Consider $V = U/W$ a subquotient of $H = H^+ \oplus H^-$. We then get a Lagrangian subbundle $F \subset \mathcal{O}(V \oplus V^*)$ over $P^1$. You want to construct the canonical resolution of $F$:

$$0 \to F \to \mathcal{O} \otimes H^0(F^* \otimes (-1)) \to \mathcal{O}(1) \otimes H^0(F^* \otimes 1) \to 0$$

where $d = \dim(H)$ and $n = \dim(V)$. My guess is that $H^0(F^* \otimes (-1)) = H$, $H^0(F^* \otimes 1) = U \oplus W^\perp$.

Let's discuss simpler cases: $U = H$ or $W = 0$. $W = 0$ is the case where $V = \mathcal{O} \subset \mathcal{O}^* = H$, i.e. all modes of the circuit are external. You are used to working with $U = H$ and the map $p : H \to W$. Consider this case. On $H$ you have $A_s = s\pi_+ \oplus s^{-1}\pi$.

$$A_s^\oplus = s\pi_+ \oplus s^{-1}\pi_-= \text{ on } H = H^+ \oplus H^-$$

$$A_s^{-1} = s^{-1}\pi_+ \oplus s\pi_-$$

$$\rho A_s^\rho \pi^4 = s^{-1}(\rho \pi_+ \pi^4) + s(\rho \pi_- \pi^4) \text{ on } W^\perp = \bigoplus \omega_{\rho}$$

Where are you? You have better go back to case $V = U = \mathcal{O} \hookrightarrow H = \mathcal{O}^*$.

$$A_s = s\pi_+ + s^{-1}\pi_- \text{ on } H^+ \oplus H^-$$

$A_s^{-1} = s^{-1}\pi_+ + s\pi_-$

$A_s^{-4} = s^{-1}(\pi_+ \pi^4) + s^{-1}(\pi_- \pi^4)$ on $V$.

$A_s^4$ is a quadratic form in $V$, gives us a map $V \to V^* = V^*$, so we have $F_s = (\pi^* A_s^4)V \subset V^*$.

To understand this, you use sp. thm. for $\pi_+^* i$:

$$i^* A_s^4 = \bigoplus \omega \frac{s + s^{-1}\omega^2}{1 + \omega^2} \pi_\omega \text{ on } V = \bigoplus \omega V_\omega$$
What happens is that everything here is a direct sum. All you have is a subspace \( U \) in \( H = H^+ \oplus H^- \), i.e. an rep. of dihedral group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). For you, however, cases to look at

\[
U = V = \begin{pmatrix} 1 \\ -1 \end{pmatrix} H^+ \subset \frac{H^+}{H^-} = \mathbb{R}^2
\]

\[
V = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \mathbb{R} \subset \mathbb{R}^2
\]

Wait. \( V = \mathbb{R} \). \( i: V \to H \) \( i = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{1+\omega^2}} \).

And what a mess.

Take \( H = \mathbb{R}^2 \). Take \( H = \mathbb{R}^2 \). \( V = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mathbb{R} \)

quad. form is \( s \mapsto s_1^2 + s_2^2 \omega^2 \). \( s = (s_1, s_2, \omega^2) \).

normalize at \( s = 1 \) to be canonical form to get \( s \mapsto \frac{s_1^2 + s_2^2 \omega^2}{1 + \omega^2} \).

Then \( F_s = \begin{pmatrix} 1 \\ s_1 + s_2 \omega^2 \\ 1 + \omega^2 \end{pmatrix} \mathbb{R} \subset \mathbb{R}^2 = V \oplus V^* \). \( \frac{Ls^2 + 1}{Cs^2 + 1} \).

They have poles at \( s = 0, \infty \). \( Z_s = \frac{s_1 + s_2 \omega^2}{1 + \omega^2} \).

You have \( F = \mathcal{O}(-2) \).

You want to embed \( F_s \) into \( \mathbb{R}^3 = V \oplus H \). You have chosen two maps \( F_s \to \mathbb{R} \) from a 3 dual space.

There should be a better viewpoint. Let's try something Lagrangian. Start with \( H \) and \( \mathbb{Q} \) on \( H^+ \oplus H^- \). Then what do we get? This quad.
gives a Lagr. subbundle \( F = F_+ \oplus F_- = (H^+ \oplus H^-) \). \( F_+ = \text{graph}\ (1) \subset \mathcal{O}(H^+ \oplus H^*) \).

\( F_+ = \mathcal{O}(-1) \Rightarrow H^+ \).
Abstract question: Given quadratic form $g$ on $H$ with certain non-deg. properties, then get induced q.f. on any subquotient. Can you formulate this symplectically? First question is: given Lagrangian subspace of a symplectic vector space and an isotropic subspace, when does the symplectic vector space split naturally, e.g. symplectic quotient by an isotropic subspace.

Take

$$0 \to Y \to X \to X/Y \to 0$$

$$0 \leftarrow Y^* \leftarrow X^* \leftarrow Y^0 \leftarrow 0$$

So an interesting point is that given a Lagrangian subspace $Y$ and the quotient $X/Y$ are dual, so you get another symplectic space $Y \oplus X/Y$. So to $F$ and the quotient bundle $\mathcal{O} \otimes (V^* \otimes V)$ are self-dual.

I just learned that a Lagrangian subspace $Y$ and the quotient $X/Y$ are dual, so that the even graded $Y \oplus X/Y$ is naturally symplectic.

Next a Lagrangian subspace by itself has no intrinsic notion of positivity. But what is positive? Once $V \oplus V^* = X$ is chosen, i.e. Lagrangian subspace + lag. complement, then another to any Lagrangian $Y$ complementary to both both $V, V^*$ is described by a non-deg. quadratic form $g: V \to V^*$, so it has a signature. Can you really get to the bottom of the situation.
You have to analyze things further. Initially, you have $0 < W < U < H = H^+ \oplus H^-$.

You get $Q_s = \sigma Q_+ \oplus s^{-1} Q_-$, a pos. quad form on $H$.

You find the induced g.f. on $U/W = V$. This gives a Lagrangian subbundle $F$ of $\mathcal{O}(V^* \oplus V)$.

The problem is to find, compute, describe the canonical resolution.

\[ 0 \to F \to \mathcal{O} \otimes H^0(F^*)^* \to \mathcal{O}(1) \otimes H^0(F^{-1})^* \to 0 \]

Jean drives Cindy to work at 10:45. Then to Bislett Village, I'm alone with Anne-Marie at 11:00.

So how to proceed? I propose to use a symplectic approach. One has $F \subset \mathcal{O}(H \oplus H^*)$ to begin with, and ends with $F \subset \mathcal{O}(V \oplus V^*)$.

What's the geometry? From quad form viewpoint, you have

\[ U \to H \]
\[ \downarrow \]
\[ W/W \to H/W \]

\textbf{Question:} Should you be looking at orthogonal subspaces for the quadratic form? Thus you have $Q$ on $H^+ \oplus H^-$ singular at $s = 0, \infty$ but the singularities separate in the direct sum. So you can form $W_s^0$. You can take $0 < W < U < H$ and split this filtration using $Q_s$. So what.

If $A_s = s \sigma_+ \oplus s^{-1} \sigma_-$ on $H^+ \oplus H^-$

then $W_s^0 = A_s^{-1}(W^+)$. Some understanding might be achieved in this fashion.

But you would like a symplectic approach passing from $F$ Lagrangian in $\mathcal{O}(H \oplus H^*)$ to $F$ Lagrangian in $\mathcal{O}(V \oplus V^*)$, where $V$ subquot of $H$. 
You have to recall understand this stuff with subquotients. Consider

\[ 0 \rightarrow W \xrightarrow{\iota} H \xrightarrow{p} H/W \rightarrow 0 \]

What is the best way, a good way to describe including a quad form to the subquotient \( \psi = U/W \). Simplest formula

\[ Q(\psi) = \inf_{\omega \in W} Q(\psi + \omega) \]

Stationary value

So there a way to translate stationary value, critical value into Lagrangian subspace terms? Pull-back, intersect?

I should be able to settle this. Basically, you have \( 0 < W < U < H \), \( V = U/W \), \( A : H \rightarrow H^* \) and you end up with \( A_1 : V \rightarrow V^* \). Process

\[ \begin{array}{ccc}
U/W & \xleftarrow{\phi_i} & U \\
\downarrow{\phi_i} & & \downarrow{\phi_i} \\
(U/W)^* & \xleftarrow{\phi_i} & U^* \\
\end{array} \]

\[ \begin{array}{ccc}
H & \xrightarrow{p} & H/W \\
\uparrow{A^{-1}} & & \uparrow{(pA^T)^{-1}} \\
H^* & \xrightarrow{p^T} & (H/W)^* \\
\end{array} \]
So you've described the process of inducing $A : H \to H^*$ to a $A_1 : V \to V^*$ where $V$ is a subquotient, namely

$$A_1 = (p_i (c^t A_i)^{-1} p_i)^{-1} = c^t (p A^{-1} p^t)^{-1} c$$

$$U \xleftarrow{i} H$$
$$\xrightarrow{P} \xrightarrow{P}$$
$$U/W \xleftarrow{i} H/W$$

Now might there be a symplectic version of this formula? You have a Lagrangian subspace $Y$ of the symplectic space $X$. What's the symplectic analog of a subquotient $U/W$ of $V$? Try a symplectic subquotient $Y/Z$ where $Z \subset Y \subset X$ are symplectic subspaces, i.e. the restriction of the symplectic form on $X$ to $Y$ and $Z$ is non-degenerate.

12:13 Alicia returns.

Suppose $X = H \oplus H^*$, does a subquotient of $H$ determine a symplectic quotient of $X$? Probably not. You have $0 < W < U < H$

$$0 < U < W \subset H^*$$

But there's a notion of symplectic quotient, starting from an isotropic subspace. Above you have using definition of symplectic quotient.
Let's start with $W < H$.

$H \oplus H^* \rightarrow H/W \oplus W^*$

Work abstractly. Given $X$ symplectic consider symplectic flag manifold, $G = KAN$

Iwasawa decomp for $Sp_n(R)$. I think that $KnB = T$ and $B = TAN$. $SL_2(R) = SO(2) \times \mathbb{R}^+ \times \mathbb{R}^+$

I think that the symplectic flag manifold consists of flags of isotropic subspaces. Count dimensions

$\dim Sp_n = 2n^2 + n = \frac{n^2 + n}{2}$

$K = U_n$ complex symm. in $n \times n$ that

symplectic flag man. has $\dim = \dim (K/T) = n^2$ if $T = O(1)^n$

$2n-1 + 2n-3 + \cdots + 3 + 1 = \frac{n^2}{2}$

$F_1 \subset F_2 \subset \cdots \subset F_n$

Assume this is true, i.e. that the only interesting symplectic quotients arise from isotropic subspaces.

