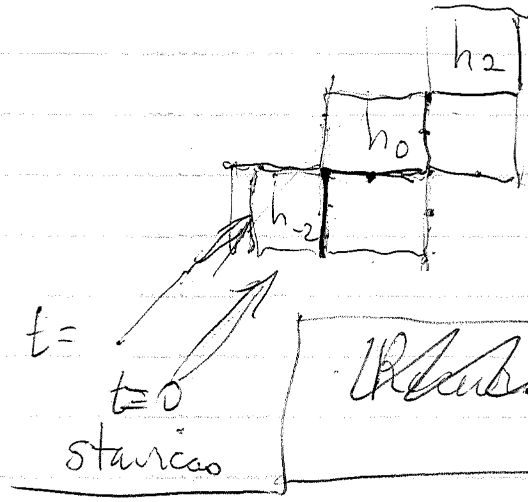


You should work out the descent idea, where you introduce a square root of u, this should reduce things to ladder systems.

~~Recursion relation~~

E grid space with parameters $(h_n)_{n\in\mathbb{Z}}$

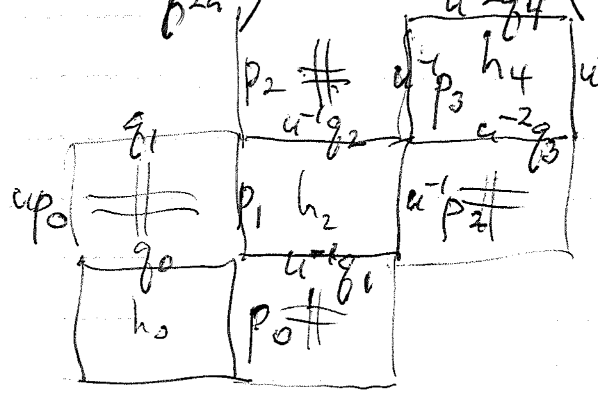
$\psi \in (E/(A-u)E)^*$ How to get started

The ~~basic~~ idea you want to develop is that introducing $u^{1/2}$, ~~should be i.e.~~ i.e. forming $\mathbb{C}[u^{1/2},u^{-1/2}]\otimes_{\mathbb{C}[u,u^{-1}]}E$ should yield ~~an interesting~~ a grid space with all odd $h_n=0$. I think this is clear on the eigenfunction level, equivalently on $E/(A-u)E$.

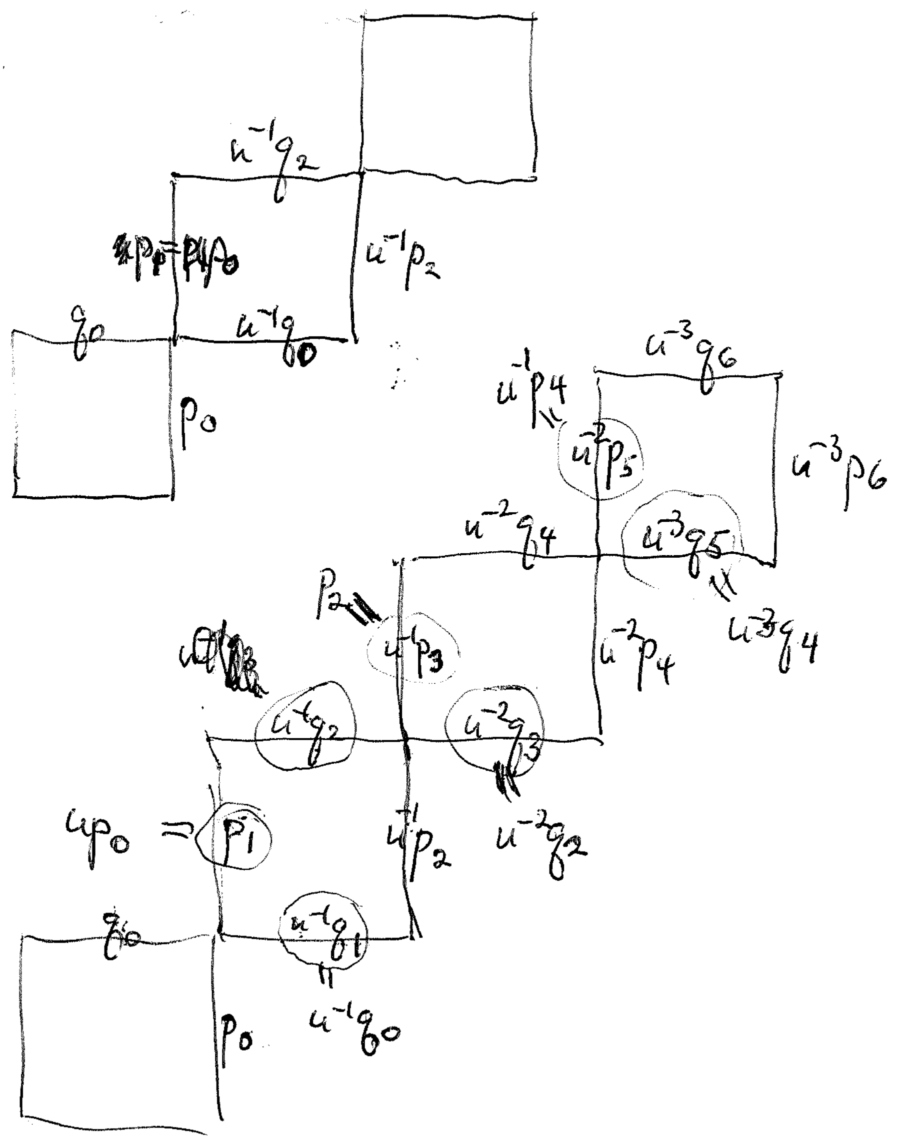
Begin with the other direction, namely suppose you have $h_{odd}=0$

$$\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix} = g(h_{2n})\begin{pmatrix}u&0\\0&1\end{pmatrix}\begin{pmatrix}p_{2n-1}\\ q_{2n-1}\end{pmatrix} = g(h_{2n})\begin{pmatrix}u^2&0\\0&1\end{pmatrix}\begin{pmatrix}p_{2n-2}\\ q_{2n-2}\end{pmatrix}$$

$$\begin{pmatrix}u^{-n}p_{2n}\\ u^{-n}q_{2n}\end{pmatrix} = g(h_{2n})\begin{pmatrix}u&0\\0&u^{-1}\end{pmatrix}\begin{pmatrix}u^{-(n-1)}p_{2n-2}\\ u^{-(n-1)}q_{2n-2}\end{pmatrix}$$



orthogonality is clearer



$$\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_2 \end{pmatrix} = g(h_2)\begin{pmatrix} up_0 \\ u^{-1}q_0 \end{pmatrix} = g(h_2)\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} u^{-2}p_4 \\ u^{-2}q_4 \end{pmatrix} = g(h_4)\begin{pmatrix} p_2 \\ u^{-2}q_2 \end{pmatrix} = g(h_4)\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} u^{-1}p_2 \\ u\, p_2 \end{pmatrix}$$

$$\begin{pmatrix} u^{-3}p_6 \\ u^{-3}q_6 \end{pmatrix} = g(h_6)\begin{pmatrix} u^{-1}p_4 \\ u^{-3}q_4 \end{pmatrix} = g(h_6)\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} u^{-2}p_4 \\ u^{-2}q_4 \end{pmatrix}$$

Back to ~~Q~~ coeff system on Q-tree and Green's fns. Acyclic coeff system.

$$\begin{tikzcd}[column sep=small]
\cdots \oplus M_{-1} & \oplus & M_0 & \oplus & M_1 & \oplus & M_2 \oplus \cdots \\
& M_{-1,0} \arrow[ul] \arrow[ur] & & M_{01} \arrow[ul] \arrow[ur] & & M_{12} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

You assume acyclic, i.e. $\partial : C_1 \xrightarrow{\sim} C_0$.

Look at $\partial^{-1} M_0 = Z_1(T, 0)$ 1-chains which are cycles except at 0.

Get a decomp of M_0 corresp to $Z_1(T,0) = Z_1(T_{\geq 0}, 0) \oplus Z_1(T_{\leq 0}, 0)$

In your situation you ~~...~~ should have dim $M_\sigma = 2$, and there's a λ parameter to worry about. Do you have any hope?

Go back to transmission + reflection. Think of unit segments of transmission line of different impedances or with ~~simple~~ with constant (indep. of frequency) transfer matrices at junctions.

$$-\partial_x E = \partial_t I \qquad \partial_t E + \partial_x I = 0$$

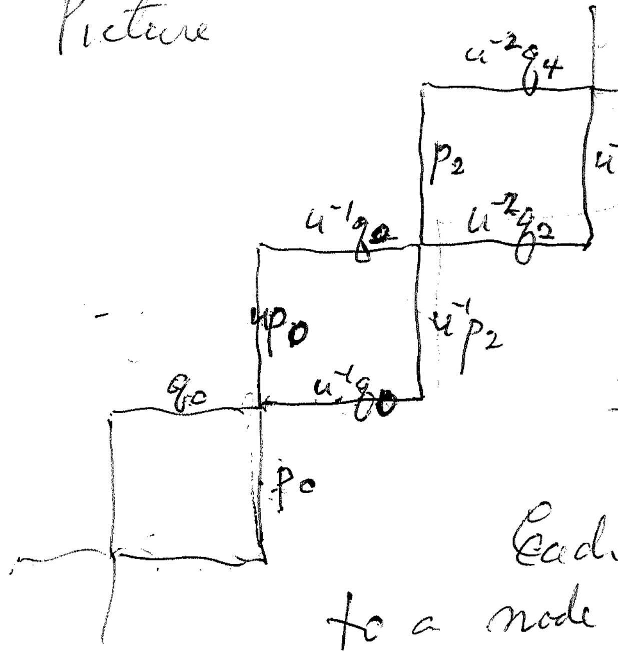
$$-\partial_x I = \partial_t E \qquad \partial_t I + \partial_x E = 0$$

$$(\partial_t + \partial_x)(E + I) = 0 \qquad E + I = A e^{-s(\cancel{+x+t})}$$ right mov. outgoing

$$(\partial_t - \partial_x)(E - I) = 0 \qquad E - I = B e^{s(x+t)}$$ left mov.

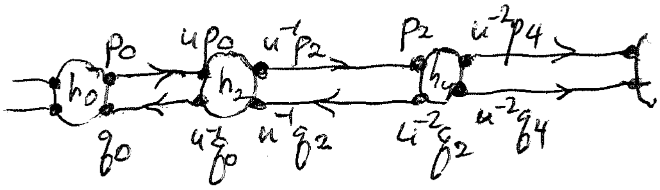
Join segments of transmission line

Picture



of grid space with $h_{odd} = 0$.

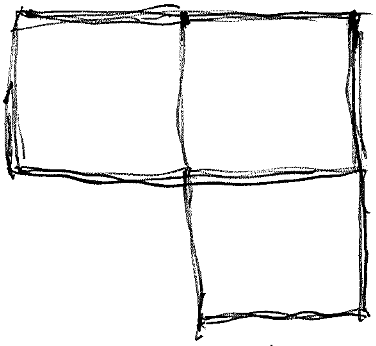
Corresponding "ladder"



Each edge in the picture corresponds to a node in the ~~ladder~~ ladder. The unitary operator u is given by the arrows in the latter. The circle h_{2k} stands for the 2-dim'l Hilbert space corresponding to a square in the picture. Note that these two dim'l spaces are orthogonal. The operator takes a p edge say p_0 to up_0 which splits into a reflected component const $u^{-1}q_0$ and a transmitted component $u^{-1}p_2$. Thus u moves p-edges up and q-edges down.

Now let's formulate the problem you'd like to solve. Start with sequence $(h_n)_{n \in \mathbb{Z}}$, form grid space E as usual. Look at the ~~eigenvectors~~ eigenfunctions $\varphi \in (E/(\lambda - u)E)^*$. Such a φ.

How to describe a $\varphi \in E^*$? by its effect on a basis. Consider



One problem: given a grid space E form the base extension

$$L(u^{1/2}) \otimes_{L(u)} E$$

Is there a natural way to make this a grid space?

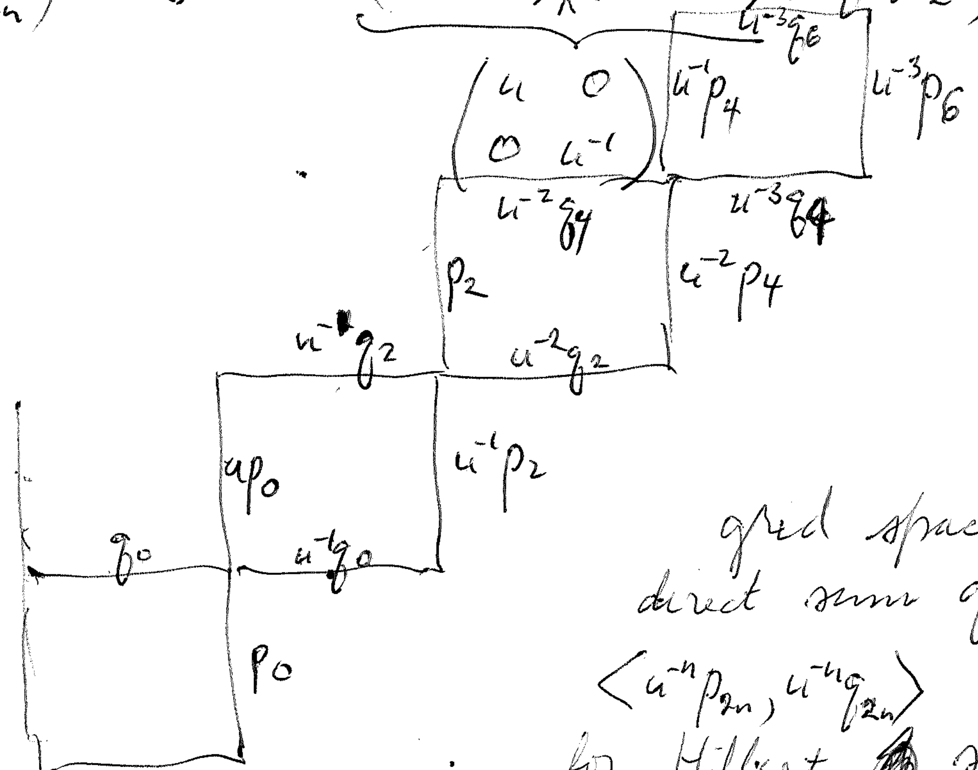
Review some of the ideas yesterday.

~~[illegible] Ladder setup~~

Ladder case: $h_{odd} = 0$.

$$\cancel{u^{-n}\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix} = g(h_{2n})\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix} u^{-n}\begin{pmatrix}p_{2n-1}\\ q_{2n-1}\end{pmatrix}}$$

$$u^{-n}\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix} = g(h_{2n})\underbrace{\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix} u^{-1} \begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}}_{\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix}} u^{-n+1}\begin{pmatrix}p_{2n-2}\\ q_{2n-2}\end{pmatrix}$$



grid space
direct sum of 2 planes
$\langle u^{-n}p_{2n}, u^{-n}q_{2n}\rangle$ $\perp$
for Hilbert structure

Relate resolvent $\frac{1}{\lambda - u}$ on $\mathcal{H}$
to eigenfunctions.

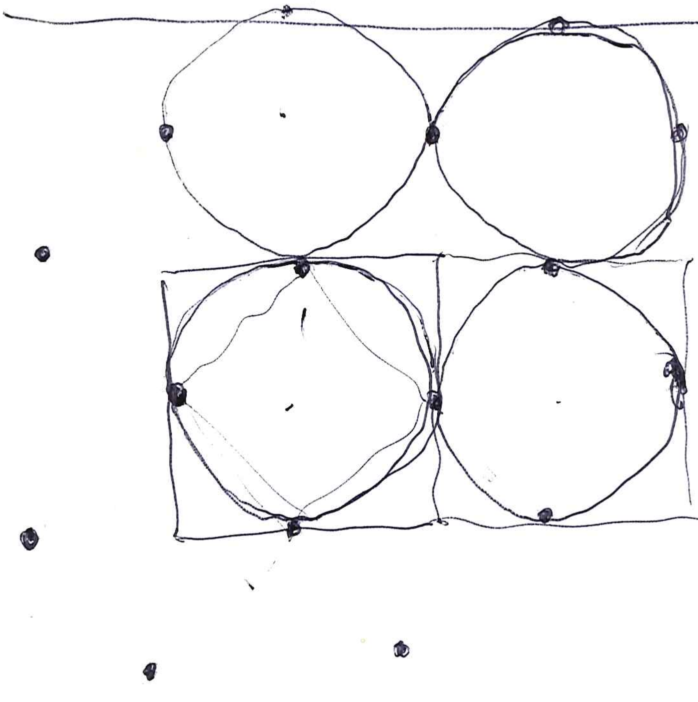
Fix E grid space as usual – it's a free module over $A = \mathbb{C}[u, u^{-1}]$ of rank 2 with various distinguished bases $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ $n \in \mathbb{Z}$ connected by recursion relation

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

For each $\lambda \in \mathbb{C}^\times$ get $W_\lambda = \left(E/(\lambda - u)E\right)^*$, space of eigenfunctions.

Use idea that E is essentially equivalent to the family ~~of~~ $\lambda \longmapsto E/(\lambda - u)E$. The A module E should be essentially equivalent to the vector bundle over $\mathbb{C}^\times$ with fibres $\lambda \longmapsto E/(\lambda - u)E$.

Let $A' = \mathbb{C}[u^{1/2}, u^{-1/2}]$ be "double cover" of A.



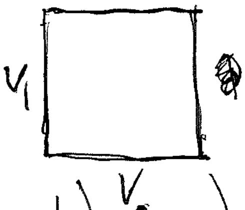
$$\mu = \frac{1}{k}\left(\frac{1-k^2}{k\lambda - 1} + 1\right)$$
$$= \frac{-k+\lambda}{k\lambda - 1} = \frac{\lambda - k}{k\lambda - 1}$$
$$= \begin{pmatrix} 1 & -k \\ k & -1 \end{pmatrix}(\lambda)$$

Let E be a grid space, it's a free module of rank 2 over $A = \mathbb{C}[u, u^{-1}]$. Let $A' = \mathbb{C}[u^{1/2}, u^{-1/2}]$ be the "double cover" of A, and let $E' = A' \otimes_A E$. Then E' is a free module of rank 2 over A'. Is E' naturally a grid space?????

