January 29, 2000

Grid spaces. Begin with a basic operation on 2×2 matrices. Let p, q, p', q' be elements of a vector space related by

$$(1) \qquad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}.$$

Assuming $d \neq 0$ these relations may be written

$$(2) \qquad \begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} p' \\ q \end{pmatrix}.$$

This gives an operation on 2×2 matrices with $d \neq 0$:

$$(3) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$$

which is idempotent. Note that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant $=1$ iff the transformed matrix has $=$ diagonal entries.

Let us think of p, q, p', q' as variables related as above. (More precisely, regard p, q, p', q' as linear functions on the dual space.) One then has the equivalence

$$(4) \qquad |p|^2 - |q|^2 = |p'|^2 - |q'|^2 \iff |p|^2 + |q'|^2 = |p'|^2 + |q|^2$$

whence

$$(5) \qquad \boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1) \iff \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \in U(2)}$$

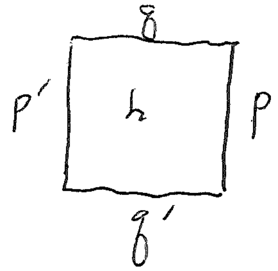
In this way the transform yields a diffeomorphism of $U(1,1)$ (as manifold) with the open subset of $U(2)$ where $d \neq 0$.

Special case: Let $h \in \mathbb{C}$ satisfy $|h| < 1$. Then

$$(6) \qquad \begin{pmatrix} p \\ q \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} p' \\ q' \end{pmatrix} \iff \begin{pmatrix} p \\ q' \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar{h} & k \end{pmatrix}\begin{pmatrix} p' \\ q \end{pmatrix}$$

where $k = \sqrt{1-|h|^2}$. We use the following picture

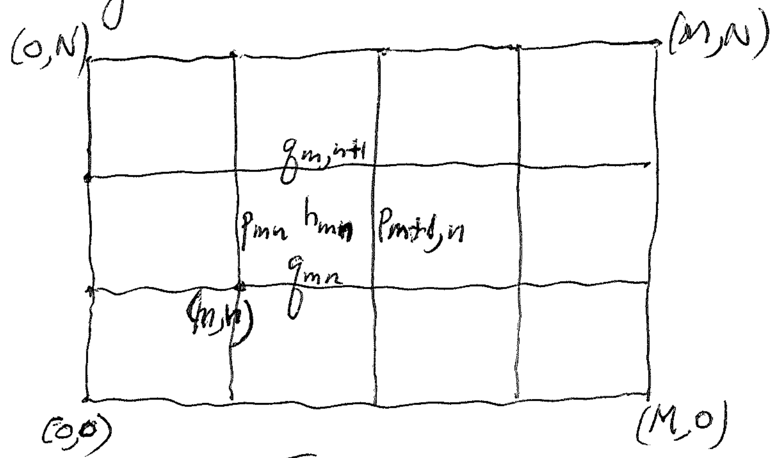
(7)



to indicate a 2-diml vector space V spanned by vectors p, q, p', q' satisfying these relations. Note that the two edges issuing from any corner are a basis of V. Also there is on V a well-defined positive definite (resp. nondegenerate indefinite) hermitian bilinear form such that $\{p, q'\}$ and $\{p', q\}$ (resp. $\{p, q\}$ and $\{p', q'\}$) are orthonormal bases (resp. orthogonal bases normed so that the horizontal edges have norm 1 and vertical edges have norm -1).

We now put such squares together to form a grid.

(8)



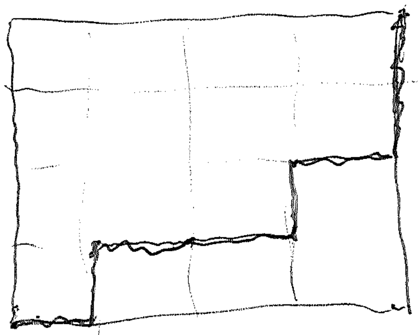
The grid space E corresponding to this picture is the vector space generated by the edges $p_{m,n}, q_{m,n}$

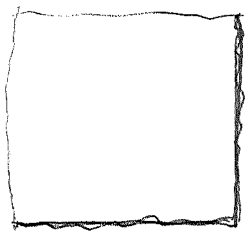
[redacted] subject to the relations

$$\begin{pmatrix} p_{m+1,n} \\ q_{m,n+1} \end{pmatrix} = \frac{1}{k_{mn}} \begin{pmatrix} 1 & h_{mn} \\ \bar{h}_{mn} & 1 \end{pmatrix} \begin{pmatrix} p_{m,n} \\ q_{m,n} \end{pmatrix}$$

for $0 \leq m < M$, $0 \leq n < N$. Here we have $p_{m,n}$ for $0 \leq m < M$, $0 \leq n \leq N$ and $q_{m,n}$ for $0 \leq m \leq M$, $0 \leq n < N$, so we have [redacted] $M(N+1) + (M+1)N = 2MN + M + N$ generators and $2MN$ relations, so $\dim E \geq M+N$.

On the other hand it is easily seen that any exhaustive increasing staircase:





may be pushed 1 square at a time to without changing [redacted] its span in E. Also any edge lies in some exhaustive increasing staircase, so that any such staircase [redacted] (which has $M+N$ edges) spans E. $\therefore$

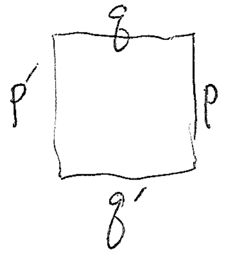
a) $\dim E = M+N$, and any exhaustive increasing staircase is a basis. Also for decreasing.

b) E has a well-defined positive definite herm. form such that any exhaustive increasing staircase is an orthonormal basis. (This is because transitions between adjacent bases are given by unitary matrices). Same for indefinite nondegenerate hermitian form, and exhaustive decreasing staircases.

(better: ascending + descending staircases).

Remark that the preceding holds more generally for grids with arbitrary $u(1,1)$ matrices in the squares.

Question about whether E has a well-defined volume element when the squares have $SU(1,1)$ matrices. Sign problems: If $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix}$



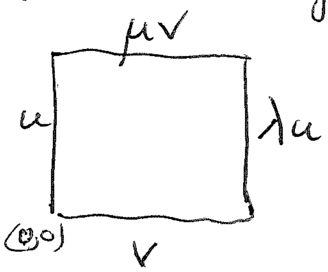
$$q \wedge p = (cp' + dq') \wedge (ap' + bq') = (cb - da)\, p' \wedge q'$$

so you seem to need $\det = -1$.

Infinite constant-coefficient grid spaces.

Consider the doubly-infinite integer lattice point grid in the plane, let E be the grid space with generators the edges and relations the squares where h_{mn} is a constant h for all $m,n \in \mathbb{Z}\times\mathbb{Z}$. Then the additive group $\mathbb{Z}\times\mathbb{Z}$ acts on E; let λ (resp. μ) be the shift one step to the right (resp. up). Then E becomes a module over the group ring $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}] = \mathbb{C}[\lambda,\mu,\lambda^{-1},\mu^{-1}]$, call this A. Let $v = p_{00}$, $u = q_{00}$, whence $p_{mn} = \lambda^m \mu^n v$, $q_{mn} = \lambda^m \mu^n u$, so that the free vector space with the edges for generators is the free A-module with generators v, u. The space of relations among these edges is the A-submodule generators the 2 relations for the square:

(1)



namely $\begin{pmatrix} \lambda u \\ \mu v \end{pmatrix} = \frac{1}{k} \begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$

Thus E is the A-module gen. by u, v subject to the relations

(2) $$\begin{cases} (k\lambda - 1)u = hv \\ (k\mu - 1)v = \bar{h}u \end{cases}$$

We now construct an isomorphism of the grid space

E with the space $\mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}] = B$ of rational functions which are regular outside of $\{0, \infty, k, k^{-1}\}$. Define operators λ, μ on B and elements $u, v \in B$ by

(3)
$$\begin{aligned} \lambda &= \text{mult. by } z \\ \mu &= \text{mult. by } \frac{z-k}{kz-1} \\ v &= 1 \\ u &= \frac{h}{kz-1} \end{aligned}$$

Note that λ, μ are invertible on B, so that B becomes a module over the group ring A. Check the relations:

$$(k\lambda - 1)u = (kz-1)\frac{h}{kz-1} = h = hv$$

$$(k\mu - 1)v = \left(k\frac{z-k}{kz-1} - 1\right)1 = \frac{\cancel{kz} - k^2 \cancel{-kz} + 1}{kz-1} = \bar{h}\frac{h}{kz-1} = \bar{h}u$$

Thus we get a unique map from E to B defined by (3); it is the unique A-module map sending v to $1 \in B$.

To show this map is an isom, note first that the relations (2) imply that the relation

$$(k\lambda - 1)(k\mu - 1) = 1 - k^2$$

holds in E, since the difference kills both u, v and hence E. In particular $k\lambda - 1$ is an invertible operator on E, and

$$\mu = \frac{1}{k}\left(\frac{1-k^2}{k\lambda - 1} + 1\right) = \frac{\lambda - k}{k\lambda - 1}.$$

Since μ is invertible on E, also $\lambda - k$ is invertible. Since the operators $\lambda, \lambda - k, k\lambda - 1$ on E are invertible E becomes a B-module with z acting as λ. Then we get a map from B to E by acting on v. The map $B \longrightarrow E$ is surjective since $Av = Bv$ contains $u = \frac{h}{k\lambda - 1} v$. Hence the map $E \longrightarrow B$ is an isomorphism since $B \twoheadrightarrow E \longrightarrow B$ is the identity.

Here another proof for the isomorphism using the positive definite inner product on E. Let $\bar{E}$ be completion of E with respect to this inner product. The operators λ, μ clearly preserve this inner product, hence extend uniquely to unitary operators on $\bar{E}$. Note that the relation $\mu(k\lambda - 1) = \lambda - k$ on E extends to $\bar{E}$ by continuity; also $(k\lambda - 1)$ is invertible by geometric series and $k < 1$. Thus $\mu = \frac{\lambda - k}{k\lambda - 1}$ on $\bar{E}$.

Recall that $\lambda^m v$ is an orthonormal set in E, so its closed span in $\bar{E}$ will be isomorphic to $L^2(S^1, \frac{dz}{2\pi i z})$ via $\lambda^m v \longleftrightarrow z^m$. Then $\bar{E} = \overline{\mathbb{C}[\lambda, \lambda^{-1}] v} \cong L^2(S^1)$ because this closed subspace is stable under μ.

The important point is that E embeds in $\bar{E}$ as we have a pos. def. inner product on E. Then E is described by its image which is the span of the elements $\lambda^m \mu^n \begin{matrix} u \\ \text{or} \\ v \end{matrix}$, and coincides with $B = \mathbb{C}[z, z^{-1}, (z-k)^{-1}, (kz-1)^{-1}] \subset L^2(S^1)$.

<u>January 30, 2000</u>

Calculation of the indefinite hermitian bilinear form, denote it $IH(\xi,\xi')$, on E, which we have seen is isomorphic to $\mathbb{C}[z,z^{-1},(z-k)^{-1},(kz-1)^{-1}] = B$ via the map $f(z) \longmapsto f(\lambda)v$. By translation invariance ~~[illegible]~~ of IH we have

$IH(\lambda^m\mu^n\xi,\xi') = IH(\xi,\lambda^{-m}\mu^{-n}\xi')$, whence

$IH(a\xi,\xi') = IH(\xi,a^*\xi')$, where $*$ is the (anti-linear) involution on $A=\mathbb{C}[\lambda,\lambda,\mu,\mu^{-1}]$ such that $\lambda^*=\lambda^{-1}$, $\mu=\mu^{-1}$, $c=\bar{c}$ for $c\in\mathbb{C}$. Consequently

$$IH(f(\lambda)v, g(\lambda)v) = IH(v, (f^*g)(\lambda)v) \qquad f,g\in B$$

so we only need to calculate the linear functional

$f(z) \longmapsto IH(v, f(\lambda)v)$ on B.

Recall that a linear function from E to a vector space V is the same as a V-valued solution of the grid equations:

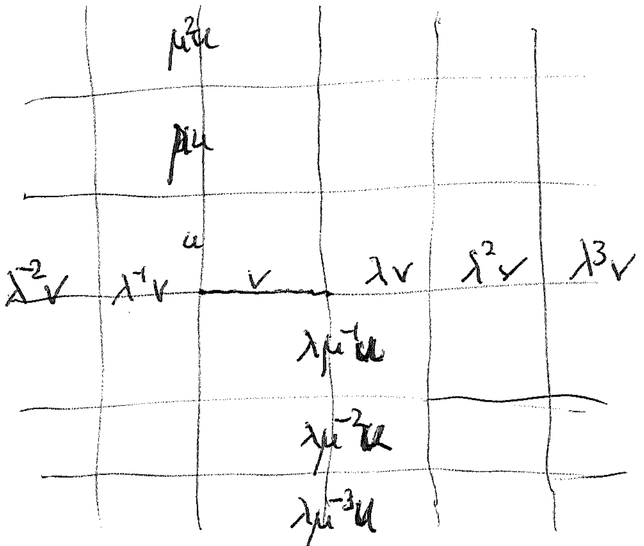
$$\begin{pmatrix}\psi^1_{m+1,n}\\ \psi^2_{m,n+1}\end{pmatrix} = \frac{1}{k}\begin{pmatrix}1 & h\\ \bar{h} & 1\end{pmatrix}\begin{pmatrix}\psi^1_{m,n}\\ \psi^2_{m,n}\end{pmatrix}$$

where $\psi^1_{m,n}$ and $\psi^2_{m,n}$ are the values of the linear function on p_{mn}, q_{mn}.

$$\boxed{\text{Residue Formula: } IH(v,f(\lambda)v) = \operatorname{Res}_{\{0,k^{-1}\}}\left(f(z)\frac{dz}{2\pi i z}\right) \text{ for } f\in B}$$

It suffices to check this formula on a basis of B.

Here's a picture of the basis we seek:



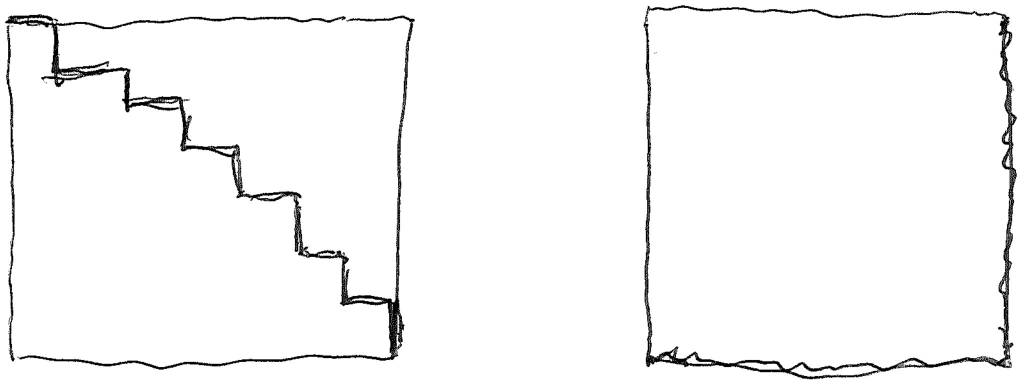
namely

$$(\lambda^m v)_{m\in\mathbb{Z}}$$

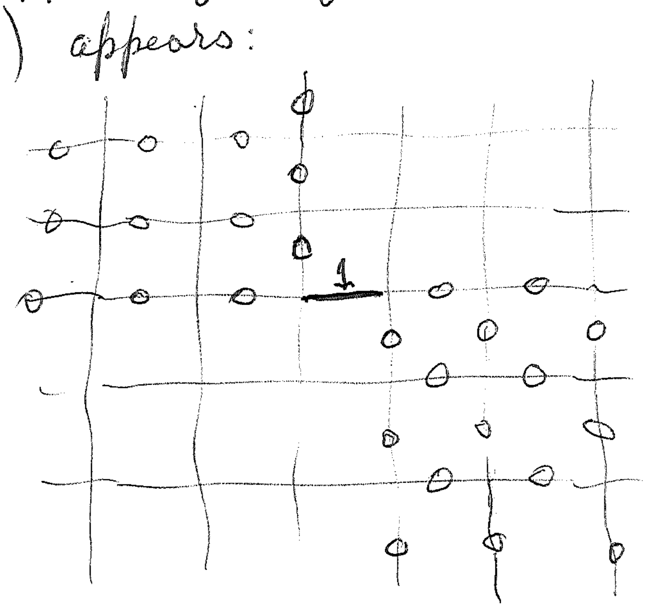
$$(\lambda\mu^{-n}u)_{n\geq 1}$$

$$(\mu^n u)_{n\geq 0}$$

It is easy to see this is a basis for E by using that the subspace spanned by the grid vectors in a square admits descending and ascending bases



Because two grid vectors are $\perp$ wrt IH when one is to the upper left of the other, the linear functional $IH(v,\sim)$ appears:



where the values on the other edges can be found by using the grid equations

To prove the formula for IH we only have to check the formula on our basis, i.e.:

$$\delta_m = (v|\lambda^m v) = \operatorname{Res}_{[redacted]\{0,k^{-1}\}}\left(z^m \frac{dz}{2\pi i z}\right)$$ OK since the diffℓ regular at k^{-1}.

$$0 \overset{n\geq 1}{=} (v|\lambda\mu^{-n}u) = \operatorname{Res}_{\{0,k^{-1}\}}\left(\cancel{\tfrac{1}{z}}\,\frac{(kz-1)^{+n-1}}{(z-k)^n}\,h\,\frac{dz}{2\pi i \cancel{z}}\right)$$ diffℓ is regular at $0, k^{-1}$

$$0 \overset{n\geq 0}{=} (v|\mu^n u) = \operatorname{Res}_{\{0,k^{-1}\}}\left(\frac{(z-k)^n h}{(kz-1)^{n+1}}\,\frac{dz}{2\pi i z}\right)$$

$$= \operatorname{Res}_{\{0,k^{-1},k\}}\left(\text{———}\right)$$ since the diff is regular at k

$$= -\operatorname{Res}_{\{\infty\}}\left(\text{—}\right)$$ sum of res $= 0$

$$= 0$$ since the diffℓ is regular at ∞.

(for the last point we can also use the integral over [redacted] a large circle).

<u>January 31, 2000</u>

<u>Continuous limits</u> of the constant h grid.