First case

$\subseteq \begin{array}{c} \text{subspace trick} \quad \omega(M, X) = 0 \Rightarrow x \in M \\
X \leftarrow M \rightarrow Y \\
W \oplus H^* \rightarrow \downarrow U^* \\
N \leftarrow \rightarrow W \\
U^* \downarrow \\
Z \end{array}$
Suppose there is notion of symplectic quotient: pass to superlag subspace, divide by null subspace. Alternative is to divide by an isotropic subspace and pass to subspace where skew form is well defined.

**Question:** Given two subspaces \( W \subset U \), is there some way to relate \( U/W \oplus (U/W)^* \) to \( X? \) Suppose \( U = \text{Lagrangian} \). Then \( U^0 \hookrightarrow X \)

\[
0 = U^0/U
\]

It seems that there is an angle here that is not a consequence of symplectic philosophy. Namely, if \( W \subset U \) are isotropic subspaces of \( X \) symplectic, then there doesn't seem to be a natural symplectic space with max isotropic subspace \( U/W \). Simpler: If \( U \) is isotropic in \( X \), there doesn't seem to be a natural symplectic space with maximal isotropic subspace \( U \).

**IDEA:** maximal isotropic subspaces are related to the boundary of the symmetric space - you vaguely recall picking a polarization, describing another polarization via a contraction which you can diagonalize via the action of \( U \), leading to eigenvalues \( 0 < \lambda_1 \leq \ldots \leq \lambda_n < 1 \). You can let intervals of the \( \lambda_i \) tend to 1 at different rates.
To back up p581 question: Given symplectic $X$, a Lagrangian subspace, and an isotropic subspace, can you say anything? (Motivation: Quadratic form on $V$ which is non-deg on $W$ defines splitting: $V = W \oplus W^*$.)

This is insufficient information, e.g., the isotropic subspace can be extended to a Lagrangian subspace, and generically two Lag subspaces describe $X$ in hyperbolic form.

Example: If $X = H \oplus H^*$, $W \subset H$, then $W \oplus W^*$ is Lagrangian in $X$. Suppose Lag subspace is graph of $Q : H \rightarrow H^*$

$$W \subset H \rightarrow H/W \downarrow \quad W^* \subset H^* \subset W^*$$

So $F_Q (W \oplus W^*) = \{(Qh) \in W^* \}$ i.e., $\langle h, Qh \rangle = 0$.

This in this example non-degenerate on $W \Leftrightarrow \exists$ graph $Q$ and $W \oplus W^*$ are $X$.

How can I handle $W \subset U \subset H$?

$H \oplus H^*$

$W \oplus W^*$

$U \oplus U^*$

$W \oplus U^*$

$U \oplus W^*$

$W \oplus U^*$

$U \oplus W^*$

two Lag complements for $F$
Main problem: Given \( W \subset U \subset H = H^+ \oplus H^- \) get \( F \subset \mathcal{O} \otimes (V \oplus V^*) \) Lagrangian subbundle. \( r = \text{rk}(F) = \text{dim}(V) \leq d = \text{deg}(F) = \text{dim} H \), in minimal situation.

All this has to be checked carefully and written out. The problem is to construct the "canonical" resolution of \( F \):

\[
0 \to F \to \mathcal{O} \otimes H^0(F^* \otimes H^0(F^-)^* \to \mathcal{O} \to 0
\]

It looks like

\[
H^0(F^*)^* \cong U \oplus W^-
\]

The square

\[
\begin{array}{ccc}
U & \xleftarrow{i} & H \\
\downarrow{p_1} & & \downarrow{p} \\
U \cap W^+ & \xleftarrow{i_1} & W^+
\end{array}
\]

is fixed. You are after the graph of an operator on \( V \) and you have a formula

\[
(p_1(i^* A i)^{-1} p_1^*)^{-1} = i^* (p A^{-1} p^*)^{-1} i
\]

So what?

Feb 2. Very little progress toward finding the correspondence. So where to begin.

Let's make more precise the degree structure. Start with

\[
Z = \sum_{s + w^3 \neq 0} s \frac{z(s + w^3)}{w^3 + w^2} q_w
\]

Look at the graph of \( Z : V^* \to V \). \( F^+_3(Z) \otimes V^* \cong \bigoplus V^* \).
Then want intersection with \((\mathcal{O})^*\)
so what actually happens is we have
\[
(Z_5)^* = F_s^* \subset \bigoplus V^*
\]
and you have \((\mathcal{O})^*\)
want to know what it's like to handle a pole say at \(s = -i\omega\). We have
\[
\frac{s(1+i\omega^2)}{s^2 + \omega^2} = \frac{1+i\omega^2}{2} \left( \frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right)
\]
Can you see what happened? near \(s = i\omega\) we have
\[
Z_s = g_s + \omega' \frac{1}{s-i\omega} \quad \text{an analytic near } \omega.
\]
\[
Z_s = g_s + \omega' \frac{1}{s-i\omega} \quad \text{in other words the intersection of } F_s \text{ with } (\mathcal{O})^* \quad \text{Take } \omega = 0.
\]
Then you have \(F_s = (Z_5)^* \mapsto (\mathcal{O})^* \mapsto (\mathcal{O})^* \mapsto V^* \text{ and you want its intersection with } (\mathcal{O})^* \mapsto V^* \mapsto (\mathcal{O})^* \mapsto V^* \mapsto V \quad \text{and we have the map } V^* \xrightarrow{a' + sg_s} V \quad \text{which is why you get the degree you do.}
Let's go over carefully how to calculate the intersection of \( \{ F_s \} = F \) vector bundle over \( S^2 = \mathbb{CP}^1 \) contained in \( \mathcal{O} \otimes (V^* \oplus V) \) with the subbundle \( \mathcal{O} \otimes V^* \). This means the intersection of the map \( F \to \mathcal{O} \otimes (V^* \oplus V) \to \mathcal{O} \otimes V \) with \( 0 \) computed properly. Locally, around \( s = 0 \) (take \( z = 0 \)) this map is \( V^* \to \mathcal{O} \otimes V \), which is non-zero except at \( s = 0 \) where it is 0. Let 

\[
\begin{pmatrix} 2s \varepsilon & (g_s + \frac{\alpha}{s-i\omega}) \varepsilon \n \end{pmatrix} V^* = \begin{pmatrix} (\alpha + (s-i\omega)g_s) \varepsilon \n \end{pmatrix} V^*
\]

where 

\[
\begin{pmatrix} (2s) \varepsilon & (g_s + \frac{\alpha}{s-i\omega}) \varepsilon \n \end{pmatrix} V^* = \begin{pmatrix} (\alpha + (s-i\omega)g_s) \varepsilon \n \end{pmatrix} V^*
\]

we have to find the fibre of this at \( s = 0 \).

Split \( V^* \) into \( \ker(\alpha) \) and a complement \( \text{Im}(\alpha) \). Then in the limit you should get 

\[
(2s) V^* = (g_s + \frac{\alpha}{s-i\omega}) V^* \supset \begin{pmatrix} \text{Im}(\alpha) \n \end{pmatrix} \ker(\alpha)
\]

\[
(2s) V^* = (g_s(s-i\omega) + \frac{\alpha}{s-i\omega}) V^* \supset (a_\omega V^*)
\]

Presumably, 

\[
(2s) V^* \to \begin{pmatrix} \text{Im}(\alpha) \ker(\alpha) \oplus (a_\omega V^*) \n \end{pmatrix}
\]

But the intersection with \( (1) V \) is 

\[
(a_\omega V^*) \oplus (0)
\]

as expected.
Can we compare $s = 0, s = \infty$ in some good way?

$$H^+_s \oplus H^-_s$$

$$V = U \cap W^+ \xrightarrow{\ i \ } W^+$$

$$P_1$$

$$A_s = s \Pi_+ + s^{-1} \Pi_-$$

$$l^*_1 \left( p A^{-1}_s p^* \right) l_1 = \left( p_1 \left( l^* A_1 p^* \right) p_1 \right)^{-1}$$

$$p A^{-1}_s p^* = s^{-1} (p \Pi_+ p^*) + s (p \Pi_- p^*) = \sum \frac{s^{-1} + s \omega^2}{1 + \omega^2} \Phi(\omega)$$

$$l^*_1 \left( p A^{-1}_s p^* \right) l_1 = \sum \frac{s (1 + \omega^2)}{s^2 + \omega^2} \frac{l^* \Pi_1 \omega}{q_0^*} l_1$$

When is $H$ minimal? You have $W^+$ minimal when

$$W^+ = \text{Im} (\Phi(\omega))$$

$$l_1 : V \rightarrow \oplus (W^+ \omega)$$

$$l_1, \omega = \Phi(\omega)$$

You have $H_\omega$. Take spectral form for

$$p \Pi_+ p^*$$

$s \in S = (0, 1]$.

The thing to understand is when $H$ is minimal. This should be easy in either picture.
Remark that the "pole" frequencies, i.e. the poles of $\text{c}^*(p_+(A_S))$, are determined by the eigenvalues of $W$, whereas the "zero" frequencies, i.e. the poles of $p_+(\text{c}^*(A_S))$, are determined by $U$. Let's try to understand minimality. Now when is $H$ minimal as far as $W^+$ is concerned, i.e. when is $H$ the minimal dilation of the operator $p_+p_+^*$ to a projection. For $0 < \omega < 1$, $H_\omega$ is twice the size of $W^+_\omega$.

$$p_+p_+^* = \frac{1}{1 + \omega^2}$$

$\omega = 0$ means $p_+p_+^* = 1$

or $p_+p_+^* = 0$ or $\pi_-(W^+_\omega) = 0$

$W^+_\omega = H^+ \cap W^+$

from the poles of $\text{c}^*(p_+A_S)$ you get the minimal ($W^+$), which will give you

$$2 \text{rank } q_\omega + \text{rk } (q_\omega) + \text{rk } q_\omega$$

$0 < \omega < \infty$

$H^+ \cap (W^+) \sim H_0(W^+)$

for the claim of the minimal $H$. What about the other side?
Look at \( U = \bigoplus U_\omega \) and the eigenvalues \( \lambda^* A \lambda = s \lambda + s^{-1} \lambda^{-1} \).

Feb 3. Go back to your response function \( Z \) and see if you can construct a Hilbert space from the polar data and maybe another from the zero data. Except you must bring in \( s = 0 \) somehow into the picture.

Consider then a natural \( Z_s = \bigoplus \) and say \( Z_s \) corresponds to a measure of finite support.

Look at the moment problem. Given the moments \( \mu_n = \int x^n \, d\mu(x) \), recover \( d\mu(x) \). Stieltjes found

\[
\int \frac{d\mu(x)}{z - x} = \int \frac{d\mu}{z(1 - \frac{x}{z})} = \sum_{n \geq 0} \frac{\mu_n}{z^{n+1}}
\]

This is convergent for \( |z| > R = \) amplitude of \( \text{supp } d\mu \).

Take finite measure and construct cont. frac.