Because of universal properties we might be able to formulate the question with a specific isomorphism.

~~Starting with E~~

Constant case.



$$\psi_{mn} = \lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$$

$$\begin{pmatrix} \lambda v^1 \\ \mu v^2 \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\boxed{\begin{aligned} (k\lambda - 1) v_1 &= h v_2 \\ (k\mu - 1) v_2 &= \bar h v_1 \end{aligned}}$$

Found $E \cong \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}]$

$\lambda \longleftrightarrow z$

$\mu \longleftrightarrow \dfrac{\lambda - k}{k\lambda - 1}$

$v_1 \longleftrightarrow \dfrac{h}{k\lambda - 1}$

$v_2 \longleftrightarrow 1$

$$v_2 = \frac{\bar h}{k\mu - 1} v_1$$

$$= \frac{\bar h}{1 - k^2} (k\lambda - 1) v_1$$

$$\boxed{v_2 = \frac{1}{h}(k\lambda - 1) v_1}$$

$$u = \mu \lambda^{-1}$$

$$= \frac{\lambda - k}{k\lambda - 1} \frac{1}{\lambda} = \frac{1 - k\lambda^{-1}}{k\lambda - 1} = (-1) \frac{1 - k\lambda^{-1}}{1 - k\lambda}$$

degree 0.

So ultimately you have a map, probably a double covering

$$\mathbb{C}[u,u^{-1}] \subset \mathbb{C}[\lambda,\lambda^{-1},(\lambda-k)^{-1},(k\lambda-1)^{-1}]$$

$$u \longmapsto (-1)\,\frac{1-k\lambda^{-1}}{1-k\lambda}$$

$\lambda = 0, k^{-1}$ $\quad u = \infty$

$\lambda = \infty, k$ $\quad u = 0$

$$(-u)(1-k\lambda) = (1-k\lambda^{-1})$$

$$-u + ku\lambda = 1 - k\lambda^{-1}$$

$$-\lambda u + ku\lambda^2 = \lambda - k$$

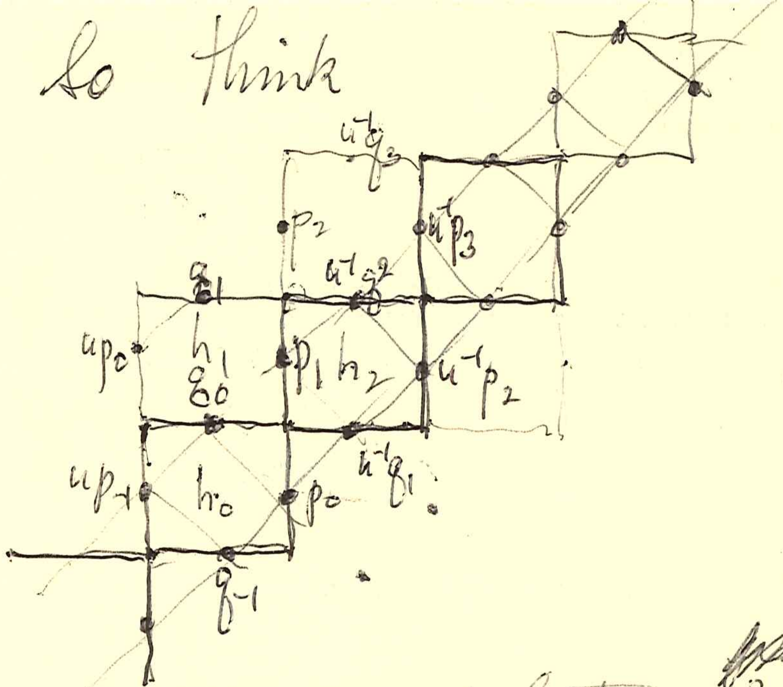
$$ku\lambda^2 - \lambda(u+1) + k = 0$$

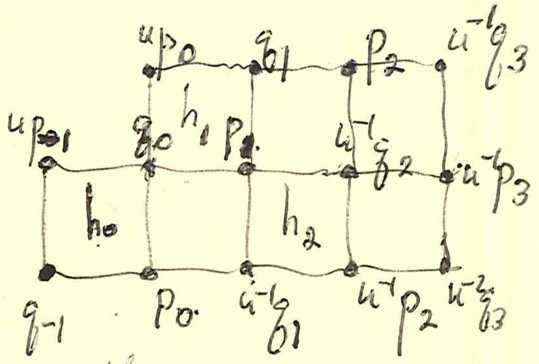
$$\lambda = \frac{u+1 \pm \sqrt{(u+1)^2 - 4k^2u}}{2ku}$$

~~28 Mar~~	~~31 Mar~~
~~3 Apr~~	~~4 Apr~~
~~25 Apr~~	~~27 Apr~~

Repeat yesterday! Constant h case, adjoining $u^{1/2}$. If what you suspect is true then there should be a simple isomorphism reflecting the one on the eigenfunction level.

So think





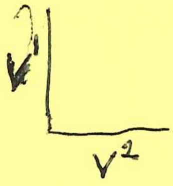
there are square here not giving relations!

Let's begin with a sequence $h = (h_n)$ and construct E the grid space. E is a free module over $A = \mathbb{C}[u, u^{-1}]$ of rank 2 with various bases. From the picture want something transversal to the time flow.

Idea: Go back to constant coeff grid and try to describe it as a module over the group of translations u_0, $u_1^2 = \mu\lambda$, $u_0^{-1} = \mu\lambda^{-1}$. You know that E a free module of rank 2 over $A = \mathbb{C}[u_0, u_0^{-1}]$.

~~[illegible]~~ E is defined by generators and relations, there's the symmetry $u \Rightarrow \mu^{-1}$ which preserves general (h_n) and another $\lambda\mu$ preserving the h's when they are constant mod 2.

Go back to Szegö not



$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$u^{-\frac{n}{2}} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n) \begin{pmatrix} u^{\frac{1}{2}} & 0 \\ 0 & u^{-1/2} \end{pmatrix} u^{-\frac{n-1}{2}} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Let $\psi \in \left(E/(\lambda-u)E\right)^*$, let $\psi_n = \psi \begin{pmatrix} u^{-\frac{n}{2}} p_n \\ u^{-\frac{n}{2}} q_n \end{pmatrix} = \lambda^{-\frac{n}{2}} \begin{pmatrix} \psi(p_n) \\ \psi(q_n) \end{pmatrix}$

Need notation

First consider the E' assoc. to

$$h'_n = \begin{cases} 0 & n \text{ odd} \\ h_{\frac{n}{2}} & n \text{ even.} \end{cases}$$

0	h_2
h_0	0

what is $W_\lambda(E')$?

an elt of $\left(E'/(\lambda-u)E'\right)^*$?

$$u^{-1} \begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = g(h_2) \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$= g(h_2) \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

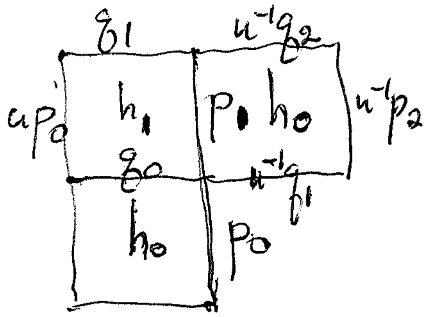
Thus $\psi \longmapsto (\psi_{2n})_{n \in \mathbb{Z}}$, $\psi \longmapsto \psi\left(u^{-n} \begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix}\right)$

gives isom of $W_\lambda(E')$ with

$$\left\{ (\psi_{2n})_{n \in \mathbb{Z}} \;\middle|\; \psi_{2n} = g(h_{2n}) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \psi_{2n-2} \quad \forall n \right\}$$

Situation: To examine the periodic case.

Consider ~~[illegible]~~ grid space E with h_{ev}, h_{odd} constant, so you have shifts u ↖ and u_1 ↗. ~~[illegible]~~ can you understand it?



$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = g(h_1)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_2 \end{pmatrix} = g(h_0)\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

Let's go back to eigenfunctions which hopefully tell us everything we need to know in order to handle the grid space

$\psi \in \left(E/(\lambda-u)E\right)^*$ can be identified with

$$\psi_{2m} = \psi\left(u^{-m}\begin{pmatrix} p_{2m} \\ q_m \end{pmatrix}\right) \qquad 2m \text{ even}$$

sat.

$$u^{-m}\begin{pmatrix} p_{2m} \\ q_{2m} \end{pmatrix} = g(h_{2m})\begin{pmatrix} 1 & \\ & u^{-1} \end{pmatrix} u^{-m+1}\begin{pmatrix} p_{2m-1} \\ q_{2m-1} \end{pmatrix}$$

$$u^{-m+1}\begin{pmatrix} p_{2m-1} \\ q_{2m-1} \end{pmatrix} = g(h_{2m-1})\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} u^{-m+1}\begin{pmatrix} p_{2m-2} \\ q_{2m-2} \end{pmatrix}$$

$$\psi_{2m} = \psi\left(u^{-m}\begin{pmatrix} p_{2m} \\ p_{2m} \end{pmatrix}\right)$$

$$\psi_{2m-1} = \psi\left(u^{-m+1}\begin{pmatrix} p_{2m-1} \\ q_{2m-1} \end{pmatrix}\right)$$

...

You hope to recover E as $A=\mathbb{C}[u,u^{-1}]$-module from the family of $W_\lambda = \left(E/(\lambda-u)E\right)^*$ somehow. ~~These are special~~ The idea: E as A-module is determined by ~~fact~~ property that ψ:

$$\psi_{2m} = \begin{pmatrix} u^{-m} p_{2m} \\ u^{-m} q_{2m} \end{pmatrix}, \quad \psi_{2m-1} = \begin{pmatrix} u^{-m+1} p_{2m-1} \\ u^{-m+1} p_{2m-1} \end{pmatrix}$$

is a universal solution

~~Consider a~~ Let $E(h)$ be the grid space assoc from $h=(h_n)_{n\in\mathbb{Z}}$ with its A-mod. structure, $A=\mathbb{C}[u,u^{-1}]$, let $A'=\mathbb{C}[u^{1/2},u^{-1/2}]$ and $E'=A'\otimes_A E$. Then for any A'-module M, an element of

$$\psi \in \mathrm{Hom}_{A'}(E',M) = \mathrm{Hom}_A(E,M)$$

is the same as $\psi_n^i \in M$, $n\in\mathbb{Z}$, $i=1,2$.

set
$$\begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix} = g(h_n)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \psi_{n-1}^1 \\ \psi_{n-1}^2 \end{pmatrix} \qquad \forall n$$

This can be rewritten, since you have the operator $u^{1/2}$ on M in ~~the~~ equivalent form as follows. Put $\phi_n = u^{-\frac{n}{2}}\psi_n$. Then the cond. on $\psi=(\psi_n^i)$ is the same as the cond.

$$\phi_n = g(h_n)\begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix}\phi_{n-1}$$

Need to subdivide $\mathbb{Z}$

....

Given $(h_n)_{n\in\mathbb{Z}} = h$ have $E(h)$ an module over $A = \mathbb{C}[u, u^{-1}]$ generators p_n, q_n $n \in \mathbb{Z}$ relations
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Let $A' = \mathbb{C}[u^{1/2}, u^{-1/2}]$, $E' = A' \otimes_A E(h)$. Then E' is the A' module with same generators (really $1\otimes p_n$, $1 \otimes q_n$) and relations. Can also describe E' by a slightly different system of generators and relations. Let m range over $\frac{1}{2}\mathbb{Z}$, ~~set p'_m~~ you want relations

$$\begin{pmatrix} p'_n \\ q'_n \end{pmatrix} = g(h_n) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{n-\frac{1}{2}} \\ q'_{n-\frac{1}{2}} \end{pmatrix}$$

$$\begin{pmatrix} p'_{n-\frac{1}{2}} \\ q'_{n-\frac{1}{2}} \end{pmatrix} = g(0) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{n-1} \\ q'_{n-1} \end{pmatrix}$$

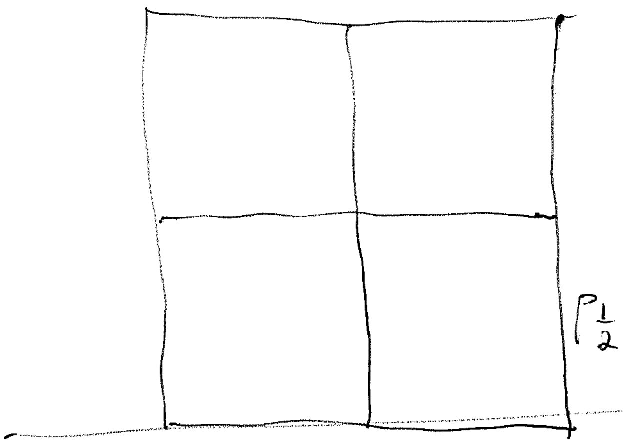
Thus define
$$\begin{cases} p'_m = p_m & \text{if } m \in \mathbb{Z} \\ p'_m = u^{1/2} p_{m-\frac{1}{2}} & \text{if } m \in \frac{1}{2} + \mathbb{Z} \\ q'_m = q_m & \text{if } m \in \mathbb{Z} \\ q'_m = q_{m-\frac{1}{2}} & \text{if } m \in \frac{1}{2} + \mathbb{Z} \end{cases}$$

Thus it seems that E' is the grid space assoc. to sequence $0\ h_0\ 0\ h_1\ 0\ h_2$

Check carefully

Before (with E) you have $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$ $\begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$

what happens?

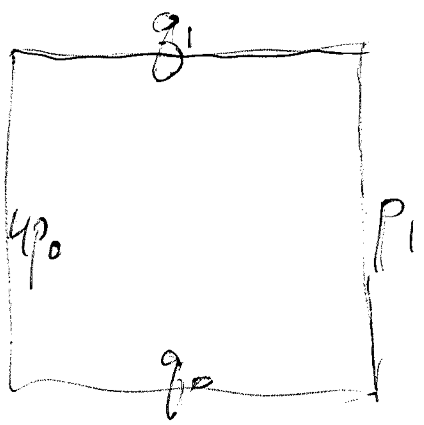


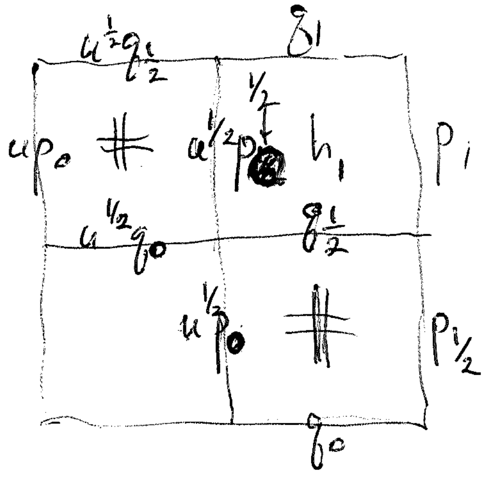
Start again: in E' you have $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ gen. with usual relations. But you also have the operator $u^{1/2}$ on E'

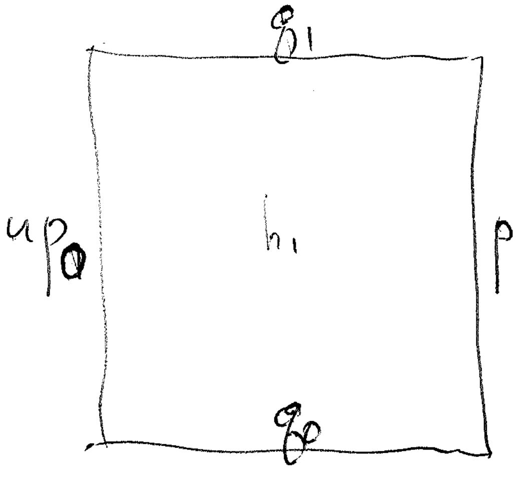
$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = g(h_1) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overset{p_{\frac{1}{2}}}{u^{1/2} p_0} \\ q_0 \end{pmatrix} \; q_{\frac{1}{2}}$$

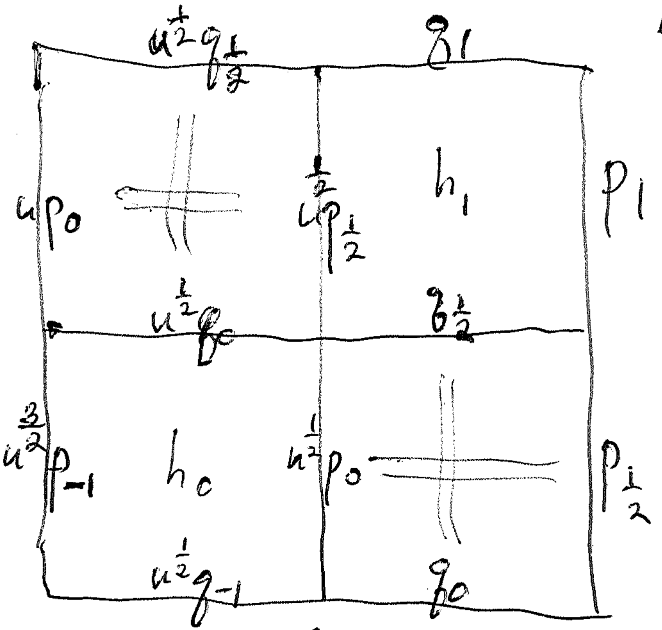
$$\begin{pmatrix} u^{1/2} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

It seems that you have an win









seems to work but could there be a better way. Galois action: sending $u^{1/2}$ to $-u^{1/2}$ preserves p_n, q_n $n \in \mathbb{Z}$

Principal E is a universal ~~sat~~ A-module equipped with p_n, q_n sat

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$A' = \mathbb{C}[u^{1/2}, u^{-1/2}]$ double cover of $\mathbb{C}[u,u] = A$. Then $A' \otimes_A E$ is ~~sat~~ universal A'-module equipped with elements p_m, q_m $m \in \frac{1}{2}\mathbb{Z}$ sat

$$\begin{pmatrix} p_m \\ q_m \end{pmatrix} = g(h_m) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{m-\frac12} \\ q_{m-\frac12} \end{pmatrix} \qquad m \in \mathbb{Z}$$

$$\begin{pmatrix} p_m \\ q_m \end{pmatrix} = \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{m-\frac12} \\ q_{m-\frac12} \end{pmatrix} \qquad m \in \tfrac{1}{2} + \mathbb{Z}$$

First step $E' = A' \otimes_A E$ is univ. A' module equipped with p_n, q_n sat basic relations.

