You are looking at $\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$. You want to construct a Morita between certain algebras. The first alg is $\mathcal{B} = \bigoplus_{F} \mathbb{C} \otimes F$, the alg version; from modules are given by $H, V$, $H_{0}^{1/2} | h_{0}^{1/2}u^{n}h_{0}^{1/2} = 0$ for $|n| \geq 2$.

$$\sum u^{n}h_{0}u^{-n} = id$$

$$H \rightarrow \mathcal{B} \otimes \mathbb{C} \rightarrow \mathbb{C}$$

$\beta f = \sum u^{n}h_{0}^{1/2}f(\cdot)$

$\xi \rightarrow (\alpha \xi)(\nu) = h_{0}^{1/2}u^{-\nu} \xi$

$\beta(\delta \xi) = \beta((\delta \xi))$

Do it clear that $\xi$ has finite support? The point is that $\delta \xi = \sum u^{n}h_{0}u^{-n} \xi$ is a finite sum, for any $\xi \in H$. This gives $\beta$ surjective, namely $\xi = \beta(\delta \xi)$.

You want to formulate the sum condition as $h_{0}^{1/2}u^{-\nu} \xi = 0$ for almost all $n$ and $\sum u^{n}h_{0}u^{-n} \xi = \xi$.

$$(\alpha \beta f)(\nu) = \sum (h_{0}^{1/2}u^{-\nu+n}h_{0}^{1/2})f(\nu)$$

Use $\mathcal{C}(\mathbb{Z}) \otimes V = \bigoplus_{\nu \in \mathbb{Z}} V_{\nu}$.

You are trying to get an algebraic Morita equivalence which should be close to the Hilbert space picture. Ultimately you get a projection, idempotent $p_{F}$ on $\mathcal{C}(\mathbb{Z}) \otimes V$, as $\mathcal{C}(\mathbb{Z})$-module. Think!

Here's what you should do. Let $A$ be Cantz's $P_{F}$: generators $p_{n}$, relations $p_{n} = \sum p_{i}p_{n-i}$, $p_{n} = 0$ for $|n| > 1$. This may $A$ is idempotent. Why? $A_{2}$}

In general $A = \sum C_{p_{n}} + \sum A_{p_{n}} = \sum p_{n}A_{p_{n}} < A_{2}$.
Let $V$ be an $A$-module. Then on $A[[r]] \otimes V$ you have a canonical identity map, a convolution of

$$(pf)(u) = \sum_m p_{m,m} f(m)$$

using notation $A[[r]] = A[[x,x^{-1}]]$, $p(z) = \sum p_n z^n$, $f(z) = \sum z^m f(m)$. You put $p : A[[r]] \otimes V$.

Now you have a functor $V \mapsto p(V \otimes A[[r]])$. (Note: When you identify elements of $V \otimes A[[r]]$ with functors $f: \Gamma \to V$ of finite support, it's irrelevant whether you put $A[[r]]$ on the left or right. You have to choose the maps which translate ops to use:

$$(L_t f)(s) = f(t^{-1}s), \quad (R_t f)(s) = f(t s)$$

Do you have a functor $V \mapsto p(V \otimes A[[r]])$?

$$(pf)(s) = \sum_t p_{s,t} f(t)$$

for a $A$-inv op.

$$_{a=t^{-1}s}^{t u = s} = \sum_a p_{a,u} f(s a)$$

Put $E(V) = p(V \otimes A[[r]])$. This is an exact functor of $V \in \text{Mod}(A)$. If $AV = 0$ i.e. all $p_s = 0$, then $p = 0$ so $E(V) = 0$.

Therefore

$$E(V) = E(A) \otimes_A V = E(A) \otimes_A V$$

So now you have $E(A) = p(A \otimes A[[r]])$. (Note...}
What is your problem? What do you have?

For every $A = P_\Gamma$ module $V$ you have a canonical projection $p_\Gamma$ on $V \otimes \mathbb{C}[\Gamma]$ as left $\Gamma$-module.

\begin{align*}
(pf)(s) &= \sum_t p(t^{-1}s)f(t) \\
(L_u(pf))(s) &= (pf)(u^{-1}s) = \sum_t p(t^{-1}u^{-1}s)f(t) \\
(p(L_u f))(s) &= \sum_t p(t^{-1}s)(L_u f)(t) = \sum_t p(t^{-1}s)f(u^{-1}t) \\
&= \sum_{u \Gamma} p((u \Gamma)^{-1}s)f(\gamma^{-1}t) = \sum_t p(t^{-1}s)f(u^{-1}t) \\
\end{align*}

Aim to find a natural ring of operators on $E(V) = p(V \otimes \mathbb{C}[\Gamma])$ as $\Gamma$-module, i.e. $L_u$ operators.

Recall: A gen. $p_\Gamma, s \in \Gamma$ \hspace{1cm} $p_\Gamma = \sum_{\Gamma} p_\Gamma \tau x^{-1}s$ \\
$s \in \mathbb{F}$

$p$ on $V \otimes \mathbb{C}[\Gamma]$ is

\begin{align*}
(pf)(s) &= \sum_t p(t^{-1}s)f(t) \\
(L_u pf)(s) &= (pf)(u^{-1}s) = \sum_t p(t^{-1}u^{-1}s)f(t) \\
(pL_u f)(s) &= \sum_t p(t^{-1}s)(L_u f)(t) = \sum_t p(t^{-1}s)f(u^{-1}t)
\end{align*}
Given an $A$-module $V$ you get the $\Gamma$-module $E(V) = p(V \otimes C[\Gamma])$ which is exact, right art, and kills $V \otimes AV = 0$.

It might help to examine the case: $\Gamma = \mathbb{Z}$, $F = \{-1, 0, 1\}$. Then $A$ has three generators $p_{-1}, p_0, p_1$, subject to the relations $\hat{p}(z)^2 - \hat{p}(z)$, where $\hat{p}(z) = z^{-1}p_{-1} + p_0 + zp_1$. For any $A$-module $V$ you have the $C[\Gamma] = C[\mathbb{Z}]$-module $E(V) = p(C[\mathbb{Z}] \otimes V)$.

Do it true that $A$ and $A^p$ are canonically isomorphic? Relations in $A$:

$$p(n) = \sum_k p(k) p(n-k) = \sum_{k+l=n} p(k)p(l)$$

becomes

$$p(n) = \sum_k p(n-k)p(k) \quad \text{in} \quad A^p$$

so the relations are preserved, which means that you have an isomorphism $A^p \rightarrow A$ by sending $p(n)$ to $p(n)$.

Next you would like to find a nice algebra of operators acting on $E(V)$ for any $V$.

**Point**: Because $\Gamma = \mathbb{Z}$ is commutative, the left and right actions $(l_f)(s) = f(e^s)$ and $(R_f)(s) = f(es)$ coincide up to $-1$, i.e. $l_f = R_{f^{-1}}$. Two examples $C[\mathbb{Z}]$. 
So what operators do you have on $E(V)$? $E(V) = \rho(C[V] \otimes V)$, so you have $\Gamma$-equivariant maps

$$E(V) \xrightarrow{\alpha} C[V] \otimes V \xrightarrow{\beta} E(V)$$

Think of $C[Z]$ as the ring of Laurent poly.

$\rho(z) = \sum_{|n| \leq 1} z^n p_n$ acts on $C[V] \otimes V = V \otimes C[V, u^{-1}]$

$\beta$ is determined by $\beta_0: V \rightarrow E(V)$

$$\beta f = \sum_n u^n \beta_0 f(n)$$

$\alpha$ is determined by $\alpha_0: E(V) \rightarrow V$: $(\alpha_0)(n) = \alpha_0 u^{-n} \xi$

$\beta \alpha \xi = \sum_n u^n \beta_0 \alpha_0 u^{-n} \xi$ \hspace{1cm} $\beta_0 \alpha_0 = h_0$. 

So what operators do you get on $E(V)$? You have $Z$ actions and this $h_0$. 

Recap: Given $\rho(z) = z^2 p_1 + z^0 p_0 + z^{-1} p_1$, $\rho(1) = 1$, you construct $E(V) = \rho(C[Z] \otimes V)$, on which you have the mult. group $\{z^n\}$ acting and the operator $h_0 = \beta_0 \alpha_0$. You have on $E(V)$ a module structure over $\sum_{|n| \leq 1} \Gamma = \Gamma$, $\beta_0$ is a locally unitary $\Gamma$-module and $E(V)$ is a locally unitary $\beta_0$-module. In the converse direction you suppose given $E$ with $\Gamma$ action and $h_0$ satisfying $h_0 u^n h_0 = 0$ for $|n| > 2$ and $\sum u^n h_0 u^{-n} = 1$. I think you have to make a reality assumption for $h_0: E_0 \rightarrow E_0$. 


Recap. \( V \) an \( A \)-module, \( \lambda V \) a \( \mathbb{L} \)-module, \( \tilde{\beta}(z) = z \tilde{p}_1 + \tilde{z} \tilde{p}_0 + \tilde{z} \tilde{p}_1 \), a Laurent poly family of projections. \( E = \hat{\beta}(\mathbb{L}[z,z^{-1}] \otimes V) \). Then \( E \) is a \( \mathbb{L} \)-module equipped with \( \alpha_0 : E \to V \), \( \beta_0 : V \to E \), linear maps \( \lambda \) whence an operator \( h_0 = \beta_0 \alpha_0 \) on \( E \), and this satisfies \( \sum_{n \in \mathbb{Z}} u^n h_0 u^{-n} = \text{id} \) on \( E \).

So \( E \) is a \( \mathbb{B} \)-module where \( \mathbb{B} = \mathbb{C}[z^{-1}] \) is an algebra with local left and right units. You have converse? Yet \( \beta \) is a \( \mathbb{B} \)-module \( E \), forgotten the support condition, but this is involving \( \mathbb{L} \).

\[
E \xrightarrow{\alpha} \mathbb{C}[z,z^{-1}] \otimes V \xrightarrow{\beta} \mathbb{B} \xrightarrow{\hat{\beta}} E
\]

\[
(f : \Gamma \to V) \mapsto \sum_{n} u^n (\beta_0 f(n))
\]

\[
\xi \mapsto (\alpha \xi)(n) = \alpha_0 u^{-n} \xi
\]

\[
\beta \xi = \sum_{n} u^n \beta_0 \alpha_0 u^{-n} \xi = \xi
\]

\[
(\alpha \beta f)(n) = \sum_{m} (\alpha_0 u^{-n+m} \beta_0) f(m)
\]

\[
p_{-m} \text{ which is } 0 \text{ for } |m| > 2
\]

So now what about the converse. Given \( E \) with \( \Gamma \) action and \( h_0 \)

\[
\Lambda = \mathbb{C}[z, z^{-1}] \mathbb{C}[\tilde{\Gamma}] \otimes V \text{ has projection } \tilde{\beta} = z \tilde{p}_1 + \tilde{z} \tilde{p}_0 + \tilde{z} \tilde{p}_1. \text{ The } \mathbb{L} \text{-poly module} \Lambda \otimes V \text{ gen. by \( V \) has a certain kind of splitting}
\]

\[
E = E(V) = \hat{\beta}(\Lambda \otimes V) \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E
\]
Let \( p = \alpha \beta \) and \((\hat{p} f)(n) = \sum_m p(n-m) f(m)\).