First examine the case where both x, y directions become continuous. The discrete grid equations are

$$(1)\qquad \begin{pmatrix} \psi^1_{m+1,n} \\ \psi^2_{m,n+1} \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar{h} & 1 \end{pmatrix}\begin{pmatrix} \psi^1_{mn} \\ \psi^2_{mn} \end{pmatrix}$$

Use spacing $\varepsilon \to 0$, $x = m\varepsilon$, $y = n\varepsilon$, and let $h = b\varepsilon$. Then to first order in ε we have

$$\begin{pmatrix} \psi^1_{xy} + \varepsilon\partial_x\psi^1_{xy} \\ \psi^2_{xy} + \varepsilon\partial_y\psi^2_{xy} \end{pmatrix} \simeq \begin{pmatrix} 1 & b\varepsilon \\ \bar{b}\varepsilon & 1 \end{pmatrix}\begin{pmatrix} \psi^1_{xy} \\ \psi^2_{xy} \end{pmatrix}$$

$$(2)\quad \text{i.e.}\quad \boxed{\begin{aligned} \partial_x\psi^1 &= b\psi^2 \\ \partial_y\psi^2 &= \bar{b}\psi^1 \end{aligned}}$$

. Now look for exponential solutions

$$(3)\quad \psi_{xy} = e^{i\xi x + i\eta y}\begin{pmatrix} u \\ v \end{pmatrix}, \text{ whence } \boxed{\begin{aligned} i\xi u &= bv \\ i\eta v &= \bar{b}u \end{aligned}}$$

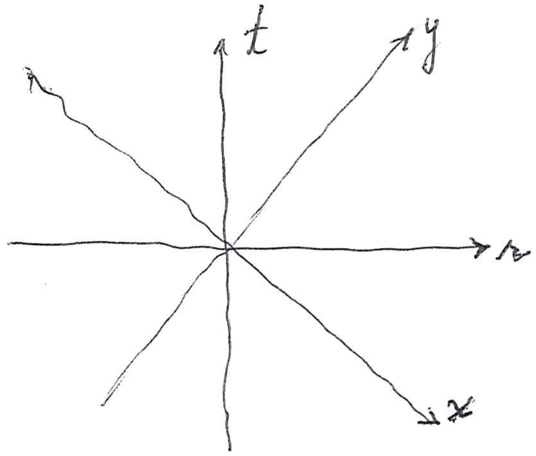
$$(4)\quad \text{which implies that } \boxed{-\xi\eta = |b|^2}.$$

What seems puzzling at this point is the fact that the DE (2) does not appear to have a skew-adjoint form. Let's compare what happens for the massive Dirac equation on the line:

$$(5)\qquad \partial_t\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} \partial_r & m \\ -m & -\partial_r \end{pmatrix}\begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

m real, usually > 0, can also use $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} m$

Introduce characteristic coordinates:

(6) 

$$t = -x + y$$
$$r = x + y$$

$$\partial_x f(t,r) = (\partial_t f)(-1) + \partial_r f$$
$$\partial_y f(t,r) = \partial_t f + \partial_r f$$

$$\partial_x = -\partial_t + \partial_r$$
$$\partial_y = \partial_t + \partial_r$$

(7) so
$$\boxed{\begin{aligned} -\partial_x \psi^1 &= m\psi^2 \\ \partial_y \psi^2 &= -m\psi^1 \end{aligned}}$$
as
$$\begin{pmatrix} \partial_t - \partial_r & 0 \\ 0 & \partial_t + \partial_r \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix} \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}$$

But this has the form (2), in particular $(-i\xi)(i\eta) = -m^2$ or $-\xi\eta = m^2$, so there's no puzzle any more.

Now take $m = 1$ + change sign of ψ^1, whence the grid equations are $\boxed{\partial_x \psi^1 = \psi^2, \ \partial_y \psi^2 = \psi^1}$ and the exponential solutions are

$$\psi_{xy} = e^{i\xi x + i\eta y} \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{where} \quad \begin{aligned} i\xi u &= v \\ i\eta v &= u \end{aligned}$$
$$\text{whence} \quad -\xi\eta = 1 .$$

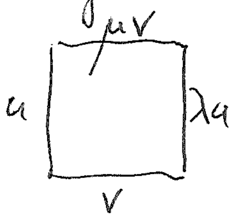
The general solution should be

$$\psi_{xy} = \int e^{i(\xi x - \xi^{-1} y)} \begin{pmatrix} \frac{1}{i\xi} \\ 1 \end{pmatrix} (?) \, \frac{d\xi}{2\pi}$$

where (?) is a suitable function (distribution) on the ξ-line.

February 1, 2000

Consider a constant grid where instead of $\frac{1}{k}\begin{pmatrix}1 & h\\ \bar h & 1\end{pmatrix}$ a general $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in U(1,1)$ is used. The grid space is the A-module generated by u, v subject to the relations



$$\begin{pmatrix}\lambda u\\ \mu v\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} \qquad \boxed{\begin{aligned}(\lambda - a)u &= bv\\ (\mu - d)v &= cu\end{aligned}}$$

and it has natural positive definite and indefinite hermitian inner products preserved by translation. Recall

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\Delta & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\bar d & \bar c\\ c & d\end{pmatrix}$$

where $\Delta = ad - bc$ has $|\Delta| = 1$, and $|d|^2 - |c|^2 = 1$.

Assume $|c| \neq 0$ to avoid the trivial case where the horizontal + vertical directions are uncoupled. Then

$$\mu = d + \frac{bc}{\lambda - a} = \frac{d\lambda - ad + bc}{\lambda - a} = \frac{d\lambda - \Delta}{\lambda - \Delta\bar d}$$

i.e. $\boxed{\mu = \begin{pmatrix}d & -\Delta\\ 1 & -\Delta\bar d\end{pmatrix}(\lambda)}$. To show that any fractional linear transf. in $U(1,1)/\text{scalars} \simeq SU(1,1)/\{\pm\}$ arises from suitable d, c, Δ as above.

$$\begin{vmatrix}d & -\Delta\\ 1 & -\Delta\bar d\end{vmatrix} = -\Delta(|d|^2 - 1) = -\Delta|c|^2.$$

Let $e^{i\varphi}|d| = d$, $e^{2i\theta} = -\Delta$. Then

$$(*)\qquad \frac{e^{-i\theta}}{|c|}\begin{pmatrix} d & -\Delta \\ 1 & -\Delta\bar{d} \end{pmatrix} = \begin{pmatrix} e^{-i\theta+i\phi}\frac{|d|}{|c|} & e^{i\theta}\frac{1}{|c|} \\ e^{-i\theta}\frac{1}{|c|} & e^{i\theta-i\phi}\frac{|d|}{|c|} \end{pmatrix}$$

is in $SU(1,1)$. A general element of $SU(1,1)$ has the form $\begin{pmatrix} e^{i\alpha}r & e^{i\beta}s \\ e^{-i\beta}s & e^{-i\alpha}r \end{pmatrix}$ with $\alpha,\beta \in \mathbb{R}$, $r,s>0$, $r^2-s^2=1$.

The matrix $(*)$ above can be obtained by putting $|c|=\frac{1}{s}$, $|d|=\frac{r}{s}$, $\theta=\beta$, $\phi=\alpha+\beta$.

February 27, 2000

Inverse scattering in the Hilbert space setting. Let $\beta(z)$ be bounded measurable on S^1 with $\|\beta\|_\infty < 1$. Then $f \mapsto \beta f$ is a contraction on $L^2 = L^2(S^1)$, so one has a Hilbert space H obtained by glueing two copies of L^2 together via β. Writing $\xi_+, \xi_- : L^2 \to H$ for the two embeddings, the inner product in H is

$$\|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

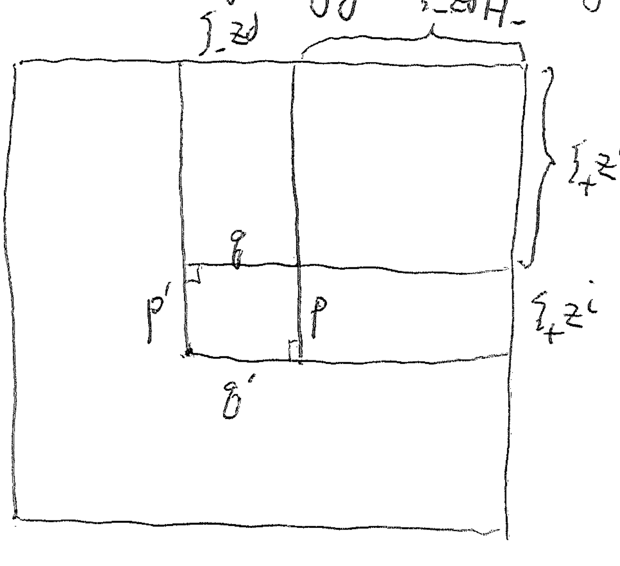
Since $(1-\gamma)^{-1} = \sum_{n\geq 0} \gamma^n$ and $\|(1-\gamma)^{-1}\| \leq \sum \|\gamma\|^n = \frac{1}{1-\|\gamma\|}$ for a contraction γ, e.g. $\gamma = \begin{pmatrix} 0 & \bar\beta \\ \beta & 0 \end{pmatrix}$, the norm^2 on H is equivalent to the direct sum norm^2 on $L^2 \oplus L^2$, so that $\xi_+ L^2 + \xi_- L^2$ is already complete, therefore it $= H$.

As $\beta \cdot$ commutes with $z\cdot$, there is a unitary operator u on H given by $u(\xi_+ f + \xi_- g) = \xi_+ zf + \xi_- zg$.

Construction of the grid space E. One has a bifiltration consisting of subspaces

$$\xi_+ z^i H_+ + \xi_- z^j H_- \subset H$$

which increase if i decreases or j increases. These subspaces are closed because H and $L^2 \oplus L^2$ have the same topology. The grid vectors arise by Gram-Schmidt:

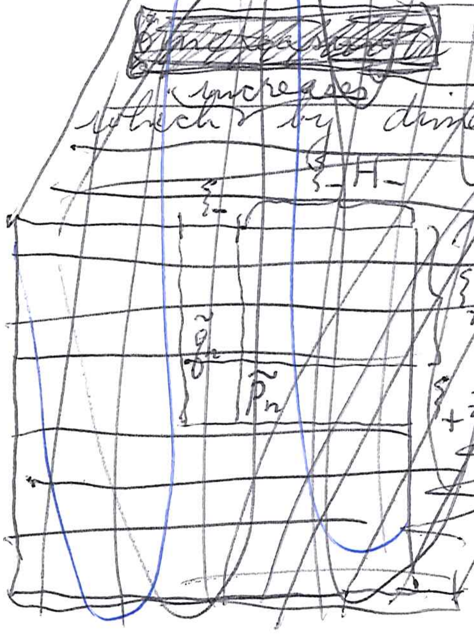


$$H_+ = \overline{\mathbb{C}[z]} \subset L^2$$
$$H_- = \overline{z^{-1}\mathbb{C}[z^{-1}]} \subset L^2$$

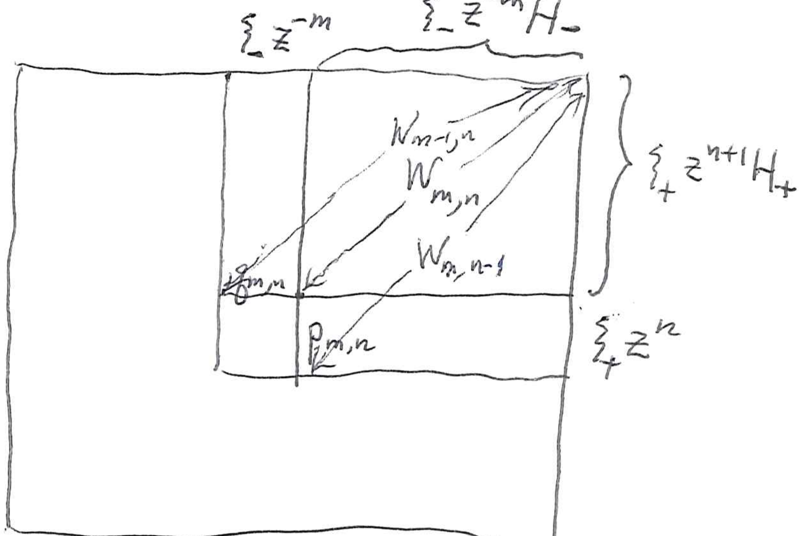
$p = \frac{\tilde p}{\|\tilde p\|}$ where $\tilde p$ is the component of ξz^i perpendicular to $\xi_+ z^{i+1} H_+ + \xi_- z^j H_-$.

Similarly for q, p', q'.

~~A better way to proceed is perhaps to first consider the filtration by the subspaces $\xi_+ z^n H_+ + \xi_- H_-$ which increases by dimension 1 as n decreases by 1~~



Let begin the construction of the grid space associated to β again. Given $(m,n) \in \mathbb{Z}\times\mathbb{Z}$ consider the picture



and let $W_{m,n}$ be the (closed) subspace

$$W_{m,n} = \xi_+ z^{n+1} H_+ + \xi_- z^{-m} H_-$$

Since $\mathbb{C}\,\xi_+ z^n + W_{m,n} = W_{m,n-1}$ and $\mathbb{C}\,\xi_- z^{-m} + W_{m,n} = W_{m-1,n}$ we can split $\xi_+ z^n$ (resp $\xi_- z^{-m}$) lying in and perpendicular to $W_{m,n}$, defining elements (which are $\neq 0$ as $\xi_+ z^n, \xi_- z^{-m} \notin W_{m,n}$)

$$\tilde{p}_{mn} \in \left(\xi_+ z^n + W_{mn}\right) \cap W_{mn}^{\perp} \qquad p_{mn} = \frac{\tilde{p}_{mn}}{\|\tilde{p}_{mn}\|}$$

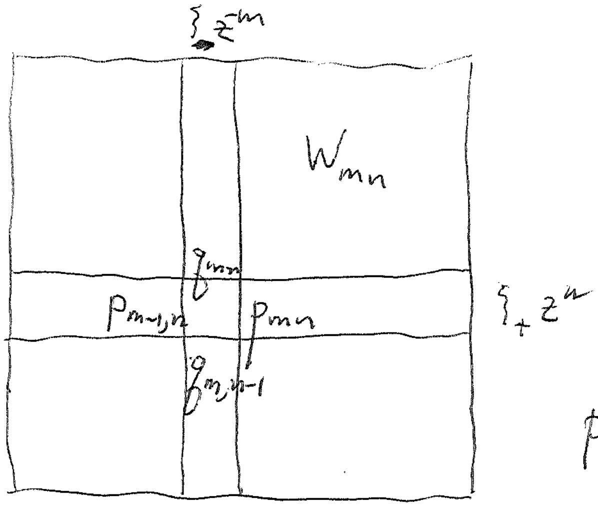
$$\tilde{q}_{mn} \in \left(\xi_- z^{-m} + W_{mn}\right) \cap W_{mn}^{\perp} \qquad q_{mn} = \frac{\tilde{q}_{mn}}{\|\tilde{q}_{mn}\|}$$

Since $uW_{m,n} = W_{m-1,n+1}$ it is clear that

$u(\tilde{p}_{mn}) = \tilde{p}_{m-1,n+1}$ and similarly for $\tilde{q}_{mn}, p_{mn}, q_{mn}$.

February 28, 2000

$$H = \xi_+ L^2 + \xi_- L^2 \qquad \|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}$$



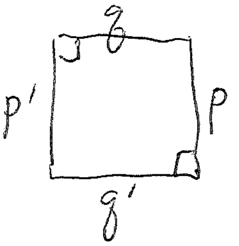
$$W_{mn} = \xi_+ z^{n+1} H_+ + \xi_- z^{-m} H_-$$

$$\tilde p_{mn} = \left(\xi_+ z^n + W_{mn}\right) \cap W_{mn}^\perp$$

$$\tilde q_{mn} = \left(\xi_- z^{-m} + W_{mn}\right) \cap W_{mn}^\perp$$

$$p_{mn} = \frac{\tilde p_{mn}}{\|\tilde p_{mn}\|} \qquad q_{mn} = \frac{\tilde q_{mn}}{\|\tilde q_{mn}\|}$$

Basic lemma: Given a 2 dim'l Hilbert space with unit vectors p, q, p', q' such that $p' \perp q$, $q' \perp p$.



Assume $\mathbb{C}p \neq \mathbb{C}q$ (equiv. $\mathbb{C}q' \neq \mathbb{C}p'$). Then

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p' \\ q' \end{pmatrix} \qquad \begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ in $U(1,1)$. If

$a > 0$ and $d > 0$ (we say p, p' and q, q' <u>positively related mod</u> q', p')

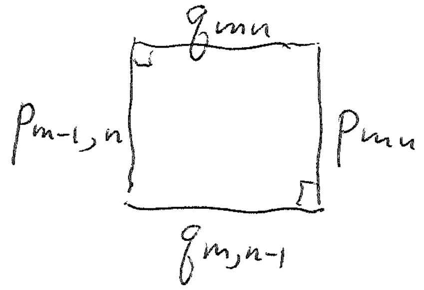
or $a' > 0$ and $d' > 0$ (we say p', p and q', q <u>positively related mod</u> q, p)

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the form $\frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}$, $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & -h \\ -\bar h & 1 \end{pmatrix}$ $\quad |h| < 1$, $k = (1 - |h|^2)^{1/2}$

In fact $h = (q|p)$, since $p' = \frac{1}{k} p - \frac{h}{k} q \implies$

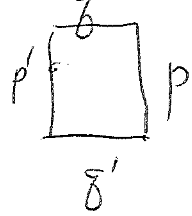
$$0 = (q|p') = \left(q \middle| \frac{1}{k} p - \frac{h}{k} q\right) = \frac{1}{k}\left((q|p) - h\right),$$

We want to apply this lemma to the square



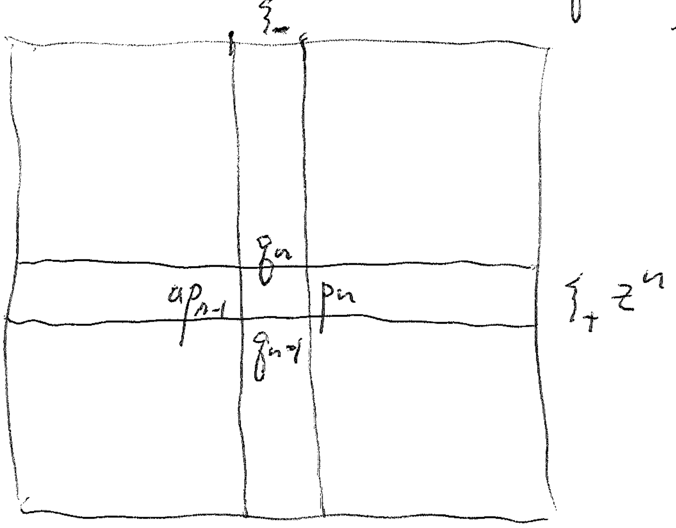
We have
$$\tilde p_{m-1,n} = \tilde p_{m,n} - q_{mn}(q_{mn}|\tilde p_{m,n})$$
$$\tilde q_{m,n-1} = \tilde q_{m,n} - p_{mn}(p_{mn}|\tilde q_{mn})$$

Clearly the left side Q of the first eqn. is $\perp W_{mn}$ and to q_{mn}, hence it is $\perp$ to $\mathbb{C}q_{mn} + W_{mn} = W_{m-1,n}$. Also $Q \equiv \tilde p_{mn} \equiv \xi_+ z^n \mod W_{m-1,n}$, so $Q = \tilde p_{m-1,n}$. Sim. for $\tilde q_{m,n-1}$. Thus we have the case



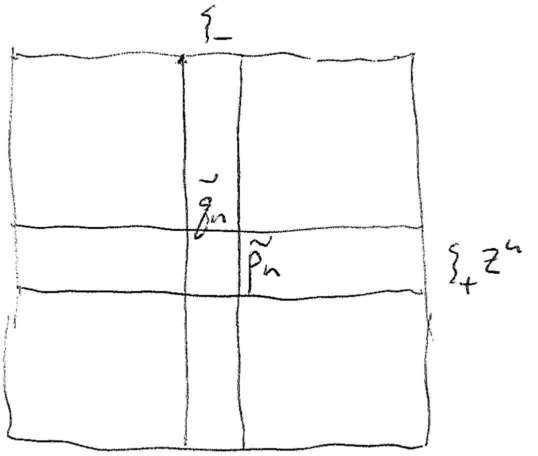
where p', p pos. related mod q
q', q ——— mod p

