691 Feb 21 Cayley transform of some sort between $U(p,q)$ and $U(p+q)$. Apparently this is a game played with 2×2 matrices, maybe eventually the Gelfand Re- quasi-determinants - Plücker coordinates. It amounts to describing a subspace as a graph in two ways. Maybe your observation with length one complexes and general invertible 2×2 matrices. 4 objects and 8 maps.

$$\begin{tikzcd} X_1 \arrow[d] & X_2 \\ Y_1 & Y_2 \end{tikzcd}$$

Let's get it straight
You want to go between
an isom $X_1 \oplus Y_1 \quad X_2 \oplus Y_2$
and an isom $X_1 \oplus X_2 \quad Y_1 \oplus Y_2$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a + \beta c & \alpha b + \beta d \\ \gamma a + \delta c & \gamma b + \delta d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & Y_1 \arrow[d] & & \\ 0 \arrow[r] & X_1 \arrow[r] & X_1 \oplus X_2 \arrow[r] \arrow[d] & X_2 \arrow[r] & 0 \\ & & Y_2 \arrow[d] & & \\ & & 0 & & \end{tikzcd}$$

$$\xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \quad X_1 \oplus X_2$$

$$\begin{tikzcd}
 & Y_1 \arrow[d, shift left] \arrow[dr, "\gamma"] & \\
X_1 \arrow[ur, "a"] \arrow[r, shift right] \arrow[dr, "c"'] & V \arrow[l, shift right] \arrow[u, shift left] \arrow[d, shift left] \arrow[r, shift right] & X_2 \arrow[l, shift right] \\
 & Y_2 \arrow[u, shift left] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

4 short exact sequences with splitting. But the new point ~~without~~ arises when one of the maps is an isom

You start with the isom $\begin{matrix} X_1 \\ \oplus \\ X_2 \end{matrix} \xrightarrow{\sim} \begin{matrix} Y_1 \\ \oplus \\ Y_2 \end{matrix}$ and assume some component is invertible, say $X_1 \xrightarrow{c} Y_2$, then you find the "~~opposite~~ parallel" component $\gamma : Y_1 \to X_2$ is invertible, explicit inverse given by "quasi-det". Then get ~~[illegible]~~ map $\begin{matrix} X_1 \\ \oplus \\ Y_1 \end{matrix} \longleftarrow \begin{matrix} X_2 \\ \oplus \\ Y_2 \end{matrix}$

$$\begin{matrix} X_1 \\ \oplus \\ X_2 \end{matrix} \xleftarrow{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}} \begin{matrix} Y_1 \\ \oplus \\ Y_2 \end{matrix} \xleftarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{matrix} X_1 \\ \oplus \\ X_2 \end{matrix}$$

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & Y_1 \arrow[d] \arrow[dr, dashed, "\gamma"] & & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[dr, dashed, "c"'] & V \arrow[r] \arrow[d] & X_2 \arrow[r] & 0 \\
 & & Y_2 \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Assume $c^{-1} \exists$

$$y_2 = c x_1 + d x_2$$
$$x_1 = c^{-1}(y_2 - d x_2)$$
$$y_1 = a c^{-1}(y_2 - d x_2) + b x_2$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -c^{-1}d & c^{-1} \\ b - ac^{-1}d & ac^{-1} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "c"'] & X \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d, dashed] & 0 \\
0 & Y_2 \arrow[l] & Y \arrow[l] & Y_1 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Point: Given a map $X \xrightarrow{f} Y$, a submodule $X_1 < X$, a quotient module $Y \twoheadrightarrow Y/Y_1$ such that $X_1 < X \xrightarrow{f} Y \twoheadrightarrow Y/Y_1$ is an isomorphism, then there are canon splittings

$$X = X_1 \oplus f^{-1}(Y_1)$$

$$Y = f(X_1) \oplus Y_1$$

respected by f.

So when one starts with $X = X_1 \oplus X_2$, $Y = Y_1 \oplus Y_2$, i.e. a choice of complements, you need "triangular" matrices to shift to the canonical splittings.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}\begin{pmatrix} & \\ & \end{pmatrix}$$

$$\begin{array}{cccc} Y_2 & f(X_1) & X_1 & X_1 \\ \oplus & \oplus & \oplus & \oplus \\ Y_1 & Y_1 & f^{-1}(Y_1) & X_2 \end{array}$$

What you need now is ~~the~~ to add quadratic forms to the situation.

Seems to be the basic mechanism that you go through when solving equations. So now see if you can understand the link between $U(p+q)$ and $U(p,q)$. Let us begin with a unitary transformation $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and suppose some entry invertible, say c so that we have a related transf. $\begin{pmatrix} y_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} y_2 \\ x_2 \end{pmatrix}$

then $\overset{\text{left out}}{\|y_1\|^2} + \overset{\text{right out}}{\|y_2\|^2} = \overset{\text{left in}}{\|x_1\|^2} + \overset{\text{right in}}{\|x_2\|^2}$

translates to $\overset{\text{left out}}{\|y_1\|^2} - \overset{\text{left in}}{\|x_1\|^2} = \overset{\text{right in}}{\|x_2\|^2} - \overset{\text{right out}}{\|y_2\|^2}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \begin{pmatrix} y_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} y_2 \\ x_2 \end{pmatrix}$$

$$x_1 = c^{-1}(y_2 - d x_2)$$
$$y_1 = ac^{-1}(y_2 - dx_2) + bx_2$$

$$\begin{pmatrix} y_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} ac^{-1} & b - ac^{-1}d \\ c^{-1} & -c^{-1}d \end{pmatrix} \begin{pmatrix} y_2 \\ x_2 \end{pmatrix}$$

I'm puzzled by the sign. Consider potential scattering

$$(-\partial_x^2 + V)\, u = k^2 u$$

$x \ll 0$ $\qquad\qquad$ $x \gg 0$

$\overset{\text{left in}}{e^{+ikx}} \longleftrightarrow Ae^{ikx} + \overset{\text{right in}}{Be^{-ikx}}$

$e^{-ikx} \longleftrightarrow Ce^{ikx} + De^{-ikx}$

Wronskian constant $\Longrightarrow$ $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = 1$

reality $\Longrightarrow$ $C = \bar{B}$, $D = \bar{A}$ when k is real

scattering picture $e^{i(kx - \omega t)}$ right moving

$\overset{\text{left out}}{\frac{1}{D} e^{-ikx}} \longleftrightarrow \overset{\text{right out}}{\frac{C}{D} e^{+ikx}} + \overset{\text{right in}}{e^{-ikx}}$

$\overset{\text{left in}}{e^{ikx}} - \overset{\text{left in}}{\frac{B}{D} e^{-ikx}} \longleftrightarrow \overset{\text{right out}}{\underbrace{\left(A - \frac{BC}{D}\right)}_{\frac{1}{D}} e^{ikx}}$

scattering matrix $\begin{pmatrix} \frac{C}{D} & \frac{1}{D} \\ \frac{1}{D} & -\frac{B}{D} \end{pmatrix}$ is unitary and complex symmetric

695 Start again with a unitary transf.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\|y\|^2 = \|gx\|^2 = \|x\|^2$

$x^* g^* g x = x^* x$

$\Rightarrow g^* g = 1.$

between $Y_1 \oplus Y_2$ and $X_1 \oplus X_2$. Thus

$\|y_1\|^2 + \|y_2\|^2 = \|x_1\|^2 + \|x_2\|^2$

Assume c invertible whence can solve

$$\begin{pmatrix} y_1 \\ x_1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} y_2 \\ x_2 \end{pmatrix}$$

$\ni \quad \|y_1\|^2 - \|x_1\|^2 = \|x_2\|^2 - \|y_2\|^2$

Another possibility

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

unitary

$\|y_1\|^2 + \|y_2\|^2 = \|x_1\|^2 + \|x_2\|^2$

assume a invertible

$$\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} [redacted] \; y_1 \\ x_2 \end{pmatrix}$$

$\|x_1\|^2 - \|y_2\|^2 = \|y_1\|^2 - \|x_2\|^2$ YES.

Thus the order [redacted] is important. [redacted] Maybe the Cramer rule sign applies

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696 So where do you go next? You want to understand simple 2 ports. You scattering data $X \underset{b}{\overset{a}{\rightleftarrows}} Y$. ~~dim(X)~~

First case: $X = 0$. Then you must give a unitary isom $V \xrightarrow{\sim} W$ between the incoming states and outgoing states. In practice $V = \mathbb{C}^2 = V_1 \oplus V_2$ and $W = \mathbb{C}^2 = W_1 \oplus W_2$ are split, so you are given a unitary $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ relating incoming and outgoing states. ~~The transmission of~~

Goal: You want to be able to handle the graph $\mathbb{Z} \subset \mathbb{R}$ with a translation invariant "circuit" with transmission line segments linking vertices, a given 2-port used to connect segments at each vertex. The idea is then that the port, -

The picture: There will be a transfer matrix for the edges. This will be in $U(1,1)$ depending analytically on z

transmission line eqns.

$$\partial_x E + \partial_t I = 0$$

$$\partial_{\cancel{x}} E + \partial_x E = 0$$

$E_x \quad E_{x+dx}$ $\quad T$

$$E_x - E_{x+dx} = l\,dx\,\partial_t I_x$$

$$I_x - I_{x+dx} = c\,dx\,\partial_t E_x$$

$$(\partial_x + \partial_t)(E + I) = 0 \qquad E + I = Ae^{s(-x+t)}$$

$$(\partial_x - \partial_t)(E - I) = 0 \qquad E - I = Be^{s(x+t)}$$

$$\boxed{\begin{matrix} E_0 + I_0 = A & E_1 + I_1 = Ae^{-s} \\ E_0 - I_0 = B & E_1 - I_1 = Be^{s} \end{matrix}}$$

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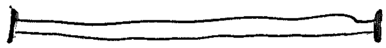
$$\begin{pmatrix} E_0 + I_0 \\ E_0 - I_0 \end{pmatrix} = \begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} E_1 + I_1 \\ E_1 - I_1 \end{pmatrix}$$

