Try to do some work 15.05

I think I might review what needs to be done before you can finish. You can go back to Keltch, but the next month should make much of your brain so where to start. Maybe review

What was the Klein-Leningrad existence of objects about? Inverting isomorphism. You idea Ext(T, M) = 0 \( j = 0,1 \) \( \Rightarrow \) Hom(\( \cdot \), M) injective makes birth kernel + cokernel come to T. Let's try to figure out what might have been meant.

So consider the effect of \( \delta \) in \( E \leq 1 \). Suppose we go back to the example you know. Coherent sheaves on \( \mathbb{P}^1 \) \( T = \mathcal{O} \oplus \mathcal{O}(1) \)

\( \mathcal{A} = \text{Coh}(\mathbb{P}) \). Then I have positive and negative sheaves. So what do you know about the sheaf situation? If you take \( T = \mathcal{O} \oplus \mathcal{O}(1) \)

then \( \text{Hom}(T, F) = H^0(F) \oplus H^0(F(-1)) \)

\( \text{Ext}^1(T, F) = H^1(F) \oplus H^1(F(-1)) \)

Certainly \( H^0(F) = 0 \Rightarrow H^0(F(-1)) = 0 \)

and \( H^1(F(-1)) = 0 \Rightarrow H^1(F) = 0 \).

So what? There are sheaves \( F \) with \( H^0(F) = H^1(F) = 0 \). So what you want is to understand \( F = F^+ \oplus F^- \).

Reg she closed under quotients of extensions. They are like torsion modules. Reg: regular vs closed under extras + subobjects. They are like torsion-free modules.
What does this relate to? Condition $\text{Hom}(T,F) = 0$ closed under submodules and extensions. Condition $\text{Ext}'(T,F) = 0$ leads to a closed category of adjoints. Assuming $T$ has projective dimension $\leq 1$. Maybe the real issue is existence of adjoints. So suppose that $\mathcal{A}$ is a full subcategory of $\mathcal{A}$. What does it mean for the inclusion $\mathcal{A}^\mathbb{C} \subset \mathcal{A}$ to have the appropriate adjoint?

Let's ask for the following: $\forall M \in \mathcal{A}^\mathbb{C}, M^\# \in \mathcal{A}$ with kernel and cokernel in $\mathcal{A}$ and $M^\# \in \mathcal{A}$. At this point things would be much easier if I had a copy of $[GL]$. So suppose $\text{Hom}(\mathcal{A}(T,F) \neq 0$. Consider $0 \to F^+ \to F \to F^- \to 0$

For each sheaves on $\mathcal{A}^\mathbb{C}$, I don't see any interesting $\mathcal{A}^\mathbb{C}$ and ask for you might as well take $\mathcal{A}$. Then for $s = 0$

$H^0(F) = H^1(F) = 0 \implies F = \mathcal{O}(-1) \otimes V$

So its much similar. But then maybe $F \mapsto \mathcal{O} \otimes H^0(F)$ is the appropriate adjoint.
Anyway I didn't get much further. So let's think a bit. The question is whether $S$ is always a dense subcategory. What is $\text{Hom}_M(N, M)$ all $M$ such that $\text{Ext}_*(M, N) = 0$ for all $N$ such that $\text{Ext}_*(S, N) = 0$ $S \subseteq I$. If $S = \text{Hom}_M(N, M)$? Also in general $S^+$ and $I^+$ are closed under extensions.

$$S^+ = \{ N \mid \text{Ext}_*(S, N) = 0 \ \forall S \in I \}.$$ 

$$0 \to N' \to N \to N'' \to 0$$

$$\to \text{Hom}(S, N') \to \text{Hom}(S, N) \to \text{Hom}(S, N'') \to \text{Hom}(S, N'')$$

$$\to \text{Ext}(S, N) \to \text{Ext}(S, N'') = 0$$

If $\text{projdim} S \leq 1$, then get

so you see that nothing General case you get $N', N'' \in I \implies N \in I$

$N \in I$ and $N' \in I \implies N' \in I$.

