-1} s) = \varphi(u^{-1} sv) \)

\( (R_v L_u \varphi)(s) = (L_u \varphi)(sv) = \varphi(u^{-1} sv) \).

Another example is \( \Theta \in \mathcal{L}(V) \) where \( (\Theta \varphi)(s) = \Theta \varphi(s) \). \((L_u \Theta \varphi)(s) = (\Theta \varphi)(u^{-1} s) = \Theta \varphi(u^{-1} s) = \Theta L_u \varphi\) but \( \Theta \) also commutes with \( R_v \).

\[(\Theta R_v \varphi)(s) = \Theta (R_v \varphi)(s) = \Theta \varphi(sv) \]
\[(R_v \Theta \varphi)(s) = (\Theta \varphi)(sv) = \Theta \varphi(sv) \).

Put these together.

\[\sum \Theta_v R_v\] finite sum, i.e. \( \Theta_v \neq 0 \) in finite set.

So apparently the

Recall. \( H, \Gamma, h_1 \geq 0 \), \( h_s = sh_1 s^{-1} \), \( \sum h_s = 1 \) on \( H \).

\[ V = \frac{1}{h_1^{1/2}} H \]

\[ s V = \frac{1}{h_s^{1/2}} H \]

\[ H \rightarrow \bigoplus_{s \in \Gamma} s V \]

\[ \sum \frac{1}{h_s^{1/2}} V = L^2(\Gamma, V) \]

\( (\xi \varphi)(s) = h_1^{1/2} s^{-1} \xi \)

\[ ||\xi \varphi||^2 = \sum_s \frac{1}{h_s^{1/2}} ||\xi||^2 = ||\xi||^2 \]

\( (\xi, s h_1 s^{-1} \xi) = (\xi, h_s \xi) \)
\[ \alpha^* \varphi = \sum_s h^{1/2} \varphi(s) \]

\[ (L_t \varphi)(s) = (\alpha^* \varphi)(t^{-1} s) = h^{1/2} s^{-1/2} \alpha(t^{-1} s) \]

\[ (R_u \varphi)(s) = \varphi(s u) \]

\[ (L_t L_u \varphi)(s) = (L_u \varphi)(t^{-1} s) = \varphi(u^{-1} t^{-1} s) \]

\[ (R_t (R_u \varphi))(s) = (R_u \varphi)(s t) = \varphi(s t u) = (R_t R_u \varphi)(s) \]

\[ [L_t, R_u] = 0 \quad \Theta \in L(V) \quad (\Theta \varphi)(s) = \Theta \varphi(s) \]

\[ [L_t, \Theta] = [R_u, \Theta] = 0 \]

So you get operators \( \Theta R_u \) on \( L^2(\Gamma, V) \) commuting with left action:

\[ L(V) \otimes \mathbb{C}[\Gamma] \rightarrow L^2(\Gamma, V) \]

\[ \Theta \otimes s \rightarrow \Theta R_s \]

This notation is clearly what Joachim uses.

Now where are we? You have this projection \( \alpha^* \) on \( L^2(\Gamma, V) \), and you want to get it in the image of the map above. This should involve the overlap condition:

\[ h_s h_t^{1/2} = 0 \quad \text{for} \quad s^{-1} t \notin \mathcal{F} \]

\[ H \xrightarrow{\alpha} L^2(\Gamma, V) \xrightarrow{\alpha^*} H \]

\[ \xi \rightarrow (\alpha^* \xi)(s) = h^{1/2} s^{-1/2} \]

\[ (\alpha^* \varphi)(s) = h^{1/2} s^{-1} \sum_t t h^{1/2} f(t) = \sum_t (h^{1/2} s^{-1} t h^{1/2}) f(t) \]

\[ = 0 \quad \text{for} \quad s^{-1} t \notin \mathcal{F} \]

\[ (\alpha^* \varphi)(u^{-1} s) = \sum_t h^{1/2} s^{-1} u t h^{1/2} f(t) \]

\[ (\alpha^* \varphi)(s) = \sum_t h^{1/2} s^{-1} t h^{1/2} f(t) \]
So far we have reached the formula

$$(\alpha x^*) f(t) = \frac{h_{1/2}^2 s^{-1}}{t} \sum \mathcal{O} h_{1/2} f(t)$$

$$= \sum \left( h_{1/2} s^{-1} h_{1/2} \right) f(t) = \sum \left( h_{1/2} \ast u h_{1/2} \right) f(stu)$$

So something is not clear.

Alternative. Maybe use a different embedding.

$$H \xrightarrow{\alpha} L^2(\mathbb{R}_+ V)$$

$$\xi \xrightarrow{} (\alpha \xi)(s) = h_{1/2} s^{\xi}$$

$$(R_t (\alpha \xi))(s) = (\alpha \xi)(st) = h_{1/2} s^{\xi}$$

$$(\alpha(t \xi))(s) = h_{1/2} s^{\xi}$$

$$R_t \alpha = \alpha t$$

$$(R_t R_{tu} \varphi)(s) = (R_{stu} \varphi)(st) = \varphi(stu) = (R_{stu} \varphi)(s)$$

$$(R_t (\alpha \xi))(s) = (\alpha \xi)(st) = h_{1/2} s^{\xi} = \alpha(t \xi)(s)$$

$$(R_t R_{tu} (\alpha \xi))(s) = R_{stu} (\alpha \xi)(st) = (\alpha \xi)(stu) = R_{stu} \alpha$$

should write $$R_t (\alpha) = \alpha t$$. Then $$(R_t (\alpha \xi))(s) = h_{1/2} s^{\xi}$$

$$(R_t (\alpha(\xi)))(s) = (\alpha(\xi))(st) = h_{1/2} s^{\xi} = \alpha(t \xi)(s)$$

$$(R_{tu} (\alpha_\xi))(s) = R_{tu} (\alpha(t \xi)) = \alpha(st \xi) = R_{stu} (\alpha(\xi))$$

$$(R_u (R_t \varphi))(s) = \varphi(stu) = \varphi(satu) = (R_{stu} \varphi)(s)$$
Analyze the situation. You have a notation $H$ for the situation. On the $H$ side you have $\Gamma$ acting and $h_1 > 0$ such that $\sum h_5 = 1$ on $H$. What about the $V$ side? Try for something intrinsic, i.e. like the subspaces $\mathbb{R}^V$, the translates of $V$ under $\Gamma$. Take the free case where the translates are orthogonal.