So consider a transmission line between $x = 0, 1$.



$$\xi^+ = E + I \qquad \text{fn. of } x$$
$$\xi^- = E - I$$

$$(\partial_x + s)\,\xi^+ = 0 \qquad \xi^+ = A e^{-sx}$$
$$(\partial_x - s)\,\xi^- = 0 \qquad \xi^- = B e^{sx}$$

$$\boxed{\begin{pmatrix} \xi^+_0 \\ \xi^-_0 \end{pmatrix} = \begin{pmatrix} e^{s} & 0 \\ 0 & e^{-s} \end{pmatrix} \begin{pmatrix} \xi^+_1 \\ \xi^-_1 \end{pmatrix} \qquad \text{transfer matrix}}$$

What about the scattering matrix? Incoming coords are ξ^+_0, ξ^-_1 outgoing are ξ^-_0, ξ^+_1

$$\begin{pmatrix} \xi^-_0 \\ \xi^+_1 \end{pmatrix} = \begin{pmatrix} e^{-s} & \\ & e^{-s} \end{pmatrix} \begin{pmatrix} \xi^-_1 \\ \xi^+_0 \end{pmatrix}$$

Probably better to write

$$\begin{pmatrix} \xi^+_1 \\ \xi^-_0 \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-s} & 0 \\ 0 & e^{-s} \end{pmatrix}}_{\text{unitary for } \mathrm{Re}(s) = 0.} \begin{pmatrix} \xi^+_0 \\ \xi^-_1 \end{pmatrix}$$

~~usual singularities~~ also decaying in $\mathrm{Re}(s) > 0$.

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$$\text{out}\begin{pmatrix}\xi_1^+\\ \xi_0^-\end{pmatrix} = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi_0^+\\ \xi_1^-\end{pmatrix}\text{in}$$

assume a^{-1} exists, i.e. $a \neq 0$, in which case from $|a|^2+|b|^2=1$ $|b|^2+|d|^2=1$, get $d \neq 0$

a, d are transmission coeffs.

b, c — reflection coeffs

$$\begin{pmatrix}\xi_0^+\\ \xi_0^-\end{pmatrix} = \begin{pmatrix} a^{-1} & -a^{-1}b\\ ca^{-1} & d-ca^{-1}b\end{pmatrix}\begin{pmatrix}\xi_1^+\\ \xi_1^-\end{pmatrix}$$

$$\xi_1^+ = a\,\xi_0^+ + b\,\xi_1^- \qquad \xi_0^+ = a^{-1}(\xi_1^+ - b\,\xi_1^-)$$

$$\xi_0^- = c a^{-1}(\xi_1^+ - b\,\xi_1^-) + d\,\xi_1^-$$

$$\det\left(a^{-1}\begin{pmatrix}1 & -b\\ c & ad-bc\end{pmatrix}\right) = a^{-2}\,ad = \frac{d}{a}$$

$$\begin{pmatrix} a & b\\ -\bar{b} & \bar{a}\end{pmatrix} \rightsquigarrow \begin{pmatrix} a^{-1} & -\frac{b}{a}\\ -\frac{\bar{b}}{a} & \bar{a}+\frac{|b|^2}{a}\end{pmatrix} = \begin{pmatrix}\frac{1}{a} & -\frac{b}{a}\\ -\frac{\bar{b}}{a} & \frac{1}{a}\end{pmatrix}$$

699 Rather than proceed with formulae - better to

$$\begin{pmatrix} a & b \\ -\bar{b}\zeta & \bar{a}\zeta \end{pmatrix} \longmapsto \begin{pmatrix} a^{-1} & -a^{-1}b \\ -\bar{b}\zeta a^{-1} & \bar{a}\zeta + \bar{b}\zeta a^{-1} b \end{pmatrix}$$

$|\zeta| = 1.$

$$g = \begin{pmatrix} \frac{1}{a} & -\frac{b}{a} \\ -\frac{\bar{b}}{a}\zeta & \frac{\cancel{|a|^2+|b|^2}\ 1}{a}\zeta \end{pmatrix}$$

Suppose $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $g^* \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

i.e. $g^{-1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} = \begin{pmatrix} \alpha^* & -\gamma^* \\ -\beta^* & \delta^* \end{pmatrix}$

$$\frac{1}{\Delta}\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \qquad \left(\alpha^* = \frac{\delta}{\Delta} \quad \gamma^* = \frac{\beta}{\Delta} \quad \beta^* = \frac{\gamma}{\Delta} \quad \delta^* = \frac{\alpha}{\Delta} \right)$$

$$\Delta = \det(g) = \frac{\zeta}{a^2}(1+|b|^2) = \zeta\frac{\bar{a}}{a}$$

$\alpha^* = \frac{1}{\bar{a}}$ $\qquad \frac{\delta}{\Delta} = \frac{\zeta}{a} \Big/ \frac{\zeta\bar{a}}{a} = \frac{1}{\bar{a}}$

$\beta^* = -\frac{\bar{b}}{\bar{a}}$ $\qquad \frac{\gamma}{\Delta} = \left(-\frac{\bar{b}}{a}\zeta\right) \Big/ \frac{\zeta\bar{a}}{a} = -\frac{\bar{b}}{\bar{a}}$

$\gamma^* = -\frac{b}{\bar{a}}\bar{\zeta}$ $\qquad \frac{\beta}{\Delta} = \left(-\frac{b}{a}\right) \Big/ \frac{\zeta\bar{a}}{a} = -\frac{b}{\bar{a}\zeta}$

$\delta^* = \frac{1}{\bar{a}}\bar{\zeta}$ $\qquad \frac{\alpha}{\Delta} = \frac{1}{a} \Big/ \zeta\frac{\bar{a}}{a} = \frac{1}{\bar{a}\zeta} = \frac{\bar{\zeta}}{\bar{a}}.$

So now we have a transfer matrix which is the product of a $U(1,1)$ matrix (arbitrary) times $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$.

Then what's of interest is the mapping

$$\zeta \longmapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}(\zeta)$$

on the Riemann sphere.

Actually you should describe determinant 1 : $SU(1,1)$

$$\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} \alpha^* & -\gamma^* \\ -\beta^* & \delta^* \end{pmatrix} \qquad \begin{matrix} \delta = \alpha^* \\ \gamma = \beta^* \end{matrix}$$

i.e. you have $\begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$ $\quad |\alpha|^2 - |\beta|^2 = 1$.

We are basically interested in the map on the R.S., Replace z by $z^{1/2}$, and work modulo center

$$\begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}(\zeta) = \frac{z\zeta - h}{1 - \bar{h} z \zeta}$$

Simpler: You have $\zeta \longmapsto z\zeta$ and then

$$\zeta \longmapsto \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix}(\zeta) = \frac{\zeta - h}{1 - \bar{h}\zeta} \qquad \text{with } |h| < 1.$$

What about the spectrum? ~~There should~~ The point is that the segmented transmission line

has a response which is an analytic function of z, for $|z| < 1$ with unitary values for $|z| = 1$?

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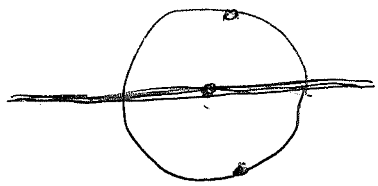
$$S(z) = \begin{pmatrix} 1 & -h \\ -\bar{h} & 1 \end{pmatrix} \begin{pmatrix} z S(z) \end{pmatrix}$$

$$S = \frac{zS - h}{1 - \bar{h} z S}$$

$$S - \bar{h} z S^2 = zS - h$$

$$0 = \bar{h} z S^2 + (z-1) S - h$$

$$S = \frac{-(z-1) \pm \sqrt{(z-1)^2 + 4\bar{h} z h}}{2\bar{h} z}$$



$\cos\theta$

$z^2 - 2\lambda z + 1$

$$z^2 - 2z + 1 + 4|h|^2 z$$

$$= z^2 + 2\underbrace{\left(2|h|^2 - 1\right)}_{\text{~~ranges between~~ } (-1,1)} z + 1$$

roots: $z = -\left(2|h|^2 - 1\right) \pm \sqrt{\left(2|h|^2-1\right)^2 - 1}$

$$0 = \bar{h}\left(z^{1/2} S\right)^2 + \left(z^{1/2} - z^{-1/2}\right)\left(z^{1/2} S\right) - h$$

Probably can suppose $0 < h < 1$.

$$\left(z^{1/2} S\right)^2 + \frac{z^{1/2} - z^{-1/2}}{h}\left(z^{1/2} S\right) - 1$$

Something is funny because $h = 0$ seems to yield $S(z) = z S(z)$

702 Feb 22. The idea is to couple transmission line segments in ~~a sensible~~ a translation invariant way to get some analog of a Schroedinger equation with periodic potential. spectrum is continuous but split into bands.

Go over the theory $\left(-\partial_x^2 + V(x)\right)\psi = \lambda\psi$

$V(x+2\pi) = V(x)$. Let $\Psi(\lambda, x, x_0)$ be the propagator

$$\left(-\partial_x^2 + V(x)\right)\Psi(\lambda, x, x_0) = \lambda\,\Psi(\lambda, x, x_0)$$

$$\Psi(\lambda, x_0, x_0) = \mathrm{id}.$$

Then $\Psi(\lambda, 2\pi, 0)$ controls the propagation. This is a matrix in $SL_2(\mathbb{R})$ (Wronskian).

~~You ought to be able to construct~~

Point: For each λ you get 2 plane of eigenfunctions. What ~~are the~~ about the spectrum as an unbdd s.a. op. on $L^2(\mathbb{R})$?