Also $S \cap S^+ = 0$. The good situation should be that $S + S^+ = A$ in some sense. In all this it seems that $S$ is tame. Therefore you don't understand the relation with sheaves on $P_1$. 
where can you look next? What about the 'Self and stuff'? Selfand situation: You have a quiver and you reverse the arrows going into a sink. The indecomposables don't change. Ex.

Here you have

\[ M_0 \rightarrow M_1 \rightarrow \cdots \]

and you can split off the cokernel of \( M_0 \oplus M_1 \rightarrow M_0 \) from the indecomposables.

I should review all this stuff, all these examples. In general suppose you have a single sink. This example is related to the basic idea is that you can split off the cokernel of

\[ \bigoplus_{y} M_y \rightarrow M_x \]

Then you are left with a surjection, so \( M_x \) is determined by the kernel, call it \( K_x \). So you replace \( M \) by \( \{ M_y \mid y \neq x \} \) together with the \( K_x \) at \( x \) and the maps \( K_x \rightarrow M_y \) for each \( y \rightarrow x \). So you have a new functor of the new quiver when arrows going to \( x \) are reversed.

Let's see if we can recall how it works.

You have an example. Let \( P_y \) be the projective module generated by \( k \) at \( y \). Then \( P_y(y) = \bigoplus_k y \rightarrow y' \).

This should be paths from \( y \) to \( y' \)
So if $x$ is a sink, then what?

$a \rightarrow b \rightarrow c \rightarrow d \rightarrow e \rightarrow f$

$k \rightarrow 0 \rightarrow 0
k = k \rightarrow 0, k = k = k
0 \rightarrow k \rightarrow 0
0 \rightarrow k = k \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow k$

Let $M$ be a module, i.e., a rep. of groups. Find:

I want to get the cokernel $M_x/\oplus M_y$. What's important is $k_y$ at 0 elsewhere. Then $k_y$ should be a quotient of $P_y$. In the example:

$a \rightarrow y \rightarrow x \rightarrow z \rightarrow k_x = P_x$

$k_x \rightarrow 0 \rightarrow k$

$k_c \rightarrow 0 \rightarrow k$

$p_x \rightarrow 0 \rightarrow k$

$k_c : k \rightarrow p_a : k \rightarrow 0$

Instead look at:

$k_a : k \rightarrow 0 \rightarrow k \rightarrow 0$,

$k_c : k \rightarrow 0 \rightarrow k \rightarrow 0$,

$k_a : k \rightarrow 0 \rightarrow k \rightarrow 0$.

Exact:

$0 \rightarrow P_c \rightarrow P_a \rightarrow k_a \rightarrow 0$

$0 \rightarrow \text{Hom}(k_a, M) \rightarrow M_a \rightarrow M_c \rightarrow \text{Ext}^1(k_a, M) \rightarrow 0$

So what happens? Basically, you have a source $a$ - it gives rise to

$0 \rightarrow \oplus P_b \rightarrow P_a \rightarrow k_a \rightarrow 0$

Now somehow I need to

[Signature]
Think of this as torsion theory. $\text{Hom}(k_a, M)$ - this gives the part to split off. So the free subcategory might be $M$ supported at the source $a$. Torsion and torsion free. Closed means what?

$0 \rightarrow \text{Hom}(k_a, M) \rightarrow M_a \rightarrow \bigoplus_{b} M_b \rightarrow \text{Ext}^1(k_a, M) \rightarrow 0$

$M \in \mathcal{S}^+$ means $M_a = \bigoplus_{b} M_b$. So it clear that $\mathcal{S}^+ \supseteq \mathcal{A}/\mathcal{S}$, I guess so. The only point here seems to be that $\mathcal{S}^+$ is the modules for the quiver with the source $a$. So if true what about changing the arrows. Start with $M$ and split off the submodule supported at $a$. Then you end up with $0 \rightarrow M_a \rightarrow \bigoplus_{b} M_b \rightarrow \text{Ext}^1(k_a, M) \rightarrow 0$.

To explain the relation. You start with $M$ a module for the quiver in which $a$ is a source. You get an equivalence of cats between $M$ torsion-free $M$ for this quiver and cotorsion-free $N$ for the other quiver. Actually we are dealing with idempotent ideals in this case. What are the simple objects? Is it clear that the vertices correspond to simple objects? Suppose given $M$. No. Interesting.

representations of a tensor algebra. So if you stay away from looks then OK.