So we have now constructed a grid space E ~~starting~~ from β. If $h_{mn} = (q_{mn}|p_{mn})$ is the parameter describing the m,n square, one has $h_{mn} = h_{m-1,n+1}$, so the grid space E is one arising from the discrete Dirac equation obtained by restricting to squares with $m=0$:



$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar h_n & 1 \end{pmatrix}\begin{pmatrix} up_{n-1} \\ q_{n-1} \end{pmatrix}$$

Next we do the orthogonal projections explicitly so that we can estimate the h_n in terms of β.



$$\tilde p_n = \xi_+ z^n(1-\overset{\overset{zH_+}{\cup}}{f}) + \xi_-(-\overset{\overset{H_-}{\cup}}{g})$$

$$\int \begin{pmatrix} z^{n+1}H_+ \\ H_- \end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} z^n(1-f) \\ -g \end{pmatrix} = 0$$

$$\int \begin{pmatrix} zH_+ \\ H_- \end{pmatrix}^* \begin{pmatrix} 1 & z^{-n}\bar\beta \\ \beta z^n & 1 \end{pmatrix} \begin{pmatrix} 1-f \\ -g \end{pmatrix} = 0$$

Let $\varepsilon_+ : zH_+ \hookrightarrow L^2$, $\varepsilon_- : H_- \hookrightarrow L^2$ be the inclusions. The above condition on f, g is

$$\begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & \varepsilon_-^* \end{pmatrix} \begin{pmatrix} 1 & z^{-n}\bar\beta \\ \beta z^n & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_+ f \\ \varepsilon_- g \end{pmatrix} = \begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & \varepsilon_-^* \end{pmatrix} \begin{pmatrix} 1 & z^{-n}\bar\beta \\ \beta z^n & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

or
$$\begin{pmatrix} 1 & T_n^* \\ T_n & 1 \end{pmatrix} \begin{pmatrix} f_n \\ g_n \end{pmatrix} = \begin{pmatrix} 0 \\ \varepsilon_-^*(\beta z^n) \end{pmatrix} \qquad T_n = \varepsilon_-^* \beta z^n \varepsilon_+$$

so
$$\boxed{\begin{aligned} f_n = -T_n^* g_n, \quad g_n &= (1-TT^*)^{-1}\varepsilon_-^*(\beta z^n) \\ f_n &= -T^*(1-TT^*)^{-1}\varepsilon_-^*(\beta z^n) \end{aligned}}$$

we have $\|T_n\| = \|T_n^*\| \le \|\beta\|_\infty < 1$, $\quad \|(1-TT^*)^{-1}\| \le \dfrac{1}{1-\|\beta\|_\infty^2}$

Let $\beta_n = (\xi_- \mid \xi_+ z^n) = \int \beta z^n$ so that

$\beta = \sum_{k\in\mathbb{Z}} \beta_k z^{-k}$. Then $\varepsilon_-^*(\beta z^n) = \varepsilon_-^* \sum_{k\in\mathbb{Z}} \beta_k z^{-k+n} = \sum_{k>n} \beta_k z^{-k+n}$,

$$\boxed{\begin{aligned} \|\varepsilon_-^*(\beta z^n)\|^2 = \sum_{k>n} |\beta_k|^2, \qquad \|g_n\| &\le \frac{1}{1-\|\beta\|_\infty^2}\Big(\sum_{k>n}|\beta_k|^2\Big)^{1/2} \\ \|f_n\| &\le \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty^2}\Big(\sum_{k>n}|\beta_k|^2\Big)^{1/2} \end{aligned}}$$

Since $\|\beta\| = \|\beta \cdot 1\| \le \|\beta\|_\infty \|1\| = \|\beta\|_\infty$, one has $\sum |\beta_k|^2 < \infty$, so $\sum_{k>n} |\beta_k|^2 \longrightarrow 0$ as $n \to +\infty$.

Thus $u^{-n}\tilde{p}_n = \xi_+(1-f_n) + \xi_- z^{-n}(-g_n) \longrightarrow \xi_+$. Similar equations hold for $\tilde{q}_n = \xi_+ z^n(-\phi_n) + \xi_-(1-\psi_n)$ with $\phi \in zH_+$, $\psi \in H_-$.

$$(T_n = \varepsilon^* \beta z^n \varepsilon_+)$$

$$\int \begin{pmatrix} z^{n+1}H_+ \\ H_- \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} z^n(-\phi_n) \\ 1-\psi_n \end{pmatrix} = 0 \rightsquigarrow \begin{pmatrix} 1 & T_n^* \\ T_n & 1 \end{pmatrix} \begin{pmatrix} \phi_n \\ \psi_n \end{pmatrix} = \begin{pmatrix} \varepsilon_+^*(z^{-n}\bar{\beta}) \\ 0 \end{pmatrix}$$

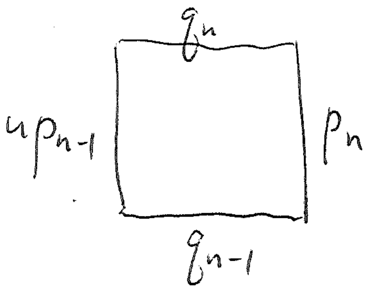
so
$$\boxed{\psi_n = -T_n\phi_n, \quad \phi_n = (1 - T_n^* T_n)^{-1} \varepsilon_+^*(z^{-n}\bar{\beta}), \quad \psi_n = -T_n(1-T_n^*T_n)^{-1}\varepsilon_+^*(z^{-n}\bar{\beta})}$$

$$\bar{\beta} = \sum_{k \in \mathbb{Z}} \bar{\beta}_k z^k \qquad \varepsilon_+^*(z^{-n}\bar{\beta}) = \varepsilon_+^*\Big(\sum_{k\in\mathbb{Z}} \bar{\beta}_k z^{-n+k}\Big) = \sum_{k>n} \bar{\beta}_k z^{-n+k}$$

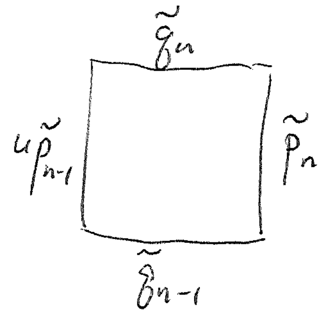
$$\boxed{\|\varepsilon_+^*(z^{-n}\bar{\beta})\|^2 = \sum_{k>n} |\beta_k|^2}$$

same kind of ests for $\|\phi_n\|, \|\psi_n\|$.

Next, formula for $\|\tilde{p}_n\|, \|\tilde{q}_n\|$. Consider



and



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One has
$$\tilde{q}_n = \tilde{q}_{n-1} + (\text{const})\tilde{p}_n \quad \text{so} \quad \tilde{q}_n \equiv \tilde{q}_{n-1} \mod(\mathbb{C}p_n)$$
$$\tilde{p}_n = u\tilde{p}_{n-1} + (\text{const})\tilde{q}_n \quad \text{so} \quad \tilde{p}_n \equiv u\tilde{p}_{n-1} \mod(\mathbb{C}q_n)$$

$$\begin{pmatrix} up_{n-1} \\ q_{n-1} \end{pmatrix} = \frac{1}{k_n}\begin{pmatrix} 1 & -h_n \\ -\bar{h}_n & 1 \end{pmatrix}\begin{pmatrix} p_n \\ q_n \end{pmatrix} \quad \text{so} \quad \begin{aligned} up_{n-1} &\equiv \frac{1}{k_n} p_n \mod(\mathbb{C}q_n) \\ q_{n-1} &\equiv \frac{1}{k_n} q_n \mod(\mathbb{C}p_n) \end{aligned}$$

$$\|\tilde{p}_n\| p_n = \tilde{p}_n \equiv u\tilde{p}_{n-1} = \|\tilde{p}_{n-1}\| u p_{n-1} \equiv \|\tilde{p}_{n-1}\| \frac{1}{k_n} p_n$$

mod $\mathbb{C}q_n$, so you get $\boxed{\|\tilde{p}_n\| = \|\tilde{p}_{n-1}\|\frac{1}{k_n}}$

Similarly $\boxed{k_n\|\tilde{q}_n\| = \|\tilde{q}_{n-1}\|}$. Maybe a better way to see this is as follows. One has

$q_n = h_n p_n \overset{\perp}{+} k_n q_{n-1}$, so the ~~[illegible]~~ projection of q_n orthogonal to $\mathbb{C}p_n$ is $k_n q_{n-1}$, hence the projection of $\tilde{q}_n = \|\tilde{q}_n\| q_n$ orthog. to $\mathbb{C}p_n$ is $\|\tilde{q}_n\| k_n q_{n-1}$, but this projection is $\tilde{q}_{n-1}$, so $\boxed{\|\tilde{q}_{n-1}\| = \|\tilde{q}_n\| k_n}$. Similarly from $p_n = k_n u p_{n-1} \overset{\perp}{+} h_n q_n$, you see the proj of $\tilde{p}_n = \|\tilde{p}_n\| p_n$ $\perp$ to $\mathbb{C}q_n$ is $\|\tilde{p}_n\| k_n u p_{n-1} = u\tilde{p}_{n-1}$, whence $\boxed{\|\tilde{p}_n\| k_n = \|\tilde{p}_{n-1}\|}$. Then find

$$\|\tilde{q}_n\| = k_{n+1}\|\tilde{q}_{n+1}\| = \prod_{j>n} k_j \|\xi_-\| = \prod_{j>n} k_j$$

So
$$\boxed{\|\tilde{p}_n\| = \|\tilde{q}_n\| = \prod_{j>n} k_j}$$

Next $$h_n = (q_n|p_n) = \frac{(\tilde{q}_n|\tilde{p}_n)}{\|\tilde{q}_n\|\|\tilde{p}_n\|} = \frac{(\tilde{q}_n|\tilde{p}_n)}{\|\tilde{p}_n\|^2}$$

$$(\tilde{q}_n|\tilde{p}_n) = (\xi_-|\tilde{p}_n) = \int \begin{pmatrix}0\\1\end{pmatrix}^* \begin{pmatrix}1 & \bar{\beta}\\ \beta & 1\end{pmatrix}\begin{pmatrix} z^n(1-f_n)\\ -g_n\end{pmatrix}$$

$$= \int \beta z^n(1-f_n) - \cancel{g_n} = \underbrace{\int \beta z^n}_{\beta_n} - \int \beta z^n f_n$$

But $\left|\int \beta z^n \overset{zH_+}{f_n}\right| = \left|\int \Sigma_-^*(\beta z^n) f_n\right| \le \|\Sigma_-^*(\beta z^n)\| \dfrac{\|\beta\|_\infty}{1-\|\beta\|_\infty^2} \|\Sigma_-^*(\beta z^n)\|$

so
$$\boxed{\left|\int \beta z^n f_n\right| \le \frac{\|\beta\|_\infty}{1-\|\beta\|_\infty^2} \sum_{k>n} |\beta_k|^2}$$

$$\|\tilde{p}_n\|^2 = (\xi_+ z^n | \tilde{p}_n) = \int \begin{pmatrix} z^n \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix} \begin{pmatrix} z^n(1-f_n) \\ -g_n \end{pmatrix}$$

$$= \int 1 - \cancel{\overset{zH_+}{f_n}} - z^{-n}\bar{\beta} g_n = 1 - \int z^{-n}\bar{\beta} \overset{H_-}{g_n}$$

$$1 - \|\tilde{p}_n\|^2 = \left|\int \Sigma_+^*(z^{-n}\bar{\beta}) g_n\right| \le \frac{1}{1-\|\beta\|_\infty^2}\left(\sum_{k>n} |\beta_k|^2\right)$$

Now assemble

$$\|\tilde{p}_n\|^2 h_n = (\tilde{g}_n | \tilde{p}_n) = \beta_n - \int \beta z^n f_n$$

$$\|\tilde{p}_n\|^2 h_n = h_n - h_n \int z^{-n}\bar{\beta} g_n$$

$$|\beta_n - h_n| = \left|\int \beta z^n f_n - h_n \int z^{-n}\bar{\beta} g_n\right|$$

$$\le \left(\frac{\|\beta\|_\infty}{1-\|\beta\|_\infty^2} + \frac{1}{1-\|\beta\|_\infty^2}\right) \sum_{k>n} |\beta_k|^2$$

$$\boxed{|\beta_n - h_n| \le \frac{1}{1-\|\beta\|_\infty} \sum_{k>n} |\beta_k|^2}$$

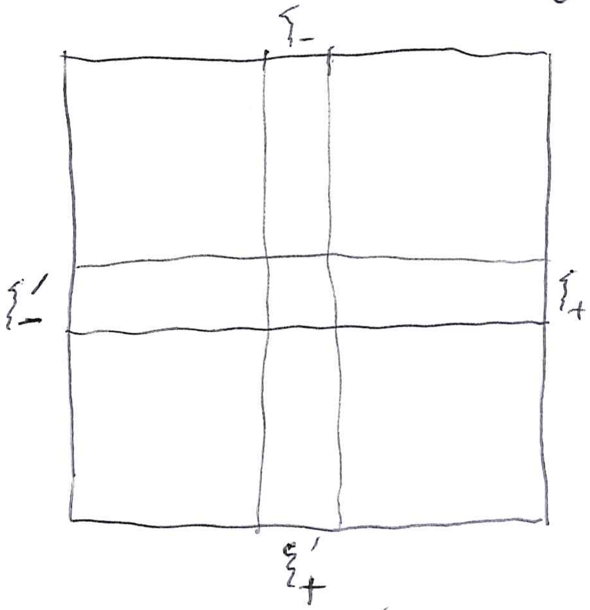
Assume $\beta_n = O\left(\frac{1}{n^\alpha}\right)$ as $n \longrightarrow +\infty$

$$\sum_{k>n} |\beta_k|^2 = O\left(\int_n^\infty \frac{1}{x^{2\alpha}} dx\right) = O\left(\frac{1}{n^{2\alpha-1}}\right)$$

so $h_n = O\left(\frac{1}{n^{\alpha}}\right) + O\left(\frac{1}{n^{2\alpha-1}}\right) = O\left(\frac{1}{n^{\alpha}}\right)$

for $\alpha \geq 1$. This is enough to show that β smooth $\Longrightarrow$ h_n rapidly decreasing. Except we have to first handle h_n as $n \to -\infty$.

Construction of ξ'_-, ξ'_+. ξ'_- up to now is the projection of ξ_+ orthogonal to the subspace $\xi_+ zH_+ + \xi_- L^2$. Let



$$\xi'_+ = \xi_+ \underset{H_+ \ni}{f} + \xi_- \underset{L^2 \ni}{g}, \quad \text{where } f(0) > 0.$$

$$\int \begin{pmatrix} zH_+ \\ L^2 \end{pmatrix}^* \begin{pmatrix} 1 & \beta^* \\ \beta & 1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = 0$$

$$\Longrightarrow \quad g = -\beta f, \quad \varepsilon_+^*(f + \beta^* g) = 0$$

whence $\varepsilon_+^*\left((1-|\beta|^2)f\right) = 0$. How do we know these equations can be solved? One reason is because ~~[illegible]~~ orthogonal projection onto the closed subspace $\xi_+ zH_+ + \xi_- H_-$ is defined in the Hilbert space H. Or putting $f = 1-\phi$, $g = -\psi$ the equations to solve become

$$\begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta^* \\ \beta & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_+ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \beta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & T^* \\ T & 1 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ \beta \end{pmatrix}$$

with $T = \beta \varepsilon_+ : zH_+ \to L^2$, $T^* = \varepsilon_+^* \bar{\beta}$.

Reconstruction of the scattering matrix

$$\begin{pmatrix} \xi_+ \\ \xi'_+ \end{pmatrix} = \begin{pmatrix} \frac{a}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix} \begin{pmatrix} \xi'_- \\ \xi_- \end{pmatrix}$$

from the reflection coefficient $\beta = \frac{b}{d}$. Given $\beta(z)$ on S^1 satisfying $\|\beta\|_\infty < 1$, expand in F.S:

$$\log(1-|\beta|^2) = \sum_{n\in\mathbb{Z}} a_n z^n, \qquad \bar{a}_n = a_{-n}$$

and put $h = \frac{1}{2}a_0 + \sum_{n\geq 1} a_n z^n$, $\delta = e^h$.

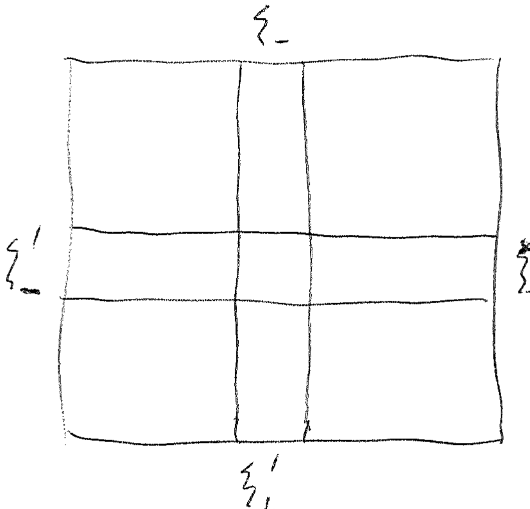
Properties of δ: $|\delta|^2 = 1-|\beta|^2$, δ extends to a nonvanishing analytic function on the unit disk. The scattering matrix is then

$$\begin{pmatrix} \delta & \beta \\ -\bar{\beta}\frac{\delta}{\bar{\delta}} & \delta \end{pmatrix}$$

since it's a $U(2)$-valued function. One has also

$$\delta(0) = e^{\frac{1}{2}a_0} = \exp \tfrac{1}{2}\int \log(1-|\beta|^2)\frac{d\theta}{2\pi}.$$

Let's correlate with the asymptotics. Given



$$H = \xi_+ L^2 + \xi_- L^2, \quad \|\xi_+ f + \xi_- g\|^2 = \int \begin{pmatrix} f \\ g \end{pmatrix}^* \begin{pmatrix} 1 & \bar{\beta} \\ \beta & 1 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}$$

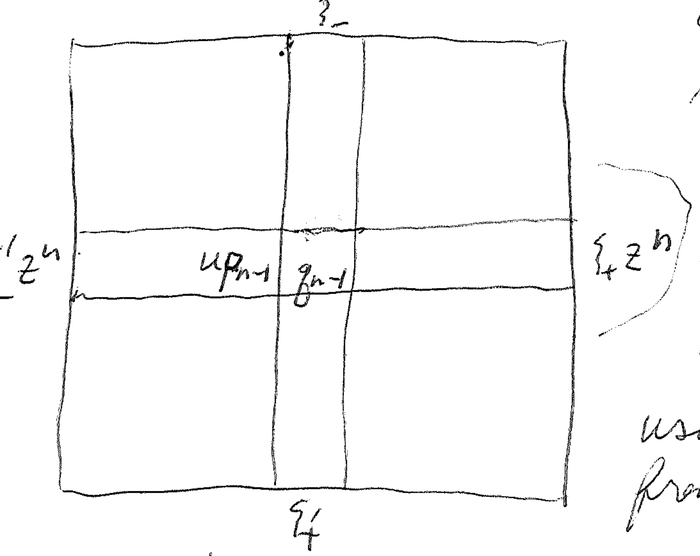
put
$$\begin{cases} \xi'_- = (\xi_+ - \xi_-\beta)\delta^{-1} = \xi_+ d - \xi_- b \\ \xi'_+ = (-\xi_+\bar{\beta} + \xi_-)\bar{\delta}^{-1} = -\xi_+\bar{b} + \xi_-\bar{d} \end{cases}$$

and check that they have the right properties, e.g. $\xi'_- \perp (\xi_+ z H_+ + \xi_- L^2)$ and that $\xi'_- \equiv \xi_+ d(0)$ modulo this subspace with $d(0) > 0$. Since $\delta(0) = \frac{1}{d(0)}$ is the norm of the projection of $\xi_+ \perp$ to this subspace, we have the formula

$$\boxed{\begin{aligned}\delta(0) = \prod_{j\in\mathbb{Z}} k_j &= \exp \tfrac{1}{2}\sum_j \log\left(1-|h_j|^2\right)\\ &= \exp\ \tfrac{1}{2}\sum_j \log\left(1-|\beta|^2\right)\end{aligned}}$$