What's important is whether the eigenfunctions are bounded, and that depends on whether the transfer matrix $\Psi(\lambda, 2\pi, 0)$ is elliptic or hyperbolic. I guess you see instantly that there is no discrete spectrum, just because the product of the eigenvalues is ± 1, so either both have abs. value 1, or one has abs. value <1 and the other >1.

$$g^* \varepsilon g = \varepsilon \quad \Rightarrow \quad \overline{\det(g)}\,\cancel{\det(\varepsilon)}\,\det(g) = \cancel{\det(\varepsilon)}$$

$$\Rightarrow |\det(g)| = 1.$$

Work on the lecture.

$$H = H^- \oplus X \oplus H^+ \qquad \begin{aligned} uH^+ &\subset H^+ \\ u^{-1}H^- &\subset H^- \end{aligned}$$

$i : X \hookrightarrow H$ inclusion

$i^* i = 1$. $\quad ii^* =$ orth. proj. onto X.

Let $\gamma = i^* u i : X \longrightarrow X$. Claim

$$\begin{cases} i^* u^n i = \gamma^n & n \geq 0 \\ i^* u^{-n} i = (\gamma^*)^n & n \geq 0 \end{cases}$$

Proof: $\quad u^{-1}H^- \subset H^- \Longrightarrow H^- \subset uH^-$

$u(H^-)^\perp = (uH^-)^\perp \subset (H^-)^\perp$, i.e.

u carries $X \oplus H^+$ into itself, and u carries H^+ into H^+, so u induces an operator γ on $X \simeq (X \oplus H^+)/H^+$, and u^n induces γ^n for all $n \geq 1$. $\gamma^n = i^* u^n i$. Other half: $(\gamma^*)^n = (i^* u^n i)^* = i^* u^{-n} i$.

Now look at $H' = \overline{\sum_{n \in \mathbb{Z}} u^n X} \subset H$

Claim

$$L^2(S^1, X; d\mu) \xrightarrow{\sim} H' \quad ?$$

At some point you want to start with a γ. First replace γ by $r\gamma$ where $0<r<1$ so can assume $\|\gamma\| < 1$, in which case $1 - z^{-1}\gamma$ etc are invertible.

Consider $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} X \simeq \bigoplus_{n \in \mathbb{Z}} z^n X = H_0$

with inner product.

$$\langle z^p \xi, z^q \xi' \rangle = \langle \xi, z^{q-p} \xi' \rangle$$

$$= \begin{cases} \langle \xi, \gamma^{q-p} \xi' \rangle & q-p \geq 0 \\ \langle \xi, (\gamma^*)^{p-q} \xi' \rangle & q-p \leq 0. \end{cases}$$

Then

$$\langle \xi, z^n \xi' \rangle_{H_0} = \begin{matrix} \langle \xi, \gamma^n \xi' \rangle_X & n \geq 0 \\ \langle \xi, (\gamma^*)^{-n} \xi' \rangle_X & " \end{matrix}$$

$$\langle f(z) \xi, g(z) \xi' \rangle = \langle \xi, f(z)^* g(z) \xi' \rangle$$

$$= \int \frac{d\theta}{2\pi} f(z)^* g(z)\, z^{-n} \left\langle \xi, \begin{pmatrix} \gamma^n \\ \gamma^{*-n} \end{pmatrix} \xi' \right\rangle$$

$$= \int \frac{d\theta}{2\pi} f(z)^* g(z) \langle \xi, \mu(z) \xi' \rangle$$

$$\mu(z) = \sum_{n \geq 0} z^{-n} \gamma^n + \sum_{n > 0} z^n (\gamma^*)^n$$

$$= \frac{1}{1 - z^{-1}\gamma} + \frac{z\gamma^*}{1 - z\gamma^*}$$

$$= \frac{1}{1 - z\gamma^*} \Big(\underbrace{\cancel{z\gamma^*}(\cancel{1} - z^{-1}\gamma) + 1 - \cancel{z\gamma^*}}_{1 - \gamma^*\gamma} \Big) \frac{1}{1 - z^{-1}\gamma}$$

$$= \frac{1}{1 - z\gamma^*} (1 - \gamma^*\gamma) \frac{1}{1 - z^{-1}\gamma}$$

So what happens? You have

$$\langle F(z), G(z)\rangle_{H'} = \int \frac{d\theta}{2\pi} \left\langle F(z), \frac{1}{1-z\gamma^*}(1-\gamma^*\gamma)\frac{1}{1-z^{-1}\gamma} G(z)\right\rangle$$

$$= \int \frac{d\theta}{2\pi}\left\langle \sqrt{1-\gamma^*\gamma}\,\frac{1}{1-z^{-1}\gamma}F(z), \sqrt{1-\gamma^*\gamma}\,\frac{1}{1-z^{-1}\gamma}G(z)\right\rangle$$

so you find an isometry.

$$\boxed{\begin{aligned} L^2(S^1, X) &\xrightarrow{\ \sim\ } \overline{\bigoplus_{n\in\mathbb{Z}} u^n X} \\ F(z) &\longmapsto (1-\gamma^*\gamma)^{1/2}(1-z^{-1}\gamma)^{-1}F(z) \end{aligned}}$$

~~If $\|\gamma\| < 1$.~~

Back to

$$H = H^- \oplus X \oplus H^+$$

$$L^2(S^1, V) \cdots \oplus u^{-1}V \oplus V \oplus X \oplus W \oplus uW \oplus \cdots \longrightarrow L^2(S^1, W)$$

$$\uparrow$$

$$L^2(S^1, X)$$

What are the pieces?

$u = ba^{-1}$

$$X \overset{a}{\underset{b}{\rightrightarrows}} Y \qquad \begin{array}{c} aX \oplus \mathrm{Ker}(a^*) \\ \cong \\ \mathrm{Ker}(b^*) \oplus bX \end{array}$$

$$\gamma = i^* u i = a^* b a^{-1} a = a^* b : X \to X$$

$$\gamma^* = b^* a : X \to X.$$

$\mathrm{Ker}(1-\gamma^*\gamma)$?

$$\gamma^*\gamma x = x \Rightarrow \|\gamma x\| = \|x\|.$$

$$\|a^* b x\| = \|bx\| \iff bx \in aX$$

706 So you are confused about $1-\gamma^*\gamma$?

$$\begin{array}{ccc} aX & \oplus & W \\ & \| & \\ V' & \oplus & bW \end{array} \qquad \gamma = a^* u c = a^* b$$

First – dilate a s.a. contraction A: $\quad A = A^*$, $\; A^2 \le 1$.
to an involution. $\quad F = \begin{pmatrix} A & \sqrt{1-A^2} \\ \sqrt{1-A^2} & -A \end{pmatrix}$

Next – dilate a contraction γ. to a unitary

$$\begin{pmatrix} \sqrt{1-\gamma^*\gamma} & \gamma^* \\ \gamma & -\sqrt{1-\gamma\gamma^*} \end{pmatrix}\begin{pmatrix} \sqrt{1-\gamma^*\gamma} & \gamma \\ \gamma^* & -\sqrt{1-\gamma\gamma^*} \end{pmatrix}$$

$$\cancel{\begin{pmatrix} \gamma & -\sqrt{1-\gamma\gamma^*} \\ \sqrt{1-\gamma^*\gamma} & \gamma^* \end{pmatrix}}$$

So the simplest seems to be

$$\begin{pmatrix} \sqrt{1-\gamma^*\gamma} & \gamma^* \\ \gamma & -\sqrt{1-\gamma\gamma^*} \end{pmatrix}$$

Suppose you have $X' \xleftarrow{\gamma} X$ a contraction
and you want to dilate to a unitary, ~~seem~~ simplest
seems to add

$$\begin{array}{ccc} X' & \xleftarrow{\gamma} & X \\ & \sqrt{1-\gamma^*\gamma} & \\ X & \xleftarrow{\gamma^*} & X' \end{array}$$

Yes.

So how does this compare to $\quad \begin{array}{ccc} V & & aX \\ \oplus & \swarrow & \oplus \\ bX & & W \end{array}$

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$$\begin{tikzcd} V\oplus X \arrow[rr, "\sim"] \arrow[dr] & & X\oplus W \arrow[d] \\ X \arrow[u] \arrow[r, "a"] \arrow[rr, bend right, "\gamma"'] & Y \arrow[r, "b^*"] & X \end{tikzcd}$$

~~Note~~ Recall: ~~quotient~~ complete $X\oplus X = \{ax_1 + bx_2\}$

with $\|ax_1+bx_2\|^2$

$= \|x_1\|^2 + \|x_2\|^2 + (x_1, \underbrace{a^*b}_{\gamma^*} x_2) + (x_2, \overbrace{b^*a}^{\gamma} x_1)$

$\gamma = b^*a$

$= \|x_1 + \gamma x_2\|^2 + \|\sqrt{1-\gamma^*\gamma}\, x_2\|^2$

~~$= \|$~~

$$\|ax_1+bx_2\|^2 = \|x_1\|^2 + \|x_2\|^2 + (ax_1, bx_2) + (bx_2, ax_1)$$

$$= \|x_1\|^2 + \|x_2\|^2 + (\gamma x_1, x_2) + (x_2, \gamma x_1) + \|\gamma x_1\|^2 - \|\gamma x_1\|^2$$

$$= \|\sqrt{1-\gamma^*\gamma}\, x_1\|^2 + \|\gamma x_1 + x_2\|^2 \qquad \boxed{\gamma = a^*b}$$

$$\|x_1 + \gamma^* x_2\|^2 + \|(1-\gamma\gamma^*)^{1/2} x_2\|^2$$

This corresponds to $Y = V\oplus X = X\oplus W$

Next $\|(x_n)\|^2 = \|\sum z^n x_n\|^2 = \sum_{n\le m} (x_n, \gamma^{m-n} x_m) + \sum_{n>m} (\gamma^{n-m} x_n, x_m)$

$$\|x_0 + zx_1 + z^2x_2\|^2 = \|x_0\|^2 + (x_0, \gamma x_1) + (x_0, \gamma^2 x_2) + (\gamma x_1, x_0) + \|x_1\|^2 + (\gamma^2 x_2, x_0)$$

$$-\|x_0 + \gamma x_1 + \gamma^2 x_2\|^2$$

$u(ax) = bx$

$= aa^*bx + (1-aa^*)bx$

$= ~~\|x_0\|^2 + (\gamma x_1, x_0) + (\gamma^2 x_2, x_0)~~$

$(x_0, \gamma x_1) \quad \|x_1\|^2 - \|\gamma x_1\|^2 \quad (\gamma x_2, x_1) - (\gamma^2 x_2, \gamma x_1)$