You think indexing by $s^{-1}$ might help.

$$\ell^2(\Gamma, V) \overset{\beta}{\to} H \quad (f: \Gamma \to V) \mapsto \sum_{s} h_5 f(s)$$

$$\mathbb{R}^V \quad (\beta^* s, f) = \mathbb{R} (\beta^* s, \sum_{s} h_1^{1/2} s^{-1/2} f(s))$$

$$= \sum_{s} (\beta^* s, f(s)) = (\beta^* s, f(s)) + \ldots$$

$$(\beta^* s) = h_1^{1/2} s^{-1/2}$$

$$\therefore \quad (\beta^* s) = h_1^{1/2} s^{-1/2}$$

$$\sum_{s} (\beta^* s) = \sum_{s} h_1^{1/2} s^{-1/2} f(s)$$

$$= s^{-1/2} \sum_{s} s^{1/2} h_1^{1/2} f(s)$$

$$= (s \mapsto \sum_{t} h_1^{1/2} s^{-1/2} h_1^{1/2} f(t))$$

So if you take $\varepsilon = -1$, then you have something in convolution form, namely

$$(pf)(s) = \sum_{t} \left( h_1^{1/2} s t^{-1} h_1^{1/2} \right) f(t)$$
In progress made. Given $H, \Gamma, h_i$, etc.

\[ \begin{align*}
H & \xrightarrow{\alpha} \bigoplus S \nu \xrightarrow{\beta} H \\
\xi & \longmapsto (s \mapsto h_i^{1/2} s \xi) \\
\sum \left( s \mapsto h_i^{1/2} s \xi \right) & \xrightarrow{f} \sum s^{-1} h_i^{1/2} f(s)
\end{align*} \]

Composite \( \xi \longmapsto \sum s^{-1} h_i^{1/2} f(s) = \xi \).

\[ \alpha(t \xi) = (s \mapsto h_i^{1/2} s t \xi) \]

Two actions of $\Gamma$ on $L^2(\Gamma, \nu)$:

- Left action \( (tf)(s) = f(t^{-1} s) \)
- Right action \( (tf)(s) = f(st) \)

Use right action $\Sigma = 1$ \( (\alpha(t \xi)) \xi ) \)

Then \( \alpha(t \xi) = h_i^{1/2} s^{-1} \xi \\
(t(\alpha \xi))(s) = (\alpha \xi)(t^{-1} s) = h_i^{1/2} (t \xi)^{-1} \)

Left action \( t \beta f = \sum s h_i^{1/2} f(s) \)

- Right action \( t \beta f = \sum s h_i^{1/2} f(t^{-1} s) = \beta(tf) \)

\[ (\alpha \beta f) = \alpha \left( \sum \frac{1}{t} h_i^{1/2} f(t) \right) = h_i^{1/2} s^{-1} \sum \frac{1}{t} h_i^{1/2} f(t) \]
Confused again.

\[
H \xrightarrow{\beta} L^2(\Gamma, V) \xrightarrow{\alpha} H \\
\alpha \beta \xi = \sum_s s^{\xi} h_1^{1/2} s^{-\xi} \xi = \sum_s s^{\xi} h_1^{1/2} s^{-\xi} \xi = \xi.
\]

\[
\alpha \beta \xi = \sum_s s^{\xi} h_1^{1/2} s^{-\xi} \xi = \sum_s s^{\xi} h_1^{1/2} s^{-\xi} \xi = \xi.
\]

\[
\alpha \beta \xi = (s \mapsto h_1^{1/2} s^{-\xi} \xi) = R_\xi \alpha \xi = R_\xi (s \mapsto h_1^{1/2} s^{-\xi} \xi)
\]

\[
\alpha \beta \xi = \sum_s s^{\xi} h_1^{1/2} \xi (f(st))(s) = \sum_s s^{\xi} h_1^{1/2} f(st) = \sum_s s^{\xi} h_1^{1/2} f(s) = R_\xi f(st)
\]

Conclusion: Let \( \Gamma \) act on \( L^2(\Gamma, V) \) via \((tf)(s) = f(st)\) i.e. \( R_\xi \)

\[
\text{Assembly map.} \quad H, \Gamma, h, h > 0, \quad V = h_1^{1/2} H
\]

\[
L^2(\Gamma, V) \xrightarrow{\beta} H \\
\beta f = \sum_{s \in \Gamma} s^{\xi} h_1^{1/2} f(s)
\]

\[
(\beta f, \xi) = \sum_s \langle f(s), h_1^{1/2} s^{-\xi} \xi \rangle
\]

\[
(\beta^* \xi)(s) = h_1^{1/2} s^{-\xi} \xi
\]

\[
\beta^* \xi = \sum_{s \in \Gamma} s^{\xi} h_1^{1/2} h_1^{1/2} s^{-\xi} \xi = \xi
\]
$t \beta(t) = \sum_s t s^{\epsilon} h_1^{\epsilon} f(s) = \sum_s t s^{\epsilon} h_1^{\epsilon} f(s)$

$= \sum_s t (t^{-\epsilon} h_1^{\epsilon} f(t^{-\epsilon})) = \beta_{t f}(s)$

$\beta_{t f}(s) = \sum_s t^{-\epsilon} s^{\epsilon} h_1^{\epsilon} f(s)$

$\beta_{t f}(s) = \sum_s t^{-\epsilon} s^{\epsilon} h_1^{\epsilon} f(s) = \sum_s \left( \frac{1}{t^{\epsilon}} \right) h_1^{\epsilon} f(ut) = \beta_{R_t f}(s)$

You want to take $\epsilon = -1$ to get convolution form.

$\begin{cases} 
\epsilon = +1 & t \beta(t) = \beta_{h_t f} \\
\epsilon = -1 & t \beta(t) = \beta_{R_t f} 
\end{cases}$

$\beta_{T^s f}(s) = h_1^{\epsilon} s^{-\epsilon} \sum_t s^{\epsilon} h_1^{\epsilon} f(t)$

$= \sum_t s^{\epsilon} h_1^{\epsilon} s^{-\epsilon} h_1^{\epsilon} f(t)$

$= \sum_t \left( h_1^{\epsilon} s t^{-\epsilon} h_1^{\epsilon} f(t) \right)$

$\begin{align*}
\alpha: H & \to \ell^2(\Gamma, V) \\
\alpha x f &= \sum_s s^{-\epsilon} h_1^{\epsilon} f(s)
\end{align*}$

$\begin{align*}
(\alpha x f)(s) &= h_1^{\epsilon} s^{-\epsilon} f(s) \\
(\alpha x f, \xi) &= \sum_s \left( s^{-\epsilon} h_1^{\epsilon} f(s), \xi \right)
\end{align*}$

$\begin{align*}
\alpha x \xi &= \sum_s s^{-\epsilon} h_1^{\epsilon} s^{-\epsilon} f(s) \xi = \sum_s h_1^{\epsilon} s^{-\epsilon} \xi \\
\alpha x \alpha x f &= \left( \alpha \left( \sum_t s^{-\epsilon} h_1^{\epsilon} f(t) \right) \right) = \sum_t h_1^{\epsilon} s t^{-\epsilon} h_1^{\epsilon} f(t)
\end{align*}$

$\begin{align*}
\alpha x \xi &= \xi \\
(\alpha x f)(s) &= h_1^{\epsilon} s^{-\epsilon} f(s) \\
(\alpha x f)(s) &= \sum_t h_1^{\epsilon} s t^{-\epsilon} h_1^{\epsilon} f(t)
\end{align*}$

If $\epsilon = 1$,

$\text{(tf)(s)} = (R_t f)(s) = f(st)$. Then $t \alpha x f = \sum_s s^{-\epsilon} h_1^{\epsilon} f(s)$

$= \sum_s s^{-\epsilon} h_1^{\epsilon} f(st^\epsilon t) = \sum_u u^{-\epsilon} h_1^{\epsilon} f(u t) = \alpha x R_t f$
\( x(t \xi)(s) = h^{\frac{1}{2}} h_t s t \xi = (\xi)(st) = (R_t \xi)(s) \)

\[ x^* = R_{t^*} \]

\[ x t^* = R_t x^* = R_t R_{t^*} \xi = R_{t + a} \xi. \]

so it all seems to work. One way to check this is to look at operators on \( C[\Gamma] \otimes \mathbb{V} \) which commute with \( R_t \) operators. This contains linear comb of operators \( \mathcal{L} \).

