What can you do tomorrow? ??

You are now free to work out ch3's formulas.

Reflection positivity: Suppose given a f.d. Hilbert space $V$ and a family $\phi(t) \in \mathcal{L}(V)$ for $t > 0$, such that $\phi(0) = 1$, $\phi(t + t') = \phi(t) \phi(t')$.

Free field theory > Gaussian
Think about real stochastic process

\[
\begin{pmatrix}
(\xi e^* + A^*) & X & (\epsilon) \\
\oplus & \epsilon & Y
\end{pmatrix}
\]

Review yesterday's formulas.

\[
\begin{align*}
\epsilon &= \frac{1}{2} (a + b) \\
A &= \frac{1}{2} (a - b) \\
(\xi e^* + A^*) \xi &= \frac{1}{2} (a + b) = \frac{1}{2} (1 + 6^* a) \\
(\xi e^* + A^*) A &= \frac{1}{2} (a - b) = \frac{1}{2} (1 - 6^* a)
\end{align*}
\]

I think you should work in $Y \oplus Y$ as much as possible. Unitary picture $\|y_1\|_2^2 - \|y_2\|_2^2$

\[
W = \begin{pmatrix} a \\ b \end{pmatrix} X < Y, \\
W_0 = W \oplus \text{Ker}(a^*), \\
\text{Ker}(b^*)
\]

What is interesting? $(\frac{1}{2}) Y$ is isotropic for the hermitian form when $|x| = 1$.

$W_0 \cap (\frac{1}{2}) Y$ is a line consisting of $y_1 = ax + v^+$, $y_2 = bx + v^-$ where $2y_1 = y_2$.

$S(2) : V^+ \to V$ solution of $(9z^2 - b)x = -2v^+ + v^- \Rightarrow S(2) z v^+ = v^-$

\[
(1 - bb^*)(1 - 2ab^*)^{-1}
\]
Another description is \( W^0 = \{ (y_1, y_2) | a^* y_1 = b y_1 \} \). So \( W^0 \cap (1, 2) Y = \{ y \in Y | (z^{-1} a^* - b^*) y = 0 \} \) is \( \text{Ker} \{ (a^* - b^*) : Y \rightarrow X \} \) for \( |z| = 1 \).

Is there a simple way to see that
\[
y \mapsto y(z) = (e^z, (1 - z e A b^*)^{-1} y)
\]
gives an isometric embedding \( Y \rightarrow L^2(S^1) \).

Go back to \( W = \langle (\varepsilon, A) X \rangle \subset Y \), isotropic for \( \langle (y_1, y_2), (-1, 0)(y_1, y_2) \rangle \).

i.e. \( (\varepsilon x', Ax) = (Ax', \varepsilon x) \), \( W^0 = \{ (y_1, y_2) | (y_1, Ax) = (y_2, \varepsilon x) \} \).

\[
W^0 \cap (1, 2) Y = \{ (\lambda y_1, y_2) | (\lambda y_1, Ax) = (y_2, \varepsilon x), \lambda \in \mathbb{R} \}
\]
\[
\lambda (\varepsilon - A^*) y = 0, \quad \forall x
\]

In the J-matrix picture you have \( u^\lambda = \begin{pmatrix} u_1^\lambda \\ u_{\text{odd}}^\lambda \end{pmatrix} \) defined

\[
(\lambda I_{n+1} - \tilde{A}) u^\lambda = \begin{pmatrix} u_1^\lambda \\ \vdots \\ \lambda u_{n+1}^\lambda \end{pmatrix}
\]

Recall
\[
u_1^\lambda = 1
\]

\[
a_1 u_2^\lambda = (\lambda - b_1) u_1^\lambda
\]

\[
a_2 u_3^\lambda = (\lambda - b_2) u_2^\lambda - a_1 u_1^\lambda
\]

\[
a_{n-1} u_n^\lambda = (\lambda - b_{n-1}) u_{n-1}^\lambda - a_{n-2} u_{n-2}^\lambda
\]

\[
a_n u_{n+1}^\lambda = (\lambda - b_n) u_n^\lambda - a_{n-1} u_{n-1}^\lambda
\]
You want to derive a spectral representation for the element of $\mathbb{R}$. Review simplest version.

\[
\begin{align*}
\lambda \in \mathbb{R} & \quad \lambda \in \mathbb{R} \\
\mathbf{e}_{n+1} & \quad \mathbf{e}_{n+1} \\
\otimes & \quad \otimes \\
\mathbb{C} & \quad \mathbb{C}
\end{align*}
\]

\[
\begin{align*}
Y & \quad (\mathcal{A} - \lambda \mathcal{I}) Y \\
\mathbf{e}_{n+1} & \quad \mathbf{e}_{n+1} \\
\oplus & \quad \oplus \\
\mathbb{C} & \quad \mathbb{C}
\end{align*}
\]

\[
y = (\mathcal{A} - \lambda \mathcal{I}) \mathbf{e}_{n+1} \\
\mathbf{e}_{n+1}^* & \quad \mathbf{e}_{n+1}^* \\
(\mathcal{A} - \lambda \mathcal{I})^{-1} & \quad (\mathcal{A} - \lambda \mathcal{I})^{-1}
\]

\[
g(\lambda) = \mathbf{e}_{n+1}^* (\mathcal{A} - \lambda \mathcal{I})^{-1} y
\]

$g(\lambda)$ is a rational function with $n$ simple poles, the eigenvalues of $\mathbf{e}_{n+1}^* \mathcal{A}^{-1} \mathbf{e}_{n+1}$.

\[
(\mathcal{A} - \lambda \mathcal{I}) x_n + \tilde{g}(\lambda) \mathbf{e}_{n+1} = y
\]

Go back to $u^\lambda = \begin{pmatrix} u_1^\lambda \\ u_2^\lambda \\ \vdots \\ u_{n+2}^\lambda \end{pmatrix}$, $\mathbf{A}_{n+1} u^\lambda = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_{n+2}^\lambda \end{pmatrix}$.

\[
(\lambda - \mu) (u^\mu, u^\lambda) = (u^\mu, \lambda u^\lambda) - (\mu u^\mu, u^\lambda) = (u^\mu, (\lambda - \mu) u^\lambda) - (\mu u^\mu, u^\lambda)
\]

\[
= u_{n+1}^\lambda a_{n+1} u_{n+2}^\mu - a_{n+1} u_{n+2}^\mu u_{n+1}^\lambda
\]

\[
= a_{n+1} \begin{vmatrix} u_{n+1}^\lambda & u_{n+2}^\mu \\ u_{n+2}^\mu & u_{n+1}^\lambda \\ (\lambda - \mu) & 1 \\ (\mu - \lambda) & 1 \end{vmatrix}
\]

\[
(u^\lambda) \in W_0^\mu (\lambda) \mathbb{C} \quad (\mu, \lambda u^\lambda) = (A x, u^\lambda)
\]

\[
(u^\mu, (-1)^n (\lambda u^\lambda)) = (\lambda - \mu) (u^\mu, u^\lambda) = a_{n+1} \begin{vmatrix} u_{n+1}^\mu & u_{n+2}^\lambda \\ u_{n+2}^\lambda & u_{n+1}^\mu \\ (\lambda - \mu) & 1 \\ (\mu - \lambda) & 1 \end{vmatrix}
\]
basic problem: Given an operator \( A \) with cyclic vector \( \xi \), to embed \( Y \) into \( \mathbb{C} \) and transforms \( Y \) to rational functions, injective \((A-A)^{-\xi} = (\lambda^1 + \lambda^2 A + \lambda^3 A^2 + \cdots)^{-\xi}\)

So \[ \frac{\partial}{\partial \lambda} \frac{f(A)}{(A-A)}^\xi = f(A)^\xi \]

\[ \frac{\partial}{\partial \lambda} f(x) \mu^x (A-A)^{-\xi} = \mu^x f(A)^\xi \]

To back to \( W = (\xi|A) \times (\xi|A) \quad W^0 \cap (\xi|A) = C(\lambda^1) \)

\[ \begin{pmatrix} \frac{\partial}{\partial \lambda} \\ \frac{\partial}{\partial \lambda} \end{pmatrix} (x,y) = (0, 1)(x,y) = (Ax, y) \]

\[ y \in \text{ker}(\lambda \xi - A^*) \]

suppose you pick a line in \( W^0/W \times \mathbb{R}\) such that \( \mu^x \) maps to \( L \) in \( W^0/W \).

This gives an extension of the partial operator. Example \( L = (\lambda^x|\lambda^x) \).

What is the spectrum of the resulting operator? You seem to have a line which should be related to the remaining \( \lambda \) such that \( (\lambda^x|\lambda^x) \) maps to \( L \) in \( W^0/W \).

Have \( W = (\xi|A) \times (\xi|A) \quad W^0 \cap (\xi|A) = C(\lambda^1) \)

\( C \mu^x = \text{ker}(\lambda \xi - A^*) \). A line \( L \) in \( W^0/W \), corresponds to an extension of \( \lambda \xi - A^* \) to an operator on \( Y \) which has eigenvalues.
so you should get a pencil of divisors of degree n+1. Another description: you have to distinguish between the \( \mathbb{P}^1 \) and \( \mathbb{P}^1(W/W) \).

There is a natural map \( \lambda \mapsto \ker (\lambda e^* - A^*) \cong W^n(\lambda' Y) \) which sends real axes to isotropic lines in \( W/W \) degree.

Philosophy: A line in \( W/W \) is a kind of boundary condition to be added in order to obtain a well-defined operator \( A \) having a spectrum, resolvent, etc. When \( W \) is enlarged to the graph of this operator then the resolvent is linked to the way \( A^* \) and \( \lambda Y \) interact.

\[
\begin{align*}
(\lambda Y) + (\lambda Y) &= \{ (\lambda y) \mid \lambda y = \Lambda y \}
\end{align*}
\]

To insert

\[
\begin{array}{c}
Y \xrightarrow{\lambda} Y \\
Y \xrightarrow{-\Lambda + \lambda} Y
\end{array}
\]

\[
\begin{array}{c}
Y \xrightarrow{\lambda} \Lambda Y \\
Y \xrightarrow{(-\Lambda - 1)} Y
\end{array}
\]

\[
\begin{align*}
(\Lambda \lambda)^{-1} &= \begin{pmatrix} 2 \lambda - 1 \\ -\Lambda - 1 \end{pmatrix} \frac{1}{\lambda - \Lambda}
\end{align*}
\]

What are the natural questions? You really are missing the appropriate viewpoint. First question is how to describe the spectrum. For each \( \lambda \) you have this line \( W^n(\lambda') Y \) mapping to \( W^n W \) and the line \( L \to W^n W \). So the spectrum is described
as though \( X \) s.c. this is not transverse i.e. if \( W/W = L \), then \( W^0 \cap (1)Y \to W/Y \) vanishes, so we have a dual section of the sub line bundle

\[
0 \to \ker (\lambda^* - A^*) \to Y \xrightarrow{\lambda^* A^*} X \to 0
\]

degree \(-n\)

so the spectrum is a divisor of degree \( n \).

Approach \( W = (e^1) \times \subseteq Y \), lines in \( W^0/W \)

except for the line \( (e_{n+1}) \subseteq Y \)

are the same as the extensions of \( (e^1) \times \) to \( (\tilde{A})X \)

where \( \tilde{A} : Y \to Y \supseteq \epsilon^* (\tilde{A})Y \)

\( \epsilon^* \tilde{A} = A^* \)

\( \tilde{A} \varepsilon = A \).

In terms of \( \epsilon_1 \in \mathbb{C} \)

J-matrix

\[
\tilde{A} = \begin{pmatrix}
0 & 1 & 0 & \cdots \\
1 & a_1 & a_2 & \cdots \\
0 & a_2 & a_3 & \cdots \\
& \vdots & \ddots & \ddots \\
& & & a_n \end{pmatrix}
\]

So extensions are described by \( b_{n+1} \in \mathbb{C} \), hermitian

\( \tilde{A} \) real.

The interesting point is to extend lines in \( W/W \) in the form \( W^0 \cap (1)Y = \ker \omega (\mu \epsilon^* - A^*) \).

What this means is you will choose \( b_{n+1} \) to be

the coefficient arising from \( \delta (\mu^k) \).

This means \( \tilde{A} u^k = \mu u^k \)

and this is to be \( \mu u^k \)

\[
e_{n+1} (\tilde{A} u^k) = a_n u^k + b_{n+1} u^k
\]

\( a_n u^k + (b_{n+1} - 1) u^k \to 0 \).
So we fix the boundary condition so that $u_i$ is an eigenfunction under
\[ \tilde{A} u_i = i u_i. \]
At the same time,

Now that we have $\tilde{A}$ which is nearly hermitian you will get a spectral representation.

Once you know $\tilde{A} - \tilde{A}^*$ which is essentially the imaginary part of $b_{n+1}$
\[ i = -\frac{a_n u_i}{u_{n+1}}. \]

back to self-positivity. Try to understand the simple harmonic oscillator.