How does this proceed?

\[
f(z) = \sum \frac{a_j}{\lambda_j - z} + a_\infty \frac{z}{z}
\]

\[
f(z) = a_\infty \frac{z}{z} + \frac{1}{z - a + bi}
\]

\[
\frac{1}{2i} \left( \frac{1}{x - z} - \frac{1}{x - \bar{z}} \right) = \frac{x - \bar{z} - x + \bar{z}}{2i (x - z)(x - \bar{z})} = \frac{\text{Im}(z)}{(x - a)^2 + b^2}
\]
\[ f(z) = \int \frac{d\mu(x)}{x-z} + b_1 + a_1 z \]

\[ = b_1 + a_1 z + \frac{1}{b_2 + a_2 z} \]

\[ \text{Im}(z) > 0 \implies \text{Im} \left( -\frac{1}{z} \right) > 0 \]

\[ f_1(z) = a_1 z - b_1 - \frac{1}{a_2 z + b_2 - \frac{i}{a_3 z + b_3}} \]

\[ f_1(z) = \begin{pmatrix} a_1 z & -b_1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} f_2(z) = \frac{(a_1 z - b_1) f_2 - 1}{f_2} \]

\[ \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix} = \begin{pmatrix} a_1 z & -b_1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \eta_1 \end{pmatrix} \]

\[ \xi_0 = (a_1 z - b_1) \xi_1 + \eta_1 \]

\[ \eta_0 = \xi_1 \]

\[ \xi_0 - (a_1 z - b_1) \xi_1 + \xi_2 = 0 \]

\[ \xi_0 + b_1 \xi_1 + \xi_2 = a_1 \xi_1 \]

So you have a positive diagonal matrix \((a_1, a_2, \ldots)\) and a symmetric \(T\) matrix \(T_1\), leading to real eigenvalues, \(\xi_1, \xi_2, \ldots\), leading to \(\eta_1, \eta_2, \ldots\) roots.
Review what's happening. You start with an \( f(z) \in \mathbb{C}(z) \) such that \( f(\mathbb{R}) \subset \mathbb{R} \cup \infty \), \( f(\text{Im } z > q) \subset \{ \text{Im } z > q \} \), and constructed its continued fraction rep.

\[
f(z) = a_1z - b_1 - \frac{1}{a_2z - b_2 - \frac{1}{\cdots}}
\]

where \( a_1 > 0, a_2 > 0, \ldots, a_n > 0, b_1, \ldots, b_n \in \mathbb{R} \).

From this, I get Jacobi system.

Feb 4 Go over Jacobi matrix.

Start first with \( f(z) \in \mathbb{C}(z) \), \( f(\mathbb{R} \cup \infty) \subset \mathbb{R} \cup \infty \), \( \text{Im } f(\mathbb{R}) \), \( f(\text{Im } z > c) \subset \{ \text{Im } z > c \} \).

\[ f(z) = a_1z - b_1 \]

Let \( d = \text{degree } f \), \( f(z) = f(2^d) = a_1z - b_1 + O(\frac{1}{z}) \).

- If \( a_1 \neq 0 \), then \( \deg g_1 = d - 1 \).
- If \( a_1 = 0 \), then \( \deg g_1 = d \)
  
  But \( g_1(\infty) = 0 \), \( \frac{a_1z - b_1}{g_1} \) has a pole at \( z = \infty \).

\[ f_1 = a_1z - b_1 - \frac{1}{a_2z - b_2} \]

The goal should be the underlying "Hilbert space structure" for these formulas. Apparently de Branges has completely worked this out at least for rank 1. Program should be to find the appropriate setup for the algebraic case.

analyse carefully, begin.
What are some of the variables? There is an eigenvalue parameter, call it \( \lambda \), or \( s \) or \( z \). A frequency parameter, which places us over \( \mathbb{CP}^1 \) for the algebraic stuff. There is also a circle \( \mathbb{P}(\mathbb{C}) \) and the disk on either side.

In the moment problem situation, you treat \( \lambda = \infty \), \( s = \infty \) specially, moreover this is a point on the distinguished \( \mathbb{P}(\mathbb{C}) \).

A key idea is to employ a simple 2 ports, a simple 2-port should be of degree 1 in the eigenvalue parameter, examples? for LC circuits, the simple ports are described by

\[
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
\]

with \( a > 0 \). What about moment problem?

\[
f(z) = a_2 - b_1 - \frac{1}{a_2 z - b_2 - \frac{1}{a_3 z - b_3 - \frac{1}{\ldots}}}
\]

\[
\xi_0 = a z - b_1 - \eta_1 = a z - b_1 - \frac{1}{(a_2 z - b_2) f_1 - \frac{1}{\eta_1}} = (a_2 z - b_2) f_1 - \frac{1}{\eta_1}
\]

\[
\begin{pmatrix}
\xi_0 \\
\eta_0
\end{pmatrix} = \begin{pmatrix}
a z - b_1 & -1 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\eta_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
\xi_0 \\
\eta_0 \\
\xi_1
\end{pmatrix} = \begin{pmatrix}
a z - b_1 & -1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix}
\]

\[
\xi_0 = (a z - b_1) \xi_1 - \xi_2
\]

\[
\xi_0 + b_1 \xi_1 + \xi_2 = a z \xi_1
\]
So the simple 2 ports have the form
\[
\begin{pmatrix}
az - b & -1 \\ 1 & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
aw - b & -1 \\ 1 & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
\frac{(az - b)w - 1}{w} = az - b - \frac{1}{w}
\end{pmatrix}
\]

Now you want to look at a product of these
\[
\begin{pmatrix}
a_{2z} - b_2 & -1 \\ 1 & 0
\end{pmatrix}\begin{pmatrix}
a_{1z} - b_1 & -1 \\ 1 & 0
\end{pmatrix} = \begin{pmatrix}

\end{pmatrix}
\]

\[
\begin{pmatrix}
(a_{2z} - b_2)(a_{1z} - b_1) - 1 & -a_{2z} + b_2 \\ a_{1z} - b_1 & -1
\end{pmatrix}
\]

There are a lot of minuses making this ugly.

But it's clear you are iterating
\[
\xi_{i+1} = (a_{1z} - b_1)\xi_i - \xi_{i-1}
\]

So
\[
\begin{align*}
\xi_0 &= 0 \\
\xi_1 &= 1 \\
\xi_2 &= a_{1z} - b_1 \\
\xi_3 &= (a_{2z} - b_2)(a_{1z} - b_1) - 1
\end{align*}
\]

Therefore, \((a_{2z} - b_1)\) raises degree \((inz)\) by 1.
Thus in \(SL_2(\mathbb{R})\) for \(z\) real.
\[
\begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
The key idea is probably the point evaluator. In any case there is a finite amount of data to get straight.

Point: Given the inner product in polys $\mathbb{R}[s]$, so if you look at polys of degree $< n$, you have an even $T$ Jacobi matrix, a measure with support on $n$ points, you have a standard way to "close" the partial helm.

operator $T \rightarrow \hat{T}$ to a hermitian operator. Probably what you need to finish the picture is the point evaluator formula.

one project - correlate $J$-matrix and cont. fr. exp.

Start with $f(s) = a_1 s \frac{1}{a_2 s} \frac{1}{a_3 s}$.

$$f_1(s) = a_1 s + \frac{1}{f_2(s)} = \left( I \cdot a_1 s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

Thus if $\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \left( a_1 s \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}$ $\Rightarrow \xi_1 = \eta_2$

and $\xi_0 = \eta_1$

and so we end up with

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} a_1 s \xi_1 + \xi_2 \\ \xi_1 \\ \xi_1 \end{pmatrix}$$

In general then we end up this way with the standard skew-adjoint operator $\sigma \rightarrow \sigma^{-1}$ where $\sigma$ is the shift, and there is the pos. s.a. operator $a$. Then to associate understand the operator

$$\sigma = \sigma^{-1} - a s$$

Alternate notation $f_1(\omega) = a_1 \omega \frac{1}{f_2} = \begin{pmatrix} a_1 \omega & -1 \\ 1 & 0 \end{pmatrix}$

$\xi_0 = a_1 \omega \xi_1 - \xi_2$ or $\xi_0 + \xi_2 = a_1 \omega \xi_1$
Focus upon the increasing family of Hilbert spaces that you get from the orthogonal polynomials you have a p.f. 

\[ f_1 = \begin{pmatrix} 1 & a_1 \omega \\ 0 & a_2 \omega - 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_2 \\ f_1 \end{pmatrix} \]

\[ = \begin{pmatrix} a_1 \omega & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f_2 \\ f_1 \end{pmatrix} \]

leads to 

\[ \xi_0 + \xi_2 = a_1 \omega \xi_1 \]

\[ \xi_{n-1} + \xi_{n+1} = a_n \omega \xi_n \]

You want something smooth...
Here is something occurring to me. Recall that the version leads to a standard skew-symmetric \( \sigma \), roughly \( \sigma = \sigma^{-1} \) and a positive definite \( \sigma = (\sigma_\alpha) \). What's rather nice is the picture of coupled 2 ports that emerges. You have the standard type symplectic structure coupled with the diagonal terms. Diagonal terms are:

\[
Z_0 = L_s + Z_1, \quad \frac{1}{Z_0} = C_s + \frac{1}{Z_1}, \quad \frac{1}{Z_0} = \frac{1}{C_s + \frac{1}{Z_1}}.
\]

\[
Z_0 = \begin{pmatrix} 1 & L_s \\ 0 & 1 \end{pmatrix}, \quad Z_0 = \begin{pmatrix} 1 & 0 \\ C_s & 1 \end{pmatrix} Z_1
\]

\[
P = \text{net power in} = E_0 I - E_1 I.
\]
This seems like an interesting point, but perhaps not really important. It seems advisable now to forgo the electrical picture because the inductive capacitance distinction seems not to be basic. Just odd versus even in numbering.

\[ E_0 I_0 - E_1 I_1 = (E_0 - E_1) I_1 \]

\[ E_1 I_1 - E_2 I_2 = E_1 (I_1 - I_2) \]

Write in terms of \( \xi_0 = E_0 \), \( \xi_1 = I_1 \), \( \xi_2 = E_2 \).

It seems the skew form is not really skew symmetric because of edge effects.

\[ \sum p_i \Delta q_i = \sum p_i (\xi_i - \xi_{i+1}) - \sum \Delta \phi \]

Feb 6. Yesterday you ended with confusion over the symplectic business. Go over some of the ideas. First, you wrote the continued fraction in the \( s \) variable

\[ f = \frac{a_1 s + \frac{1}{f_2}}{1} \]

leading to

\[ \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} = \begin{pmatrix} a_1 s & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \]

or

\[ \xi_0 = a_1 \xi_1 + \xi_2 \]

\[ \xi_0 - \xi_2 = s a_1 \xi_1 \]

Thus ignore boundary effects

\[ \xi_{n-1} - \xi_n = s a_n \xi_n \]

\[ (s - s^{-1}) \xi = s(\xi) \]
where \((\sigma^{\dagger})_{n} = \xi_{n-1}\) is the forward shift. What I liked about this is the combination of the skew symmetric operator \(\sigma - \sigma^{-1}\) and the positive symmetric operator \(a\), which means we have a harmonic oscillator structure, phase space picture. (Also write \(a^*\) instead of \(a\) to handle the Toeplitz (half space) version).