You want \((\alpha \xi) (n) = \alpha_n \xi^1, \beta f = \sum_m \xi^m \beta_0 f(m)\),

then \( \beta \alpha \xi = \sum_n \xi^m \beta_0 \alpha_n \xi = \xi \).

\[
(\alpha (\beta f))(n) = \sum_m \alpha_n \xi^m \beta_0 f(m)
\]
\[
(\hat{\alpha} f)(n) = \sum_m \alpha_n f(n-m)
\]

\[
\sum_n (\hat{\alpha} f)(n) z^n = \sum_n \sum_m \alpha_n \hat{f}(m) z^{n-m} = \hat{f}(z)
\]

\[
\hat{f}(z) = \hat{f}(1/z)
\]

What is going on? What's important is \( E = \mathbb{C} \otimes \mathbb{C} \Gamma \).

This means (ultimately because \( \Gamma \) has local (right?) units) that there is a natural \( \Gamma \)-action on \( E \), also a partition of unity \( \sum h_n = 1 \) \( \Gamma \)-equivariant \( h_n u^m \xi^i \) satisfying independence condition \( h_n h_m = 0 \) \( |n-m| > 2 \).

Your problem here is to factor \( h_0 \) into \( \alpha \beta_0 \). Is it possible to take \( V = H \) somehow? The problem is to get \( \alpha \) defined.

\[
H \rightarrow V \quad \text{when does it extend to} \quad H \rightarrow \Lambda \otimes V \quad \Gamma \text{-linear}
\]
\[ \alpha : H \rightarrow V \quad \xi : \emptyset \rightarrow \{ \{ f, \gamma \} \rightarrow V \} \]

\[
\begin{align*}
(\alpha \cdot \xi)(s) &= \alpha \cdot \xi^{-1} \\
(\alpha \cdot t \xi)(s) &= \alpha \cdot s \cdot t \xi \\
(L \cdot \alpha \xi)(s) &= (\alpha \xi)(t^{-1} s) = \alpha \cdot s^{-1} t \xi 
\end{align*}
\]

When does \( \alpha \) have finite support? Need \( \forall \xi \in H \) that \( \alpha \) sees only finitely many \( s \)’s by \( \alpha \cdot \xi = \alpha \cdot \xi^{-1} \).

But \( H \) is generated by \( h_0 \cdot H \), \( h_0 \). Maybe this was what lead to \( kh_0 = h_0 \).

So you have \( h_0 = \sum_{n \geq 1} h_n h_0 \) because \( h_n h_0 = 0 \) for \( |n| \geq 2 \). This gives then \( h_0 = kh_0 \) with \( k = \sum_{n \geq 1} h_n \).

And so you factor \( h_0 = kh_0 \):

So you take \( h_0 = \beta_0 \alpha_0 \) \( \alpha_0 = h_0 \), \( \beta_0 = k \).

Let \( E \) be a \( B \)-module such that \( E = \text{BE} \).

We use that \( B \) has local left units. This has to be written out at some point. So where do we begin?

We start with \( E \) \Gamma action, \( h_0 \) such that \( \sum_{n \geq 1} h_n \cdot u^n = 1 \).

Try doing the independence first, namely \( h_0 \cdot u^n h_0 = 0 \) for \( |n| \geq 2 \). The completeness condition amounts to \( \sum_{n \geq 1} h_n \cdot h_i = h_i \) because \( E = \text{BE} \) implies \( B = \sum h_i B \).

That \( E = \sum h_i E \), so it seems the completeness condition + indep. amounts to \( kh_0 = h_0 \) the rest.
should follow from group action. What actually happens. You have the vector space $E$ with $T$-action, operator $h_0$ satisfying

$$h_0 u_n h_0 = 0 \quad \text{for} \quad |n| \geq 2$$

$$\sum_{|n| \leq 1} h_n h_0 = h_0$$

so now take $\alpha_0 = h_0$, $\beta_0 = \sum_{|n| \leq 1} h_n$. Then define

$$E \xrightarrow{\alpha} \Lambda \otimes E \xrightarrow{\beta} E$$

$$(f: \Gamma \to E) \longmapsto \beta f = \sum_n u_n \beta_0 f(n)$$

$$\xi \longmapsto (\alpha 0)(\xi)(n) = \alpha_0 u_n \xi$$

$\beta$ is well-defined because $f$ has finite support.

Given $E$, factor $h_0: E \overset{=}{} E$ $\alpha_0 = h_0$, $\beta_0 = k_0$ $\rightarrow$ $E$

$\beta_0 \alpha_0 = k_0 h_0 = h_0$ then you get

$$E \xrightarrow{\alpha} \Lambda \otimes E \xrightarrow{\beta} E$$

$$(f: \Gamma \to E) \longmapsto \beta f = \sum_n u_n \beta_0 f(n)$$

$$\xi \longmapsto (\alpha\xi)(n) = \alpha_0 u_n \xi$$

$\beta\xi = \sum_n u_n \beta_0 \alpha_0 u_n \xi = \xi$

Check $\xi$ has finite support. You assume $E = \mathbb{B} E \Rightarrow E = \sum_n h_n \mathbb{B} E = \sum_n h_n E$

$$(\alpha_0 h_m \xi)(n) = \alpha_0 u_n u_m h_0 u_m \xi = \frac{h_0 u_n u_m h_0 u_m \xi}{\text{zero for } |m-n| \geq 2}$$
Thus \((\alpha \beta f)(n) = \sum_m (\alpha_u u^{-m} \beta_0) f(m)\)

\[ p(n) = \alpha_0 u^{-n} \beta_0 = h_0 u^{-n} k_0 \]

\[ = h_0 u^{-n} (h_1 + h_0 + h_1) \]

This projection doesn’t have the same support.

Let’s see if there is a way to get an equivalence of module categories. It seems that you want to take the inductive limit with respect to \(F\). The first module category should consist of vector spaces \(V\) equipped with a Laurent polynomial projection \(\hat{p}(z) = \sum_n z^n p(n)\) such that \(p(n) \in \mathbb{Z}(V)\) and \(\hat{p}(z)^2 = \hat{p}(z)\) with nil and nils of

The second module category should consist of \(F\)-modules \(E\) such that \(\exists F \subset \mathbb{F}\) such that \(h_0 u^n h_0 = 0\) for \(n \notin F\) and such that \(\xi = \sum_{n \in \mathbb{Z}} u^n h_0 u^{-n} \xi\) for all \(\xi \in E\).
Program: To set up an equivalence of module categories. The first consists of vector spaces $V$ equipped with a Laurent polynomial projection, odd degree $p(z) = \sum_{n \in F} z^n p(n)$, $p(n) \in \text{End}(V)$.

\[ \hat{p}(z)^2 = \hat{p}(z), \quad \hat{p}(n) = \sum_{m} p(m) p(n-m) \]

Let a projection $p$ on $\Lambda \otimes V^m$, $\Lambda = \mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots]$, $(pf)(n) = \sum_{m} p(n-m) f(m)$, i.e., $(pf)^n = p^n f^n$

\[ E(V) = p(\Lambda \otimes V) \]

Second kind of modules consists of vector spaces $E$ with $\Gamma = \mathbb{Z}$ action equipped with an $h_0 \in \text{End}(E)$ satisfying (i) $h_0 u^n h_0 = 0$ for $n \in F$; some finite subset of $\Gamma$, (ii) $\sum_{n} h_n \xi = \xi$ $\forall \xi \in E$, where $h_n = u^n h_0 u^{-n}$; this condition means $\{ n \mid h_n \xi \neq 0 \}$ is finite $\forall \xi$.

Condition (ii) $\Rightarrow$ $E = \sum u^n h_0 E$.

Perhaps you write the conditions differently, namely

(a) $E = \sum u^n h_0 E$

(b) $\{ n \mid h_n u^n h_0 \neq 0 \}$ is finite $F$

This implies $h_n u^n h_0 \xi = u^n h_0 u^{-n+m} h_0 \xi = 0$ for $m \neq 0$

$\Rightarrow \sum h_n \xi$ is a finite sum $\forall \xi$. 

\[ \]
(a) \( E = \sum_n u^n h_0 E \)  
(b) \( h_0 u^n h_0 = 0 \) for finite \( n \) and \( F \) finite.
(c) \( \sum h_n \xi = \xi \) finite for all \( \xi \in E \)

then get \( k_0 = \sum_{n \in F} h_n \) \( k_0 h_0 = h_0 \)

2nd module type: \( V \) with \( \Gamma = \mathbb{Z} \)-action (operators \( u^n, n \in \mathbb{Z} \)) and \( h_0 \in \text{End}(E) \) satisfying:
(a) \( E = \sum_n u^n h_0 E \)
(b) \( F = \{ n \mid h_0 u^n h_0 \neq 0 \} \) is finite.
(c) \( \sum h_n h_0 = h_0 \)

\( \Rightarrow \forall \xi \epsilon E \) \( \sum h_n \xi \) is def \( t = 3 \).

hence, \( V, \phi(x) \) have \( E(V) = \rho(\Lambda \otimes V) \)

\[ E \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E \]

\( (\psi \phi)(n) = \alpha \psi \beta \phi \)

\( h_0 \phi = \phi \)

\( (\rho \psi \phi)(n) = \sum_{m} \alpha \psi \beta \phi \)
Details of Moita equivalence:

1st kind of module is $V \otimes V$ with Laurent poly projection

$$\hat{p}(z) = \sum z^n p_n$$

$p_n \in \text{End}(V)$, $p_n = 0$ \text{ for } $n \notin F$

$$\hat{p}^2 = \hat{p}$$

equiv. \hspace{1cm} $p_n = \sum_{m} p_{n-m} p_m$

$\hat{p}$ \text{ corresponds to a projection } p \text{ in the Laurent poly module } \Lambda \otimes V$

$\Lambda = \mathbb{C}[x, x^{-1}] = \mathbb{C}[z, z^{-1}]$

$(f : \Gamma \rightarrow V)$ \hspace{1cm} $(pf)(n) = \sum_{m} p(n-m) f(m)$ \hspace{1cm} $\hat{p} f = \hat{p} f$