Review: \( W = (E) X \subset Y \)
\( W^0 \cap W \) is 2-dim. \( W^0 = \{(y_1, y_2) \mid \begin{array}{c} (y_1, A x) = (y_2, e^x) \forall x \\ (y_1)_1 = e^x, y_1 = A x \end{array} \}

i.e. \( y_1 = e^x, y_2 = A x \)

Suppose \( e_{n+1} \) is a unit vector gen. \( \ker(e^x) = (e^x) \).

\[ \ker A^* \oplus \ker e^x \quad \text{i.e.} \quad e^x A x = 0 \quad \text{and} \quad A^* e^x = 0 \]

Any line \( L \) in \( W^0 / W \) corresponds to a \((n+1)-\)subspace of \( Y \) to \( \text{cent } W, \) \( L \)

Look at \( \rho_1: V \to Y \) either \( \rho_1 V \subset e^x \)

\[ \ker(\rho_1|V) \text{ is a line in } Y \text{ cent in } W^0 \text{ i.e. } (e^x)^{-1}. \]
or $p_i : V \rightarrow Y$ and then $V$ is graph of $A : Y \rightarrow Y$ must have $\tilde{p}_A \subset W$, i.e., $(p_A, Ax) = (\tilde{A}x, \tilde{y})$, for $x, y$. Everything about $\tilde{A}$ is determined by $\tilde{A}^* e^* n_{n+1} = 0 + i e^* n_{n+1} \in Y$

but you want the de Branges picture, which is based on a specific choice for $\tilde{A}$, namely he uses the line $\text{Im} \{ W^0 (\mu) Y \rightarrow W^0 / W \}$ where $\mu = i$. So $\tilde{p}_A = W \oplus \mathbb{C}(i u, i v)$ $\tilde{A} u, v = cu, v$

which means $\det (\tilde{A} - A)$ has the factor $\tilde{A} - i$. Now $u$ is killed $\mu e^* A^*$. In general

I think I want to use $\tilde{e}^* (\tilde{A} - \tilde{A})^{-1} y$ for the isometric embedding. Need to know about $\tilde{A} - \tilde{A}^*$. $\tilde{e}^* (\tilde{A} - \tilde{A}^*) = A^* - (\tilde{A} \tilde{e})^* = A^* - A^* = 0$, also

$\tilde{e}^* (\tilde{A} - \tilde{A}^*) \tilde{e} = \tilde{A} - \tilde{A} = 0$. Can you relate $\tilde{A}$ to the stuff before

$$\begin{align*}
\varepsilon &= \frac{1}{2} (a + b) \\
A &= \frac{1}{2} (a - b) \\
-\varepsilon^* A^* &= -i b^* \\
\varepsilon^* + A^* &= i b^* \\
(\varepsilon^* + A^*) \varepsilon &= c b^* \frac{1}{2} (a + b) = \frac{1}{2} (1 + b^* a) \\
(\varepsilon^* + A^*) A &= c b^* \frac{1}{2} (a - b) = \frac{1}{2} (1 - b^* a) \\
-\varepsilon (\varepsilon^* + A^*) &= \frac{1}{2} (1 + b^* a) \\
\varepsilon^* + A^* A &= 1
\end{align*}$$
Start with $\mu$ on $\mathbb{R}$ prob. measure, when scalar product on $O[\lambda]$. Restrict to $Y = F_{n+1}$, poly. of degree \leq n. Let $(e_i)_{A} F_{n+1}$ be a reproducing kernel. What can you say about point evaluation.

$$y(\alpha) = (e_\alpha, y) \in Y = F_{n+1}.$$ 

$$\left(e_\alpha, (A - \alpha)x\right) = 0 \quad \forall x$$

so

$$\left(A^* - \bar{\alpha}\right)e_\alpha, x) = 0 \quad \forall x \in F_n$$

Confused.

$$Y = F_{n+1} = C1 + C\lambda + \ldots + C\lambda^n$$

$$X = F_n = C1 + C\lambda + \ldots + C\lambda^n$$

$$y(\alpha) = (e_\alpha, y)$$

defined $e_\alpha \in Y$ for $\alpha \in C$.

Then

$$\left(e_\alpha, (\lambda - \alpha)x\right) = 0$$

so $e_\alpha \perp (\lambda - A)x$

so

$$\left(A^* - \bar{\alpha}\right)e_\alpha = 0$$

Further thoughts:

Let $p_1, \ldots, p_{n+1}$ be the orthogonal polynomials.

Then

$$e_\alpha = \sum_i p_i(\alpha) \cdot p_i$$

so that

$$\left(e_\alpha, y\right) = \sum_i p_i(\alpha) \left(p_i, y\right)$$

$$\left(e_\alpha, p_i\right) = \sum_i p_i(\alpha) \delta_{ij} = p_j(\alpha).$$

$$\int e(\alpha, \lambda) y \, d\mu(\lambda) = y(\alpha)$$

$$\int e(\alpha, \lambda) y(\lambda) \, d\mu(\lambda) = y(\alpha)$$

$$\int e(\alpha, \lambda) \, d\mu(\lambda) = \frac{e(\alpha, \lambda)}{e(\alpha, \lambda)}$$

$$\int e(\alpha, x) \, d\mu(x) = \int y(x)$$

$$\int d\mu(x) e(x, x') e(x', x) = e(x, x)$$
Review. \( W = (\mathbb{A}) X \subseteq W^0 = \{(y_1, y_2) \mid (y_1, A x) = (y_2, \mathbb{A} x) \ \forall x\} \)

Consider a line \( V/W \) in \( W^0/W \) where \( W \subset V \subset W^0 \). Assuming \( V \cap \text{Ker}(\rho_1: W^0 \to Y) = 0 \), then \( \rho_1: V \to Y \) so \( V = (\mathbb{A}) X \{ (\mathbb{A} x) \} \)

where \( \mathbb{A} x = A x \) and \( (y_1, A x) = (\mathbb{A} y_1, \mathbb{A} x) \ \forall x, y \)

Again, \( \mathbb{A}^* \mathbb{A} = A^* \) is an inner product. Note that \( A^* e = A \), so \( (A^* - \mathbb{A}) e = 0 \) \( \Rightarrow \ e^*(A^* - \mathbb{A}) = 0 \). \( \text{Ker}(e^*) \)

\[
W^0 \cap (\mathbb{A}) Y = \{ (y_1, \mathbb{A} x) \mid (y_1, A x) = (\mathbb{A} y_1, \mathbb{A} x) \ \forall x \}
\]

i.e. \( (Ae^* - A^*) y = 0 \)

\[
W^0 \cap (\mathbb{A}) Y = (\mathbb{A}) \text{Ker}(\rho e^* - A^*) = (\mathbb{A}) \text{Ker}(\rho e^* - A^*)
\]

What would you like to do? Couple to a transmission line. Look at Hardy space

\[
H = L^2(\mathbb{R}, \frac{d\omega}{2\pi}) = H^+ \oplus H^-\]

You would like to make \( H^- \oplus Y \oplus H^+ \) a self-adjoint op. combining \( A e^{-1} \) with \( Y \) mult by \( \omega \) on \( H^- \). Work with subspaces \( \mathbb{H}^- \oplus \mathbb{Y} \oplus \mathbb{H}^+ \).

What happens with \( H^+ \)?

\[
W^+ = (\mathbb{A}) D_{\omega} \subset H^+
\]

\[
(w^+) = \{(f_1, f_2) \mid (f_1, \omega f_2) = (f_2, f) \ \forall f, f_2 \}
\]

Keep a trying.

\[
W^+ = (\omega) D_{\omega} \subset \left( H^+ \right)^\frac{1}{2}(L - i) \left( H^+ \right)^{\frac{1}{2}}
\]

\[
\left( \begin{array}{cc}
1 - i & i \\
 1 & i
\end{array} \right) = \left( \begin{array}{cc}
A & i \omega \\
-i & i
\end{array} \right) \left( \begin{array}{cc}
a & b \\
-b & a
\end{array} \right)
\]

\[
\left( \begin{array}{cc}
a & b \\
 b & a
\end{array} \right) = \left( \begin{array}{cc}
 -i & 1-i \\
-1 & i
\end{array} \right) \left( \begin{array}{cc}
 a & b \\
 b & a
\end{array} \right)
\]
\[
\begin{align*}
(a) & \quad x = s + iA \\
b & \quad \xi + iA \\
1 - iw & \quad 1 + iw \\
(1 - iw)Dw & = (1 + iw)Dw
\end{align*}
\]

\[
(1 - iw)Dw = H^+ \\
(1 + iw)Dw = (w - i)Dw
\]

kernel of evaluation at \( w = i \). So \( W^+ \) has a

Review. Take \( (a)X \subset Y \) and \( (b)Dw \subset H^+ \)

\[
(1) Dw = \left( \frac{(1 + w^2)^{1/2}}{w(1 + w^2)^{1/2}} \right) H^+ \subset H^+ \oplus H^+
\]

\[
W = (a)X \subset Y \quad W^0 = W \oplus \text{Ker}(a^t) \oplus \text{Ker}(b^t)
\]

\[
W'' + x = F(t) \quad \text{has solution}
\]

\[
x = \int_{-\infty}^{t} G(t - t')F(t')dt' + \frac{1}{2} \left( \frac{\partial^2}{\partial t} + 1 \right) G(t) = \delta(t)
\]

\[
G(t) = \begin{cases} \sin t & t > 0 \\ 0 & t < 0 \end{cases}
\]

\[
\text{vanishing as } t \to -\infty
\]

and general solution

\[
x = \text{Re}(Ae^{-it}) + \int_{-\infty}^{t} \sin (t - t')F(t')dt'
\]

What happens as \( t \to \infty \)

\[
\int_{-\infty}^{t} \sin (t - t')F(t')dt' + \int_{0}^{t} \text{Re}(-ie^{i(t-t')})F(t)dt' = \int_{0}^{\infty} \text{Re}(-ie^{iu})F(t-u)du
\]
for $t \gg 0$, dry up 11:00, then 30 min

$x(t) = \int_{-\infty}^{\infty} \text{Re}(-ie^{it-t'})F(t')dt'$

$= \text{Re} \left[ -ie^{it} \int_{-\infty}^{\infty} e^{-it'}F(t')dt' \right]$

$H = wa^*a$

$H = \frac{1}{2m}(p^2 + \frac{1}{2} \omega^2 \hat{g}^2)$

$(w^2 - ip)(w^2 + ip)$

$[w^2, \hat{g}] = \omega \hat{g}^2 - i\omega \hat{g}$

$(a^*, a) = \left[ \frac{w^2 + ip}{2k\omega}, \frac{w^2 - ip}{2k\omega} \right] = \frac{i}{2k}\omega$

$\omega \frac{w^2 - ip}{2k\omega}$

$H = \frac{p^2}{2m} + \frac{1}{2} k \omega \hat{g}^2$

$H = \frac{p^2}{2m} + \frac{m}{2a^*} \omega^2 \hat{g}^2$

$= \hbar \omega \left( \frac{ip}{\sqrt{2m\omega}} + \frac{ma^*}{2k\omega} \right) \left( \frac{-ip}{\sqrt{2m\omega}} + \frac{ma^*}{2k\omega} \right) = \hbar \omega (H - \frac{1}{2})$

$[\frac{ip}{\sqrt{2m\omega}}, \frac{ma^*}{2k\omega}] = \frac{1}{2k} \hbar = \frac{1}{2}$

Suppose $H = wa^*a$

$\hat{g} = a + a^*$

$\langle 0 | e^{-itH} \hat{g} | 0 \rangle = \langle 0 | e^{-itw^2a^*a} | 0 \rangle$

$t = -it$

$e^{-i(-it)H} = e^{-itH}$
Review: $W = (a \otimes b) \otimes Y \otimes Y$

$L^2(\mathbb{R}, \frac{dx}{2\pi}) = H^- \oplus H^+$

Handy facts:
- $\varepsilon = \frac{1}{2}(a+b)$
- $A = \frac{1}{2}(a-b)$
- $\varepsilon - iA = a$
- $\varepsilon + iA = b$

To understand:

\[
\begin{align*}
D_\omega & \rightarrow H^+ \oplus (1-i) H^+ \\
& \oplus H^- \rightarrow H^+
\end{align*}
\]

$a = 1-i\omega$

$b = 1+i\omega$

$ba^{-1} = \frac{1+i\omega}{1-i\omega}$

Idea is that $1+i\omega$ vanishes at $\omega = i$ so that $(1+i\omega)D_\omega$ should have codimension 1.

You need $L^2(S^1)$

\[
\begin{align*}
2 & = \frac{1+i\omega}{1-i\omega} = \frac{-\omega + i}{\omega + i} \\
1 & = \int \left| \frac{1}{\omega + i} \right|^2 \frac{d\omega}{2\pi} = \int \frac{d\omega}{1+\omega^2} \\
& = \frac{\arctan \omega}{\pi} \bigg|_{-\infty}^{\infty} = \frac{\pi}{2} = 1
\end{align*}
\]

Go back to:

$H = H^- \oplus Y \oplus H^+$

Let's return to $H = \cdots \oplus u^iV^- \oplus aX \oplus V^+ \oplus uV^+ \oplus \cdots$

... $\oplus u^iV^- \oplus V^- \oplus bX \oplus uV^+$

and try for the hermitian analogues. So what to do next? Do there a simple way to describe $H$?

**Suppose you try to generalize $\gamma^* u^i j = (\gamma^* u^j)^n$ for $n>0$ to the continuous case.**
Look for H with \( u_t = e^{th} \) form.

\[ j^* u_t j = (j^* u_j)^t \quad \text{for } t \geq 0. \]

Maybe it works?

\[
j \sum_{n \in \mathbb{Z}} z^n j^* u_n j = \sum_{n > 0} (z^{-j})^n + \sum_{n < 0} (z^{j})^n
\]

\[
= \frac{1}{1 - z^{-j}} + \frac{z^j}{1 - z^j}
\]

\[
= \frac{1}{1 - z^{-j}} \left( \frac{1 - z^j}{1 - z^j} \right) \frac{1}{1 - z^j}
\]

Analogue is

\[
\int_{-\infty}^{\infty} e^{-j t} j^* u_t j dt
\]

\[
\int_{-\infty}^{0} e^{-j t} e^{\beta t} dt + \int_{0}^{\infty} e^{-j t} e^{-\beta^* t} dt
\]

\[
\left[ \frac{e^{-(\omega + \beta) t}}{-i \omega + \beta} \right]_{-\infty}^{\infty} + \left[ \frac{e^{-(\omega + \beta^*) t}}{-i \omega + \beta^*} \right]_{-\infty}^{\infty}
\]

\[
= \frac{1}{i \omega - \beta^*} - \frac{1}{i \omega + \beta} = \frac{1}{i} \left( \frac{1}{\omega + i \beta} - \frac{1}{\omega - i \beta^*} \right)
\]

\[
= \frac{1}{\omega - i \beta^*} \left( \frac{\omega - i \beta^* - (\omega + i \beta)}{i} \right) \frac{1}{\omega + i \beta} = \frac{1}{\omega - \alpha^*} (-i(\alpha - \omega)) \frac{1}{\omega - \alpha}
\]

Put \( \beta = i \omega \)
Can you see this somehow. The idea is to produce \( \mathcal{A} \) nearly hermitian operator directly from (8).