\(\sigma - \sigma^{-1}\) is a standard type skew symm. op. You want to link it to the coupling of 2 ports. This brings in power somehow. You were identifying power \(P\) somehow with the symplectic form.

But power is a quadratic function on

\[ f(s) = a_{1}s + \frac{1}{f_{2}} = \left(\begin{array}{c} a_{1}s \\ 0 \end{array}\right) \]  

\[ \left(\begin{array}{c} \xi_{0} \\ \xi_{1} \end{array}\right) = \left(\begin{array}{cc} a_{1}s & 1 \\ 1 & 0 \end{array}\right) \left(\begin{array}{c} \xi_{0} \\ \xi_{1} \end{array}\right) \]  

\[ \xi_{0} = s a_{1} \xi_{1} + \xi_{2} \]

\[ \xi_{n-1} - \xi_{n+1} = s a_{n} \xi_{n} \]

which means studying the operator \((\sigma - \sigma^{*})(\xi) = sa_{1}\)

\(\sigma - \sigma^{*}\) skew symm, a pos symmetric, which means we have a harmonic oscillator, at least if \(\sigma - \sigma^{*}\) is non-degenerate which should be true in even degrees. Certainly true for \(n > 1\).

Therefore I want to use the idea that \(\sigma - \sigma^{*}\) is a standard coupling.
I want to make something out of the symplectic coupling idea. Actually, you mean Hamilton's Principle. Hamilton's principle says the classical motion in time is stationary for the action \( \int \rho d\mathbf{q} - H dt \), thus

\[
\delta \int_{t_0}^{t_1} (\rho \dot{\mathbf{q}} - H) dt = \int \left( \delta \rho \dot{\mathbf{q}} + \rho \delta \dot{\mathbf{q}} - \delta \rho \frac{\partial H}{\partial \mathbf{q}} - \delta \dot{\mathbf{q}} \frac{\partial H}{\partial \dot{\mathbf{q}}} \right) dt
\]

\[
= \int \left( \delta \rho \left( \dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{p}} \right) + \left( -\mathbf{p} \dot{\mathbf{q}} - \frac{\partial H}{\partial \mathbf{q}} \right) \delta \mathbf{q} \right) dt
\]

\[= \left[ \rho \delta \mathbf{q} \right]_{t_0}^{t_1} = 0
\]

\[\Rightarrow \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}
\]

and

\[\left( \rho \delta \mathbf{q} \right) (t_1) = \left( \rho \delta \mathbf{q} \right) (t_0)
\]

Is there any significance to this last fact? What is the meaning of \( \rho \delta \mathbf{q} \)? What does it mean? \( \rho \delta \mathbf{q} \) is a bilinear form. \( \rho \delta \mathbf{q} \) is a quadratic function of \((\mathbf{p}, \mathbf{q}) \in V^* \oplus V^* \). Fundamentally, it defines the duality between \( V^* \) and \( V^* \), i.e., between position and momentum space. The condition says this duality is preserved under time evolution. This implies that the symplectic structure is preserved on \( V^* \oplus V \). How can I phrase things? This is strange. Suppose I consider two symplectic spaces in split form \( V^* \oplus V^* \) and \( V^* \oplus W^* \) and suppose I give an isomorphism \((a, b) \mapsto W \oplus W^* \). Then the meaning of \( \dot{\mathbf{q}} = \rho \delta \mathbf{q} \), i.e., \( \dot{\mathbf{q}} = \ddot{\mathbf{q}} + \rho \delta \mathbf{q} \), hence \( \mathbf{q} \in V, \mathbf{p} \in V^* \), \( \mathbf{q} \in W, \mathbf{p} \in W^* \)

\[\begin{bmatrix} \mathbf{q}' \\ \mathbf{p}' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}
\]

\[\dot{\mathbf{q}}' = \dot{\mathbf{q}} + \mathbf{p} \dot{\mathbf{t}} \]

\[\dot{\mathbf{p}} = \rho \delta \mathbf{q}
\]

and certainly \( \rho \delta \mathbf{q} \) for \( \text{dim} = 1 \).
Support \( H = 0 \) so that \( p^2 \) constant in time

\[
\int_{t_0}^{t_1} (p \dot{q} - H(p, q)) \, dt = 0.
\]

No help.

Am I right about Hamilton's principle wrong?

I have

\[
\int_{t_0}^{t_1} (p \dot{q} - H(p, q)) \, dt = A
\]

The action functional in the space \((\mathbb{R}^n, p(x))\) of paths

\[ [t_0, t_1] \rightarrow \mathbb{R}^n. \]

This action functional is a quadratic functional on the path spaces. Stationary means

\[
8A = \frac{\partial A}{\partial q} \frac{\partial}{\partial q} q + \frac{\partial A}{\partial p} \frac{\partial}{\partial p} p = 0
\]

Keep \( q(t) \) fixed, what is \( \frac{\partial A}{\partial p} \)?

\[
8A = \int_{t_0}^{t_1} \delta p (\dot{q} - \frac{\partial H}{\partial p}) \, dt
\]

Keep \( p(t) \) fixed, \( t \),

\[
8A = \int_{t_0}^{t_1} \delta p (\dot{q} - \frac{\partial H}{\partial p}) \, dt
\]

\[
= \left[ p \delta q \right]_{t_0}^{t_1} - \int_{t_0}^{t_1} (\dot{q} + \frac{\partial H}{\partial \dot{q}}) \delta q \, dt
\]

I guess what might happen is that \( p \) might jump at the endpoints.

The problem is clear, namely, the action functional is a quadratic functional of the path in phase space, so it depends only on the symmetrization of \( A \).

Symmetrize

\[
\int p \dot{q} \, dt = \int \dot{H} \, dt
\]

to get

\[
\frac{1}{2} \int (p_1^2 + p_2^2) \, dt - \frac{1}{2} \int (H(p_1, q_1) + H(p_2, q_2)) \, dt
\]

\[
\delta \left( p_1 \dot{q}_1 + p_2 \dot{q}_2 \right) = -\frac{1}{2} (p_1^2 + p_2^2) \bigg|_{t_0}^{t_1} - \frac{1}{2} \int \left( \dot{p}_1 \dot{q}_1 + \dot{p}_2 \dot{q}_2 \right) - H(p_1, q_1) - H(p_2, q_2) \]
What you've decided to do is to try to sort out this business of the symplectic coupling on the discrete level. How does this work? How can I do this? How can you proceed? What are the basic ideas? Let's start with the coupling idea—the change from a quadratic form to a symplectic transformation. This is fairly basic. We have $V \oplus W$ configuration space for the part and a quadratic form on $V \oplus W^*$, i.e., an isomorphism $V \oplus W \cong V^* \oplus W^*$ whose graph is maximal isotropic for the symplectic form $\omega_\text{can} \oplus (-\omega_\text{can})$ on $(V \oplus V^*) \oplus (W \oplus W^*)$. In good case this maximal isotropic subspace is also the graph of an symplectomorphism $V \oplus V^* \cong W \oplus W^*$.

Formulas:

\[
\begin{pmatrix}
1 & 0 \\
\beta & 1 \\
0 & \beta^t \\
\alpha & \gamma
\end{pmatrix}
\begin{pmatrix}
\alpha & \beta^t \\
\beta & \gamma
\end{pmatrix}
= (0 \ 1)
\begin{pmatrix}
1 & 0 \ \\
\beta & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
W \oplus V^* \\
V \oplus W^*
\end{pmatrix}
\begin{pmatrix}
V \\
W^*
\end{pmatrix}
\]
So you have

\[
\begin{pmatrix}
\alpha & \beta^t \\
\beta & \gamma
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
-\beta^t \gamma & \beta^t \\
\beta - \alpha \beta^t & \alpha \beta^t
\end{pmatrix}
\begin{pmatrix}
\beta^t \\
-\alpha \gamma
\end{pmatrix}^{-1}
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\beta^t \\
-\alpha \gamma
\end{pmatrix}^{-1}
\begin{pmatrix}
\beta^t \\
-\alpha \gamma
\end{pmatrix}^{-1}
\]

Some things to consider: This map goes from symmetric matrices to symplectic matrices, so it's a kind of Cayley transform.

Look at the ideal case.

\[
\begin{pmatrix}
\alpha & \beta \\
\beta & \gamma
\end{pmatrix}
\rightarrow
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
-\frac{\alpha}{\beta} & \frac{1}{\beta} \\
\beta - \alpha \beta & \alpha
\end{pmatrix}
\begin{pmatrix}
\beta & \gamma
\end{pmatrix}^{-1}
\begin{pmatrix}
\alpha & \beta \\
\beta & \gamma
\end{pmatrix}^{-1}
\begin{pmatrix}
\beta & \gamma
\end{pmatrix}^{-1}
\]

Sign is wrong.

\[
\text{Symplectic: } \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad g^T J g = J
\]

\[
g^{-1} = J g^T J^{-1}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}
\]

Look at C.T.

\[
g = \frac{1 + X}{1 - X}, \quad g^T = \frac{1 + X^t}{1 - X^t}
\]

\[
\frac{1 - X}{1 + X} = g^{-1} = J^{-1} g^T J = \frac{1 + J X^t J}{1 - J X^t J}
\]

\[
J^{-1} X^t J = -X, \quad X^t J = -J X = (X^t J)^t
\]

Thus \( X = S J \) where \( S \) symmetric,

\[
x^t = -J S = -J^X J
\]
$g = \begin{pmatrix} 1 + x \\ 1 - x \end{pmatrix} \Rightarrow g^T = \begin{pmatrix} 1 + x^t \\ 1 - x^t \end{pmatrix}$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Symp} \Rightarrow g^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a^T & c^T \\ b^T & d^T \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c^T & -a^T \\ b^T & -d^T \end{pmatrix} = \begin{pmatrix} -d^T & -b^T \\ -c^T & -a^T \end{pmatrix}$

so suppose $g = \begin{pmatrix} 1 + x \\ 1 - x \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}(x)$, then $x = \begin{pmatrix} 1 & -1 \end{pmatrix}(x) = \frac{g - 1}{g + 1}$

$\begin{pmatrix} 1 + x \\ 1 - x \end{pmatrix} \begin{pmatrix} J - \frac{J^T x^T J}{1 - x^T J} \end{pmatrix} = \frac{1 + J^T x^T J}{1 - x^T J} \Rightarrow g^{-1} = \frac{1 - x}{1 + x}$

so want $J^T x^T J = -x$, i.e. $x \in \text{Lie Sp}$

$x^T J + Jx = 0 \Rightarrow (Jx)^T = x^T (-J) = Jx$

Thus $\text{Lie Sp} = \mathfrak{sp}$ is the space of symm. matrices, via $x \mapsto J^x x = h \Rightarrow x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{sp} \iff \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & -a \\ b & -d \end{pmatrix}$

so symm. i.e. $a = -a^T$, $b = b^T$, $c = c^T$

$h = \begin{pmatrix} x & \beta \\ \beta & \gamma \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x & \beta^T \\ \beta & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & \beta^T \\ -\alpha & -\beta^T \end{pmatrix} = \begin{pmatrix} 1 + x \\ 1 - x \end{pmatrix} \mapsto \begin{pmatrix} 1 + J^{-1}h \\ 1 - J^{-1}h \end{pmatrix} = (1 - J^{-1}h)^{-1}(1 + J^{-1}h) = (J - h)^{-1}(J + h)$

So it seems that the correspondence you want is not the Cayley transform. Work it out for $SL_2(\mathbb{R}) = \text{Sp}_{2}(\mathbb{R})$. Is it involved with transmission, scattering?
Anyway what happens?