\( (L_t \Theta f)(s) = \Theta f(st) \)

and really is the tensor product alg \( C[\Gamma] \otimes \text{End}(V) \).

What's left?

Go back over things. \( H, \Gamma, h_t > 0, \sum h_s = 1 \) on \( H \)

\( V = \mathbb{H}^2 \mathbb{H}, \quad x: H \to \ell^2(\Gamma, \mathbb{V}) \)

\[ x^* = \text{id} \]

\[ (\alpha x^* f)(t) = \sum_{s} \left( h^{\frac{1}{2}} s^{-\frac{1}{2}} h_t \right) f(t) \]

\[ s^{\frac{1}{2}} \]

two possibilities: \( \varepsilon = 1 \)

\[ h_t^2 s^{-\frac{1}{2}} h_t \quad \text{right invariant} \]

\[ \varepsilon = -1 \]

\[ h_t^{\frac{1}{2}} s^{-1} h_t^{\frac{1}{2}} \quad \text{left} \]

Check. \( \varepsilon = 1 \)

\( (x(t \xi))(s) = h^{\frac{1}{2}} h_t s t \xi = (\xi)(st) = (R_t (\xi))(s) \)

\( \varepsilon = -1 \)

\( (x(\xi))(s) = h^{\frac{1}{2}} s^{-1} h_t \xi = (\xi)(t^{-1} s) = (L_t (\xi))(s) \)

\( \varepsilon = 1 \)

\[ (\alpha x^* f)(s) = \sum_{t} \left( h^{\frac{1}{2}} s t^{-1} h_t^{\frac{1}{2}} \right) f(t) \]

\[ (st^{-1})^{-1} = t^{-1} s^{-1} \]

Supports. \( h_s h_t = s h_s t h_t t^{-1} = 0 \)

\[ h_s^{-1} h_t = 0 \Rightarrow h^{\frac{1}{2}} s^{-1} h_t^{\frac{1}{2}} = 0 \]
So you learn a little, namely that the projectors are supported in a lift (resp. right) invariant tube depending on your choice of notation. Let's try to reach Cuntz's notation.

\[ B = \bigoplus \mathcal{E} \times \Gamma \] is a C*-algebra (nonunital) whose reps on a Hilb. space \( H \) (satisfying \( BH = H \)) should be equivalent to a \( \Gamma \)-action 

\[ h_i \in \mathcal{F} \text{ s.t. } \sum h_i = 1 \]

and also \( h_i^* h_i = 0 \text{ for } s \in \mathcal{F} \). You get a projection in \( B \).

What is happening? Hill: reps of \( B \) should be the same as data \( H, \Gamma, h_i \) s.t. \( \sum h_i = 1 \), \( h_i^* h_i = 0 \text{ for } s \in \mathcal{F} \).

Question: What is \( pH \)? Look at a projection in a \( \Gamma \)-graded algebra \( B = \bigoplus_{s \in \Gamma} B_s \)

\[ p = \sum_s p_s \text{, } p_s \in B_s \]

\[ p^2 = \sum_{s,t} p_s p_t = \sum_a \sum_{s,t=a} p_s p_t \]

\[ p_H = \sum_{s,t=a} p_s p_t = \sum_s p_s p_s \]

Amazing how what you don't understand looks to be \( B = \bigoplus \mathcal{E} \times \Gamma \). You believe that a \( \Gamma \)-Hilb. rep. of \( B \) is given by the data \( H, \Gamma, h_i \) such that \( h_i^* h_i = 0 \text{ for } s \in \mathcal{F} \) and \( \sum_s h_i^* h_i = 1 \).

Hence you know that \( \sum_s h_i^* h_i \text{ is idempotent. What is } p \text{ in } H \)?

First step. Let's try to eliminate \( V \). This should be easy because \( V \) can be any subspace between \( h_i^* H \) and \( H \).
Let $H$ be a Hilbert space with unitary action of $\Gamma$ and operator $h^{1/2} > 0$ such that $\sum_{s} s h_{s} s^{-1} = 1$ and $h^{1/2} s h^{1/2} = 0$ for $s \in F$. Let $p = \sum_{s \in F} h^{1/2} s h^{1/2}$.

Then
\[
p^{2} = \sum_{s, t} h^{1/2} s h^{1/2} h^{1/2} t h^{1/2} = \sum_{s, t} h^{1/2} (s h_{s}^{-1}) s h_{s} h^{1/2} t h^{1/2} = \sum_{s, t} h^{1/2} (s h_{s}^{-1}) s h_{s} h^{1/2} t h^{1/2} = \sum_{s} h^{1/2} (s h_{s}^{-1}) s h_{s} h^{1/2} t h^{1/2} = h^{1/2} \sum_{u} u h^{1/2} = p.
\]

$p$ is an element of $E_{\sum_{s} s} \times \Gamma$, hence an operator on $H$. $p$ should be the orthogonal projection onto $V = \frac{1}{h^{1/2}} H$.

\[
p H = h^{1/2} \sum_{u} u h^{1/2} H = h^{1/2} H = \frac{1}{h^{1/2}} H
\]

\[
p h^{1/2} = \sum_{s} h^{1/2} s h^{1/2}
\]

Given $H$, $\Gamma$, $h^{1/2} > 0$ and $h^{1/2} s h^{1/2} = 0$ for $s \in F$.

\[
\sum_{s \in F} h^{1/2} s h^{1/2} = 1
\]

Put $p_{s} = h^{1/2} s h^{1/2}$. Then
\[
\sum_{s \in F} h^{1/2} s h^{1/2} p_{t} = \sum_{s \in F} h^{1/2} s h_{s} h^{1/2} t h^{1/2} = \sum_{s} h^{1/2} s h_{s} t h^{1/2} = h^{1/2} u h^{1/2} = p_{u}
\]

You get a $\Gamma$-graded projection $\pi$ with values in $h^{1/2} \mathbb{L}(H) h^{1/2}$.
Continue with trying to set up a Morita equivalence $E$ generators $h_s$ relations $h_s h_t = 0$ $s \neq t$ \[ h_s = \sum_t h_t h_s = \sum_t h_s h_t \]

$\Gamma$ action $s h_s s^{-1} = h_s$

Set up Morita equivalence for Hilb. representations.

$E_{\Sigma F} \times \Gamma$

$H, \Gamma, h_\Gamma \geq 0$, $\sum s h_s s^{-1} = 1$

Keep to simple situations. Go back to $\mathbb{Z}$ case, you want to understand what happens when you change from $C_c(\mathbb{R})$ to Cuntz's $E_{\Sigma F}$. Recall $E_{\Sigma F} = C_c(\mathbb{R})$

Now you know that $C_c(\mathbb{R}) \times \mathbb{Z}$ is Morita equivalent to $C(\mathbb{R}/\mathbb{Z})$.

