Sept 1. Back to mathematics a little things to do. Review Cuntz construction to see how much you have forgotten. Uniform algs of Roe. General case Γ operating on X loc. comp.

Special case: $\mathbb{Z}$ acting on $\mathbb{R}$. C^* alg not quite $C_c(\mathbb{R}) \rtimes \mathbb{Z}$. ~~Actually you look at continuous~~

$C_c(\mathbb{R})$ continuous functions comp. support on $\mathbb{R}$

You recall looking first at the principal bundle $\mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ constructing a "line bundle" E over the base $\mathbb{R}/\mathbb{Z}$ with fibre the alg group alg $\mathbb{C}[\mathbb{Z}] \simeq \mathbb{C}[z,z^{-1}]$. Get a f.g. projective $C_c(\mathbb{R})$ module over the alg torus $C(\mathbb{R}/\mathbb{Z}) \otimes \mathbb{C}[\mathbb{Z}]$, sections of E over $\mathbb{R}/\mathbb{Z}$ with proper support.

But it turns out that a better gadget is the cross product $C_c(\mathbb{R}) \rtimes \mathbb{Z}$, I mean the algebraic cross product. This nonunital ring should contains an idempotent

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To spend some time on Cuntz(?) constructions. Apparently periodicity (Bott) can be proved, say relies upon an equivalence of some sort between two circles of Voevodsky. Specifically you use an isomorphism between $C(0,1)$ = cont fns on $(0,1)$ vanishing at ∞ and cont fn. on S^1 vanishing at the basepoint. $\mathbb{G}_m$ circle versus the singular curve ∞. To explore these ideas.

Another idea, ~~[illegible]~~ raised by Baum after Cuntz's talk at Durham, ~~[illegible]~~ why does the finite support condition not conflict with the fact that the Bott class on S^2 is not algebraic.

Let's review the situation ~~[illegible]~~ examined after Durham. $\Gamma = \mathbb{Z}$ $E\Gamma = \mathbb{R}$ $B\Gamma = \mathbb{R}/\mathbb{Z}$. The basic idea of the assembly map for Γ. You have a principal bundle $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ with group $\mathbb{Z}$ acting by translations. ~~[illegible]~~ ~~Let's not yet~~

What is essential? Geometrically, the fibre bundle over the circle $\mathbb{R}/\mathbb{Z}$ with fibre $\mathbb{C}[\mathbb{Z}]$ the group ring. ~~[illegible]~~ You get a space of continuous sections which is is $C_c(\mathbb{R})$, cont functions with compact support.

Basic object is $C_c(\mathbb{R})$ or $C(\mathbb{R})$ with $\mathbb{Z}$ acting by translation. ~~[illegible]~~

Cuntz says that $C(\mathbb{R}) \rtimes \mathbb{Z}$ is Morita equiv. to $C(\mathbb{R}/\mathbb{Z})$ quite generally.

Your problem is to ~~[illegible]~~ understand why $C(\mathbb{R})$ is a finite projective $C(\mathbb{R}/\mathbb{Z}) \otimes_{\mathbb{Z}}$ ~~[illegible]~~ module, as well as a finite projective $C(\mathbb{R}) \rtimes \mathbb{Z}$ module. How to make this clear.

$A = C(\mathbb{R})$ cont fns on $\mathbb{R}$ vanishing at ∞

$A \rtimes \mathbb{Z}$ C^*alg obtained by adjoining u unitary to A satisfying $u^n a = \sigma^n(a) u^n$ $\quad (\sigma^n a)(x) = a(x-n)$

[illegible] To understand why

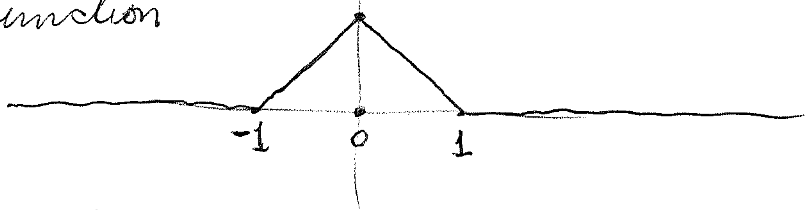
A is a fin. gen. proj $B = A \rtimes \mathbb{Z}$ module

Make things a bit more algebraic: $A = C_c(\mathbb{R})$.

The basic idea here is to use a partition of unity on $S^1 = \mathbb{R}/\mathbb{Z}$. ~~you want A to exist~~

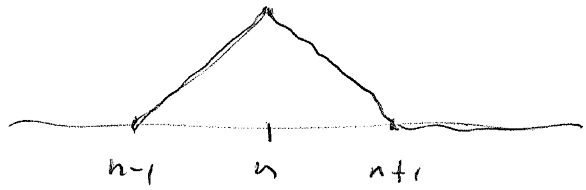
You need to construct B-module maps from A to B, as well as ~~[illegible] module maps~~ B-mod. maps $\tilde{B} \longrightarrow A$, i.e. generators for A.

Now A has a nice generator, namely the function



$h_0(x)$

Why? $h_n(x) = h_0(x-n)$



$1 = \sum h_n(x)$. Roughly $\underbrace{A \cdot \mathbb{C}[\mathbb{Z}]}_{B} h_0(x) = A$.

$B\, h_0(x) =$ [illegible] $A \sum_{n \in \mathbb{Z}} \mathbb{C}\, u^n * h_0 =$ [illegible]

$A \sum_n \mathbb{C}\, h_n = A$ at least in the ~~finite~~ compact support case. So you have $\tilde{B} \longrightarrow\!\!\!\!\rightarrow A$ a B module map. ~~In the other hand you~~

Next you need to produce enough B-mod maps $A \longrightarrow B$. Here you use use partition of 1, i.e.

cont periodic function of x vanishing at coset, say $\mathbb{Z}$. $|\sin(\pi x)|$. What is the point? Let $\phi(x) = |\sin(\pi x)|$, consider $A \xrightarrow{\phi\cdot} A$. This factors thru space of cont functions comp. support vanishing on the subset $\mathbb{Z}$. So what happens? Denote this space by D. It is a B-submodule of A, in fact a $\mathbb{Z}$-graded B-modules. $D = \bigoplus_{n\in\mathbb{Z}} C((n, n+1))$

$= \bigoplus_{n\in\mathbb{Z}} u^n C((0,1))$

Review: First you have A, a suitable ring of functions on $\mathbb{R}$, acted on by translation grp $\mathbb{Z}$. Basic result is M. eq. of $A \rtimes \mathbb{Z}$ with $C(S^1)$. Begin with $A = C(\mathbb{R})$, $B = A \rtimes \mathbb{Z}$ as C^*-algs. A is a left B-module and a right A-module, bimodule.

$$(fu^n)\cdot g\,\mathbb{1} \qquad\qquad \mathbb{1} = \text{constant function } 1.$$

$$\shortparallel$$

$$f(u^n{*}g)\,u^n\mathbb{1} = f\cdot(u^n{*}g)\,\mathbb{1}$$

You get involved with multipliers. Think, think, think.