Basic object must be Lagrangian subspaces

The quadratic form $\omega$ on $V$ is the same as $V^*$ transversal to $V^*$. Lagrangian subspaces of $V \oplus V^*$ by the graph construction.

$$V \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (1, 0) = (1, 0)$$

$$= -g + g^t$$

dim Lagrangian $= \frac{n(n+1)}{2}$

Let $X, Y$ be symplectic with $\omega_X \oplus \omega_Y$. Lagrangian for $\Gamma \subset X \oplus Y$ trans.

To $X, Y$. $\Gamma$ graph of $f: X \rightarrow Y$. Then $f$ is a symplectic iso $\subset \Gamma$ Lagrangian.

$$\omega_X(f(x_1), f(x_2)) = \omega_X(x_1, x_2)$$

$$\omega_X(x_1, x_2) = \omega_Y(fx_1, fx_2) = 0$$

i.e. $$(x, y)$$ is Lagrangian.

$$X = \mathbb{R}^2 = V \oplus V^*$$

$$Y = W \oplus W^* = \mathbb{R}^2$$

$$X \oplus Y = \begin{pmatrix} V \oplus W & V^* \oplus W^* \end{pmatrix}$$

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$
\[ J = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \]

\[ \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -d^t c & d^t \\ a - bd^t c & bd^t \end{pmatrix} \]

It looks like we want
\[ \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} a & bd^t c \\ b^t d^t & bd^t \end{pmatrix} \]

Is it clear that \[ \begin{pmatrix} -d^t c & d^t \\ a - bd^t c & bd^t \end{pmatrix} \] is symmetric when \[ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} \]

\[ d^t b = b^t d \]

\[ a^t a = a^t c \]

\[ a - bc^t (d^t)^{-1} = (d^{-1})^t \]

\[ (d^{-1})^t = d^t \]

Go back to C.T. but on \[ V \oplus V^* \] A quadratic function on \[ V \oplus V^* \] is \[ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \] and function on \[ V \oplus W \] is
Get thoroughly confused. The theory might be simpler? How can you possibly do this?

I guess I would like to understand the successive coupling arising from continued fractions.

Here's what we did above: \[ X = W \overset{(a \ b)}{\circlearrowleft} W^* \]

Symplectic is

\[
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}^{-1} = \begin{pmatrix} d^{\ast} & b^{\ast} \\
  -c^{\ast} & a^{\ast}
\end{pmatrix}
\]

\[
\begin{pmatrix} a & b \\
  c & d
\end{pmatrix} \begin{pmatrix} d^{\ast} & b^{\ast} \\
  -c^{\ast} & a^{\ast}
\end{pmatrix} = \begin{pmatrix} ad^{\ast} - bc^{\ast} & ab^{\ast} + bd^{\ast} \\
  cd^{\ast} - dc^{\ast} & da^{\ast} - cb^{\ast}
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
  0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix} d^{\ast} & b^{\ast} \\
  -c^{\ast} & a^{\ast}
\end{pmatrix} \begin{pmatrix} a & b \\
  c & d
\end{pmatrix} = \begin{pmatrix} d^{\ast}a + b^{\ast}c & d^{\ast}b + b^{\ast}d \\
  -c^{\ast}a + a^{\ast}c & a^{\ast}d - c^{\ast}b
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
  0 & 1
\end{pmatrix}
\]

form \( X \oplus Y \) with \( \omega_x \oplus (-\omega_y) \)

\[
X \oplus Y = \begin{pmatrix}
  V & V \\
  V^* & W \\
  W^* & W
\end{pmatrix} \sim \begin{pmatrix}
  V & V \\
  V^* & W \\
  W^* & W
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
  0 & 1 \\
  a & b
\end{pmatrix} \begin{pmatrix}
  
\end{pmatrix}
\]

\[
\begin{pmatrix}
  V_1 & V_2 \\
  V_1^* & V_2^*
\end{pmatrix} \begin{pmatrix}
  
\end{pmatrix} \begin{pmatrix}
  W_1 & W_2 \\
  W_1^* & W_2^*
\end{pmatrix} = \begin{pmatrix}
  \omega_1 & \omega_2 \\
  \omega_1^* & \omega_2^*
\end{pmatrix} \begin{pmatrix}
  
\end{pmatrix} \begin{pmatrix}
  \omega_1 & \omega_2 \\
  \omega_1^* & \omega_2^*
\end{pmatrix} = -\omega_1 \omega_2 + \omega_1^* \omega_2^* - \omega_1 \omega_2 - \omega_1^* \omega_2^*
Go backwards

$$
\begin{pmatrix}
1 & 0 \\
0 & \alpha & \beta^* \\
\beta & \gamma \\
\end{pmatrix}
\begin{pmatrix}
V \\
V^* \\
W \\
\end{pmatrix}
= 
\begin{pmatrix}
V \\
V^* \\
W \\
\end{pmatrix}
$$

something went wrong because you want

$$
\beta = d^{-1} \\
\begin{pmatrix}
\alpha & \beta \\
0 & 1 \\
\end{pmatrix}
= 
\begin{pmatrix}
\beta^{-1} & -\beta^{-1} \gamma \\
0 & 1 \\
\end{pmatrix}
$$

try transmission line approach

$$
\frac{d}{dx} E_x = \frac{1}{\varepsilon} \frac{d}{dt} I_x \\
-\frac{d}{dx} E_x = \frac{1}{\mu} \frac{d}{dt} I_x \\
\frac{1}{\varepsilon} \frac{d^2 E_x}{dx^2} = -\frac{1}{\mu c^2} \frac{d^2 I_x}{dt^2}
$$

$$
E_{ix} - E_i = L_i \frac{d}{dt} I_i \\
I_i - I_{i+1} = C_i \frac{d}{dt} E_i \\
E_{i-\Delta x} - E_i = (L_i \Delta x) \frac{d}{dt} I_i \\
I_x - I_{x+\Delta x} = (C \Delta x) \frac{d}{dt} E_x \\
\frac{d^2 E_x}{dx^2} = -\frac{1}{\mu c^2} \frac{d^2 I_x}{dt^2}
$$
\[
\begin{align*}
(E) & = (I) e^{\omega (x-t)} \\
-\partial_x (E e^{st}) & = \lambda s (I e^{st}) \\
-\partial_x (I e^{st}) & = cs (E e^{st})
\end{align*}
\]
Now I want to study a 2-port from the viewpoint of scattering.

\[ \partial_x E + \kappa \partial_t I = 0 \]
\[ \partial_x I + \kappa \partial_t E = 0 \]

solutions

\[
\begin{pmatrix}
E \\
I
\end{pmatrix}(x,t) = e^{s(t+x)} \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} A + e^{s(t-x)} \begin{pmatrix} 1 \\ \kappa \end{pmatrix} B
\]

\[
\begin{pmatrix}
E \\
I
\end{pmatrix} = \begin{pmatrix} 1 \\ -\kappa \end{pmatrix} C + \begin{pmatrix} 1 \\ \kappa \end{pmatrix} D
\]

Study 2 port.

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} \rightarrow \begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

quadratic form

\[
P_s = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}
\]

Response

\[ I = P_s E \]

\[
\begin{pmatrix} I_0 \\
I_1
\end{pmatrix} = P_s \begin{pmatrix} E_0 \\
E_1
\end{pmatrix}
\]

There is some confusion because \( P \) ? I haven't really explained the duality between current and voltage space.

Duality is given by the quadratic function.

Consider an example of a ladder circuit.

\[ E_0 I_0 = E_1 I_1 \]

\[ E_p - E_1 = Ls I_0 \]

\[ I_0 + I_1 = C_s E_1 \]
So we have 4 dual spaces with coordinates \( E_0, I_0, E_1, I_1 \), 2 equations:

\[
\begin{align*}
E_0 - E_1 &= L s I_0 \\
I_0 + I_1 &= C s E_1
\end{align*}
\]

get 2 dual subspaces for any \( s \).

In the response, check \( F_s \) is Lagrangian. What is the skew form, power into circuit?

\[
\begin{pmatrix}
E_0' \\
I_0' \\
E_1' \\
I_1'
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
E_0 \\
I_0 \\
E_1 \\
I_1'
\end{pmatrix}
= E_0 I_0' - E_0 I_0 \\
E_1 I_1' - E_1 I_1
\]

Power into circuit is a quadratic function

\[
\begin{pmatrix}
E_0 \\
I_0 \\
E_1 \\
I_1
\end{pmatrix}
\begin{pmatrix}
E_0' \\
I_0' \\
E_1' \\
I_1'
\end{pmatrix}
\rightarrow E_0 I_0 + E_1 I_1
\]

which sets up a duality between \( E_0 \) space and \( I_0 \) space.

and then you take the corresponding symplectic form.

Already I should do this for a 1-port.

\[
\begin{pmatrix}
E \\
I
\end{pmatrix}
\begin{pmatrix}
E' \\
I'
\end{pmatrix}
\rightarrow E E' \quad \text{for each } s \quad \text{get } F_s = (Z_s)^R \subset \mathbb{R}^2
\]

so I get a line bundle over the \( s \) plane \( \rightarrow \)

start again: Go back to

\[
\begin{align*}
E_0 I_0 \\
E_1 I_1
\end{align*}
\]

You have for each \( s \) a 2 dual subspace of 4 space with words \( E_0 I_0, E_1 I_1 \). So you have a correspondence between voltage space + current space, and a comp between \( E_0 I_0 \) and \( E_1 I_1 \). Heuristically, these correspondences should be isomorphisms. Find formulas.
\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix} 1 & Ls \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_1 \\
I_1
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ Cs & 1 + Ls \end{pmatrix} \begin{pmatrix} E_1 \\
-I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_1 \\
-I_1
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Cs & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix} 1 & -Ls \\ -Cs & 1 + Ls^3 \end{pmatrix} \begin{pmatrix} E_0 \\
I_0
\end{pmatrix}
\]

\[
E_1 = (Cs)^{-1}(I_0 + I_1)
\]

\[
E_0 = E_1 + LsI_0 = (Cs)^{-1} + LsI_0 + (Cs)^{-1}I_1
\]

\[
\begin{pmatrix}
E_0 \\
E_1
\end{pmatrix} = \begin{pmatrix} (Cs)^{-1}Ls & (Cs)^{-1} \\ (Cs)^{-1} & (Cs)^{-1} \end{pmatrix} \begin{pmatrix} I_0 \\
I_1
\end{pmatrix}
\]

\[
\text{det} = Lc^{-1}
\]

\[
I_0 = (Ls)^{-1}E_0 - (Ls)^{-1}E_1
\]

\[
I_1 = CsE_1 - (Ls)^{-1}E_0 + (Ls)^{-1}E_1
\]

\[
= (Cs + (Ls)^{-1})E_1 - (Ls)^{-1}E_0
\]

\[
\begin{pmatrix}
I_0 \\
I_1
\end{pmatrix} = \begin{pmatrix} (Ls)^{-1} & -(Ls)^{-1} \\ -(Ls)^{-1} & Cs + (Ls)^{-1} \end{pmatrix} \begin{pmatrix} E_0 \\
E_1
\end{pmatrix}
\]

\[
\text{det} = cL^{-1}
\]

So what should the formulas be?