$(x_0, \gamma^2 x_2) \quad (x_1, \gamma x_2) - (\gamma x_1, \gamma^2 x_2) \quad \|x_2\|^2 - \|\gamma^2 x_2\|^2$

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$$x_0 + ux_1 + u^2x_2 + \cdots$$
$$-(x_0 + \gamma x_1 + \gamma^2 x_2 + \cdots) \quad \in W \oplus uW \oplus u^2W$$
$$= ?$$

So think. You need a notation for $i(ux) - i(\gamma x) \in \operatorname{Ker}(a^*) = W$.

i.e $\quad a^*b$

$$\underbrace{u(ix)}_{bx} = i\gamma x + px$$
$$aa^*bx + (1-aa^*)bx$$

$$u^2 - \gamma^2 = (\gamma+p)^2 - \gamma^2 = \gamma p + p\gamma + p^2$$
$$= p\gamma + up$$

$$u^2x_2 = \gamma^2 x_2 + p(\gamma x_2) + u(px_2)$$

you want to identify p with $(1-\gamma^*\gamma)^{1/2} : X \longrightarrow (1-\gamma^*\gamma)^{1/2}X$ eventually.

So think. You have

$$ux = \gamma x + px \quad \in X \oplus W$$
$$u^2x = u\gamma x + upx$$
$$= \gamma^2 x + p\gamma x + upx \quad \in X + W + uW$$
$$u^n x_n = \gamma^n x_n + p\gamma^{n-1}x_n + up\gamma^{n-2}x_n + \cdots + u^{n-1}px_n$$

$$\sum_{n\geq 0} u^n x_n = \sum_{n\geq 0} \gamma^n x_n + p\sum_{n\geq 1}\gamma^{n-1}x_n + up\sum_{n\geq 2}\gamma^{n-2}x_n + \cdots$$

$$= \underbrace{\sum_{n\geq 0}\gamma^n x_n}_{\in X} + \underbrace{p\sum_{n\geq 0}\gamma^n x_{n+1}}_{\in W} + \underbrace{up\sum_{n\geq 0}\gamma^n x_{n+2}}_{uW} + \cdots$$

||

~~contr~~ Contraction of $h : X \to X'$

factor? $X \xrightarrow{a} Y \xrightarrow{b^*} X'$ where $a^*a = 1$, $b^*b = 1$.

$$\|ax + bx'\|^2 = \|x\|^2 + \|x'\|^2 + (ax, bx') + (bx', ax)$$
$$= \|x\|^2 + \|x'\|^2 + (hx, x') + (x', hx)$$

So let Y be the completion of $X \oplus X'$ with r.t. this inner product. Have obvious isom. embeddings $a : X \to Y$, $b : X' \to Y$.

$$(ax_1 + bx_1', ax_2 + bx_2') = (x_1, x_2) + (x_1', x_2') + (x_1', hx_2) + (hx_1, x_2')$$

How does this construction compare with a Grassm. one

$$\begin{pmatrix} \sqrt{1-h^*h} \\ h \end{pmatrix} X \subset \begin{matrix} X \\ \oplus \\ X' \end{matrix}$$ basically the same.

$$a = \begin{pmatrix} \sqrt{1-h^*h} \\ h \end{pmatrix} : X \hookrightarrow \begin{matrix} X \\ \oplus \\ X' \end{matrix} \xrightarrow{(0\ \ 1)} X'$$

---

what to do for lecture.

$H = H^- \oplus X \oplus H^+$ o.d.s $\quad u(H^+) \subset H^+$, $u^{-1}(H^-) \subset H^-$

Let $i : X \to H$ be the inclusion
$i^* : H \to X$ orth proj onto X.
$h = i^* u i$ compression of u to X

Claim: $i^* u^n i = \begin{cases} h^n & n \geq 0 \\ (h^*)^{-n} & n \leq 0. \end{cases}$

Proof. u carries $X \oplus H^+ = (H^-)^\perp$ into itself
H^+

so u induces an operator $\bar{u}$ on the quotient space $X \oplus H^+ / H^+$. namely $h = i^* u i$. Clearly $\overline{u^n} = \bar{u}^n$.

7ΦD $\quad \dim(X) = 1. \quad |h| < 1. \quad \xi_0 \in X \quad |\xi_0| = 1$

Smallest closed subspace of H containing X $\cap$ and stable under u, u^{-1}. Cyclic unitary representation of $\mathbb{Z}$

There's a Radon measure $d\mu$

$$L^2(S^1, d\mu) \simeq \overline{\bigoplus_{n\in\mathbb{Z}} u^n X} \subset H$$

$$z^n \longmapsto u^n \xi_0$$
$$f(z) \longmapsto f(u)\xi_0$$

$$(f,g) = \int \overline{f(z)}\, g(z)\, d\mu$$

$$\int z^n\, d\mu = (\xi_0, u^n \xi_0) = \begin{cases} h^n & n \geq 0 \\ \bar{h}^{-n} & n < 0. \end{cases}$$

$$d\mu = \underbrace{\left(\sum_{n\geq 0} z^{-n} h^n + \sum_{n>0} z^n \bar{h}^n\right)}_{\frac{1}{1-z^{-1}h} + \frac{z\bar{h}}{1-z\bar{h}}} \frac{d\theta}{2\pi}$$

$$\frac{1}{1-z^{-1}h} + \frac{z\bar{h}}{1-z\bar{h}} = \frac{1-|h|^2}{(1-z\bar{h})(1-z^{-1}h)}$$

$$\begin{tikzcd} L^2(S^1, V) \arrow[r, hook] & H & L^2(S^1, W) \arrow[l] \\ & L^2(S, d\mu) \arrow[u] & \end{tikzcd}$$

711 Try again to get a clean lecture.

h: contraction operator $X \longrightarrow X'$

$$a = \begin{pmatrix} \sqrt{1-h^*h} \\ h \end{pmatrix} \qquad b^* = (1 \;\; 0)$$

$$X \xrightarrow{a} \begin{matrix} X \\ \oplus \\ X' \end{matrix} \xrightarrow{b^*} X' \qquad b^*a = h$$

$$\|ax+bx'\|^2 = \|x\|^2 + \|x'\|^2 + (ax, bx') + (bx', ax)$$

$X \oplus X'$

$$\|x \oplus x'\|^2 \overset{\text{def}}{=} \|x\|^2 + \|x'\|^2 + (hx, x') + (x', hx)$$

$$= \|x + h^*x'\|^2 + (x', (1-hh^*)x')$$

$$= \|x' + hx\|^2 + (x, (1-h^*h)x)$$

$$Y = X \oplus \underbrace{(1-\gamma\gamma^*)^{1/2}X'}_{\simeq X'/\mathrm{Ker}(1-\gamma\gamma^*)^{1/2}}$$

Let's think about what happens over a graph.

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$$H = H^- \oplus X \oplus H^+ \qquad u(H^+) \subset H^+ \qquad u^{-1}(H^-) \subset H^-$$

$$H = \overline{\sum_{n\in\mathbb{Z}} u^n X}$$

$\iota : X \hookrightarrow H$

$$\iota^* u^n \iota = \begin{cases} h^n & n \geq 0 \\ (h^*)^{-n} & n \leq 0 \end{cases}$$

712 <u>Coupling</u>: You want to do things over the $PSL_2(\mathbb{Z})$ tree equivariantly.

For each valence 3 vertex of the tree we will get a little Hilbert space X equipped with contraction h acted on by $\mathbb{Z}/3$. We have to divide corresponding V, W into three pieces permuted cyclically. For each valence 2 vertex you have another (X, h) with $\mathbb{Z}_2$ acting so splitting of V, W.

Let's try to make this more explicit.

Each edge has two vertices I think I want Hilbert space assigned to each edge and to each vertex. At a vertex we have a three port and at an edge we have a 2 port. You now have an understanding of these ports in terms of partial unitaries. A partial unitary ~~will give is~~ is: $\overset{d}{X} \overset{a}{\underset{b}{\rightrightarrows}} \overset{d+n}{Y}$, 3 port means $n = 3$. Think of $\mathrm{Ker}(a^*)$ as the "out" gate and $\mathrm{Ker}(b^*)$ as the "in" gate. I think I want these gates to split according to incident edges. So what's basically going on is that at each vertex I will get a 3×3 unitary operator relating the in and out states at the edges at and then each edge we have a 2×2 unitary matrix relating in and out states at the two ends.

 Anyway, put the structure together
Whatever a port "is" it amounts to something you should be able to pin down. You want to pin down a port.

Things you are assuming to be true.

1) After coupling the components belonging to the edges and vertices you end up with a Hilbert space equipped with a unitary operator. Any n-port consists of n - gates, each gates gives you an incoming and outgoing line. When you couple two gates you identify the outgoing line of one to the incoming line of the other, and the reverse. This means the Hilbert space "over" the graph consists of the $Y = X \oplus W$ space for each vertex and for each edge? Wait. Think about connecting partial unitaries to obtain a unitary operator. It seems clear you can do this easily.

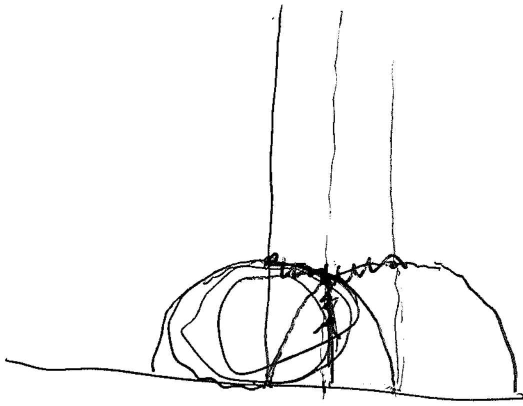
2) You should be able to understand the spectrum of the unitary operator by Gelfand type eigenvectors. By understand I mean seeing whether the spectrum is continuous or discrete. I'm looking for eigenvectors which decay on one side of an edge.