A'-module means A-module equipped with op $u^{1/2}$ whose square is u.

Should be clear that an A'-module with (p_n, q_n) $n \in \mathbb{Z}$ sat $*$ same as A'-module with p_m, q_m sat. $*'$

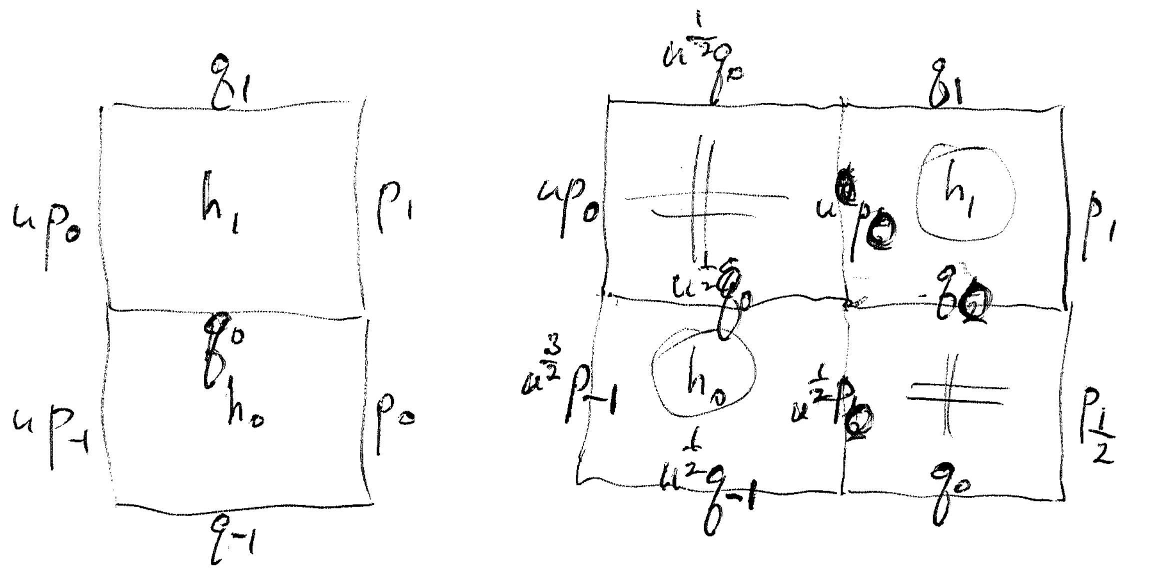
Improvement: The A'-module $A' \otimes_A E$ ~~[illegible]~~ universal with p_n, q_n $n \in \mathbb{Z}$ is the A'-module equipped with p'_m, q'_m sat

$$\begin{pmatrix} p'_m \\ q'_m \end{pmatrix} = g(h'_m) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{m-\frac{1}{2}} \\ q'_{m-\frac{1}{2}} \end{pmatrix}$$

where $h'_m = h_m \qquad m \in \mathbb{Z}$

$\qquad\; h'_m = 0 \qquad m \in \frac{1}{2} + \mathbb{Z}$

NOT CLEAR



$p_{\frac{1}{2}} = u^{1/2} p_0$

$q_{\frac{1}{2}} = q_0$

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = g(h_1) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{\frac{1}{2}} \\ q_{\frac{1}{2}} \end{pmatrix} = g(h_1) \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{1/2} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

Repeat yesterday insight

E grid space assoc. to $(h_n)_{n\in\mathbb{Z}}$ it's an A-module defined by generators + relations. Szegö form of gen. + relations.

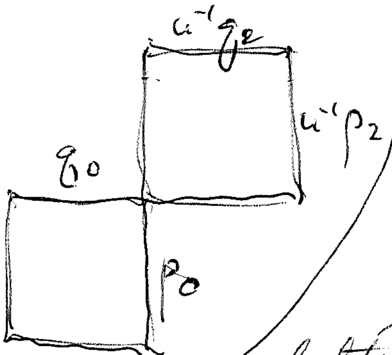
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

"$t=0$" form.

$$u^{-n}\begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix} = g(h_{2n})\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} u^{-n+1}\begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix}$$

$$u^{-n+1}\begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix} = g(h_{2n-1})\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} u^{-n+1}\begin{pmatrix} p_{2n-2} \\ q_{2n-2} \end{pmatrix}$$

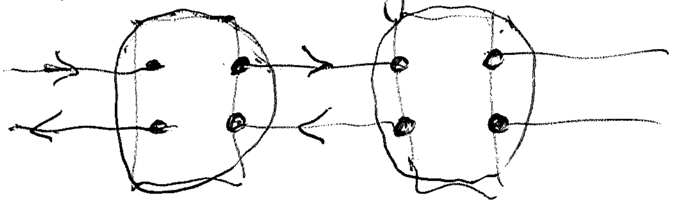
Try combining these ~~things~~



Idea: Avoid depending on a specific notation. ~~Most ...~~ The ladder situation seems better to handle things, so you may want a notation adapted to it.

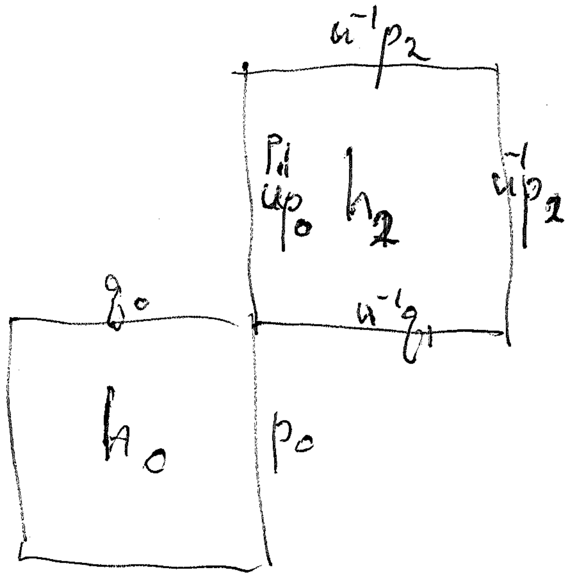
Handle the general case as fix points of $\mathbb{Z}/2$ on a ladder. What about the $-$ eigenspace?

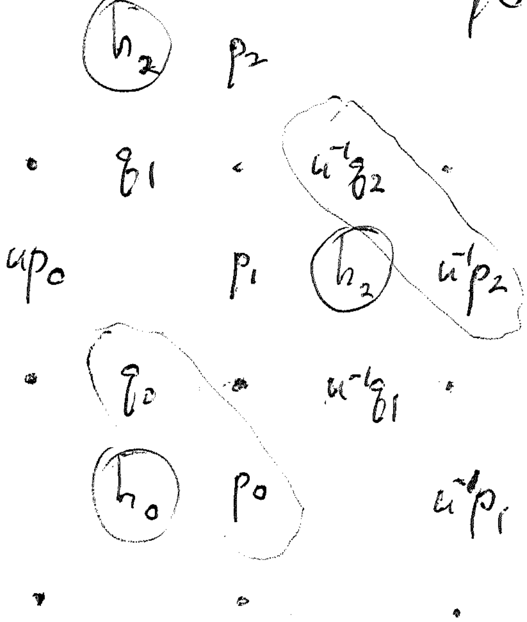
Work out a good notation for a ladder



Each box is a 2 dim'l vector space with 4 grid vectors, box labelled by $n \in \mathbb{Z}$

YES!





Maybe you want to keep track of n, the position coordinates

$$\rho_n = \begin{pmatrix} p_n^1 \\ p_n^2 \end{pmatrix} = \begin{pmatrix} u^{-n} p_{2n} \\ u^{-n} q_{2n} \end{pmatrix} \qquad \rho_n = \begin{pmatrix} \rho_n^1 \\ \rho_n^2 \end{pmatrix}$$

Philosophy: $E = E(h)$ $\qquad h = (h_n)_{n \in \mathbb{Z}}$

E module over A free of rank 2.

$A' = \mathbb{C}[u^{1/2}, u^{-1/2}]$, let $E' = A' \otimes_A E$, E' is universal for solutions with values in an A' module

$$u^{-\frac{n}{2}} \begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} u^{-\frac{n}{2}} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

$$= g(h_n) \begin{pmatrix} u^{1/2} & 0 \\ 0 & u^{-1/2} \end{pmatrix} u^{-\frac{(n-1)}{2}} \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix}$$

Alt. Approach. Form $E' =$ grid space assoc. to (h'_n), $h'_n = \begin{cases} 0 & n \text{ odd} \\ h_{n/2} & n \text{ even} \end{cases}$

$$\begin{pmatrix} p'_{2n} \\ q'_{2n} \end{pmatrix} = g(h'_{2n}) \begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{2n-1} \\ q'_{2n-1} \end{pmatrix}$$

$$= g(h'_{2n}) \begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{2n-2} \\ q'_{2n-2} \end{pmatrix}$$

$$\begin{pmatrix} p'_{2n} \\ q'_{2n} \end{pmatrix} = g(h_n) \begin{pmatrix} (u')^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{2n-2} \\ q'_{2n-2} \end{pmatrix}$$

so you get ~~an embedding~~ a A-module map

$$E \longrightarrow E'$$

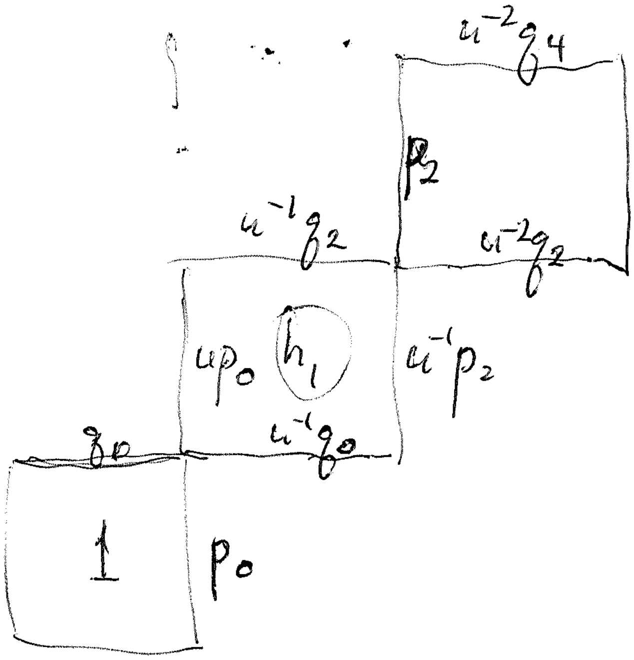
$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} \longmapsto \begin{pmatrix} p'_{2n} \\ q'_{2n} \end{pmatrix}$$

which induces isom $A' \otimes_A E \xrightarrow{\sim} E'$

Example: suppose $1 = h_0, 0, h_2, 0, h_4, \dots$ then you have a

$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} z \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \frac{1}{R} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} z^2 \\ 1 \end{pmatrix} = \frac{1}{R} \begin{pmatrix} z^2 + h \\ \bar{h} z^2 + 1 \end{pmatrix} \qquad \begin{pmatrix} p_4 \\ q_4 \end{pmatrix} = \frac{1}{R} \begin{pmatrix} z^2(z^2+h) \\ \bar{h} z^2 + 1 \end{pmatrix}$$

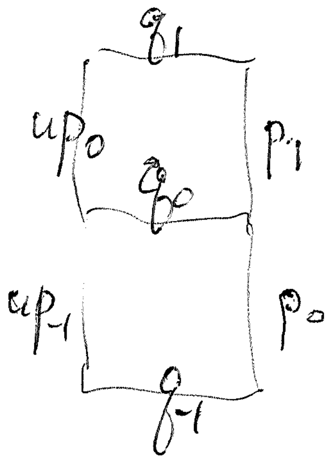


Thus

$q_{2n} = q_{2n-1}$ is a poly in z^2

$p_{2n} = z^{2n} +$ lower even degree

$q_{2n-1} = z^{2n-1} +$ lower odd degree.



$$\begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{1}{k_1}\begin{pmatrix} 1 & h_1 \\ \bar{h}_1 & 1 \end{pmatrix}\begin{pmatrix} up_0 \\ q_0 \end{pmatrix}$$

presence of grading implies $h_1 = 0$, more generally h_1 is odd in some sense.

Repeat: Set up the basic isomorphism.

$A = \mathbb{C}[u, u^{-1}]$, $\quad A' = \mathbb{C}[u^2, u^{-2}]$

$E = E(u, h)$ where $\quad h_n = 0 \quad n$ odd
$\qquad\qquad\qquad\qquad\quad = h'_{\frac{n}{2}} \quad n$ even

$E' = E(u^2, h')$

E is the A-module gen. by $\begin{matrix} p_n \\ q_n \end{matrix}$ satisf

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \qquad \forall n$$

i.e. $\begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix} = g(h'_{2n})\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix}$, $\quad \begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix} = \begin{pmatrix} u p_{2n-2} \\ q_{2n-2} \end{pmatrix} \quad \forall n$

Last pair can be used to eliminate odd p, q's, get E_1 (A mod 2) gens $\begin{matrix} p_{2n} \\ q_{2n} \end{matrix}$ subj to

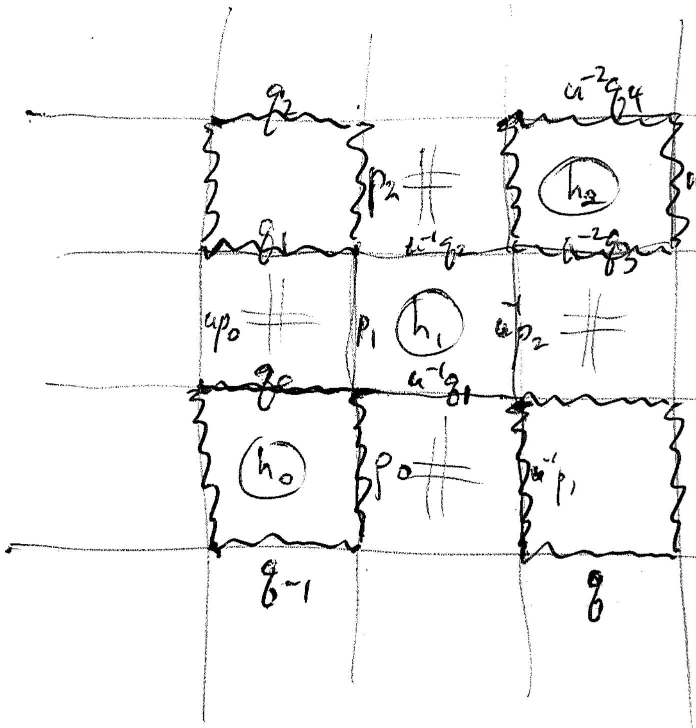
$$\begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix} = g(h_{2n}) \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{2n-2} \\ q_{2n-2} \end{pmatrix}$$

But E' is the A' module gen. by $\begin{pmatrix} p'_n \\ q'_n \end{pmatrix}$ subj. to

$$\begin{pmatrix} p'_n \\ q'_n \end{pmatrix} = g(h'_n) \begin{pmatrix} u^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p'_{n-1} \\ q'_{n-1} \end{pmatrix}$$

so you get $\quad E \cong A \otimes_{A'} E'$

$$\begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix} \longmapsto \begin{pmatrix} p'_n \\ q'_n \end{pmatrix}$$



spot even part. Squiggled edges. What do you want to see here?

It's clear now that ~~you~~ get an embedding $E(u^2, h')$ into $E(u, h)$

namely $\begin{pmatrix} p'_n \\ q'_n \end{pmatrix} \longmapsto \begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix}$ $\qquad$ Check.

$$\parallel$$

$$g(h'_n) \begin{pmatrix} u^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p'_{n-1} \\ q'_{n-1} \end{pmatrix} \longmapsto g(h_{2n}) \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{2n-2} \\ q_{2n-2} \end{pmatrix}}^{\begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix}}$$

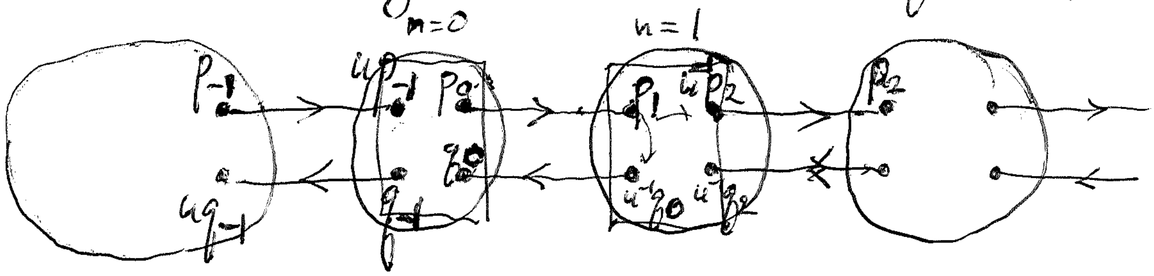
Question: What kind of $d\mu$ on S^1 yields $h_{odd} = 0$? $h_n = (q_n | p_n)$

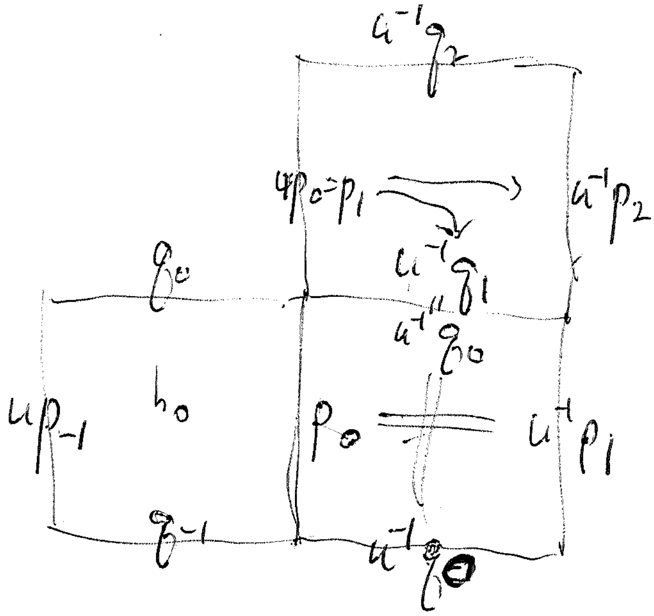
enough that q_n even $\forall n$, p_n parity n.