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_1 \\
-I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_0 \\
E_1
\end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} I_0 \\
I_1
\end{pmatrix}
\]
\[ \begin{align*}
I_0 &= cE_1 - dI_1 \\
I_0 + dI_1 &= cE_1 \\
E_1 &= c^{-1}I_0 + c^{-1}dI_1 \\
E_0 &= a(c^{-1}I_0 + c^{-1}dI_1) - bI_1 \\
&= ac^{-1}I_0 + (ac^{-1}d - b)I_1
\end{align*} \]

\[
\begin{pmatrix}
E_0 \\
E_1
\end{pmatrix} =
\begin{pmatrix}
ac^{-1} & ac^{-1}d - b \\
c^{-1} & c^{-1}d - b
\end{pmatrix}
\begin{pmatrix}
I_0 \\
I_1
\end{pmatrix}
\]

Reminded of Legendre transform - poor man's F.T. or L.T.

\[ V^* \]

\[ F \rightarrow \hat{F} \]

\[ \hat{F}(\lambda) = \text{stationary value of} \]

\[ \langle \lambda, v(\lambda) \rangle - F(v) \quad \forall \in V \]

Critical values \( \Lambda \): \( \lambda = F'(v) \) solve for \( v = v(\lambda) \)

and then \( \hat{F}(\lambda) = \langle \lambda, v(\lambda) \rangle - F(v(\lambda)) \). So what.

\[ F = \log \det \]

\[ \begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

\[ \lambda E_1 - \lambda I_1 = I_0 + dI_1 \]

\[ E_0 = ac^{-1}I_0 + (ac^{-1}d)I_1 \]

\[ E_0 = aE_1 - bI_1 \]

\[ bI_1 = -E_0 + aE_1 \]

\[ I_1 = -b^{-1}E_0 + b^{-1}aE_1 \]

\[ I_0 = cE_1 - d(-b^{-1}E_0 + b^{-1}aE_1) \]

\[ = db^{-1}E_0 + (c - db^{-1}a)E_1 \]

\[ \begin{pmatrix}
I_0 \\
I_1
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{c} & \frac{1}{d} \\
-\frac{b}{c} & \frac{1}{c}
\end{pmatrix}
\begin{pmatrix}
E_0 \\
E_1
\end{pmatrix} \]

\[ (cd)^t = ac^{-1}d - b ? \]

\[ l = ac^{-1}dct - bct \]

\[ = ad^t - bct \]
There are problems with $E_0, I_0 = I, E_1$

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
E_1 \\
I_1
\end{pmatrix} \implies \begin{pmatrix}
E_0 \\
E_1
\end{pmatrix} = \begin{pmatrix}
ax^{-1} & (c^{-1})^t \\
c^{-1} & c^{-1}d
\end{pmatrix} \begin{pmatrix}
I_0 \\
I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
I_0 \\
I_1
\end{pmatrix} = \begin{pmatrix}
db^{-1} & -(b^{-1})^t \\
-b^{-1} & b^{-1}a
\end{pmatrix} \begin{pmatrix}
E_0 \\
E_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix}
Ls^{-1} & -Ls \\
Ls & (Ls)^{-1}
\end{pmatrix} \begin{pmatrix}
E_0 \\
E_1
\end{pmatrix}
\]

singular.

Feb 7

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

Assume $\beta$ invertible, and solve for \( \begin{pmatrix} E_0 \\ I_0 \end{pmatrix} \) in terms of \( \begin{pmatrix} E_1 \\ I_1 \end{pmatrix} \)

\[
\beta^{-1} E_1 = \beta I_0 + \beta^{-1} \gamma I_1
\]

\[
I_0 = \beta^{-1} E_1 - \beta^{-1} \gamma I_1
\]

\[
E_0 = \alpha (\beta^{-1} E_1 - \beta^{-1} \gamma I_1) + \beta^{-1} I_1
\]

\[\begin{pmatrix}
\alpha & \beta^t \\
\beta & \gamma
\end{pmatrix} = \begin{pmatrix}
ac^{-1} & c^{-1} \\
c^{-1} & c^{-1}d
\end{pmatrix}
\]

\[\begin{pmatrix}
\alpha \beta^{-1} & \alpha \beta^{-1} \gamma - \beta^t \\
\beta^{-1} \gamma & -\beta^{-1} \gamma
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha \beta^{-1} & \alpha \beta^{-1} \gamma - \beta^t \\
\beta^{-1} \gamma & -\beta^{-1} \gamma
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha & b \\
c & d
\end{pmatrix}
\]

\[
\lambda = ac^{-1} = \alpha \beta^{-1} (\beta^t)^{-1} = \lambda
\]

\[
\beta = c^{-1} = (\beta^t)^{-1} = \beta
\]

\[
\gamma = c^{-1} d = (\beta^{-1}) \beta^t \gamma = \gamma
\]
\[
\begin{pmatrix}
\alpha & \beta^- \\
\beta & \gamma
\end{pmatrix} \xrightarrow{\text{claim}} 
\begin{pmatrix}
\alpha \beta^- & \alpha \beta^- \gamma - \beta^- \\
\beta^- & \beta^- \gamma
\end{pmatrix} = 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

Claim: \[ (a, b, c, d) \text{ is o.s.} \]

\[
(a, b)^{-1} = 
\begin{pmatrix}
dt & -b \\
-c & at
\end{pmatrix}
\]

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = 
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

You have to straighten out the symp. stuff later.
Today's lecture

Continuous limit of ladder network

\[ \frac{E_x}{\Delta x} I_x = \frac{E}{\Delta x} I + \frac{E_x}{\Delta x} \]

\[ L = \Delta x \frac{d}{dx} \]

\[ C = \Delta x \]

\[ \frac{1}{\beta} \]

\[ \frac{\partial^2 E}{\partial x^2} = -L \frac{\partial^2 I}{\partial t^2} = L \frac{d}{dt} \frac{d^2 E}{dx^2} \]

wave eqn.

speed \[ \frac{1}{\sqrt{LC}} \]

take \( L = 1 \)

\[ E = f(x+t) + g(x-ct) \]

time def. est.

\[ \frac{\partial^2 E}{\partial x^2} = s^2 E \]

\[ E = (e^{sx} + e^{-sx}) e^{st} \]

\[ \begin{pmatrix} E \\ I \end{pmatrix} = e^{s(x+t)} \begin{pmatrix} 1 \\ -c \end{pmatrix} A + e^{s(-x+t)} \begin{pmatrix} 1 \\ c \end{pmatrix} B \]

\[ \text{incoming from right} \]

\[ \text{outgoing to right} \]

\[ \begin{pmatrix} E_o, I_o \end{pmatrix} \]

\[ \frac{1}{Z} = \frac{E_o}{I_o} = \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \]

\[ \frac{A}{B} = \begin{pmatrix} c & -1 \\ -c & 1 \end{pmatrix} \begin{pmatrix} -2 \end{pmatrix} = \frac{t + 2c + 1}{c Z + 1} = \frac{Z + l}{Z - l} \]

reflection coeff.

\[ \frac{Z - l}{Z + l} \]

Examine 2-port

\[ \begin{pmatrix} E_o \\ E_l \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \begin{pmatrix} I_o \\ I_l \end{pmatrix} \]

\[ \begin{pmatrix} E_o \\ I_o \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_l \\ -I_l \end{pmatrix} \]
\[ \begin{align*}
\beta I_0 + \gamma I_1 &= E_1 \\
I_0 &= \beta' E_1 - \beta' \gamma I_1 \\
E_0 &= \alpha (\beta' E_1 - \beta' \gamma I_1) + \beta I_1
\end{align*} \]

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} =
\begin{pmatrix}
\alpha \beta' & \alpha \beta' (\gamma - \beta) \\
\beta' & \beta' (\gamma - \beta)
\end{pmatrix}
\begin{pmatrix}
E_1 \\
-I_1
\end{pmatrix}
\]

\[ c E_1 = a I_1 + I_0 \]
\[ E_1 = c' d I_0 + c' d I_1 \]
\[ E_0 = a c^{-1} I_0 + a c^{-1} d I_1 - b I_1 \]

\[
\begin{pmatrix}
E_0 \\
E_1
\end{pmatrix} =
\begin{pmatrix}
a c^{-1} + \sqrt{a^2 c d - b} \\
0 \\
0 \\
-c^{-1} c d
\end{pmatrix}
\begin{pmatrix}
I_0 \\
I_1
\end{pmatrix}
\]

symmetric
determinant: \[ \frac{b}{c} \]

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} =
\begin{pmatrix}
\alpha \beta' & \alpha \beta' (\gamma - \beta) \\
\beta' & \beta' (\gamma - \beta)
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix} =
\begin{pmatrix}
a c^{-1} & a c^{-1} d - b \\
0 & c^{-1} c d
\end{pmatrix}
\]

given \( E \) covers between \((a \ b) \in SL_2 \cap c^{-1} E\)
and \((\alpha \beta)\) symm. + \(\beta' E\).
\[
\begin{align*}
E_1 & = \begin{pmatrix} x & y \\ z & d \end{pmatrix} = \begin{pmatrix} ac^{-1} & ac^{-1}d-b \\ c^{-1} & c^{-1}d \end{pmatrix} \cdot \begin{pmatrix} I_0 \\ I_1 \end{pmatrix} \\
E_0 & = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac^{-1} & ac^{-1}s-b \\ c^{-1} & c^{-1}s \end{pmatrix} \cdot \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}
\end{align*}
\]

Is there a possibility that somewhere in this algebra lurks a convoluted kernel to describe composition of operators?

So on now to scattering which involves a similar transformation:
\[
\begin{align*}
E_1 &= Ae^{sx}(1) + Be^{-sx}(-c) \\
E_0 &= \begin{pmatrix} 1 & 1 \\ -c & c \end{pmatrix} A \\
E_1 &= \begin{pmatrix} 1 & -c \\ c & 1 \end{pmatrix} B \\
E_0 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} x > 0.
\end{align*}
\]

Value at \( x = 0 \) of the solution of the transmission line equations for \( x < 0 \)

\[
\begin{align*}
\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_1 \\ -I_1 \end{pmatrix}
\end{align*}
\]

But we want a different basis. What am I trying to do?
Consider instead \( V \) of compact support.  