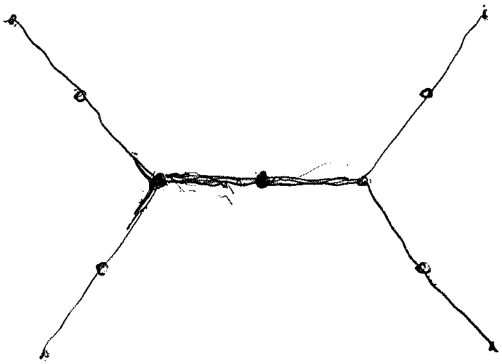
What is the eigenvector equation? Let's see.
You are seeking the Hilbert space. This consists of an X space for each vertex

 ~~[illegible]~~

You need to organize things. How? You first have to describe the Hilbert space and the unitary operator. It would be nice if the Hilbert space turns out to be some sort of space of L^2 cochains on the graph.

Picture of graph





At each 3 valent vertex I think of there being a partial unitary ~~[illegible]~~ $V \oplus X \xrightarrow{\sim} X \oplus W$ with $\mathbb{Z}/3$ action, and V, W split into three parts permuted cyclically. So ~~[illegible]~~ V, W split according to the edges issuing from v. ~~It seems we can think~~ ~~At the~~ At a 2-valent vertex you have a partial unitary with $\mathbb{Z}/2$ action, YES.

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Anyway what you learned yesterday was to fix the frequency and calculate the corresponding eigenvectors. Strange business. Once you fix a frequency an n-port becomes a unitary matrix.

linear graph.

2-port consists of 2 gates (gate = polarized 2-diml Hilbert space) and an isomorphism of the underlying pseudo-hermitian spaces.

Consider two polarized Hilbert spaces which are isomorphic as pseudo-Hilbert spaces. Equivalently a polarized Hilbert space with another polarization. The other polarization is the same as a strict contraction operator.

$$\begin{array}{c} H^+ \oplus H^- \\ \cup \\ W \end{array} \qquad W = \Gamma_h = \begin{pmatrix} 1 \\ h \end{pmatrix} H^+$$

where $\|hx\| < \|x\|$ $x \neq 0$.

$$\begin{array}{c} H^+ \\ \oplus \\ H^- \end{array} = \begin{array}{c} W \\ \oplus \\ W^\perp \end{array}$$ orth in the pseudo unmorph.

In the end you get

$$\begin{pmatrix} 1/\sqrt{1-h^*h} & h^*/\sqrt{1-hh^*} \\ h/\sqrt{1-h^*h} & 1/\sqrt{1-hh^*} \end{pmatrix}$$

?

So suppose your edge is

Start again. State space at the left end of the edge is $H^+ \oplus H^-$ and at the right ~~end~~ is $W^+ \oplus W^-$ and the edge at the fixed freq is an isom. $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ preserving the pseudo-hermitian product:

$$\|\xi^+\|^2 - \|\xi^-\|^2 = \|\eta^+\|^2 - \|\eta^-\|^2$$

which can be rewritten

$$\|\xi^+\|^2 + \|\eta^-\|^2 = \|\eta^+\|^2 + \|\xi^-\|^2$$

There are confusing aspects here. I have trouble with the signs. Also I would like to see the big unitary operator, at least it's eigenfunctions.

Look at a 2 port with transmission

What is a two port. It has two ~~ends~~ gates, each gate is a pol. Hilb space $H^+ \oplus H^-$, $W^+ \oplus W^-$. H^+ W^+ are ~~the space of~~ incoming spaces on the left + right, H^- and W^- are the outgo-

~~Energy is not lost, $\|\xi^+\|$~~ One gives a bijection between incoming + outgoing states:

$$H^+ \oplus W^+ \simeq H^- \oplus W^-$$

Energy is preserved $\|\xi^+\|^2 + \|\eta^+\|^2 = \|\xi^-\|^2 + \|\eta^-\|^2$.

Can be rewritten

$$\|\xi^+\|^2 - \|\xi^-\|^2 = -\|\eta^+\|^2 + \|\eta^-\|^2$$

net energy in at left — net energy out at the right

Assume good transmission: map $H^+ \longrightarrow W^-$ is an isomorphism.

We have a glimpse of cohomology.
Each edge gives a diagram

$$\begin{tikzcd} & \Gamma \arrow[dl, "\backsim"'] \arrow[dr, "\sim"] & \\ H^+ \oplus H^- & & W^+ \oplus W^- \end{tikzcd}$$

So at least on the linear graph we have a local system, and the only issue is the L^2 character of the cochains, and ~~what happens here~~ this reduces to the eigenvalues of the transfer matrix.

Next look at trivalent graphs to see what can be said. At a vertex you have six quantities, parameters, linked by three relations. Ultimately I should try to describe the state space.

Start from the viewpoint of a global Hilbert space with unitary operator. I think it is ~~completely~~ clear that ~~such~~ a pair exists starting from any 2 port with $\mathbb{Z}/2$ action and any 3 port with $\mathbb{Z}/3$ action. This Hilbert space would ~~appear to have~~ arise by placing the 3 port on each vertex and the 2 port on each edge.

$$\mathbb{C}[G]\otimes_A Y_A = \mathbb{C}[G]\otimes_A \big(X_A \oplus \overbrace{W_A}^{\mathbb{C}[A]\otimes W}\big)$$

$$= \big(\mathbb{C}[G]\otimes_A X_A\big) \oplus \mathbb{C}[G]\otimes W$$

$$\mathbb{C}[G]\otimes_A Y_A = \mathbb{C}[G]\otimes_A (V_A \oplus X_A) = \big(\mathbb{C}[G]\otimes V\big) \oplus \mathbb{C}[G]\otimes_A X_A$$

718 $A=\mathbb{Z}/3$, $B=\mathbb{Z}/2$, A similar partial unitary arises from the edges

$$\mathbb{C}[G]\otimes_B Y_B = \mathbb{C}[G]\otimes_B (X_B \oplus V_B)$$
$$= \mathbb{C}[G]\otimes_B X_B \oplus \mathbb{C}[G]\otimes V$$
and also $\simeq \mathbb{C}[G]\otimes W \oplus \mathbb{C}[G]\otimes_B X_B$

same V, W but interpreted with outgoing ~~&~~ incoming reversed. Thus I seem to get a nice Hilbert space with unitary. Why? You have two partial unitaries which can be glued together.

Examples. Consider the case of an iterated 2-port. You ~~can't~~ have then? In general you ought to ~~do things geom~~ describe things geometrically. For the $SL_2(\mathbb{Z})$-tree this means some sort of coefficient system on the tree. What did I describe above?

There are two types of vertices G/A G/B and edges G/C. What is the Hilbert space? You give

$Y_A = X_A \oplus W_A \simeq V_A \oplus X_A$ acted on by $A=\mathbb{Z}/3$

such that $V_A = \mathbb{C}[A]\otimes_C V_C$, $W_A = \mathbb{C}[A]\otimes_C W_C$;

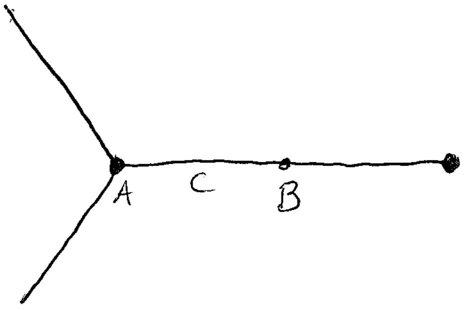
V, W associated to the distinguished edge. Similarly from dist. B vertex

give $Y_B = X_B \oplus (\mathbb{C}[B]\otimes V) = (\mathbb{C}[B]\otimes W)\oplus X_B$

719 You ~~xxxxx~~ have

$$\mathbb{C}[G]\otimes_A Y_A = \mathbb{C}[G]\otimes_A X_A \oplus \mathbb{C}[G]\otimes_B Y_B$$

I want to work this out very carefully.



At A we have $Y_A = X_A \oplus \mathbb{C}[A]\otimes_C W_C = \mathbb{C}[A]\otimes_C V_C \oplus X_A$

where V_C = incoming states to A from C

W_C = outgoing ~~from~~ A to C

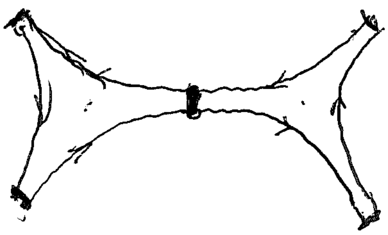
At B you have $Y_B = X_B \oplus$ not correct yet

~~Do not try to handle the gener~~

In general you could have Y_A, Y_C, Y_B

but instead you want

Feb 26 - ~~xxxxxxx~~ After much pacing the good picture of the tree seems to be



for handling the situation where the 3 port is just simple connections appropriate to a ribbon graph. To treat this as a perturbation where the perturbation occurs at the junctions The free situation is a simple infinite shift for each element of $P_1(\mathbb{Q})$.

The first point is the Hilbert space with $\Gamma = PSL_2(\mathbb{Z})$ action. There is a basis element for each side of the junction, Γ is acting simply transitively on this set. So we have $\ell^2(\Gamma)$ with Γ acting on the left and the translation $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on the right — this is the unperturbed operator. The perturbation takes place at the junction. You want to replace the perfect transmission to allow for some reflection.