Maybe it's just $d\mu$ is invariant under $\sigma : z \longmapsto -z$. Then F_n is invariant under this involution, also $F_n \ominus F_{n-1}$ leading to $q_n \in F_n \ominus zF_{n-1}$, $q_0(0) > 0$ so $q_n^\sigma = q_n$ i.e. q poly in z^2. etc. How does the converse work?

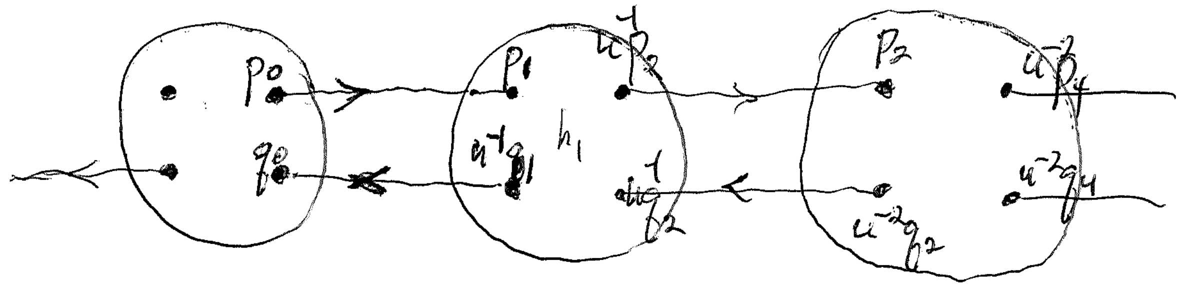
What next?

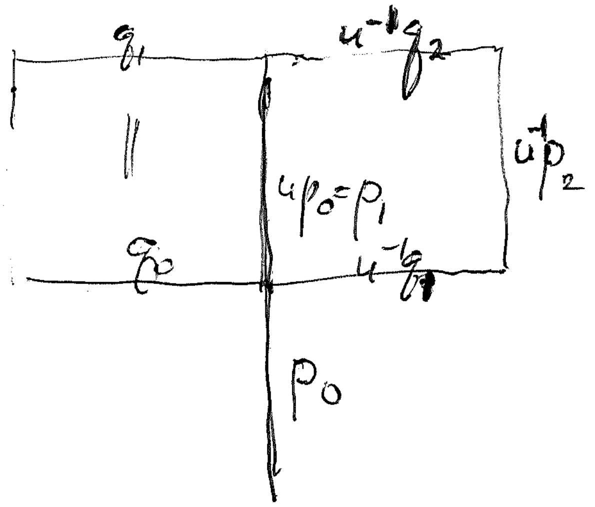
Let's study a ladder carefully.





ladder





$$\begin{pmatrix} p_2 \\ q_2 \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} up_1 \\ q_1 \end{pmatrix}$$

$$\begin{pmatrix} p_2 \\ q_1 \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} up_1 \\ q_2 \end{pmatrix}$$

$$\begin{pmatrix} up_1 \\ u^{-1}q_2 \end{pmatrix} = \begin{pmatrix} k & -h \\ \bar{h} & k \end{pmatrix}\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_1 \end{pmatrix}$$

The question is whether you can simplify the notation

Problem: Clean up identification of ~~the~~ u grid space with $h_{odd} = 0$ with a ladder. ~~[illegible]~~

Given E with usual gen. p_n, q_n ~~satisfying~~ subject to

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = g(h_n)\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix} \quad \forall n. \quad \text{Let} \quad v_n = \begin{pmatrix} u^{-n}p_{2n} \\ u^{-n}q_{2n} \end{pmatrix}.$$

~~that v_n satisfy $v_n = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}u^{-n}$~~

~~Then $v_n = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} u^{-n}\begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix} = g(h_{2n})\begin{pmatrix} u & 0 \end{pmatrix}$~~

Put
$$V_n = \begin{pmatrix} u^{-n} p_{2n} \\ u^{-n} q_{2n} \end{pmatrix}.$$
Then
$$V_n = u^{-n}\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix} u^{-n} \underbrace{\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix}}_{\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{2n-2} \\ q_{2n-2}\end{pmatrix}}$$

$$\boxed{V_n = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} V_{n-1} \quad \forall n}$$

Conversely given V_n satisfying put
$$\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = u^n \begin{pmatrix} V_n^1 \\ V_n^2\end{pmatrix} \qquad \begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix} = \begin{pmatrix} u p_{2n-2} \\ q_{2n-2}\end{pmatrix} \qquad \forall n.$$

Then
$$\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} u\, u^{n-1}\begin{pmatrix} V_{n-1}^1 \\ V_{n-1}^2\end{pmatrix}$$
$$= g(h_{2n})\begin{pmatrix} u^2 & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{2n-2} \\ q_{2n-2}\end{pmatrix}$$
$$= g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix}$$

Clearer is to put $\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = u^n \begin{pmatrix} V_n^1 \\ V_n^2\end{pmatrix}$,

$$\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix} = \begin{pmatrix} u^{n} V_{n-1}^1 \\ u^{n-1} V_{n-1}^2\end{pmatrix}.$$
Then $\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{2n-2} \\ q_{2n-2}\end{pmatrix}$

and
$$\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = u^n\begin{pmatrix} V_n^1 \\ V_n^2\end{pmatrix}$$
$$= u^n g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix} V_{n-1}^1 \\ V_{n-1}^2\end{pmatrix} = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\underbrace{\begin{pmatrix} u^n V_{n-1}^1 \\ u^{n-1} V_{n-1}^2\end{pmatrix}}_{\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix}}$$

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$$V_n = u^{-n}\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} \implies V_n = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix} u^{-n}\underbrace{\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix}}_{\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{2n-2} \\ q_{2n-2}\end{pmatrix}}$$

$$\implies V_n = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} V_{n-1}$$

Conversely given V_n sat put

$$\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = u^n V_n \qquad \begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix} u^{n-1} V_{n-1}$$

Then $\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} p_{2n-2} \\ q_{2n-2}\end{pmatrix}$ | ~~$\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = u^n g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} V_{n-1} = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} u^n V^1_{n-1} \\ u^{n-1} V^2_{n-1}\end{pmatrix}$~~

$$\begin{pmatrix} p_{2n} \\ q_{2n}\end{pmatrix} = u^n g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} V_{n-1}$$

$$= g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\underbrace{\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix} u^{n-1} V_{n-1}}_{\begin{pmatrix} p_{2n-1} \\ q_{2n-1}\end{pmatrix}}$$

Logic maybe is that given gen ψ_{2n}, ψ_{2n-1} sat

$$\psi_{2n} = g(h_{2n})\begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\psi_{2n-1}, \quad \psi_{2n-1} = \begin{pmatrix} u & 0 \\ 0 & 1\end{pmatrix}\psi_{2n-2}$$

eliminate ψ_{odd} to have gen. ψ_{2n} subj to

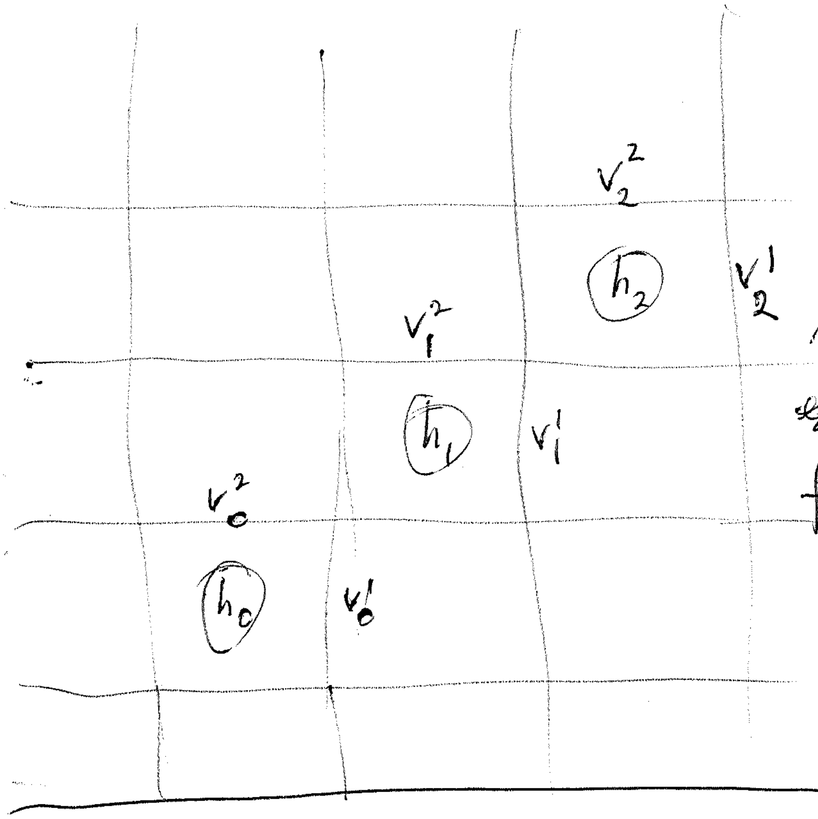
$$\psi_{2n} = g(h_{2n})\begin{pmatrix} u^2 & 0 \\ 0 & 1\end{pmatrix}\psi_{2n-2}$$

Now you need to get used to ladder notation

$$V_n = g(h_n)\begin{pmatrix} u & 0 \\ 0 & u^{-1}\end{pmatrix} V_{n-1}$$

You want a good picture of this for calculations.

Idea was For each n have the two dim'l space V_n spanned by V_n^1, V_n^2 and these are orthogonal. The angle is h_n.



What you need to clarify is the Greens function, ~~that~~ really the relation between ~~eigenfns~~ the Greens function and eigenfunctions

need suitable notation. ~~Basis vectors~~

Think about eigenfunctions $\psi \in (E/(\lambda-u)E)^*$

$$\psi_n = g(h_n)\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\psi_{n-1}$$

$$\psi_n = \psi\left(u^{-n}\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix}\right) = \lambda^{-n}\psi\begin{pmatrix}p_{2n}\\ q_{2n}\end{pmatrix}$$

In scattering ~~situation~~ $p_{2n} \sim \lambda^{2n}\xi_+$

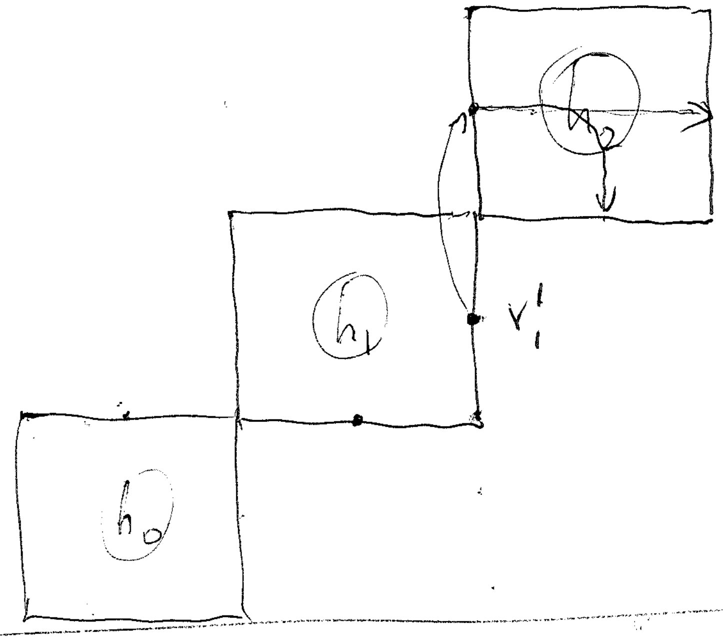
$q_{2n} \sim \xi_-$

so $\psi_n \sim \begin{pmatrix}\lambda^{n}\,\psi(\xi_+)\\ \lambda^{-n}\,\psi(\xi_-)\end{pmatrix}$. ~~[illegible]~~

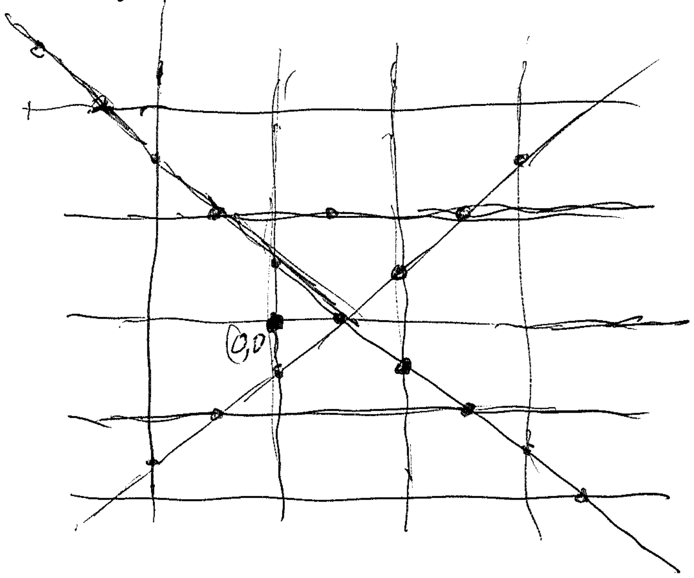
where do you start?

Eigenfn = ~~[illegible]~~ $\psi \in (E/(\lambda-u)E)^*$

equiv. to (ψ_n^i) sat $\psi_n = g(h_n)\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\psi_{n-1}$



Latest idea is to set up coordinates in $\frac{1}{2}\mathbb{Z} \times \frac{1}{2}\mathbb{Z}$, each point corresp. to a vertex, edge, or square.



e.g. $\mathbb{Z}\times\mathbb{Z}$ = vertices

$\frac{1}{2}\mathbb{Z}\times\mathbb{Z}$ = ~~horizontal~~ edges

$\mathbb{Z}\times\frac{1}{2}\mathbb{Z}$ = vertical edges

$\frac{1}{2}\mathbb{Z}\times\frac{1}{2}\mathbb{Z}$ = squares.

You might want to label ?

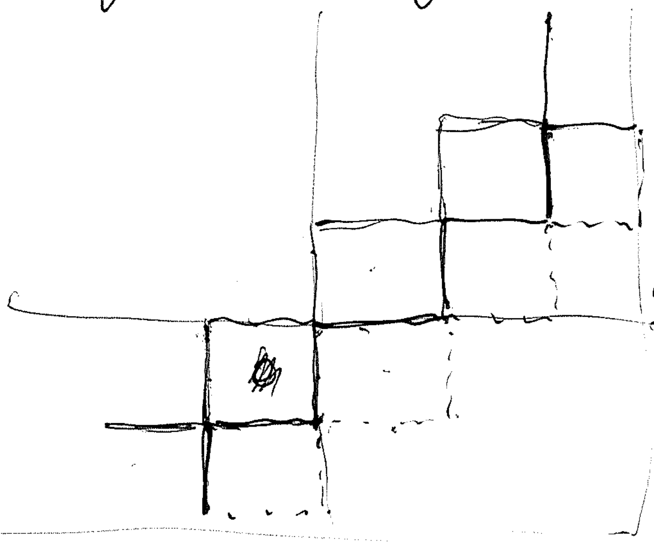
First handle ~~case~~ free case where $h_n = 0, \forall n$

Have a $\nwarrow$ time direction.

Every edge has a position and time coord, t = constant gives an ascending staircase. ~~t=~~ ~~constant gives~~

Principles: time action on squares, edges and vertices. ~~Descendent~~ What is a fundamental domain?

Confused again

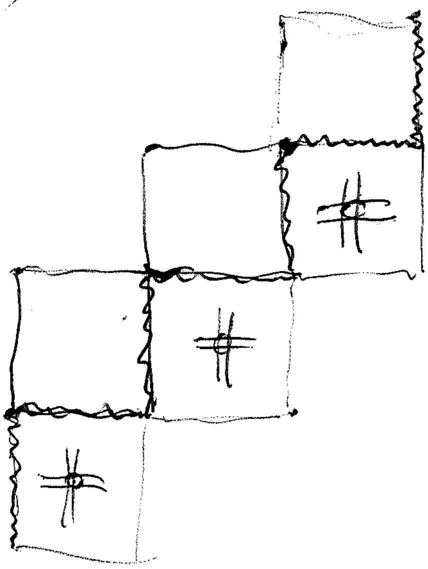


do general grid case fund. domain
$t=0$ ~~for~~ line for
the squares.

Wait, for an ascending staircase S, ~~…~~ this is an orth basis for E. Two lines of squares on either side of S. If you combine these lines you get a strip such that u is an isom from bottom staircase to the top. Now ~~…~~ What happens when one line of squares has h's $=0$.