If $F$ is fixed, then these are the same as modules over the identity. \hspace{1cm} $\Lambda \hat{\otimes} V = \Lambda$

2nd kind of module is a $\otimes V$ with $\Gamma$ action and an operator $h_0$ on $E$ following hold. Put

$$h_0 = u^n - u^{-n} \text{.}$$

You want $h^n_0 h_0 = 0$ \text{ for } $n \in F$ \text{ (F-modules)}

Also

$$h_0 h_0 = h_0 u^n - h_0 u^{-n}$$

Thus

$$h_n h_0 = 0 \text{, } h_0 h_{-n} = 0 \text{, } h_0 u^n h_0 = 0$$

Want

$$\sum h_n \xi = \xi \text{ for all } \xi \in E \text{.}$$

This means that $h_n \xi = 0$ for only fin. many $n$.

$$\sum \sum_{h_n} h_0 = h_0$$

Example:

$$E(V) = \rho(\Lambda \otimes V)$$

$\rho : \Lambda \otimes V \rightarrow E$

$(\rho f)(\xi) = \alpha \xi u^n$

$$\sum_{h_n} h_0 = \beta_0 \alpha_0$$

$$h_0 = \beta_0 \alpha_0$$

$$h_n = u^n \beta_0 \alpha_0 u^{-n}$$

$E \otimes \Lambda \otimes V \rightarrow E$

$(\rho f)(\xi) = \sum u^n \beta_0 f(n)$
How to organize? Left-right problem. Suppose you start with a $P=A$ module $V$, $\hat{p}=\sum_{n \in F} z^np(n)$. You know that $E(V) = p(\Lambda \otimes V)$ is an exact right functor of $V$, so that $0 \rightarrow AV \rightarrow V \rightarrow V/AV \rightarrow 0$ leads to $0 \rightarrow E(AV) \rightarrow E(V) \rightarrow E(V/AV) \rightarrow 0$. Also $E(V) \subset E(\hat{A}) \otimes_A V$ by restriction. $E(\hat{A})$ is clearly a flat $A^\text{op}$ module.

Given $V$ you put $E(V) = p(\Lambda \otimes V)$ and let $\alpha: E(V) \hookrightarrow \Lambda \otimes V$ be the inclusion. $\beta: \Lambda \otimes V \twoheadrightarrow E(V)$ be induced by $p$. $\Lambda \otimes V \twoheadrightarrow E(V) \hookrightarrow \Lambda \otimes V$ do these something...
Things learned yesterday. When defining $E = \rho(A \otimes V)$ you get

$$E \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E \quad \beta \circ \alpha = \text{id}_E$$

so any $\xi \in E$ has the form $\xi = \beta f$ with $f : \Gamma \to V$ of finite support. 

$$E = \sum \xi \otimes \beta.$$

In fact $\xi = \rho(x \xi)$, so $x \xi$ is a minimal choice for $f$ such that $\xi = \beta f$.

**Review of GNS construction** $\Gamma(p : A \to B)$. Here use unital setting, so that $p$ is linear $p(1) = 1$.

You associate to $p : A \to B$ the category of $(M, N, i, j)$, where $M$ is an $A$-module, $N$ a $B$-module, $i : N \to M$, $j : M \to N$ are linear satisfying $ja \in \rho(a) N$.

In particular $j_1 = \text{id}_N$, so that $N$ is determined by $(M, i, j)$. I recall that the object $(M, N, i, j)$ is equivalent to $M$ with its natural module structure over $\Gamma(p : A \to B) = A \oplus A \otimes B \otimes A$ semi-direct

product.

where $a \otimes b \otimes c \mapsto ab \otimes c$ on $M$, and $A \otimes B \otimes A$
has mult. \((q'_1 \otimes b_1 \otimes q'_2) (q'_2 \otimes b_2 \otimes q'_3) = q'_1 \otimes b_1 (q_2 q'_2) b_2 \otimes q'_3\).

Given a \(B\)-module \(N\), the different ways of "dilating" \(N\) to a \(\Gamma\)-module \(M\) are equivalent to factoring: \(m \mapsto (a' \mapsto \rho(a'm))\)

\[
\begin{array}{ccc}
A \otimes N & \longrightarrow & M \\
\otimes a \otimes n & \longrightarrow & a \cdot n
\end{array}
\]

the canonical map \(a \otimes n \mapsto (a' \mapsto \rho(a'a) n)\) from \(A \otimes N\) to \(\text{Hom}(A, N)\). A minimal choice for \(M\), namely, the image of this canonical map.

\[
\text{So back to } \hat{\rho}(z) = \sum_{n \in \mathbb{F}} z^n \rho(n) e \otimes V/\rho^2 = \rho
\]

\[
p = \text{corresp. projection on } \Lambda \otimes V
\]

\[
E = p(\Lambda \otimes V), \quad E \xrightarrow{\rho} \Lambda \otimes V \xrightarrow{\rho} E
\]

You know that

\[
\beta f = \sum_n u^n \beta_0 f(n), \quad (\alpha \beta f)(m) = \sum_n \beta_0 (\alpha^{-m-n} \beta_0) f(m)
\]

\[
\beta = \sum_n u^n \beta_0 u^{-n} = \text{id}_E
\]

New idea: We know that \(E(V) = \rho(\Lambda \otimes V)\) is an exact functor from \(A = \mathbb{P}_F\)-modules (i.e. vector spaces \(V\) equipped with Laurent poly projection \(\hat{\rho}(z)\), support in \(F\)) to \(\Gamma = \mathbb{Z}\)-modules with appropriate equivariant partition of \(1\).

Moreover \(E(V) = 0\) where \(AV = 0\), i.e. all \(p_n = 0\).

Now \(A\) is idempotent, so there is a unique nil and cod nil \(A\)-module nil isom. to \(V\), namely

\[
\text{Im} \{ AV \rightarrow V/AV \} = \text{Im} \{ A \otimes V \rightarrow \text{Hom}_A(A, V) \}
\[
\begin{align*}
0 & \rightarrow A \rightarrow \bar{A} \rightarrow 0 \\
0 & \rightarrow \text{Hom}_A (C, V) \rightarrow \text{Hom}_A (\bar{A}, V) \rightarrow \text{Hom}_A (A, V) \\
0 & \rightarrow AV \rightarrow V \rightarrow \text{Hom}_A (A, V) \\
0 & \rightarrow AV \rightarrow AV \\
A \otimes V & \rightarrow V \rightarrow \text{Hom}_A (A, V)
\end{align*}
\]

Look at

\[
\begin{array}{ccc}
E & \xrightarrow{\alpha_0} & V \\
\downarrow & & \downarrow \beta_0 \\
\alpha_0 E & \xrightarrow{\beta_0 \alpha_0} & E
\end{array}
\]

This says that if we factor \( E \xrightarrow{h_0} E \) through a vector space \( V \), then \( V \) will be larger than \( h_0 E \).

Here's what you should do? You want to start from the \( E \) side - a \( \Gamma \)-module with \( h_0 \) leading to an equiv. partition \( \sum h_n = 1 \). You need to check.

\[\text{(Handwritten notes and diagrams)}\]
Still trying for Morita equivalence. 

Given $V$ with projective $p \in \Lambda \otimes \text{End}(V)$, $p = \sum u^n \rho_n$. 

get \[ E = p(\Lambda \otimes V) \] \[ \implies E \xrightarrow{\alpha} \Lambda \otimes V \xrightarrow{\beta} E \] \[ \xrightarrow{\gamma} \] \[ E \] \[ \beta = p \] 

You know that $E(V)$ depends only on $V$ up to nil ideal, so $V$ can be replaced by $\text{Im} \{ AV \to V_A \}$.

Start with \[ V, p \in \Lambda \otimes \text{End}(V), p^2 = p \]

$E(V) = p(\Lambda \otimes V)$ is an $\Gamma$-module with \[ h_0 = \beta_0 \alpha_0 : E \to E \] such that (i) $h_0 u^n h_0 = 0$ for $u \in F$

(ii) \[ \sum h_0 \beta_0 = \beta_0 \sum \frac{u^n \beta_0}{h_0} \frac{1}{h_0} \]

(Review: \[ (\alpha \beta f)(n) = \alpha_0 u^{-n} \beta_0 \sum \frac{u^n \beta_0}{h_0} f(n) = \sum \frac{(\alpha_0 u^{-n+1} \beta_0) f(n)}{h_0} \]

\[ (\alpha \beta f)(n) = \alpha_0 u^{-n} \beta_0 \sum u^n \beta_0 f(m) = \sum \frac{(\alpha_0 u^{-n+1} \beta_0) f(m)}{h_0} \]

Since \[ p(n) = \alpha_0 u^{-n} \beta_0, \] if $F = \{ n \mid p(n) \neq 0 \}$, then you have \[ \alpha_0 u^{-n} \beta_0 = 0 \] for $n \notin F = \text{Supp}(p)$

hence \[ h_0 u^{-n} h_0 = 0 \]
Now, still using \((V, \beta)\) let us try to understand what happens as you replace \(V\) by smaller things. You fix \(E\) first of all.

You have \(E \xrightarrow{\alpha_0} V \xrightarrow{\beta_0} E\).

\[ \beta_0 \circ \alpha_0 = 1_E \]

Question: Is \(\alpha_0 E\) an \(A\)-subm.

What do you know?

\[ h_0 u^n h_0 = 0 \quad n \notin F. \]
\[ \alpha_0 u^n \beta_0 = 0 \quad n \notin F. \]

\[ \sum_{n} \alpha_0 u^n \beta_0 = \sum_{n} \rho(n) \]

\[ \sum_{n \in F} u^n h_0 u^{-n} h_0 = \sum_{n \in F} h_n h_0 = h_0 \]

\[ \beta_0 = \sum_{n \in F} u^n \beta_0 \alpha_0 u^{-n} \beta_0 = \sum_{n \in F} h_n \beta_0 = \left( \sum_{n \in F} h_n \right) \beta_0 \]

\[ \alpha_0 = \sum_{n} \alpha_0 u^n \beta_0 \alpha_0 u^n = \sum_{n \in F} \alpha_0 h_{-n} = \alpha_0 \left( \sum_{n \in F} h_{-n} \right) \]
Here is the point. Let $E = E(V)$ where $V$ is an $A = \mathbb{P}^1$-module i.e. a vector equipped with idempotent $p \in \Lambda \otimes \text{End}(V)$, $p = \sum_{n \in \mathbb{F}} u^n \otimes p(n)$.

You have $\alpha = \text{id}$, $p = p(n)$, and $f(n) = \alpha u^{-n} g(n)$.