<u>Smooth model</u> $\quad A =$ Schwarz space on $\mathbb{R}$

Digress for refresher course on Poisson summation fmla.

$$f(x) = \int e^{ix\xi}\,\hat{f}(\xi)\,\frac{d\xi}{2\pi} \qquad \hat{f}(\xi) = \int e^{-ix\xi} f(x)\,dx$$

$$g(x) = \sum_{n\in\mathbb{Z}} f(x-n) \qquad \text{periodic } 1$$

$$g(x) = \underbrace{\sum_{n\in\mathbb{Z}} e^{2\pi i x n}\,\hat{g}(n)} \qquad \hat{g}(n) = \int_0^1 e^{-2\pi i x n} g(x)\,dx$$

$$\hat{g}(m) = \int_0^1 e^{-2\pi i m x} g(x)\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 e^{-2\pi i m x}\, e^{2\pi i n x} \quad ?$$

$$g(x) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x}\,\hat{g}(n) \qquad \hat{g}(n) = \int_0^1 e^{-2\pi i n x} g(x)\,dx$$

$$\hat{g}(m) = \int_0^1 e^{-2\pi i m x} \sum_{n\in\mathbb{Z}} f(x-n)\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 e^{-2\pi i m x} f(x-n)\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_{-n}^{-n+1} e^{-2\pi i m(x+n)} f(x)\,dx$$

$$= \sum_{n\in\mathbb{Z}} \int_{-n}^{-n+1} e^{-2\pi i m x} f(x)\,dx = \int_{-\infty}^{\infty} e^{-2\pi i m x} f(x)\,dx$$

$$= \hat{f}(m)$$

~~Start~~ again:

$f(x)$ Schwartz function.

$$g(x) = \sum_{n\in\mathbb{Z}}' f(x+n) \qquad \text{period } 1.$$

$$g(x) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x}\,\hat{g}(m)$$

$$\hat{g}(m) = \int_0^1 e^{-2\pi i m x} \sum_{n\in\mathbb{Z}}' f(x+n)\, dx$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 e^{-2\pi i m \overset{y-n}{x}} f(\overset{y}{x+n})\, \overset{dy}{dx}$$

$$= \sum_n \int_n^{n+1} e^{-2\pi i m (y-n)} f(y)\, dy$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i m y} f(y)\, dy = \hat{f}(~~2\pi~~ m)$$

Poisson summ. says

$$\boxed{\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} e^{2\pi i m x}\,\hat{f}(2\pi m)}$$

Now you have to place this in the appropriate context

~~Now you have to place~~

Go back to ? $f(x)$ Schwartz (most general is continuous vanishing at ∞). ~~Then~~

Go back to Mor. equiv. of $A\rtimes\mathbb{Z}$ with $C(\mathbb{R}/\mathbb{Z})$

In any case consider ~~[illegible]~~ $\mathcal{S}(\mathbb{R})$ and the map

$$f(x) \longmapsto \sum_n e^{iny} f(x+n) = F(x,y)$$

What are you doing here? Take f on $\mathbb{R}$, restrict to $x+\mathbb{Z}$ to get a sequence $n \longmapsto f(x+n)$ then take F.T. of this sequence to get a function on the circle of $y \in \mathbb{R}/2\pi\mathbb{Z}$

$$F(x, y+2\pi) = F(x,y)$$

$$F(x+1, y) = \sum_n e^{iny} f(1+x+n)$$

$$= \sum_n e^{i(n-1)y} f(x+\cancel{1}+n-\cancel{1}) = e^{-iy} F(x,y)$$

To find a way to organize all of this.

Start again. Poisson sum formula stuff

$$f(x) \in \mathcal{S} \quad \longmapsto \quad \sum_{n \in \mathbb{Z}} f(x+n) = g(x)$$

g is periodic smooth so has F.S. expansion

$$g(x) = \sum_m e^{2\pi i m x} \underbrace{\int_0^1 e^{-2\pi i m y} g(y)\, dy}$$

$$\sum_{n \in \mathbb{Z}} \underbrace{\int_0^1 e^{-2\pi i m y} f(y+n)\, dy}$$

$$\int_n^{n+1} e^{-2\pi i m (y-n)} f(y)\, dy$$

$$g(x) = \sum_m e^{2\pi i m x} \int_{-\infty}^{\infty} e^{-2\pi i m y} f(y)\, dy$$

$$\boxed{\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{m \in \mathbb{Z}} e^{2\pi i m x} \hat{f}(2\pi m)}$$

Best way to view this might be

$$f(x) = \frac{1}{2\pi} \int e^{i\xi x} \hat{f}(\xi)\, d\xi$$

$$\sum_{n \in \mathbb{Z}} f(x+n) = \int \sum_{n \in \mathbb{Z}} e^{i\xi(x+n)} \hat{f}(\xi) \frac{d\xi}{2\pi}$$

$$= \int e^{i\xi x} \underbrace{\sum_{n \in \mathbb{Z}} e^{i\xi n}}_{2\pi \sum \delta(\xi - 2\pi m)} \hat{f}(\xi) \frac{d\xi}{2\pi} = \sum e^{i 2\pi \xi m} \hat{f}(2\pi m)$$

Repeat. $f(x) = \int e^{ix\xi} \hat{f}(\xi) \frac{d\xi}{2\pi}$

$$\sum_{n\in\mathbb{Z}} f(x+n) = \sum_n \int e^{i(x+n)\xi} \hat{f}(\xi) \frac{d\xi}{2\pi}$$

$$= \int e^{ix\xi} \underbrace{\sum_n e^{in\xi}}_{\sum_{m\in\mathbb{Z}} 2\pi\delta(\xi - 2\pi m)} \hat{f}(\xi) \frac{d\xi}{2\pi}$$

$$\boxed{\sum_{n\in\mathbb{Z}} f(x+n) = \sum_{m\in\mathbb{Z}} e^{ix2\pi m} \hat{f}(2\pi m)}$$

To identify $\mathcal{S}(\mathbb{R})$ with smooth sections of "the" line bundle of degree $+1$ over $\mathbb{T}^2$.

$$f(x) \longmapsto \sum e^{+2\pi i n y} f(x - n) = F(x,y)$$

$$\begin{cases} F(x,y+1) = F(x,y) \\ F(x+1,y) = e^{+2\pi i y} F(x,y) \end{cases}$$

What about the converse direction?