It seems important to treat grid space as a module over $A = \mathbb{C}[u,u^{-1}]$, i.e. to look at solutions of the grid equations (these are linear maps $\psi: E \longrightarrow V$ in general) which are A-module maps, in which case $\psi(u\xi) = u\psi(\xi)$. So let's consider ~~a~~ $E(h)$ where ? need convention about the h's. ~~…~~ h is a function on the squares, ~~…~~ stationary wrt u. At some point you want to pick a ~~…~~ fundamental domain for the action of u. This is provided exactly by an ascending staircase together with diagonal lines of squares on either side. There's no duplication except for the two outer staircases [illegible] linked by a

So now adjust your notation to fit the fundamental domain. There are various ~~edges~~ and ~~squares~~, I should say generators and relations, to consider



Now you are looking at the case where one of the two diagonals of squares has $h=0$, say the lower

You know E ~~[illegible]~~ is free of rank 2 over A. Obvious choices ~~[illegible]~~ for a basis

Want eigenfunctions, their relation to $\frac{1}{\lambda - a}$

~~Eigenfunctions | you need a diff~~

Ideas: tree, Green's functions on,

~~[illegible]~~ main problem to be solved today is control of the Hilbert space of a ladder

~~[illegible]~~

Discuss philosophy. Given $h = (h_i)$ ~~you~~ define a ~~[illegible]~~ $E(h)$

~~[illegible]~~ You start with 1

Philosophy. infinite square grid with h parameters given in each square leads to a grid space, ~~generators~~ edges = generators, two relations per square, structures: pos. def and indef herm. forms, possibly a det. ~~...~~

when h stationary wrt u: grid sp E is ~~...~~ an $A = \mathbb{C}[u,u^{-1}]$-module, free of rank 2, ~~seems natural to introduce~~ What happens to the ~~...~~ structures? conjugation, is there a Wronskian? ~~...~~ $su(1,1)$ picture linear over A.

$$g \in u(1,1) \quad \text{when} \quad g^*\begin{pmatrix}1&0\\0&-1\end{pmatrix}g = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$g = \begin{pmatrix}a&b\\c&d\end{pmatrix}$$

i.e. when $g^{-1} = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}\bar a&\bar c\\\bar b&\bar d\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix} = \begin{pmatrix}\bar a&-\bar c\\-\bar b&\bar d\end{pmatrix}$

If $\det(g)=1$ $\quad \begin{pmatrix}d&-b\\-c&a\end{pmatrix}$ $\quad$ ∴ $g = \begin{pmatrix}a&b\\\bar b&\bar a\end{pmatrix}$ $\quad |a|^2-|b|^2=1$

$$g \in u(1,1) \Rightarrow \overline{\det g}\,(-1)\det g = -1 \quad ∴ \det g \in S^1$$

~~...~~ V complex vector space with conjugation σ and a hermitian form $H(v_1,v_2)$. ~~...~~ Restrict H to V^σ, then $\mathrm{Re}\,H$, $\mathrm{Im}\,H$ are symm, anti-symm, forms on V^σ. ∴ Herm form on V equivalent to a pair S, A on V^σ. Clear also by choosing ~~...~~ basis for V^σ, use as $\mathbb{C}$-basis for V, get herm matrix whose Re Im part is symm., anti-symm.

For dim $V = 2$ interesting case is where $\underline{A \text{ is a volume}}$, $S = 0$

$$H(v,v') = v^*\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}v'$$

$$= \begin{pmatrix}\bar v_1 & \bar v_2\end{pmatrix}\begin{pmatrix}-iv_2'\\ iv_1'\end{pmatrix} = -i\left(\bar v_1 v_2' - \bar v_2 v_1'\right) = -i\begin{vmatrix}\bar v_1 & v_1'\\ \bar v_2 & v_2'\end{vmatrix}$$

What is the missing point?

$H(\sigma v, v')$ is $\mathbb{C}$ bilinear. Maybe, it's the relation between 3 structures:

- conjugation σ or real structure on V
- hermitian form $H(v,v')$ such that $H(\sigma v, v')$ is alternating
- volume = alternating complex bilinear form. (usually non-deg).

Suppose
$$H(v,v') = v^*(\underset{\text{real symm}}{S} + i\underset{\text{real anti-symm.}}{A})v' \qquad \sigma v = \bar v$$

$$H(\sigma v, v') = v^t(S+iA)v' \qquad \mathbb{C} \text{ bilinear}$$

$$\overline{H(\sigma v, v')} = v^*(S - iA)\,\overline{v'}$$
$$= v'^*(S+iA)\,\bar v$$
$$= H(v', \sigma v) \qquad \text{just herm. symm}$$

$$S = 0 \iff \underbrace{H(\sigma v, v)}_{v^t S v} = 0 \quad \forall v$$

Grid space - rank 2 module over A (functions on S^1)

First general case if possible. What do you get from the $SU(1,1)$ matrices belonging to the squares.

First over a point

$\dim_{\mathbb{C}} V = 2$. There's a standard relation between

σ conjugation

H hermitian form $\ni$ $H(\sigma v, v) = 0$

ω volume

$\omega(\sigma v, \sigma v') = \overline{\omega(v,v')}$.

Choose a real basis, so that $H(v,v') = v^*(S + iA)v'$

Then $H(\sigma v, v') = v^t(S+iA)v = 0 \iff S = 0$.

$$i\underbrace{v^tAv'}_{\omega(v,v')} = H(\sigma v, v')$$

example: $V = \mathbb{C}^2$, $\sigma v = \bar{v}$, $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\omega(v,v') = v^tAv' = v^t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}v' = v_2v_1' - v_1v_2' = -\begin{vmatrix} v_1 & v_1' \\ v_2 & v_2' \end{vmatrix}$$

and
$$H(v,v') = iv^*\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}v'$$
$$= -i\begin{vmatrix} \bar{v}_1 & v_1' \\ \bar{v}_2 & v_2' \end{vmatrix}$$

Now ~~but~~ Consider grid space with u

Wait take $\begin{pmatrix}p\\ q\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}p'\\ q'\end{pmatrix}$

and suppose you have a conjugation σ on E ~~[illegible]~~. Then

$$\boxed{\begin{aligned}\begin{pmatrix}\sigma p\\ \sigma q\end{pmatrix} &= \frac{1}{k}\begin{pmatrix}1 & \bar h\\ h & 1\end{pmatrix}\begin{pmatrix}\sigma p'\\ \sigma q'\end{pmatrix}\\ \text{so}\quad \begin{pmatrix}\sigma q\\ \sigma p\end{pmatrix} &= \frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}\begin{pmatrix}\sigma q'\\ \sigma p'\end{pmatrix}\end{aligned}}$$

$$\begin{pmatrix}u^{-n}p_{2n}\\ u^{-n}q_{2n}\end{pmatrix} = g(h_{2n})\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix}\begin{pmatrix}u^{-n+1}p_{2n-2}\\ u^{-n+1}q_{2n-2}\end{pmatrix}$$

OKAY How to define σ on E? You must interchange horizontal and vertical edges. Translations

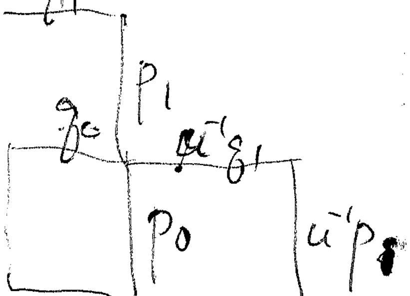
$$\sigma(u^{-n}p_{2n}) = \sigma(u^{n}\,u^{-2n}p_{2n}) = u^{-n}q_{2n}$$

$$\sigma(u^{-n}q_{2n}) = u^{n}\,\sigma(q_{2n}) = u^{n}\,u^{-2n}p_{2n} = u^{-n}p_{2n}$$

~~σp_n~~

$$\sigma(q_n) = u^{-n}p_n \qquad \begin{pmatrix}p_n\\ q_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar h_n & 1\end{pmatrix}\begin{pmatrix}u & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}p_{n-1}\\ q_{n-1}\end{pmatrix}$$

$$\sigma(p_n) = u^{-n}q_n$$



$$\begin{pmatrix}u^{-n}q_n\\ u^{-n}p_n\end{pmatrix} = \frac{1}{k_n}\begin{pmatrix}1 & \bar h_n\\ h_n & 1\end{pmatrix}\begin{pmatrix}u^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u^{-n+1}q_{n-1}\\ u^{-n+1}p_{n-1}\end{pmatrix}$$

$$\begin{pmatrix}q_n\\ p_n\end{pmatrix} = \underline{\qquad\qquad}\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix}\begin{pmatrix}q_{n-1}\\ p_{n-1}\end{pmatrix}$$

Interesting - might conjugation help ~~transformation~~ you define grid space. u might be a product of two conjugations. In fact it should be since there is some sort of dihedral group generated by σ and u.

E rank 2 free A-module with various bases. You want Wronskian, roughly a generator for $\Lambda^2_A E$, distinguished in some way, so you bases such that ~~transitions~~ transfer matrices have det 1. Should be related to the center of a σ. Go back to

$$\left|\begin{array}{ll} \sigma & \text{conj} \\ H & \text{herm. symm. form} \\ \omega & \text{volume} \end{array}\right.$$

H is intrinsic to the grid space. But you're confusing global + local H. Local H is a hermitian form over A, namely a hermitian form on ~~each eigenspace~~ the bundle of eigenspaces. Let's ~~[illegible]~~ work on making this precise.

$$W_\lambda = \left(E/(\lambda-u)E\right)^*$$

$$\psi_n = g(h_n)\begin{pmatrix}\lambda & 0 \\ 0 & 1\end{pmatrix}\psi_{n-1} \qquad \psi_n = \begin{pmatrix}\psi(p_n) \\ \psi(q_n)\end{pmatrix}$$

$$E_\lambda = E/(\lambda-u)E \quad \text{has bases} \begin{pmatrix}p_n \\ q_n\end{pmatrix} \bmod (\lambda-u)E$$

$$\Lambda^2 E_\lambda \ni p_n \wedge q_n$$

What are you doing? You might as well work with $\Lambda^2 E$, an A-module. You should be able to see the herm. form H on E over A

global $H = \int$ local H.
where local H ~~expressed~~ defined via conjugation σ and $\wedge$ = Wronskian.

Let's say $\wedge : E \times E \longrightarrow \Lambda^2_A E \xrightarrow[(\omega)]{\sim} A$
is the Wronskian. An A-bilinear altern. form

You feel already certain that E has an intrinsic $SU(1,1)$ structure over A in particular a real structure (given by any σ) and a volume ω.

~~Choose~~ $E = Ap_0 \oplus Aq_0$ basis $u^n p_0$, $u^n q_0$

~~Suppose you have~~

Global H says $u^n p_0$, $u^n q_0$ mutually orth with norm^2 = $+1$, -1.

Local H. $\underset{\sim}{H}(fp_0 + gq_0, \text{"})= |f|^2 - |g|^2$

So $\int \underset{\sim}{H}_{loc} \frac{d\theta}{2\pi} = H_{glob}$

Look at $\mathbb{C}^2$ with $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad -\begin{vmatrix} v_1 & v_1' \\ v_2 & v_2' \end{vmatrix}$

i.e. $H(v, v') = v^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v'$

$\sigma v = \sigma \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \bar{v}_2 \\ \bar{v}_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \bar{v}$

$H(\sigma v, v') = \bar{v}^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} v'$

$= v^t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} v' = (v_1 \;\; v_2) \begin{pmatrix} -v_2' \\ v_1' \end{pmatrix} = -v_1 v_2' + v_2 v_1'$

$$E = Ap_0 + Aq_0 \qquad H_{glob}(fp_0 + gq_0) = \int (|f|^2 - |g|^2)\frac{d\theta}{2\pi}$$

H_{loc} depends on conjugation σ and "volume" ω

$$H_{loc}(v, v') = \frac{\sigma v \wedge v'}{\omega} \begin{cases} \in \Lambda^2_A E \\ \in A. \end{cases} \qquad \begin{aligned} \sigma p_0 &= q_0 \\ \sigma q_0 &= p_0 \end{aligned}$$

$$H_{loc}(fp_0 + gq_0, \;{}^{\shortparallel}\;) = \frac{(\bar f q_0 + \bar g p_0) \wedge (fp_0 + gq_0)}{\omega}$$

$$= (|f|^2 - |g|^2)\,\frac{q_0 \wedge p_0}{\omega}$$

Today - systematic treatment of eigenfunctions and Green's functions.

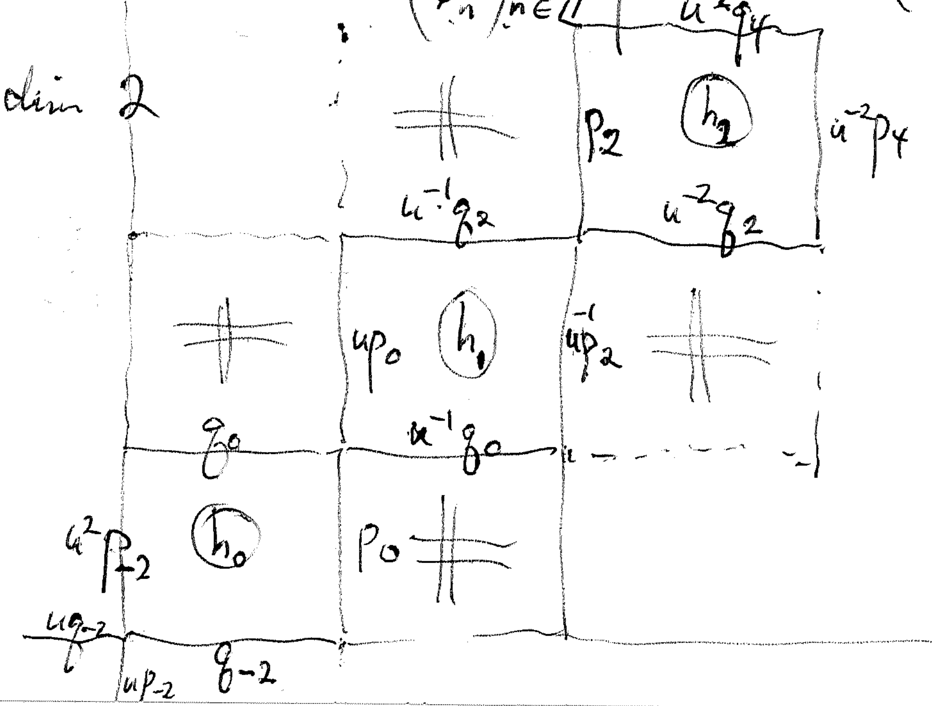
E = A-module with gen. $u^{-n}p_{2n}$, $u^{-n}q_{2n}$ $\quad n \in \mathbb{Z}$

subject to

$$\begin{pmatrix} u^{-n}p_{2n} \\ u^{-n}q_{2n} \end{pmatrix} = g(h_n)\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}\begin{pmatrix} u^{-n+1}p_{2n-2} \\ u^{-n+1}q_{2n-2} \end{pmatrix}$$

$$W_\lambda = \left(E/(\lambda - u)E\right)^* \simeq \left\{ \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}_{n \in \mathbb{Z}} \;\middle|\; \psi_n = g(h_n)\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}\psi_{n-1} \quad n \in \mathbb{Z} \right\}$$

W_λ dim 2



Repeat what's important

$$W_\lambda = \left(E/(\lambda-u)E\right)^* = \left\{ \begin{pmatrix} \psi_n^1 \\ \psi_n^2 \end{pmatrix}_{n\in\mathbb{Z}} \;\middle|\; \psi_n = \overbrace{g(h_n)\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}}^{\alpha_n} \psi_{n-1} \right\}_{n\in\mathbb{Z}}$$

$\in \mathbb{C}^2$

You have a recursion relation given by a chain of 2x2 matrices: $\alpha_n = g(h_n)\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix}$. Can pose the G fn. question, i.e. solving the inhomogeneous equation (?) except at one site.

What do you ~~kno~~ want to prove?, understand.

<u>Assertion</u>: For $|\lambda| \neq 1$ W_λ sp into lines L_λ^-, L_λ^+ consisting respectively of eigenfunctions decaying at $n \to -\infty$, $+\infty$.

~~Assertion~~

The problem is to link $\frac{1}{\lambda-u}$ with W_λ, the rough idea is that $\frac{1}{\lambda-u}\xi$ is roughly an eigenvector for u, at least

$$\left.(\lambda-u)\left(\frac{1}{\lambda-u}\xi\right)\right| = 0 \quad \text{on } \xi^\perp$$

so you should be able to use partial unitary ideas. Take $\xi = p_0$

First go over the eigenvector equation. Given u unitary on H, Y closed subspace of H, $X = u^{-1}Y \cap Y$, $\xi \in H$ an eigenvector $u(\xi) = \lambda\xi$.

$$\mathcal{H} = Y \oplus Y^\perp$$
$$= X \oplus V^+ \oplus Y^\perp = uX \oplus V^- \oplus Y^\perp$$
$$\xi = x_1 + \underbrace{v^+ + \eta}_{\in X^\perp} = ux_2 + v^- \oplus \eta$$
$$u\xi \in ux_1 + u(X^\perp) \qquad \lambda\xi = u(\lambda x_2) + \underbrace{\lambda(v^- + \eta)}_{\in (uX)^\perp}$$
$$\in ux_1 + (uX)^\perp.$$

1. $u\xi = \lambda\xi \implies ux_1 = u(\lambda x_2) \implies x_1 = \lambda x_2$

So putting $x_2 = x$ you get

$$\lambda x + v^+ = ux + v^-$$
$$(\lambda - u)x = -v^+ + v^-$$

$a : X \hookrightarrow Y \qquad ua = b \qquad (\lambda a - b)x = -v^+ + v^-$

~~Basic~~ Point: $\forall y$ can solve $(\lambda a - b)x = -v^+ + y$ for $|\lambda| > 1$ and —— $(\lambda a - b)x = -y + v^-$ for $|\lambda| < 1$.

$$\underbrace{Y \xrightarrow{\begin{pmatrix} a^* \\ \varepsilon_+^* \end{pmatrix}} \begin{matrix} X \\ \oplus \\ V^+ \end{matrix} \xrightarrow[\quad]{\overset{a - \lambda^{-1}b}{(a \;\; \varepsilon_+)}} Y}_{1 - \lambda^{-1}ba^*}$$

$$aa^* + \varepsilon_+\varepsilon_+^* = 1.$$
$$\begin{pmatrix} a^* \\ \varepsilon_+^* \end{pmatrix}(a \;\; \varepsilon_+) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

above is review of partial ~~unitary~~ isometry formalism.

So $$\begin{matrix} X \\ \oplus \\ V^+ \end{matrix} \xrightarrow[\sim]{\begin{pmatrix} a^* \\ \varepsilon_+^* \end{pmatrix}(1 - \lambda^{-1}ba^*)^{-1}} Y$$

Can you apply this to p_0?

Get back to stumbling block, namely, to relate $\frac{1}{\lambda-u}$ or $\hat{E}$ to Green's functions(s) G_λ

Recall what G_λ is. $W_\lambda = (E/(\lambda-u)E)^*$, recall that $E/(\lambda-u)E$ is universal for solutions to the grid eqns with eigenvalue λ. For $|\lambda|\neq 1$ you should know that W_λ splits into lines $L_\lambda^{\pm}$ of appropriate decay. Then $G_{\lambda m}$

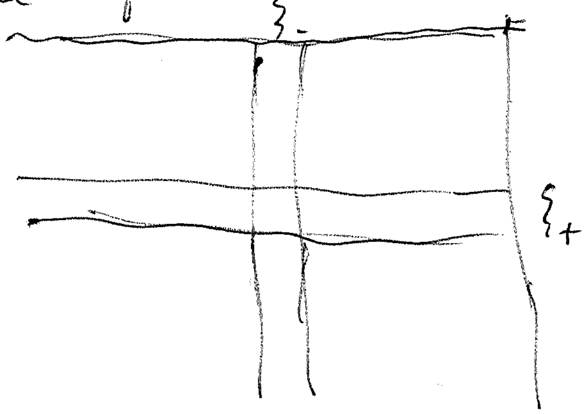
Idea: Work out the details in the scattering situation.