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This time I know about localizing subcats. How about derived categories? What next? What might happen to the derived categories? Take a source, a sink. Can we find a functional resolution? Try a sink maybe first. I think I need an examples.

Basic projectives:

\[ P_a : \xrightarrow{k} \quad P_b : \xrightarrow{k} \quad P_c : \xrightarrow{k} \]

\[ Q : \xrightarrow{k} \quad 0 \rightarrow P_c \rightarrow P_a \oplus P_b \rightarrow Q \rightarrow 0 \]

\[ 0 \rightarrow \text{Hom}(Q, M) \rightarrow M_a \oplus M_b \rightarrow M_c \rightarrow \text{Ext}^1(Q, M) \rightarrow 0 \]

What I want is some module \( T \) probably \( P_a \oplus P_b \oplus Q \) which will give us via \( \text{Hom}(T, -) \)

\( M_a, M_b \) and the submodule \( \ker(M_a \oplus M_b \rightarrow M_c) \).

Thus \( \text{Hom}(T, M) = M_a \oplus M_b \oplus \text{Hom}(Q, M) \). What is \( \text{End}(T) \)?

\[ \text{Hom}(T, T) = \begin{pmatrix}
\text{Hom}(P_a, P_a) & \text{Hom}(P_a, P_b) & \text{Hom}(P_a, Q) \\
\text{Hom}(P_b, P_a) & \text{Hom}(P_b, P_b) & \text{Hom}(P_b, Q) \\
\text{Hom}(Q, P_a) & \text{Hom}(Q, P_b) & \text{Hom}(Q, Q)
\end{pmatrix} \]

\[ = \begin{pmatrix}
k & 0 & k \\
0 & k & k \\
0 & 0 & k
\end{pmatrix} \]
In general, given a sink \( x \), what do you do?

\[ a \rightarrow b \]

The column of \( P_b : 0 \rightarrow k \)

\[ P_a : k \rightarrow k \oplus k \]

I want to control \( M_a \oplus M_a \rightarrow M_b \), so I take

\[ 0 \rightarrow P_b \rightarrow P_a \oplus P_a \rightarrow Q \rightarrow 0 \]

\[ k \oplus k \rightarrow k \]

Confusion:

\[ a \rightarrow b \]

\[ M_a \rightarrow M_b \oplus M_c \]

\[ 0 \leftarrow k_a \leftarrow P_a \leftarrow P_b \oplus P_c \rightarrow 0 \]

But if you try to take \( k_a \oplus P_b \oplus P_c \), then you get wrong ends ring.

So you need to understand the difference between \( a \leftarrow b \) and \( a \rightarrow b \).

2nd:

\[ M_a \leftarrow M_b \oplus M_c \]

\[ 0 \rightarrow P_a \rightarrow P_b \oplus P_a \rightarrow Q \rightarrow 0 \]

\[ (k < 0) \]

\[ (k > 0) \]
Source versus sink.

Source a, then the M with support a form a Serre subcat of all M. I generated by ka so \( S = M \) such that \( \text{Ext}^i(k_a, M) = 0 \).

Here things are localizing - inclusion \( i : A \rightarrow A \) has a right adjoint.

Case of a sink a. Again M with support a should form a Serre subcat. In general if I pick any vertex, then those M supported on that vertex form a Serre subcat closed under products. Why?

Condition is \( \text{Hom}(P_0, M) = 0 \) for \( b \neq a \) and \( P_0 \) projective.

Two Serre subcats.

What about \( W = 0 \) get \( V = V \).

As well as Other \( V = 0 \) get negative type.

Yesterday I got one idea on the plane - namely to use Serre subcats in com. with selfdual business. Given a quiver, let \( A = \text{abel. cat of its reps} \) (modules). Then you have these proj. \( P^a \) for each vertex \( \text{Hom}(P^a, M) = M_a \). \( P^a \) has basis the paths going from \( a \) to \( b \).

Then \( E(M|_a = 0) \) is a Serre subcat closed under prods \( \text{Hom}(P^a, M) \) so you can know \( \Lambda \) is belocalizing. Functions \( M \rightarrow M \) have. You want...
to understand \( M_j \) in the case of a source and a sink.