Suppose $F(x,y)$ given smooth on $\mathbb{R}\times\mathbb{R}$ satisfying these periodicity. Put

$$f(x,n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-2\pi i n y} F(x,y)\, dy$$

Then $f(x+1,n) = f(x,n-1)$

$$\int e^{-2\pi i x y} F(x,y)$$

$e^{-2\pi i xy} F(x,y)$ periodic in x.

$$e^{-2\pi i xy} F(x,y) = \sum_n e^{2\pi i n x} \int_0^1 e^{-2\pi i n x'} e^{-2\pi i x' y} F(x',y)\, dx'$$

$$F(x,y) = \sum_n e^{2\pi i n y} f(x-n)$$

$$\therefore \quad f(x-n) = \int_0^1 e^{-2\pi i n y} F(x,y)\, dy$$

$$e^{-2\pi i xy} F(x,y) = \sum_n e^{2\pi i (-xy + ny)} f(x-n)$$

$$= \sum_n e^{2\pi i (-x+n) y} f(x-n)$$

$$\int_0^1 e^{-2\pi i xy} F(x,y)\, dx = \sum_n \int_n^{n+1} e^{2\pi i (-x) y} f(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-2\pi i x y} f(x)\, dx = \hat{f}(y)$$

$\int_0^1 F(x,y)\, dy = f(x)$	$\int_0^1 e^{-2\pi i xy} F(x,y)\, dx = \hat{f}(y)$

Next, back to the Morita equivalence $A \rtimes \mathbb{Z} \sim C(\mathbb{R}/\mathbb{Z})$, which perhaps should hold in the smooth situation, i.e. $A = \mathcal{S}(\mathbb{R})$. This should be pretty straight forward. dual pair? So how does this work? $A = Q \otimes_B P$, $B = P \otimes_A Q$

First situation. $C(\Gamma) \rtimes \Gamma$. Instead of $\mathbb{R}$ consider $x + \mathbb{Z}$, form cross product of $C(x+\mathbb{Z})$ (functions on $x+\mathbb{Z}$ vanishing at ∞) with $\mathbb{Z}$ acting by translation. $A = C(\Gamma)$, ~~an A-module~~ as a ring A is a completion of $\bigoplus_{\mu \in x+\mathbb{Z}} \mathbb{C} e_\mu$, where e_μ are orth. idempotents, A ~~lies~~ between the direct sum and direct product. A good (firm!) module for A. is a graded vector space wrt $x+\mathbb{Z}$. A good module for $B = A \rtimes \mathbb{Z}$ is a graded module with compatible translation actions and this reduces to a single component, B Mor eq to $\mathbb{C}$.

How? $A \rtimes \mathbb{Z} = C(x+\mathbb{Z}) \rtimes \mathbb{Z}$ basis $e_\mu z^n$, $A =$ diagonal matrices, rapidly decreasing matrices? kernel $k(m,n)$. So how do we manipulate this?

$$A = C(x+\mathbb{Z}) = \bigoplus_{\mu \in x+\mathbb{Z}} \mathbb{C} e_\mu = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} e_{x+m}$$

$$B = C(x+\mathbb{Z}) \rtimes \mathbb{Z} = \bigoplus_{m,n \in \mathbb{Z}} \mathbb{C} e_{x+m} u^n$$

You have to decide left + right KK stuff uses right B modules

This shouldn't be too essential. A ~~firm~~ good right module M over B should have the form

$$M = \bigoplus M e_{x+m} = M e_x \otimes \mathbb{C}[\mathbb{Z}]$$

$$B = A \otimes \mathbb{C}[\mathbb{Z}] \quad \text{with} \quad u^n e_\mu = e_{\mu+n} u^n$$

$$M = M \otimes_B B = M \otimes_A A \otimes_{\mathbb{Z}} \mathbb{C}[\mathbb{Z}] \quad ?$$

$\underbrace{M \otimes_A A}$ $\otimes \mathbb{C}[\mathbb{Z}]$ with further relations from $\mathbb{Z}$ actions

$= M$

so take

So what is going on? Let M be a right module over B such that $M = MA$

~~Start~~ again. $A = \bigoplus_{m \in \mathbb{Z}} \mathbb{C} e_{x+m}$ e's orth idemp.

$$B = A \otimes \mathbb{C}[\mathbb{Z}] \quad \text{with} \quad u^n e_\mu = e_{\mu+n} u^n.$$

B is a non unital ring which should turn out to be double infin. fin. support matrices. There should be an idempotent e_x in B such that

$$B e_x = A, \quad e_x B = e_x \mathbb{C}[\mathbb{Z}] \quad \text{and}$$

$$e_x B \otimes_B B e_x \overset{OK}{=} e_x B e_x = \mathbb{C}. \qquad B e_x B = B$$

~~$e_{x+m} u^n e_x = e_{x+m} e_{x+n} u^n$~~

$$e_x e_{x+m} u^n = \begin{cases} 0 & m \neq 0 \\ e_x u^n & \end{cases}$$

$$e_{x+m} u^n e_x = e_{x+m} e_{x+n} u^n = \begin{cases} 0 & m \neq n \\ e_{x+n} u^n & m = n \end{cases}$$

$\therefore\ Be_x = \bigoplus_m \mathbb{C} e_{x+m} u^m \qquad e_x B = \bigoplus_n \mathbb{C} e_x u^n$

$$e_x u^n e_{x+m} u^m = e_x e_{x+n+m} u^{n+m} = \begin{cases} 0 & n+m \neq 0 \\ e_x & n+m = 0 \end{cases}$$

$$e_{x+m} u^m e_x u^n = e_{x+m} e_{x+m} u^{m+n} = e_{x+m} u^{m+n}$$

Do this ~~in~~ greater generality. Γ discrete gp. elt $s,t \in \Gamma$. $A = \bigoplus \mathbb{C} e_s$ idemp.
$B = A \rtimes \Gamma$. Let M be a B^{op}-module. Q:
M B^{op} firm $\Leftrightarrow$ M firm over A^{op} ?

$MB = MA$ $\qquad$ M firm.

~~First~~ Point A is an ideal in B? No
point ~~[illegible]~~ maybe is that $AB = B$
$\therefore\ ABM = BM$?

Consider in more generality Γ discrete $A = \bigoplus_{s \in \Gamma} \mathbb{C} e_s$
e_s $s \in \Gamma$ ~~[illegible]~~ comm. idempotents. ~~Then firm A mod~~
Then A-mod M s.t. $AM = M$ same as Γ graded vector spaces. $M = \bigoplus_{s \in \Gamma} M_s$. Take $A \rtimes \Gamma =$
$A \tilde{\otimes} \mathbb{C}[\Gamma]$ basis $e_s t$ $\quad s,t \in \Gamma$. mult.

$e_s t\, e_{s_1} t_1 = e_s e_{ts_1} t t_1$. $\qquad$ $t e_s = e_{ts} t$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t e_{t^{-1}s} = e_s t$

So suppose M a B-module such that $BM = M$, then $ABM = AM$. But $AB = B$ so $AM = BM = M$. $B = \Gamma A = A\Gamma$, so

$$AB = A^2\Gamma = A\Gamma = B \quad \text{and} \quad BA = \Gamma A^2 = \Gamma A = B$$

So let M be a B module such that $BM = M$ equivalently $AM = M$. Then you have $M = \bigoplus_{s\in\Gamma} e_s M$ with $t : e_s M \xrightarrow{\sim} e_{ts} M$

so $M = \bigoplus_{t\in\Gamma} t e_1 M$. The Morita equiv. should go from V over $\mathbb{C}$ to $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} V$ equipped with the natural Γ grading and Γ translations.