Next explain how to control the h_n arising from β as $n\to-\infty$. We describe H as $\xi'_-L^2+\xi'_+L^2$ with

$$\boxed{\|\xi'_-f+\xi'_+g\|^2 = \int \begin{pmatrix} f\\ g\end{pmatrix}^* \begin{pmatrix} 1 & \gamma\\ \bar\gamma & 1\end{pmatrix}\begin{pmatrix} f\\ g\end{pmatrix}}$$



where $\gamma = -\frac{c}{d} = -\bar\beta\frac{\delta}{\bar\delta}$. We construct the grid space using the filtration increasing from the lower left corner, i.e. $\xi'_- z^n H_- + \xi'_+ z^{-m} H_+$. This is the same grid space as one gets from the increasing filtration from the upper right corner. One filtration is the orthogonal filtration of the other, so the one diml subspaces (corresponding to grid edges) obtained by splitting one filtration coincide with the one dimensional subspaces of the other filtration. One can check that the unit vectors in these lines are the same, essentially because $\delta(0)>0$. For example

$$(\xi_-|\xi'_+) = \int \begin{pmatrix} 0\\ 1\end{pmatrix}^* \begin{pmatrix} 1 & \bar\beta\\ \beta & 1\end{pmatrix}\begin{pmatrix} -\bar\beta\,\bar\delta^{-1}\\ \bar\delta^{-1}\end{pmatrix} = \int (1-|\beta|^2)\,\bar\delta^{-1} = \int \delta = \delta(0)$$

I did the analogous calculations for the $\xi'_-L^2+\xi'_+L^2$ setting and found

$$\boxed{\left|h_n+\bar\gamma_n\right| \le \frac{1}{1-\|\gamma\|_\infty}\sum_{k<n}|\gamma_k|^2}$$

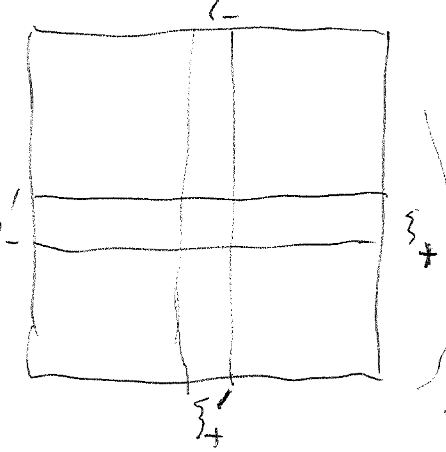
Check the sign. To first order in (h_n) the transfer, scattering matrices are

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & \sum_n h_n z^{-n} \\ -\sum_n \bar h_n z^n & 1 \end{pmatrix} \qquad S = \begin{pmatrix} 1 & \sum h_n z^{-n} \\ -\sum \bar h_n z^n & 1 \end{pmatrix}$$

so $\bar\gamma = -\sum h_n z^{-n}$ and $\tilde\gamma_n = \int \bar\gamma z^n = -h_n$ to first order. Or, $\bar\gamma = -\beta \frac{\bar\delta}{\delta} \approx -\beta$ to first order.

Finally if β is smooth, then so is $\log(1-|\beta|^2)$, so it Fourier coefficients are rapidly decreasing. The same is true for $h = \frac{1}{2}a_0 + \sum_{n\geq 1} a_n z^n$, so $h(z)$ is a smooth function on the closed unit disk analytic for $|z|<1$. $\therefore$ $\delta = e^h$ is smooth nonvanishing on $|z|\leq 1$ and analytic for $|z|<1$, and $\gamma = -\bar\beta \frac{\delta}{\bar\delta}$ is a smooth function on S^1 with $\|\gamma\|_\infty = \|\beta\|_\infty$. Then we conclude that $|h_n|$ is rapidly decreasing as $n \to -\infty$.

Next, take up the IH form picture. Begin with the asymptotic structure assoc. to a disc. DE with



(h_n) decaying sufficiently. Thus we have a completed grid space H which is a Hilbert space with pos. def. inner product $\| \ \|^2$ and indefinite hermitian inner product $IH(\ \ )$. The four sides of this square give four closed subspaces $\xi_\pm L^2$, $\xi'_\pm L^2$ and H is the direct sum of the two subspaces connected by any vertex, thus giving 4 descriptions of H as $L^2 \oplus L^2$ related by

$$\begin{pmatrix}\xi_+\\ \xi_-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi'_+\end{pmatrix} \qquad \begin{pmatrix}\xi_+\\ \xi'_+\end{pmatrix} = \begin{pmatrix}\frac{1}{d} & \frac{b}{d}\\ -\frac{c}{d} & \frac{1}{d}\end{pmatrix}\begin{pmatrix}\xi'_-\\ \xi_-\end{pmatrix}$$

transfer   scattering

$\|\xi_+ f + \xi'_+ g\|^2 = \|f\|^2 + \|g\|^2$, sim for $\|\xi'_- f + \xi_- g\|^2$

$IH(\xi_+ f + \xi_- g) = \|f\|^2 - \|g\|^2$, sim for $IH(\xi'_- f + \xi_- g)$

We will calculate the description (incoming picture) $\xi'_- f + \xi_- g$. Thus $H = \xi'_- L^2 + \xi_- L^2$ with

$$IH(\xi'_- f + \xi_- g) = IH\big((\xi_+ d - \xi_- b)f + \xi_- g\big)$$
$$= IH\big(\xi_+(df) + \xi_-(-bf+g)\big)$$
$$= \|df\|^2 - \|-bf+g\|^2 = \int \begin{pmatrix} f\\ g\end{pmatrix}^* \begin{pmatrix} |d|^2-|b|^2 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix} f\\ g\end{pmatrix}$$

$$\therefore \boxed{IH(\xi'_- f + \xi_- g) = \int \begin{pmatrix} f\\ g\end{pmatrix}^* \begin{pmatrix} 1 & \bar{b}\\ b & -1\end{pmatrix}\begin{pmatrix} f\\ g\end{pmatrix}}$$

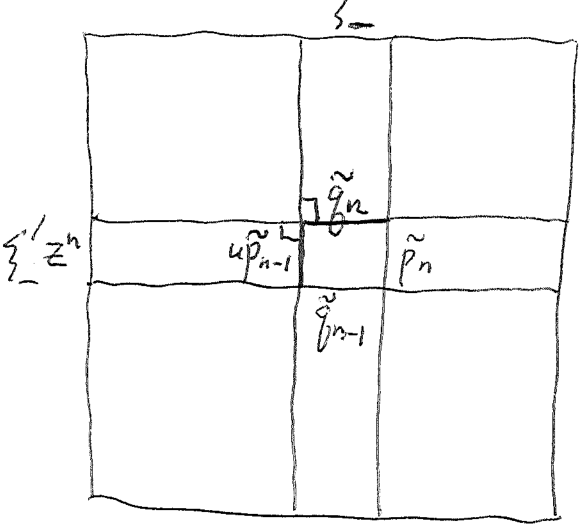
Observe $\begin{pmatrix} 1 & \bar{b}\\ b & -1\end{pmatrix}$ at $z \in S^1$ has the eigenvalues $\pm(1+|b|^2) = \pm|d|$, where $|d| \geq 1$, so there's a polarization of H, splitting orthogonal wrt both $\| \ \|^2$ and IH, such that $IH > 0$ on one and < 0 on the other. Thus H equipped with IH is a Krein space.

Observe that the description of H that we have given depends only on b. But d, as well as

$$\begin{pmatrix} \xi_+ \\ \xi_+' \end{pmatrix} = \frac{1}{d}\begin{pmatrix} 1 & b \\ -\bar{b} & 1 \end{pmatrix}\begin{pmatrix} \xi_-' \\ \xi_- \end{pmatrix},$$

can be obtained from b by splitting $\log(1+|b|^2) = h + \bar{h}$, h holom. in the disk, $h(0) \in \mathbb{R}$, then $d = e^h$.

Next to construct the grid space (and disc. DE) using orthogonal projection with respect to IH applied to the increasing filtration from the upper left, namely the subspaces $\xi_-' z^m H_+ + \xi_- z^m H_+$. Picture:



Let $\tilde{q}_n$ be the projection of $\xi_- \perp$ to $\xi_-' z^{n+1} H_+ + \xi_- z H_+$ and let $u\tilde{p}_{n-1}$ be the projection of $\xi_-' z^n \perp$ to this subspace

$$u\tilde{p}_{n-1} = \xi_-' z^n(1 - \overset{zH_+}{f}) + \xi_-(-\overset{zH_+}{g})$$

$$\tilde{q}_n = \xi_-' z^n(-\phi) + \xi_-(1-\psi)$$

orthog. condition is

$$\int \begin{pmatrix} z^{n+1}H_+ \\ zH_+ \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix}\begin{pmatrix} z^n(1-f) & z^n(-\phi) \\ -g & 1-\psi \end{pmatrix} = 0$$

$$\begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & \varepsilon_+^* \end{pmatrix}\begin{pmatrix} 1 & z^{-n}\bar{b} \\ bz^n & -1 \end{pmatrix}\begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} = \begin{pmatrix} \varepsilon_+^* & 0 \\ 0 & \varepsilon_+^* \end{pmatrix}\begin{pmatrix} 1 & z^{-n}\bar{b} \\ bz^n & -1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} 1 & T_n^* \\ T_n & -1 \end{pmatrix}\begin{pmatrix} f_n & \phi_n \\ g_n & \psi_n \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_+^*(z^{-n}\bar{b}) \\ \varepsilon_+^*(bz^n) & 0 \end{pmatrix}}$$

$T_n = \varepsilon_+^* b z^n \varepsilon_+$
Toeplitz operator on zH_+

Notice that $\begin{pmatrix}1 & T^*\\ T & -1\end{pmatrix} = \varepsilon + A$, $A = \begin{pmatrix}0 & T^*\\ T & 0\end{pmatrix}$

is self adjoint, and invertible since $(\varepsilon + A)^2 = \varepsilon^2 + \varepsilon A + A\varepsilon + A^2 = 1 + A^2$, so

$$(\varepsilon + A)^{-1} = (\varepsilon + A)(1+A^2)^{-1} = \varepsilon(1+A^2)^{-1} + A(1+A^2)^{-1}$$

where $1+A^2$ is invertible since its spectrum is ≥ 1, (apply functional calculus, case of the function $\frac{1}{x}$). Note that $-\frac{1}{2} \leq \frac{x}{1+x^2} \leq \frac{1}{2}$, so $\|A(1+A^2)^{-1}\| \leq \frac{1}{2}$, in particular

$$\|T(1+T^*T)^{-1}\| \leq \frac{1}{2}.$$

It's clear now that IH restricted to any of subspaces $\xi'_- z^n H_+ + \xi_- z^m H_+$ is non-degenerate, so the filtration admits an orthogonal splitting into lines corresponding to edges in the grid. However, although these lines are nondegenerate (non-isotropic) for IH, it is not clear that IH is >0 on vertical edges, resp. <0 on horizontal edges.

Calculate. It's enough to treat $n=0$.

$$IH(\widetilde{up}_{-1}, \widetilde{up}_{-1}) = IH(\xi'_-, \widetilde{up}_{-1}) = \int \begin{pmatrix}0\\1\end{pmatrix}^* \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}1-f\\ -g\end{pmatrix}$$

$$= \int 1 - \underbrace{f}_{zH_+} - \bar b g = 1 - \underbrace{\int \bar b g}$$

$\int \bar b g = (b|g) = (b|\varepsilon_+ g) = (\varepsilon_+^*(b)|g)$. But $f = -T^*g$

$-g + T(-T^*g) = \varepsilon_+^*(b) \Rightarrow g = -(1+TT^*)^{-1}\varepsilon_+^*(b)$ so

$-\int \bar b g = \left(\varepsilon_+^*(b) \middle| (1+TT^*)^{-1}\varepsilon_+^*(b)\right) \geq 0$, since $(1+TT^*)^{-1} \geq 0$

Thus $IH(u\tilde{p}_{-1}) \geqslant 1$. In fact, can do more.

$$-(\varepsilon_+^*(b) \mid g) = ((1+TT^*)g \mid g) = \|g\|^2 + \|f\|^2$$

so $$\boxed{IH(u\tilde{p}_{-1}) = 1 + \|g\|^2 + \|f\|^2 = \|u\tilde{p}_{-1}\|^2.}$$

Similarly $IH(\tilde{q}_0) = IH(\tilde{q}_0, \tilde{q}_0) = IH(\xi_-, \tilde{q}_0)$

$$= \int \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* \begin{pmatrix} 1 & \bar{b} \\ b & -1 \end{pmatrix} \begin{pmatrix} -\phi \\ 1-\psi \end{pmatrix} = \int b(-\phi) - 1 + \cancel{\psi}^{zH_+} = -1 - \int b\phi$$

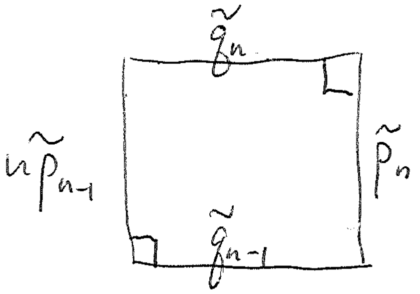
and $$\int b\phi = (\bar{b} \mid \phi) = (\bar{b} \mid \varepsilon_+ \phi) = (\varepsilon_+^*(\bar{b}) \mid \phi),$$

$$T\phi = \psi, \quad \phi + T^*\psi = \varepsilon_+^*(\bar{b}), \quad (1+T^*T)\phi = \varepsilon_+^*(\bar{b}), \text{ so}$$

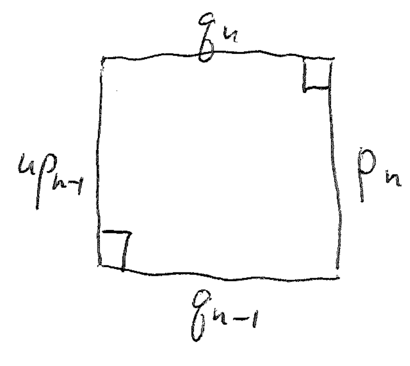
$$\int b\phi = ((1+T^*T)\phi \mid \phi) = \|\phi\|^2 + \|\psi\|^2. \quad \text{Thus}$$

$$\boxed{IH(\tilde{q}_0) = -(1 + \|\phi\|^2 + \|\psi\|^2) = -\|\tilde{q}_0\|^2}$$

Next examine the square



which normalized is



We know that the projection of $u\tilde{p}_{n-1} \perp_{IH} \mathbb{C}q_n$ is $\tilde{p}_n$

$\tilde{q}_n \perp \mathbb{C}u\tilde{p}_{n-1}$ is $\tilde{q}_{n-1}$.

Also we know up_{n-1}, q_{n-1} and p_n, q_n are orthogonal bases for the two-dim'l spaces with $IH = 1$ for the p's and $IH = -1$ for q's. So

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} up_{n-1} \\ q_{n-1} \end{pmatrix} \quad \text{with } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1)$$

hence
$$\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} up_{n-1} \\ q_n \end{pmatrix} \quad \text{with } \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2)$$

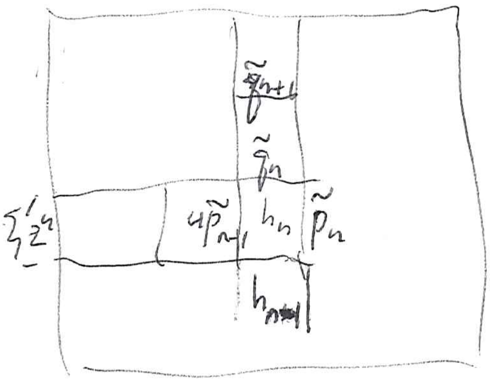
But
$$u\tilde{p}_{n-1} = \tilde{p}_n + (?)q_n \Rightarrow \alpha > 0$$
$$\tilde{q}_n = (?)up_{n-1} + \tilde{q}_{n-1} \Rightarrow \delta > 0$$

hence $\alpha = (1-|\beta|^2)^{1/2} = \delta$ so $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ has the form $\begin{pmatrix} k_n & h_n \\ -\bar{h}_n & k_n \end{pmatrix}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has the form $\frac{1}{k_n}\begin{pmatrix} 1 & h_n \\ \bar{h}_n & 1 \end{pmatrix}$.

Compare
$$\left.\begin{aligned} up_{n-1} &= k_n^{-1} p_n - \frac{1}{k_n} h_n q_n \\ q_n &= \frac{\bar{h}_n}{k_n} up_{n-1} + \frac{1}{k_n} q_{n-1} \end{aligned}\right) \text{ with } \left(\begin{aligned} u\tilde{p}_{n-1} &= \tilde{p}_n + (?)q_n \\ \tilde{q}_n &= (?)up_{n-1} + \tilde{q}_{n-1} \end{aligned}\right.$$

and you conclude $\|u\tilde{p}_{n-1}\| = k_n \|\tilde{p}_n\|$, $\|\tilde{q}_n\| = k_n \|\tilde{q}_{n-1}\|$.

Thus referring to we have



$$\|\tilde{p}_{n-1}\| = \frac{1}{k_{n-1}} \|\tilde{p}_{n-2}\| = \frac{1}{k_{n-1}k_{n-2}} \|\tilde{p}_{n-3}\| \cdots$$
$$\|\tilde{q}_n\| = \frac{1}{k_{n+1}} \|\tilde{q}_{n+1}\| = \frac{1}{k_{n+1}k_{n+2}} \|\tilde{q}_{n+2}\| \cdots$$

Claim:
$$\boxed{\|\tilde{p}_{n-1}\| = \prod_{j<n} \frac{1}{k_j}, \quad \|\tilde{q}_n\| = \prod_{j>n} \frac{1}{k_j}}$$

Suffices to show $\tilde{q}_n \to \xi_-$ as $n \to +\infty$
$u^n \tilde{p}_n \to \xi'_-$ as $n \to -\infty$.