On the left you have two basic solutions \( e^{ikx}, e^{-ikx} \) and on the right also \( e^{ikx}, e^{-ikx} \).

Get SL\(_2\) matrix:

\[
\begin{pmatrix}
\psi \\
\psi^*
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix}
\phi \\
\phi^*
\end{pmatrix}
\]

In the left:

\[
e^{ikx} \quad \Leftrightarrow \quad Ae^{ikx} + Be^{-ikx} \quad e^{ikt}
\]

\[
e^{-ikx} \quad \Leftrightarrow \quad Ce^{ikx} + De^{-ikx}
\]

\[
-\frac{B}{D} e^{-ikx} \quad \Leftrightarrow \quad -\frac{BC}{D} e^{ikx} - Be^{-ikx}
\]

\[
\frac{1}{D} e^{-ikx} \quad \Leftrightarrow \quad \frac{c}{D} e^{ikx} + e^{-ikx}
\]

Scattering matrix is something like:

\[
\begin{pmatrix}
\frac{B}{D} & \frac{1}{D} \\
\frac{1}{D} & \frac{c}{D}
\end{pmatrix}
\]
be intelligent. have solution on the left
\[ e^{s(x+t)} \begin{pmatrix} 1 \\ -1 \end{pmatrix} h_1 + e^{s(-x+t)} \begin{pmatrix} 1 \\ 1 \end{pmatrix} h_2 \]
boundary values
\[
\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}
\]
and we have solution by on the right
\[ e^{sx} \begin{pmatrix} 1 \\ -1 \end{pmatrix} k_1 + e^{-sx} \begin{pmatrix} 1 \\ 1 \end{pmatrix} k_2 \]
with values at \( x = 0 \).
\[
\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}
\]
Then
\[
\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E_1 \\ I_1 \end{pmatrix}
\]
so
\[
\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}
\]
Find another approach. Look at the incoming solution
\[
\begin{pmatrix} E_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]
\[
\begin{pmatrix} E_0 \\ I_0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}
\]
Basically you should only have to consider the 1-sided case of scattering.

\[
\begin{align*}
(E)_{x,t} &= e^{s(x+t)} \begin{pmatrix} 1 \end{pmatrix} A + e^{s(-x-t)} \begin{pmatrix} -1 \end{pmatrix} B
\end{align*}
\]

Here \(A, B\) are vectors, so at \(x = 0\) (\(t = 0\))

\[
\begin{pmatrix} E \\ I \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}
\]

If the transmission line is connected to an \(n\)-port with input \(Z\), then we have

\[
\begin{pmatrix} E \\ I \end{pmatrix} = (Z)^* \begin{pmatrix} E \\ I \end{pmatrix}
\]

So

\[
\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} (Z)^* = \begin{pmatrix} Z+1 \\ Z-1 \end{pmatrix} (Z+1)
\]

So the scattering operator is \(\frac{Z-1}{Z+1}\), essentially the Cayley transform. Recall that

\[
Z = \sum \frac{s(1+\omega^2)}{\delta^2+\omega^2} a_\omega\]

\[
\frac{1+\omega^2}{2} \left( \frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right)
\]

Looks like we should worry about \(\mathbb{U}(1)\) Siegel UHP.

Note
The point is that for the transmission line you have probably chosen a inner product on $V$ the voltage space.

\[
\begin{align*}
\partial_x E + l \partial_t I &= 0 \\
\partial_x I + c \partial_t E &= 0 \\
\partial_x^2 E &= \frac{1}{c^2} \partial_t^2 E = 0 \\
E &\in V \text{ dim } n \\
I &\in V^* \text{ dim } n.
\end{align*}
\]

\[b : V^* \to V \quad c : V \to V^* \quad \text{pos. def. q.s.} \]

even for $n = 1$. \(E^* \subset V, V^* \text{ dual but not canon. iso.}\)

Conclude that from an invariant viewpoint, you might as well suppose \(l = c = 1\). So what do you learn? The reflection coeff in operator on $V = V^*$.

\[
V^* \xrightarrow{2-l} V \overset{(2+l)^{-1}}{\xrightarrow{2+l}} V^*
\]

But now see if you can get

\[
g = \frac{z-1}{2+1} \\
2s = \sum_{\omega} \frac{s(1+\omega^2)}{\omega} > 0.
\]

\[
\left\| (2+i) \omega \right\|^2 = \left( \frac{z^2 \sigma}{s} \right)^2
\]

\[
(\omega, (2+i) \omega) = \left\| \omega \right\|^2 + \frac{1}{2} \sum_{\omega} \frac{s(1+\omega^2)}{\omega^2 + \omega^2} \left( \omega, \omega \right) > 0
\]

\[
\frac{1+\omega^2}{2} \left( \frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right)
\]

\[
\text{Re} \left( \frac{1}{s-i\omega} \right) = \frac{1}{2} \left( \frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right) = \frac{1}{2} \frac{s+i\omega}{(s-i\omega)(s+i\omega)}
\]

\[
\text{Re}(s) = \frac{1}{2} \frac{s+i\omega}{1 + \omega^2 + 4s^2 - 4s^2} = \frac{1}{2} \frac{s+i\omega}{1 + \omega^2 + 4s^2 - 4s^2} = \text{Re}(s)
\]
\[
\begin{align*}
\text{Re} \left( \frac{1}{s-i\omega} + \frac{1}{s+i\omega} \right) &= \frac{1}{2} \left( \text{Re} \left( \frac{1}{s-i\omega} \right) + \text{Re} \left( \frac{1}{s+i\omega} \right) \right) \text{?} \\
\text{Re} \left( \frac{1}{s-i\omega} \right) &= \frac{1}{2} \left( \frac{s+5}{s^2 + \omega^2} \right) \\
&= \frac{\text{Re}(s)}{\text{Re}(s)^2 + \text{Im}(s)^2 - 2 \text{Re}(s) \text{Im}(s) s + \omega^2} \\
&= \frac{\text{Re}(s)}{\text{Re}(s)^2 + (\text{Im}(s)-\omega)^2} \\
&= \frac{\text{Re}(s)}{\text{Re}(s)^2 + (\text{Im}(s)-\omega)^2} \\
&= \frac{\text{Re}(s)}{\text{Re}(s)^2 + (\text{Im}(s)-\omega)^2}
\end{align*}
\]

Feb 10: What can you do about lecture?

n-fold transmission line \((E, I) \in V\)

\[
\begin{align*}
\partial_x E + \partial_t I &= 0 \\
\partial_x I + \partial_t E &= 0 \\
(\partial_x + \partial_t)(E + I) &= 0 \\
(\partial_x - \partial_t)(E - I) &= 0 \\
B e^{s(-x+t)} \\
E + I &= f(x-t) \\
E - I &= g(x+t) \\
A e^{s(x+t)} \\
\frac{1}{2} \left[ \begin{array}{cc} 1 & 1 \\ +1 & -1 \end{array} \right] \left( \begin{array}{c} E \\ +1 \\ -1 \end{array} \right) &= \left( \begin{array}{c} B \\ A \end{array} \right) \\
\left( \begin{array}{c} E \\ +1 \end{array} \right) &= \frac{1}{2} \left[ \begin{array}{cc} 1 & +1 \\ -1 & -1 \end{array} \right] \left( \begin{array}{c} B \\ A \end{array} \right) \\
\left( \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right) \left( \begin{array}{c} B \\ A \end{array} \right) &= -Z \\
S = \frac{B}{A} &= \left( \begin{array}{cc} +1 & +1 \\ -1 & -1 \end{array} \right) (-Z) = \frac{-Z+1}{-Z-1} \frac{2-1}{2+1}
\end{align*}
\]
$$S(s) = \frac{Z_s - 1}{Z_s + 1}$$

$$Z_s = \sum_{0 \leq \omega < \infty} \frac{\overline{s(1+i\omega)^2}}{s^2 + \omega^2} \alpha_\omega$$

\[\begin{align*}
&\text{vec}^n (\mathbf{u}) (Z_s + 1) \mathbf{v} = \|\mathbf{u}\|^2 + \sum_{0 \leq \omega < \infty} \frac{s(1+i\omega)^2}{s^2 + \omega^2} (\mathbf{v}, \alpha_\omega \mathbf{u}) \\
&\text{if } \mathbf{v} \neq \mathbf{0} \text{ for } \text{Re}(s) > 0 \text{ at least one } \omega > 0.
\end{align*}\]

\[\begin{align*}
&\text{taken } \text{Re}(s) > 0 \text{ into } \text{Re}(s) > 0.
\end{align*}\]

$$(Z_s + 1)^{-1} \mathbf{I}$$ for $\text{Re}(s) > 0$. $S(s)$ is analytic for $\text{Re}(s) > 0$.

Another point: $S \in \mathbb{R}^{\infty \times \infty}$

$$Z_s = \sum_{\omega < \infty} s \alpha_\omega$$

$h(s)$ analytic at $\infty$. $h(\infty) = 0$

$$\frac{Z_s - 1}{Z_s + 1} = \frac{h + s\alpha_\infty - 1}{h + s\alpha_\infty + 1} = 1 - \frac{2}{s\alpha_\infty + 1 + h}$$

$p > 0$

$$a_\infty = \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

$$s\alpha_\infty + 1 + h = \begin{pmatrix} \frac{sp + 1}{x} & \frac{x}{x} \\ \frac{x}{x} & \frac{x}{x} \end{pmatrix}$$

\[\begin{align*}
&\begin{pmatrix} 1 & 0 \\ -ca^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -a \end{pmatrix} \\
&\begin{pmatrix} a & b \\ 0 & -ca^{-1}b + d \end{pmatrix} \begin{pmatrix} 1 & -a \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d-ca^{-1}b \end{pmatrix}
\end{align*}\]
Be sure to scat stemming.

An idea: you know that connecting an n-port to an n-fold transmission line transforms the impedance \( Z_0 \) to its Cayley transform \( \frac{Z-I}{Z+I} = S \):

\[
\begin{align*}
\partial_x E + \partial_t I &= 0 \\
\partial_x I + \partial_t E &= 0 \\
(\partial_x + \partial_t)(E+I) &= 0 \\
(\partial_x - \partial_t)(E-I) &= 0
\end{align*}
\]

Here take \( V = V^* \), \( l = c = 1 \).

\[
\begin{align*}
E+I &= A e^{s(x+t)} \\
E-I &= B e^{s(x+t)}
\end{align*}
\]

\[
S = \frac{B}{A} = \frac{E_0 + I_0}{E_0 - I_0} = \frac{1}{\frac{I_0}{E_0}} = -Z = \frac{Z-1}{Z+1}
\]

This \( S \) is a meromorphic function on \( \Re(s) > 0 \) unitary-valued on the boundary.

It might be nice to find a good proof of this.

There is also this reality condition \( Z(s)^* = Z(\overline{s}) \) together with \( Z(-s) = -Z(s) \), which implies \( Z(s)^* = -Z(-\overline{s}) \) so that \( Z(s) \) is skew hermitian on \( i \mathbb{R} \). It has its C.T. is unitary for \( s \in i \mathbb{R} \).