Try the left Γ-module $\mathbb{C}[\Gamma]$ and the finite supp dual $C(\Gamma)$ = fns. $\Gamma \to \mathbb{C}$ fin. supp. under mult.

These are rings, whereas for a mor. eq you want a dual pair.

Start with Γ form B the ring with basis $e_s t$ $s, t \in \Gamma$. etc whose firm modules are $M = \bigoplus M_s$ Γ-graded v.s. with comp. Γ action $tM_s \subset M_{ts}$.

Functors are $M \longmapsto e_1 M$ $\quad \mathcal{M}(B) \rightleftarrows \mathcal{M}(\mathbb{C})$

$\mathbb{C}[\Gamma]\otimes V \longleftarrow V$

left B right $\mathbb{C}$ bimodule is $\mathbb{C}[\Gamma]$ $(= Be_1\ ?)$

right B left $\mathbb{C}$ —————— $e_1 B$

Start with a good B-module M

write $M = \mathbb{C}[\Gamma]\otimes e_1 M$

$B = \mathbb{C}[\Gamma]\otimes\mathbb{C}[\Gamma]$

Look at B to see if it arises from a dual pair in an obvious way. ~~...~~ Basis for B of te_s, mult is $te_s\, t_1 e_{s_1} = tt_1\, e_{t_1^{-1}s}\, e_{s_1}$

$$= \begin{cases} 0 & t_1^{-1}s \neq s_1 \\ tt_1 e_{s_1} & t_1^{-1}s = s_1 \end{cases}$$

too confusing

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try right modules

$M = Me_1 \otimes \mathbb{C}[\Gamma]$

$B = Be_1 \otimes \mathbb{C}[\Gamma]$

$\bigoplus_{s\in\Gamma}\mathbb{C}e_s s \quad \bigoplus_{t\in\Gamma}\mathbb{C}e_1 t$

basis B: $e_s t$

$e_s t e_1 = e_s e_t t$

$$= \begin{cases} 0 & s\neq t \\ e_s s & s=t \end{cases}$$

$e_s s e_1 = e_s e_s s = e_s s$

$Be_1 = \bigoplus_{s\in\Gamma}\mathbb{C}e_s s$, $\quad e_s s = s e_1$

$e_1 B = \bigoplus_{t\in\Gamma}\mathbb{C}e_1 t$

$$\langle e_1 t, \overset{e_s s}{\overset{\|}{s}} e_1 \rangle = e_1 t s e_1 = e_1 e_{ts} ts = \begin{cases} 0 & t \neq s^{-1} \\ e_1 & t = s^{-1} \end{cases}$$

Review again! Γ discrete group, two rings

$\mathbb{C}[\Gamma]$ the group ring with basis $\{t \in \Gamma\}$ relation $st = st$

$A = C(\Gamma)$ the ring of functions with fin. support, has basis $\{e_s, s \in \Gamma\}$

relations $e_s e_t = \delta_{st} e_s$

$B = C(\Gamma) \otimes \mathbb{C}[\Gamma]$ basis $e_s t$

relations $e_s t \; e_{s'} t' = e_s e_{ts'} tt'$

$$B = A\Gamma = \bigoplus_{t \in \Gamma} At = \Gamma A \qquad A^2 = A$$

$$BA = B, \quad AB = B$$

$M = MA \implies M = M\Gamma = MA\Gamma = MB$

$M = MB \implies MA = MBA = MB = M.$

Suppose then M is a right B-module $\ni$ $M = MA$ equiv. $M = MB$. Then $M = \bigoplus_s Me_s$ is Γ-graded

and $me_s \longmapsto me_s t$

Better $M = \bigoplus_s M_s$ where $M_s = Me_s$

Moreover $e_s t = t e_{t^{-1}s} \implies (Me_s)t \subset Me_{t^{-1}s}$

$M_s t = M_{t^{-1}s}.$

Look at left B module $M = \bigoplus_s M_s$

$M_s = e_s M$ and $te_s = e_{ts} t$

$tM_s = e_{ts} tM = e_{ts} M = M_{ts}$

What is your goal? $B = A \rtimes \Gamma$ should be a $P \otimes Q$ for some pairing. Working with B-mods

$M \longmapsto e_1 M$ $\qquad$ $\mathcal{M}(B) \rightleftarrows \mathcal{M}(k)$

$\Gamma \times V \qquad V$

$k\Gamma$

$M \longmapsto e_1 M = Q \otimes_B M$

$\mathcal{M}(B) \rightleftarrows \mathcal{M}(k)$

$k\Gamma \otimes V \longleftarrow\!\shortmid V$

$P \otimes_k V$

$P = ?\ Be_1$

$Q = e_1 B$

P has basis $te_1 = e_t t$

Q ———— $e_1 s = s e_{s^{-1}}$

pairing $\langle e_1 s, te_1 \rangle = e_1 ste_1 = e_1 e_{st} st = \begin{cases} 0 & st \neq 1 \\ e_1 & st = 1. \end{cases}$

Start again: Γ disc. gp. $P = k\Gamma$, $Q = k\Gamma$

pairing $Q \otimes_k P \longrightarrow k$

$(q, p) \longmapsto \int qp$

$s \quad t \longmapsto \begin{cases} 0 & st \neq 1 \\ 1 & st = 1 \end{cases}$

$B = \overbrace{Be_1}^{P} \otimes \overbrace{e_1 B}^{Q}$

Try again. Γ discrete group, ~~B finite support~~

$A = \bigoplus \mathbb{C} e_s \qquad e_s e_t = \delta_{st} e_t$ describes Γ graded modules

$B = A \rtimes \Gamma$ describes Γ graded modules with compatible Γ ~~grading~~ Γ-action $sM_t = M_{st}$

$M = \oplus M_s \qquad M_s = e_s M$

$t M_s = M_{ts}$, so that $M = \oplus e_s M \simeq \bigoplus_s s e_1 M$

$\simeq \mathbb{C}[\Gamma] \otimes e_1 M$: This is the basic M. eq.

$$\mathcal{M}(B = \oplus \mathbb{C} e_s \rtimes \Gamma) \underset{\longleftarrow}{\overset{M \mapsto e_1 M}{\longrightarrow}} \mathcal{M}(k)$$

$$\mathbb{C}[\Gamma] \otimes V \longleftarrow\!\shortmid V$$

$M \mapsto e_1 B \otimes_B M \qquad e_1 B$ basis $e_1 s$, $s \in \Gamma$

$e_1 B = \oplus e_1 e_s \mathbb{C}[\Gamma] = e_1 \mathbb{C}[\Gamma]$

$B e_1 = (\oplus \mathbb{C} e_s t) e_1 \qquad e_s t e_1 = e_s e_t t = \begin{matrix} 0 & s \neq t \\ e_s s & s = t \\ \parallel & \\ s e_1 & \end{matrix}$