Recall $C_c(\mathbb{R}) \times \mathbb{Z} = C_c(\mathbb{R} \times \mathbb{Z}) = C_c(\mathbb{R} \times \mathbb{R} / \mathbb{Z})$

$\leftarrow C_c(\mathbb{R}) \otimes C_c(\mathbb{R}/\mathbb{Z})$. You want a noncommutative generalization. Other points:

$C(\mathbb{R}/\mathbb{Z})$ is unital, so that $C_c(\mathbb{R})$ is a finite proj. right (resp. left) module over the cross product $C_c(\mathbb{R}) \times \mathbb{Z}$.

It should be the image of a projection. You get a projection by choosing $k \in C_c(\mathbb{R})$ s.t. $\pi_k(k) = 1 \in C(\mathbb{R}/\mathbb{Z})$ then you have $P \otimes_A Q \Rightarrow P \uparrow$

$A$ is a summand of $P \overset{Q}{\otimes} A$

so $Q \overset{P \otimes A}{\longrightarrow} P \otimes_A Q$

You see a problem looming in the noncomm. setting: the analog of $A$ is not unital.
Something you've forgotten is when \( \mathbb{I} \) is a flat \( A \)-module. Let \( A \) be a left ideal in \( R \) itself. When is \( R/A \) a flat \( R \)-module? Special case:

When \( R/A \) is proj. \( R \)-module:

\[
0 \rightarrow A \rightarrow R \rightarrow R/A \rightarrow 0
\]

\( \exists x \in R \) such that \( Ax = 0 \), \( x \in A \)

so \( (x-1)x = 0 \) \( \forall \) \( x \neq x^2 \). Put \( e = 1-x \).

Better:

\( \exists e \in A \) such that \( ee = e \quad \forall e \in A \). \( A = Re \Rightarrow Ae = A \)

Summary so far: A left ideal of \( R \) is \( R \)-injective, then \( R/A \) is \( R \)-projective \( \iff A = Re \) \( \exists e \) such that \( ee = e \).

All:

\( \exists e \in \mathbb{I} \) \( \Rightarrow A(1-e) = 0 \).

When is \( R/A \) \( R \)-flat?

when \( \forall q_1 \ldots q_n \exists a \in I \) \( \forall i q_i(1-a) = 0 \)

\( \forall q_1 \exists a \in I \) \( q_1(1-a) = 0 \)

Ind:\n
\( a_i(1-a') = 0 \quad i = 1, \ldots, n-1 \)

Choose \( a'' \) \( a_n(1-a')(1-a'') = 0 \)

then \( a_i(1-a) = 0 \)

\( a = a' + a'' - a' a'' \).

So what's next.
matrix criterion for flatness.

\[ R \rightarrow R^n \rightarrow R^m \]

\[ R \xrightarrow{a} R \xrightarrow{(x_j)} R^m \]

\[ \alpha \in A \implies \exists (x_j) \quad a(x_j) = 0 \]

\[ 1 \equiv \sum x_j \bar{x}_j \mod A \]

\[ \forall a \in A \quad \exists \ x_j \in R, \ ax_j = 0 \]

\[ \exists y_j \in R, \ 1 - \sum x_j y_j \in A \]

\[ \exists a', \ 1 - a' = \sum x_j y_j \]

\[ \implies a(1 - a') = \sum ax_j y_j = 0 \]

So the condition: \( \forall a \exists a' \quad a(1 - a') = 0 \)

Local right identities is equivalent to \( R/A \)

left flat whenever \( A \) embedded in \( R \) as left ideal.

\[ 0 \rightarrow A \rightarrow R \xrightarrow{R/A} \rightarrow 0 \]

\[ 0 \rightarrow M \otimes_A A \rightarrow M \rightarrow M/MA \rightarrow 0 \]

\[ M \text{ finite} \iff M = MA. \]
to construct a Module equivalence. \( E_F \times \Gamma \)

\[ C = E_F \text{ gen. } h_s^{1/2}, \text{ s } \in \Gamma \Rightarrow h_s^{1/2}h_t^{1/2} = 0 \text{ for } s \neq t \]

and \( h_s = \sum_t h_t h_s = \sum_t h_s h_t \), which implies \( E_F \)

has local left + right identities. This implies that a module \( M \) over \( E_F \) is flat \( \Leftrightarrow E_M \) \( M = M \).

i.e. \( \sum h_s M = M \). From \( M = C \otimes \hat{C} \) you should get an action of \( \Gamma \) on \( M \)? Why?

Time to do this carefully. Suppose \( A \) nonunital with \( \Gamma \) action, \( B = A \times \Gamma = A \otimes C[\Gamma] \). Extension:

\[
\frac{B}{A \times \Gamma} \rightarrow \frac{A \times \Gamma}{C[\Gamma]}
\]

semi direct product type extension. flat \( B \)-modules \( = \) \( B \)-flat \( R \)-modules. \( \hat{B} \) has local identities, \( \Rightarrow f,R/B \) is \( R \) flat. OK

\( \Gamma \) module \( M \) is flat \( \Leftrightarrow BM = M \).

Wait: You know \( C = E_F \) has local identities, what about \( B = C \times \Gamma \) ? \( \Rightarrow \sum_s E_s \) finite, so \( \exists q \in (\mathbb{Z})^C \Rightarrow 0. \) Close

so \( a_{\Gamma} \otimes (1-c)E_s = 0 \). Close

so a \( B = C \times \Gamma \) module \( M \) is flat iff \( \sum_s h_s = 1 \) on \( M \)

To see calculation, start with a flat \( B \)-module \( H \), so you have \( H, \Gamma, h_1^{1/2} \), \( \sum h_s h_t = 0 \text{ s } \in F \\text{ and } \sum h_s h_t = 1 \text{ on } M \).

\[
\begin{array}{ccc}
H & \xrightarrow{\nabla} & C[\Gamma] \otimes H \\
\{f : \Gamma \rightarrow H\} & \xrightarrow{\beta(f)} & H \\
\text{fun. support} & & \text{fun. support}
\end{array}
\]

\[
(\beta(f))(s) = h_1^{1/2} f(s)\]

\[
\beta x = 1 \text{ on } H
\]
\[ t \beta(t) = \sum_s s \beta(t) f(s) = \sum_s s h_{1/2} f(s) = \sum_s s h_{1/2} (\beta t f)(s) = \beta t f(s) \]

\[(\alpha \beta f)(s) = h_{1/2}^{-1} \sum_t t h_{1/2} f(t) = \sum_t (h_{1/2}^{-1} h_{1/2}) f(t) \]

for \( s \cdot t \notin F \)

What can you say? \( \alpha \beta \) is a projection on \( C[F] \otimes H \)
in fact on \( C[F] \otimes h_{1/2} H \).