Philosophy: Why do you know the decaying lines in W_λ exist? You should have a proof via convergence of the Schur expansion, which uses non Euclidean distances. This should relate to convergence of Neumann (geometric) series.

Scattering situation

Picture of resolvent

Clear on $L^2(S^1)$



u = mult by z

$\frac{1}{\lambda-u}$ = mult by $\frac{1}{\lambda-z}$

Given p_n, say p_0 express in scattering picture

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} d^2 & -b^2 \\ -c^2 & a^2 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\frac{1}{\lambda-u}p_0 = \xi_+\frac{d^2}{\lambda-z} + \xi_-\frac{-b^2}{\lambda-z}$$

~~Simplest to use~~

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \underbrace{\begin{pmatrix} a^r & b^r \\ c^r & d^r\end{pmatrix}}_{\in \begin{pmatrix} zH_- & H_- \\ zH_+ & H_+ \end{pmatrix}} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix} \qquad \begin{matrix} 1 & h z^{-1} \\ h z & 1 \end{matrix}$$

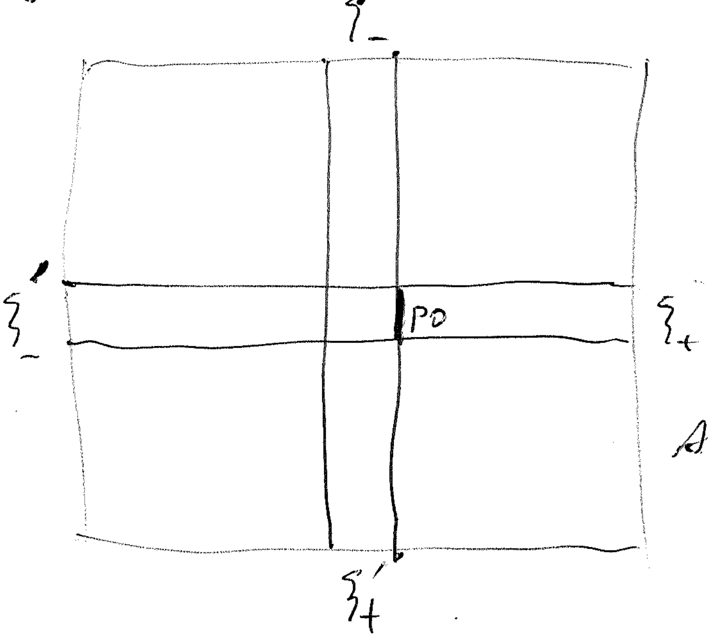
$b^r \in H_-$

$$\frac{1}{\lambda-u}\, p_0 = \xi_+ \frac{d^r(z)}{\lambda - z} + \xi_- \frac{(-b^r)(z)}{\lambda - z}$$

$$= -\xi_+\left(\frac{d^r(\lambda)}{z-\lambda} + \underbrace{\frac{d^r(z)-d^r(\lambda)}{z-\lambda}}_{H_+}\right) - \xi_-\underbrace{\left(\frac{b^r(z)}{\lambda - z}\right)}_{H_-}$$

Up to now you've done orth proj to get p_n, q_n.

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \frac{1}{d}\underbrace{\begin{pmatrix} d^r & b^l \\ -c^r & d^l \end{pmatrix}}_{\in\begin{pmatrix} H_+ & H_+ \\ zH_+ & H_+ \end{pmatrix}}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$



Ao $\frac{1}{\lambda - z}\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$?

Still stuck on Green's functions.
~~Do~~ Work them out in the scattling case

~~Quite~~ Repeat the calculation of the Green's function for $\partial_t \tilde{\psi} = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix} \tilde{\psi}$ $m(x)$ decaying enough.

$$\psi = \begin{pmatrix} e^{xs} & 0 \\ 0 & e^{-xs} \end{pmatrix} \phi$$

assume $\tilde{\psi} = \psi e^{st}$

$$s\psi = \begin{pmatrix} \partial_x & -m \\ \bar{m} & -\partial_x \end{pmatrix} \psi.$$

$$s\psi = \left(\varepsilon \partial_x + \begin{pmatrix} 0 & -m \\ \bar{m} & 0 \end{pmatrix} \right) \psi$$

$$\varepsilon \partial_x \psi = (s - V)\psi \qquad \partial_x \psi = (\varepsilon s - \varepsilon V)\psi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} \psi$$

$$\psi = \overset{g_x}{e^{x\varepsilon s}} \phi \qquad g_x = e^{x\varepsilon s}$$

$$\partial_x g\phi = \begin{pmatrix} s & m \\ \bar{m} & -s \end{pmatrix} g\phi$$

$$\psi = g\phi \qquad \partial_x \psi = \overset{g\varepsilon s}{\overset{\|}{(\partial_x g)}} \phi + g\, \partial_x \phi$$

$$\overset{\|}{(\varepsilon s - \varepsilon V) g\phi} \qquad g = \begin{pmatrix} e^{xs} & 0 \\ 0 & e^{-xs} \end{pmatrix}$$

$$g\, \partial_x \phi = -\varepsilon V g\phi = \begin{pmatrix} 0 & m e^{-2xs} \\ \bar{m} e^{2xs} & 0 \end{pmatrix} \phi$$

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so check $\partial_t \psi(x,t) = \begin{pmatrix} \partial_x & -m \\ +\bar{m} & -\partial_x \end{pmatrix} \psi(x,t)$ $\begin{pmatrix} s & +m \\ \bar{m} & -s \end{pmatrix}$

$$\psi(x,t) = \psi(x,s)\, e^{st}$$

$$s\psi = (\varepsilon \partial_x + V)\psi$$

$$\partial_x \psi = (\varepsilon s - \varepsilon V)\psi$$

$$\overset{g\varepsilon s}{(\partial_x g)}\phi + g\partial_x \phi = (\cancel{\varepsilon s} - \varepsilon V) g\phi$$

$$\partial_x \phi = g^{-1}(-\varepsilon V) g\, \phi$$

$$\left(s - \varepsilon\partial_x - \varepsilon V\right)G(x,x') = \delta(x-x')I$$

$$\Phi(\infty,-\infty) = \Phi(\infty,0)\,\Phi(0,-\infty)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}\begin{pmatrix} a^\ell & b^\ell \\ c^\ell & d^\ell \end{pmatrix}$$

$$\partial_x\phi = \begin{pmatrix} 0 & m e^{-2sx} \\ \bar{m} e^{2sx} & 0 \end{pmatrix}\phi$$

Anyway $\quad G^<(x,x') = e^{\varepsilon s x}\,\Phi(x,-\infty)\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}$

As $x \to -\infty \quad \Phi(x,-\infty) \longrightarrow I$

$$e^{\varepsilon s x}\,\Phi(x,-\infty) \sim \begin{pmatrix} e^{\varepsilon s x} & 0 \\ 0 & e^{-\varepsilon s x} \end{pmatrix}$$

Assume $\mathrm{Re}(s) > 0 \quad e^{\varepsilon s x}$ decays as $x \to -\infty$

$$G^<(x,x') = e^{\varepsilon s x}\,\Phi(x,-\infty)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix}$$

$$G^>(x,x') = e^{\varepsilon s x}\,\Phi(x,+\infty)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$$

$x'=0$ first

$$\left(G^>(0^+,0) - G^<(0^-,0)\right)(-\varepsilon) = I$$

$$G^>(0^+,0) = \underbrace{\Phi(0,+\infty)}_{\begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} -b^r \\ a^r \end{pmatrix}\begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix}$$

$$\underbrace{-G^<(0^-,0)}_{-\varepsilon} = \Phi(0,-\infty)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \end{pmatrix} = \begin{pmatrix} a^\ell \\ c^\ell \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \end{pmatrix}$$

$$-\varepsilon = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d^r & -b^r \\ -c^r & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} \qquad \begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}\begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

$$\begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} \qquad \begin{pmatrix} \xi'_- \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ \frac{c}{a} & \frac{1}{a} \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}\begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \frac{1}{a}\begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix}$$

$$\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix}^{-1}(-\varepsilon)$$

$$= \frac{1}{a}\begin{pmatrix} -a^r & b^r \\ +c^l & a^l \end{pmatrix}$$

$$G^{>}(x, x') = e^{\varepsilon s x}\begin{pmatrix} -b^r_x \\ a^r_x \end{pmatrix}\frac{1}{a}\left(c^l_{x'}, a^l_{x'}\right)e^{-\varepsilon s x'}$$

$$G^{<}(x, x') = e^{\varepsilon s x}\begin{pmatrix} a^l_x \\ c^l_x \end{pmatrix}\frac{1}{a}\left(a^r_{x'}, -b^r_{x'}\right)e^{-\varepsilon s x'}$$

So what's up? You want to study G-function in discrete case. Let's begin ~~xxxx~~ by studying the scattering situation ~~xxxx~~ Concerned about eigenfunctions.

$$\psi_n = g(h_n)\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\psi_{n-1}$$

Same as $\psi : E \longrightarrow \mathbb{C} \ni$ ~~xxx~~ $\psi u = \lambda\psi$

$\psi_n = \psi\begin{pmatrix}p_n\\ q_n\end{pmatrix}$. Let Ψ denote propagator

$$\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} = \Psi(n,-\infty)\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

$$\Psi_{\cancel{m}}(n,-\infty) = \prod_{n\geq m} \frac{1}{K_m}\begin{pmatrix}1 & h_m z^{-m}\\ \bar h_m z^m & 1\end{pmatrix}$$

$$= T\exp\sum\begin{pmatrix}0 & h_m z^{-m}\\ \bar h_m z^m & 0\end{pmatrix}$$

slightly different

$$\begin{pmatrix}u^{-n}p_n\\ q_n\end{pmatrix} = \frac{1}{K_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z^n & 1\end{pmatrix}\begin{pmatrix}u^{-n+1}p_{n-1}\\ q_{n-1}\end{pmatrix}$$

$$\Psi(n,-\infty) = \begin{pmatrix}a_n^{\leq} & b_n^{\leq}\\ c_n^{\leq} & d_n^{\leq}\end{pmatrix} = \frac{1}{K_n}\begin{pmatrix}1 & h_n z^{-n}\\ \bar h_n z & 1\end{pmatrix}$$

$$\Psi(n,\infty) = \begin{pmatrix}a_n^{>} & b_n^{>}\\ c_n^{>} & d_n^{>}\end{pmatrix} \times \begin{pmatrix}a_n^{<} & b_n^{<}\\ c_n^{<} & d_n^{<}\end{pmatrix}$$

Let's try to understand how things work. Do calculations in this ~~notion~~ notation. Because of the scattering hypothesis things should be simple to control?

The aim is to link the resolvent $\frac{1}{\lambda - u}$ on $\hat{E} = \mathcal{H}$ with the Green's function assoc. to $\psi_n = g(h_n)\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\psi_{n-1}$

(Recall $\psi_n = \psi\begin{pmatrix} u^{-n} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} \lambda^{-n}\psi(p_n) \\ \psi(q_n) \end{pmatrix}$ ~~where~~

$\psi \in \left(E/(\lambda-u)E\right)^*$.

What is the Green's function supposed to be? You have the site n associated to p_n, q_n. Actually this should be a special case of 1st order linear recursion relation

$$\psi_n = \alpha_n \psi_{n-1}$$

Idea: Study by introducing diagonal matrix. Introduce variable $\alpha = (\alpha_n)$ z. There is a simple direct or inverse system here you want to understand

Repeat. Suppose given V_n $\mathbb{Z}$ functor defined on the poset $\mathbb{Z}$:

$$\cdots \longrightarrow V_n \xrightarrow{f_n} V_{n+1} \longrightarrow$$

$$\varinjlim V_k \longrightarrow \prod V_k \xrightarrow{1-f} \prod V_k$$

$$0 \longrightarrow V_n \xrightarrow{(f^n_k)} \prod_{k \geq n} V_k \longrightarrow \prod_{k > n} V_k \xrightarrow{?} 0$$

You need a suitable framework, preferably abstract for handling this

How to get started. Functor on $\mathbb{Z}$

$n \mapsto V_n$ together with $f_n : V_n \to V_{n+1}$

some natural things to consider are $\varinjlim_n V_n$

$\varprojlim_n V_n$. A nice way to describe $\varinjlim_n V_n$

is to form $\bigoplus_{n\in\mathbb{Z}} t^n V_n$ module over $\mathbb{C}[t]$

and then $\varinjlim V_n = \bigoplus_{n\in\mathbb{Z}} t^n V_n \Big/ (t-1) \bigoplus_{n\in\mathbb{Z}} t^n V_n$

and something you can also do is to localize

$M(V) = \bigoplus t^n V_n$ $\qquad M(V) \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}[t,t^{-1}]$

which should be $= (\varinjlim V_n) \underset{\mathbb{C}}{\otimes} \mathbb{C}[t,t^{-1}]$.

Somewhere in this algebra should lie some standard tools of an analysist.

So wh[illegible] it is not [illegible] to

Repeat: Consider $\to V_n \xrightarrow{f} V_{n+1} \xrightarrow{f} V_{n+2} \to$

i.e. a graded vector space $\bigoplus_{n\in\mathbb{Z}} V_n$ with an an operator f of degree $+1$. You get a difference equation $v_n = f v_{n-1}$ $\forall n$ which you can solve in the increasing n direction. ~~There's also alternate~~ ~~First case~~

Form $M = \bigoplus t^n V_n$ graded module over $\mathbb{C}[t]$ which is equivalent graded module $V = \bigoplus_{n\in\mathbb{Z}} t^n V_n$ over $\mathbb{C}[t]$ is the same as a system of vector spaces

$$\longrightarrow V_{n-1} \longrightarrow V_n \longrightarrow V_{n+1} \longrightarrow \qquad \text{Alg.}$$

~~[illegible]~~ Questions. Objects. Such a V determines a quasi-coh. sheaf on $\mathbb{P}^1$

Can you make ~~[illegible]~~ something interesting with $\mathbb{Z}$ tree? Before this you need ~~[illegible]~~ spectrum, λ,

Go back to V:

$$\longrightarrow V_{n-1} \xrightarrow{a} V_n \xrightarrow{a} V_{n+1} \longrightarrow$$

= graded $\mathbb{C}[t]$ module. You have $\varinjlim V_n$ and $R\varprojlim V_n$ associated to ~~[illegible]~~ V

$$0 \longrightarrow \mathbb{C}[t]\otimes V \xrightarrow{t\otimes 1 - 1\otimes a} \mathbb{C}[t]\otimes V \longrightarrow V \longrightarrow 0$$

$$0 \longrightarrow \mathbb{C}[t]\otimes\mathbb{C}[t] \xrightarrow[\text{exact}]{t\otimes 1 - 1\otimes t} \mathbb{C}[t]\otimes\mathbb{C}[t] \longrightarrow \mathbb{C}[t] \longrightarrow 0$$

$$\mathbb{C}[t,t^{-1}]\otimes_{\mathbb{C}[t]} V = V[t^{-1}] \simeq \mathbb{C}[t,t^{-1}]\otimes \varinjlim V_n$$

$$0 \longrightarrow \mathbb{C}[t,t^{-1}]\otimes V \xrightarrow{t-a} \mathbb{C}[t,t^{-1}]\otimes V \longrightarrow V[t^{-1}] \longrightarrow 0$$

~~Consider a system~~

~~$\to V_n \to V_{n+1} \to \cdots$~~

~~which leads to~~

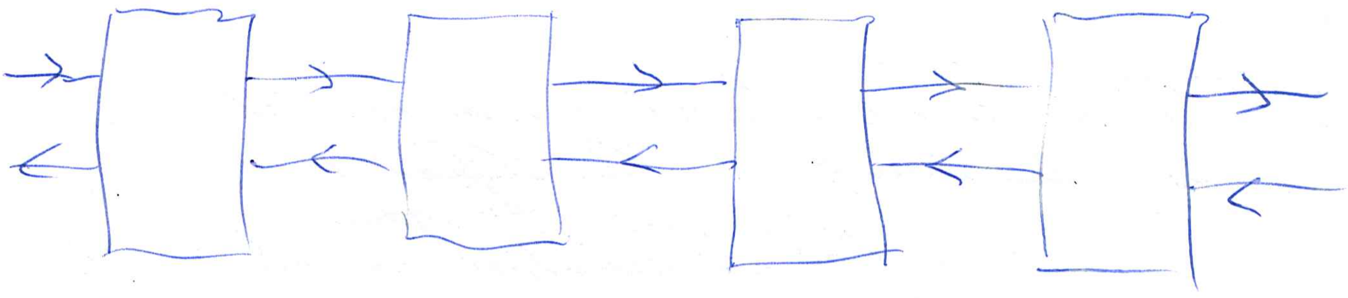
IDEA: contraction operators might help understand how to handle the arithmetic analogs of vector bundles: lattices with pos. quad. form. Perhaps also useful would be KK constructions related to Weyl von-Neumann thm. → compact perturbation of any self adjoint op to one having a basis of L^2-eigenvectors.

~~Yesterday you looked at a system~~

~~$\to V_n \to V_{n+1} \to$~~

~~which you~~

Start with ladder system



where each box is represents a 2 dim'l space with four distinguished elements related by a $SU(1,1)$ transfer matrix.

You want to view this ladder as living over $\mathbb{Z}$; a coefficient system of some sort.