So $e_1 B$ has basis $e_1 s$, $s \in \Gamma$

$B e_1$ ——— $t e_1$, $t \in \Gamma$

pairing $\quad e_1 B \times B e_1 \longrightarrow k e_1$

$e_1 s, t e_1 \longmapsto e_1 s t e_1 = e_1 e_{st} st = \begin{cases} 0 & st \neq 1 \\ e_1 & st = 1 \end{cases}$

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basis $\quad t e_1 s = t s e_{s^{-1}} = e_t ts$

So what do you have? $\quad B = (\oplus \mathbb{C} e_s) \tilde{\otimes} (\oplus \mathbb{C} t)$

$t e_s = e_{ts} t \qquad e_1 B \simeq \bigoplus_t \mathbb{C} e_1 t \qquad B e_1 = \langle e_s t e_1 \rangle$

$= \oplus \mathbb{C} e_s s e_1$

$B e_1$ basis $e_s s = s e_1 \qquad \langle e_1 t, s e_1 \rangle = \delta_{t, s^{-1}} e_1$

$e_1 B_1$ basis $e_1 t = t e_{t^{-1}}$

Doesn't get better. Next go back to $\Gamma = \mathbb{Z}$

acting on $A = C_c(\mathbb{R})$

$B = Be_1 \otimes e_1 B$

Be_1 basis $te_1 = e_t t$

$e_1 B$ — $e_1 s$

$e_s s \; e_t t$

~~$s e_1 t e_1 = s e_1 e_t t$~~

Can you identify the B modules Be_1 and $A = \bigoplus \mathbb{C} e_s$?

Think of A as the ring of functions on the group Γ

a left B-module (finite) is a ~~graded~~ Γ-graded module with compatible Γ action, a vector with Γ action, Γ grading and rule $tM_s = M_{ts}$. e.g. $\mathbb{C}[\Gamma] = \bigoplus \mathbb{C}s$

what about $A = \mathbb{C} e_s$

~~[illegible]~~ $\mathbb{C}[\Gamma]$ Still puzzled.

$$A = C(\mathbb{Z}) = \bigoplus_{s \in \Gamma} k e_s$$

$$B = A \rtimes \Gamma = \bigoplus_{s,t} k e_s t \qquad e_s t \, e_{s_1} t_1 = e_s e_{ts_1} t t_1$$

$$B = A \rtimes \Gamma = \bigoplus_{s \in \Gamma} A s \qquad s a = {}^s a s, \quad s\, {}^{s^{-1}} a = a s$$

$B e_1$ $\quad e_s t e_1 = e_s e_t t = \begin{cases} 0 & s \neq t \\ e_s t & s = t. \end{cases}$

~~$e_s t e_1$~~

$$e_t t e_1 = {}^t e_1 e_1 = te_1 = e_t t$$

Be_1 has basis $e_t t = te_1$ $\quad t \in \Gamma$

$e_1 B$ ~~[illegible]~~ has basis $e_1 t$ $\quad t \in \Gamma$

multiplication.

$e_s t\, e_{s_1} t_1$ is zero unless $s = t s_1$

in which case the prod. is $e_s t t_1$

$$B = \bigoplus_{s\in\Gamma} k e_s \otimes \bigoplus_{t\in\Gamma} kt \qquad \{e_s t \text{ basis for } B\}$$

$$t e_s = e_{ts} t$$
$$t e_{t^{-1}s} = e_s t$$

$$B e_1 = \bigoplus_{t\in\Gamma} k\, t e_1$$

$$\langle e_1 s, t e_1 \rangle = \begin{cases} 0 & st \neq 1 \\ e_1 & st = 1. \end{cases}$$

$$e_1 B = \bigoplus_{s\in\Gamma} k\, e_1 s$$

Check. $(s e_1 t)(s' e_1 t') =$

$$f(x,y) * g(x',y') = \int f(x,y)\, g(-y, y')$$

To find a better version. Go back to $f(x) \in A$

$A \rtimes \mathbb{Z}$. $\qquad (f u^m)(g u^n) = f(x) g(x+m)\, u^{m+n}$

Anyway what happens. Look at $A = C_c(\mathbb{R})$, form $B = A \rtimes \mathbb{Z}$, try to understand Meq. ~~What does~~ You can do something locally over $\mathbb{R}/\mathbb{Z}$. For each coset $x + \mathbb{Z}$ you have ~~yield~~ ? Better, look at a small interval K around $x + \mathbb{Z}$ i.e. $[x-\varepsilon, x+\varepsilon] + \mathbb{Z}$ $\quad 0 < \varepsilon < \frac{1}{2}$ then A is replaced by A_K which ~~still~~ has both

$\mathbb{Z}$ action and $\mathbb{Z}$-grading. ~~You seem to get~~ ~~[illegible]~~

~~Anyway the point is $\mathbb{Z}$ graded~~

~~If you~~

~~The idea is~~ Aim to understand $A \rtimes \mathbb{Z}$

$A = C_c(\mathbb{R})$, why $A \rtimes \mathbb{Z}$ is Meq to $C(S^1)$

First step is case of $A = C_c(x+\mathbb{Z})$ functions finite support on the coset. You have $\mathbb{Z}$ action on A, what you need is the complementary grading

---

Aim to learn about $A \rtimes \mathbb{Z}$, $A = C_c(\mathbb{R})$

$M = C_c(\mathbb{R})$ with $\mathbb{Z}$ acting by translation. ~~this is not a graded~~ You want a compatible $\mathbb{Z}$-grading. For example look at

$M_{x+\mathbb{Z}} = \text{Ker}\{C_c(\mathbb{R}) \longrightarrow C_c(x+\mathbb{Z})\}$ i.e $f \in C_c(\mathbb{R})$ $f(x+n) = 0$ $\forall n$. This a subspace of $C_c(\mathbb{R})$ stable under A mult and $\mathbb{Z}$ translation. It has ~~a nat~~ an obvious $\mathbb{Z}$-grading compatible with translation. When you have such a grading, you have an induced module of the form $\mathbb{C}(\mathbb{Z}) \otimes V$, where V is a $\mathbb{C}(S^1)$ module.

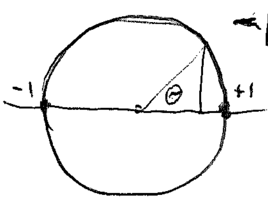
This is not very clear language. But you should be able to make it more precise. Geometrically ~~it is over~~ things can be viewed over the circle. You have a simple space S^1 covered by closed intervals and everything is nice over S^1.

perhaps sheaf theory ideas are useful.

What is your aim? Understand the ~~nature of~~ Morita equivalence $C_c(\mathbb{R}) \rtimes \mathbb{Z} \sim C(\mathbb{R}/\mathbb{Z})$.

List ideas. ~~Picture of modules over $C_c(\mathbb{R}) \rtimes \mathbb{Z}$~~

Involves covering the circle, partition of unity.