Alternative notation:
\[ \beta(t) = \sum s^{-1} h_{1/2} f(s) \]
\[ t \beta(t) = \sum \frac{1}{s} h_{1/2} f(st) = \sum t s h_{1/2} f(st) = (\beta R_t f)(s) \]

\[(\alpha \beta f)(s) = h_{1/2}^{-1} \sum_t t h_{1/2} f(t) = \sum_t (h_{1/2}^{-1} h_{1/2}) f(t) \]

for \( st^{-1} \notin F \)

\[ f(s) = g(s^{-1}) \]

\[ (\alpha \beta f)(s) = \sum_t h_{1/2} (s^{-1} h_{1/2} f(t) = \sum_t h_{1/2} (st^{-1}) h_{1/2} f(t) \]

\[ (\alpha f)(s) = \sum_s s h_{1/2} f(s) = \sum_s h_{1/2} s^{-1} h_{1/2} f(s) \]

\[ h_{1/2} s^{-1} h_{1/2} \quad h_{1/2} (s t^{-1}) h_{1/2} \]

\[ h_{1/2} s t^{-1} h_{1/2} \quad h_{1/2} (s^{-1} t) h_{1/2} \]

Still confused. Try to focus upon the problem. The idea is that a firm \( \beta = C \times F \)-module \( H \) seems to amount to a type of module, namely, a vector space \( V \) equipped with operators.
\[
p_s = h_{1/2} s h_{1/2} \quad \sum_s p_s p_s^{-1} t = \sum_s h_{1/2} s h_{1/2} t h_{1/2} = h_{1/2} t h_{1/2} = pt.
\]

So there's a certain ring \( R \) given by element \( p_s, s \in \Gamma \) subject to \( p_s = 0 \) \( s \notin F \) and \( \sum_s p_s p_s^{-1} = pt \), \( \forall t \). This ring \( R \) is clearly idempotent as each generator is quadratic expression of the others. So what next??

\[C = \mathbb{E}_F \text{ generators } h_s, s \in \Gamma, \text{ etc.}\]

\( C \) has local left and right ideals, so a \( C \)-module \( H \) from \( \text{iff } CH = H \). Let \( B = C \times \Gamma \), extn.

\[
C \times \Gamma \rightarrow \tilde{C} \times \Gamma \rightarrow \tilde{C} \tilde{\Gamma}
\]

but \( C \times \Gamma \) should have local units also. So form \( H \), \( B \)-module, should amount to a \( \Gamma \)-module \( H \) with \( h_{1/2} \times h_{1/2} = h_{1/2} h_{1/2} = 0 \) \( s \notin F \), \( \sum s h_s s^{-1} = 1 \).

Write this carefully sometime

Anyway you can form

\[
H \xrightarrow{\alpha} C[C \times \Gamma] \times H \xrightarrow{\beta} H
\]

\[
\alpha \left( \tau \right)(s) = h_{1/2} s^{-1} t \frac{3}{2}
\]

\[
\beta \alpha = \text{id}_H
\]

\[
\left( \alpha \beta \right) \left( t \right)(s) = \sum_s h_{1/2} s^{-1} t h_{1/2} f(s)
\]

So you end with the function \( p_s = h_{1/2} s h_{1/2} \)

\[
\sum_s p_s p_s^{-1} t = \sum_s h_{1/2} s h_{1/2} s^{-1} t h_{1/2} = pt
\]

and \( p_s = 0 \) for \( s \notin F \).

Let \( A = \text{F} \) be the alg with a \( \text{homom. } A \rightarrow B \)

Any \( B \)-mod restricts to an \( A \)-mod \( M \) which we replace by \( A \rightarrow A \tilde{M} = h_{1/2} M \)?
You want to go backwards. Let $V$ be an $A$-module, i.e., equipped with operators $p_s \in \text{End} A$. Then an $\text{End} A$-action, you should have an idempotent operator commuting with $\Gamma$-action.

$$C[\Gamma] \otimes V = \{ f : \Gamma \to V \text{ fin. supp} \}$$

Suppose you have $k : \Gamma \to \text{End}(V)$ fin. supp.

$$(k_2 f)(s) = \sum_t k(s^{-1} t) f(t)$$

$$(k_1 k_2 f)(s) = \sum_t k(s^{-1} t) f(u^{-1} t) = \sum_t k(s^{-1} u t) f(t)$$

$$(k_2 k_1 f)(s) = \sum_t k(u^{-1} s^{-1} t) f(t)$$

$$C[\Gamma] \otimes \text{End}(V) \to \text{End}(C[\Gamma] \otimes V)$$

Composition $(k_1, k_2 f)(s) = \sum_t k_1(s^{-1} t)(k_2 f)(t)$

$$= \sum_t k_1(s^{-1} t) \sum_u k_2(u^{-1} t) f(u)$$

So $f \mapsto k_2 f$ is idempotent if $k_1 k_2 = k_2$. 

$$\sum k_1(x) k_2(y) = (k_1 \ast k_2)(s^{-1} u)$$

So now given $V$ with $p_s \in \text{End}(V)$, set $\Gamma$ satis support + idemp. ends, then $$(p \ast f)(s) = \sum_t p(s^{-1} t) f(t)$$
Suppose \( H \) is a \( B = C \times \Gamma \), \( C = \mathbb{E}_\mathcal{F} \) modulo some \( \mathcal{F} \) module \( \mathcal{F} \). Which is then \( \mathcal{F} \)?

Then we get \( H \xrightarrow{\alpha} C[\Gamma] \otimes H \xrightarrow{\beta} H \) where \( \beta f = \sum s h_s f(s) \) and \( \{ f: \Gamma \to H \text{ finite supp.} \} \). \( \alpha f(s) = \sum \frac{h_s}{s} s^{-1} t h_t f(s) \) for \( t \leq s \).

So you go from \( H \) with operators \( s, h_s \), i.e. \( H \) as \( B \) module to \( H \) with operators \( p_s = h_s^{1/2} h_s^{1/2} \) satisfying \( \sum_s p_s p_s^{-t} = \sum_s h_s^{1/2} h_t s^{-1} t h_t^{1/2} = p_t \) for \( t \leq s \).

If \( p_s = 0 \) if \( s \notin \mathcal{F} \). If \( A = P_F \) is the universal ring, then get \( H \) as \( A \)-module.

i.e. have homomorphism \( A \to B \) with \( p_s \to h_s^{1/2} h_s^{1/2} \).

Next you go in opposite direction.)

Start with an \( A \)-module \( V \) and use \( p_s \) as above to use the \( p_s \) to define a projection on \( C[\Gamma] \otimes A \).

Find \( p^2 = p \) and \( \frac{1}{2} p \).

Recover the algebraic viewpoint, where you avoid \( h_s^{1/2} \). \( \sum \frac{h_s}{s} \) for \( s \notin \mathcal{F} \).

Write in terms of \( h_s, h_t \), set \( h_t h_s = 0 \) for \( s \notin \mathcal{F} \).

\( h_s h_s = 0 \) if \( s \notin \mathcal{F} \). \( h_t h_s = 0 \) for \( t \notin \mathcal{F} \). \( h_s = \sum_{t \in \mathcal{F}} h_t h_s \).

\( k = \sum_{t \in \mathcal{F}} h_t \) for \( t \notin \mathcal{F} \).

\( (\alpha \beta f)(s) = k s^{-1} \).

\( (\alpha \beta f)(s) = \sum \frac{h_s}{s} s^{-1} t h_t f(s) \) for \( t \leq s \).

\( (\alpha \beta f)(s) = k s^{-1} f(s) \) for \( s \notin \mathcal{F} \).