We expect to find a self adjoint operator \( H \) such that

\[
\frac{d}{dt} e^{itH} = e^{itA} \quad \text{Im}(\lambda) \geq 0
\]

so

\[
\frac{d}{dt} \frac{1}{\omega - H} \psi = \frac{1}{\omega - \lambda}
\]

do what?

\[
W = \mathcal{E} \times \mathcal{A} \quad W^0 = \{(y_1, y_2) \mid (y_1, \mathcal{A}x) = (y_2, x) \quad \forall x\}
\]

and

\[
W^0 \cap (1) = \{ (y, y) \mid y \in \text{Ker}(\lambda e^{-\mathcal{A}^*} - \mathcal{A}) \}
\]

Please note the exception to the unitary picture:

\[
W = (a, b) X \oplus Y
\]

\[
W^0 = \{(y_1, y_2) \mid (y_1, a x) = (y_2, b x) \quad \forall x\}
\]

we have \( a^* y_1 = b^* y_2 \)

given \( (y_1) \in W^0 \) let \( x = a^* y_1 \). Then

\[
(y_1, y_2) - (a^* y, y_2) = (y_1', y_2') \quad \text{where} \quad a^* y_1' = 0
\]

\[
b^* y_2' = 0.
\]

\[\begin{align*}
W^0 &= W \oplus \text{Ker} a^* \oplus \text{Ker} b^* \\
L_2 &= W^0 \cap (1) = \{ (y_1, y_2) \mid a^* y = b^* y \}
\end{align*}\]

0 \rightarrow L_2 \rightarrow Y \xrightarrow{q - b^*} X \rightarrow 0

0 \rightarrow W^0 \rightarrow Y \oplus Y \rightarrow X \rightarrow 0

\[
Y = Y
\]
Consider \( W = \langle x \rangle \) \( x \leq y \), isotropic \( \|ax\|^2 = \|bx\|^2 \) for \( \Re y \). Find \( W^0 = \overline{\text{W} \oplus \text{Ker}(b^*)} \). What is the strategy? Now pick a line in \( W^0/W \). The one you take is \( \text{Ker}(b^*) \). Note that any line in \( W/W \) corresponds to a \( V \), \( W \subset V \subset W^0 \). When is \( V \) a graph? Look at \( p : V \to Y \) Wait. You know that \( W \ominus W = \text{Ker}(a^*) \oplus \text{Ker}(b^*) \).

\[ V \ominus W \text{ is a line } < \text{Ker}(a^*) \ominus \text{Ker}(b^*) > \]

\[ V \ominus W \text{ is a line } < \text{Ker}(a^*) \ominus \text{Ker}(b^*) > \]

\[ \text{So the compact} \]

\[ \text{Let's review the situation in the unitary (U) case.} \]

\[ \text{You have } Y \text{ a } n+1 \text{-dim Hilbert space, an } n \text{-dim v.s. } X \]

\[ \text{and maps } a, b : X \to Y \implies a^* - b : X \to Y \text{ all } z \in \mathbb{C}^\infty \]

\[ ||a(x)|| = ||b(x)|| \text{ all } x. \]

\[ \text{Get } ||x|| \text{ on } X \implies a^* = b^* = 1. \]

\[ H_{ax} = \langle aX \oplus V^+ \oplus uV^+ \oplus V \ominus bX \oplus \rangle \]

\[ y = a^*y + \pi^+ y \]

\[ u^* y = b a^* y + u \pi^+ y \]

\[ = a^* b a^* y + \pi^+ b a^* y + u \pi^+ y \]

\[ u^2 y = a^* (b a^*)^2 y + \pi^+ (b a^*)^2 y + u \pi^+ (b a^*) y + u^2 \pi^+ y \]

\[ y = u^{-N} \{ a^* (b a^*)^N y + u \pi^+ (b a^*)^N y + \ldots \} \]

\[ \text{So move into functions} \]

\[ y \mapsto \pi^+ y + u^1 \pi^+ (b a^*) y + u^2 \pi^+ (b a^*)^2 y + \pi^+ (l - z^{-1} b a^*) y \]
\[ \begin{align*}
(az-b)z^\frac{1}{a^*} + ee^* &= 1 - z^\frac{1}{ba^*} \\
(a^2-b^2)z^\frac{1}{a^*} + ee^* &= 1 - z^\frac{1}{ba^*}
\end{align*} \]

How do you use, organize, these ideas?

You want to deform

\[ \begin{align*}
Y &\xrightarrow{X} \frac{e^*}{e^*} \\
V^+ &\oplus \frac{(az-b)e}{V^-}
\end{align*} \]

How to organize? You might work with

\[ \begin{align*}
W &= (q)X, (q)Y, W^0 = W \oplus (V^+) \\
&\oplus (V^-) \\
\end{align*} \]

Your previous success is based upon the splitting

\[ \begin{align*}
Y &= X \oplus V^+ \quad \text{or} \quad Y = \frac{X}{V^-} \\
\end{align*} \]

You somehow use the

\[ \begin{align*}
W &= W \oplus \text{Ker}(a^*) \\
&\oplus \text{Ker}(b^*) \\
\end{align*} \]

The basic spectral representation arises from the splitting

\[ \begin{align*}
Y &\xrightarrow{X} \frac{e^*}{e^*} \\
V^- &\oplus \frac{(az-b)e}{V^-}
\end{align*} \]

Leads to solution of

\[ (a^2-b)z^- = -y + \hat{y}(z)e \]

\[ \hat{y}(z) = \left( \begin{array}{c}
\frac{e^*}{e^*} \\
\end{array} \right) (1 - z^\frac{1}{ba^*})^{-1} y \]

But then you can prove that

\[ \int |\hat{y}(z)|^2 \frac{d\theta}{2\pi} = \|y\|^2 \]

It's this residue trick you need to understand.
76 better. Yes. How might you prove it. First you might take 2 elements \( y, y' \in Y \) and somehow understand
\[
(y', (1-\bar{z}b^*)(1-e^*(1-\bar{z}b^*)^{-1}y))
\]
trick

\[
\frac{1}{1-\bar{z}y^*} + \frac{zg^*}{1-zg^*} = \frac{1}{1-z^*y^*} (1-\bar{z}^*y^* + zg^* + (1-zg^*)) \frac{1}{1-zg^*}
\]

\[
\frac{1}{1-z^*y^*} (1-zg^*) \frac{1}{1-zg^*} = \frac{A}{1-z^*y^*} +
\]

invariant approach - see next 2 pages.

\( P' = PT \), where \( T \) is 2-plane equipped with pseudoscalar product.

\( \otimes Y \) has pseudoscalar product, canonical sequence

\[
0 \rightarrow \mathcal{O}(-1) \otimes Y \rightarrow \mathcal{O} \otimes T \otimes Y \rightarrow \mathcal{O}(1) \otimes Y \rightarrow 0
\]

\( L^2 \) sections of \( \mathcal{O}(-1) \) over the real \( P' \) should form a Hilbert space in an intrinsic way, hence also \( L^2 \) sections of \( \mathcal{O}(-1) \otimes Y \). So what else happens? You now want to proceed to spectral representation. You need to choose a line in \( W^0/W \) and specify conjugate (or adjoint) line. Corresponds to choose \( W < V < W^0 \) and its annihilator \( V^0 \). Interested in \( V \neq V^0 \).
Invariant approach. $T$ 2 dimensional space with hermitian form of signature 1, -1. $T \otimes Y$ Hilbert spaces. $T \otimes Y$ is kernel space. Have basic exact sequence

$$0 \to \mathcal{O}(-1) \otimes Y \to T \otimes Y \to \mathcal{O}(1) \otimes Y \to 0$$

false $W$ isotropic in $T \otimes Y$ for the pseudo scalar product, get $W^0/W$. How is this $K$-mod?

Example: $T = \mathbb{C}^2$, $||y||^2 = |z_1|^2 - |z_2|^2$

$T \otimes Y = Y \oplus Y^0 \quad W^0 = \{ (y_1, y_2) \mid y_1y_2 = b^*y_2 \}$

$W^0 = \{ (y_1, y_2) \mid a^*y_1 = b^*(y_2) \} \sim K_{\mathbb{C}}(a^* - \overline{z}b^*)$

$W^0(\frac{1}{2})Y = \{ (\frac{y_1}{z_2}, y_2) \mid a^*y_1 = b^*(y_2) \}$

So get $\mathcal{O} \otimes W^0 \to \mathcal{O}(1) \otimes Y$ which should be $\mathcal{O} \otimes W^0 \to \mathcal{O}(1) \otimes Y$ provided $\mathcal{O} \otimes W^0$ and $\mathcal{O}(-1) \otimes Y$ intersect transversally, i.e. $W^0 + (\frac{1}{2})Y = \mathbb{C}^2$. How is this related to $W \cap (\frac{1}{2})Y = 0$?

There should be some relation between $W^0 = \{ (y_1, y_2) \mid a^*y_1 = b^*(y_2) \}$

$W^0(\frac{1}{2})Y = \{ (\frac{y_1}{z_2}, y_2) \mid a^*y_1 = b^*(y_2) \}$

If no bound states

$W \cap (\frac{1}{2})Y = \{ (x_1, x_2) \mid bx = 2ax \}$

Annihilator relation ship

$(W^0 + (\frac{1}{2})Y) = W \cap (\frac{1}{2})Y$

So how do you proceed at this point?

We know that $W^0/W$ has induced pseudoscalar product. Suppose it has anis, we have $O(n)$ case.

Picking a line
\[
(W^0 \cap (1/2)Y)^\circ = W + (1/2^\cdot)Y
\]

\[
\begin{align*}
0 & \rightarrow W \xrightarrow{(g)} Y \xrightarrow{\text{quot}} ? \\
& \quad \downarrow \left( \frac{1}{2^\cdot} \right) \\
& \quad \uparrow \sigma
\end{align*}
\]

What you need to understand? You choose \( V \), \( W \subset V \subset W^0 \| V/W \) line in \( V^+ \). There is a conjugate line \( L^\circ \).

\[
\left( \left( \frac{1}{2} \right) Y \right)^\circ = \left\{ \begin{array}{l}
(y_1, y_2) \\
(y_1, y) = (y_2, Fg) \quad \forall y_2
\end{array} \right.
\]

\[
= \left( F^* \right) Y
\]

In the intrinsic picture, you have chosen \( V \), \( W \subset V \subset W^0 \). This should yield a spectrum namely the complement of those points of \( \text{G C P} \mathbb{T} \) such that \( W \) and \( O(-1) \otimes Y \) are complementary. Somehow you want to associate embed \( Y \) in \( L^2 \) sections on the real circle. If you take the standard picture where \( V/W = \ker a^* \) or \( \ker b^* \) you know what to do. So one way to proceed would
be to try to arrange this by choosing the coordinates. Suppose given \( W \subset T \otimes Y \) isotropic, and \( V, W < V < W^0 \). Choose a polar of \( T = \begin{pmatrix} 0 & k \end{pmatrix} \begin{pmatrix} k & 0 \end{pmatrix} \) so \( W^0 = W \oplus \text{ker} k \). The V intersects to give a line \( L \subset \text{ker} k \). Actually we can restrict the hermitian form to \( V \) so that it vanishes on \( W \) and has a sign \( >0 \) or \( <0 \) on \( V/W \). In fact there's a map to \( V/W \) is a line with scalar product.

If you have \( V \oplus \mathbb{G}_m \otimes Y \to T \otimes Y \) for \( x \) not in the spectrum and then your quotient nice \( V/W \), what equations are you solving?

\[
\begin{pmatrix} 1 \end{pmatrix} Y + \begin{pmatrix} 1 \\ -k \end{pmatrix} Y \to Y
\]

So it seems that I get an element of \( V/W \) for any triple \( (z, y_1, y_2) \) and choose \( z \).

\[
\begin{array}{c}
(z-1) : \begin{pmatrix} 1 \\ 1 \end{pmatrix} Y \to Y \\
Y \to Y
\end{array}
\]

So given \( Y \)

\[
\mathbb{O} \otimes V \to \mathbb{O}(1) \otimes Y
\]

\[
\mathbb{O}(V/W)
\]
You have \( W < V < W^0 < T \otimes Y \) and \( \phi(-1) \otimes Y \rightarrow \phi(T \otimes Y) \rightarrow \phi(1) \otimes 1 \rightarrow 0 \).

The hermitian form on \( T \otimes Y \) restricts to 0 on \( W \) so you get a 1-dim quotient \( V/W \) with pos. def. herm. form. Spectral transform, namely go from \( Y \) to functions on the real \( \mathbb{RP}^1 \subset \mathbb{RP}^T \).

Spectrum = where \( (\frac{1}{z})Y \) and \( V \) are not complementary off spectrum. Assume \( V = (\frac{1}{z})Y \)

\[ \left( \begin{array}{c}
\frac{1}{z} \\
\frac{1}{z}
\end{array} \right) Y + \left( \begin{array}{c}
\frac{1}{z} \\
\frac{1}{z}
\end{array} \right) Y = Y \]

\[ \iff \quad (z - 1) \left( \begin{array}{c}
\frac{1}{z} \\
\frac{1}{z}
\end{array} \right) = (z - 1) : Y \rightarrow Y \quad \text{is an isom.} \]

Anyway

\[ \left( \begin{array}{cc}
1 & 1 \\
1 & 2
\end{array} \right)^{-1} = \left( \begin{array}{cc}
z & -1 \\
-1 & 1
\end{array} \right) (z - 1)^{-1}. \]

At the moment you have a map from \( (\frac{y_1}{y_2}) < Y \) to

\[ \left( \begin{array}{c}
1 \\
1
\end{array} \right) (z - 1) (z - 1)^{-1} \left( \begin{array}{c}
y_1 \\
y_2
\end{array} \right) \]

\[ = \frac{z (2 - \theta)^{-1} y_1 - (2 - \theta)^{-1} y_2}{y_1 - y_2} = (2 - \theta)^{-1} (zy_1 - y_2) \]

\[ \text{This is the element of } V \]

which is to be projected onto \( V/W \).

It's worth looking for a proof that

\[ y \mapsto \pi (2 - \theta)^{-1} y = \hat{y}(\theta) \]

is unitary embedding

\[ \hat{y}(\theta)^* y(\theta) = \hat{y}(\theta)^* (2 - \theta)^{-1} \pi \pi^* (2 - \theta)^{-1} y \]
\[(1-z^*)^{-1} + z^* (1-z^*)^{-1} \]
\[
= (1-z^*)^{-1} (1-z^*) z^* Y + (1-z^*) (1-z^*)^{-1} \]
\[
= (1-z^*)^{-1} (1-z^*) (1-z^*)^{-1} \]

Do there an intrinsic way to do this?

\[ V \subset T \otimes Y \cong \ell_0 \otimes Y \]

So you get an \( \ell_0 \) as a map

\[ Y \rightharpoonup (T/\ell_0)^* \otimes V \rightharpoonup (T/\ell_0)^* \otimes Y \]

You want this for \( \ell_0 \) is real

Roughly \( \tilde{y}(z) = \pi (z-\bar{z})^{-1} y = \pi \tilde{z}^{-1} (1-\bar{z}^*)^{-1} y \)

is analytic \( \mathbb{D} \) on \( \mathbb{D} \) and outside \( \{z = 1\} \).