On $E$ you have $\Gamma$ action and operator $h_0 = \beta_0 \alpha_0$ properties: $h_0 u^{-n} h_0 = \beta_0 (\alpha_0 u^{-n} \beta_0) \alpha_0 = 0$ for $n \notin \mathbb{F}$.

So now what to expect. It's clear that any factor injection $E \hookrightarrow W \hookrightarrow E$ of $h_0$ should lead to a direct embedding of $E$ into $\Lambda \otimes W$. You need to derive your idea is to factor $h_0 = \beta_0 \alpha_0$ as

$$ x_0 E = \alpha \beta_0 (\Lambda \otimes V) = \left\{ \alpha \beta_0 \sum_{n \in \mathbb{F}} u^n \beta_0 f(n) \mid f \in \Lambda \otimes V \right\}. $$

$$ \sum_{n \in \mathbb{F}} \rho(n) f(n) $$

$10. x_0 E = A V = \sum_{n \in \mathbb{F}} \rho(n) V$. Similarly

$$ \beta_0 V = V / \ker(\beta_0 : V \to E) \quad \ker(\beta_0) = \{ v \mid \alpha \beta_0 v = 0 \} = \{ v \mid \alpha_0 u^{-n} \beta_0 v = 0, v \alpha_0 u^{-n} \beta_0 v = 0 \} $$

$\ker(\beta_0) = \ker(\beta_0) \otimes \Lambda \otimes V$.
Review the formulas. Let $E$ be a $\Gamma$ module equipped with an operator $h_0$ satisfying $h_0 u^n h_0 = 0$ for $u \in F$. Let $\beta_0 : V \to E$ inductively so that $h_0 = \beta_0 \beta_0$. Let $\alpha_0, \beta_0$.

$V \xrightarrow{\beta_0} E \xrightarrow{\alpha_0} V \quad p(n) = \alpha_0 u^{-n} \beta_0$?

$h_0 E \xrightarrow{h_0} h_0 E$?

$h_0 : E \xrightarrow{h_0} E$ \quad $\sum h_n = 1$ \quad $h_0 u^n h_0 = 0$ \quad $u \in F$

$E \xrightarrow{\alpha_0 = h_0} h_0 E \xrightarrow{\beta_0} E \xrightarrow{\alpha_0} h_0 E \quad p(n) = \alpha_0 u^{-n} \beta_0$

$h_0 = \beta_0 \alpha_0 \quad V$

$\sum \beta_0$ \quad $h_0 u^{-n} h_0 = (\beta_0 (\alpha_0 u^{-n} \beta_0) \alpha_0$ \quad surj.

So you find $p(n) = 0 \Leftrightarrow h_0 u^{-n} h_0 = 0$.

The other point is that $V$ is "round." next. I have to prove this.
\[ V \overset{\beta_0}{\longleftarrow} E \overset{\alpha}{\longrightarrow} \Lambda \otimes V \]
\[ (\alpha \beta_0 \omega)(v) = (\alpha \omega \beta_0)v = p(\omega) \]
\[ \{ \omega \in V \mid p(\omega) = 0 \ \forall \alpha \} = 0. \]

Now I know everything should work.

**Question** to be explored is whether your old GNS picture in the modular ring context generalizes to the partition of 1 situation. So what can you do next?

You should finish the Morita equivalence.

V a \( P \) module, i.e. a vector space equipped with a \( P \) projection \( p \) on \( \Lambda \otimes V \).

\[ E = E(V) = p(\Lambda \otimes V) \]

exact functor of \( V \) which kills nil \( P \) modules.

\[ E \stackrel{\alpha = \text{id}}{\longrightarrow} \Lambda \otimes V \overset{\beta = p}{\longrightarrow} E \]

\[ \{ f : \Gamma \to V \} \longmapsto \sum_{\text{fin supp}} u^n \beta_0 f(u) \]

so the next thing that comes

Consider general \( \Gamma \), \( \Lambda = C(\Gamma) \).

You want to use the \textit{mult} action \( (R_t f)(s) = f(st) \).

Want \( p \in \otimes \text{End}(V) \) acts on \( \Lambda \otimes V \) commutes with \textit{it} mult.
\[ \Lambda \otimes \text{End}(V) = \text{End}(V) \otimes \Lambda \]

To act on \( \Lambda \otimes V \), look at operators on \( \Lambda \otimes V \) commuting with \( R_t \).

Try \( \Lambda \otimes V \) with operators \( \text{End}(V) \otimes \Lambda \) commuting with \( R_t \).

Interesting:

\[ R_t^{-1}(u \otimes s) = u \otimes st^{-1} \]

Puzzle:

\[ \Lambda \otimes V = \{ f: \Gamma \to V \mid f \text{ fin support} \} \]

\[ \sum_{s \in \Gamma} f(s) \otimes s \]

\[ \sum_{s \in \Gamma} f(s) \otimes bs = \sum_{s \in \Gamma} f(t^{-1}s) \otimes s \]

\[ (lt^{-1})f(s) \]

So if you identify \( \Lambda \otimes V \) with \( \{ f: \Gamma \to V \mid f \text{ fin supp} \} \), then \( t \cdot f \) is \( lt \cdot f \). Next you want a proj. \( P \) on \( \Lambda \otimes V \) commuting with the \( \Gamma \)-action. This...
should be given by a left invariant kernel: \\
\[ (pf)(s) = \sum_{t} p(t^{-1}s)f(t) \] which can write \\
\[ = \sum_{t} p((st)^{-1}s)f(st) = \sum_{t} p(t^{-1})f(s^{-1}t) \]

\[ p = \sum_{t} p(t^{-1})R_{t} \in \text{End}(V) \otimes \Lambda \]

\[ \text{so what happens at this point?} \]

You assume \[ p(t^{-1}) \] supported in \( F \) and \( p^{2} = p \), define \( E(V) = p(\Lambda \otimes V) \)

\[ \begin{array}{ccc}
E & \xrightarrow{\alpha} & \Lambda \otimes V \\
\beta & \downarrow & \downarrow V \\
\beta & \searrow & \downarrow 1 \\
\end{array} \]

\[ \alpha^{\otimes f}(s) = \alpha^{s^{-1}t} \]

\[ (\alpha^{t})^{(s)} = (\alpha^{s^{-1}t})^{(t^{-1}s)} \]

\[ \alpha(\beta^{t})(s) = \beta(\alpha^{s})(t^{-1}) \]

\[ \alpha(t^{-1}i)(s) = (\alpha^{s})(t^{-1}s) \]

\[ \alpha(\beta^{t})(1) = (\alpha^{s})(s) \]

Get \[ \beta^{\otimes f} = \sum_{s} s^{\beta_{i}}\alpha^{s^{-1}t} \]

\[ \sum_{s} h_{s} = 1. \]

\[ (\alpha^{\beta^{t}}f)(s) = (pf)(s) = \sum_{t} (\alpha_{i}s^{-1}t)\beta_{i}f(t) \]
f(1) = 0 \quad s = 1
\begin{align*}
\text{so there's no problem} & \\
(p f)(s) &= \sum_t p(t^s) f(t) \\
(\alpha \beta f)(s) &= \frac{\sum_t (\alpha \beta^{-1} s^t \beta_1) f(t)}{p(t^s)}
\end{align*}

\[ h_s \cdot \chi \delta^{-1} = \left( \frac{\beta \sigma \delta^{-1}}{p(s)} \right) \chi \delta^{-1} \]

\[
\Lambda = C[G], \quad \text{given } (V, \rho)
\]

V vector space, \( \rho \) is a left invariant idempotent operator on \( \Lambda \otimes V \):

\[
(p f)(s) = \sum_t p(t^s) f(t)
\]

where \( p(s) \in \text{End}(V) \) and \( p(s) = 0 \quad s \notin F \).

\[
E = \rho(\Lambda \otimes V), \quad E \xleftarrow{\chi^{-1}} \Lambda \otimes V \xrightarrow{\beta \rho} E
\]

\[
\beta f = \sum_s s \beta_1 f(s)
\]

because \( \beta \) is \( G \)-mod. map.

\[
(\alpha \beta_1 : V \rightarrow \Lambda \otimes V), \quad \gamma_1 = 1 \otimes \nu
\]

Given \( \xi \in E \) let \( \alpha \xi = \sum_s \sigma(\alpha \xi)(s) \).

\[
\xi_1 : \Lambda \otimes V \rightarrow V \quad \text{be } f_1 = f(1), \quad \text{then and let}
\]

\[
f_1 \chi = \alpha_1 : E \rightarrow \Lambda \otimes V \rightarrow V.
\]

Then \( (\alpha \xi)(s) = f_1 s^{-1} \chi \xi \) = \( f_1 \alpha s^{-1} \xi = \alpha_1 s^{-1} \xi \).

\[
p(s) = \alpha_1 s^{-1} \beta
\]

\[
\sum_s s \beta_1 \alpha s^{-1} \xi = \sum_s h_s \xi
\]
so what happens? \( h_i s^{-1} h_i = \beta_i (h_i s^{-1} \alpha_i) \beta_i \) [Boxed] \( = 0 \) [Boxes] \( s \neq F \)

What to do? All kinds of things.
You need the Morita equivalence, you want the Morita context in particular the dual pair over \( A = F \).
There is some duality game to be made explicit.

Things to review. \( \Gamma \)-graded vector spaces

\[ V = \bigoplus_{s \in \Gamma} V_s. \]

Q: What is a homogeneous operator \( T \)?

Logical procedure: Use \( \otimes \)

\[ (V \otimes W)_s = \bigoplus_{t \in \Gamma} V_t \otimes W_u. \]

Idea: Preferred direction - operators on left or right.

If you want \( \otimes \) a lift module

Then take \( V = \{ \square \} \)

Have \( \otimes \) for \( \Gamma \)-graded modules

\[ (V \otimes W)_s = \bigoplus_{t \in \Gamma} V_t \otimes W_u \]

Now take \( V_t = \begin{cases} \square & t = t_0 \\ 0 & \text{otherwise} \end{cases} \)

Then \( (V \otimes W)_s = V_{t_0} \otimes W_{t_0^{-1}s} = W_{t_0^{-1}s} \).
left operator of degree $\alpha$ on $W$ such as

$$T : W \to W$$

needs more clarification. Basic idea should be a left $\Gamma$-graded module $N$ over a

$\Gamma$-graded ring $B$: $B_s \otimes N_t \to N_{st}$

return now to $C[\Gamma] \otimes V$.

Go back to $E$ a $\Gamma$-module with $h, \alpha$ ---

simplified case: $E \to V \to E$ with

$$\beta_1 \xi_1 \beta_1 = \begin{cases} 0 & s \neq 1 \\ \text{id}_V & s = 1 \end{cases}$$

and also $\sum s \beta_1 \alpha_1 s^{-1} = \text{id}_E$

then you should get $\beta_1$ extends to $\beta$ and $\alpha_1$ extends to $\alpha$.