In fact \( Z(it) \) is \( i \) times a real symm. matrix, but I don't think this means very much.

You seem to have some problem relating the \( s \) variable to the \( z \) variable describing the disk. These are the variables describing the Riemann sphere. The difference
This must be understood better. It's not obvious to formulate definitively about what is the interior in this picture. The program for the moment should be:

1. Handle scattering operators.
2. Analyze operators.
3. Compute the exterior.
4. Find the interior.

Do what served the exterior and extend to the interior. We are interested in loops, and the point might be that after

- the scattering operators are handled, the exterior is computed. The point might be that after

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- the scattering operators are handled, the exterior is computed. The point might be that after
Let's review. I consider a port LC network inside, use frequency parameters \( s \), so that time dependence is \( e^{st} \), "real" frequencies are \( s = \omega - i \eta \). There's a response function, which properly speaking is a vector sub-bundle \( F^\perp (V \oplus V^*) \) over the Riemann sphere, thus \( F^\perp \subset V \oplus V^* \). This is one theorem, which you must get into a good form. How? Ideally, you should derive it from varying the proof deriving the structure of the impedance \( Z \). What form might this take?

You start with \( H = H_+ \oplus H_- \) polarized Euclidean space, with operator \( s \pi_+ \oplus s \pi_- \), and then suitably induce this operator to a subquotient \( V = U/W \), \( W < U < H \)  .

Maybe good to see if this induction can be easily described in vector bundle terms.

There's an alternate viewpoint consisting of coupling the port to a trans. line and looking at the scattering operator \( S = \frac{Z-1}{Z+1} \). Supposedly \( F^\perp \) is the vector bundle over the Riemann sphere associated to the clutching function \( S \).

(Consider.

Is there a de Branges theory arising from \( \mathcal{E} \) curves, the idea being to generalize from a Hilbert space of polynomials? )

Scattering should be simpler. Instead of \( \mathcal{E} \), you look at \( (E-I)(E+I) \), let's start with \( 2^\dagger : V \rightarrow V^* \) we?
Still unclear. You have voltage space $V$, current space $V^*$, and $Z_s^{-1}: V \rightarrow V^*$ symmetric.

Initially you think of $V$ and $s$ as real, so $s \mapsto Z_s^{-1}$ can be viewed as a map to symmetric from the Riemann $s$' sphere. If $Re(s) > 0$, then $Re(Z_s^{-1}) > 0$. So you have a rational map into the Siegel UHP from $Re(s) > 0$, and the boundary $Re(s) = 0$ goes to Lagrangian subspaces.

You have the above symplectic approach.

Next change to scattering picture.

$Z = \frac{E}{I} \mapsto \frac{E-I}{E+I} = \frac{2-1}{2+1} = s$

$$F_s = \left( \begin{array}{c} Z_s \\ 1 \end{array} \right) V \subset V \oplus V$$

$$\left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) F_s = \left( \begin{array}{c} Z_s - 1 \\ Z_s + 1 \end{array} \right) V = \left( \begin{array}{c} s \\ 1 \end{array} \right) V \subset \bigoplus V$$

Start again. Consider a point with voltage space $V$, current space $V^*$. Look at response. This is for each $s$ a Lagrangian subspace $F_s \subset V \oplus V^*$. This is the symplectic picture. But there is also the scattering picture.

You start with $W < U < H^+ \oplus H^-$, $H$ polarized Euclidean space, then form $s||h+||^2 + s||h||^2$ you get an induced form on $V = U/W$, whose form we have analyzed. Toles $w > 0$. You understand a lot, but the details are incomplete. You hope the scattering picture is
The scattering picture requires an isomorphism \( V \cong V^* \) and it's natural to take \( c \) to be \( Z_1 \). What then is \( E \), \( I \) in the same space \( V \). Let's understand scattering operators for simple circuits.

\[
E_0, I_0, E_1, I_1, \quad E_0 - E_1 = LsI_0, \quad I_0 = -I_1
\]

So minded the 4-dim space of \((E_0, I_0, E_1, I_1)\), you have the 2-plane satisfying \( I_0 = I_1 \), \( E_0 - E_1 = LsI_0 \). But we want to use the coordinates \( E_0 \pm iI_0 \) and \( E_1 \pm iI_1 \).

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
= \begin{pmatrix}
1 \\
-1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
= \begin{pmatrix}
1 & Ls \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_1 \\
-I_1
\end{pmatrix}
= \begin{pmatrix}
1 & -Ls \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
\]

\[
E+I = e^{s(-x+t)} A
\]
\[
E-I = e^{s(x+t)}
\]

Let's use

\[
E_0 \rightarrow E_1, \quad I_0 \rightarrow I_1, \quad I_0 = -I_1, \quad E_0 - E_1 = LsI_1
\]

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
= \begin{pmatrix}
1 & Ls \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & Ls \\
0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
= \begin{pmatrix}
1 & Ls \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix}
= \begin{pmatrix}
1 & 1 & Ls \\
0 & 1 & -1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]
\[
\begin{align*}
\begin{pmatrix}
E_0 + I_0 \\
E_0 - I_0
\end{pmatrix}
&= \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix}
\begin{pmatrix}
L_+ \\
0
\end{pmatrix}
\begin{pmatrix}
-1 & 1 \\
-1 & 1
\end{pmatrix}
\begin{pmatrix}
E_1 + I_1 \\
E_1 - I_1
\end{pmatrix} \\
&= \frac{1}{2}
\begin{pmatrix}
1 & 1 + L_+ \\
1 & L_-
\end{pmatrix}
\begin{pmatrix}
-1 & 1 \\
-1 & 1
\end{pmatrix} \\
&= \frac{1}{2}
\begin{pmatrix}
-2 - L_+ & L_+ \\
- L_+ & L_+ - 2
\end{pmatrix}
\begin{pmatrix}
E + I = e^{s(-x+t) \text{ const.}} \\
E - I = e^{s(x+t) \text{ const.}}
\end{pmatrix}
\end{align*}
\]

\[
\begin{pmatrix}
E_0 + I_0 \\
E_0 - I_0
\end{pmatrix}
_{\text{out}}
= \begin{pmatrix}
1 + \frac{1}{2}L_+ & -\frac{1}{2}L_+ \\
\frac{1}{2}L_+ & 1 - \frac{1}{2}L_+
\end{pmatrix}
\begin{pmatrix}
E_1 + I_1 \\
E_1 - I_1
\end{pmatrix}
_{\text{inc.}}
\]

This lies in \( SU(1,1) \) for \( s \in \mathbb{R} \).

\[
\begin{pmatrix}
a & b \\
b & \bar{a}
\end{pmatrix}
\Rightarrow \ |a|^2 - |b|^2 = 1.
\]

What's the relation between \( SU(1,1) \) and \( U(2) \)

\[
\begin{align*}
(E_0 + I_0) &= \alpha (E_1 + I_1) + b (E_1 - I_1) \\
(E_0 - I_0) &= \overline{b} (E_1 + I_1) + \overline{\alpha} (E_1 - I_1)
\end{align*}
\]

\[
\begin{align*}
(E_1 + I_1) &= \frac{1}{\alpha} (E_0 + I_0) - \frac{b}{\alpha} (E_1 - I_1) \\
(E_0 - I_0) &= \frac{\overline{b}}{\alpha} (E_0 + I_0) + \frac{\overline{\alpha}}{\alpha} (E_1 - I_1)
\end{align*}
\]
\[
\begin{pmatrix}
\frac{E_0 + I_0}{E_0 - I_0}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{a} & -\frac{b}{a} \\
\frac{b}{a} & \frac{1}{a}
\end{pmatrix}
\begin{pmatrix}
E_0 + I_0 \\
E_1 - I_1
\end{pmatrix}
\]

in the example.

\[
\begin{pmatrix}
\frac{1}{1 + \frac{1}{2} l s} & \frac{\frac{1}{2} l s}{1 + \frac{1}{2} l s} \\
\frac{\frac{1}{2} l s}{1 + \frac{1}{2} l s} & \frac{1}{1 + \frac{1}{2} l s}
\end{pmatrix}
\]

identity if \( s = 0 \).

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

if \( s = \infty \).

This doesn’t look very helpful.

Next try

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} \rightarrow \begin{pmatrix} E_1 \\
I_1
\end{pmatrix}
\]

\[
E_0 = E_1 \\
I_0 - I_1 = \alpha s E_1
\]

\[
\begin{pmatrix}
E_0 \\
I_0
\end{pmatrix} = \begin{pmatrix} 1 & 0 \\
\alpha s & 1
\end{pmatrix}
\begin{pmatrix}
E_1 \\
I_1
\end{pmatrix}
\]

\[
\begin{pmatrix}
E_0 + I_0 \\
E_0 - I_0
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix} 1 & 0 \\
\alpha s & 1
\end{pmatrix} \begin{pmatrix} 1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix} \frac{1}{2} l s \\
1 + \frac{1}{2} l s
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1 + \alpha s}{2} & \frac{\alpha s}{2} \\
\frac{-\alpha s}{2} & \frac{2 - \alpha s}{2}
\end{pmatrix}
\begin{pmatrix}
\frac{a}{2} & \frac{b}{2} \\
\frac{-\frac{1}{2} \alpha s}{2} & \frac{1 - \frac{1}{2} \alpha s}{2}
\end{pmatrix}
\]
\[
\begin{pmatrix}
E_1 + I_1 \\
E_0 + I_0
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{1 + \frac{1}{2}Cs} & -\frac{1}{2}Cs \\
-\frac{1}{2}Cs & \frac{1}{1 + \frac{1}{2}Cs}
\end{pmatrix}
\begin{pmatrix}
E_0 + I_0 \\
E_1 - I_1
\end{pmatrix}
\]

Would it help to do a 1-port.

\[Z = Ls + \frac{1}{Cs} = \frac{LCs^2 + 1}{Cs}\]

\[\frac{Z - 1}{Z + 1} = \frac{LCs^2 + 1}{LCs^2 + 1} - 1 = \frac{LCs^2 - Cs + 1}{LCs^2 + Cs + 1}\]

As a check note that roots of num. are

\[s = \frac{C \pm \sqrt{C^2 - 4LC}}{2LC}\]

If \(C^2 - 4LC \leq 0\), then \(\text{Re}(s) = \frac{E}{2LC} > 0\)

\[> 0\], then \(\frac{C + \sqrt{C^2 - 4LC}}{2LC} > 0\)

and the prod of the roots is \(\frac{1}{LC} > 0\)

other root also > 0

In the example, \(\det = \frac{a}{a} = \frac{1 - \frac{1}{2}Cs}{1 + \frac{1}{2}Cs}\) or \(\frac{1 - \frac{1}{2}Ls}{1 + \frac{1}{2}Ls}\)

again root is \(s = \frac{a}{C}\) or \(\frac{a}{L}\) in RHP.

So where to start?
Your aim is to control the vector bundle.
Roughly