\( (\alpha \beta f)(s) = k s^{-1} f(s) \).
\[ \beta \xi = \sum_{s} \frac{h_{s}}{s^{1}} s^{-1} \xi = \sum_{s} h_{s} = \xi \]

\[
(ax^{f})(s) = \sum_{t} k s^{-1} h_{s} f(t) \\
p_{s} = k s^{-1} h_{s} 
\]

so for all \( s \) in \( F \), you have \( p_{s} k = p_{s} \), \( p_{s} = k h_{s} \)

\[ p_{1} = \sum_{s} p_{s} p_{s-1} = \sum_{s} k s h_{s} k s^{-1} h_{s} = \sum_{s} k s h_{s} s^{-1} h_{s} = k h_{1} \]

se F

\[
\beta f = \sum_{s} k s h_{s} f(s) \\
(x_{f})(s) = h_{s} s^{-1} \xi
\]

\[
(ax^{f})(s) = \sum_{t} h_{1} s^{-1} t k f(t) \\
p_{s} = h_{s} k
\]

\[ \sum_{s} p_{s} p_{s-1} = \sum_{s} h_{s} k h_{s} s^{-1} t k
\]

Check things carefully. Recall \( \beta f = \sum_{t} k s h_{s} f(t) \)

\[ p_{s} > 0 \rightarrow s \in F \Rightarrow s \in F \]

\[ \sum_{s} k s h_{s} t^{-1} \]

Suppose \( p_{s} = h_{s} k \)

\[ k = \sum_{t \in F} \beta h_{s} t^{-1} \]

\[ p_{s} = h_{s} \sum_{t \in F} \beta h_{s} t^{-1} = \sum_{t \in F} h_{s} \beta h_{s} t^{-1} \]

\[ k = \sum_{s \in F} h_{s} \\
k h_{1} = \sum_{s \in F} h_{s} h_{1} t^{-1} \\
\]

\[ h_{s} h_{1} \neq 0 \Rightarrow s^{-1} t \in F \]

\[ k h_{1} = \sum_{s \in F} s h_{s} s^{-1} h_{1} \]
Let \( K \neq F \) consider \((\sum_{s \in K} s h_i)^2\)

\[
= \sum_{s \in K} s h_i t h_i = \sum_{s \in K} s h_i s^{-1} s t h_i = \sum_{s \in K} \sum_{t \in K} s h_i s^{-1} t h_i
\]

\[
= \sum_{t \in K} (\sum_{s \in K} s h_i) t h_i = \sum_{t \in K} k t h_i
\]

Start again. \( h_s^t h_t = 0 \) \( s^{-1} t \not\in K \) \( K = K^{-1} \) cont. 0.

\[
\sum_{s \in K} s h_i \sum_{t \in K} t h_i = \sum_{s \in K} \sum_{t \in K} s h_i t h_i = \sum_{s \in K} \sum_{t \in K} s h_i t h_i
\]

\[
= \sum_{s \in K} \sum_{t \in K} s h_i s^{-1} s t h_i = \sum_{s \in K} \sum_{s \in K} s h_i s^{-1} u h_i, \quad u \in F, i \neq 0 \Rightarrow s^{-1} u \in K
\]

\[
= \sum_{s \in K} s h_i s^{-1} u h_i, \quad (s, u) \in K \times KK
\]

\[
- \sum_{u \in KK} (\sum_{s \in K} s h_i) u h_i
\]

\[
= \sum_{u \in KK} k u h_i
\]

\( h_s^t h_t = 0 \) \( s^{-1} t \not\in F \) \( \not\Rightarrow t \in F \)

\[
\sum_{s \in F} s h_i \sum_{t \in F} t h_i = \sum_{(s, t) \in F \times F} s h_i s^{-1} s t h_i
\]
\[
\sum_{s \in F} \sum_{t \in F} \frac{yh_1}{sh_1} = \sum_{s \in F} \sum_{t \in F} \frac{sh_1}{th_1} = \sum_{s \in F} \sum_{t \in F} \frac{1}{t} \Rightarrow t \in F
\]

\[
= \sum_{s \in F} \sum_{u \in F} h_5 \cdot u h_1 = \sum_{u \in F} \sum_{s \in F} h_5 \cdot u h_1 = \sum_{u \in F} k \cdot u h_1
\]

\[
B \subseteq \Gamma, \quad h_5 = sh_1 s^{-1}, \quad h_5 h_t = 0 \quad s \neq t \notin F
\]

\[
h_1 = \sum_{s \in F} h_5 \cdot h_1 = \sum_{s \in F} h_5 h_1
\]

\[
k = \sum_{s \in F} h_5
\]

\[
k h_1 = 0
\]

\[
h_5 = \sum_{s \in F} h_5
\]

\[
k h_1 = 0
\]

\[
p_s = h_1 s k
\]

\[
\alpha f) = \sum_{s \in F} s k h_1 s^{-1} \Rightarrow \sum_{s \in F} h_5 = \Xi.
\]

\[
(\alpha f) = \sum_{s \in F} h_5
\]

\[
\beta_0 = \sum_{s \in F} h_5 k = \Xi
\]

\[
(\alpha f) = \sum_{s \in F} s k = f(t)
\]

\[
p_s = h_1 s k
\]

\[
\sum_{s \in F} p_s = \sum_{s \in F} h_1 s k h_1 s \cdot t h_1
\]

\[
\sum_{s \in F} p_s = h_1 t k = p_t
\]

\[
\Rightarrow s \neq t \in F
\]

---

You are confused. Go over things again. \( C = \sum_{s \in F} \frac{h_5}{s} \neq \frac{h_5}{s} \in \Gamma \) relations \( h_5 h_t = 0 \) for \( s \neq t \notin F \).

\[
\text{Equation: } \sum_{s \in F} h_5 t^{-1} = h_5 s \quad (h_5 = \sum_{s \in F} h_5 t^{-1})
\]

\[\beta = C \times \Gamma, \quad C \] has local identities \( (1 - \sum_{t \in F} h_t) h_5 = 0 \)

\[\text{same should be true for } \beta \]

\[C = \sum_{s \in F} h_5 C \quad \beta = C \times \text{circ} = \sum_{s \in F} h_5 B
\]

**Question:** What's the meaning of \( \sum_{s \in F} h_5 s k = \sum_{s \in F} p_s ? \)
This is a projection \[ \sum_{s, t} h_{s, t, \epsilon} k = \sum_{s, t} h_{s, t, \epsilon} k \]
\[ = \sum_{s, t \neq u} h_{s, t, \epsilon} k = \sum_{s, u} h_{s, u} k = \sum_{u} h_{u} k \]

\[ p = \sum_{s} h_{s} k \quad kp = p \]

\[ p = \sum_{s} h_{s} k \]
\[ \sum_{t \in F} s \in F \]
\[ = \sum_{t \in F} \sum_{u \in F} h_{u} k \]
\[ = \sum_{t \in F} \left( \sum_{s \in F} h_{s} s \right) \]
\[ = \sum_{t \in F} \left( \sum_{s \in F} h_{s} s \right) \]

\[ \sum_{t \in F} h_{t, \epsilon} = \sum_{t \in F} h_{s, t, \epsilon} = \sum_{t \in F} h_{s} = \sum_{t \in F} h_{t} \]