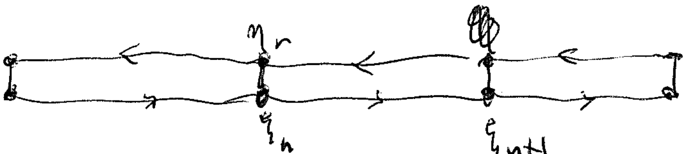
Because the operator you are constructing is automatically unitary — Jacobi type etc. There should be a response function associated to one side of the graph, and it should satisfy a functional equation.

$$f(z) = g\left(f(z)^2\right) \qquad g \in U(1,1)$$

?

end up with a cubic equation? missing z corresponding to a shift.

Now I have to get the formulas straight. Begin with the line $\mathbb{Z}$ acting on $\mathbb{R}$. Picture



unperturbed situation is $\ell^2(\mathbb{Z})^2 = L^2(S^1)^2$ with the unitary operator $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$

721 Thinking of ~~a copy one do~~ $\mathbb{C}$ at each vertex in this picture. Maybe it would help to use U, V maybe U_n, V_n. Keep trying.

What you want is two 2-ports connected in alternating fashion $\alpha\beta\alpha\beta\alpha\beta\ldots$

Let's give an orth basis ξ_n, η_n $n \in \mathbb{Z}$. ~~Basic~~ ~~perturbation~~ unperturbed $\begin{cases} u_0 \xi_n = \xi_{n+1} \\ u_0 \eta_n = \eta_{n-1}. \end{cases}$

perturbed operator will involve transmission and reflection

$$u\xi_n = \tau\xi_{n+1} + \rho\eta_n \qquad |\tau|^2+|\rho|^2=1.$$

$$u\eta_{n+1} = ?\,\xi_{n+1} \quad ?\,\eta_n$$

$$u\eta_n = ?\,\xi_n + ?\,\eta_{n-1}$$ <u>won't work.</u>

~~Let's start with H, a~~

In the end you want to start with a transfer matrix in $U(1,1)$ and produce a unitary operator on $\ell^2(\mathbb{Z})^{\oplus 2}$ commuting with translation. Picture: The transfer matrix yields a 2-port which can be coupled periodically to itself. (Look up $\boxed{\text{Quantum Hall Effect - Belessard}}$)

2-port transfer matrix $\begin{pmatrix} \sqrt{1-|h|^2} & h \\ \bar{h} & \sqrt{1-|h|^2} \end{pmatrix}$

$$Y = aX \oplus W$$
$$\parallel$$
$$V \oplus bX$$

in order to complete you must give an isom $W \simeq V$ with $\mathbb{Z}$-action

722 Choose basis ξ_0 for
Is Y a fund. domain?
Basically you have $Y = aX \oplus W$
$= V \oplus bX$
and you take a copy of Y
at each vertex: $\oplus z^n Y$
then you define u to be the isom $ba^* : aX \to bX$
direct sum with z times the isom of W with V
This means we have

$$\begin{matrix} X & & V & & X \\ \oplus & = & \oplus & \nearrow & \oplus \\ W & & X & & zW \end{matrix}$$

couple periodically.

Start carefully. The problem above is that it's one-diml.

Start again. You need a 2 port to begin and you propose one with $X = 0$. So all you have is a $U(1,1)$ matrix, between the left states and the right states. But then when you couple you need to identify the left and the right states. So in the end you have ~~identified the~~ constructed a flat bundle over the circle - cylinder construction. What's strange is that we (seem to) have a Hilbert space with unitary operator made out of ~~these~~ the sections of this bundle

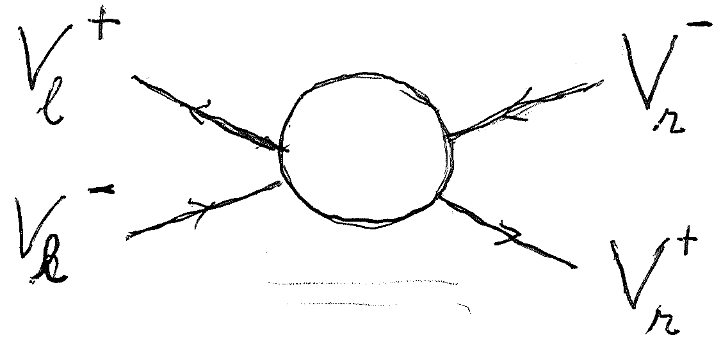
723 Start with a unitary matrix ~~having~~ invertible transmission. This should be ~~[illegible]~~ a self isomorphism of the same hermitian $\mathbb{Z}_2$ graded space

Really you are looking at $g \in \underline{U(2)}$, because you must identify the in and out spaces in order to proceed further. Presumably it should not matter if I conjugate by a diagonal matrix. ~~So that everything should depend ultimately~~

~~Presumably it should not~~

Start again with a 2-port indep of freq.

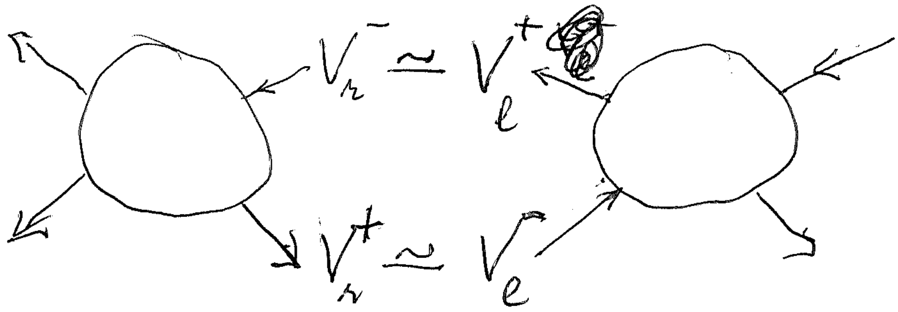
left state space right state space



$$\|\xi_l^+\|^2 + \|\xi_r^+\|^2 = \|\xi_l^-\|^2 + \|\xi_r^-\|^2 \qquad \text{cons. of eng}$$

$$-\|\xi_l^+\|^2 + \|\xi_l^-\|^2 = \|\xi_r^+\|^2 - \|\xi_r^-\|^2$$

Now couple these to form a chain



724 Now what you need is to construct the Hilbert space + unitary operator arising from this stuff.

Coupling two circuits yields another 2 port given by the product of transfer matrices. V^+, V^-.

I've run into a problem, namely, you seem ~~to~~ need an X around before iterating a 2 port yields an interesting Hilbert space and unitary operator

Consider the process of connecting 2 ports. Recall that a 2-port consists of a partial unitary

$$V^- \oplus X \simeq X \oplus V^+$$

together with decompositions

$$V^+ = V^+_l \oplus V^+_r$$

$$V^- = V^-_l \oplus V^-_r$$

To couple $\overbrace{\text{these}}^{\text{two of}}$ you take direct sum superscripts 1,2

$${}^1V^-_l \oplus {}^1X \oplus {}^1V^-_r \simeq {}^1V^+_l \oplus {}^1X \oplus {}^1V^+_r$$

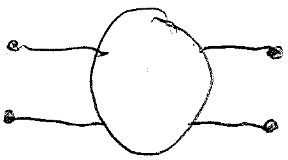
$${}^2V^-_l \oplus {}^2X \oplus {}^2V^-_r \simeq {}^2V^+_l \oplus {}^2X \oplus {}^2V^+_r$$

725 You need isomorphisms

$$^1V_r^+ \simeq {}^2V_l^- \qquad {}^1V_r^- \simeq {}^2V_l^+$$

From this you should be able to get an isom.

$$^1V_l^- \oplus {}^1X^* \oplus {}^2X^* \oplus {}^2V_r \qquad ??$$



What do you think happens? A 2-port is like a R.S. with 2 boundary components

A boundary component can be identified with a pseudo hermitian plane.

Coupling requires glueing boundary components corresponds to composing transf. matrices.

Somehow you have to understand what's going on

You need to do things ~~[illegible]~~ on a line.

The aim of ~~the~~ glueing is to obtain an analytic function of z whose zeroes are on the unit circle. This means essentially

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exhibiting a unitary op with disc. spectrum

726 You want to construct a unitary op by glueing partial unitaries together. Finite dimensional partial unitary have degree = dim X.

I need good examples. If I want inf. dim unitary then I need "infinite degree". So I need at least to start with dim(X) = 1 and glue copies of this together. Understand this carefully.

~~So I want to look at the simplest~~

So I want to self-couple a two port so as to obtain a ~~constant~~ unitary with $\mathbb{Z}$-symmetry. So start with the 2-port

$$Y = aX \oplus W = V \oplus bX$$

Here W, V have dim 2, X has dim 1, so $V \cap W$ is 1-diml. But W, V are split. Really $V = V^+$, $W = V^-$ where $V^+ = V^+_\ell \oplus V^+_r$ $V^- = V^-_\ell \oplus V^-_r$ and to couple one needs to identify the left and right gates: $V^-_r \simeq V^+_\ell$ $V^+_r \simeq V^-_\ell$

Now this situation may seem special, but you should be able to see a Hilbert space and unitary operator in all of this.

~~Look: The Hilbert~~

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727 You start with $X \rightrightarrows Y$ and want to couple two copies.

Consider Schroedinger equation periodic potential.

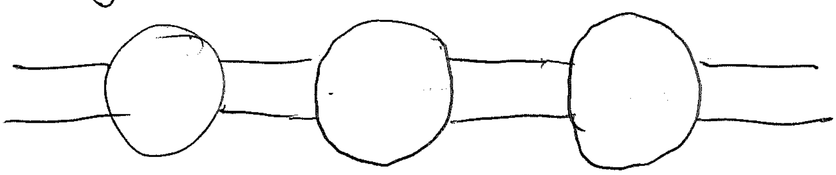
You need to understand the s.a. Schroed. operator versus the transfer matrix, Bloch eigenfunctions, Q. Hall eff.

Let's see what you understand, or can construct.

~~You consider zilch~~ The real

Try the following: Begin with a 2-port with invertible transmission, ~~take a 2-port~~ ?