~~[illegible]~~ The idea I had was to combine various things, basically take constructions from algebra but add inner products, spectrum, λ,

recursion relation $\psi_n = g_n \psi_{n-1}$

system $\longrightarrow V_{n-1} \xrightarrow{g_n} V_n \longrightarrow V_{n+1} \longrightarrow$

algebraically a system $\uparrow$, i.e. functor defined on the poset $\mathbb{Z}$, has associated $\varinjlim$ and $\varprojlim$, possibly derived functors. This alg seems to be trivial in the case when all g_n are isos. But it is not trivial, there is this spectrum idea,. The system is equivalent to a graded $\mathbb{C}[t]$-module $V = \bigoplus_{n\in\mathbb{Z}} V_n$. Then you can localize

$$V_{(t^{-1})} = \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} V$$

turns out

$$= \mathbb{C}[t,t^{-1}] \otimes \varinjlim V_n$$

so you have a family of quotients

$$V/(\lambda - t)V$$

all ~~of which are~~ isomorphic to $\varinjlim V_n$. Somehow this brings λ into play

Further idea: Acyclic system on the $\mathbb{Z}$ tree

$$(*) \quad \begin{tikzcd}[column sep=small] V_0 & & V_1 & & V_2 \\ & V_{01} \arrow[ul] \arrow[ur] & & V_{12} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

You would like to fit together ~~this~~ such a system and the ladder. Although: there's a problem with the spectrum maybe.

To get started, take the ladder, which comes with λ, λ^{-1}. Notice that ~~the~~ a $\mathbb{Z}$ tree system gives a graded K-module

$$\begin{tikzcd} \bigoplus_n V_n \\ \bigoplus_n V_{n,n+1} \arrow[u, shift left, "a"] \arrow[u, shift right, "b"'] \end{tikzcd}$$

so that automatically you have a notion of spectrum, the λ for which $\lambda a - b$ is invertible, and for such λ you get a splitting. Because of the grading ~~that λ outside the sp~~ the spectrum should be invariant under $\mathbb{C}^*$. The splitting should essentially be the G fn.

The first observation is that a system on the $\mathbb{Z}$-tree:

$$\begin{tikzcd}[column sep=small] & V_0 & & V_1 & & V_2 \\ V_{-1,0} \arrow[ur] & & V_{01} \arrow[ul] \arrow[ur] & & V_{12} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

yields a K-module

$$\bigoplus_{n\in\mathbb{Z}} V_{n,n+1} \overset{a}{\underset{b}{\rightrightarrows}} \bigoplus_{n\in\mathbb{Z}} V_n ,$$ whence you can define λ to be outside the spectrum when $\lambda a - b$ is invertible, ~~and then~~ you should have the desired splitting of V_n $\forall n$. Because of the grading the non spectrum is closed under $\mathbb{C}^\times$. I think the $\mathbb{Z}$-tree system splits canonically into two parts, the former having $a \simeq$ and ba^{-1} locally nilpotent, the latter —— $b \simeq$ —— ab^{-1} nilpotent. Note that if $a \simeq$, then you have a system $\longrightarrow V_0 \xrightarrow{b} V_1 \xrightarrow{b} V_2 \xrightarrow{b} \cdots$ and you want $\lambda - b$ to be invertible on $\bigoplus V_n$ which should mean b loc. nilp.

Next consider the latter. Consider the latter as a recursion relation

$$\psi_n = g(h_{2n}) \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \psi_{n-1}$$

universal case is $\quad \psi_n = \begin{pmatrix} u^{-n} p_{2n} \\ u^{-n} q_{2n} \end{pmatrix}$

$$u^{-n}\begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix} = g(h_{2n}) \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} u^{-n+1} \begin{pmatrix} p_{2n-2} \\ q_{2n-2} \end{pmatrix}$$

V_n spanned by ψ_n^1, ψ_n^2 ?

The problem is to translate the recursion relation into a $\mathbb{Z}$ tree system, or maybe into just a K-module. First ψ

---

analyze recursion relation. Suppose $h_{2n} = 0$ all n

$$\psi_n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \psi_{n-1} \qquad \text{special case} \quad \psi_n^1 = \lambda \psi_{n-1}^1$$

universal situation is $E = \mathbb{C}[u,u^{-1}]$ with $\psi_n^1 = u^n$

Check this: Given a $\mathbb{C}[u,u^{-1}]$ module M, then a sequence $\psi_n^1 \in M$ $\ni$ $\psi_n^1 = u\psi_{n-1}^1$ corresp. 1-1 with the elt $\psi_0^1 \in M$, namely $\psi_n^1 = u^n \psi_0^1$

Next. for $\psi_n = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \psi_{n-1}$, then $\psi_n^1 = u^n \psi_0^1$

$$\psi_n^2 = u^{-n} \psi_0^2$$

gives 1-1 corresp. between $\left(\psi_n\right)_{n \in \mathbb{Z}}^{\text{solns in } M}$ and (ψ_0) in M.

~~Maybe you have~~ You want to fit the ladder situation to the $\mathbb{Z}$ tree, in particular to get a K-module corresponding to the ladder

So you need to specify V_n $V_{n,n+1}$

The V_n are generators, $V_{n,n+1}$ relations

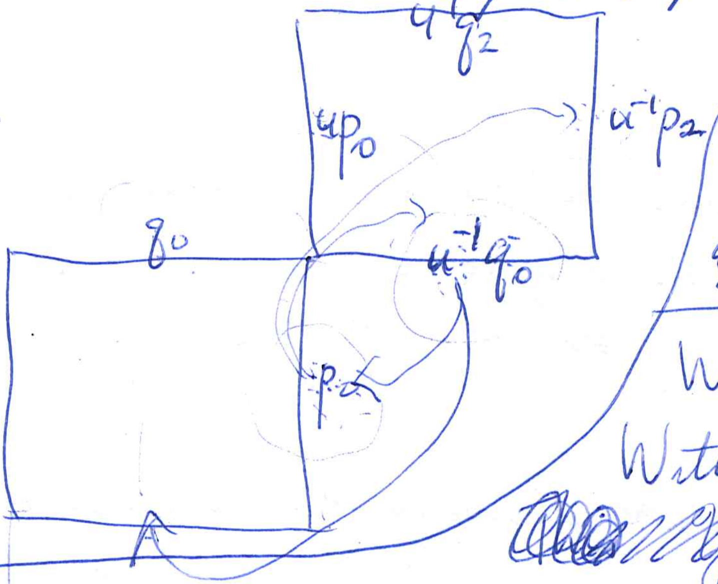
generators are $u^{-n} p_{2n}$, $u^{-n} q_{2n}$

$p_0, q_0, u^{-1}p_2, u^{-1}q_2$

$$\begin{pmatrix} u^{-1} p_2 \\ u^{-1} q_2 \end{pmatrix} = g(h_2) \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$$

$$\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_2 \end{pmatrix} = \frac{1}{k_2}\begin{pmatrix} 1 & h_2 \\ \bar{h}_2 & 1 \end{pmatrix} \cancel{\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}} \begin{pmatrix} up_0 \\ u^{-1}q_0 \end{pmatrix}$$

$$\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_0 \end{pmatrix} = \begin{pmatrix} k_2 & h_2 \\ -\bar{h}_2 & k_2 \end{pmatrix} \begin{pmatrix} up_0 \\ u^{-1}q_2 \end{pmatrix}$$



Recall how a ladder looks: orthogonal 2-planes associated to each site, 2-planes ~~[illegible]~~

Where to start? With grid space E. ~~[illegible]~~

1 space diml wave equation Cauchy problem. Disc 1-diml space leads to a ~~[illegible]~~ ~~grid space~~ Picture: of a line

$t=0$ two coords q_0 at each site ~~x~~.
There's a grid space E of states with compact support. A dual space E' of distributions = linear functionals on E. And $\mathcal{H} = \hat{E}$ between the two: $E \subset \hat{E} \subset E'$

What actually takes place, what is E' exactly. E is grid space hence isomorphic as A-module to $A^{\oplus 2}$. Get nice basis for E: $u^n p_0$, $u^n q_0$

If you use orth basis for E say asc staircase, then the nature of E, E' becomes clearer.

E is grid space, a free A module of rank 2

Wronskian pairing should mean E isom to $E^\vee = \mathrm{Hom}_A(E, A)$.

There seems to be some Wronskian angle to the Green's function. e.g. for $(\partial_x^2 - V)\psi = 0$

~~$\psi^<(x, x')$ decaying soln. to left~~

~~$\psi^>(x, x')$ soln.~~

$\psi^<(x)$, $\psi^>(x)$ solns decaying to left, right

$$G^<(x, x') = B\psi^<(x) \qquad x < x'$$

$$G^>(x, x') = A\psi^>(x) \qquad x > x'$$

$$(G^> - G^<)(x', x') = 0 \Rightarrow A\psi^>(x') - B\psi^<(x') = 0$$

$$(\partial_x G^> - \partial_x G^<)(x', x') = 1 \Rightarrow A(\partial_x\psi^>)(x') - B(\partial_x\psi^<)(x') = 1$$

$$\begin{pmatrix} \psi^> & \psi^< \\ \partial_x\psi^> & \partial_x\psi^< \end{pmatrix}\begin{pmatrix} A \\ -B \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} A \\ -B \end{pmatrix} = \frac{1}{W}\begin{pmatrix} \partial_x\psi^< & -\psi^< \\ -\partial_x\psi^> & \psi^> \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{W}\begin{pmatrix} -\psi^< \\ \psi^> \end{pmatrix}$$

Looks like change to $(-\partial_x^2+V)\psi=\delta$

$$G(x,x')=\frac{\psi^>(x^>)\psi^<(x^<)}{W(\psi^>,\psi^<)}$$

~~$(k^2+\partial_x^2)\psi=0 \qquad \psi=e^{ikx},\ e^{-ikx}$~~

$$(-\partial_x^2+m^2)\psi=0 \qquad \psi^>=e^{-mx},\ \psi^<=e^{+mx}$$

$$G=\frac{e^{-mx^>}\,e^{+mx^<}}{\begin{vmatrix} e^{-mx} & e^{+mx} \\ -me^{-mx} & +me^{+mx}\end{vmatrix}}=\frac{e^{-mx^>}\,e^{mx^<}}{+2m}$$

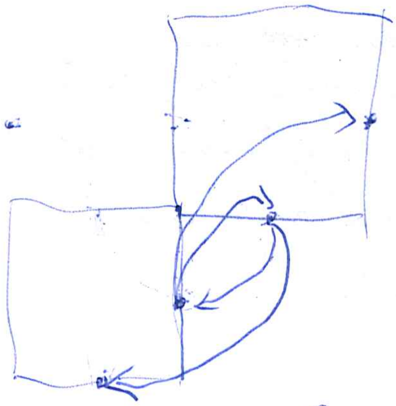
Try again to understand $\frac{1}{\lambda-u}$ on $\hat{E}=\mathcal{H}$. You want to relate this operator applied to $\xi_0\in E$ to a Green's function. Idea of G is that given λ you have W_λ a 2-diml space of eigenfns. containing decaying ones.

$\psi=\left(\xi_0\frac{1}{\lambda-u^{-1}}\Big|\ -\right)$. This is a linear fnl on E. almost an eigenfunction since

$$\psi((\lambda-u)\xi)=\left(\xi_0\frac{1}{\lambda-u^{-1}}\Big|(\lambda-u)\xi\right)=(\xi_0|\xi)$$

which seems extremely simple if you are working with an orthonormal basis. $\lambda\psi(\xi)-\psi u(\xi)=$

So what next? Main problem lies with conjugation.



Digress. Go back to $(\mathcal{H}, u, \xi_+)$ $\quad \|\xi_+\| = 1$

$\xi_- = u\xi_+$. $\quad c_h = u(aa^* + \xi_+ h \xi_+^*) = ba^* + \xi_- h \xi_+^*$

~~Can you interpret the missing ...~~

Basically here you remove ξ_+ from the domain of u to get the partial isometry ba^*?

Question: Can you link $\frac{1}{\lambda - u}\xi$ to a partial eigenvector?

Go back to the ~~start~~ begining where you have $(\mathcal{H}, u, \xi_0)$ and you form $\frac{1}{\lambda - u}\xi_0$. Is this related to a partial ~~unitary~~ isom. You can form the partial isom. $X = \xi_0^\perp$, $a : X \hookrightarrow \mathcal{H}$
$b = ua : X \longrightarrow (u\xi_0)^\perp \subset \mathcal{H}$. The eigenv. eqn. is

$$(\lambda - u)ax = -v_+ + v_- \qquad \mathcal{H} = aX \oplus V_+$$
$$= bX \oplus V_-$$

$$\xi = ax_1 + v_+ \qquad u\xi = bx_1 + uv_+$$
$$= bx_2 + v_- \qquad \lambda\xi = \lambda bx_2 + \lambda v_-$$

proj onto bX

$\lambda x_2 = x_1$

$\lambda a x + v_+ = b x + v_-$ $\quad (\lambda a - b)x = -v_+ + v_-$

$x \perp \xi_0$? $\quad (\lambda - u)x = -\xi_0 + S(\lambda)u\,\xi_0$

$$x + \frac{1}{\lambda - u}\xi_0 = \frac{S(\lambda)u}{\lambda - u}\xi_0$$

$$x = \frac{S(\lambda)u - 1}{\lambda - u}\xi_0$$

Go over things. Fix ξ_0 form $\frac{1}{\lambda - u}\xi_0$

---

grid space E, u, ξ_0 grid vector

$\left(\frac{1}{\lambda - u^{-1}}\xi_0 \,\middle|\, -\right) = \psi$, $\quad \psi$ linear

functional on E such that $\quad \psi = \xi_0^* \frac{1}{\lambda - u}$

whence $\quad \psi \circ (\lambda - u) = \xi_0^*$

$H = aX \oplus \mathbb{C}\xi_0$ $\quad b = au$

$$\underbrace{(\lambda - u)x}_{(\lambda a - b)x} = \xi - \tilde{\xi}(\lambda)\xi_0$$

$$Y \xrightarrow{\begin{pmatrix} a^* \\ \xi_0^* \end{pmatrix}} \begin{pmatrix} X \\ \mathbb{C} \end{pmatrix} \underset{(a - \lambda^{-1}b \;\; \xi_0)}{\overset{(a \;\; \xi_0)}{\longrightarrow}} Y$$

$$\mathrm{id} \quad \frac{\begin{pmatrix} x \\ \tilde{\xi}(\lambda) \end{pmatrix}}{}$$

$$y \mapsto \begin{pmatrix} a^* \\ \xi_0^* \end{pmatrix} \frac{1}{1 - \lambda b a^*} y$$

$(a - \lambda^{-1}b)a^* + \xi_0\xi_0^* = 1 - \lambda b a^*$ $\quad (a - \lambda^{-1}b)x + \tilde{\xi}(\lambda)\xi_0 = y$

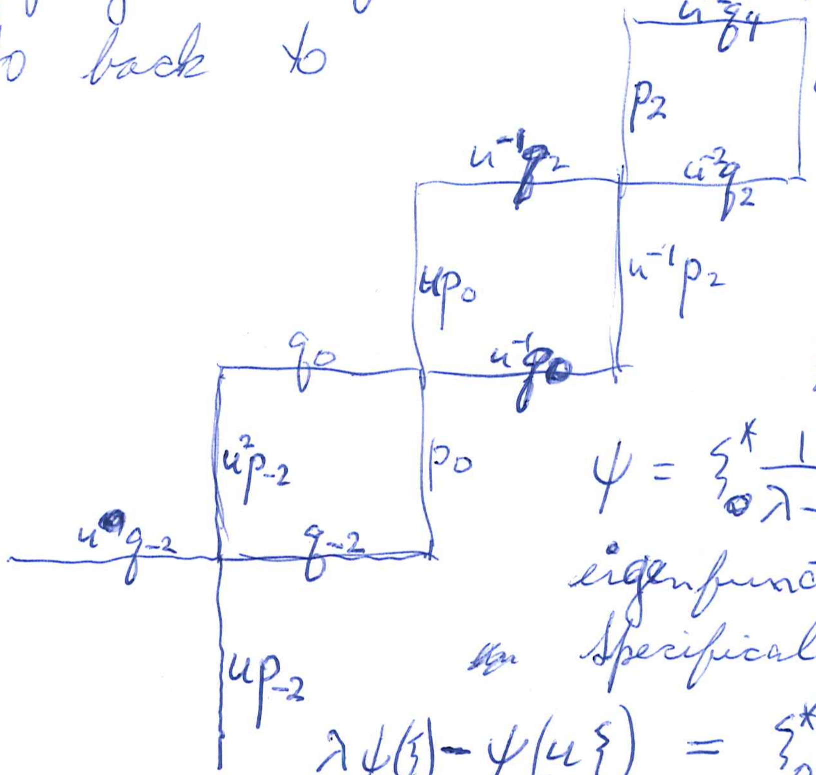
$\mathcal{H}, u, \xi_0$ $\quad \dfrac{1}{\lambda - u} = \sum_{n\geq 0} \dfrac{u^n}{\lambda^{n+1}} = -\sum_{n<0} \dfrac{u^n}{\lambda^{n+1}}$

$$\xi_0^* \frac{1}{\lambda-u}\xi = \sum_{n\geq 0} \lambda^{-n-1}(u^{-n}\xi_0 | \xi) \qquad |\lambda|>1$$

$$= -\sum_{n<0} \lambda^{-n-1}(u^{-n}\xi_0 | \xi) \qquad |\lambda|<1.$$

So the point is that $\xi_0^* \dfrac{1}{\lambda - u}$ is ~~...~~ the projection of $\mathcal{H}$ onto ~~the~~ $\overline{\mathbb{C}[u,u^{-1}]\xi_0}$

Go back to



So you pick a ξ_0 and consider

$\psi = \xi_0^* \dfrac{1}{\lambda - u}$. Almost an eigenfunction with eigenvalue λ. ~~...~~ Specifically

$$\lambda\psi(\xi) - \psi(u\xi) = \xi_0^*\xi$$

so now take

Apparently you missed the fact that $\xi_0^* \frac{1}{\lambda-u}$, a map from $\mathcal{H}$ to analytic functions of λ, $|\lambda| \neq 1$. should be equivalent to the orthogonal projection of $\mathcal{H}$ onto the cyclic rep $\overline{\mathbb{C}[u,u^{-1}]\xi_0}$ gen. by ξ_0. Why?

$$\xi_0^* \frac{1}{\lambda-u} \xi = \sum_{n \geq 0} \lambda^{-n-1} \underbrace{\xi_0^* u^n \xi}_{(u^{-n}\xi_0 | \xi)} \qquad |\lambda| > 1$$

$$\xi_0^* \frac{1}{\lambda-u} \xi = \xi_0^* \left(-\sum_{n \geq 0} \lambda^n u^{-n-1} \right) \xi$$

$$= -\sum_{n \geq 0} \lambda^n (u^{n+1}\xi_0 | \xi) \qquad |\lambda| < 1$$

$$= \sum_{n < 0} \lambda^n (u^{n+1}\xi_0 | \xi) \qquad |\lambda| < 1$$

$$\xi_0^* \frac{1}{\lambda-u} \xi = \frac{1}{\lambda} \sum_{n \geq 0} \frac{1}{\lambda^n} (u^{-n}\xi_0 | \xi) \qquad |\lambda| > 1$$

$$= -\sum_{n \,\text{[redacted]}\, 0} \lambda^{-n} (u^{-n+1}\xi_0 | \xi) \qquad |\lambda| < 1$$

I don't know quite how to write this in a way to remember.