I guess what's intriguing is inner product between different \( \pi (z-\bar{z})^{-1} y \). In the \( J \)-matrix case what's interesting is the pairing between \( u^a \) and \( \bar{u}^a \). This seems to involve \( W \) and \( W^0 \).

In the \( J \)-matrix case you have \( \langle u^a, \bar{u}^a \rangle = \frac{1}{\mu - \lambda} \)

entire in \( J \)-hermitian form applied to \( \ell_0 \).

Which will vanish when \( \lambda = \mu \) because the lines \( L_1, L_2 \) are orth.

So this leads us to \( \ell_0 \) ignore \( V \) and concentrate on the family of \( L_\lambda \subset W_0/W \)

\[ L_\lambda = W_0 \cap \ell_0 \otimes Y \]

Given two: \( \sim \), \( \sim' \)

\[ (T/\ell_0)^* \otimes Y \rightarrow 0 \]

looks like \( L_0 = O(-m_0) \).
So you have two of these lines $L_0, L_\omega$. Clea

You bring this discussion to an end.

$W = T \otimes Y \supset l_2 \otimes Y$

$W^0 (l_2 \otimes Y) = L_2$ maps into $W^0/W$

Main ideas? From the data $T, Y, W$ you seem to get the 2-dual $W^0/W$ (Kronecker space) and this sub-line bundle $L$ of $O \otimes W^0/W$ over $P^1$ with certain adjointness properties. Let's describe this as well as we can.

$W = \begin{pmatrix} a \\ b \end{pmatrix} X \quad W^0 = W \oplus \text{Ker} a^* \oplus \text{Ker} b^*$

$W^0 (l_2 \otimes Y) \quad (y) = (ax) + (v^+)$

i.e. $z (ax + v^+) = bx + v^-$

$(az - b)x = -2v^+ + v^-$

So the image of $L_2$ in $W^0/W = \text{Ker} (a^x) \oplus \text{Ker} (b^y)$ consists of $(v^+)$ such that $-2v^+ + v^- \in V$.

$s(z) (2v^+) = v^-$

Notice that the degree of $2s(z)$ is $n+1$.

Next go back to $W \subset T \otimes Y \supset l_2 \otimes Y$

$L_2 = W^0 (l_2 \otimes Y) \hookrightarrow W^0/W$. Pencil of hyperplane sections of degree $n+1$. 
Intrinsically you have $L \cong O(n-1)$ embedded in $O(W/w)$. If you take quotient lines of $W/w$ (or lines) then you get sections maps $L \rightarrow O^{+}$ i.e.,

divisors of degree $n+1.

W/w$ has hermitian form so $P(W/w)$ has a real projective line. Now the form on $W/w$ pulls back to the one which is restriction of given herm. form $\Omega$. But

$$\begin{pmatrix} y_1 \\ z_1 \\ \overline{z_1} \\ y_2 \\ z_2 \end{pmatrix} \rightarrow \begin{pmatrix} (1-|z_1|^2)(|y_1|^2) \end{pmatrix}, \quad \Omega \rightarrow L_2 \text{ from } P(T) \to P(W/w),$$

preserves real $PT$ and disks.

Let's start now with $(\sigma)X < X$, equifixed with

$$\begin{pmatrix} (y_1, y_2) \\ (-1,0)(y_1, y_2) \end{pmatrix} = (y_1, y_2) - (y_2, y_1)$$

$W$ isotropic means $(\varepsilon X', A=X) = (AX', \varepsilon X)$

i.e. $A^*\varepsilon = \varepsilon A^*$ (end of scalar product on $X$).

Intrinsic version: T 2-diml Krein, $Y$ $n+1$ diml Hill, $T \otimes Y$ then $2n+2$ diml Krein, $W = \text{h diml intro. in } T \otimes Y$, $W/w$ then 2-diml Krein, this is the port on-terminal. So $\omega \in PT$, $L_{w}$ corresp. line in $T$, assume $W \cap (L_{w} \otimes Y) = 0$

$w$ (no bound states), $L_{w} = L_{\bar{w}}$, $\overline{w} = \text{reflection of } w$ through the real $P'$ given by the $L_{w}$ added null lines for the Krein form, so $W \cap (L_{w} \otimes Y) = 0 \iff W^0 + L_{w} \otimes Y = T \otimes Y$

as $w$ varies $L_{w} = W \cap (L_{w} \otimes Y)$ is a line subbundle of $W$, $0 \rightarrow L_{w} \rightarrow W \rightarrow T/L_{w} \otimes Y \rightarrow 0$, so $\{L_{w}\} \cong \mathcal{O}(-n-1)$. Also $\text{Fund } E_{L_{w}} \rightarrow W/w$ gives an alg. map $PT \rightarrow P(W/w)$, this covered by a line bundle $L \rightarrow \mathcal{O}(-1)$ with that Krein form on $W$ compatible with
Krein forms since the Krein form on $W^0$ descends to $W/W$. $Z$ is the response function. It preserves the null circles and the $+$, $-$ disks, has degree $n+1$.

To get spectral rep for elements of $Y$ choose $V
\begin{align*}
W < V < W^0
\end{align*}
so that $V/W$ is a real line, then $V/W$ is a positive line. Get spectrum of $\omega_1 : V \ni (l_{\omega_{1}}Y) \neq 0$ off the spectrum get $V \ni (l_{\omega_{1}}Y) = 0$, this true for $\omega_1 \neq 0$.

For real Krein form on $V$ is $< 0$. and on $l_{\omega_{1}}Y$ is $\geq 0$.

So split in LHP. Off spectrum we have $V \ni \infty$. and $V \ni V/W$, so we get $V \ni \infty \Rightarrow \infty \ni V \ni V/W$.

Spectral rep... $V/W$ neg line in $W/W$

$V/W$ corr. $b$ pos line. Choose $V \ni l_{\omega_{1}}Y = 0$

for $\text{Im}(\omega_{1}) \geq 0$ because the Krein form on $V-W$ is $\leq 0$

and $\geq 0$ on $l_{\omega_{1}}Y$. Thus $V \ni \infty \Rightarrow \infty \ni Y$ in closed LHP.

What's important, what do I want to emphasize?

Close Krein space $T \otimes H$.

What is the response of a trans line? Here $H$

is infinite dual. A trans line is the sum of a shift and its adjoint. Look at one. For you

Support $Y$ Hilbert space with $\mathcal{S}$ such that $\mathcal{S}^* \mathcal{S} = 1$

and $\text{Ker}(\mathcal{S}^*)$ one dim. OK.

$W = (1)Y \ni Y$. \begin{align*}
W^0 = (\mathcal{S}^*)Y
\end{align*}

Then \begin{align*}
W \ni W^0 = W \oplus (\mathcal{K} \mathcal{S}^*Y)
\end{align*}

\begin{align*}
(y)_{(\mathcal{S}^*)} = (y, \frac{\mathcal{S}^*y}{y})
\end{align*}

$z - \mathcal{S}^*Y \ni 0 \Rightarrow (z \mathcal{S}^* - 1)y = 0$

$z \ni Y$ for $|z| < 1$. 

Spectrum

\[ (1 - z^{-5}) y = 0 \Rightarrow y = 0 \text{ for } |z| < 1 \]

You are confused. You probably should review what happens when \( W = \) ? For a partial unitary \( \begin{pmatrix} a \\ b \end{pmatrix} X < y \) there is a complete picture for \(|z| \neq 1\), namely two spectral representations associated to the contractions \( ba^* \) and \( ab^* \):

\[
W^0 = W \oplus \text{Ker} \, a^* + \text{Ker} \, b^* \cap \begin{pmatrix} 1 \\ z \end{pmatrix} y = (a x) + (b x) + (u^+) + (u^-) = \begin{pmatrix} y \\ z y \end{pmatrix}
\]

\[
z(a x + u^+) = (b x + u^-)
\]

\[
\langle a z - b \rangle x = -z u^+ + u^-
\]

\[ |z| < 1: \quad u^+ = (1 - a b^*) (1 - z a b^*)^{-1} u^+ \]

\[ |z| > 1: \quad z u^+ = (1 - a q^*) (1 - z^{-1} b a^*)^{-1} u^- \]

Suppose \( \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) with \( s^* s = 1 \).

\[
W^0 = \begin{pmatrix} 1 \\ s \end{pmatrix} y + \begin{pmatrix} 0 \\ \text{Ker}(s^*) \end{pmatrix}
\]

ab^* = s^*

ba^* = s

In general you find the response is a map \( V^+ \to V^- \) for \(|z| < 1\). And a map \( V^- \to V^+ \) for \(|z| > 1\).

So in the shift case the response line is \( (0) x \) for \(|z| < 1\), and \( (0) x \) for \(|z| > 1\).

You need to make sense of transam line theory...
Truncation line is direct sum of in and out.

\[ V^+ \oplus u V^+ \oplus i V^+ \oplus \ldots \]

\[ V^- \oplus u^{-1} V^- \oplus u^{-2} V^- \oplus \ldots \]

Intrinsic picture of \( \text{alteration} \) is a \( W \subset T \otimes Y \) such that \( W/W \) is one dimensional, sign of kernel form or \( W/W \) gives in or out type.

To find out polarize \( T \):

\[ W = (a) x \subset \gamma \supset (1) Y = k_x \otimes Y. \]

If \( \dim 1 \), either \( \ker a^x \) or \( \ker b^x \), so if \( \dim 1 \), either \( \ker a^x \) or \( \ker b^x \), so \( \ker b^x \) is either \( \ker a^x \) or \( \ker b^x \), so either \( \ker a^x \) or \( \ker b^x \) is zero.

Assume sign = - on \( W/W \), so \( W/W = \ker b^x \). Then \( a^x \) is a rim, so \( W = (1) Y C \) where \( g^x Y = 1 \) and \( \ker a^x \) is 1.

\[ W^0 = (g^x) Y. \]

Ask now about response. When is \( W^0 + (1) Y = Y \)?

If \( \psi \overset{(g^x)}{\sim} W^0 \subset (2^0 - 1) Y \), then \( Y \) is an rim, i.e., \( 1 - g^x \) is invertible. True for \( |z| < 1 \), get spectral embedding

\[ \psi \overset{(g^x)(1 - g^x)^{-1}}{\sim} W^0 \rightarrow W^0/W = \ker a^x \]

that is \( g^{-1} \rightarrow (1 - g^x)(1 - g^x)^{-1} g \)

Problem: When you discussed response you first asked \( W \supset (1) Y = 0 \) and then found varying line \( L \) in \( W^0 \) whose image in \( W^0/W \) gives the response function. Spectrum not discussed until \( V \) chosen. Here there are only two choices for \( V \), namely \( V = W^0 \) or \( V = W \).

Properties of \( 1 - g^x \) for \( |z| > 1 \)? Does it have kernel?

\[ g^x(a_0, a_1, \ldots) = (a_1, a_2, \ldots) \] \[ = z^{-\infty} a_0, \quad a_2 = z^{-1} a_1, \ldots \]

and this sequence is in \( l^2 \). So spectrum for \( V = W^0 \) is the closed disk \( \{ |z| > 1 \} \).
Now take \( V = W = (\frac{1}{y}) Y < \frac{Y}{y} \)

\[
Y \vdash (\frac{1}{y}) Y \leq \frac{Y}{y} \quad \text{is invertible for } |y| > 1.
\]

But there is no line to project \((\frac{1}{y}) (z - 1)^{-1} Y\) into.

Summarize. Considering \( W = (\frac{1}{y}) Y \leq \frac{Y}{y} \), \( W^* = (\frac{1}{y}) Y \)

where \( g^* g = 1 \), \( \ker g^* \dim 1 \). First study the response, i.e. the intersection \( L_2 = W \cap (\frac{1}{y}) Y \).

\[
L_2 = (g^*)^* \ker (1 - g^* g : Y \rightarrow Y).
\]

Case 1. \( |\frac{1}{y}| > 1 \). In this case \( L_2 \) has \( \dim 1 \)

for all \( |\frac{1}{y}| > 1 \) including \( \infty \), and the line \( L_2 \) projects onto \( W^0 / W \).

Case 2. \( |\frac{1}{y}| < 1 \). In this case \( L_2 = 0 \)

Response function for a transmission line. is the dipole or polarized Klein space.

\[
\begin{align*}
\mathbb{V}^+ & \oplus V^- & \oplus V^+ & \oplus u V^+ \\
\begin{align*}
|y|^2 \\
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\mathbb{V}^+ & \oplus V^- & \oplus V^+ & \oplus u V^+ \\
\begin{align*}
-|y_2|^2 \\
\end{align*}
\end{align*}
\]

\( W \) is the graph of the arrows so that \((ax)^+ = V^- \uparrow (V^-)

\( bX, bX^+ = V^+ \downarrow (V^+) \) (observe signs are wrong)

\[
W^0 / W = (\phi^-) \quad L_2 = (\frac{y}{2y}) \in W^0. \quad \text{Suppose } |\frac{1}{y}| < 1
\]

start with \( e V^+ \), then you get

\[
\begin{pmatrix}
\varepsilon^1 + \varepsilon^2 u_1^2 + \varepsilon^3 u_2^2 + \cdots \\
\varepsilon + \varepsilon^2 u_1^2 + \varepsilon^3 u_2^2 + \cdots
\end{pmatrix} \in L_2
\]

provided \( |\frac{1}{y}| > 1 \). And a similar element starting from
\[ \eta \in V - \varepsilon \eta + \varepsilon^2 \eta + \varepsilon^3 \eta \in L^2 \text{ provided } |\varepsilon| < 1. \]

These only possible for \( L^2 \) \( (|\varepsilon| \neq 1, \text{ image of former is } V^- \oplus V^+ \text{ in } W^0/W \) and image of the latter is \( V^- \oplus V^+ \text{ so the response function is} \]

\[
Z_\varepsilon = \begin{cases} 
V^+ & \text{for } |\varepsilon| > 1 \\
V^- & \text{for } |\varepsilon| < 1.
\end{cases}
\]

Next couple a transmission line to a 1-port of type \( C(n) \). You do this by means of an isomorphism between the terminals. Actually you take the direct sum of the \( V \)-Hilbert spaces and the direct sum of the \( W \)'s with a sign change on the Krein form. Then you need a maximal isotropic subspace of \( W^0/W \oplus W^0/W \), so there should be degenerate couplings. The dimensions are funny NO & Krein is not \( U(2,2) \) dim 4, Lagrangian subspace describ by unitaries \( U(2) \) dual.