You believe that \( A/A_a \) is in the cat. of modules on the quiver with the vertex a deleted. In general, \( A \) is an abel. cat with a small proj generator \( \oplus P_r \). So we have all \( P_r \) \( r \in Q_0 \). \( Q = Q - \{ a \} \).

\[ \text{Hom}(P_r, -) \] kills \( I_a \). In fact this is an exact functor kills the face subcat, so \( \oplus P_r \) should be a proj gen. of \( A/I \).

Notice that \( \oplus P_r \subseteq I \) since it inverts \( I \).

Thus for a quiver \( Q \), \( A/Q \), \( Q = Q - \{ a \} \), and we get three subcats realizing this quotient cat. \( I_a \) from, \( I_a \) closed, reduced.

Reduced means no submodule, no quotient module supported at \( a \).

\[ \begin{array}{c}
\xymatrix{ X \ar[r]^a & Y }
\end{array} \]

\[ \begin{array}{c}
\xymatrix{ M_X \ar[r] & M_a \ar[r] & M_Y }
\end{array} \]

Reduced.

If \( M_a \to M_Y \) not surj then \( 0 \to k \to 0 \) is a submodule.

If \( M_X \to M_a \) not surj.

\[ \begin{array}{c}
\xymatrix{ 0 \ar[r] & k \ar[r] & 0 }
\end{array} \]

Is a quotient module.

What might \( k \) mean??
Check this.

\[ M_x \to M_a \to M_y \]
\[ M_x \Rightarrow M_a \Rightarrow M_y \]

\[ M_x \xrightarrow{k} M_a \xrightarrow{k} M_y \]
\[ N_x \xrightarrow{k} N_a \xrightarrow{k} N_y \]

This is an extra worry. Kernel is \( k \).

So much for this example. Next consider a source.

\[ a \xrightarrow{\circ} x \]

\[ M_a \Rightarrow M_x \]

Reduced (\( M_a \rightarrow M_x \times M_x \))
\[ M_a = 0 \]

Otherwise \( k = 0 \), in mod.

Reduced means just that \( M_a = 0 \).

Form is the same, probably.

Closed should probably mean: \( M_a \Rightarrow M_x \times M_x \)

Check:
\[ M_a \Rightarrow M_x \]
\[ M_x \times M_a \Rightarrow M_x \]

So, OKAY.
This should also work with:
\[ M_x \rightarrow M_y \]
\[ M_a \rightarrow M_y \]

no quotient mod of \( \delta \) means \( M_a = 0 \).

so reduced + firm x same + mean \( M_a = 0 \).

closed:
\[ M_a \Rightarrow M_x \]
\[ M_x \Rightarrow M_y \]
\[ M_x \times M_y \Rightarrow M_x \times M_y \]

What will the general picture be? You should have:

\[ a \rightarrow \square \rightarrow a \leftrightarrow \square \]

So what does reduced mean: a submodule \( k^a \) means a non-zero element of \( M_a \) killed by outgoing arrows.

a quotient \( k^a \) means a linear form on \( M_a \) killing incoming arrows. Thus red means \( M_a \) gets by incoming arrows and cogen by outgoing arrows.

\[ \bigoplus M_x \rightarrow M_a \rightarrow \bigcap M_y \]
\[ x \rightarrow a \]

What is firm mean: probably that:

\[ \bigoplus M_x \rightarrow M_x \rightarrow \bigcap M_y \]
\[ x \rightarrow a \]

and closed means prob.

\[ M_a \rightarrow \bigcap M_y \]

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It might be pretty messy to remove an internal vertex. What can you do next?

Now go back to the hitting idea.

Wait: [GL] K-theory argument. What is $S_a$?

$S_a$ = all module supported at $a$. There's a problem when there is a map from $a$ to $a$.

So what happens. You have paths from $a$ to $a$, but only to arrows from $a$ to $a$ matter. Thus if $M$ has support at $a$, then only a loop $\bullet$ will act non-trivially -- other paths give 0. If we avoid loops then the Serre subcategory $S_a$ are all $\rightarrow$ equivalent to $C$.