Return to p98 for

$$f_n + T_n^* g_n = 0\,, \quad g_n = -(1+T_n T_n^*)^{-1}\,\mathcal{E}_+^*(bz^n)$$
$$f_n = T_n^*(1+T_nT_n^*)^{-1}\,\mathcal{E}_+^*(bz^n)$$

$$\|\mathcal{E}_+^*(bz^n)\| = \Big\|\mathcal{E}_+^*\Big(\sum_{k\in\mathbb{Z}} b_k z^{-k+n}\Big)\Big\| = \Big\|\sum_{k<n} b_k z^{-k+n}\Big\| = \Big(\sum_{k<n}|b_k|^2\Big)^{1/2}$$

so
$$\|g_n\| \le \underbrace{\|(1+T_nT_n^*)^{-1}\|}_{\le 1}\;\underbrace{\|\mathcal{E}_+^*(bz^n)\|}_{\text{goes to zero as } n\to-\infty}$$
$$\|f_n\| \le \underbrace{\|T_n^*(1+T_nT_n^*)^{-1}\|}_{\le \frac{1}{2}}\;\|\mathcal{E}_+^*(bz^n)\|$$

Similarly
$$T_n\phi_n = \psi_n\,, \quad \phi_n = (1+T_n^*T_n)^{-1}\,\mathcal{E}_+^*(z^{-n}\bar b)$$
$$\psi_n = T_n(1+T_n^*T_n)^{-1}\,\mathcal{E}_+^*(z^{-n}\bar b)$$

$$\|\mathcal{E}_+^*(z^{-n}\bar b)\| = \Big\|\mathcal{E}_+^*\Big(\sum_{k\in\mathbb{Z}} \bar b_k z^{k-n}\Big)\Big\| = \Big\|\sum_{k>n}\bar b_k z^{k-n}\Big\| = \Big(\sum_{k>n}|b_k|^2\Big)^{1/2}$$

so $\phi_n, \psi_n \longrightarrow 0$ as $n\to+\infty$.

Next, control of h_n.

$$IH(q_n, u p_{n-1}) = IH\Big(q_n, \cancel{\tfrac{1}{k_n}p_n} - \tfrac{h_n}{k_n}q_n\Big) = \frac{h_n}{k_n} \quad \text{as } IH(q_n) = -1.$$

$$IH(\tilde q_n, u\tilde p_{n-1}) = IH(\xi_-, u\tilde p_{n-1}) = \int \begin{pmatrix}0\\1\end{pmatrix}^* \begin{pmatrix}1 & \bar b\\ b & -1\end{pmatrix}\begin{pmatrix}z^n(1-f_n)\\ -g_n\end{pmatrix}$$
$$= \int bz^n(1-f_n) + g_n = \underbrace{\int bz^n}_{b_n} - \int bz^n f_n$$

$$\int bz^n f_n = (\bar b z^{-n}\,|\,f_n) = (\bar b z^{-n}\,|\,\mathcal{E}_+ f_n) = (\mathcal{E}_+^*(z^{-n}\bar b)\,|\,f_n)$$
$$= \big(\mathcal{E}_+^*(z^{-n}\bar b)\,\big|\,T_n^*(1+T_nT_n^*)^{-1}\,\mathcal{E}_+^*(bz^n)\big)$$

$$\therefore\ \left|\int b z^n f_n\right| \leq \left\|\Sigma_+^*(z^{-n}\bar{b})\right\| \left\|T_n^*(1+T_nT_n^*)^{-1}\right\| \left\|\Sigma_+^*(bz^n)\right\|$$
$$\leq \frac{1}{2}\Big(\sum_{k>n}|b_k|^2\Big)^{1/2}\Big(\sum_{k<n}|b_k|^2\Big)^{1/2}$$

Lastly $\quad IH(\tilde{g}_n, u\tilde{p}_{n-1}) = \underset{\prod_{j<n}\frac{1}{k_j}}{\|\tilde{p}_{n-1}\|}\ \underbrace{IH(g_n, up_{n-1})}_{\frac{h_n}{k_n}}\ \underset{\prod_{j>n}\frac{1}{k_j}}{\|\tilde{g}_n\|}$

yielding

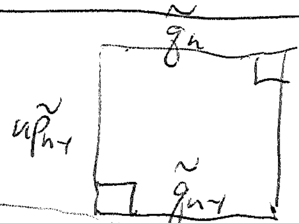
$$\boxed{\left|\frac{h_n}{\prod_{j\in\mathbb{Z}} k_j} - b_n\right| \leq \frac{1}{2}\Big(\sum_{k>n}|b_k|^2\Big)^{1/2}\Big(\sum_{k<n}|b_k|^2\Big)^{1/2}}$$

~~[illegible]~~ Note that

$$\prod_{j\in\mathbb{Z}}\frac{1}{k_j} = d(0) = \exp\left\{\frac{1}{2}\int \log\left[1+|b|^2\right]\right\}$$

so that this product, which initially depends on knowing the h_n, is actually given by the above formula in terms of b.

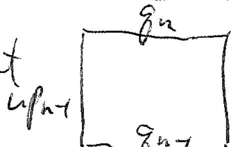
Improved argument. Start with



$\tilde{g}_n - c_1 u\tilde{p}_{n-1} = \tilde{g}_{n-1}$

$u\tilde{p}_{n-1} - c_2\tilde{g}_n = \tilde{p}_n$

Normalizing you get



where p_n, g_n and up_{n-1}, g_{n-1} are orthonormal bases (with signs) wrt IH. $\therefore$ $\begin{pmatrix}p_n\\ g_n\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}up_{n-1}\\ g_{n-1}\end{pmatrix}$ with $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in U(1,1) \implies \begin{pmatrix}p_n\\ g_{n-1}\end{pmatrix} = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}up_{n-1}\\ g_n\end{pmatrix}$ with $\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in U(2)$.

But the two relations with the $\sim$ elts imply g_{n-1}, g_n positively related modulo $\mathbb{C}up_{n-1} \implies \delta > 0$ and similarly $\alpha > 0$.

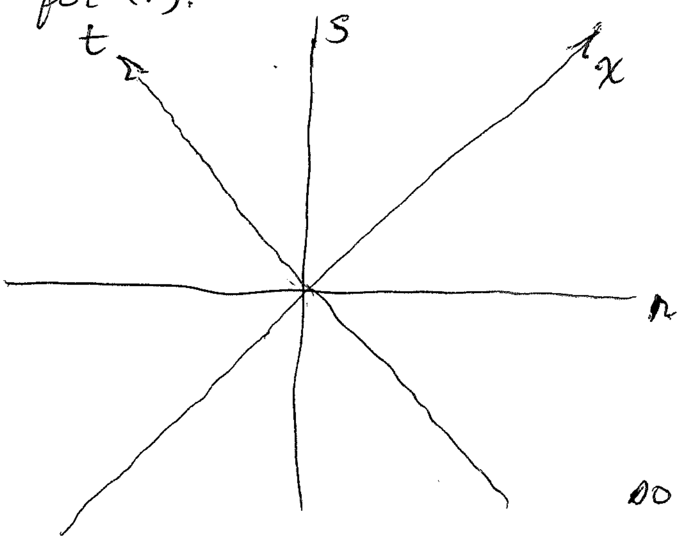
March 16, [redacted]00

To study the Dirac equation

$$(1) \qquad \partial_t \psi = \begin{pmatrix} \partial_x & ih \\ i\bar{h} & -\partial_x \end{pmatrix} \psi$$

with $\psi = \begin{pmatrix} \psi^1(x,t) \\ \psi^2(x,t) \end{pmatrix}$ is a 2-component function on space-time and h is constant $\neq 0$. This D.E. should describe the continuous version of a grid space with constant h-parameters. By a constant ~~phase~~ phase factor on ψ^1 one can suppose $h > 0$; as the spectrum is $\omega^2 = k^2 + |h|^2$, $h = |h|$ [redacted] can be viewed as the mass. To simplify take $h = 1$.

It's natural to introduce characteristic coords. for (1).



$$\partial_r = -\partial_t + \partial_x$$
$$\partial_s = \partial_t + \partial_x$$

[redacted]

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial r}\right)^{1} + \frac{\partial f}{\partial t}\left(\frac{\partial t}{\partial r}\right)^{-1}$$
$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\left(\frac{\partial x}{\partial s}\right)^{1} + \frac{\partial f}{\partial t}\left(\frac{\partial t}{\partial s}\right)^{1}$$

so $\quad x = r + s \qquad r = \frac{x-t}{2}$

$\quad\ \ t = -r + s \qquad s = \frac{x+t}{2}$

Then $\begin{aligned} (\partial_t - \partial_x)\psi^1 &= i\psi^2 \\ (\partial_t + \partial_x)\psi^2 &= i\psi^1 \end{aligned}$ becomes $\boxed{\begin{aligned} -\partial_r \psi^1 &= i\psi^2 \\ \partial_s \psi^2 &= i\psi^1 \end{aligned}}$

Since the DE is translation invariant, look [redacted] for exponential solutions: $\psi(r,s) = e^{i(r\rho + s\sigma)}\hat{\psi}$.

Then $\begin{aligned} -\rho\hat{\psi}^1 &= \hat{\psi}^2 \\ \sigma\hat{\psi}^2 &= \hat{\psi}^1 \end{aligned}$ so $(1 + \rho\sigma)\hat{\psi} = 0$

$$\hat{\psi} = \begin{pmatrix} 1 \\ -\rho \end{pmatrix} \hat{\psi}^1$$

Thus the spectrum is the curve $\sigma = -\rho^{-1}$, and the general solution of the DE which is a linear combination of these oscillatory exponential solutions has the form

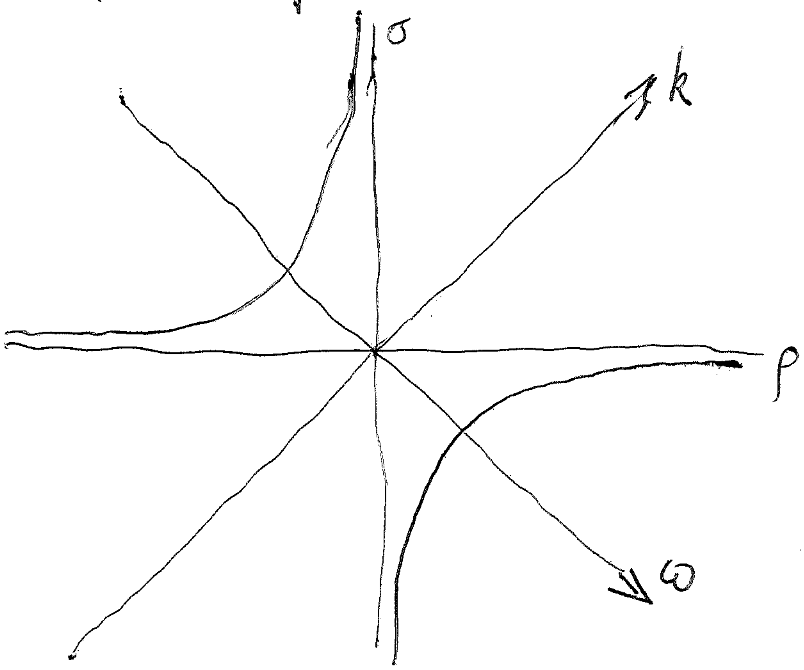
$$\psi(r,s) = \int_{-\infty}^{\infty} e^{i(r\rho - s\rho^{-1})} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

where f is a distribution. In (x,t) terms we have

$$\psi(x,t) = \int_{-\infty}^{\infty} e^{i\left(\frac{x-t}{2}\rho - \frac{x+t}{2}\rho^{-1}\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho$$

$$* \quad \boxed{\psi(x,t) = \int_{-\infty}^{\infty} e^{i\left(\left(\frac{\rho-\rho^{-1}}{2}\right)x - \left(\frac{\rho+\rho^{-1}}{2}\right)t\right)} \begin{pmatrix} 1 \\ -\rho \end{pmatrix} f(\rho)\, d\rho}$$

Put $k = \frac{\rho - \rho^{-1}}{2}$, $\omega = \frac{\rho+\rho^{-1}}{2}$ whence $\rho = \omega + k$, $\rho^{-1} = \omega - k$ and the spectrum is $\omega^2 = k^2 + 1$.



Discuss next solving the Cauchy problem. Recall that the Dirac equation being hyperbolic can be solved for given initial data on a non characteristic curve. We look at the cases $\cancel{t}=0$ and $x=0$.

Given $\psi_0(x)$ the solution $\psi(x,t)$ of $\partial_t\psi = \begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}\psi$ with $\psi(x,0)=\psi_0(x)$ is

$$\psi(x,t) = e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}}\psi_0(x)$$

Using the Fourier Transform
$$\psi_0(x) = \int e^{ikx}\hat{\psi}_0(k)\frac{dk}{2\pi}$$
$$\hat{\psi}_0(k) = \int e^{-ikx}\psi_0(x)\,dx$$

we have

$$\psi(x,t) = \int \frac{dk}{2\pi} e^{t\begin{pmatrix}\partial_x & i \\ i & -\partial_x\end{pmatrix}} e^{ikx}\hat{\psi}_0(k)$$
$$= \int \frac{dk}{2\pi} e^{ikx} e^{it\begin{pmatrix}k & 1 \\ 1 & -k\end{pmatrix}}\hat{\psi}_0(k)$$

Now $A=\begin{pmatrix}k & 1 \\ 1 & -k\end{pmatrix}$ has $A^2=(k^2+1)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}$. Putting $\omega=\sqrt{k^2+1}$ one has

$$e^{itA} = \sum_{n\geq 0}\frac{(-1)^n}{(2n)!}\underbrace{t^{2n}A^{2n}}_{(\omega t)^{2n}\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}} + \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}t^{2n+1}A^{2n}\,iA$$

$$\boxed{\begin{aligned} e^{itA} &= \cos(\omega t)\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} + i\frac{\sin(\omega t)}{\omega}\begin{pmatrix}k & 1 \\ 1 & -k\end{pmatrix} \\ &= e^{i\omega t}\underbrace{\frac{1}{2\omega}\begin{pmatrix}\omega+k & 1 \\ 1 & \omega-k\end{pmatrix}}_{\text{proj on the } +\omega \text{ eigenspace of } A} + e^{-i\omega t}\underbrace{\frac{1}{2\omega}\begin{pmatrix}\omega-k & -1 \\ -1 & \omega+k\end{pmatrix}}_{\text{proj on the } -\omega \text{ eigenspace of } A}\end{aligned}}$$

$$\boxed{\psi(x,t) = \int\frac{dk}{2\pi}\left(e^{i(kx+\omega t)}\frac{1}{2\omega}\begin{pmatrix}\omega+k & 1 \\ 1 & \omega-k\end{pmatrix} + e^{i(kx-\omega t)}\frac{1}{2\omega}\begin{pmatrix}\omega-k & -1 \\ -1 & \omega+k\end{pmatrix}\right)\hat{\psi}_0(k)}$$

Next rewrite the DE $\partial_x \psi = \begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix} \psi$

and suppose given $\psi_0(t) = \int e^{i\omega t} \hat{\psi}_0(\omega) \frac{d\omega}{2\pi}$. The solution of the DE with $\psi(0,t) = \psi_0(t)$ is

$$\psi(x,t) = e^{x\begin{pmatrix} \partial_t & -i \\ i & -\partial_t \end{pmatrix}} \psi_0(t) = \int e^{i\omega t} e^{ix\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}} \hat{\psi}_0(\omega) \frac{d\omega}{2\pi}$$

Let $B = \begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}$, $k = \sqrt{\omega^2 - 1}$ so that $B^2 =$

$$\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} = \begin{pmatrix} \omega^2-1 & 0 \\ 0 & \omega^2-1 \end{pmatrix} = k^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ Then

$$e^{ixB} = \sum_{n \geq 0} \frac{(-1)^n}{(2n)!} x^{2n} k^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n \geq 0} \frac{(-1)^n}{(2n+1)!} x^{2n+1} k^{2n} \, iB$$

$$\boxed{\begin{aligned} e^{ixB} &= \cos(kx)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\frac{\sin(kx)}{k}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix} \\ &= e^{ikx}\frac{1}{2k}\begin{pmatrix} k+\omega & -1 \\ 1 & k-\omega \end{pmatrix} + e^{-ikx}\frac{1}{2k}\begin{pmatrix} k-\omega & 1 \\ -1 & k+\omega \end{pmatrix} \end{aligned}}$$

So

$$\boxed{\psi(x,t) = \int_{-\infty}^{\infty} e^{i\omega t}\left(\cos(kx)\hat{\psi}_0(\omega) + i\frac{\sin(kx)}{k}\begin{pmatrix} \omega & -1 \\ 1 & -\omega \end{pmatrix}\hat{\psi}_0(\omega)\right)\frac{d\omega}{2\pi}}$$

Notice that $k = \sqrt{\omega^2 - 1}$ is imaginary when $|\omega| < 1$, so that in general $\psi(x,t)$ is not a linear combination of oscillatory exponentials. So this solution seems to be outside the solution $*$ on p105. To be more specific suppose $\psi_0(t) = I$, i.e. $\hat{\psi}_0(\omega) = 2\pi\delta(\omega) I$, then

$$\psi(x,t) = \begin{pmatrix} \cosh x & -i\sinh x \\ i\sinh x & \cosh x \end{pmatrix}$$ which grows exponentially.

Let's go over some linear algebra related to Wronskians, quantization.

Let V be a complex vector space, and σ be a conjugation on V so that V = the complexification of the real vector space V^σ.

Let $B_{sym}(v, v')$ be a symmetric (complex) bilinear form on V satisfying the reality condition $B_{sym}(\sigma v, \sigma v') = B_{sym}(v, v')$. Then $B_{sym}(\sigma v, v')$ is a hermitian symmetric form on V, i.e. sesqui-linear, and

$$\overline{B_{sym}(\sigma v, v')} = B_{sym}(v, \sigma v') = B_{sym}(\sigma v', v).$$

If $B_{sk}(v, v')$ is a skew-symmetric bilinear form on V satisfying the same reality condition, then $i B_{sk}(\sigma v, v')$ is hermitian symmetric:

$$\overline{i B_{sk}(\sigma v, v')} = -i B_{sk}(v, \sigma v') = i B_{sk}(\sigma v', v).$$

Combining these cases we get a hermitian form

$$H(v, v') = (B_{sym} + i B_{sk})(\sigma v, v')$$

~~satisfying the reality condition: $H(\sigma v, \sigma v') = \overline{H(v, v')}$.~~ NO

Conversely any hermitian form on V ~~which is real in the above sense~~ can be expressed uniquely by a (B_{sym}, B_{sk}) as above. Take a basis for V^σ over $\mathbb{R}$ and use it as a basis of V over $\mathbb{C}$. Then H is given by a hermitian matrix:

$$H(v, v') = v^* A v'$$

and any hermitian matrix is uniquely the sum of a real symmetric matrix plus i times a real skew-symmetric matrix.

Harmonic oscillator algebra (previous work 03/27/99)

Let V be a f.d.v.s., let B be a bilinear form on V. We identify an element $v \in V$ with the linear map $\mathbb{C} \to V$ sending 1 to v. Then $v^t : V^* \to \mathbb{C}^* = \mathbb{C}$ is the map evaluating a linear functional on v. B can be identified with the linear map $B : V \to V^*$ such that the value of B on the pair (v_1, v_2) is

$$B(v_1, v_2) = v_1^t B v_2 : \mathbb{C} \xrightarrow{v_2} V \xrightarrow{B} V^* \xrightarrow{v_1^t} \mathbb{C}$$

Let's begin with the bosonic case. Let $\omega : V \xrightarrow{\sim} V^*$, $\omega^t = -\omega$ be a symplectic form on V. Associated to (V, ω) we have the symplectic Lie algebra

$$sp(V, \omega) = \{ X \in V \otimes V^* = End(V) \mid X^t \omega + \omega X = 0 \}$$

and the Weyl algebra $W(V) = Weyl(V, \omega)$ which is generated by a universal linear map $\phi : V \to W(V)$, $v \mapsto \phi_v$ satisfying the CCR:

$$[\phi_{v_1}, \phi_{v_2}] = v_1^t \omega v_2$$

We have the isomorphism

$$\begin{aligned} sp(V, \omega) &\xrightarrow{\sim} S^2 V^* \\ X &\longmapsto \omega X \end{aligned}$$

In effect, $(\omega X)^t = X^t \omega^t = -X^t \omega$, so $X \in sp(V, \omega)$ iff $H = \omega X$ is symmetric.