By habit you want

$$E \xrightarrow{x} V \quad \text{and} \quad E \xrightarrow{x} \text{Hom}(C[\Gamma], V)$$

so what happens?

$$\text{Hom}(E, \text{Hom}(C[\Gamma], V)) = \text{Hom}_{C[\Gamma]}(E, V)$$
Start with a $\Gamma$-module $E$ together with linear maps $E \xrightarrow{\alpha} V \xrightarrow{\beta} E$ where $V$ is just a vector space. $\beta$ extends uniquely to a $\Gamma$-map $\beta : \Lambda \otimes V \to E$. $\alpha$ extends to a $\Gamma$-map $\alpha : E \to \text{Hom}(\Lambda, V)$. $\beta(s \otimes v) = s \cdot \beta(v)$, $(\alpha s)(s) = \alpha(s) \cdot s$. Why $\text{Hom}_\Gamma(\Lambda \otimes V, E) = \text{Hom}(V, E)$?

$\text{Hom}_\Gamma(E, \text{Hom}(\Lambda, V)) = \text{Hom}(\Lambda \otimes E, V) = \text{Hom}(E, V)$

You want a canonical map in $\text{Hom}_\Gamma(\Lambda \otimes V, \text{Hom}(\Lambda, V))$.

You get the dual product map:

You want a canonical map in $\text{Hom}(\Lambda \otimes V, V)$.

$\text{Hom}(V, \text{Hom}(\Lambda, V))$
\[ \text{Check } \chi(t \circ \nu)(s) = \delta_i(st)v \]

\[ \text{Hom}_A(A \otimes V, \text{Hom}(A, W)) \]

\[ \text{Hom}(V, \text{Hom}(A, W)) \]

\[ \text{Hom}(V \otimes V, \text{Hom}(A, W)) \]

\[ \text{Hom}(A \otimes V, \text{Hom}(A, W)) \]

\[ \text{Hom}(A, W) \]

\[ \text{Hom}(V, \text{Hom}(A, W)) \]

\[ \text{Hom}(V \otimes V, \text{Hom}(A, W)) \]

\[ \text{Hom}(A \otimes V, \text{W}) \]

You reach a situation familiar from GNS. \[ A = C[G], \quad B = \text{End}(V), \quad M = E, \quad N = V \]

\[ E \xrightarrow{\chi_i} V \xrightarrow{\beta_i} E \]

\[ M \xrightarrow{\iota} N \xrightarrow{_{\text{const}}} M \]

So in the GNS you have \[ j a i(n) = p(a) \phi(n) \]. Recall that a

\[ \text{B module } N, \text{ the possible } M \text{ correspond to } \]

A module factorizations \[ A \otimes N \xrightarrow{\text{A module map}} M \xrightarrow{\text{Hom}(A, N)} \]

\[ a \otimes n \mapsto a[n] \]

\[ (a' \mapsto p(a'a)n) \]

\[ (a' \mapsto p(a'a')n) \]

\[ p: A \rightarrow B \text{ linear} \]
In GNS you have: \[ E \otimes V \xrightarrow{\beta} E \]

Map \( \beta \):

\[ \beta(a) = x_1 a x_1^{-1} \]

Take simplest case \( \rho(s) = \delta'(s) = \begin{cases} 0 & s \neq 1 \\ 1 & s = 1 \end{cases} \)

\( s \otimes v \mapsto s \beta_1 v \mapsto (s') \mapsto \delta'(s's) v' \)

\[ C = \begin{cases} 0 & s' \neq s^{-1} \\ 1 & s' = s^{-1} \end{cases} \]

\[ \sum t \otimes v \mapsto (\sum t \delta^{-1} v) (s) = \begin{cases} 0 & t \neq s^{-1} \\ v_{s^{-1}} & t = s^{-1} \end{cases} \]

Things should become clearer using the GNS formula:

\[ E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \]

\( \beta_1 \) extends to an \( A \)-map \( \beta : A \otimes V \rightarrow E \)

\( \alpha : E \rightarrow \text{Hom}(A, V) \)

Com. \( \alpha_1 \beta : A \otimes V \rightarrow E \rightarrow \text{Hom}(A, V) \)

\( \alpha_1 \beta (a \otimes \sigma) = (a' \mapsto (\chi a' a_1 \beta_1) \sigma) \rho(a) \)
Notice for the $E$, that elements of $V$ give you generators for $E$, while elements of $V^*$ give you $A$-module maps $E \to A^*$. No, you are close to something madcap, at least you have a candidate for $\varphi$ of $A^*$.

\[ A \otimes V \xrightarrow{\varphi} \text{Hom}(A, V) \]

\[ \beta \downarrow \alpha \]

\[ E \]

In the group case $A = C[G]$, the $\varphi$ map is injective and the support hypothesis say the image of $\varphi$ is contained in the image of $\beta$. This $\alpha$ lifts back to a uniquely $\beta$ back thru $\varphi$.

\[ A \otimes V \]

\[ \beta \]

\[ E \]

\[ \beta \varphi = \text{identity} \]

\[ \beta \beta = \text{idempotent} \]

Not very clear.

Instead look at $A = C[F]$. Case. The $\varphi$ map arises from $A \to \text{End}(V)$ linear $p(a) = x, a \beta$. There's an obvious choice when $A = C[G]$.
Namely \( \rho(s) = s(\delta) = \begin{cases} 0 & s \neq 1 \\ 1 & s = 1 \end{cases} \).

Why this map? \( \rho(s) = \delta, \beta \mapsto \rho(s) = s(\delta) \)

corresponds to orthogonality.

\[
\begin{array}{c}
\text{C}[\Gamma] \otimes V \\
\downarrow \delta \otimes v \\
\text{Hom}(C[\Gamma], V) \\
(\delta \mapsto s(\delta) v)
\end{array}
\]

Use this as standard:

\[
\sum_s s \otimes f(s) \quad \mapsto \quad \left( \sum_s s^{-1} \otimes f(s) \right)
\]

\[\delta_{s^{-1}}(t) = 1 \quad \text{for} \quad t = s^{-1}\]

What do you learn?

Review: You are using GNS ideas to find nice formulas. GNS idea: let \( E \) be an \( A \)-module, \( V \) a vector space, linear maps

\[
E \xrightarrow{\alpha} V \xrightarrow{\beta} E \quad \text{and} \quad A \otimes V \xrightarrow{\gamma} E \quad \text{to} \quad \text{Hom}(A, V)
\]

there's a minimal map \( \rho: A \longrightarrow \text{End}(V) \)

\[\rho \text{ linear map } A \longrightarrow \text{End}(V) \]

\[\text{Image of } A \otimes V \xrightarrow{\rho} \text{Hom}(A, V) \]
so far you look at

\[ V \xrightarrow{f} E \xrightarrow{x} V \]

\[ \rho(a) = \alpha_1 \alpha_2 \beta_1 \]

next look at \[ E \xrightarrow{\pi} V \xrightarrow{x} E \]

\[ A \otimes V \xrightarrow{\pi} E \]

\[ E \xrightarrow{\pi} \text{Hom}(A, V) \]

You need \[ \# \] so restrict to \[ A = C[\Gamma] \]

where there should be canonical maps \[ \# \] of the form \[ \rho = \delta = (s \mapsto \begin{cases} 0 & \text{if } s = 1 \\ \pi & \text{otherwise} \end{cases} \]

\[ \sum t \otimes v \mapsto (s \mapsto \delta_s (st)v) = \sum \delta_{t^{-1}} v \]

so you get for \[ C[\Gamma] \otimes V \xrightarrow{=} \text{Hom}(C[\Gamma], V) \]

\[ \sum t \otimes f(t) \mapsto \sum_{t} \delta_{t^{-1}} f(t) \]

\[ \# \left( \sum_{t} t \otimes f(t) \right)(s) = \sum_{t} \delta_{t^{-1}}(s)f(t) = f(s^{-1}) \]

Let \[ \Phi = \# : C[\Gamma] \otimes V \rightarrow \text{Hom}(C[\Gamma], V) = \text{Map}(\Gamma, V) \]

\[ \Phi \left( \sum_{t} t \otimes f(t) \right)(s) = \sum_{t} \delta_{t}(st)f(t) = f(s^{-1}) \]

\[ \Phi \left( u \sum_{t} t \otimes f(t) \right)(s) = \Phi \left( \sum_{t} t \otimes f(u^{-1}t) \right)(s) = f(u^{-1}s^{-1}) \]
Again, \( A \otimes V \rightarrow \text{Hom}(A, V) \)

\[
\begin{array}{c}
\beta \\
\alpha
\end{array} \rightarrow a_\beta, b_\beta, \xi \mapsto (a' \mapsto a_\beta a_\alpha \xi)
\]

\( \alpha \beta : a \otimes \sigma \mapsto (a' \mapsto (a_\beta a_\alpha \xi) \sigma) \)

\[
\rho(a \alpha) \overline{(a_\beta a_\alpha \xi) \sigma}
\]

\( C[\Gamma] \otimes V \rightarrow \text{Hom}(C[\Gamma], V) = \text{Map}(\Gamma, V) \)

\[
\tau \otimes \sigma \mapsto (s \mapsto \delta_\tau(st) \sigma)
\]

\[
\delta_\tau = \sum_{t \tau} \delta_\tau(t) \sigma
\]

\[
\sum_{t \tau} \delta_\tau(t) \sigma
\]

Thus left mult on \( C[\Gamma] \) correspond to \( R_\tau \) on \( C_\tau(\Gamma) \)

So it seems that the good embedding of \( C[\Gamma] \) into \( \text{Map}(\Gamma, C) \), better the good identification between \( C[\Gamma] \) and \( C_\tau(\Gamma) \) is

\[
\sum_{s} \delta_s(\tau) \rightarrow f
\]

Look at \( t \) action

\[
t \sum_{s} \delta_s(\tau) s^{-1} f(s) = \sum_{s} s^{-1} f(st)
\]
Let us now see how $\alpha, \beta$ look.

You want to lift $\alpha: E \to \text{Hom}(A, V)$ back into $A \otimes V$.

\[ C[\Gamma] \otimes V \xrightarrow{\beta} E \]
\[ \sum_{s \in \Gamma} s \alpha_{s^{-1}} \xrightarrow{\beta} \sum_{s} s \beta_{s} x_{s^{-1}} \]
\[ s \mapsto (s \mapsto \alpha_{s^{-1}}) \]

If you start with $V$ and the operators $\alpha, \beta \in \text{End}(V)$, then you have the maps:

\[ C[\Gamma] \otimes V \xrightarrow{\beta} \Gamma \]
\[ t \otimes s \]
\[ (s \mapsto \beta(tst^{-1})) \]

and you take $E$ to be the image, so that $\beta$ surjective and $\alpha$ injective. Then you want $\alpha E = \beta(C[\Gamma] \otimes V)$ to consist of fin. support fun. need $\alpha_{s} \otimes \beta_{s}$ for supp $\Gamma$.