Recall the situation. The problem is to understand coupling a partial unitary to a transmission line. The result is a unitary operator, only thing we can ask is the spectral measure arising from a convenient cyclic vector.
Review. When you couple a 1-port to a transmission line you obtain a Hilbert space and unitary operator. There are two cycloctic vectors and a less obvious one: form the set. The obvious ones are $V^+$, $V^-$ associated cycloctic measure is $\frac{d\theta}{2\pi}$. These are related by $S(\theta)$; factoring $S = p/q$ leads to less obvious ones. Zeros of $q$ are outside $S'$.

Question: what is the Branges function when this coupling can be understood in terms of the response functions. Given two 1-ports if you couple them the spectrum is given by appropriate difference of the response functions.

$$E = \mathcal{L}(\omega)I \quad I = \mathcal{C}(\omega)E$$

You want to bring in power

$$P = EI$$

$$\int EI \, dt = \int \mathcal{L}II \, dt = \frac{1}{2} L I^2 + \text{const}$$

$$\int EI \, dt = \int E^2 \, dt = \frac{1}{2} C E^2$$

You would like to take 2-dim space of $E(I)$, allow $E, I$ to be complex; define hermitian form $\langle E | I \rangle$.

There has to be an intelligent way to handle this somehow. You want to fit the situation into a TOY somehow.
Consider $E = \mathbb{I} \odot I$ and $E = L(-i\omega)I$

power is? You have $\mathbb{T}$ equipped with hermitian form. You would like to associate a 2-dim space to each edge.

Have basic 2-plane of $(E_I)$ and for is the line $(E_I(-i\omega))$, hermitian form.

\[
\begin{pmatrix}
(E_1) \\
(E_2)
\end{pmatrix}
\begin{pmatrix}
0 & i \\
-i & 0
\end{pmatrix}
\begin{pmatrix}
(E_2) \\
(E_1)
\end{pmatrix} = i(E_1 I_2 - I_1 E_2)
\]

Maybe you need Kähler stuff.

What you do in the real Lagrangian case for an LC circuit. Basic space is $\mathbb{C} \otimes \mathbb{C}$, with hyperbolic skew form. This is the sum of hyperbolic planes for each edge. Skew form is

\[
\left< \begin{pmatrix}
(E_1) \\
(E_2)
\end{pmatrix} \right| = E_1 I_2 - E_1 E_2
\]

and any line is of course isotropic. Now for frequency $\omega$ you want the line $(L \omega)$.

What's the relation between the skew form and the hermitian form?

Skew form $(x_1')(-1)(x_1)$

\[
\frac{1}{2} \left( \begin{array}{c}
\bar{z}_1' \\
\bar{z}_2'
\end{array} \right) \left( \begin{array}{cc}
-1 & 1 \\
1 & 1
\end{array} \right) = \bar{z}_1' \bar{z}_2 - \bar{z}_2' \bar{z}_1
\]

TextNode 1

Of $z' = z$

$$\bar{z}_1' \bar{z}_2 - \bar{z}_2' \bar{z}_1 = 2 \Re \left( \bar{z}_1 \frac{1}{2} \bar{z}_2 \right)$$ seems to be type(4,4).
Try harder. First point is that a symplectic space when complexified is naturally a Krein space. Why? Equivalence between hermitian and skew herm. forms. Take $\mathbb{R}^2$ and a skew form $\omega \left( \left( \begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array} \right), \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \right) = \left( \begin{array}{c} x_1 \\ x_2 \\ y_1 \\ y_2 \end{array} \right), \left( \begin{array}{c} \bar{x}_1 \\ \bar{x}_2 \\ \bar{y}_1 \\ \bar{y}_2 \end{array} \right) \right) \dagger \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = a \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \text{ a real.}

Extend sesquilinear to the complexification

$\omega \left( \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right), \left( \begin{array}{c} y_1 \\ y_2 \end{array} \right) \right) = a \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \text{ skew herm.}

If you multi. by $i$ then $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Define the problem is to fit LC circuits into your abstract framework. $T$, should be specified. $\mathbb{R}^2$ with volume $\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = x^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, complexified becomes $\mathbb{C}^2$ with $\begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}$.

You want $\frac{1}{i} \begin{pmatrix} 1 \\ i \end{pmatrix} \omega = \frac{\omega - \bar{\omega}}{i} > 0$ for $\omega \in \mathbb{H}$. In the case of an inductance, you have $\begin{pmatrix} E_1 & E_2 \\ I_1 & I_2 \end{pmatrix}$, hence herm. form $\frac{1}{i} \begin{pmatrix} E_1 \\ I_1 \\ E_2 \\ I_2 \end{pmatrix}$ on $\mathbb{C}^2$.

$L \omega = \left( \begin{array}{c} L(-i \omega) \\ 0 \end{array} \right) \begin{pmatrix} \begin{pmatrix} L + (i \omega) \\ -L(i \omega) \end{pmatrix} \end{pmatrix} = L \begin{pmatrix} \hat{E} \ \hat{I} \end{pmatrix} \begin{pmatrix} \hat{E} \\ \hat{I} \end{pmatrix}$.
\[ L_\omega = \begin{pmatrix} 1 \\ C(\mathrm{i} \omega) C(-\mathrm{i} \omega) \end{pmatrix} = C \begin{pmatrix} \omega - \mathrm{i} \omega \\ \omega + \mathrm{i} \omega \end{pmatrix} = -C \mathrm{i} (\omega + \bar{\omega}) \]

Try the line
\[ L_\mathrm{s} = \begin{pmatrix} L_\mathrm{S} \\ 1 \end{pmatrix} C \begin{vmatrix} L_\mathrm{S} & L_\mathrm{S} \\ 1 & 1 \end{vmatrix} = L(\bar{\mathrm{s}} - \mathrm{s}) \]

\[ L_\mathrm{s} = \begin{pmatrix} 1 \\ C_\mathrm{s} \end{pmatrix} C \begin{vmatrix} 1 \\ C_\mathrm{s} \end{vmatrix} = C(\bar{\mathrm{s}} - \mathrm{s}) \]

This is a pain. How do I proceed to organize this? Concentrate on what you have as \( C^1 \oplus C^1 \).

Pairing between factors, important is for each \( \bar{s} \) a subspace \( N_\mathrm{s} \subset C^1 \). There is an \( L \)-part.

This is the direct sum of \( L \cdot C \) situations
\[ N_\mathrm{s}^{\text{ind}} = \begin{pmatrix} L_\mathrm{s} \\ 1 \end{pmatrix} C \quad N_\mathrm{s}^{\text{cop}} = \begin{pmatrix} 1 \\ C_\mathrm{s} \end{pmatrix} C^1 \oplus C^1 \]

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = i \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = \omega - \bar{\omega} \]

\[ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \sqrt{x_1 y_2 + x_2 y_1} \quad 1\bar{s} + \bar{s} = \bar{s} + \bar{s} \]

Now you have to divide.

Now you understand \( T \).

It seems that I need another ingredient \( T \) is fixed; say \( T = C^2 \) with \( \begin{pmatrix} x_1 & 0 & 1 \\ x_2 & 0 & i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \).
$l_5 = (1) C = l_5 \circ = \{ (y_1) \mid (1) \times (0) (y_1) = 0 \}$

$y_2 + 5y_1 = 0$

$y_2 = -5y_1$

So what do you get? So what do you get? There is a difficulty here. It seems that all we get in the case $\text{dim}(Y) = 1$ is $\pi$?

What do you need or want?

LC correct to what extent is it a harmonic oscillator - it should be except for $\omega = 0, \infty$. Discuss modes of the homogeneous system. Have real space $C^1 \oplus C_1$ by symplectic structure given by natural pairing $C^1 \times C_1 \rightarrow R$. (determined up to sign)

$L_5$ have impedance subspace $F_5 \subset C^1 \oplus C_1$

Make quadratic forms so $W = SC^1 \oplus V^1$.

$F_5$ is Lagrangian. Another Lag. subspace consists of $s \neq W$ not transverse to $F_5$. But you need $s$ complex. To basically you need to complexify phase space.

Try again. Basic is

$\left\{ \Gamma_5 \right\}$ subspace bundle of $C^1 \oplus V$

\[ \text{Analogous to Hermitian forms: equivalent on a } C \oplus V \text{.} \]

Hermitian bilinear form $H(x,y)$

Real symmetric $S(x,y)$ on underlying real vs. $\sigma$ $S(x,y) = S(y,x)$

Real skew-symmetric $A(x,y)$

Real quadratic form $Q(x)$

$Q(ix) = Q(x)$

$Q(ix) = \bar{Q}(x)$

$H(x,y) = S(x,y) + iA(x,y)$

Real + imaginary parts.
\[ S(x, y) + iA(x, y) = i(S(x, y) + iA(x, y)) \]

\[ A(x, iy) = S(x, y) \]

\[ A(y, ix) = -A(x, y) = -A(x, iy) = A(x, y) \]

A skew \( \iff \) S symm.

\[ Q(x) = S(x, x) = A(x, ix) \]

\[ S(x, y) \]

"H(x, y)"

Suppose \( V \) is the complexification of \( V_r \).

Choose basis: \( V = \mathbb{C}^n \) \( \mathfrak{a} = \mathbb{R}^n \). A hermitian bilinear \( H(x, y) \)

Same as herm. matrix which splits into a real symm. matrix + i times skew symm. matrix.

Point is that \( H(x, y) \) is determined by sesquilinearity to \( x, y \in V_n \) and then \( H(x, y) = \frac{S(x, y)}{\text{real symm.}} + i \frac{A(x, y)}{\text{real skew symm.}} \)

\[ H(x, y) + H(x, y)^\dagger = 2H(x, y) \]

\[ H^\dagger(x, y) + H(y, x) = 2H(y, x) \]

\[ H(x, y) + H(y, x)^\dagger = 2H(y, x) \]

\[ H(x, y) + H(y, x) = 2H(y, x) \]

\[ S = 0 \iff H(x, x) = S(x, x) = 0 \ \forall x \in V_r \]

Thus equivalence between skew symm. forms on \( V_r \) and herm. forms on \( V_c \) such that \( V_n \) is isotropic.

Back to LC circuit. You have \( C_1 \oplus C_1 \) = phase space real with a symplectic form (up to \( \pm \)). Complexification then has natural K"ahler form.

Notice that any real subspace isotropic for \( A \) is a isotropic for \( H \) since \( S(x, x) = 0 \) for \( x \in V_r \).
You have to look at $V_n = \{(E) \in \mathbb{R}^{22}\}$ skew form is

$$(E_1)(D)(E_2) = E_1 I_2 - I_1 E_2 = \begin{vmatrix} E_1 & E_2 \\ I_1 & I_2 \end{vmatrix}$$

The corresponding hermitian form should be

$$(E_1)^* (-i 0)(E_2) = i \begin{vmatrix} \bar{E}_1 & E_2 \\ \bar{I}_1 & I_2 \end{vmatrix}$$

when $(E) = \begin{pmatrix} E_x \\ I_x \end{pmatrix}$ d = 3, 2.

$$i \left( \bar{E} I - \bar{I} E \right) = 2 \text{Im} (\bar{I} E)$$

Do what next? Impedance $\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} Ls \\ I \end{pmatrix}$

$$i \begin{vmatrix} Ls \bar{I} & Ls I \\ I & I \end{vmatrix} = i L (\bar{s}-s)|I|^2 = 2 \text{Im}(s) L |I|^2$$

$$i \begin{vmatrix} \bar{E} & \bar{E} \\ C \bar{E} & C \bar{E} \end{vmatrix} = i C |E|^2 (\bar{s}-s) = C |E|^2 (-2 \text{Im}s)$$

Think real $i L \frac{\bar{I}}{I} = 0$ and before the symplectic reduction.

$E_L = \omega L \frac{\bar{I}}{I}$

$CE_C = \frac{\bar{I}}{I}$

$L = \frac{\bar{I}}{I}$ and $E_L = -E_C$
Try to understand what you can. Show the eigenvalues are purely imaginary. Involves real spaces and it involves Siegel UHP with positive real part. Consider symmetric Lagrangian subspace.

\[ 0 \to \mathbb{C}^\circ \to \mathbb{C}' \to \mathbb{C}_1 \to 0 \]

\[ 0 \to \mathbb{C}/\mathbb{Z}_1 \to \mathbb{C}_1 \to \mathbb{Z}_1 \to 0 \]

When \( W = \mathbb{C}^\circ + \mathbb{Z}_1 \), trans. to \( \mathbb{F}_N^\circ \).

\[ \frac{\mathbb{C}^\circ}{\mathbb{Z}_1} \cap \left( \frac{1}{\mathcal{N}_s} \right) \mathbb{C}^1 \subseteq \mathbb{W}_N \]

The intersection is \( \{ w \in \mathbb{C}^\circ \mid \mathcal{N}_s w \in \mathbb{Z}_1 \} \). What argument to give that this can't happen unless \( \text{Re}(s) = 0 \). The argument is by self-pairing. You take.

In this situation, you have for \( x \in \mathbb{C}^1 \) say \( x = (x_C, x_L) \) that

\[ \left( \begin{array}{c} x_L^* \\ x_C \end{array} \right) \mathcal{N}_s \left( \begin{array}{c} x_L \\ x_C \end{array} \right) = x_L^* \frac{1}{L_s} x_L + x_C^* C_s x_C \]
Example. Let $V$ be a complex vector space, let $V^\dagger = \text{anti dual} = \text{dual with opposite complex structure}$, have pairing $V^\dagger \otimes V \rightarrow \mathbb{C}$ which is sesquilinear.

\[ \langle t^* | v \rangle \] 
\[ t \in V^\dagger, \ v \in V. \]

So make hermitian symmetric so on $V^\dagger \oplus V$ you have a sesquilinear form, namely $(t_1, v_1), (t_2, v_2) \mapsto \langle t_1, v_2 \rangle$ which you can symmetrize in forms.

\[ H((t_1, v_1), (t_2, v_2)) = \langle t_1, v_2 \rangle + \langle t_2, v_1^* \rangle \]

linear in $t_2$ anti linear in $v_1^*$

Given $V, W$ $\mathbb{C}$-vector spaces, let $F(v, w)$ be sesquilinear: linear in $w$, anti-linear in $v$, e.g. $V = \mathbb{C}^n$, $W = \mathbb{C}^m$, $x \in \mathbb{C}^{m \times n}$ matrix

\[ F(v, w) = v^* x w. \]

Let another sesq. form $G(w, v) = \overline{F(v, w)} = w^* x^* v$ and then

\[ H((w_1, v_1), (w_2, v_2)) = F(v_1, w_2) + G(w_1, v_2) \]

\[ = v_1^* x w_2 + w_1^* x^* v_1 = (v_1^* x^* v_1) (0 \ x \ 0) (v_1^* x^* v_1) \]

so there is a natural analogue if you push a Hilbert space structure, then

so it seems that there is a Kreinian analogue of symplectic.