Do slowly. You take a 2-port



and couple it to itself periodically. Does this give you a Hilbert space & unitary operator commuting with $\mathbb{Z}$ translation? I have been assuming this is obvious. There should not be a problem. Each port gives a partial unitary

$$V \oplus X \simeq X \oplus V^+$$

You then have a partial unitary:

$$\bigoplus z^n V^- \oplus \bigoplus z^n X \simeq \bigoplus z^n X \oplus \bigoplus z^n V^+$$

and you have an identification $\bigoplus z^n V^- \simeq \bigoplus z^n V^+$ coming from the splitting $V^- = V^-_\ell \oplus V^-_r$ $\quad V^+ = V$

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This works even if $X = 0$.

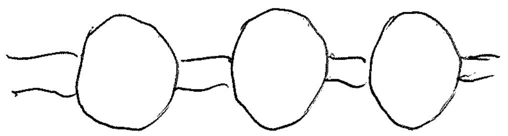
728 ~~Anyway of what next?~~

Somewhere here there is a way to handle a constant unitary. You start with $V^- \simeq V^+$

Feb 27. Coupling.

You have to set up a proper notation.

The idea is to take a 2 port and connect it to itself ~~infinitely~~ periodically.



so as to obtain something with $\mathbb{Z}$-symmetry. This should be an analog of a Schroedinger equation with periodic potential.

Begin by coupling two copies:

$$V_\ell^- \oplus \overset{L}{V_r^-} \oplus X \xrightarrow[\sim]{\theta} X \oplus V_\ell^+ \oplus V_r^+$$

$$\underset{M}{zV_\ell^-} \oplus zV_r^- \oplus zX \xrightarrow[\sim]{z\theta z^{-1}} zX \oplus zV_\ell^+ \oplus zV_r^+$$

We are given $V_r^+ \simeq zV_\ell^-$, $V_r^- \xrightarrow{\sim} zV_\ell^+$

Take direct sum you get

$$V_\ell^- \oplus zV_r^- \oplus L \oplus M \oplus X \oplus zX \simeq$$

$$X \oplus zX \oplus M \oplus L \oplus V_\ell^+ \oplus zV_r^+$$

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Point is that you will have

729 Point: You ~~[illegible]~~ take the direct sum of two partial unitaries, then extend the partial unitary by using the boundary identifications. ~~The important thing to note is~~ Notice that the ~~[illegible]~~ partial unitary will be defined on $X \oplus zX$ $\oplus$ the ~~[illegible]~~ connecting gate, ~~[illegible]~~ which is $V_r^+ \simeq zV_l^-$ $\oplus$ $V_r^- \simeq zV_l^+$. ~~Note that partial unitary is too~~

So what do I do in general. You are trying to put something ~~the universal cover~~ over a circle. ~~You want shift operator~~

How to set things up over $\mathbb{R} = \tilde{S}^1$? Basically you put something over $[0,1]$ and give clutching data at the ends. ~~In our~~ What does this mean in our case. You start with the partial unitary

$$Y = aX \oplus V^+ = V^- \oplus bX$$

(notice order different from preceding page)

$$V^+ = V_l^+ \oplus V_r^+ \qquad V^- = V_l^- \oplus V_r^-$$

glueing: $V_r^+ \simeq zV_l^-$, $V_r^- \simeq zV_l^+$

I think you want

$$\bigoplus z^n Y = \left(\bigoplus z^n aX\right) \oplus \left(\bigoplus z^n V^+\right)$$
$$= \left(\bigoplus z^n V^-\right) \oplus \left(\bigoplus z^n bX\right)$$

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730 ~~Does it have any eigenfunctions~~

The important point is that we produce a unitary operator because then it ~~should~~ might be possible to get a functional equation.

$$L^2(S^1,Y) = L^2(S^1,aX) \oplus L^2(S^1,V_l^+) \oplus L^2(S^1,V_r^+)$$

$z\cdot$ (given iso) $\qquad z^{-1}$ (given iso)

$$= L^2(S^1,V_l^-) \oplus L^2(S^1,V_r^-) \oplus L^2(S^1,bX)$$

take $X=0$. Then we have two orthogonal splittings

$$L^2(S^1,Y) = L^2(S^1,V_l^+) \oplus L^2(S^1,V_r^+)$$

$$= L^2(S^1,V_r^-) \oplus L^2(S^1,V_l^-)$$

$$Y = V_l^+ \oplus V_r^+ = V_r^- \oplus V_l^-$$

These are two orthogonal splittings and in addition you give isos. $V_l^+ \xrightarrow{\sim} V_r^-$ $\quad V_r^+ \xrightarrow{\sim} V_l^-$. Count dimensions $S^2\times S^2\times S^1\times S^1$ dim $=6$. Act on it by $U(2)$. NO! ~~Choose orth bases for V_l^+, V_r^+; then~~

If you fix the orth decomp. $Y = V_l^+ \oplus V_r^+$, then to give another orth. decomp $Y = V_r^- \oplus V_l^-$ together w unit isos. $V_l^+ \xrightarrow{\sim} V_r^-$ $\quad V_r^+ \xrightarrow{\sim} V_l^-$ is the same as

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giving a unitary isom $Y \xrightarrow{\sim} Y$.

731 So take a 2×2 unitary matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and then we get a unitary on $L^2(S^1)^{\oplus 2}$ commuting with z, should be a unitary matrix function, namely $\begin{pmatrix} az & b \\ c & dz^{-1} \end{pmatrix}$. Since this commutes with z there should be simultaneous "eigenvectors" essentially, namely eigenvectors for the above matrix.

$$\lambda^2 - (az + dz^{-1})\lambda + (ad-bc) = 0$$

Suppose $ad - bc = 1$. $\begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix}$

$$\lambda^2 - (az + \bar a z^{-1})\lambda + 1 = 0.$$

suppose $a = 1$. the roots are $\lambda = z, z^{-1}$.

The unitary $\begin{pmatrix} az & b \\ c & dz^{-1} \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$

Next look at the 1-port given by the half-line ∂. What is the response function? Use the same eigenfunctions.

Wait first take $0 < a < 1$. Then for $|z| = 1$ $\frac{az + \bar a z^{-1}}{2}$ is real in $[-a, a]$ and $\lambda = e^{\pm i\theta}$ where $\cos\theta = \frac{az + \bar a z^{-1}}{2}$.

As z ranges over S^1, λ

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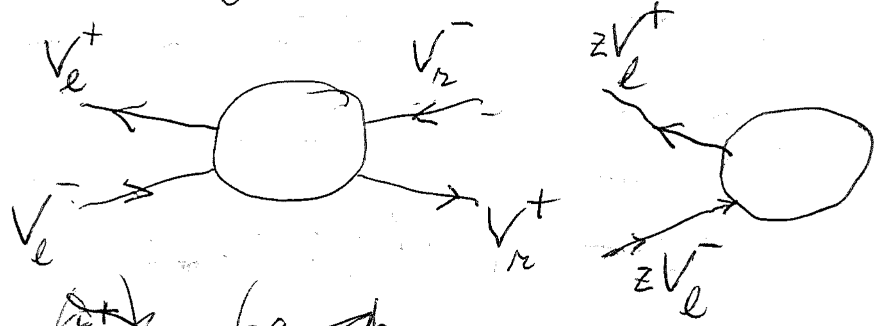
We have a curve: (λ, z) satisfying

$$\lambda^2 - (az + \bar a z^{-1})\lambda + 1 = 0$$

describing the simultaneous "eigenfunctions" for z and $\begin{pmatrix} az & b \\ -\bar b & \bar a z^{-1} \end{pmatrix}$.

To finish the picture you should check results via the transfer matrix. You have the unitary matrix



$$\begin{pmatrix} \xi_l^+ \\ \xi_r^+ \end{pmatrix} = \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \begin{pmatrix} \xi_r^- \\ \xi_l^- \end{pmatrix}$$

$$\xi_l^+ = \left(a + \frac{|b|^2}{\bar a}\right)\xi_r^- + \frac{b}{\bar a}\xi_r^+$$

$$\xi_l^+ = a\xi_r^- + b\xi_l^-$$

$$\xi_r^+ = -\bar b\xi_r^- + \bar a\xi_l^- \implies \xi_l^- = \frac{1}{\bar a}\xi_r^+ + \frac{\bar b}{\bar a}\xi_r^-$$

$$\begin{pmatrix} {}_0\xi_l^- \\ {}_0\xi_l^+ \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar a} & \frac{\bar b}{\bar a} \\ \frac{b}{\bar a} & \frac{1}{\bar a} \end{pmatrix} \begin{pmatrix} \xi_r^+ \\ \xi_r^- \end{pmatrix} = \begin{pmatrix} \frac{1}{\bar a} & \frac{\bar b}{\bar a} \\ \frac{b}{\bar a} & \frac{1}{\bar a} \end{pmatrix} \begin{pmatrix} z\,{}_1\xi_l^- \\ z^{-1}\,{}_1\xi_l^+ \end{pmatrix}$$

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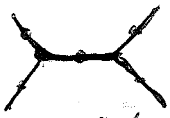
$$\frac{1-|b|^2}{\bar a^2} = \frac{a\bar a}{\bar a^2} = \frac{a}{\bar a}$$

$$\begin{pmatrix} \frac{z}{\bar a} & \frac{\bar b}{\bar a} z^{-1} \\ \frac{b}{\bar a} z & \frac{1}{\bar a} z^{-1} \end{pmatrix}$$