So you get

\[ E \xrightarrow{\alpha} C[\Gamma] \otimes V \xrightarrow{\beta} E \]
\[ \sum_{s \in \Gamma} s \alpha_{s^{-1}} \xrightarrow{\beta} \sum_{s} s \beta_{s} x_{s^{-1}} \]

So you find that your formulas for $\alpha$ and $\beta$ are correct.
The interesting point seems to be

The embedding

\[ C[\Gamma] \otimes V \rightarrow V \]

\[ \sum_{s \in \Gamma} s \otimes f(s^{-1}) \rightarrow (f: s \mapsto f(s)) \]

\[ \sum_{s \in \Gamma} s \otimes f(s^{-1}) \]

\[ \sum_{s} s \otimes f(s^{-1}t) \]

\[ (R_t f) : s \mapsto f(st) \]

Question: From \[ e[\Gamma] \otimes V \xrightarrow{\gamma} E \xrightarrow{\alpha} \text{Hom}(C[\Gamma], V) \]

you get generators for \( E \)

and you get \( \Gamma \)-maps \( E \rightarrow C[\Gamma] \) from linear functionals on \( V \). So there should be an obvious class of finite rank operators on \( E \), e.g.

You still have to get the Morita equivalence. At the moment you have \( E \) with \( \Gamma \)-action and \( h_1 : E \otimes C[\Gamma] \rightarrow E \) such that

\[ h_1 \circ h_1 = 0 \quad \text{for } s \in F \]

\[ \sum_{s \in \Gamma} s h_1 s^{-1} = 1 \quad \text{on } E \]

\[ h_1 = \sum_{s \in F} \beta_1(s) \mathbf{1} \]

hence \( 0 = h_1 \mathbf{1} = \sum_{s \in F} \beta_1(s) \mathbf{1} \mathbf{1} \Rightarrow \mathbf{1} \beta_1(s) = 0 \quad s \in F \]

get

\[ E \xrightarrow{\alpha} V \otimes \xrightarrow{\beta} E \]

\[ \sum_{s \in F} s \otimes \beta(s) \mathbf{1} \mathbf{1} = \sum_{s \in F} s \beta(s) \mathbf{1} \mathbf{1} = \mathbf{1} \]
The formulas

\[
C[\Gamma] \otimes V \rightarrow \text{Hom}(C[\Gamma], V)
\]

\[
t \otimes v(t) \rightarrow (s \mapsto \delta_s(t) v(t))
\]

\[
\sum_{t \in T} \alpha t^{-\frac{1}{2}} = \sum_{t \in T} \alpha \delta_s(t) v(t) = \alpha s^{-1}
\]

\[
\alpha \beta \sum_{t} t \delta_s(t) = \alpha \sum_{t} t \delta_{s^1} v(t) = \sum_{s} s \otimes \sum_{t} (\alpha s^{-1} t \beta) v(t)
\]

\[
\beta \sum_{t} t \delta_s(t) = \beta \sum_{t} t \delta_s(t) = \sum_{t} t \beta_{s^{-1}} v(t)
\]

Is there a way to do this using tensor distributive ideas?
Morita equivalence

\[ E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \]

\[ \text{A-mod} \]

\[ \beta_1 \text{ extends (uniquely?)} \]

\[ \text{Hom}_A (\tilde{A} \otimes V, E) = \text{Hom}_A (V, \text{Hom}_A (\tilde{A}, E)) \]

\[ \alpha_1 \text{ coextends} \]

\[ \text{Hom}_A (E, \text{Hom}_A (\tilde{A}, V)) = \text{Hom}_A (\tilde{A} \otimes_A E, V) \]

---

What is happening with the group ring.

The group ring is embeddable

\[ A \otimes V \xrightarrow{\beta} E \]

\[ \tilde{A} \otimes V \xrightarrow{\beta} E \]

\[ \tilde{A} \otimes V \xrightarrow{\beta} \text{End}(V) \]

---

\[ \tilde{A} \text{ of } A \]

You have to decide on a notation. Two induced modules.

\[ C[\tilde{A}] \otimes V \text{ and } \text{Hom}(C[\tilde{A}], V) \]

\[ V^\tilde{A} = \{ f: \tilde{A} \rightarrow V \} \text{ with } (tf)(s) = f(st) \]

\[ R_+ \text{ action} \]

What about \[ C[\tilde{A}] \otimes V \], you have two choices:

\[ \sum_{s \in \tilde{A}} sf(s) \text{ or } \sum_{s \in \tilde{A}} sg(s) = \sum_{s \in \tilde{A}} s^{-1} g(s) \]

\[ R_+ \text{ action} \]

If finite support

\[ t \sum_{s} sf(s) = \sum_{s} tsf(s) = \sum_{s} s f(t^{-1} s) \]

\[ t \sum_{s} sg(s) = \sum_{s} ts^{-1} g(s) = \sum_{s} (st^{-1})^{-1} g(s) = \sum_{s} s^{-1} g(st) \]

\[ R_+ \text{ action} \]
There is a canonical map \( C[\Gamma] \otimes V \rightarrow V^\Gamma \) arising from the embedding \( V \hookrightarrow V^\Gamma \) \( v \mapsto \delta_v \).

You have

\[ C[\Gamma] \otimes V \xrightarrow{f^1} V \xrightarrow{\iota_1} V^\Gamma \]

\( \iota_1(v) \) is the function \( \delta_1(s) \sigma = \begin{cases} v & \text{if } s=1 \\ 0 & \text{otherwise} \end{cases} \)

\[ t \otimes \sigma \xrightarrow{\hat{f}} (s \mapsto \delta_1(st) \sigma) \]

\[ \hat{f}(t^0) (s) = f(st) \sigma = \delta_1(st) \sigma \]

\[ \hat{f} \left( \sum_t s^{-1} f(t) \right) = \sum_t \delta_1(st^{-1}) f(t) = f(s) \]

**Idea now:** use \( C[\Gamma] \otimes V \rightarrow V^\Gamma \)

\[ \sum_s s^{-1} f(s) = \sum_s (st^{-1})^{-1} f(s) \]

Action given by \( f \mapsto R f \)

\[ t \sum_s s^{-1} f(s) = \sum_s (st^{-1})^{-1} f(s) \]

Recall \( C[\Gamma] \otimes V \rightarrow V^\Gamma \)

\[ t \otimes \sigma \xrightarrow{\hat{f}} (s \mapsto \delta_1(st) \sigma) \]

\[ \sum_t t^0 u_t \xrightarrow{\hat{f}} (s \mapsto \sum_t \delta_1(st) u_t) = (s \mapsto u_{s^{-1}}(s)) \]

\[ \hat{f}(t) \sigma = \delta_1(st) \sigma \]
Assume \( \left( \sum_{t} t u_{t}(s) \right)(s) = f(s) \), then

\[
\sum_{t} s^{-1}(st) u_{t} = \sum_{s^{-1}} u_{s^{-1}} \Rightarrow \sum_{t} t u_{t}^{-1} \quad \text{for} \quad f(s) = u_{s^{-1}}
\]

\[
f(t) = u_{s^{-1}}
\]

\[
f(t) = u_{s^{-1}}
\]

What this means is that I should write \( \sum_{s^{-1}} f(s) \) for an elt of \( \mathbb{C} [T] \otimes V \), which is the correct elt of \( V \). Next what?

\[
E \xrightarrow{\alpha} V \xrightarrow{\beta} E
\]

\[
\sum_{s^{-1}} s^{-1} \xi_{s} \in V \oplus V
\]

Given \( \sum_{s^{-1}} f(s) \in \mathbb{C} [T] \otimes V \), find elt of \( E \).
\[ E \xrightarrow{\alpha_1} V \xrightarrow{\beta_1} E \]
\[ \alpha_2 = \sum_{s} s \alpha_1 s^{-1} \alpha_2 \]
\[ \beta \sum_{s} f(s) \]

\[ E \xrightarrow{\alpha} \text{Hom}(C[G], V) \xrightarrow{\beta} E \]
\[ \xi \mapsto (s \mapsto s \alpha_2 \xi) \]
\[ \sum_{s} s \otimes s^{-1} \xi \mapsto \sum_{s} s \beta_1 s^{-1} \xi \]

At the moment you have
\[ E \xrightarrow{\alpha} C[G] \otimes V \xrightarrow{\beta} E \]
\[ \sum_{s} s \otimes s^{-1} \alpha_2 \xi \mapsto \sum_{s} s \beta_1 s^{-1} \xi \]
\[ \alpha \beta (\sum_{t} t \otimes f(t^{-1})) = \sum_{s} \sum_{t} s \otimes s^{-1} \alpha_2 \beta f(t^{-1}) \]

Continue. Let's agree to write an all of \( C[G] \otimes V \) in the form \( \sum_{s} s \otimes f(s) \) where for finite support

Then
\[ \beta \sum_{s} s^{-1} \alpha_2 f(s) = \sum_{s} s^{-1} \beta_1 f(s) \]
\[ \alpha \left( \sum_{s} s^{-1} \beta_1 f(s) \right) = \sum_{t} t^{-1} \otimes \alpha_2 t \sum_{s} s^{-1} \beta_1 f(s) \]
\[ \alpha \beta \sum_{s} s^{-1} f(s) = \sum_{t} t^{-1} \otimes \sum_{s} (\alpha_2 t s^{-1} \beta_1) f(s) \]
\[ \beta \gamma = \sum_s s \gamma \beta_s \]

\[ (\alpha \beta f)(s) = \sum_t (\alpha_s \beta_t) f(t) \]

\[ (\alpha \beta f)(s^{-1}) = \sum_t (\alpha_s \beta_t^{-1}) f(t) \]

**Standard formulae**

\[ \Lambda \otimes V = \sum_s s \otimes f(s) \]

\[ \beta (\sum_s s \otimes f(s)) = \sum_s s^{-1} \beta s f(s) \]

\[ \alpha \otimes \sum_s s \otimes f(s) = \alpha \sum s \otimes f(s) \]

\[ (\alpha \beta f)(s) = \sum_t (\alpha_s \beta_t) f(t) \]
formulas shouldn't matter too much.

(i) $E$ is a $\Gamma$-module equipped with a linear operator $h_1$ such that $(i)$ $h_1 s h_1 = 0$ for $s \in F$.