Back to LC circuit. Begin with space of $(E) \in \mathbb{C}^2$ and the Hermitian bilinear form

\[ (E_1) (0 \ I) (E_2) = E_1 I_2 + I_1 E_2 = 2 Re (E_1 I_1) \]

\[ E_2 = E_1, I_2 > I_1. \]
\[
\begin{pmatrix}
L^* & 0 & 1 \\
1 & 0 & 1 \\
C_s & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
L & L^* & L
\end{pmatrix}
= L (s + s^*) = L (2 \text{Im} s)
\leq 1 = 9.35 \text{PP}
\]

Okay, let's check. You have the above standard hermitian form on $C^1 \oplus C_1$, namely $E_1 + iE = 2 \text{Re}(EI)$, with
\[
\begin{pmatrix}
E_1^* & 0 & 1 \\
I_1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
E_2 \\
I_2
\end{pmatrix}.
\]

Next have subspace $Z^* \subset C_1$ which is isotropic, and the annihilator is $sC_1 + C_1$, so there should be no problem with $sC_1 \oplus Z$, being maximal isotropic.

Summary: You have done some progress toward linking LC circuits to the invariant version of unitaries.

Consider LC network. Up to now you have studied the "configuration space" viewpoint, namely, you pick say the voltage space $C^1$ as config. space. you have a quadratic (hermitian) form on $C^1$ depending on $s$?

Let's get this straight. Start with a real situation and then complexify. Real situation is a vector space $D$, the dual space $D^*$, a quadratic form $Q_s$ on $D$ depending on $s$, a subspace of $D$ spectrum of those $s$ such that $Q_s$ is nondegenerate.

LC network. You are used to a "configuration space" version.
You have a real voltage function space \( C_1 \) and dual current function space \( C_1^* \). The impedance of the edges yields a map \( \phi: C_1 \to C_1^* \) for real \( s \). The direct sum of types is \( \phi = (Ls)_{C_1} \oplus (C_s)_{C_1^*} \).

Your configuration space is a real space together with a quadratic form \( Q_s(E) \) which is the direct sum of your configuration space is a real space.

Check: \( (E, I) = (E, Q_s(E)) = s^{-1}(E, L^tE) + s(E, C_E) \).

So you have a real vector space split into \( V_L \oplus V_C \) and \( Q_s = s^{-1}(E, L^tE) + s(E, C_E) \) for \( Q \).

Next we have a subquotient corresponding mode potential part of \( V \) - restrict to conservative, ordered potential supported voltage functions, and divide by \( \pi \)-potential supported on the \( \pi \)-vertices. Look at the induced quadratic form \( \tilde{Q} \) on subquotient.

Real Quadratic form version.

Complex hermitian version: Exactly the same, namely \( V \) is a complex vector space split into \( V_L \oplus V_C \) with hermitian forms on \( V \) and \( C \) on \( V \).

You want to organize, merge two themes:

- Analysis of partial unitaries - here you encounter isotropic subspaces in a Krein space. In this theory there is a Hilbert space \( Y \) around LC networks. Somehow this is adapted a symplectic or Krein viewpoint.

You need to double - hermitian forms become isotropic subspaces.
Complify an LC network. Originally \( E, I \) are real functions of \( t \).

Given \( E(t), I(t) \) with compact support, then

\[
\int E(t) I(t) \, dt = \text{power into the edge}
\]

\[
\int_{-\infty}^{\infty} E(\omega) I(-\omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{I(\omega)} E(\omega) \, d\omega
\]

I guess my point is that complex phase space is \( C^1 \oplus C^{-1} \) and it carries a natural hermitian inner, hyperbolic type; also skew-hermitian multiplicity by \( i \).

The impedance of each edge yields a subbundle \( N_0 \subset C^1 \oplus C_1 \) direct sum of either \((1, 0)\) or \((0, 1)\) for \( \Re(s) = 0 \) this subspace should be isotropic.

The quotient bundle \( s \rightarrow C^1 \oplus C_1 / N_0 \) is holm. + pure of type \( \mathbb{C}(1) \), so we can identify \( C^1 \oplus C_1 \) with the space of holm. section. If we split \( C^1 \oplus C_1 \) \( \text{inter} \) should get a canonical isomorphism

\[ T \otimes Y \rightarrow C^1 \oplus C_1 \]

Okay

Tolip polarized Hilbert space \( U_+ \oplus U_- \)

Maybe all that's involved is changes
to take \( Y \) to be a Krein space and
then \( \text{Tensor product} \) should have the tensor product.
You need some improvement.

Begin with a complex vector space $\Omega$ and direct sum $D = \Omega \oplus \Omega^*$, where $\Omega^*$ is the anti-dual, so we have a sesquilinear pairing $\Omega \otimes_{\mathbb{R}} \Omega \rightarrow \mathbb{C}$. Define hermitian form on $D$ by

$$H((\lambda, \lambda'), (\lambda, \lambda')) = -\langle \lambda, \lambda' \rangle + \langle \lambda, \lambda' \rangle = 0$$

Consider a graph $(\frac{1}{T}) \Omega \subset D$. When is this isotropic?

$$H((\frac{tx}{T}, \frac{tx'}{T}), (\frac{tx}{T}, \frac{tx'}{T})) = -\langle T_{x', x} \rangle + \langle T_{x', x} \rangle = 0$$

means $T : \Omega \rightarrow \Omega^*$

$$T^* : \Omega^* \rightarrow \Omega^*$$

defined by

$$\langle T^* x, x' \rangle = \langle T x', x \rangle$$

means $T$ is hermitian, i.e., $\langle T_{x', x} \rangle$ herm. symmetric in $x' x$.

$\Omega$ complex v.s. $\Omega^*$ anti-dual, a map $T : \Omega \rightarrow \Omega^*$ is equivalent to a sesquilinear form $H(\lambda, \lambda') = \langle T_{x', x} \rangle$

$T : \Omega \rightarrow \Omega^*$ same as $T : \Omega \rightarrow \Omega^*$ $T(cx) = cT(x)$

$T^* : \Omega^* \rightarrow \Omega^*$

Assume $\Omega$ is a Hilbert space so that one has a canonical inner product $\Omega \rightarrow \Omega^*$. Then $\Omega \oplus \Omega^* = \Omega \oplus \Omega$ equipped with the skew herm. form $(x_1^\ast)(-I_0)(y_1^\ast) - I_0(x_2^\ast)(y_2^\ast) = \langle y_1^\ast T x - y_2^\ast x, (T^\ast y_1^\ast y_2^\ast) \rangle = 0$ for

$$\Rightarrow y_2 = T^\ast y_1$$
Hilbert space \( \Omega = \Omega^+ \oplus \Omega^- \) and we equip it with the hermitian operator

\[
\begin{pmatrix}
\omega & 0 \\
0 & \omega^{-1}
\end{pmatrix} = \omega \pi_+ - \omega^{-1} \pi_-
\]

Then for each \( \omega \in \mathbb{P}^1 \), you have a subspace of \( \Omega \), namely the graph of this operator which is isotropic with the canonical skew-holomorphic form when \( \omega \) is real. The point to make perhaps is that you get a holomorphic subbundle over \( \mathbb{P}^1 \) of \( \mathcal{O}(\Sigma \oplus \Sigma) \).

Know that this holomorphic subbundle is pure of type \( \mathcal{O}(-1) \). Things you already know:

\[
0 \to \Gamma_\omega \to \mathcal{O} \otimes \Sigma \to \mathcal{Q}_\omega \to 0
\]

\[
\mathcal{Q} \otimes \Sigma^2 \text{ canon. isom. to } \Gamma^\ast (\mathbb{P}^1, \mathcal{Q}) .
\]

**Review.** Start with a Hilbert space \( X \), form double \( X \) with hermitian form \((x_1^\ast)(0\ 1)(x_1')\ x_2^\ast(-1\ 0)(x_2')\)

If \( T : X \to X \) is linear then

\[
( (1)X )^0 = \{ (x_2') | (x_2^\ast)(-1\ 0)(x_2') = 0 \}
\]

\[
x_2^\ast x_2' = (Tx)^\ast x_2' \quad \forall x
\]

\[
x_2' = Tx_1'
\]

So that \((1)X \) is isotropic \( \iff T = T^\ast\).

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} = \begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]
Consider the double \( X \oplus X \). Then \((\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) X \). Then \((\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) X \) preserves the skew-hermitian form:

\[
\begin{pmatrix} x_1^* & x_2^* \\ -x_2^* & -x_1^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x_1)
\]

and it carries \((\begin{bmatrix} 1 \\ 0 \end{bmatrix}) X\) into \((\begin{bmatrix} 1 \\ 0 \end{bmatrix}) X\).

So it seems that

\[
(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}) \colon X \quad \longrightarrow \quad X
\]

is an automorphism of the skew-hermitian form and it carries \((\begin{bmatrix} 1 \\ 0 \end{bmatrix}) X\) into \((\begin{bmatrix} 1 \\ -1 \end{bmatrix}) X = (\begin{bmatrix} 1 \\ -1 \end{bmatrix}) X\). This tells me how to treat an LC circuit in the framework of \( T \otimes Y \) where \( T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) is a standard skew-hermitian form.

\( l = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}) C \) and \( Y \) is a Hilbert space.

\[
(\begin{bmatrix} E_1 \\ I_1 \end{bmatrix}) \in \mathbb{C}^2
\]

The skew-hermitian form is

\[
(\begin{bmatrix} E_1 \\ I_1 \end{bmatrix}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\begin{bmatrix} E_2 \\ I_2 \end{bmatrix})
\]

Impedance line is

\[
(\begin{bmatrix} 1 \\ C \omega \end{bmatrix}) C
\]

\[
= \begin{bmatrix} E_1 & E_2 \\ I_1 & I_2 \end{bmatrix}
\]

Still not clear whether if I restrict to real frequencies, what's the problem. Maybe you should use time evolution.

\[
(\begin{bmatrix} 1 & 0 \\ I & 0 \end{bmatrix}) = (\begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix})(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = 5 + \bar{5}
\]

In the end you get \( H(0, \omega) = 2 \Re(\omega) \).
Consider the behavior described by \( E(t), I(t) \) satisfying \( I(t) = CE(t) \). Power is \( E(t)I(t) = C\dot{E}\dot{E} = \frac{1}{2} \dot{E}^2 \) so \( \int_{-\infty}^{\infty} E(t)I(t) dt = \frac{1}{2} \int_{-\infty}^{\infty} \dot{E}^2 dt \).

So the energy going in between times \( a \) and \( b \) is.

Perhaps you should think of \( E(t), I(t) \) as \( t < 0 \) and of exponential growth at \( t \to +\infty \) which is appropriate for LT.

Frequency analysis.

For \( I \) where \( E(\omega), I(\omega) \) are complex amplitudes satisfying \( I(\omega) = C(-i\omega)E(\omega) \). Power generally is

\[
\int_{-\infty}^{\infty} E(t)I(t) dt = \int_{-\infty}^{\infty} E(\omega)I(-\omega) \frac{d\omega}{2\pi}
\]

\[
= \int_{0}^{\infty} \frac{d\omega}{\pi} \left( E(\omega)I(-\omega) + E(-\omega)I(\omega) \right) \text{Re}(E(\omega)I(\omega)).
\]

For \( I(\omega) = C(-i\omega)E(\omega) \), \( \text{Re}(E(\omega)I(\omega)) = \text{Re}(-i\omega)C|E(\omega)|^2 \)

\( = 0 \) for \( \omega \) real corresponding to \( \int_{-\infty}^{\infty} E(t)I(t) dt = 0 \) if \( C, I \) have comp. support.

So the picture is the following. An edge yields a 2-dim complex space of \( \mathbb{E} \) equipped with a hermitian form \( \text{Re} \text{E}^\dagger \text{E} \approx \frac{1}{2} (\text{E} \text{I} + \text{I} \text{E}) \)

\[
= (\mathbb{E})^* (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) (\mathbb{E}).
\]

At frequency \( \omega \), \( (\mathbb{E})^* (\mathbb{I}) \) is restricted to lie on the line \( \begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix} \)

which is isotropic for this hermitian form. In general \( (\mathbb{I})^* (\mathbb{C}^\dagger) \) is isotropic for \( (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) (\mathbb{E})^* (\mathbb{I}) \)

\( E \in \mathbb{C}^\dagger \Leftrightarrow (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})^*(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}) (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = (1 T^*)(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = T^* T \)

vanishes, i.e. \( T \) should symmetric. This shows symmetric feature is fixed:

\[
C_{E(-i\omega)}^* = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{C(-i\omega)} \end{pmatrix}
\]
For an L-edge the same except the line is \((L, -i\omega)C\).

\[ g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ g^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^* \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \]

\[ = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \]

If \(ad - bc = 1\), this means \(g \in SL_2(\mathbb{R})\), so the group of such \(g\) contains \(SL_2(\mathbb{R})\) and \(c(1, 0)\). So if you are interested in 2-dim \(V\) with inner form of type \((1, -1)\), then any two are isom. \(\otimes \sigma\) to \(g\) is \(SL_2(\mathbb{R})T \subset GL_2(\mathbb{C})\).