\[ \text{So over the above. Recall } h_{s, t, \epsilon} = 0 \quad s \notin F \]
\[ h_{s} h_{t, \epsilon} = 0 \implies s \notin F \]
\[ h_{s} h_{t, \epsilon} \neq 0 \implies s \notin t \in F \]

\[ h_{s} = \sum_{t \in F} h_{t, s} \]
\[ h_{s} = \sum_{t \in F} h_{t, s} \]
\[ f = \sum_{t \in F} h_{t} \]

\[ H \xrightarrow{\alpha} C[G] \otimes H \xrightarrow{\beta} H \]
\[ (f : \Gamma \rightarrow H) \xrightarrow{\text{fun. supp}} \beta f = \sum_{s} s h_{s, t} \]

\[ \xi \mapsto (\xi')_{s} = h_{s, t, \epsilon} \]

\[ \text{(\beta f)(s)} = \sum_{t} (h_{s, t, \epsilon} f(t) \]

\[ \beta \xi = \sum_{s} s h_{s, t} \]

\[ (\alpha f)(s) = \sum_{t} (h_{s, t, \epsilon} f(t) \]

\[ \sum_{s} p_{s} p_{s, t} = \sum_{s} h_{s} h_{s, t, \epsilon} = h_{t, k} = p_{t} \]

\[ p_{s} p_{s, t} \neq 0 \implies s \in F \text{ and } s', t \in F \]
You would like to show \( h_k = h_1 \)?

\[ p = \sum_{s \in F} h_{sk} \quad p_{h_1} \]

\[ h_k = \sum_{t \in F} h_t h_t = \sum_{s \in F} h_s h_s \]

\[ \sum_{s \in F} h_s h_s = h_1 \]

\[ \sum_{s \in F} h_{sk} = h_{sk} \]

\[ = \sum_{s \in F} h_s \sum_{t \in F} h_t = \sum_{s \in F} \sum_{t \in F} h_s h_t = \sum_{s \in F} \sum_{t \in F} h_s h_t = \left( \sum_{u \in F} h_u \right)^2 \]

\[ = \sum_{s \in F} \sum_{u \in F} h_s h_t = \left( \sum_{s \in F} h_s \right)^2 \]

\[ = \sum_{s \in F} h_s \sum_{t \in F} h_t = \sum_{s \in F} h_s h_t = p \]

\[ k p = \sum_{s \in F} k h_{sk} = \sum_{s \in F} h_{sk} = p \]

\[ p_{h_1} = h_{sk} h_1 = h_{sh_1} \]

\[ p = \sum_{s \in F} h_s h_t \]

Prove that \( k h_1 = h_1 \) implies \( h_{sk} h_1 = h_{sh_1} \)

**Assumptions:** \( h_s h_t = 0 \) for \( s \neq t \) if \( F \) and all \( h_s h_1 = 0 \) for \( s \neq F \).

\[ \sum_{s \in F} h_{sk} = p \]

\[ \sum_{s \in F} h_s h_t = \sum_{s \in F} h_s h_t k = \sum_{s \in F} h_s h_t k = \]
\[ p^2 = \sum_{s, t} h_s t \sum_{u, e} h_u \quad \text{if } h_s t h_u \neq 0 \Rightarrow s e F, u e F, s t u e F \]

To use \( \sum_{s, t} h_s t h_{s t} \):

\[ p^2 = \sum_{t e F} h_t k = p \]

\[ p = \sum_{t} h_t \sum_{u e F} h_u \]

\[ p = \sum_{s} h_s k = \sum_{s} h_s \sum_{t e F} h_t t^{-1} = \sum_{s e F} h_s \sum_{t e F} h_t t^{-1} \]

\[ p = \sum_{s e F} h_s k = \sum_{s e F} \sum_{t e F} h_s h_t = \sum_{s e F} \sum_{t e F} h_s h_{s t} t^{-1} \]

\[ = \sum_{t e F} \sum_{s e F} h_{s t} h_t t^{-1} = \sum_{t e F} \sum_{s e F} h_s h_t t^{-1} = \left( \sum_{s e F} h_s \right)^2 \]

\[ p \sum_{t e F} h_t = \sum_{s e F} \sum_{t e F} h_s k h_t = \left( \sum_{s e F} h_s \right)^2 = p. \]

So you have an op \( \mathbf{g} = \sum_{s e F} h_s \) such that \( p = g^2 \)

\[ 3 \mathbf{g} = \mathbf{g}^2 \]

Roots of \( \lambda^3 - 1 = 0 \) are \( \lambda = 0, 1, -1 \)

If you use \( \mathbf{p} \) the projections to split the space into \( p = 0 \) and \( p = 1 \). Then you expect the choice of \( \mathbf{F} \) may be relevant.

Can you prove \( h_t k = h_t \)?

\[ p g = p \]
\[ p = \sum_{s} h_{sk} = \sum_{s} \sum_{t \in F} h_{sht} t^{-1} \]
\[ = \sum_{t \in F} \sum_{s} h_{sht} t^{-1} = \sum_{t \in F} \sum_{s} h_{sht} = \left( \sum_{s} h_{s} \right)^2 \]
\[ p = g^2 \]
\[ p^2 = q^2 \to q^3 = p^2 = g^2 \]
Characteristic poly is \( \lambda^3 - \lambda^2 = \lambda^2 (\lambda - 1) \) which means a splitting of the module into \( q = 1 \) eigenspace and \( a = 0 \) eigenspace.

\[ p = \sum_{s} k_{sh_{l}} = \sum_{s} \sum_{t \in F} h_{sht} = \sum_{t \in F} \sum_{s} h_{sht} = \left( \sum_{s} h_{sht} \right)^2 \]
\[ = \sum_{t \in F} \sum_{s} h_{sht} \]
\[ p = \sum_{s} k_{sh_{l}} \sum_{t \in F} h_{sht} = \sum_{s} \sum_{t \in F} k_{sh_{l}} h_{sht} \]

\[ \sum_{s} \sum_{t \in F} k_{sh_{l}} h_{sht} = \sum_{t \in F} \sum_{s} k_{sh_{l}} h_{sht} = \sum_{t \in F} \sum_{s} k_{sh_{l}} h_{sht} \]
\[ = \sum_{t \in F} \sum_{s} k_{sh_{l}} h_{sht} \]
\[ p = \sum_{s \in \Gamma} k \sum_{t \in F} \left( h_{st} h_{t^{-1}} \right)^2 t h_{t^{-1}} = \sum_{t \in F} \sum_{s \in \Gamma} t h_{s} t^{-1} h_{t} = \sum_{t \in F} \sum_{s \in \Gamma} t h_{s} t^{-1} h_{t} = \left( \sum_{t \in F} t h_{t} \right)^2 = q^2 \]

\[ p q^2 = \sum_{s \in \Gamma} k s h_{s} \sum_{t \in F} t h_{t} = \sum_{s \in \Gamma} \sum_{t \in F} k s h_{s} t h_{t} \]

\[ \sum_{t \in F} \frac{h_{st} h_{t^{-1}} h_{u}}{s^{-1} u} = \sum_{s \in \Gamma} \sum_{t \in F} k h_{s} s h_{t} \]

\[ = \sum_{u \in \Gamma} \sum_{s \in \Gamma} k h_{s} u h_{t} = \sum_{u \in \Gamma} k u h_{t} = p \]

\[ p q^3 = q^2 = p \]