(ii) $\sum h_s = 1$ where $h_s = s h_1 s^{-1}$.

Thus $\mathbb{C}[\Gamma] \otimes V \xrightarrow{\beta} E$

\[ s \otimes 1 \mapsto s \beta_1 \sigma \]

Next let $\alpha_1 : E \to h_1 E = V$, let

\[ \alpha : E \to \mathbb{C}[\Gamma] \otimes V \]

\[ \xi \mapsto \sum s \otimes \alpha_1 s^{-1} \xi \]

well defined because can suppose $\beta = t \beta_1$, since $h_1 s^{-1} t \beta_1 \alpha_1 = 0$ for $s^{-1} t \in F$.

May better notation for elements of $\mathbb{C}[\Gamma] \otimes V$ is

$\sum s f(s)$. Then $\beta \sum s f(s) = \sum s \beta_1 f(s)$

$\mathbb{C}[\Gamma] \otimes V \xrightarrow{\rho} \text{Hom}(\mathbb{C}[\Gamma], V)$

It seems there is some idea hidden here

\[ \rho(s) = f \sum \delta_1(s) \]
You had a good viewpoint this morning. Namely, special cases \( \Gamma = 1 \) or \( F = \{1\} \).

For \( \Gamma = 1 \), you have \( E = E \). Ask for a \( \Gamma \)-graded proj on \( V \). Answer is \( P_i \) a \( \pi_i \), such that \( p_i^2 = p_i \). Corresp \( E \) should be the image of \( p_i \).

Given \( \Gamma \) group consider a \( \Gamma \)-graded projection \( p(s) \) on \( V \) supported on \( F = \{1\} \). Get simply a

\[
\text{Free case. When is a } \Gamma \text{-module } E \text{ free?}
\]

When there exists a \( \Gamma \)-linear operator \( h_i \) on \( E \), such that the projection onto the generator subspace \( V \), which satisfies disjointness \( h_i s h_j = \{0 \} \) \( s \neq 1 \)

and completeness \( \sum s h_i s^{-1} = 1 \). Try to weaken a bit. 2nd cond. \( \Rightarrow \sum s h_i E = E \), so any elt \( s \) of \( E \) is a linear comb. of \( h_i s \). Put \( V = h_i E \). \( \sum s V = E \).

Next. \( s h_i s^{-1} h_j = \sum s h_i s^{-1} h_j \neq 0 \) for \( s \neq 1 \), and

\[
\sum \phi_i = \sum \phi_i s^{-1} h_j \phi_i = \frac{1}{2} \phi_i h_j \phi_i = \phi_i h_j \phi_i \quad h_i^2 = h_i.
\]

Look from \( E \) end. Let \( F = \{1\} \) so that \( h_i h_j = 0 \) \( s \neq 1 \).

Look from graded projection viewpoint.

\[
p(s) = \sum_{s \neq 1} p(s) p(u)
\]

\[
p(1) = p(1) \lquad \text{Philosophy: Think of } E \text{ as just a } \Gamma \text{-module with } h_i \text{ satisfying the two conditions } h_i h_j = 0.
\]
\( E \rightarrow \text{module, } h, c-\text{lin. of an } E \)
\( \forall \bar{s} \in E \quad \{ s \mid h, sh_1 \neq 0 \} \) is finite

\( \sum s \bar{s}^{-1} = \bar{s} \)

First case \( \{ s \mid h, sh_1 \neq 0 \} = \{ 1, \bar{s} \} \)
that is \( h, sh_1 = 0 \) for \( s \neq 1 \).

\( E \rightarrow \text{module, } h, c-\text{lin. operator on } E \)

1. \( \{ s \mid h, sh_1 \neq 0 \} = \bar{s} \) is finite
2. \( \forall \bar{s} \in E \quad \{ s \mid h, sh_1 \neq 0 \} \) is finite and
\( \sum s \bar{s}^{-1} = \bar{s} \)

First case to consider is \( \{ s \mid h, sh_1 \neq 0 \} = \{ 1, \bar{s} \} \).

d.e. \( h, sh_1 = 0 \) for \( s \neq 1 \).

Better might be to note that \( sh_1^* h, sh_1 = 0 \) for \( s \neq 1 \).

Thus \( \bar{s} = sh_1 \) are mutually annihilating projectors, and \( \text{tr} \bar{s} = \text{id}_E \).
Put $V = h_i E$. Then $\Sigma h_s$.  

Simplify the argument.  

$sh_s^{-1}h_i = 0 \quad s \neq 1$

$$\sum h_s^{-1}h_i = h_i$$

for $s \neq 1$.

Wait.  

$$\sum h_s^{-1}h_i = \sum sh_s^{-1}h_i = 0$$

for $s \neq 1$.

So you have anislocating projectors, whose sum is the identity.  

$h_s E = sh_i E$.  

$E = \bigoplus V_s \quad V_s = h_s E = sV_i$

Now you are in a good situation.

**Conclusion**: Given a $\Gamma$-module $E$ and a C-linear map $h_i$ on $E$ so $h_i h_j = 0$ for $s \neq 1$ and all $i$, if $\{s | h_s^{-1}h_i \neq 0\}$ is finite and $F = \sum h_s^{-1}h_i$, then $E = \bigoplus V_s$ where $V_s = \bigcap_{s \in F} h_s E = sV_i$.  

$h_s = sh_i^{-1}$ is the projector, killing $V_i$ etc.

Next to consider the general case: $E$ $\Gamma$-module.  

$h_i C$-linear + $\{s | h_s^{-1}h_i \neq 0\}$ is finite; call this set $F$.  

and $\forall \gamma \in E$  

$$\sum h_s \gamma = \gamma$$  

(sum assumed finite)

Then the basic idea should be to...  

Again put $V_s = sV_i \quad V_i = h_i E$

Not true that $h_s h_i = 0$ for $s \neq t$.  

But $h_s h_i = sh_i^{-1}h_i \neq 0 \iff t \in F$.  

$\therefore$
Now there's a problem about the symmetry. To some extent you can replace $F$ by $F_0 F^{-1}$.

Let's organize this:

\[
V = h_i E \xrightarrow{\beta_i = \text{inj}} V \xrightarrow{\beta_i = \text{inj}} E
\]

\[
0 = h_i s^{-1} \beta_i h_j = \beta_i (\alpha_j s^{-1} t \beta_j) \alpha_j \quad \text{iff} \quad \alpha_i s^{-1} t \beta_i = 0
\]

Do you get a $\Gamma$-graded projection on $V$?

\[
p(s) = \alpha_i s^{-1} \beta_i, \quad p(t) p(t^{-1}s) = \alpha_i t^{-1} \beta_i \alpha_i
\]

\[
\beta_i = \text{inclusion of } h_i E \text{ into } E
\]

\[
\beta_s = \quad \text{sh}_i E = h_s E \text{ into } E
\]

\[
\begin{array}{ccc}
E & \xrightarrow{\beta} & E \\
\otimes & \quad & \otimes
\end{array}
\]

\[
\begin{align*}
0 & = \sum s h_i s^{-1} \beta_i = \beta_i & \forall \beta_i \in \text{End}_E (V) \\
\text{and} \quad h_i s h_j = 0 & \end{align*}
\]

Put $V = h_i E$ then $h_i = \beta_i = \text{inj}$. Let $\alpha_i = h_i$, $\beta_i = \text{inj}$.

\[
h_i \beta_i = \beta_i = \frac{\beta_i (\alpha_i s^{-1} t \beta_i) \alpha_i}{\text{inj}}
\]
So we have the fibration
\[ V = E \rightarrow E \]

More things around \( E \) as well as

- \( h_0 = h_{-1} = 0 \)
- \( h_3 = \sigma^3 h_0 \)
- \( h_4 = \sigma^4 h_0 \)

Next notation. \( E \) is a \( T \)-module, \( h : E \rightarrow \text{End}(E) \)

\[ \text{act:} \quad \{ \theta \} h_{\theta} \neq 0 \}
\]

For some \( \sum h_3 = \text{id} \), where \( h_3 = \sigma h_0 \).

\[ \xrightarrow{	ext{act}} \]

What to do?

\[ V_0 = \sigma V_1 = \sigma h_0 V_1 = \sigma h_0 E \]

E = E \rightarrow \bigoplus V_0 = E \]
In this situation, you have the maps

\[ E \xrightarrow{\alpha} \bigoplus V_s \xrightarrow{\beta} E \]

\[ \alpha_s^\beta \quad \text{for} \quad s \in S \]

\[ \beta(\eta_s) = \sum_s \beta_s \eta_s = \sum_s \beta_s \eta_s \]

Here \((\eta_s)\) is a finite supp. map from \(\Gamma\) to \(E\) such that \(\eta_s \in V_s = \mathbb{1} V_s\).

Now the above situation is \(\Gamma\)-equivariant. Basically, it amounts to the family of canonical factorizations

\[ h_s : E \xrightarrow{\alpha_s} V_s \xrightarrow{\beta_s} E \]

\[ h_s = \beta_s \alpha_s = 5 \beta_s \alpha_s \]

of \(h_s\) for each \(s\) and the fact that these maps produce

\[ \sum_s h_s = \text{id}_E\]

Your viewpoint here is supposed to ignore the group, to somehow describe a partition of unity in an operator sense. All that matters is the family \((h_s)\), the local finiteness, \(\sum h_s = 1\), \(\{t \mid h_s h_t = 0\} \quad \text{etc.}\)

What about order? How much can you prove from these assumptions? Local finiteness completeness.
So now you are looking at Cuntz’s noncommutative simplicial complexes really.

A first question might be to see what happens with a finite index set.

\[ h_1 h_2 = 0 \quad \iff \quad h_1 h_2 = 0 \]

What do you want? Consider \( \bigoplus \langle h_0, \ldots, h_n \rangle \) relation \( \sum \lambda_j = 1 \).

You want a non-unital alg.

\[(h_0 + h_1) h_j = h_j \]

Let \( E \) be a vector space equipped with operators \( x, y \) such that \( (x + y) x = x \) and \( (x + y) y = y \) on \( E \).

Also \( E = x E + y E \). Thus \( x + y = 1 \) on \( E \).

\[(x + y)(x + y) = x + y \]

central

Let \( R = \langle x, y \rangle \) ring of operators on \( E \) gen. by \( x, y \). Then \( x + y = 1 \) in \( R \), so \( R \) is commutative.