$-1 \leq \cos\theta \leq 1$. rings

cont functions of $\mathbb{R}$, comp. supp, vanishing on $2\pi\mathbb{Z}$

$2\pi\left(\mathbb{Z}+\frac{1}{2}\right)$

Start again: To understand well the M. eq. of $C_c(\mathbb{R}) \rtimes \mathbb{Z}$ and $C(\mathbb{R}/\mathbb{Z})$.

Put $B = C_c(\mathbb{R}) \rtimes \mathbb{Z} = \left\{ \sum_{n\in\mathbb{Z}} f_n u^n \,\middle|\, f_n \in C_c(\mathbb{R}) \right.$

$$f_m u^m \, g_n u^n = f_m (\sigma^m * g_n) \, u^{m+n}$$

where $(\sigma^m * g_n)(x) = g_n(x-m)$. Is there a nice way to write this.

~~$F(x,m) * G(x,n) = \sum_m F(x,m) G(x-m,n)$~~

$F : \mathbb{R} \times \mathbb{Z} \longrightarrow \mathbb{C}$

$b = \sum F(\cdot \qquad ?$

An element of B has the form $\sum_m f(x,m) u^m$ (variable: x)

$$\sum_m f(x,m) u^m \sum_n g(x,n) u^n = \sum_{m,n} \Big(f(x,m) g(x-m,n) \Big) u^{m+n}$$

$$\sum_m \sum_n f(x,m)\, g(x-m, n-m)\, u^{m+n-m}$$

$$= \sum_n \left(\sum_m f(x,m)\, g(x-m, n-m) \right) u^n$$

$$\sum_m f(x, m+n)\, g(x-m-n, n-m-n)\, u^n$$

$$\|$$

$$\sum_m f(x, -m+n)\, g(x+m-n, m)\, u^n \qquad \square$$

yes this

Not any clearer.

---

maybe a good idea is to look at an equivalence of groupoids and to show the corresponding algebras are Morita ~~[illegible]~~ equivalent. So let's consider our two groupoids, the first ~~is the~~ has object set $\mathbb{R}$ and maps ~~existing~~ given by the ~~at~~ action of $\mathbb{Z}$ by translation, the second has S^1 for object set and only the identity morphisms. So now all you have to do is to spell out what ~~happens~~ what the usual –

Return to the idea that an equivalence between groupoids gives rise to a Morita equivalence between the corresponding algebras. You want to examine a top. situation, the topological groupoids given by a discrete group acting on a top space (locally compact). Specifically $\mathbb{R}$ with $\mathbb{Z}$ acting by translations.

What is your viewpoint? You want to recover the old stuff you learned from Graeme about equivariant cohomology, possibly generalized cohomology. Example: Given a partition of unity φ_α over X, ~~you form the~~ No, given a covering of X you form the geometric realization of the nerve of the covering. Nerve of the category of finite intersections U_I, more precisely, you form $\coprod U_\alpha = Y$ and take the geometric realization of the simplicial space $X \leftarrow Y \Leftarrow Y \times_X Y \Lleftarrow$; (there is some confusion about ordered simplices here).

What roughly happens in the special case of $\mathbb{Z}$ acting on $\mathbb{R}$. The topological groupoid has $O = \mathbb{R}$ and 1-morphism space $\mathbb{R} \times \mathbb{Z}$, so the nerve is

$$\mathbb{R} \Leftarrow \mathbb{R} \times \mathbb{Z} \Lleftarrow \mathbb{R} \times \mathbb{Z} \times \mathbb{Z}$$

i.e. $\mathbb{R} \times^{\mathbb{Z}} E\mathbb{Z}$. What viewpoint? You want ~~to restrict attention~~, to avoid simplicial objects. In Tohoku Grothendieck looked at Γ sheaves in the case of a discrete group action.

Think about nice Γ-sheaves. Groth viewpoint would be to look at all $\mathbb{Z}$-sheaves on $\mathbb{R}$, then use descent to get an equivalence with all sheaves on the quotient $\mathbb{R}/\mathbb{Z}$. You want something close to C^* algebras, better, you want to arrive at a Morita equivalence between C^*-algebras.

There is a problem getting started. You need the locally compact space — commutative C^*-algebra equivalence due to Gelfand.

Assoc. to a loc. comp. space X is a C^*-algebra $C(X)$, and when a disc. Γ acts on X (properly?) there is a crossproduct C^*-alg $C(X) \rtimes \Gamma$. You approach Morita equivalence in this situation encumbered by your firm modules. Now a C^*-alg A is flat both on left and right.

$$0 \longrightarrow A \longrightarrow \tilde{A} \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Tor}_1^{\tilde{A}}(\mathbb{Z}, M) \longrightarrow A \otimes_A M \longrightarrow M \longrightarrow \mathrm{Tor}_0^{\tilde{A}}(\mathbb{Z}, M) \longrightarrow 0$$

$$\mathrm{Tor}_0^{\tilde{A}}(\mathbb{Z}, M) = M/AM$$

You have M flat $\Rightarrow$ (M firm $\Longleftrightarrow$ $M = AM$). So when dealing with a C^*-alg you might want to ~~restrict~~ attention to flat firm modules, e.g. the space of sections of a vector bundle over a compact space.

Question: Is there an analog of sections of a vector bundle, vanishing at ∞? No you need something amounting to a compactification of the vector bundle. There might be lots of possibilities here.

Go back to $\mathbb{R}$ with $\mathbb{Z}$-translation action. Let L be an equivariant line bundle. Then $L \to \mathbb{R}$ descends to $L/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is a line bundle over the circle. If you are using complex bundles, then $L/\mathbb{Z}$ can be trivialized, so

$$\begin{array}{ccc} L & \longrightarrow & L/\mathbb{Z} \\ \uparrow\!\downarrow & & \uparrow\!\downarrow \\ \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} \end{array}$$

so it seems you get [redacted] a nonvanishing section of L over $\mathbb{R}$ which is equivariant for the $\mathbb{Z}$-action.

How to get started? You want to understand the Morita equivalence between the C^*-algs $C(\mathbb{R}) \rtimes \mathbb{Z}$ and $C(\mathbb{R}/\mathbb{Z})$. You have a ~~vague~~ rough idea about ~~modules~~ geometric modules for these algebras. In fact you can localize over the circle. For example if I is a closed arc of the circle then

Start again. You want to establish Morita equivalence between $C(\mathbb{R}) \rtimes \mathbb{Z}$ and $C(\mathbb{R}/\mathbb{Z})$. What's the idea? To proceed geometrically. Because $\mathbb{Z}$ acts freely on $\mathbb{R}$ with quotient the circle, ~~the~~ descent philosophy says $\mathbb{Z}$-equivariant objects over $\mathbb{R}$ should equivalent to objects over $\mathbb{R}/\mathbb{Z}$. Object here initially means sheaf, actually some sort of Mayer-Vietoris or glueing property. Ultimately we want modules over the rings in question — a sort of affine alg. geom.