The maps $v \mapsto \phi_v$, $v \mapsto \phi_v^2$ induce linear maps ϕ from V and α from $S^2 V$ to $W(V)$, resp., such that

$$[\tfrac{1}{2}\phi_{v_1}^2, \phi_v] = \phi_{v_1} \, v_1^t \omega v$$

whence ϕV is closed under $[\alpha(S^2 V), -]$, giving a map $S^2V \xrightarrow{\tilde{\alpha}} V\otimes V^*$ sending $\frac{1}{2}v_1^2$ to the operator $v_1 v_1^t \omega$. Let's check this operator lies in $sp(V)$, by showing that $\omega\circ v_1 v_1^t \omega = \omega v_1 v_1^t \omega$ is symmetric: $(\omega v_1 v_1^t \omega)^t = (-\omega) v_1 v_1^t (-\omega) = \omega v_1 v_1^t \omega$.

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cont. March 25, 2000.

First, you should straighten out some confusion resulting from your notation. Recall that want to work with linear maps, and so identify an element $v \in V$ with the linear map $\mathbb{C} \to V$, $z \mapsto zv$. Consider now an element $\lambda \in V^*$. On one hand λ is a map from V to $\mathbb{C}$ by definition of V^*; on the other hand λ is to be identified with a map from $\mathbb{C}$ to V^*. To be consistent you should write $\lambda : \mathbb{C} \to V^*$ for the map corresp. to the elt λ, and $\lambda^t : V \to \mathbb{C}$ for the linear map given by λ. With this notation (understood) one has the usual isom.

$$W \otimes V^* \xrightarrow{\sim} \mathrm{Hom}(V, W)$$
$$w \otimes \lambda \longmapsto (w\lambda^t : V \to \mathbb{C} \to W)$$

For example one has

$$V^* \otimes V^* \xrightarrow{\sim} \mathrm{Hom}(V, V^*)$$
$$\lambda_1 \otimes \lambda_2 \longmapsto \left(\lambda_1 \lambda_2^t : V \xrightarrow{\lambda_2^t} \mathbb{C} \xrightarrow{\lambda_1} V^*\right)$$

Start again with V, $\omega: V \xrightarrow{\sim} V^*$ s.t. $\omega^t = -\omega$. In $\mathrm{Weyl}(V)$ we have

$$\left[\tfrac{1}{2}\phi_{v_1}^2, \phi_v\right] = \phi_{v_1}(v_1^t \omega v) = \phi_{v_1 v_1^t \omega v}$$

which suggests the map

$$\begin{aligned} S^2 V &\longrightarrow \mathrm{End}(V) \\ \tfrac{1}{2} v_1^2 &\longmapsto v_1 v_1^t \omega \end{aligned}$$

Note that $v_1 v_1^t \omega$ is the rank 1 operator $v_1 \lambda^t$ where $\lambda = (v_1^t \omega)^t = -\omega v_1$. We know $v_1 v_1^t \omega \in sp(V)$ since $\omega v_1 v_1^t \omega$ is symmetric.

We have isomorphisms

$$\begin{aligned} S^2 V &\xrightarrow{\ \sim\ } sp(V) \xrightarrow{\ \sim\ } S^2 V^* \\ \tfrac{1}{2} v_1^2 &\longmapsto v_1 v_1^t \omega \longmapsto \omega v_1 v_1^t \omega \end{aligned}$$

To see this, polarize: $\frac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) = \overbrace{\tfrac{1}{2}(v_1+v_2)^{\otimes 2} - \tfrac{1}{2} v_1^{\otimes 2} - \tfrac{1}{2} v_2^{\otimes 2}}^{\text{in } S^2V}$ maps to $\frac{1}{2}(v_1 v_2^t + v_2 v_1^t) \in \mathrm{Hom}(V^*, V) \xleftarrow{\sim} V \otimes V$ which maps to $\frac{1}{2}(\omega v_1 v_2^t \omega + \omega v_2 v_1^t \omega) \in S^2(V^*)$. (This can be improved, see after the fermionic discussion below.)

The important point is that any symmetric $H: V \to V^*$ determines an infinitesimal symplectic operator $X = \omega^{-1} H$, which in turn determines a quadratic element of the Weyl algebra, namely, the element corresponding to $\omega^{-1} H \omega^{-1} \in S^2 V$.

Next, the fermionic version. Start with V and $H: V \xrightarrow{\sim} V^*$ non-degenerate symmetric.

Let $so(V,H) = \{X \in End(V) \mid X^tH + HX = 0\}$ be the associated orthogonal Lie algebra, and $Cliff(V,H)$ the associated Clifford algebra; it is generated by a universal linear map $v \longmapsto \psi_v$ satisfying the CAR: $\boxed{\psi_v^2 = v^tHv}$. Polarizing:

$$\psi_{x+y}^2 = (x+y)^tH(x+y) = (\psi_x+\psi_y)(\psi_x+\psi_y)$$

so

$$\boxed{\psi_x\psi_y + \psi_y\psi_x = x^tHy + y^tHx}$$

Next $[\psi_x\psi_y, \psi_v] = \psi_x(\psi_y\psi_v + \psi_v\psi_y) - (\psi_x\psi_v + \psi_v\psi_x)\psi_y$

so

$$\boxed{[\psi_x\psi_y, \psi_v] = \psi_{xy^tHv - yx^tHv}}$$

Thus the adjoint action of quadratic elements in $Cliff(V)$ preserves the space of linear elements, and we get a map

$$\Lambda^2 V \longrightarrow so(V)$$
$$x \wedge y \longmapsto xy^tH - yx^tH,$$

The image lies in $so(V) \subset End(V)$ because $Hxy^tH - Hyx^tH$ is skew-symmetric. Then we have isomorphisms

$$\Lambda^2 V \xrightarrow{\sim} so(V) \xrightarrow{\sim} \Lambda^2 V^*$$
$$x \wedge y \longmapsto xy^tH - yx^tH \longmapsto H(xy^t - yx^t)H$$

Return to the bosonic situation:

$$[\phi_x \phi_y , \phi_v] = \phi_x(y^t \omega v) + \phi_y(x^t \omega v) = \phi_{x(y^t\omega v) + y(x^t \omega v)}$$

where we have isos

$$S^2V \xrightarrow{\sim} sp(V) \xrightarrow{\sim} S^2(V^*)$$

$$xy \longmapsto xy^t\omega + yx^t\omega \longmapsto \omega(xy^t + yx^t)\omega$$

May 13, 2000

Bessel function stuff.

$i-j=-n$
$j=i+n$

$$e^{xs+ys^{-1}} = \sum_{i,j\geq 0} \frac{x^i y^j}{i!\,j!} s^{i-j} = \sum_{n\in\mathbb{Z}} s^n \Big(\sum_{\substack{i-j=n \\ i,j\geq 0}} \frac{x^i y^j}{i!\,j!} \Big)$$

$$= \sum_{n\geq 0} s^n x^n \sum_{j\geq 0} \frac{(xy)^j}{j!\,(n+j)!} + \sum_{n\geq 1} s^{-n} y^n \sum_{i\geq 0} \frac{(xy)^i}{i!\,(i+n)!}$$

$$\therefore \boxed{e^{xs+ys^{-1}} = \sum_{n\geq 0} s^n x^n J_n(xy) + \sum_{n\geq -1} s^{-n} y^n J_n(xy) \quad \text{where} \quad J_n(z) = \sum_{n\geq 0} \frac{z^n}{j!\,(j+n)!}}$$

This is a Laurent series expansion with complex variable s, so

$$x^n J_n(xy) = \oint e^{xs+ys^{-1}} s^{-n} \frac{ds}{2\pi i s} \qquad n\geq 0$$

Relation with the wave equation $\partial_t^2 u = \Delta u$ in polar coordinates (or the Laplacian in cylindrical coords.) $ds^2 = dr^2 + r^2 d\theta^2$, so ∇u has the component $\partial_r u$, $r^{-1}\partial_\theta u$ so $\|\nabla f\|^2 = \iint \big((\partial_r u)^2 + r^{-2}(\partial_\theta u)^2 \big) r\,dr\,d\theta$

$\int (\partial_r u)(r\partial_r u)\,dr = -\int u \frac{1}{r}\partial_r(r\partial_r u)\, r\,dr$, and we have

$$\boxed{\Delta u = \frac{1}{r}\partial_r(r\partial_r u) + \frac{1}{r^2}\partial_\theta^2 u}$$

Assuming θ, t dependence is exponential, [redacted] ie $u = u(r)e^{im\theta}e^{i\omega t}$, then $\Delta u = \partial_t^2 u$ becomes the Bessel DE:

$$(r\partial_r)^2 u - m^2 u = \lambda r^2 u. \qquad \lambda = -\omega^2$$

One wants solutions regular at $r=0$, say $u(r) = \sum_{n\geq 0} a_n r^n$. Recursion relation $(n^2 - m^2)a_n = \lambda a_{n-2}$, which yields

$$a_{m+2j} = \frac{\lambda}{(m+2j)^2 - m^2}\, a_{m+2j-2} = \frac{\lambda/4}{j(m+j)}$$

$$u = \sum_{j\geq 0} \frac{(\lambda/4)^j}{j!\,(m+1)\cdots(m+j)}\, r^{m+2j} \qquad \text{essentially } J_m$$

May 14, 00

$$J_n(z) = \sum_{j\geq 0} \frac{z^j}{j!\,(j+n)!} \qquad n \in \mathbb{N}$$

$$\partial_z J_n(z) = \sum_{j\geq 1} \frac{z^{j-1}}{(j-1)!\,(j+n)!} = \sum_{k\geq 0} \frac{z^k}{k!\,(k+1+n)!} = J_{n+1}(z)$$

$$\partial_z\left(z^n J_n\right) = \sum_{j\geq 0} \frac{z^{n+j-1}}{j!\,(j+n-1)!} = z^{n-1} \sum_{j\geq 0} \frac{z^j}{j!\,(j+n-1)!} = z^{n-1} J_{n-1} \qquad n\geq 1$$

Alt: $$\left(z\partial_z + n\right) J_n = \sum_{j\geq 0} \frac{(j+n)\, z^j}{j!\,(j+n)!} = J_{n-1}$$

Thus one has the recursion relations.

$$\boxed{\begin{array}{ll} \partial_z J_n = J_{n+1} & n \geq 0 \\ (z\partial_z + n) J_n = J_{n-1} & n \geq 1. \end{array}}$$

Program: You want to define the analog of grid space in the continuous case to be a certain space of analytic functions of the complex variable s for $s \in \mathbb{C} - \{0\}$.

Asymptotic derivation. Consider our representation of the discrete const coeff grid space by rational function regular off $0, \infty, k, k^{-1}$:

$$\lambda^m \mu^n \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = z^m \left(\frac{z-k}{kz-1}\right)^n \begin{pmatrix} \frac{h}{kz-1} \\ 1 \end{pmatrix}$$

[redacted] Subdivide – [redacted] replace z by $z^{\varepsilon} = e^{\varepsilon s}$, m by $\frac{x}{\varepsilon}$, n by $\frac{y}{\varepsilon}$, h by $h_\varepsilon = c\varepsilon$ and k by $k_\varepsilon = (1-|c|^2\varepsilon^2)^{1/2}$ $= 1 - \frac{1}{2}|c|^2\varepsilon^2 + O(\varepsilon^4)$. [redacted] Consider as $\varepsilon \to 0$:

$$(e^{\varepsilon s})^{\frac{x}{\varepsilon}}\left(\frac{e^{\varepsilon s} - k_\varepsilon}{k_\varepsilon e^{\varepsilon s} - 1}\right)^{y/\varepsilon}\begin{pmatrix}\frac{c\varepsilon}{k_\varepsilon e^{\varepsilon s} - 1}\\ 1\end{pmatrix}$$

$$\frac{e^{\varepsilon s} - k_\varepsilon}{k_\varepsilon e^{\varepsilon s} - 1} = \frac{1 + \varepsilon s + \frac{\varepsilon^2 s^2}{2} + O(\varepsilon^3) - 1 + \frac{1}{2}|c|^2\varepsilon^2}{\left(1 - \frac{1}{2}|c|^2\varepsilon^2\right)\left(1 + \varepsilon s + \frac{\varepsilon^2 s^2}{2}\right) + O(\varepsilon^3) - 1}$$

$$= \frac{\varepsilon s + \frac{\varepsilon^2}{2}\left(s^2 + |c|^2\right) + O(\varepsilon^3)}{\varepsilon s + \frac{\varepsilon^2}{2}\left(s^2 - |c|^2\right) + O(\varepsilon^3)}$$

$$= \frac{1 + \frac{\varepsilon}{2}\left(s + \frac{|c|^2}{s}\right) + O(\varepsilon^2)}{1 + \frac{\varepsilon}{2}\left(s - \frac{|c|^2}{s}\right) + O(\varepsilon^2)}$$

$$\therefore \left(\frac{e^{\varepsilon s} - k_\varepsilon}{k_\varepsilon e^{\varepsilon s} - 1}\right)^{y/\varepsilon} \longrightarrow \frac{e^{\frac{1}{2}\left(s + \frac{|c|^2}{s}\right)}}{e^{\frac{1}{2}\left(s - \frac{|c|^2}{s}\right)}} = e^{\frac{|c|^2}{s}}$$

Thus we obtain the limiting representation

$$\lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = e^{xs + ys^{-1}|c|^2}\begin{pmatrix}\frac{c}{s}\\ 1\end{pmatrix}$$

Put $c = 1$.

Consider next the limit where the horizontal direction becomes continuous. Here $h_\varepsilon = b\sqrt{\varepsilon}$, $k_\varepsilon = (1-|b|^2\varepsilon)^{1/2} = 1 - a\varepsilon + O(\varepsilon^2)$, $a = \frac{1}{2}|b|^2$.

Recall that the limiting v^2 is is a δ-function and is asymptotic to $\frac{v^2_\varepsilon}{\sqrt{\varepsilon}}$ where v^2_ε is a unit vector. The grid equations are

$$\begin{cases} (k_\varepsilon \lambda^\varepsilon - 1) v^1 = b\sqrt{\varepsilon}\, v^2_\varepsilon \\ (k_\varepsilon \mu_\varepsilon - 1) v^2_\varepsilon = \bar{b}\sqrt{\varepsilon}\, v^1 \end{cases}$$

rep by $e^{\varepsilon s}$

i.e.

$$\underbrace{\frac{(k_\varepsilon \lambda^\varepsilon - 1)}{\varepsilon}} v^1 = b \frac{v^2_\varepsilon}{\sqrt{\varepsilon}} \qquad \rightsquigarrow \qquad (-a+s) v^1 = b v^2$$

$$(k_\varepsilon \mu_\varepsilon - 1) \frac{v^2_\varepsilon}{\sqrt{\varepsilon}} = \bar{b}\, v^1 \qquad\qquad (\mu - 1) v^2 = \bar{b} v^1$$

Back to the continuous case: $\lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = e^{xs+ys^{-1}} \begin{pmatrix} \dot{s}^{-1} \\ 1 \end{pmatrix}$.

You want to specify the analytic functions ~~to~~ go into the grid spaces.

<u>May 16, 2000</u>

You now want to construct the continuous analog of the discrete grid space. Instead of rational functions of z this [redacted] consists of certain "entire" Laurent series in the complex variable s. [redacted] The desired grid space E is roughly, should be viewed as, the module over the translation group $\{\lambda^x \mu^y \mid (x,y) \in \mathbb{R} \times \mathbb{R}\}$ generated by the two vectors v^1, v^2 at the origin. The representation as functions of s [redacted] should be

$$\lambda^x \mu^y \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = e^{xs + ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$$

(i.e. λ^x, μ^y are resp. mult by e^{xs}, $e^{ys^{-1}}$, and v^1, v^2 are the functions s^{-1} and 1.)

E should split into $E_{hor} \oplus E_{ver}$ where E_{hor} is generated by $v^2 = 1$ under the horizontal translations $\{e^{xs}\}$, and E_{ver} is generated by $v^1 = s^{-1}$ under vertical translations $\{e^{ys^{-1}}\}$. Generated will hopefully turn out to mean of the form $\underbrace{\int e^{xs} \varphi(x)\, dx}_{\text{(case of } E_{hor})}$, where φ is an appropriate type of function with compact support, such as piecewise continuous. Similarly $\int \frac{e^{ys^{-1}}}{s^{-1}} \varphi(y)\, dy$ in the case of E_{ver}. Thus E_{hor} should consist of entire functions of s, and E_{ver} should consist of entire functions of s^{-1} vanishing at $s = \infty$. The splitting $E = E_{hor} \oplus E_{ver}$ amounts to splitting ~~and~~ entire Laurent series into powers s^n for $n \geq 0$ and for $n \leq -1$.

First recall Taylor's formula with remainder:

$$f(b) = f(0) + \int_0^b f'(x)\,dx$$

$$= f(0) - \left[f'(x)(b-x)\right]_0^b + \int_0^b f''(x)(b-x)\,dx$$

$$= f(0) + f'(0)b - \left[f''(x)\frac{(b-x)^2}{2!}\right]_0^b + \int_0^b f'''(x)\frac{(b-x)^2}{2!}\,dx$$

Hence

$$\boxed{f(b) = \sum_{j=0}^{n-1} f^{(j)}(0)\frac{b^j}{j!} + \int_0^b f^{(n)}(x)\frac{(b-x)^{n-1}}{(n-1)!}\,dx}$$

e.g.

$$\boxed{e^{bs} = \sum_{j=0}^{n-1}\frac{(bs)^j}{j!} + \int_0^b s^n e^{xs}\frac{(b-x)^{n-1}}{(n-1)!}\,dx}$$

In order to split $e^{ys^{-1}}\hat{\varphi}(s) = e^{ys^{-1}}\int e^{xs}\varphi(x)\,dx$ into horizontal + vertical, it suffices to take $\varphi(x) = \delta(x-b)$ and to split $e^{ys^{-1}}e^{bs}$.

$$\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{j\geq 0}\left(\frac{a^j}{j!\,s^j}e^{bs}\right)_{hor}$$

$$= e^{bs} + \frac{a}{1!}\,\frac{e^{bs}-1}{s} + \frac{a^2}{2!}\,\frac{e^{bs}-1-bs}{s^2} + \cdots$$

First expand in powers s^n. $\qquad n = i-j$, $i = n+j$

$$\sum_{j\geq 0}\frac{a^j}{j!\,s^j}\sum_{i\geq j}\frac{b^i s^i}{i!} = \sum_{n\geq 0} s^n \sum_{j\geq 0}\frac{a^j b^{n+j}}{j!\,(n+j)!}$$

$$\boxed{\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{n\geq 0} s^n b^n J_n(ab)}$$

On the other hand, using Taylor's thm.

$$\left(e^{as^{-1}+bs}\right)_{hor} = e^{bs} + \sum_{n\geq 1} \frac{a^n}{n!\,\cancel{s^n}} \int_0^b \cancel{s^n} e^{xs} \frac{(b-x)^{n-1}}{(n-1)!}\, dx$$

$$= e^{bs} + \int_0^b e^{xs} \sum_{n\geq 1} \frac{a^n (b-x)^{n-1}}{n!\,(n-1)!}\, dx$$

$$\boxed{\left(e^{as^{-1}+bs}\right)_{hor} = e^{bs} + \int_0^b e^{xs}\, J_1(a(b-x))\, a\, dx}$$

Next

$$\left(e^{as^{-1}+bs}\right)_{ver} = \frac{a}{1!}\frac{1}{s} + \frac{a^2}{2!}\frac{1+bs}{s^2} + \frac{a^3}{3!}\frac{1+bs+\frac{(bs)^2}{2!}}{s^3} + \cdots$$

$$= \frac{1}{s}\left(\frac{a}{1!\,0!} + \frac{a^2 b}{2!\,1!} + \frac{a^3 b^2}{3!\,2!} + \cdots\right)$$

$$+ \frac{1}{s^2}\left(\frac{a^2}{2!\,0!} + \frac{a^3 b}{3!\,1!} + \frac{a^4 b^2}{4!\,2!} + \cdots\right) + \cdots$$

$$\boxed{\left(e^{as^{-1}+bs}\right)_{ver} = \sum_{n\geq 1} s^{-n} a^n J_n(ab)}$$

Note that changing $s \mapsto s^{-1}$, $a \mapsto b$, $b \mapsto a$ doesn't change $e^{as^{-1}+bs}$ but interchanges horizontal and vertical, essentially (because the $n=0$ term $J_0(ab)$ moves).