So what's happening?

$$\xi_0^* \frac{1}{\lambda - u} \xi = \xi_0^* \left(\frac{1}{\lambda\left(1 - \frac{u}{\lambda}\right)} \right) \xi$$

$$= \xi_0^* \sum_{n \geq 0} \frac{u^n}{\lambda^{n+1}} \xi \qquad |\lambda| > 1$$

$$\frac{1}{\lambda - u} = -\frac{1}{u} \frac{1}{1 - \frac{\lambda}{u}} = -\sum_{n \geq 0} \frac{\lambda^n}{u^{n+1}} \qquad |\lambda| < 1.$$

$$\xi_0^* \frac{1}{\lambda - u} \xi = \sum_{n \geq 0} \frac{1}{\lambda^{n+1}} \left(u^{-n} \xi_0 \mid \xi \right) \qquad |\lambda| > 1$$

$$= -\sum_{n \geq 0} \lambda^n \left(u^{n+1} \xi_0 \mid \xi \right) \qquad |\lambda| < 1$$

But $u^{-n} \xi_0$ $\quad n \geq 0$ closed span =
$u^{n+1} \xi_0$ $\quad n \geq 0$ cyclic rep gen by ξ_0

So what do you know.

$\psi = \xi_0 \frac{1}{\lambda - u}$ is a linear fnl on grid space E, almost an eigenfunction since $\psi(\lambda - u)\xi = \xi_0^* \xi$ is mostly zero. Relate to partial sum.

recursion relation

$$\begin{pmatrix} u^{-n} p_{2n} \\ u^{-n} q_{2n} \end{pmatrix} = \frac{1}{k_{2n}} \begin{pmatrix} 1 & h_{2n} \\ \bar{h}_{2n} & 1 \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} u^{-n+1} p_{2n-2} \\ u^{-n+1} q_{2n-2} \end{pmatrix}$$

This relation involves

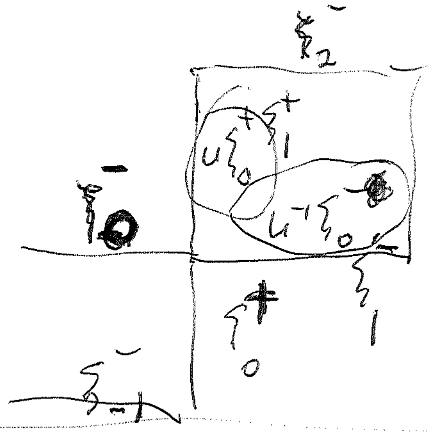
$$\begin{pmatrix} u^{-n}p_{2n} \\ u^{-n}q_{2n} \end{pmatrix} = \frac{1}{k_{2n}} \begin{pmatrix} 1 & h_{2n} \\ \overline{h_{2n}} & 1 \end{pmatrix} \begin{pmatrix} u^{-n}p_{2n-1} \\ u^{-n}q_{2n-1} \end{pmatrix}$$

$$\begin{pmatrix} u^{-n}p_{2n-1} \\ u^{-n}q_{2n-1} \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} u^{-n+1}p_{2n-2} \\ u^{-n+1}q_{2n-2} \end{pmatrix}$$

suppose $h_{2n}=0$ $\forall n$. Then ~~the description~~ inhomogeneous rec. relation sol of $\psi = \xi_0^{+*}\frac{1}{\lambda-u}$ should be easy.

$$\xi_{2n}^{+} = u^{-n}p_{2n} \qquad \xi_{2n}^{-} = u^{-n}q_{2n}$$

$$u\xi_0^+ = \xi_1^+ = \xi_2^+$$



orth. basis $\ldots \xi_0^+, \xi_1^-, \xi_2^+, \xi_3^-, \ldots$

such that
$$\boxed{\begin{aligned} u\,\xi_{2n}^+ &= \xi_{2n+2}^+ \\ u\,\xi_{2n+1}^- &= \xi_{2n-1}^- \end{aligned}}$$

$$\psi_{2n} \stackrel{\text{def}}{=} (\xi_0^+)^* \frac{1}{\lambda - u}\, \xi_{2n}^+ \qquad \text{Then} \qquad u\xi_{2n}^+ = \xi_{2n+2}^+$$

says
$$\psi_{2n+2} = (\xi_0^+)^* \frac{1}{\lambda-u}\, u\, \xi_{2n}^+$$
$$= \underbrace{(\xi_0^+)^* \frac{1}{\lambda-u}(u-\lambda)\xi_{2n}^+}_{=\,-\delta_{0,2n}} + \underbrace{(\xi_0^+)^* \frac{1}{\lambda-u}\lambda\, \xi_{2n}^+}_{\lambda\psi_{2n}}$$

$$\psi_{2n+2} = -\delta_{0,2n}$$

The simple observation is that if η is a glid vector say, then $\psi = \eta^* \frac{1}{\lambda - u}$ is almost an eigenfunction (digress: there are two ~~[illegible]~~ ways to interpret η^* either $(\eta | -)$ or $IH(\eta, -)$, ~~[illegible]~~ maybe this doesn't work for IH, because of the resolvent).

ψ almost an eigenfn in the sense that

$$\lambda\psi - \psi u = \eta^* \frac{1}{\lambda - u}(\lambda - u) = \eta^*$$

has very small support.

~~[illegible]~~

Notice that $(\lambda - {}^t u)\psi = \eta^*$ so that ψ is some ~~[illegible]~~ choice for $\frac{1}{\lambda - {}^t u}$ on the ~~[illegible]~~ space of linear functionals, at least those of the form η^* with η square integrables. You know something here, namely the kernel of $\lambda - {}^t u$ is 2-diml. ~~[illegible]~~ ~~[illegible]~~ It looks like you ~~can first~~ get some sort of Green's function on the dual space in a tautological way.

It seems that you have tautologically solved the problem of relating the resolvent operator $\frac{1}{\lambda - u}$ on $\mathcal{H}$ to the Green's function assoc. to $\lambda - u^t$ on E^*. Given $\eta^* \in E'$ with $\eta \in \mathcal{H}$, then $\psi = \frac{1}{\lambda - u^t}\eta^*$, defined to be $\eta^* \frac{1}{\lambda - u} \in E'$, obviously satisfies $(\lambda - u^t)\psi = \eta^*$ so $\eta^* \longmapsto \psi$ is the Green's function.

E grid space, E' is the dual

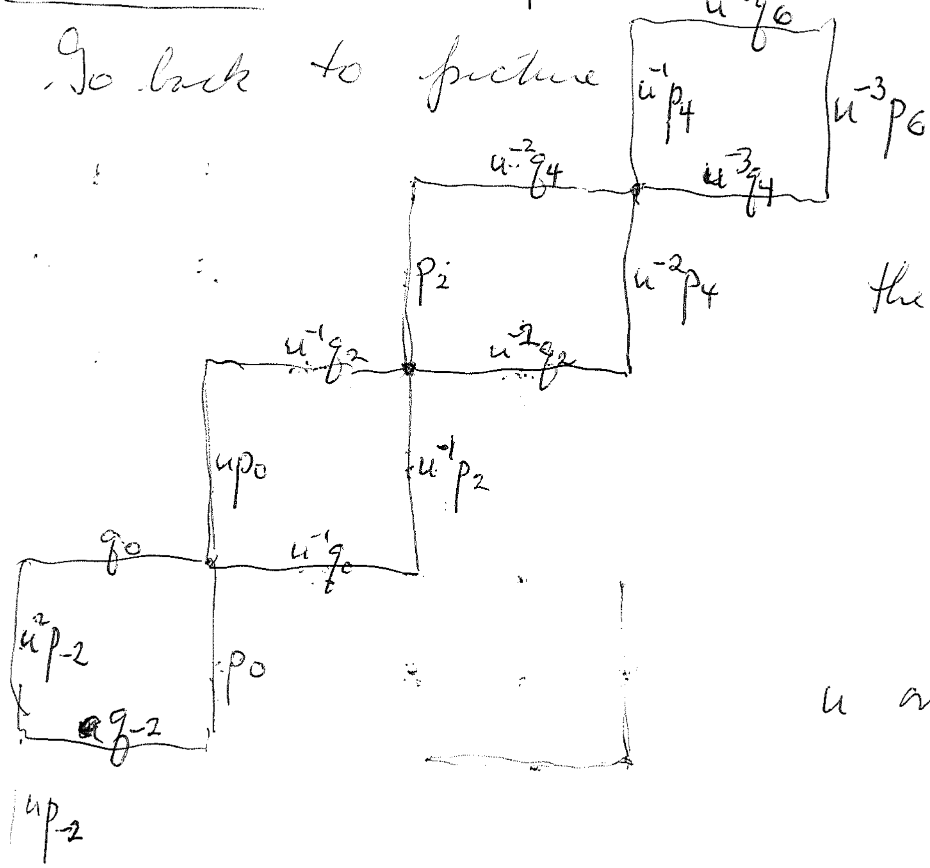
~~[illegible]~~ Given $\eta \in \hat{E} = \mathcal{H}$, then ~~[illegible]~~

$$(\lambda - u^t)\psi = \eta^*$$

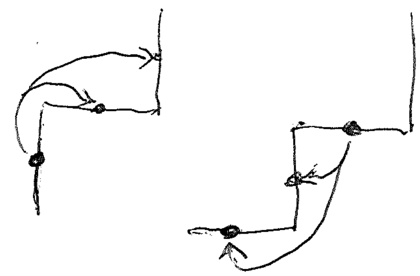
(This is the inhomog eqn in E' for the eigenvector eqn $(\lambda - u^t)\psi = 0$) has a solution ~~[illegible]~~ distinguished, namely the linear fnl

$$\psi = \eta^* \frac{1}{\lambda - u}$$

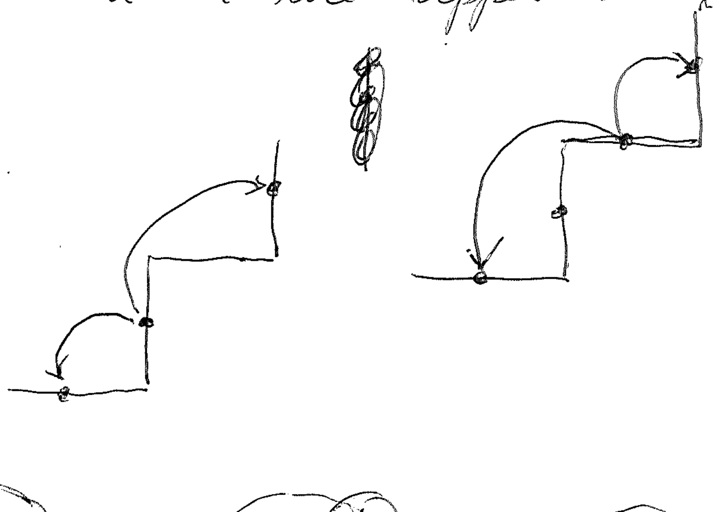
Go back to picture

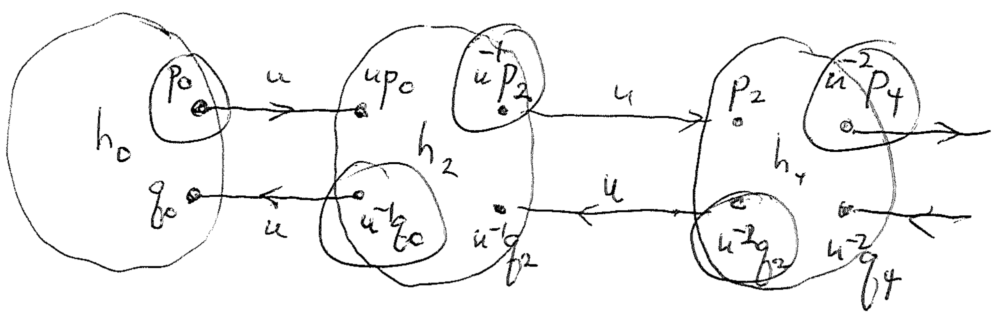


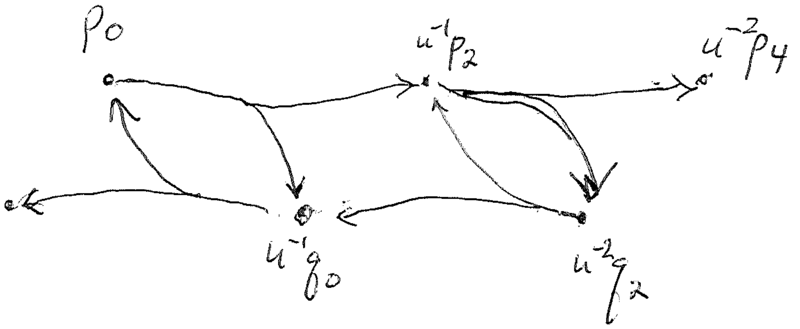
picture u on the lower staircase

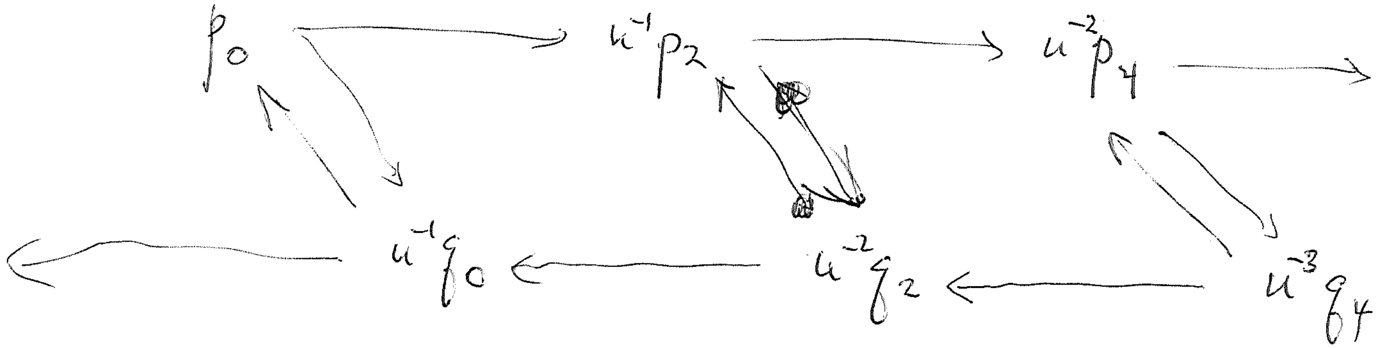


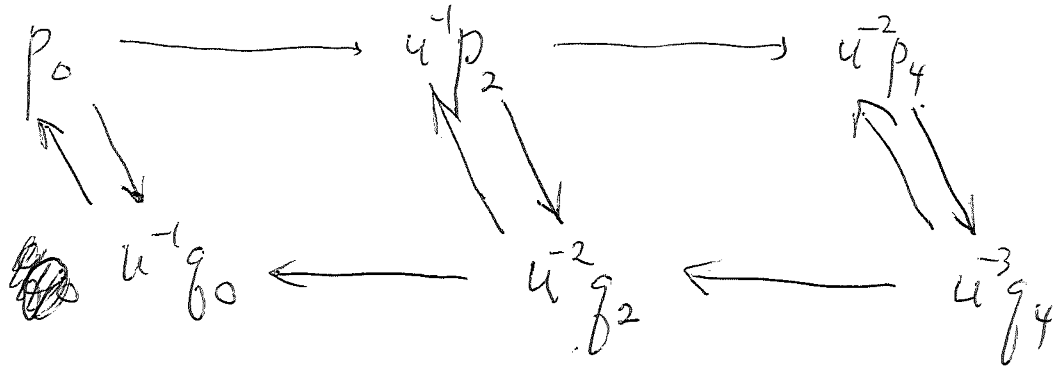
u on the upper staircase

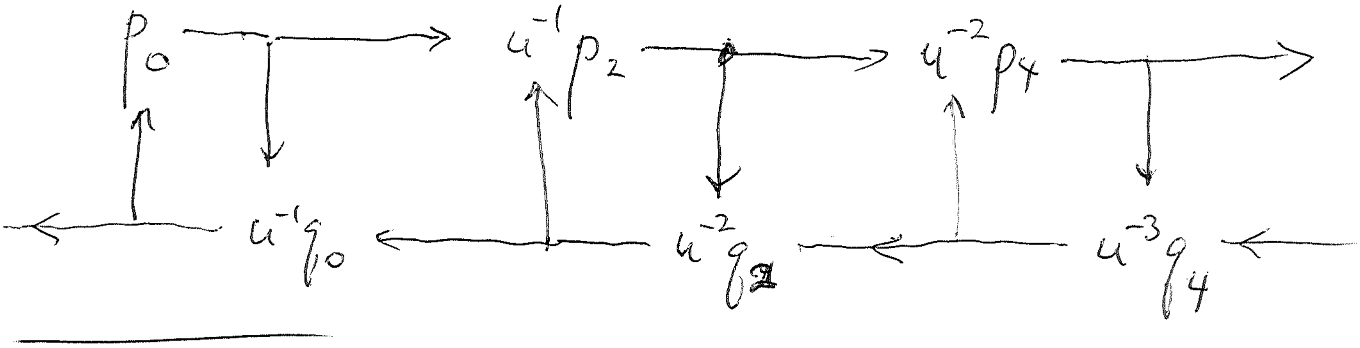












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back to the idea that partial isom.

$\eta^* \frac{1}{\lambda - u}$ maps $\mathcal{H}$ to analytic fns. of λ for $|\lambda| \neq 1$.

map should be equivalent to proj $\mathcal{H}$ onto $\overline{\mathbb{C}[u,u^{-1}]\eta}$