Look at the effect of the map $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ on modules, e.g. sections of the trivial line bundle. A vector bundle can be understood as a sheaf. You should replace any vector bundle by its module of global sections (vanishing at ∞?). A vector bundle E over $\mathbb{R}/\mathbb{Z}$ is equiv. to a fg proj $C(\mathbb{R}/\mathbb{Z})$ module. When you lift the v.b. E back to $\mathbb{R}$ you get an $\mathbb{Z}$-equivariant v.b. (π^*E) over $\mathbb{R}$. Because of your locally compact viewpoint, you don't want all sections of π^*E, rather only those vanishing at ∞. This seems to have an intrinsic meaning, because you choose a herm. metric on E over the circle, any two are bounded by each other, so the pull back metrics are equivalent, etc. — Thus you get a pull-back functor for v.b.

Review: You consider the etale top groupoid with object space $\mathbb{R}$ and morphisms given by translation action of $\mathbb{Z}$ on $\mathbb{R}$.

Review. ~~You have~~ You have

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$

$\mathbb{Z}$ acts freely on the space $\mathbb{R}$ and the ~~quotient~~ is the circle $\mathbb{R}/\mathbb{Z}$. Philosophy of descent ~~tells us~~ ~~that~~ says that objects over $\mathbb{R}/\mathbb{Z}$ should be equivalent to $\mathbb{Z}$-equivariant objects over $\mathbb{R}$. You are interested in (certain) modules - these are ~~the~~ the objects of interest.

What kind of modules. This morning on your walk you remembered Kasparov's Hilbert C^*-modules over a C^* algebra A. E is a right A-module equipped with a pairing (ξ, ξ') from $E \times E$ to A which is sesquilinear $(a\xi, \xi' a') = a(\xi, \xi')a'$, pos. $(\xi,\xi) \geq 0$, completeness of E wrt the norm $(\xi,\xi)^{1/2}$

add $(\xi', \xi) = (\xi, \xi')^*$ in A.

Example: $A = C(X)$, $E = C(X, V)$, if V is a vector bundle with hermitian product over X. Notice that the ~~pairing of the~~ inner product on V allows you to define continuous sections vanishing at ∞ on X.

Note that two herm. inner products on a vector bundle V are related by a positive def. hermitian operator ~~locally~~ bounded locally, ~~Recall~~ which has a unique pos. sqrt, which gives an isom of hermitian v.b's. Yesterday's observation: the pull back of a hermitian v.b. on $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}$ has a unique up to isomorphism herm. product, so you get a definite $C(\mathbb{R})$-module of sections over $\mathbb{R}$ vanishing at ∞.

Aim now to understand the equivalence between modules over $C(\mathbb{R}) \rtimes \mathbb{Z}$ and modules over $C(\mathbb{R}/\mathbb{Z})$.

Module = Hilbert C^*-module probably works.

At this point you have the following picture:

1) A geometric situation consisting of $\mathbb{Z}$ freely acting on $\mathbb{R}$ with quotient $\mathbb{R}/\mathbb{Z}$, and a descent equivalence between equivariant wrt $\mathbb{Z}$ hermitian v.b. over $\mathbb{R}$ and herm. v.b. over $\mathbb{R}/\mathbb{Z}$

2) A module picture of these herm. vb's:
Hilbert C^*-modules over $C(\mathbb{R}) \rtimes \mathbb{Z}$ associated to equiv. herm. vb's over $\mathbb{R}$ wrt $\mathbb{Z}$ action
Hilbert C^*-modules over $C(\mathbb{R}/\mathbb{Z})$ assoc. to herm vb's over $\mathbb{R}/\mathbb{Z}$.

Our aim is to translate the descent equivalence into a Morita equivalence between $C(\mathbb{R}/\mathbb{Z})$ and $C(\mathbb{R}) \rtimes \mathbb{Z}$. You need appropriate bimodules and a tensor product operation.

Consider pull back wrt $(\mathbb{R}, \mathbb{Z}) \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$

~~Take the true~~ The pull back functor takes V to $C(\mathbb{R}, \pi^*V)$ = space of continuous sections of π^*V vanishing at ∞. Now V is summand of a trivial v. bundle, so the bimodule should be the space of sections vanishing at ∞ of the triv. v.b. over $\mathbb{R}$, which is $C(\mathbb{R})$.

What is the ~~other~~ descent functor?

Given W a $\mathbb{Z}$-equivariant vb/$\mathbb{R}$, get module $C(\mathbb{R}, W)$ of sections of W vanishing at ∞. Now this isn't correct because it ignores the hermitian product. Take W to be ~~the~~ trivial v.b. over $\mathbb{R}$ with fibre W_0 and ~~let~~ ~~$\mathbb{Z}$ act on~~ define the $\mathbb{Z}$ action on $W = \mathbb{R} \times W_0$ to be the ~~product~~ action where $\mathbb{Z}$ acts on $\mathbb{R}$ via translation and acts on W_0 via an elt $g \in GL(W_0)$. ~~If g is not conjugate to a unitary~~. Take $W_0 = \mathbb{C}$ so $g = g \in \mathbb{C}^\times$. You want g to preserve a herm. metric ?

Start again. You want to start with the module $C(\mathbb{R}, W)$ and then recover $C(\mathbb{R}/\mathbb{Z}, W)$, where W is a vector bundle over $\mathbb{R}/\mathbb{Z}$. Use the fact that v.b.'s over $\mathbb{R}/\mathbb{Z}$ are trivial.

You want a way to recover $C(\mathbb{R}/\mathbb{Z})$ from $C(\mathbb{R})$ considered in the natural way as $C(\mathbb{R}) \rtimes \mathbb{Z}$-module and you want the recovering process to commute with multiplication by elements of $C(\mathbb{R}/\mathbb{Z})$. ~~and~~ ~~further~~

The basic idea should involve taking $f \in C(\mathbb{R})$ and summing the translates of f with respect $\mathbb{Z}$.

$$f(x) \longmapsto \sum_{n \in \mathbb{Z}} f(x+n)$$

This is not defined unless f decays sufficiently at ∞.

What is ~~the correct mechanism for~~ a good way to make sense of this. Try $C_c(\mathbb{R})$, i.e. compactly supported functions, also multipliers

~~~~ Use a partition of unity on the circle
~~~~

Lets work with functions on the circle ~~vanishing~~ ~~outside~~ supported in a proper closed arc

Better begin with the geometric picture.

You have this principal $\mathbb{Z}$-bundle $\mathbb{R} \xrightarrow{\pi} \mathbb{R}/\mathbb{Z}$ which is not trivial. Locally it is trivial;

$\pi^{-1}(U) \simeq U \times \mathbb{Z} \qquad U < \mathbb{R}/\mathbb{Z}$.