(March 17, 2000:) $\partial_x J_n(x) = \sum_{j\geq 0} \partial_x\left(\frac{x^j}{j!\,(j+n)!}\right) = \sum_{j\geq 1} \frac{x^{j-1}}{(j-1)!\,(j+n)!}$

$= \sum_{j\geq 0} \frac{x^j}{j!\,(j+n+1)!} = J_{n+1}(x)$, so

$$\int_0^b e^{xs} J_1(a(b-x))\, a\, dx = \int_0^b e^{xs} (-\partial_x) J_0(ab-ax)\, dx$$

$$= \underbrace{\left[e^{xs} J_0(ab-ax)\right]_0^b}_{-e^{bs}\cancel{J_0(0)} + J_0(ab)} + \int_0^b s e^{xs} J_0(a(b-x))\, dx$$

Thus $$\int_0^b s e^{xs} J_0\big(a(b-x)\big)\,dx$$

$$= \int_0^b e^{xs} J_1\big(a(b-x)\big)\,a\,dx + e^{bs} - J_0(ab)$$

$$= \left(e^{as^{-1}+bs}\right)_{hor} - J_0(ab) = \sum_{n\geq 0} s^n b^n J_n(ab) - J_0(ab)$$

$$= \sum_{n\geq 1} s^n b^n J_n(ab).$$ Now interchange a, b to get

$$\sum_{n\geq 1} s^n a^n J_n(ab) = \int_0^a s e^{xs} J_0\big(b(a-x)\big)\,dx$$

and replace s by s^{-1} to get

$$\boxed{\left(e^{as^{-1}+bs}\right)_{ver} = \sum_{n\geq 1} s^{-n} a^n J_n(ab) = \int_0^a \frac{e^{xs^{-1}}}{s} J_0\big(b(a-x)\big)\,dx}$$

$$\boxed{\left(e^{as^{-1}+bs}\right)_{hor} = \sum_{n\geq 0} s^n b^n J_n(ab) = e^{bs} + \int_0^b e^{xs} J_1\big(a(b-x)\big)\,a\,dx}$$

We can also write

$$\left(e^{as^{-1}+bs}\right)_{ver} = e^{as^{-1}} \int_0^a \frac{e^{-xs^{-1}}}{s} J_0(bx)\,dx$$

$$\left(e^{as^{-1}+bs}\right)_{hor} = e^{bs}\left(1 + \int_0^b e^{-xs} J_1(ax)\,a\,dx\right)$$

May 18, 2000

Consider the PDE describing a general continuous grid: $\partial_x \psi^1 = h\psi^2$, $\partial_y \psi^2 = \bar{h}\psi^1$

where $h = h(x,y)$, $\bar{h} = \bar{h}(x,y)$ are not necessarily conjugate. Integrate to get an integral equation

$$\psi^1(x,y) = \psi^1(0,y) + \int_0^x (h\psi^2)(x',y)\,dx'$$
$$\psi^2(x,y) = \psi^2(x,0) + \int_0^y (\bar{h}\psi^1)(x,y')\,dy'$$

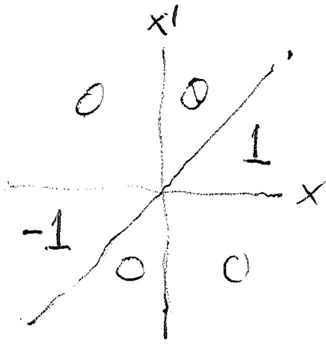
of Volterra type, which is solved as usual by iteration to get a unique solution with given $\psi^1(0,y)$, $\psi^2(x,0)$ on the characteristic lines through the origin. Recall the formulas. Put

$$D = \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_y \end{pmatrix}, \quad V = \begin{pmatrix} 0 & h \\ \bar{h} & 0 \end{pmatrix}, \quad G = \begin{pmatrix} G_x & 0 \\ 0 & G_y \end{pmatrix} \quad \text{where}$$

$$G_x f = \int_0^x f(x',y)\,dx' \quad \text{and similarly for } G_y.$$

[Note: The operator $(Gf)(x) = \int_0^x f(x')\,dx'$ which gives $(\partial_x)^{-1}$ with the initial condition "equal to zero at $x=0$" is given by the kernel

$$K(x,x') = \begin{cases} 1 & 0 < x' < x \\ -1 & x < x' < \\ 0 & \text{otherwise} \end{cases}$$



Check: $\int_x^0 (-1) f(x')\,dx' = \int_0^x f(x')\,dx'$. This operator is not translation invariant.]

Let $\varphi(x,y) = \begin{pmatrix} f^1(y) \\ f^2(x) \end{pmatrix}$ be the initial condition given on the characteristic lines but extended to the plane to satisfy $D\varphi = 0$, i.e. the unperturbed PDE. The integral equation can then be written

$$\psi = \varphi + GV\psi$$

One has $D\psi = \cancel{D\varphi} + DGV\psi = V\psi$, and also that ψ satisfies the given initial condition since φ does and since G has zero initial condition. The solution ψ is then given by the Neumann series

$$\psi = \frac{1}{1-GV}\varphi = \sum_{n\geq 0}(GV)^n\varphi.$$

March 19, 2000

Compute the fundamental solution for $\begin{cases} \partial_x\psi^1 = \psi^2 \\ \partial_y\psi^2 = \psi^1, \end{cases}$ which yields the solution with the initial conditions $\psi^1(0,y) = f^1(y)$, $\psi^2(x,0) = f^2(x)$. Apply $\mathcal{L}_x$, the Laplace transform wrt x, to the first equation getting

$$\mathcal{L}_x\psi^2 = \mathcal{L}_x(\partial_x\psi^1) = -\underbrace{\psi^1(0,y)}_{f^1(y)} + \xi\mathcal{L}_x\psi^1,$$

then apply $\mathcal{L}_y$ to get (with notation $\wedge$ for $\mathcal{L}_y\mathcal{L}_x$)

$$\widehat{\psi}^2 = -\widehat{f}^1(\eta) + \xi\,\widehat{\psi}^1.$$

Here ξ, η are the dual variables to x, y. Similarly

$$\mathcal{L}_y\psi^1 = \mathcal{L}_y(\partial_y\psi^2) = -\underbrace{\psi^2(x,0)}_{f^2(x)} + \eta\mathcal{L}_y\psi^2$$

$$\widehat{\psi^1} = -\widehat{f}^2(\xi) + \eta\,\widehat{\psi}^2$$

Now solve

$$\begin{pmatrix} 1 & -\eta \\ -\xi & 1 \end{pmatrix}\hat{\psi} = -\begin{pmatrix} \widehat{f}^2(\xi) \\ \widehat{f}^1(\eta) \end{pmatrix}$$

to get

$$\hat{\psi} = \frac{1}{-1+\eta\xi}\begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix}$$

Then $\psi(x,y)$ should be given by the inverse Laplace transform. Here $x, y > 0$ and the inverse Laplace transform is done as a contour integral running from $-i\infty$ to $i\infty$ to the right of the singularities. First integrate wrt η

$$\mathcal{L}_x\psi(\xi,y) = \int_{-i\infty}^{i\infty} \frac{d\eta}{2\pi i} e^{y\eta} \frac{1}{\xi\eta - 1}\begin{pmatrix} 1 & \eta \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\eta) \end{pmatrix}$$

(simple pole at $\eta = \xi^{-1}$)

derivative of $\xi\eta - 1$

$$= \frac{e^{y\xi^{-1}}}{\xi}\begin{pmatrix} 1 & \xi^{-1} \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\xi^{-1}) \end{pmatrix}$$

and then the inverse transform wrt ξ:

$$\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{d\xi}{2\pi i\xi} e^{x\xi} e^{y\xi^{-1}}\begin{pmatrix} 1 & \xi^{-1} \\ \xi & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(\xi) \\ \hat{f}^1(\xi^{-1}) \end{pmatrix}$$

Summarize: It seems that the solution of $\partial_x\psi^1 = \psi^2$, $\partial_y\psi^2 = \psi^1$ with initial conditions $\psi^1(0,y) = f^1(y)$, $\psi^2(x,0) = f^2(x)$ is given by

$$* \quad \boxed{\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{xs + ys^{-1}}\begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix}\begin{pmatrix} \hat{f}^2(s) \\ \hat{f}^1(s^{-1}) \end{pmatrix}}$$

where the contour is just to the right of the singularities, i.e. $s=0$, and where $x, y > 0$. ~~Push the contour to the left. You pick up the residue at $s=0$, and the rest is zero because e^{xs} decays for $x>0$. Thus~~

$$\cancel{\psi(x,y) = \oint \frac{ds}{2\pi i s} e^{xs+ys^{-1}} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix} \begin{pmatrix} \hat{f}^2(s) \\ \hat{f}^1(s^{-1}) \end{pmatrix}}$$

~~This expression for the solution makes sense for all x, y at least when the f^i have compact support. Let's check the above works. First it satisfies the DE since it's a linear combination of exponential solutions.~~

$$\cancel{\psi^2(x,0) = \int}$$

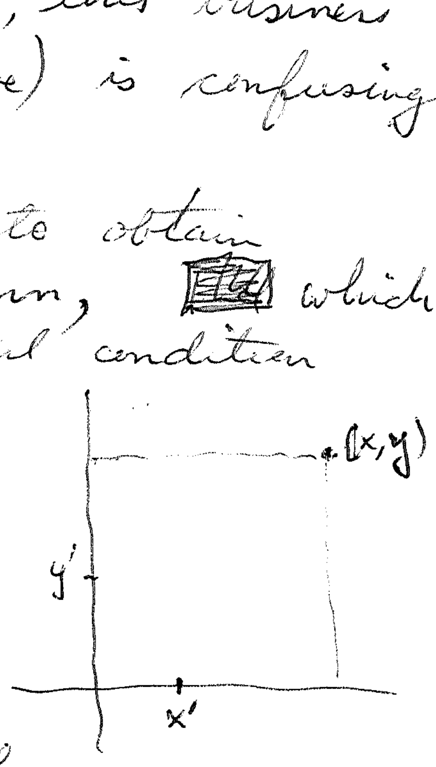
Let's check that $*$ on the previous page gives the solution. Firstly, it satisfies the DE, since it is a linear combination of exponential solutions. Also

$$\psi^2(x,0) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i \cancel{s}} e^{xs} \cancel{s} \hat{f}^2(s) = f^2(x)$$

$$\psi^1(0,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s} e^{ys^{-1}} s^{-1} \hat{f}^1(s^{-1}) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i \cancel{s}} e^{ys} \cancel{s} \hat{f}^1(s) = f^1(y)$$

So it seems to work. However, this business about moving contours (crossed out above) is confusing and prone to errors.

Let's try to use the preceding to obtain the fundamental solution of Riemann, which is the solution in the case of initial condition consisting of a δ-function.



First case: $\begin{pmatrix} f^1(y) \\ f^2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \delta(x-x') \end{pmatrix}$

where $x' > 0$ (needed to use the LT). Here $\hat{f}^1(\eta) = 0$, $\hat{f}^2(\xi) = e^{-x'\xi}$, whence

$$\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}\, e^{xs+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix} \overbrace{\hat{f}^2(s)}^{e^{-x's}}$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}\, e^{(x-x')s+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix}$$

this is zero for $x-x'<0$ — move the contour right

If $x>x'$ then we move the contour left, getting the residue at $s=0$, and the rest vanishes.

$$\psi(x,y) = \oint \frac{ds}{2\pi i s}\, e^{(x-x')s+ys^{-1}} \begin{pmatrix} 1 \\ s \end{pmatrix} = \begin{pmatrix} J_0((x-x')y) \\ J_1((x-x')y)\,y \end{pmatrix}$$

missing δ

<u>2nd case:</u> $\begin{pmatrix} f^1(y) \\ f^2(x) \end{pmatrix} = \begin{pmatrix} \delta(y-y') \\ 0 \end{pmatrix}$ where $y'>0$

so $\begin{pmatrix} \hat{f}^1(\eta) \\ \hat{f}^2(\xi) \end{pmatrix} = \begin{pmatrix} e^{-y'\eta} \\ 0 \end{pmatrix}$, and the solution is

$$\psi(x,y) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}\, e^{xs+ys^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix} \overbrace{\hat{f}^1(s^{-1})}^{e^{-y's^{-1}}}$$

$$= \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}\, e^{xs+(y-y')s^{-1}} \begin{pmatrix} s^{-1} \\ 1 \end{pmatrix} = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i s}\, e^{xs^{-1}+(y-y')s} \begin{pmatrix} s \\ 1 \end{pmatrix}$$

Again if $y-y'<0$ one gets 0, and for $y-y'>0$

$$\psi(x,y) = \oint \frac{ds}{2\pi i s}\, e^{xs^{-1}+(y-y')s} \begin{pmatrix} s \\ 1 \end{pmatrix} = \begin{pmatrix} J_1(x(y-y'))\,x \\ J_0(x(y-y')) \end{pmatrix}$$

Let's use the notation

$$\boxed{\begin{pmatrix} K(x,y|x') & K(x,y|y') \end{pmatrix} = \begin{pmatrix} J_0((x-x')y) & J_1(x(y-y'))\,x \\ J_1((x-x')y)\,y & J_0(x(y-y')) \end{pmatrix}}$$

for $\begin{cases} 0<x'<x \\ 0<y'<y. \end{cases}$

missing δ

One should also write

$$\oint \frac{ds}{2\pi i s}\, e^{xs+ys^{-1}} \begin{pmatrix} 1 & s^{-1} \\ s & 1 \end{pmatrix} = \begin{pmatrix} J_0(xy) & xJ_1(xy) \\ J_1(xy)y & J_0(xy) \end{pmatrix}$$

missing δ

for the solutions corresponding to the basis vectors v^2 (δ fn in x direction at origin) v^1 (δ function in y direction at the origin).

Discuss $J_{\pm\frac{1}{2}}$. Generalize $J_n(x) = \sum_{j\geq 0} \frac{x^j}{j!\,(j+n)!}$ to $\boxed{J_s(x) = \sum_{j\geq 0} \frac{x^j}{j!\,\Gamma(j+s+1)}}$. Then $\boxed{\partial_x J_s = J_{s+1}}$

and $(x\partial_x + s)J_s(x) = \sum_{j\geq 0} \frac{x^j (j+s)}{j!\,\Gamma(j+s+1)} = \sum_{j\geq 0} \frac{x^j}{j!\,\Gamma(j+s)} = J_{s-1}(x)$;

so $\boxed{(x\partial_x + s)J_s(x) = J_{s-1}(x)}$, equiv. $\boxed{\partial_x(x^s J_s) = x^{s-1}J_{s-1}}$;

whence DE: $\boxed{\partial_x(x\partial_x+s)J_s = J_s}$ and recursion reln. $\boxed{xJ_{s+1} + sJ_s = J_{s-1}}$

$$J_{-\frac{1}{2}}(x) = \sum_{j\geq 0} \frac{x^j}{j!\,\Gamma(j+\frac{1}{2})} \qquad \Gamma(j+\tfrac{1}{2}) = (j-\tfrac{1}{2})(j-\tfrac{3}{2})\cdots(\tfrac{1}{2})\Gamma(\tfrac{1}{2})$$

$$= \frac{1}{\sqrt{\pi}} \sum_{j\geq 0} \frac{x^j 2^{2j}}{(2j)!} = \frac{1}{\sqrt{\pi}} \sum_{j\geq 0} \frac{(2x^{1/2})^{2j}}{(2j)!} = \frac{1}{\sqrt{\pi}} \cosh(2x^{1/2})$$

$$J_{\frac{1}{2}}(x) = \sum_{j\geq 0} \frac{x^j}{j!\,\Gamma(j+3/2)} = \frac{1}{\sqrt{\pi}} \sum_{j\geq 0} \frac{x^j 2^{2j+1}}{(2j+1)!} = \frac{1}{\sqrt{\pi}} \sum_{j\geq 0} \frac{(x^{1/2}2)^{2j+1}}{(2j+1)!\, x^{1/2}}$$

so

$J_{\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}} \cosh(2x^{1/2})$	$J_{-\frac{1}{2}}(x) = \frac{1}{\sqrt{\pi}} \frac{\sinh(2x^{1/2})}{x^{1/2}}$

Note that $x = \frac{r^2}{4}$, $r = 2x^{1/2}$ in radial applications

To solve the initial value problem for $\partial_x \psi^1 = \psi^2$, $\partial_y \psi^2 = \psi^1$ with $\psi^1(0,y)$, $\psi^2(x,y)$ given, let's use the LT in the variable x:

$$\hat{\psi}(s,y) = \int_0^\infty e^{-sx}\psi(x,y)\,dx .$$

Then $\widehat{\psi^1} = \widehat{\partial_y \psi^2} = \partial_y \widehat{\psi^2}$,

$$\widehat{\psi^2} = \widehat{\partial_x \psi^1} = -\psi^1(0,y) + s\widehat{\psi^1}, \text{ so } \widehat{\psi^1} = s^{-1}\left(\psi^1(0,y) + \widehat{\psi^2}\right)$$

and

$$\boxed{\left(\partial_y - s^{-1}\right)\widehat{\psi^2} = s^{-1}\psi^1(0,y)}.$$

Hence

$$\boxed{\widehat{\psi^2}(s,y) = \int_0^y e^{(y-y')s^{-1}} s^{-1}\psi^1(0,y')\,dy' + e^{ys^{-1}}\widehat{\psi^2}(s,0)}$$

Now take the inverse LT:

$$\int_{0^+-i\infty}^{0^++i\infty} e^{xs}\, e^{(y-y')s^{-1}}\, s^{-1}\frac{ds}{2\pi i}$$

This is zero for $x<0$ as the contour can be pushed to the right where e^{xs} decays.