So \( y = 1 - x \) on \( E \)

\[ A = T \langle x, y \rangle / (x + y) x = x, \quad (x + y) y = y \]

then \( (x + y)^2 = (x + y) \) in \( A \), call \( x + y = e \). Then \( e A = A \)

so \( e \) is a left identity for \( A \).

\[ e = x + y \quad ea = a \]

\[ A = eA + (1 - e)A \]

can write \( A = A e \oplus A (1 - e) \)
Let $M$ be a left $A$-module. Then $AM = M$ if and only if $eM = M$, and in this case $M$ should be free. Check: $eA \subseteq \text{End}(A)$ because $AM = M$. Con. $AM = M \Rightarrow \text{End}(A) = eM$. Check for Morita equivalence.

$$(A, Ae) \
(Ae, eA)$$

Note $Ae = eAe$ is unital. So it seems that $A = Ae \oplus A(1-e) = A \oplus N$ where $NA = 0$.

Let $W$ be a right $A$-module. Better to take $W = A$. Split it into $Ae \oplus A(1-e)$

Is $A(1-e)$ non-zero? $x \in e$?

What happens is $Ae = A(x+y)$

$$(x+y)(x+y) = (x+y)$$

Look at $A^p$. $x(x+y) = x$ $y(x+y) = y$

$W$ an $A^p$-mod

Look at $A$ having a left unit $e$: $ea = a$. Then $A = Ae \oplus A(1-e)$ unital left unitary right nil.

$e = \sum_{i=0}^n x_i$ $e = x+y$

Can you get this to work? $A$ has two generators $x$, $y$ and the relation says the linear comb. $x+y$ should be a left idempotent. Will get some ring isomorphism.
Consider now a vector space \( X \) and non-zero element \( k \). A is quotient of monomial \( T(X) = X \oplus X \oplus \cdots \) by relations \( kx = x \) \( \forall x \in X \). How is this related to \( \mathbb{Z} \)?

Construction \( RA = \text{T-mat}(A) \) \( \mathbb{Z} \) module \( \mathbb{Z}_k \).

Keep it simple. Let \( M \) be a vector space with two operators \( k, h \) satisfying the relations \( k h = h \) and \( k k = k \). Let \( A \) be the monomial ring. \( \mathbb{Z} \) to these generators and relations. Better to have \( e, h \) satisfying \( e^2 = e \) and \( eh = h \). (This should give the same ring as the one with generators \( x, y \) subject to relations \( (x+y)x = x \) and \( (x+y)y = y \). Check \( ex = x \) and \( e(e-x) = e-x \).)

\( M \) has operators \( e, h \) \( e^2 = e, eh = h \)

You can split \( M = eM \oplus (1-e)M \). Then \( h \) can be any map from \( M \) to \( eM \).

Look at ring \( \mathbb{Z} \) \( h \in \text{gens} \) \( e, h \) \( e^2 = e \), \( eh = h \)

words in \( e, h \) what are the possibilities?

This yields a basis for the ring.
\[ A = C[h]h + C[h]e \]
\[ Ae = C[h]he + C[h]e = C[h]e \]

\[ a = fh + ge \]
\[ ha = (hf)h + (hg)e \]
\[ ea = fh + g \]

Review: You are trying to understand non-commutative rings, e.g. in dim 1, ring w. gen. \( x, y \) subject to rels.

\[(x+y)x = x, (x+y)y = y\]

In particular, you want to see that the relations do not necessarily hold.

Change notation: \( e = x+y \), \( h = x \)

The relations become
\[ eh = h \quad \text{and} \quad e(e-h) = e-h \]

Let \( A \) be the unital ring defined by these gen + rels.

Axis closed under \( h, e \) and \( e^2 = e \).

Then \( h - he \) seems to be \( \neq 0 \). To be more accurate, we should represent \( A \) on \( C[h] \otimes 2 \).

\[ a = fh + ge \]
\[ ha = (hf)h + (hg)e \]
\[ ea = fh + g \]
not convincing. Work a bit more.

A nonunital alg gen: $e, h$ rels: $eh = h, e^2 = e$

Let $A$ act on $\left( C[h] + C[e] \right)$

\[
\begin{pmatrix}
  h & e \\
  g & f
\end{pmatrix}
\begin{pmatrix}
  h & e \\
  g & f
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
  hf & e(f) \\
  hg & f(0)
\end{pmatrix}
\]

$\tilde{A} = \frac{\langle e, h \rangle}{(e^2 = e, eh = h)}$ $\tilde{A}$-modules are v.s. $W$ equipped with a splitting $W = eW \oplus (1-e)W$ and an op. $h: W \to eW$ you want to find $w \in W$ such that $h\cdot w \neq 0$.

\[
\begin{pmatrix}
eW \\
eW
\end{pmatrix}
\quad \begin{pmatrix}
eW \\
eW
\end{pmatrix}
\]

\[
h = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

clear.

Repeat $A$ gen. $x_i, i = 0, \ldots, n$ rels. $\left( \sum_{i=0}^{n} x_i \right) x_j = x_j, \quad j \geq 1, \quad e h_j = h_j, \quad e^2 = e$

The rep. of $A$ is a v.s. $W$ eq. \[ e, h_0, \ldots, h_n \] such that $h^2 = h, \quad e h_j = h_j \quad k \leq n, \quad e^2 = e$.

So you learn that both $\sum h_i h_j = h_i, \quad \sum h_j h_i = h_j$ have to be considered should be assumed to hold.
Repeat the program: You are studying a
vector space $E$ equipped with
operations of unity, i.e. a collection $\{s| h_s h_t \neq 0\}$ is finite and in the other order and completeness $\sum h_s h_t = h_t$

$e^2 = e$, $eh = h$. Then $E = \{E_s \in E \text{ such that } \sum h_s h_t = h_t \forall t\}$. Assuming $E = AE$ you get $E = \sum h_t E \Rightarrow \sum h_s = 1$ on $h_t E \forall t$ hence on $E \Rightarrow h_t \sum h_s = h_t \forall t$.

Repeat. Given $vsE$ with $h_s \neq 0$ finite $\Gamma (\sum h_s) h_t = h_t \forall t$. Assume $B$ defined by $\text{gen } h_s$, $s \in \Gamma$ s.t. $h_t \neq 0$, then $B = B^2$ so can replace $E$ by $BE = \sum h_t E$. Then get $\sum h_s = 1$ on $h_t E \forall t \Rightarrow \sum h_s = 1$ on $E$ whence $h_t \sum h_s = h_t \forall t$.

Okay. Now you want $h_s h_t = 0$ symm? Again consider a finite index set $F$ and factor $h_s = F_s \Rightarrow h_s h_t = F_s F_t \Rightarrow F_s F_t \Rightarrow F_s F_t$
\[ E \xrightarrow{\alpha} \bigoplus V_s \xrightarrow{\beta} E \]

\[(\eta_s)_{s \in \Gamma} \mapsto \sum \beta_s \eta_s \]

\[\xi \mapsto (\alpha_s \xi)_{s \in \Gamma} \mapsto \sum \beta_s \alpha_s \xi_s = \sum h_s \xi = \xi\]

So you get a projector \( \bigoplus V_s \)

\[\bigoplus V_s \xrightarrow{\beta} E \xrightarrow{\alpha} \bigoplus V_s\]

\[(\eta_s) \mapsto \sum_{t} \beta_t \eta_t \mapsto \bigoplus \sum_{t} \alpha_s \beta_t \eta_t\]

so on \( \bigoplus V_s \) we have the operator with kernel \( \alpha_s \beta_t \) \( \text{OKAY} \)

\[\alpha \beta \text{ is a projector. Does this help?}\]

\[\text{What do you want!}\]

---

If the finite case:

\[ E, \ h_s \leq \Gamma \text{ finite}, \ \sum_{s} h_s h_t = h_t \ \forall t \]

\[ \text{with } \sum_{s} h_s = e, \ \text{relate } \sum_{s} h_s h_t = e_{h_t} = h_t \ \forall t \]

\[ \text{left unit for } A \]

\[ 0 \to A \xrightarrow{\lambda} \tilde{A} \to Z \to 0 \]

\[ o \to A \otimes M \to M \to \text{image } 0 \]

\[ M \text{ projective right } A \text{ and } \]

\[ e = 1 \in M \]
So \( M(A) \) seems to be \( \text{Mod}(Ae) \). So limit the general theory.

This leads to left \( A \)-module \( A = Ae \oplus A(1-e) \) as \( A \)-module\( A = eA \) as right \( A \)-module

\[
\begin{pmatrix}
A & Ae \\
eA & eAe
\end{pmatrix}
\]

So \( P(eA = A) \)

\[
Q = Ae
\]

\[
M(A) \xrightarrow{\text{def}} \text{Mod}(Ae)
\]

\[
\begin{aligned}
M 1 & \mapsto eM = M \\
N 1 & \mapsto Ae \otimes_{\text{Ae}} N = N
\end{aligned}
\]

If \( \text{Ae-modules are the same as unitary } A \)-modules. \( \forall s \in I \)

\[
A: \begin{cases}
\text{gen. } h_s, s \in I \\
\text{rels. } eh_s = h_s \forall s \in I, \text{ where } e = \sum_{s \in I} h_s
\end{cases}
\]

An \( A \)-module is a \( V \)-is. \( M \) with

Choose a base point of \( I \), call it 1.

\[
A: \begin{cases}
\text{gen. } x_s, s \neq 1, e \\
\text{rels. } ex_s = x_s \forall s \neq 1 \\
e^2 = e
\end{cases}
\]

\[
h_s = x_s \quad s \neq 1
\]

\[
h_1 = e - \sum_{s \neq 1} x_s
\]

Then an \( A \)-module is a \( V \)-is. \( M \) equipped with \( \text{id-} \)

\[
M = eM \oplus (1-e)M
\]

and operator \( x_s: M \rightarrow eM \) for \( s \neq 1 \).

\[
e^2 = e
\]

\[
eh = h
\]

\[
M = eM \oplus (1-e)M
\]

Ask if \( x_s e = x_s \)
A \quad \text{gen } h_s \quad \forall s \in \Sigma \quad e = \sum h_s \quad \Rightarrow \quad e^2 = e

M = eM \oplus (1-e)M \quad \text{and } h_s : M \rightarrow eM \quad \text{arb. with } e

\text{question } h_s e = h_s \quad \text{or } e\in e

M = eM \oplus (1-e)M \quad h_s : M/(1-e)M \rightarrow M \quad \text{arb. with } e

\sum_h = \oplus e

There should be nothing more.

Next examine embedding of $E$ into $\bigoplus V_s$.

$V_s = h_s E \quad \alpha = h_s \rightarrow h_s E \quad \beta : h_s E \rightarrow E$

$\alpha \beta$ is a projector \hspace{1cm} \beta = \bigoplus \beta_s \alpha_s = \sum h_s = 1.

$E \xrightarrow{\alpha} \bigoplus V_s \xrightarrow{\beta} E$

\[(\alpha \beta)(v_t) = (\sum_s \beta_s \alpha_s v_t) = \]