You need the right start. ~~Consider what~~

You work over the circle $\mathbb{R}/\mathbb{Z}$, with a vector bundle of some type, so first look at what happens over a point. A point of $\mathbb{R}/\mathbb{Z}$ is a coset $y + \mathbb{Z} \subset \mathbb{R}$. You have the ring $\mathbb{C} = C(pt\ \pi(y))$ and the ring $C(y+\mathbb{Z}) \rtimes \mathbb{Z}$

You consider: functions vanishing at ∞ on the coset $y+\mathbb{Z}$. You should first understand the Morita equivalence between $C(y+\mathbb{Z}) \rtimes \mathbb{Z}$ and $\mathbb{C} = C(\pi(y))$

∴ You want to understand why $C(y+\mathbb{Z}) \rtimes \mathbb{Z} \simeq \mathcal{K}$

If BF 1493 = £28.00
then BF 2880 = £54.01

	43.80
	28
	28
	54.01
	£ 153.81

$\frac{450}{495} = \frac{x}{310.68}$ $\quad x = 282.44$

bank deposit 169.00

I received 210
40
250

$\frac{310.68}{450}$ ~~...~~ $\simeq .6904 \times 30 = 20.712$

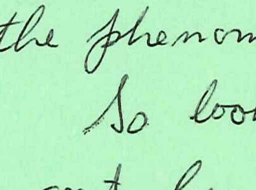
$\times 31 \quad 21.40$

So you need to understand why the C^*-alg $C(y+\mathbb{Z}) \rtimes \mathbb{Z}$ is isom to $\mathcal{K}$. Can suppose $y = 0$. $C(\mathbb{Z})$ = ring of functions on $\mathbb{Z}$ vanishing at ∞, ~~it has~~

You want to use that a C^*alg is a norm closed $*$ subalg of $\mathcal{B}(H)$ for some Hilbert space H. In the ~~present~~ case of [redacted] $C(\mathbb{Z})$, you have [redacted] $H = \ell^2(\mathbb{Z})$ and $C(\mathbb{Z})$ is the norm closure of the mult. operators by functions on $\mathbb{Z}$ of compact support. When you take $-\rtimes\mathbb{Z}$, you will get the smallest norm closure $*$alg. containing $C_c(\mathbb{Z})$ and the translation operators, and $C_c(\mathbb{Z})\rtimes\mathbb{Z}$ should be all finite support matrices on $\ell^2(\mathbb{Z})$ wrt obvious basis.

What does one know about $C(\mathbb{Z})\rtimes\mathbb{Z}$? It is graded wrt the group $\mathbb{Z}$, so the circle group acts as automorphisms.

I think you are worrying too much about C^*-algs. Most of the phenomena to be understood should be algebraic. So look at $A_c = C_c(\mathbb{R})$ the nonunital alg of compactly supp. cont. fns on $\mathbb{R}$. Consider the algebraic crossproduct $A_c\rtimes\mathbb{Z}$ which is the tensor product $A_c\otimes\mathbb{C}[\mathbb{Z}] = \bigoplus_{n\in\mathbb{Z}} A_c u^n$. I think it should be true that $A_c\rtimes\mathbb{Z}$ is Morita equivalent to $C(\mathbb{R}/\mathbb{Z})$. Why? There is a Mayer-Vietoris description of A_c. Take an open covering of the circle $\mathbb{R}/\mathbb{Z}$ by two open arcs



. Upshot will be that $A_c = A_c^+ + A_c^-$. $\mathbb{R}/\mathbb{Z} = I \cup J$ I, J contractible arcs. A_c Notation?

Best (seems to) take two different points $P, Q \in \mathbb{R}/\mathbb{Z}$ to let I, J be the complement of P, Q resp. Then $A_c^{\pm}$ resp is ideal in A_c of cont functions van. on the cosets $\pi^{-1}(P), \pi^{-1}(Q)$

~~As the best part seems~~ Now you need explicit Morita equivalence between $A_c^{\pm} \rtimes \mathbb{Z}$ and $C(S^1)^{\pm}$. Reason true is that ~~...~~ the principal bundle $\pi: \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ becomes trivial off P and off Q.

Review: ~~a~~ Consider $\pi: \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ principal bundle. to translate descent equivalence into Morita equiv.

Objects: Top group ~~...~~ $(\mathbb{R}, \mathbb{Z})$ given by $\mathbb{R}$ with $\mathbb{Z}$ action. Top gpd $\mathbb{R}/\mathbb{Z}$ with identity maps. Have continuous functor

$(\mathbb{R}, \mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z}$ which is an equivalence of some ~~...~~ sort — this is descent. Spaces over $\mathbb{R}/\mathbb{Z}$ e.g. sheaves ~~...~~ are equivalent to $\mathbb{Z}$-equivariant ~~...~~ spaces over $\mathbb{R}$. Notice this is not an equivalence of top cats because there is no functor going the other way. You can't map $\mathbb{R}/\mathbb{Z}$ into $\mathbb{R}$ appropriately. This is not said properly, but the point is that you need to introduce a covering.

Related ideas. C small category, $\mathrm{Hom}(C, \mathcal{Ab}) =$ the cat of functors $F: C \longrightarrow \mathcal{Ab}$, ~~This category~~ call them C-modules. If $\mathrm{Ob}(C)$ finite then you get a ring

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You've gotten a small idea, namely, there's a similarity between graded vector spaces ~~modules~~ $M = \bigoplus_{i \in I} M_i$ and modules over ~~a~~ a commutative C^*-algebra $C(X)$.

Why is this relevant? ~~...~~ ring with many objects (Mitchell?), ~~As~~ small additive category is a ring with many objects. ?

Let $\mathcal{C}$ be a small category, define $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{Ab})$ to be the category of $\mathcal{C}$ modules.

If $\mathcal{C}$ and $\mathcal{C}'$ are equivalent categories then $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{Ab})$ and $\underline{\mathrm{Hom}}(\mathcal{C}', \mathcal{Ab})$ are equivalent abelian categories.

Now $\underline{\mathrm{Hom}}(\mathcal{C}, \mathcal{Ab})$ has the form $\mathcal{M}(A_{\mathcal{C}})$ where $A_{\mathcal{C}}$ is an idempotent ring. $A_{\mathcal{C}}$ is the arrow ring of $\mathcal{C}$. basis = the arrows in $\mathcal{C}$ ~~relations~~ $f \in \mathcal{C}$ let $[f]$ = corresp. basis elt of $A_{\mathcal{C}}$ | relations

$$[f][g] = \begin{cases} [fg] & \text{if } fg \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

(Note the similarity with Curtis's relation $h_s h_t = 0$ if $\{s,t\}$ not a simplex)

Conclude that $\mathcal{C}, \mathcal{C}'$ equivalent $\Rightarrow A_{\mathcal{C}}, A_{\mathcal{C}'}$ are Morita equivalent.

Can you make this concrete? ~~Suppose~~

Idea: The arrow ring $A_{\mathcal{C}}$ is a "block" matrix ring indexed by $\mathrm{Ob}_{\mathcal{C}}$. For every ordered pair (x,y) of objects you have the block $\mathbb{Z}[\mathrm{Hom}(x,y)]$. You should be able to ~~make~~ make $\mathcal{C} \amalg \mathcal{C}'$ into a cat using the equivalence, then its arrow cat is the Morita context.