If $x>0$, then the contour can be moved to the left, picking up the residue at $s=0$:

$$\oint e^{xs+(y-y')s^{-1}}\frac{ds}{2\pi i s} = J_0(x(y-y'))$$

Thus

$$\boxed{\int_{0^+-i\infty}^{0^++i\infty} e^{xs+ys^{-1}}\frac{ds}{2\pi i s}\begin{pmatrix} s \\ 1 \\ s^{-1}\end{pmatrix} = \begin{pmatrix} \delta(x) + y J_1(xy)H(x) \\ J_0(xy)H(x) \\ xJ_1(xy)H(x)\end{pmatrix}}$$

$$\int_{0^+-i\infty}^{0^++i\infty} e^{xs} e^{ys^{-1}}\widehat{\psi^2}(s,0)\frac{ds}{2\pi i} = \int_0^\infty \left(\int_{0^+-i\infty}^{0^++i\infty} e^{(x-x')s+ys^{-1}}\frac{ds}{2\pi i}\right)\psi^2(x',0)\,dx'$$

$$= \psi^2(x,0) + \int_0^x y\,J_1((x-x')y)\,\psi^2(x',0)\,dx'$$

Calculation yields the following:

$$\boxed{\begin{aligned}\psi^1(x,y) &= \psi^1(0,y) + \int_0^y x J_1(x(y-y'))\,\psi^1(0,y')\,dy' \\ &\quad + \int_0^x J_0((x-x')y)\,\psi^2(x',0)\,dx' \\ \psi^2(x,y) &= \psi^2(x,0) + \int_0^x y\, J_1((x-x')y)\,\psi^2(x',0)\,dx' \\ &\quad + \int_0^y J_0(x(y-y'))\,\psi^1(0,y')\,dy'\end{aligned}}$$

This has been derived for $x > 0$ and all y, but it is easy to check that it gives the solution to the given I.V.P. for all x, y.

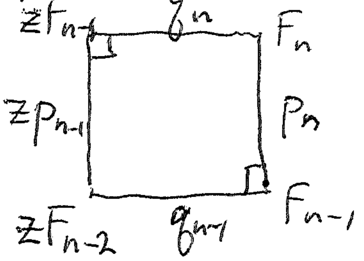
In particular for the "universal" solution $\psi(x,y) = e^{xs+ys^{-1}}\begin{pmatrix} s^{-1} \\ 1 \end{pmatrix}$ we have

$$\boxed{\begin{aligned}e^{xs+ys^{-1}}s^{-1} &= e^{ys^{-1}}s^{-1} + \int_0^y x\, J_1(x(y-y'))\,e^{y's^{-1}}s^{-1}\,dy' \\ &\quad + \int_0^x J_0((x-x')y)\,e^{x's}\,dx' \\ e^{xs+ys^{-1}} &= e^{xs} + \int_0^x y\, J_1((x-x')y)\,e^{x's}\,dx' \\ &\quad + \int_0^y J_0(x(y-y'))\,e^{y's^{-1}}s^{-1}\,dy'\end{aligned}}$$

see p 121-2.

May 23, 2000 (Alice + Emily birthday)

At last a derivation of Szegö's relations for orthogonal polys:



By ~~construction~~ p_n, zp_{n-1} have leading terms $z^n \times$ positive constant so that $p_n \equiv a\,zp_{n-1} \pmod{F_{n-1}}$ with $a>0$. Similarly $q_n \equiv d\,q_{n-1} \pmod{zF_{n-1}}$ with $d>0$. Thus we have $\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}$ with $a,d>0$.

Rewriting: $\begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$, $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{d} & \frac{b}{d} \\ -\frac{c}{d} & \frac{1}{d} \end{pmatrix}$

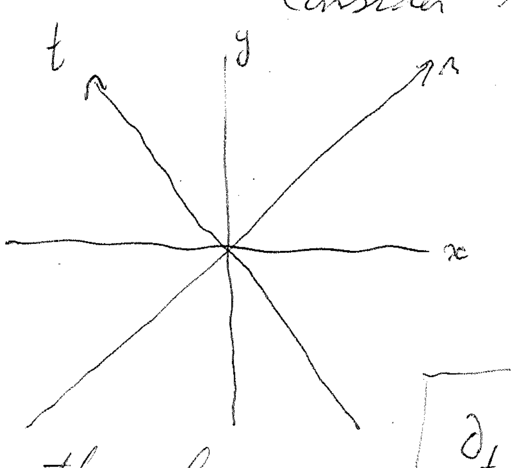
and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U(2)$ since (p_n, q_{n-1}) and (q_n, zp_{n-1}) are both orthonormal bases of $F_n \ominus zF_{n-2}$. But $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} = \frac{a}{d}$ which is both >0 and of abs. value $=1$.
$\therefore$ $a=d$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SU(2)$. But as $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1}$, one has $\begin{pmatrix} \bar\alpha & \bar\gamma \\ \bar\beta & \bar\delta \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}$, whence $\alpha = \delta = \frac{1}{d}$, $ad-bc=1$, etc. yielding the relations in the form

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \frac{1}{k}\begin{pmatrix} 1 & h \\ \bar h & 1 \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_{n-1} \end{pmatrix}, \quad \begin{pmatrix} p_n \\ q_{n-1} \end{pmatrix} = \begin{pmatrix} k & h \\ -\bar h & k \end{pmatrix}\begin{pmatrix} zp_{n-1} \\ q_n \end{pmatrix}$$

with $h=\beta$ satisfying $|h|<1$ and $k=(1-|h|^2)^{1/2}$.

June 18, 2000

Consider the wave equation



$$\partial_x = \partial_r - \partial_t \qquad \partial_y = \partial_r + \partial_t \qquad r = x+y \qquad t = -x+y$$

$$\boxed{\begin{aligned}(\partial_r - \partial_t)\psi^1 &= m\psi^2 \\ (\partial_r + \partial_t)\psi^2 &= \bar{m}\psi^1\end{aligned}}$$

where m depends on the position r but not t

other forms: $\boxed{\partial_t \psi = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix}\psi}$ $\boxed{\partial_r \psi = \begin{pmatrix} \partial_t & m \\ \bar{m} & -\partial_t \end{pmatrix}\psi}$

We want to solve the Cauchy problem on $t=0$, i.e. given $\psi_0(r) = \begin{pmatrix} \psi_0^1(r) \\ \psi_0^2(r)\end{pmatrix}$, to show there is a unique solution $\psi(r,t)$ of the wave equation such that $\psi(r,0) = \psi_0(r)$. Formally

$$\psi(r,t) = e^{tD}\psi_0 \qquad \text{where } D = \begin{pmatrix} \partial_r & -m \\ \bar{m} & -\partial_r \end{pmatrix} = \varepsilon\partial_r + V$$

Because D is formally skew-adjoint w.r.t. the usual L^2-norm $\int_{-\infty}^{\infty} (\psi^*\psi)(r)\,dr$, (the energy norm), e^{tD} is a 1-parameter group of unitary operators on the Hilbert space $\mathcal{H}$ of finite energy states. To make this Hilbert space picture precise one must show the closure of D is an unbounded skew-adjoint operator on $\mathcal{H}$. In view of the von Neumann equivalence of such operators with their Cayley transforms, this task should be essentially the same as constructing the resolvent $(s-D)^{-1}$ on $\mathcal{H}$ for $\mathrm{Re}(s) \neq 0$. Moreover, the discontinuity in the resolvent as one crosses the imaginary s axis should yield the spectral decomposition of $\mathcal{H}$ w.r.t. D. Thus one is led to study the eigenfunctions for the ordinary differential operator D on the real line: $\boxed{(s-D)\psi = 0}$, and the Green's function $\boxed{(s-D)\,G(r,r') = \delta(r-r')\,Id}$ which is the Schwartz kernel of the resolvent.

Change r to x to standardize the notation, so that $0=(s-D)\psi = (s-\varepsilon\partial_x - V)\psi$ becomes

$$\partial_x\psi = \left(\varepsilon s - \varepsilon\begin{pmatrix} 0 & -m \\ \bar m & 0\end{pmatrix}\right)\psi \quad \text{or} \quad \boxed{\partial_x\psi = \begin{pmatrix} s & m \\ \bar m & -s\end{pmatrix}\psi}$$

Treat this as a perturbation of the case $m=0$; variation of constants method yields the substitution

$$\psi = g_s\phi, \quad g_s(x) = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx}\end{pmatrix}, \quad \boxed{\partial_x\phi = \begin{pmatrix} 0 & me^{-2sx} \\ \bar m e^{2sx} & 0\end{pmatrix}\phi}$$

Let the monodromy or transfer matrix for the DE above be defined by

$$\phi(x) = \Phi(x,x')\,\phi(x')$$

for any solution ϕ. Assume m decays rapidly enough that the limits of Φ as either or both $x, x' \to \pm\infty$ exist and are nice functions of s near the imaginary axis (e.g. $\mathrm{Supp}(m)$ compact).

Let W_s be the space of eigenfunctions ψ; it is 2 dim'l and any ψ has asymptotic behavior:

$$\underset{-\infty \leftarrow x}{\begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx}\end{pmatrix}\phi(-\infty) \sim}\ \psi(x) = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx}\end{pmatrix}\Phi(x,-\infty)\phi(-\infty) \underset{x\to+\infty}{\sim} \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx}\end{pmatrix}\phi(+\infty)$$

The Green's function $G(x,x')$ should be square integrable. If $\operatorname{Re}(s) > 0$, then

$$\left\{\begin{array}{ll} e^{sx} & \text{decays as } x \to -\infty \\ e^{-sx} & \text{decays as } x \to +\infty, \end{array}\right.$$

so boundary conditions for G are $\phi(-\infty) \in \begin{pmatrix} \mathbb{C} \\ 0 \end{pmatrix}$ and $\phi(+\infty) \in \begin{pmatrix} 0 \\ \mathbb{C} \end{pmatrix}$.

$$G^{<}(x,x') = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix} \Phi(x,-\infty) \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\alpha_1 \;\; \alpha_2)$$

(Note: G is a matrix fn.)

$$G^{>}(x,x') = \begin{pmatrix} e^{sx} & 0 \\ 0 & e^{-sx} \end{pmatrix} \Phi(x,\infty) \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\beta_1 \;\; \beta_2)$$

where the α's, β's are to be determined so that

$$G^{>}(x',x') - G^{<}(x',x') = -\varepsilon$$

since $(s - \varepsilon\partial_x - V)G = \delta I$ so the jump in G is $-\varepsilon$ and $(-\varepsilon)^2 = I$.

Put

$$\Phi(\infty,-\infty) = \Phi(\infty, x)\,\Phi(x,-\infty)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix} \begin{pmatrix} a^l & b^l \\ c^l & d^l \end{pmatrix}$$

where the g^r and g^l are functions of x.

$$\Phi(x,-\infty)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a^l \\ c^l \end{pmatrix}$$

$$\Phi(x,+\infty)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a^r & b^r \\ c^r & d^r \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -b^r \\ a^r \end{pmatrix}.$$

$$-\varepsilon = G^{>}(x',x') - G^{<}(x',x')$$

$$= \begin{pmatrix} e^{sx'} & 0 \\ 0 & e^{-sx'} \end{pmatrix} \left[\begin{pmatrix} -b^r \\ a^r \end{pmatrix}_{(x')} (\beta_1 \;\; \beta_2) + \begin{pmatrix} a^l \\ c^l \end{pmatrix}_{(x')} (-\alpha_1 \;\; -\alpha_2) \right]$$

$$\therefore\ -\mathcal{E} = \begin{pmatrix} e^{sx'} & 0 \\ 0 & e^{-sx'} \end{pmatrix} \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix}$$

$$\begin{pmatrix} -\alpha_1 & -\alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \frac{1}{a} \begin{pmatrix} a^r & +b^r \\ -c^l & a^l \end{pmatrix} \begin{pmatrix} e^{-sx'} & 0 \\ 0 & e^{sx'} \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{pmatrix} = \frac{1}{a} \begin{pmatrix} a^r & -b^r \\ +c^l & a^l \end{pmatrix}_{(x')} \begin{pmatrix} e^{-sx'} & 0 \\ 0 & e^{sx'} \end{pmatrix}$$

Thus

$$G^<(x,x') = g_s(x) \begin{pmatrix} a^l(x) \\ c^l(x) \end{pmatrix} \frac{1}{a} \begin{pmatrix} a^r(x') & -b^r(x') \end{pmatrix} g_s(x')^{-1} \qquad x<x'$$

$$G^>(x,x') = g_s(x) \begin{pmatrix} -b^r(x) \\ a^r(x) \end{pmatrix} \frac{1}{a} \begin{pmatrix} c^l(x') & a^l(x') \end{pmatrix} g_s(x')^{-1} \qquad x>x'$$

Note that a, being an entry in $\Phi(\infty,-\infty)$, is independent of x. (It should have an interpretation as a Wronskian.)

Let $\tilde{G}$ denote G without the g_s factors:

$$\boxed{\begin{aligned} \tilde{G}^<(x,x') &= \begin{pmatrix} a^l \\ c^l \end{pmatrix}_x \frac{1}{a} \begin{pmatrix} a^r & -b^r \end{pmatrix}_{x'} \\ \tilde{G}^>(x,x') &= \begin{pmatrix} -b^r \\ a^r \end{pmatrix}_x \frac{1}{a} \begin{pmatrix} c^l & a^l \end{pmatrix}_{x'} \\ \left(\tilde{G}^> - \tilde{G}^<\right)(x,x) &= \begin{pmatrix} a^l & -b^r \\ c^l & a^r \end{pmatrix} \frac{1}{a} \begin{pmatrix} -a^r & b^r \\ c^l & a^l \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}}$$

You need a satisfactory explanation of these formulas.

Green's functions in the discrete case

Let $E = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} p_n$ with $u(p_{n-1}) = p_n$; it's a rank one grid space. Let ψ be an eigenfunction for u: $\psi \in (E/(\lambda - u)E)^*$, so that ψ is equivalent to a solution $\psi_n = \psi(p_n)$ of the recursion relation $\boxed{\psi_n = \lambda \psi_{n-1}}$.

Consider the corresponding inhomogeneous equation

$$\boxed{\psi_n - \lambda \psi_{n-1} = f_n.}$$

Solve à la Euler: $\hat{\psi}(z) = \sum \psi_n z^n$,

$$\hat{\psi} - \lambda z \hat{\psi} = \hat{f}, \qquad \boxed{\hat{\psi}(z) = \frac{1}{1-\lambda z} \hat{f}(z)}.$$

When $|\lambda| < 1$ it's appropriate to expand and get

$$\hat{\psi}(z) = \sum_{k \geq 0} \lambda^k z^k \hat{f}(z) \quad \text{i.e.} \quad \boxed{\psi_n = \sum_{k \geq 0} \lambda^k f_{n-k}}$$

and for $|\lambda| > 1$:

$$\hat{\psi}(z) = -\sum_{k \geq 1} (\lambda z)^{-k} \hat{f}(z) \quad \text{i.e.} \quad \boxed{\psi_n = -\sum_{k \geq 1} \lambda^{-k} f_{n+k}}$$

The Green's functions in these cases are

$$G(n, n') = \begin{cases} \lambda^{n-n'} & n \geq n' \\ 0 & n < n' \end{cases}$$

$$G(n, n') = \begin{cases} 0 & n \geq n' \\ -\lambda^{n-n'} & n < n' \end{cases}$$

Let E be the grid space with parameters $(h_n)_{n\in\mathbb{Z}}$. E is a free module over $\mathbb{C}[u,u^{-1}]$ of rank 2, so for each $\lambda\in\mathbb{C}^\times$ there is a 2-diml space W_λ of eigenfunctions $\psi\in\big(E/(\Delta-u)E\big)^*$. To describe ψ by a nice recursion relation it's convenient to ~~[illegible]~~ ~~increasing staircase basis~~ choose a square root $\lambda^{1/2}$ of λ and to put

$$\psi_n = \lambda^{-n/2}\begin{pmatrix}\psi(p_n)\\ \psi(q_n)\end{pmatrix}.$$

Then

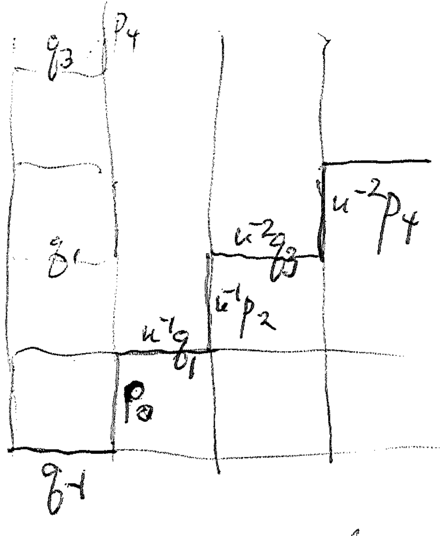
$$\psi_n = \lambda^{-n/2}\underbrace{\frac{1}{k_n}\begin{pmatrix}1 & h_n\\ \bar h_n & 1\end{pmatrix}}_{\text{call this } g(h_n)}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\psi(p_{n-1})\\ \psi(q_{n-1})\end{pmatrix}$$

$$\psi_n = g(h_n)\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\overset{\lambda^{-\frac{n-1}{2}}}{\lambda^{1/2}\lambda^{-n/2}}\begin{pmatrix}\psi(p_{n-1})\\ \psi(q_{n-1})\end{pmatrix}$$

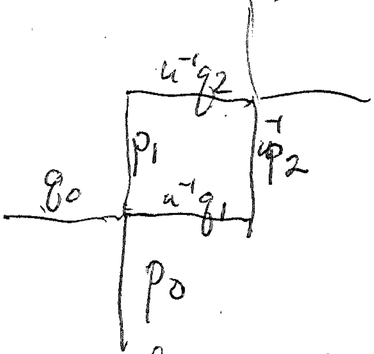
so that an eigenfunction ψ is described by

$$\boxed{\psi_n = g(h_n)\begin{pmatrix}\lambda^{1/2} & 0\\ 0 & \lambda^{-1/2}\end{pmatrix}\psi_{n-1}}$$

I obtained this nice recursion relation by starting with the increasing staircase orthonormal basis



which seems convenient for the ℓ^2 properties. You want a 2-component vector ψ_n for each n (= space coordinate) which suggests the sequence of bases (for E over $\mathbb{C}[u,u^{-1}]$): $\begin{pmatrix} p_0 \\ q_0 \end{pmatrix}$, $\begin{pmatrix} p_1 \\ u^{-1}q_1 \end{pmatrix}$, $\begin{pmatrix} u^{-1}p_2 \\ u^{-1}q_2 \end{pmatrix}$, ...



Then you have the relations

$$u^{-n}\begin{pmatrix} p_{2n} \\ q_{2n} \end{pmatrix} = \gamma(h_{2n})\begin{pmatrix} u^{-n+1} & 0 \\ 0 & u^{-n} \end{pmatrix}\begin{pmatrix} p_{2n-1} \\ q_{2n-1} \end{pmatrix}$$

$$= \gamma(h_{2n})\begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix}\gamma(h_{2n-1})\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} u^{-n+1}\begin{pmatrix} p_{2n-2} \\ q_{2n-2} \end{pmatrix}$$

which can be put in better form by introducing $u^{1/2}$. (Ultimately you might explore ~~[illegible]~~ modifying grid space so that the lines generated by p_n, q_n for n odd are well-defined, but the generators p_n, q_n depend up to sign on a choice of $u^{1/2}$.)