Let's go on to

\[ H \xrightarrow{\alpha} \Omega \bigotimes H \xrightarrow{\beta} H \]

\[ (s \mapsto f(s)) \]

\[ (\alpha \beta f)(s) = h_{s} s^{-1} f(s) \]

\[ (\alpha \beta f)(s) = h_{s} s^{-1} \sum_{t} k t f(s) = \sum_{t} \left( h_{s} s^{-1} k t f(s) \right) \]

\[ p_{s} = h_{s} k \sum_{t} p_{t} p_{t} = \sum_{t} h_{s} k h_{s} s^{-1} k t = \sum_{s} h_{s} t k \]

\[ p_{s} = h_{s} s \sum_{t \in F} h_{t} = \sum_{t \in F} h_{s} s t h_{t} \]

\[ (x \xi)(s) = h_{s} s^{-1} \xi \]

\[ \alpha t \xi = h_{t} \alpha \xi \]
Let's see if something can be done about Monte Carlo equivalence.

\[ H, \Gamma, h^{1/2}, \quad \sum h_n = 1 \text{ on } H. \]
\[ h_s h_t = \delta_{st} h_1 \]

\[ h_s h_t = \delta_{st} h_1 \]

\[ H \xrightarrow{C(\Gamma \otimes \Gamma)} H \xrightarrow{\beta} H \]

\[ (\times \beta f)(s) = \sum \left( h_s^{-1} s^{-1} t h_t^{-1} \right) f(t) \]

From this data you get

\[ \sum s h_s^{-1} f(s) = \sum s h_s^{-1} f(s) \]

\[ \sum s h_s^{-1} f(s) = \sum s h_s^{-1} f(s) \]

\[ \sum s h_s^{-1} f(s) = \sum s h_s^{-1} f(s) \]

So you find that \( H \) is your module over \( B = \mathbb{E} \sum_F \times \Gamma \). can be reconstructed from itself and the family \( \{ p_s \} \).

This seems too abstract.

So go back to \( \mathbb{Z} \) where you have a geometric picture and look at the for something noncommutative version of the \( \mathbb{Z} \)-tree.

Let's start again with \( \Gamma = \mathbb{Z} \) and \( F = \{ -1, 0, 1 \} \). The aim is to construct a "noncommutative" Monte Carlo.

Let us begin with Hilbert space representations. You have a Hilbert space \( H \) with a unitary operator \( u \) and a positive operator \( h_0 \) such that satisfying a "orthogonality condition": \( h_0 u_n h_0 = 0 \) for \( |n| > 1 \), and a generator condition:

\[ \sum_{n \in \mathbb{Z}} u_{n} h_0 u_{n} H \text{ is dense in } H. \]

No you want \( \forall x \in H \) that \( \sum h_n x = \xi. \)

Partition of unity condition.
Recap. If \( H \) is a unitary \( \delta \)-in \( H \), \( h_n = u^n h_0 u^{-n} \geq 0 \) 

an equivariant partition of 1: \( \sum h_n \Theta = 1 \) in the sense of positive hem. operators, orthogonality:

\[ h_\frac{1}{2} u h_\frac{1}{2} = 0 \quad \text{for} \quad |n| > 2. \]

First idea is GNS. You should be able to reconstruct this data from a positive definite function on \( \mathbb{Z} \) with values in operators on the image \( h_\frac{1}{2} H \).

Let's see what you should know the formulas well. Basic map:

\[
\begin{align*}
H & \rightarrow L^2(\mathbb{Z}, V) \rightarrow H \\
\{f: \mathbb{Z} \rightarrow V\} & \rightarrow \sum_{n \in \mathbb{Z}} u^n h_0 \overline{f(n)} \\
(x f)(n) & = h_\frac{1}{2} u^{-n} x f(n) \quad \text{Then} \quad \beta = \alpha^*, \quad \beta \alpha = \text{id}_H \\
\alpha f & = \sum_{n \in \mathbb{Z}} u^n h_\frac{1}{2} f(n-k) = \sum_{n} u^{k+n} h_0 \frac{1}{2} f(n) = u^k (\beta f).
\end{align*}
\]

Then \( \alpha \beta \) is a projector on \( L^2(\mathbb{Z}, V) \) commuting with translation. Think of \( L^2(\mathbb{Z}, V) \) as \( L^2 \) functions on the circle \( S^1 = \mathbb{Z} \).

What is \( \alpha \beta \)? \( (\alpha \beta f)(n) = \sum_{l \in \mathbb{Z}} h_\frac{1}{2} u^{-n} + h_\frac{1}{2} f(l) \)

and the functional calculus map only \( |k| \leq 1 \). So the \( \alpha \beta \) on \( L^2(\mathbb{Z}, V) \) is the convolution operator with kernel, corresponding to multiplication by the function

\[
\sum_{n \in \mathbb{Z}} (h_\frac{1}{2} u^{-n} + h_\frac{1}{2} u^n) f(n) \quad \text{the transform} \quad \sum e^{2\pi i n l} f(l)
\]
\((p+)(n) = \sum_{\ell \in \mathbb{Z}} p(n-\ell) f(\ell)\)

\(p(n) = h_{0,2}^{-n} h_{0,2}^{-\frac{1}{2}}\)

\[\sum (pf)(n) z^n = \sum_{n, \ell \in \mathbb{Z}} p(n-\ell) z^{n-\ell} f(\ell) z^\ell = \hat{p}(z) \hat{f}(z)\]

You want to understand exactly what arises. Keep close to condition \( F = \mathbb{S} \leq \mathbb{S} \). This should help the algebraic version. Yes, the algebraic version uses \( C[\mathbb{R}] \otimes V \), or \( C[\mathbb{R}] \otimes H \), instead of \( L^2 \).

Somewhere you have to pin down \( V \). Since

\[H \xrightarrow{\alpha} \mathbb{U} C[\mathbb{R}] \otimes V \xrightarrow{\beta} H\]

\[f \mapsto (\alpha \beta)(n) = h_{0,2}^{n/2} u^{-n/2}\]

\[f \mapsto \beta f = \sum_n u^n h_{0,2} f(n)\]

For \( \alpha \) to be defined you need \( h_{0,2}^{1/2} u^{-n/2} \) to be zero for almost all \( n \). For \( \beta \) to be onto you need \( H = \sum_{n \in \mathbb{Z}} u^n h_{0,2} H \).

\[\text{Look at } \alpha \beta \text{ to be only for } u \nmid n \text{ i.e. you take } f \text{ to have support at } k,\]

\[(\alpha \beta f)(n) = \sum_{\ell} (h_{0,2}^{1/2} u^{-n+\ell} h_{0,2}^{1/2}) \delta_{\ell-k} = h_{0,2}^{1/2} u^{-n+k} h_{0,2}^{1/2}\]

which is \( 0 \) only for \( |n-k| \leq 1 \).

So now what happens? The kind of \( \hat{p}(z) = \sum (h_{0,2}^{1/2} u^{-n} h_{0,2}^{1/2}) z^n \) you are getting are Laurent polynomial projectors.