Our structure

What I need to do is to go directly from the family of \(\begin{pmatrix} E & 0 \\ 0 & I \end{pmatrix} \in \mathbb{C}^2, \begin{pmatrix} 1 & 0 & 0 \\ 0 & I \end{pmatrix} \in \mathbb{C}^3, \begin{pmatrix} 1 \\ (L, -i\omega) \end{pmatrix} \in \mathbb{C}^4\) to a Hilbert space \(Y\), the Krein space \(T \otimes Y\) and family \(l_{\omega} \otimes Y\) where \(T = \mathbb{C}^2\), \(l_{\omega} = (i, 0), (x_1, x_2) (0, 1)(y_1, y_2)\). You need an isom \(\mathbb{C}^2 \ni \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \)

\[ H((x_1, x_2)) = (x_1, x_2)(0, 1)(x_1, x_2) = x_1 x_2 + x_2 x_1 = 2\Re(x_1 x_2)\]
\[ P = C^2, \quad C = (\begin{pmatrix} 1 & 0 \\ c(-i\omega) & 1 \end{pmatrix}, \quad H(x_1) = 2 \text{Re}(x_1 x_2) \]
\[ \sigma^c = (\begin{pmatrix} 1 & 0 \\ L(-i\omega) & 1 \end{pmatrix} \]

Consider \[ T \begin{pmatrix} c^{1/2} 0 \\ 0 c^{1/2} \end{pmatrix} \rightarrow P \begin{pmatrix} c^{1/2} 0 \\ 0 c^{1/2} \end{pmatrix} \] in the C-case:
\[ \begin{pmatrix} c^{1/2} 0 \\ 0 c^{1/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ c^{1/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & c^{1/2} \\ c^{1/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & c^{1/2} \\ c^{1/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c^{1/2} & 0 \end{pmatrix} \begin{pmatrix} 0 & c^{1/2} \\ c^{1/2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

\[ (-i\omega) C \rightarrow \begin{pmatrix} -i\omega \\ 1 \end{pmatrix} C \begin{pmatrix} 0 & L \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} -i\omega \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 & -i\omega \end{pmatrix} = (1 + i\overline{\omega})(-i\omega 1) = i(\overline{\omega} - \omega) = 2 \text{Im}(\omega) \]
\[ \begin{pmatrix} -i\omega \\ 1 \end{pmatrix} = \begin{pmatrix} i\overline{\omega} \end{pmatrix} \begin{pmatrix} 1 & -i\omega \end{pmatrix} = i\overline{\omega} - i\omega = 2 \text{Im}(\omega) \]

\[ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -i\omega \end{pmatrix} = \begin{pmatrix} -i\omega \\ 1 \end{pmatrix} \]
\[ (a \ b) = \text{scalar} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ C(-i\omega) \end{pmatrix} = C(-i\omega) + \overline{C(-i\omega)} \]
\[ \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} L(-i\omega) \\ 1 \end{pmatrix} = Ci(\overline{\omega} - \omega) = (2 \text{Im}\omega) C \]
\[ = Li\overline{\omega} - Li\omega = \overline{(2 \text{Im}\omega)} L \]
An LC circuit has 2 description configuration space: 1) polarized Hilbert space $\Omega = \Omega^+ \oplus \Omega^-$ plus a subquotient $\mathbb{F}_2/F_1$.

Start again. Begin again. Concrete model:

LC network is a graph with $C, L$ edges. Each edge has "phase space" states $(\mathbf{E}, I) \in \mathbb{C}^2$, hermitian form (power)

\[
(E^*)(1 \ 0) (E) = 2 \Re \{EI\}
\]

For an $C$-edge there is a line $d_{\omega} = (1, C(-i\omega)) \in \mathbb{C}^2 \quad \omega > 0$

For an $L$-edge $d_{\omega} = (L(-i\omega), 1) \in \mathbb{C}^2 \quad \omega > 0$

for $\omega \in S^2 \subset \mathbb{C} \cup \{\infty\}$ which is isotropic for $\omega$ real. (for this herm. form graphs \( (\cdot)^T \) is

isotropic iff $T^* = -T$.)

Each edge gives a 2-dim complex phase space equipped with herm. form type $(1, -1)$ and the family $d_{\omega}$ of lines.

\[
\begin{pmatrix}
0 & iL^{1/2} \\
-iL^{1/2} & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-i & 0
\end{pmatrix}
\begin{pmatrix}
0 & iL^{1/2} \\
-iL^{1/2} & 0
\end{pmatrix}

\begin{pmatrix}
0 & 1 \\
-i & 0
\end{pmatrix}
\]
Organize your thoughts about connecting an LC network to a transmission line.

First transmission line equation:

\[
E_{i+1} - E_i = L_i \frac{dI_i}{dt};
\]

\[
I_{i+1} - I_i = C_i \frac{dE_i}{dt}.
\]

\[
u_x \frac{\partial E}{\partial x} + \frac{\partial}{\partial t} (\rho \frac{\partial I}{\partial t}) = 0
\]

\[
\nabla \frac{\partial I}{\partial x} + \frac{\partial E}{\partial t} = 0
\]

\[
(\partial_x + \partial_t)(E + \rho I) = 0
\]

\[
(\partial_x - \partial_t)(E - \rho I) = 0
\]

\[
E + \rho I = Ae^{-s(x-ct)}
\]

\[
E - \rho I = Be^{s(x+ct)}
\]

\[
(1 - \rho)(E_{x=0}) = (A) e^{st} = \frac{-2 + \rho}{-2 - \rho} = \frac{A}{B}
\]

\[
\begin{align*}
A &= \frac{-\rho + 2}{\rho + 2} \\
B &= \frac{-2 + \rho}{-2 - \rho}
\end{align*}
\]

Typical 2 is \( Ls \frac{1}{Cs} \)

\[
S = \frac{A}{B} = \frac{Ls - 1}{Ls + 1}
\]

or \[\frac{1}{Cs} - \frac{1}{Cs + 1} \]

\[
= \frac{1 - Cs}{1 + Cs}
\]

I should take the case:

\[
\frac{E}{I} = \frac{1}{Ls + 1 + Cs} = \frac{Ls}{LCs^2 + 1}
\]

\[
S = \frac{1 - 1}{1 + 1} \left( \frac{Ls}{LCs^2 + 1} \right) = \frac{-LCs^2 + 1 - Ls}{LCs^2 + Ls + 1}
\]

\[
S = \frac{-L \pm \sqrt{L^2 - 4LC}}{2LC}
\]

mag. real part
What you need now. Take a coherent
What you need to do now is to decide
how intrinsic coupling to a transmission line
is. You have a picture of an LC network
subquotient of a polarized Hilbert space, namely
the space of 1-cochains equipped with inner
split into C + L types with the
inner product \( C \langle E \rangle^2 \) resp. \( L \langle E \rangle^2 \). The
modified form \( C \langle E \rangle^2 \) resp. \( L \langle E \rangle^2 \) induces
a hermitian (for \( s \) real) form on the subquotient
of skew-herm. (for \( s \in i\mathbb{R} \)). So you have a line with
hermitian form. For an actual circuit the line
has a basis - voltage at the external node, so the
hermitian form is \( Z_s \langle E \rangle^2 \). When you couple to
a transm
\[
\begin{array}{ccc}
C & \xrightarrow{\phi} & C^* \\
\downarrow^w & & \downarrow^c \\
C & & C
\end{array}
\]
What can you do intrinsically. You have
a line \( J \) and a sesquilinear form hermitian for real
Can form \( J \oplus J^\dagger \). First do real case. You
have a real line \( J \) and a quadratic form on it
Can form \( J \oplus J^\ast \) symplectic + graph of quad
form is isotropic. We take them. Complex case
graph of a sesqui form \( J \rightarrow J^\dagger \). For a
general subquotient of a polar. Hilb. space you
get a sesquilinear form \( Z_s (j_1, j_2) \) which is
hermitian for \( s \) real, (herm. means \( Z_s (j_1, j_2) \in \mathbb{R} \)) and
skew-herm. for \( s \in i\mathbb{R} \) (skew-herm. means \( s \cdot \text{herm.} \)).
If \( J = C^n \) then \( Z_s (j_1, j_2) = (j_1, Z_s j_2) \).
A transmission line with unit speed has an impedance which identifies
\[ \partial_x E + 2\partial_t I = 0 \]
\[ \partial_x I + \partial_t E = 0 \]
\[ (\partial_x + \partial_t)(E + pI) = 0 \]
\[ (\partial_x - \partial_t)(E - pI) = 0 \]

Solutions of frequency \( \omega \) are
\[ E + pI = A e^{-s(x-t)} \]
\[ E - pI = B e^{s(x+t)} \]

so you get
\[ \begin{pmatrix} E + pI \\ E - pI \end{pmatrix} \bigg|_{x=0} = \begin{pmatrix} 1 \\ -p \end{pmatrix} \begin{pmatrix} E \\ I \end{pmatrix} \]

so
\[ \frac{A}{B} = \frac{1}{1-p} \frac{-2+p}{2-p} = \frac{Z - p}{Z + p} \]

Lesson seems to be that the \( p \):

Structure of a 1-port complex 2-diml space equipped with a hermitian form of signature \( 1, -1 \), also a line 2 depending on \( \mathbb{G}(n) \) - in finite case \( \omega \rightarrow \omega_0 \) is a rational maps from \( \mathbb{C} \) sphere to \( 
\mathbb{P}^1 \).

Given
\[ O(-1) \otimes \mathbb{C} \rightarrow O \otimes \mathbb{C}^2 \rightarrow O(n) \]

Seems strange but something might work. Go backwards; you have \( O \rightarrow O(-1) \otimes Y \rightarrow \otimes \otimes \mathbb{C} Y \rightarrow O(1) \otimes Y \rightarrow O \), and \( W \) isotropic in \( T \otimes Y \)

Go back to symplectic case. \( T \); 2-diml symplectic \( Y \); n-diml quadratic \( T \otimes Y \) symplectic. \( W \) isotropic in \( T \otimes Y \). Assume \( W \cap O(-1) \otimes Y = 0 \) \( \forall w \), then
\[ W^0 + \mathcal{O}(-1) \otimes Y = 0 \quad \forall w \quad \Rightarrow \quad W^0 \otimes \mathcal{O}(1) \otimes Y \subseteq W^0/W \]

nice intersection

\[ \mathcal{O} \otimes W^0 \quad \xrightarrow{\text{Use } \mathcal{L}_w \text{ for } \mathcal{O}(1)} \quad \mathcal{O} \otimes \mathcal{O}(Y) \]

\[ E_w = W^0 \cap (\mathcal{L}_w \otimes Y) \]

\[ E_w \xleftarrow{\text{int}} \quad \mathcal{O} \otimes W^0 \quad \xrightarrow{\mathcal{O} \otimes \mathcal{T} \otimes Y} \quad \mathcal{O} \otimes \mathcal{O}(Y) \]

\[ \mathcal{O} \otimes (W^0/W) \]

\[ E_0 = E_0^* \]

Suppose \( Y \) n-dim \( W \) n-1 dim \( W^0 \) n+1 dim \( E_0 \) dim \( \mathcal{O}(-n) \). Can you reverse the process, namely start from \( W^0/W \) 2-dim \( \mathcal{O} \otimes W \)

\[ E^*_w \quad \xrightarrow{\text{int}} \quad \mathcal{O} \otimes \mathcal{O}(Y) \quad \xrightarrow{\mathcal{O} \otimes \mathcal{T} \otimes Y} \quad \mathcal{O} \otimes \mathcal{O}(Y) \]

\[ \mathcal{O} \otimes (W^0/W) \quad \xrightarrow{\mathcal{O} \otimes \mathcal{T} \otimes Y} \quad E^*_w \]

Try to reverse the symplectic version.

T 2-dim symplectic \( Y \) will only quadratic \( T \otimes Y \) symplectic \( W \) isotropic in \( T \otimes Y \), assume \( \mathcal{O} \otimes W \) transversal to \( \mathcal{O}(-1) \otimes Y \) over \( \mathcal{P} = \mathcal{P}^0 \), i.e. \( W \cap \mathcal{L}_w \otimes Y = 0 \) \( \forall w \)

Then \( W^0 + \mathcal{L}_w \otimes Y = T \otimes Y \) \( \forall w \) so get vector bundle
$E = \mathcal{O}_n(l_0 \otimes Y)$, $E$ should be a Lagrangian inside $\mathcal{O} \otimes \mathcal{O}^\vee$.

$$
\begin{array}{c}
\mathcal{O}(-1) \otimes Y \\
\downarrow \\
\mathcal{O} \otimes \mathcal{O}^\vee \\
\downarrow \\
\mathcal{O}(1) \otimes Y \\
\end{array}
\Rightarrow 0
$$

$n = \dim Y,
\deg E = -n,
\text{rank } E = 1
$

$\mathcal{O} \otimes (\mathcal{O}^\vee / W) = \mathcal{O} \otimes (\mathcal{O}^\vee / W^*)$

You want to reverse this process. So what do we have? How to proceed? To start with $E \hookrightarrow \mathcal{O} \otimes \mathcal{O}^\vee \rightarrow E^*$, $E$ Lagrangian.

Problem: Classify Lagrangian sub-bundles of $\mathcal{O} \otimes V$ where $V$ is a symplectic vector space. First case $\dim V = 2$. Then $L = \mathcal{O}(n)$ for some $n > 0$. So we have a line bundle $L^*$ with 2 independent sections.