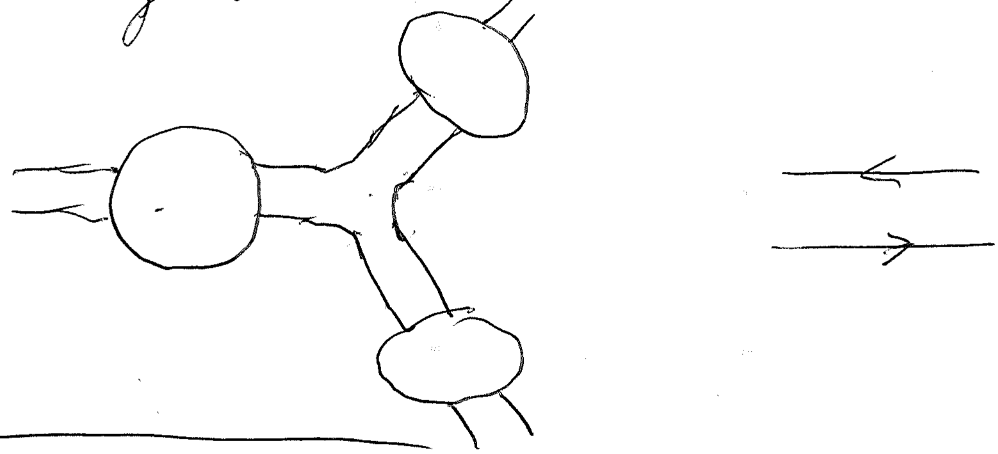
$$\lambda^2 - \left(\frac{z}{\bar a} + \frac{z^{-1}}{\bar a}\right)\lambda + \frac{a}{\bar a} = 0$$

733. I have to move on to Roe's stuff.

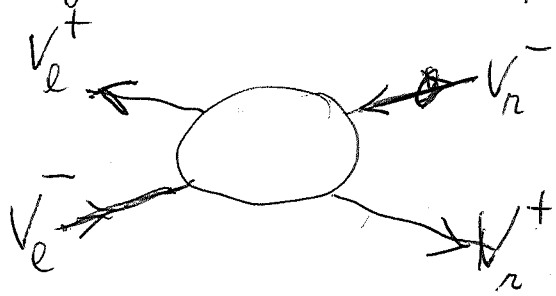
$\Gamma = PSL_2(\mathbb{Z})$ acting on tree 

You want the simplest example with scattering

Use cyclic ordering at the 3 vertices.



Go on: You need a 2 port with $\mathbb{Z}/2$ acting



$$Y = V^- \oplus aX \Rightarrow bX \oplus V^+$$

I think you want $\mathbb{Z}/2$ to act on Y preserving X, a, and b, hence also V^- and V^+. It should interchange V_r^+ and V_l^+, also V_l^- and V_r^-

If X is zero you have simply $Y = V^- = V^+$ hence ~~a unitary operator~~ two polarizations of Y namely $V_l^- \oplus V_r^-$ and $V_l^+ \oplus V_r^+$

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734 more 2-port stuff

$$Y = V^- = V^+$$

$$Y = V_\ell^- \oplus V_r^- = V_\ell^+ \oplus V_r^+$$

want $\mathbb{Z}/2$ action, ~~that~~ flips left and right. Can suppose $Y = \mathbb{C}^2$ with $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Ask when $L \perp \sigma L$?

Suppose $L = \mathbb{C}\begin{pmatrix} 1 \\ k \end{pmatrix}$, then $\sigma L = \mathbb{C}\begin{pmatrix} 1 \\ -k \end{pmatrix}$, these are $\perp$ where $\begin{pmatrix} 1 \\ k \end{pmatrix}^* \begin{pmatrix} 1 \\ -k \end{pmatrix} = (1 \ \ \bar{k})\begin{pmatrix} 1 \\ -k \end{pmatrix} = 1 - |k|^2 = 0$

so the possibilities are described by S^1.

It's time to ~~describe~~ examine 2-ports with $\mathbb{Z}/2$ symmetric. Want $X \overset{a}{\underset{b}{\rightrightarrows}} Y$ unitary reps of $\mathbb{Z}/2$.

A 2-port is more than a partial unitary - It's like a manifold with two boundary components

First discuss $X = 0$. Have $Y = V^- = V^+$ where $V^- = V_\ell^- \oplus V_r^-$, $V_\ell^+ \oplus V_r^+$. Thus Y is a herm. space with 2 orth. decompositions. Have $\mathbb{Z}/2$ action on Y preserving these decompositions

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735 Suppose $Y = \mathbb{C}^2$ with $\sigma = \begin{pmatrix}0&1\\1&0\end{pmatrix}$

Let $L = \mathbb{C}\begin{pmatrix}x\\y\end{pmatrix}$ be such that L and σL are $\perp$.

$$\begin{pmatrix}x\\y\end{pmatrix}^*\begin{pmatrix}y\\x\end{pmatrix} = x^*y + y^*x = \bar{x}y + \bar{y}x = 0$$

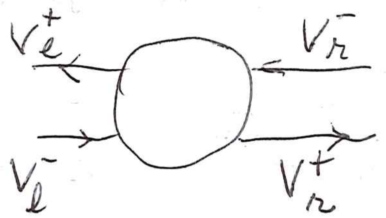
e.g. $\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$. Can suppose x real, if $x \neq 0$ then y is purely imag. So dealing with

$L = \begin{pmatrix}1\\it\end{pmatrix}\mathbb{C}$ $\quad t \in \mathbb{R}$

$$L, L^\perp = \begin{pmatrix}\cos\theta & i\sin\theta\\ i\sin\theta & \cos\theta\end{pmatrix}$$

unitary matrix $\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix}$ with a real and $b \in i\mathbb{R}$.

Life goes on.



$$\begin{pmatrix}\xi_\ell^+\\ \xi_r^+\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\xi_r^-\\ \xi_\ell^-\end{pmatrix}$$

$$\begin{pmatrix}d & c\\ b & a\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}$$ inv. under conj by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$

$\begin{pmatrix}a & b\\ b & a\end{pmatrix}$ and unitary means $|a|^2 + |b|^2 = 1$, $a\bar{b} + b\bar{a} = 0$.

can make a real by mult. by a scalar matrix.

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736 Let's now understand the effect of X. We are given

$$Y = V^{-} \oplus bX = \ldots$$

(Still you have to get a, b straight.)

Let's review the scattering business.

$$Y = aX \oplus \mathrm{Ker}(a^*) = \mathrm{Ker}(b^*) \oplus bX$$

$$= \overline{(aX + bX)} \oplus \left(\mathrm{Ker}(a^*) \cap \mathrm{Ker}(b^*)\right)$$

$$\|ax + bx'\|^2 = \|x\|^2 + \|x'\|^2 + (ax, bx') + (bx', ax)$$

$$= \|x\|^2 + \|x'\|^2 + (b^*ax, x') + (x', b^*a x)$$

Put $h = b^*a : X \circlearrowleft$. Then $aX + bX$ can be recovered from X, h as the completion of $X \oplus X$ wrt $\|x + x'\|^2 = \|x\|^2 + \|x'\|^2 + (hx, x') + (x, hx')$

$$\|hx\|^2 - \|hx\|^2$$

$$= \|x' + hx\|^2 + \|x\|^2 - \|hx\|^2$$

so find $\overline{aX + bX} \simeq X \oplus \overline{\sqrt{1 - h^*h}\, X}$

$$ax + bx' \longmapsto (hx + x') \oplus \sqrt{1 - h^*h}\, x$$

similarly find

$$\overline{aX + bX} \simeq X \oplus \overline{\sqrt{1 - hh^*}\, X}$$

$$ax + bx' \longmapsto (x + h^*x') \oplus \sqrt{1 - hh^*}\, x$$

Next discuss the scattering operator. Motivation ~~eigenvalues~~ eigenvectors for partial unitary.

737. Scattering operator? $\qquad h = b^*a$

~~[illegible]~~ $a - \zeta^{-1} b : X \longrightarrow Y$

$$\|(a - \zeta^{-1}b)x\|^2 = (x, (a^* - \bar{\zeta}^{-1}b^*)(a - \zeta^{-1}b)x)$$
$$= (x, (1 + |\zeta|^{-2} - \bar{\zeta}^{-1}h - \zeta^{-1}h^*)x)$$

$$\|(\zeta a - b)x\|^2 = \|\zeta a x\|^2 + \|x\|^2 - (\zeta a x, bx) - (bx, \zeta a x)$$
$$= \|x\|^2 + |\zeta|^2\|x\|^2 - (\zeta h x, x) - (x, \zeta h x)$$
$$+ \|\zeta h x\|^2 - \|\zeta h x\|^2$$
$$= \|x - \zeta h x\|^2 + \underbrace{|\zeta|^2(\|x\|^2 - \|hx\|^2)}_{\geq 0}$$

$$\|x - \zeta h x\| \geq \|x\| - \|\zeta h x\| \geq \|x\| - |\zeta|\|x\|$$

So if $|\zeta| < 1$ $\quad \zeta a - b$ is invertible no injective with closed image: top. isom

of
$$\begin{array}{ccc} X & \xrightarrow{\zeta a - b} & X \\ & \searrow^{\cong} & \downarrow b^* \\ & & (1 - \zeta h)X \end{array}$$

so get isom

$\mathrm{Ker}(b^*) \longrightarrow$

$$\begin{tikzcd} & \mathrm{Ker}(b^*) \arrow[d] \\ X \arrow[r, "-(\zeta a-b)"] \arrow[dr, "1-\zeta h"', "\sim"] & Y \arrow[d, "b^*"] \\ & X \end{tikzcd}$$

$Y/(\zeta a-b)X$

Point. $Y = (\zeta a-b)X \oplus \mathrm{Ker}(b^*)$ $\quad |\zeta|<1$.

What are you doing? You have a partial unitary $X \overset{a}{\underset{b}{\rightrightarrows}} Y$. It can be extended to a unitary iff $\mathrm{Ker}(a^*)$ $\mathrm{Ker}(b^*)$ are isom. You don't expect a scattering operator except in this case. How can you expect a unitary isom between $\mathrm{Ker}(a^*)$, $\mathrm{Ker}(b^*)$ otherwise.

How do you actually construct the scattering operator. Simplest method is to couple Y to a transmission line.

Try to organize the Γ case. Straighten out the Hilbert space and unitary operator. I want to look at the case where both ports have zero X. Remember that when you connect you identify two gates, but ~~they become~~ one copy appears in the Hilbert space.