$0 \rightarrow L^* \rightarrow \mathcal{O} \otimes V \rightarrow L \rightarrow 0$

Do we get a quadratic function on $H^0(L^*)^*$? deg.
113 Somehow you can fit together the canonical res. of \( O(n) \) and its dual.

\[
0 \to O(-1) \otimes S_{n-1} \to O \oplus V \to O(n) \to 0
\]

This seems fairly clear. **NO. You probably need to use two disjoint divisors of degree \( n \).**

\[
0 \to O \otimes W^0 \to O \otimes W \to O \otimes (W^*) \to 0
\]

Maybe you should begin with \( L \to O \otimes W^0 \) and construct \( W^0 \). But you observe that \( L \to O \otimes W^0 \to O \otimes Y \) must be the canonical resolution of \( L \), and then \( \Gamma(O \otimes W^0) \to \Gamma(O \otimes Y) \) will be the correspondence \( W^0 \subset \Omega \otimes Y \) of \( K \)-modules. Now use \( V \) symplectic and \( L \) Lagrangian to get

\[
0 \to L \to O \otimes V \to L^* \to 0
\]

\[
0 \to O(-1) \otimes Y^* \to O \otimes (W^0)^* \to L^* \to 0
\]

\[
O \otimes (W^0)^* = O \otimes (W^0)^*
\]

This gets canonical isom. of \( O \) the \( K \)-modules.
What happens is that $W^0 \subset \mathcal{O} \otimes Y$

is the $K$-module for $L$ and $Y^* \rightarrow \mathcal{O} \otimes (W^0)^*$

is the $K$-module for $L(1)$. Other ways

$0 \rightarrow \mathcal{O}(-1) \otimes Y^* \rightarrow \mathcal{O} \otimes (W^0)^* \rightarrow \mathfrak{z}^* \rightarrow 0$

$(W^0)^* = H^0(\mathfrak{z}^*)$

$Y^* = H^0(\mathfrak{z}^*(-1))$

$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O} \otimes W^0 \rightarrow \mathcal{O}(1) \otimes Y \rightarrow 0$

$Y = H^0(\mathfrak{z}^*(-1))$ \hspace{1cm} $H^1(\mathfrak{z}^*(-2)) = W^0$

natural duality, but

$0 \rightarrow \mathcal{O}(1) \rightarrow \mathcal{O}(1) \otimes V \rightarrow \mathcal{O} \otimes \mathfrak{z}^*(-1) \rightarrow 0$

gives $H^0(\mathfrak{z}^*(-1)) \rightarrow H^0(\mathcal{O}(1))$

Basic data - symplectic space $V$ and

Lagrangian subbundle $L$ of $\mathcal{O} \otimes V$ over $P^1$. $H^0(\mathfrak{z}) = 0$

Wing direction. Start with $Y'$ homology quadratic

form in $Y'$, $T$ 2ndind symplectic, $Y \otimes Y$ then

symplectic, $W \subset \mathcal{T} \otimes Y$ isotropic, assume $W \cap (\mathcal{L} \otimes Y) = 0$

all $\omega \in \mathcal{P}$, where $W + \mathcal{L} \otimes Y = \mathcal{T} \otimes Y$ $W$

where we get $\mathcal{L} = W^0 \cap \mathcal{L} \otimes Y \subset W/W$

$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O} \otimes W^0 \rightarrow \mathcal{O}(1) \otimes Y \rightarrow 0$

must be canonical resolution so

$Y \rightarrow H^0(\mathcal{O}(1))$

$H^1(\mathfrak{z}^*(-2)) \rightarrow W^0$

$0 \rightarrow \mathcal{O}(-1) \otimes Y^* \rightarrow \mathcal{O} \otimes (W^0)^* \rightarrow \mathfrak{z}^* \rightarrow 0$

$(W^0)^* \rightarrow \mathfrak{z}^*$

$Y^* = H^0(\mathfrak{z}^*(-1))$

$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O} \otimes V \rightarrow \mathfrak{z}^* \rightarrow 0$

$H^0(\mathfrak{z}^*(-1)) \rightarrow H^0(\mathcal{O}(1))$

$Y^* \rightarrow Y$
So let's now that this is clear you want to work in the real line.

\[ T = \mathbb{C}^2 \quad \mathfrak{l}_z = (\frac{1}{z}) \mathfrak{e}_0 \quad T \otimes Y = Y \quad Y \text{ need an isom} \quad Y^* \xrightarrow{\varphi} Y \quad \text{non degenerate sym pairing} \]

Naturally \( \varphi \) is symplectic and those subspaces which are graphs \( (\frac{1}{z})Y \) have form \( (\frac{1}{z})Y \quad F = F^* \quad Y \to Y^* \)

\( Y \) is symmetric. But to make sense of \( (\frac{1}{z}) \) you need a fixed \( \mathfrak{g} \)-f. So \( T \otimes Y = \mathfrak{g} \) have a standard form.

Real case \( T = \mathbb{R}^2 \) skew form \[
\begin{pmatrix}
(x_1) & (0 & 1) \\
(0 & 1 & 0)
\end{pmatrix}
\begin{pmatrix}
x_1' \\
x_2'
\end{pmatrix}
\]

\( Y = \text{Euclidean space} \)

\( T \otimes Y = \mathfrak{g} \)

with \[
\begin{pmatrix}
y_1 & (0 & 1) \\
y_2 & (0 & 0)
\end{pmatrix}
\begin{pmatrix}
y_1' \\
y_2'
\end{pmatrix}
= y_1y_2' - y_2y_1'
\]

\( \Gamma_2 = (\frac{1}{z})Y \) is isotropic means \( (\frac{1}{z})^t (0 1) (\frac{1}{z}) = x^t = 0 \)

In fact \( \Gamma_2^0 = \left\{ (y_1, y_2) \mid \begin{pmatrix} y_1 \\ y_2 \\ y_1 \\ y_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 \right\} = \Gamma_2^1 \)

\( y_1 y_2' - y_2 y_1' = 0 \)

So now consider \( W \) isotropic in \( Y \)

\( W = (\frac{\varepsilon}{A \mathfrak{g}})X \)

If you want \( W \) is subspace of \( \mathfrak{g} \)

\( \varepsilon = p_1 | W \quad A = p_2 | W \quad W_0 = \left\{ (y_1, y_2) \mid \begin{pmatrix} y_1 \\ y_2 \\ y_2 \\ y_1 \end{pmatrix} = 0 \right\} \)

\( (\varepsilon)^t y_2 = (Ax)^t y_1 \) or \( \varepsilon y_2 = A^t y_1 \)

\( t \) denotes \( \times \) wrt some scalar product

\( W \cap (\frac{1}{z})Y = \left\{ (\frac{\varepsilon}{A \mathfrak{g}})X \mid \varepsilon A = A^t \varepsilon \right\} \)
You need to simplify: choose orthonormal basis for $Y$ and $X$, then have solution of \((\lambda \varepsilon - A)x = 0\) \(x \in X_c\).

\[ \lambda x = Ax \]
\[ (x, Ax) = (x, A^*x) \]
\[ \lambda \|x\|^2 = (Ax, x) = (Ax, x) \]

\[ \lambda = \bar{\lambda} \]

Point \((\varepsilon x, Ax) \in \mathbb{R}^2\)

so make assumption that no bound states, this is something you test in the real setting. In particular you want $\varepsilon$ inj \((\lambda = \infty)\) \(\lambda = 0\).

More review: \(T = \mathbb{R}^2\) skew form \((x_1', x_2') \mid |x_1'| = |x_2'|\)

\(Y\) Euclidean space, \(T = Y = \mathbb{R}^2\) skew form

\[
\begin{pmatrix}
  y_1' \\
  y_2'
\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} = y_1' y_2' - y_2' y_1'
\]

\[
\Gamma_y = (\varepsilon) y \\
\Gamma^0_y = (\varepsilon^*) y
\]

\(W\) not in \(Y\) \(W = (\varepsilon) X\)

\(W = \{ (y_1, y_2) \mid (\varepsilon^* y_2 = \lambda y_1) \}

\(W \cap (\lambda) Y_c = \{ (\varepsilon x, (\lambda \varepsilon - A)x = 0 \}

0 = (\varepsilon x, (\lambda \varepsilon - A)x) = \lambda \|x\|^2 - (\varepsilon x, Ax) \quad \in \mathbb{R} \quad \lambda \in \mathbb{R}

Continue: calculate \(W^0\), \(\varepsilon\) inj, so can arrange

\[
\varepsilon^* \varepsilon = 1, \quad Y = \varepsilon X \oplus \text{Ker}(\varepsilon^*)
\]

\[
\varepsilon^* y_2 = A^* y_1
\]

Assume \(\varepsilon^* y_2 = A^* y_1\)

\[
(\varepsilon x, y_1) = (Ax, y_2)
\]

\[
(\varepsilon x, y_2) = (A^* y_2, y_1)
\]
Given \( y_1 \rightarrow (Ax, y_1) \) can be represented as \((e^x, y_2)\) uniquely with \(y_2 \in \mathcal{E}\).

Define a map \( \mathcal{E} \rightarrow X \) by requiring
\[
(Ax, y) = (e^x, e^\Theta y) = (x, \Theta y) \quad \Rightarrow \quad \Theta = A^* \]

\(\mathcal{E}\) seems to consist of \((e^x)\)?

Start with \((y_1, y_2) \in \mathcal{E}\) i.e. \(e^x y_2 = A^* y_1\)
or \((e^x, y_2) = (Ax, y_1) \quad \forall x \quad \text{but}
\[
(Ax, y_1) = (x, A^* y_1)
\]

Claim \( y_1 \) so \( y_2 = e^A^* y_1 \mod (ker e^x) \).

\[
\mathcal{E} = \begin{pmatrix} 1 \\ e^A^* \end{pmatrix} y + \begin{pmatrix} 0 \\ ker e^x \end{pmatrix}
\]

Proof: Given \((y_1, y_2) \in \mathcal{E}\) i.e. \((e^x, y_2) = (Ax, y_1) \quad \forall x \quad Then \((e^x, y_2) = (x, A^* y_1) = (e^x, e^A^* y_1) \Rightarrow y_2 = e^A^* y_1 \in ker e^x \)
\[
(y_1) = \begin{pmatrix} y_1 \\ e^A^* y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2 - e^A^* y_1 \end{pmatrix}
\]

Note \((eA^*)^* = A e^*\)
$\Pi = 1 - \varepsilon \varepsilon^*$. 

$$A\varepsilon^* = \frac{\varepsilon \varepsilon^* A\varepsilon^* + \Pi A\varepsilon^*}{\varepsilon A^* \varepsilon^*}$$

$$\varepsilon A^* = \varepsilon A^* \varepsilon^* + \varepsilon A^* \Pi$$

So it should be possible to uniquely extend the partial symm. of $\langle A \rangle$ to a symm. operator on $Y + \pi(\tilde{A}) \Pi = 0$. This gives a kind of canonical extension.

In the end you seem to get

W/W is symplectic and you have constructed a found a canonical Lagrangian subspace. In fact we have $W/W = \text{ker} \varepsilon^* + \text{ker} \varepsilon^*$

What is the answer? You have

$W = (\frac{\varepsilon}{A}) \subset Y$ and $W \subset (\frac{\varepsilon}{A}) Y \subset W^o$.

$A = A\varepsilon^* + \varepsilon A^* - (\varepsilon A^* \varepsilon^*)$

$= A\varepsilon^* + \varepsilon A^* \Pi = \Pi A\varepsilon^* + \varepsilon A^*$

Now we have a simple problem to find $W^o \cap (\frac{1}{A}) Y$

$W^o = \{ (y_1', y_2') | A^* y_1 = \varepsilon^* y_2 \}$

$W^o \cap (\frac{1}{A}) Y = (\frac{1}{A}) \text{ker} (A\varepsilon^* - A^*)$

You guess that the response function should have a simple form, like what you found for an LC network.

The basic problem here. The idea here is that the response is Lagrangian sub-bundle $L \subset \mathcal{O} \otimes (W/W)$ and since $W/W = (\text{ker} \varepsilon^*)$, $L$ should be the graph of a symmetric operator on

$I$ and resolvent of

$$\begin{pmatrix}
\varepsilon A = A \varepsilon^* & \varepsilon A^* \Pi \\
\Pi A \varepsilon^* & 0
\end{pmatrix}$$

Something like this.
\[
(a \ b) = \begin{pmatrix} a & -b \\ c & d \end{pmatrix} \quad (a \ b)^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

\[
(d-6a^2b)^{-1} = \left( \lambda - \frac{\gamma^x}{\lambda \beta} \right)^{-1}
\]

You are trying for something like inducing a quadratic form on a subspace or quotient space. To this end...

So you run into a familiar situation. Namely you take the resolvent and project into something compact into a generating subspaces. The answer is very easy. The problem is to fit it into something else.

Now what. \( \varepsilon^* \varepsilon = 1 \). Can write \( Y = \varepsilon X \oplus \ker \varepsilon^* \).

\[
\tilde{A} = \begin{bmatrix} \varepsilon^* A \varepsilon^* & \varepsilon A^* \\ \pi A \varepsilon^* & 0 \end{bmatrix}
\]

You need to organize all this stuff. How? Go back to construct \( L_0 = W \cup (\mathcal{W}) \).

\[
\text{det} \begin{pmatrix} \lambda & -\gamma \\ -\gamma^* & \lambda - \beta \end{pmatrix} = \text{tr} \begin{pmatrix} \lambda - \gamma & -\gamma \\ -\gamma^* & \lambda - \beta \end{pmatrix} = \text{tr} \begin{pmatrix} 0 & 0 \\ 0 & 8 \beta \end{pmatrix}
\]

= \text{tr} \begin{pmatrix} 1 \\ \lambda - \beta - \frac{\gamma^x}{\lambda} \end{pmatrix} 8 \beta

Note that is hermitian for \( \lambda \) real.
Given \( \sqrt{y_1} \), then \( A^* y_1 \).

You want to calculate \( W^* \) where \( W = \begin{pmatrix} A^* \\ \end{pmatrix} \) \( (\exists y_1) \in W^* \), i.e. \( A^* y_1 = \varepsilon^* y_2 \) remove \((\exists^* y_1)\) from \((y_1)\) to assume \( \varepsilon^* y_1 = 0 \). Now we have \( \forall x \quad (y_1, Ax) = (x_1, x) \) for some \( x_1 \) i.e. \( A^* (y_1, x) = (x_1, x) \).

so it seems that if \( \varepsilon^* y_1 = 0 \) then \( y_1 \)

Wait given \((y_1, y_2)\in W^* \), i.e. \( A^* y_1 = \varepsilon^* y_2 \) then \((y_1)\) satisfies \( A^* y_1 = \varepsilon^* (\varepsilon A^* y_1) \), so \( \Gamma^* = W^* \).

Also \( (\varepsilon A^*) = \Gamma^* \). A \( \varepsilon A^* \).

The point is that \( (\varepsilon A^*) Y \subset W^* \) because \( \varepsilon A^* (\exists x) = \varepsilon^* A^* x \)

\[
(y_1) = \begin{pmatrix} y_1 \\ \varepsilon A^* y_1 \end{pmatrix} + \begin{pmatrix} 0 \\ y_2-\varepsilon A^* y_1 \end{pmatrix}
\]

\[
= \begin{pmatrix} \varepsilon A A^* y_1 \\ \varepsilon A A^* y_1 \end{pmatrix} + \begin{pmatrix} y_1-\varepsilon^* y_1 \\ \varepsilon A^* y_1 \end{pmatrix}
\]

\[
A^* y_1 \in W^* \quad \text{if} \quad y_1 \in W^* \quad \text{and} \quad A^* y_1 = \varepsilon^* A^* y_1
\]

\[
A^* y_1 \in W^* \quad \text{if} \quad y_1 \in W^* \quad \text{and} \quad A^* y_1 = \varepsilon^* A^* y_1
\]

\[
A^* y_1 \in W^* \quad \text{if} \quad y_1 \in W^* \quad \text{and} \quad A^* y_1 = \varepsilon^* A^* y_1
\]