How does this affect the preceding argument, namely the picture of a red. modules. Basically any arrow follows into 4 types.

\[ a \rightarrow b \rightarrow c \rightarrow a \]

\[ b \rightarrow a \rightarrow c \rightarrow b \]

\[ a \rightarrow b \rightarrow c \rightarrow a \]

\[ b \rightarrow a \rightarrow c \rightarrow b \]

These it should be clear. It seems OK.

Question is whether the condition that there is no arrows $a \rightarrow b \rightarrow a$ is the one Lusztig imposes on his guiners.
So there remains the question of the derived category. What exactly do I learn from this stuff? Now I have $S \to A \to A/\ell$. I can ask the bimodal question. There's an idempotent ideal here. $R = \text{alg of the quiver}$. $\text{End}(\bigoplus P^x)$. Remove the vertex $a$ and see what arises. Have $P = \bigoplus P^x = P^a \bigoplus \bigoplus P^x$. So it seems we have a context. What is going on? Anyway think. You have $S \to A \to A/\ell$.

$$\text{Mod}(R/I) \to \text{Mod}(R) \to \text{Mod}(R', I)$$

Then $M(I) \cong M(R')$.

So what you need to find is an idempotent ideal in $R$ Morita equiv to $R'$. Expect maybe a Morita context $R'$.

I need to analyze this situation carefully. Basically you have $R = \text{End}(P)^\circ$ where $P = P^a \bigoplus P'$. $P$ is the projective generator for $A$. $P' = eP e$.

$R' = \text{End}(P e)^\circ = \text{Hom}_R(P e, P e)$.

Some ideal $I$. $R \pi R$. Can you make sense of bimodal? What I know about this situation is that there should be a right + left $R'$ module — a dual pair generating $R$. 
So what do I find? Split $R'$

You have split $R'$ gen. $\oplus p^x = V$. Split it into $V = V(1-c) \oplus V_e$ where $V_e^d = p^a$, $V_e = \oplus p^x$.

$R' = \text{End}(V_e \oplus V_e) = (\text{End}(V_e^d), \text{Hom}(V_e, V_e^d), \text{Hom}(V_e^d, V_e), \text{End}(V_e))$

Can look at $\text{Hom}(p^a, V_e)$ left $\text{End}(V_e)$ module.

What should $\text{Hom}(V_e, p^a) = \oplus \text{Hom}(p^a, p^a) = \oplus x^+a$ basic paths from $a \to x$.

$\text{Hom}(p^a, V_e) = \oplus \text{Hom}(p^a, p^a) = \oplus x^+a$ paths $x \to a$.

The pairings don't look too good, which is OK so what we should have is

$$
\begin{pmatrix}
\text{End}(V) & \text{Hom}(V_e, V) \\
\text{Hom}(V, V_e) & \text{End}(V_e)
\end{pmatrix}
$$

I don't see anything to learn.

Can I make any progress? The h invariant question.

I know that $R' = \text{End}(V_e)^d$ is a
path algebra.proj dim \leq 1. Does this say anything about the \( R' \) modules \( e_R \) and \( R e \)?

\[ R e = \text{Hom}(V, V) e_R = \text{Hom}(V, V) \]

Your arrows again.

\[ V = \bigoplus \mathcal{P}^x \]

Put \( R = \text{End}(V) \) to begin so \( R \) acts on the right of \( V \). Then

\[ \text{Hom}(V e, V) = e_R \]

\[ \text{Hom}(V, Ve) = \]

Set \( R = \text{End}(V)^{op} \) so that \( R \) acts to the right on \( V \). Then \( e_R \) gives \( V e_r \).

\[ \text{Hom}(V, Ve) \]

\[ \text{Hom}(V e_r) \]

\[ \text{Hom}(Ve_r, Ve) = e_R \text{Re} \]

\[ \text{Hom}(V, Ve) \]

\[ \text{Hom}(Ve, V) \]

\[ \text{Ham}(Ve_r, Ve) \]

What can you say about these as \( e_R e \) modules.

\[ \text{Ham}(Ve_r, Ve) = e_R e \text{Re} \]

\[ \text{Ham}(\mathcal{P}^x, \mathcal{P}^x) \]

as a right \( \text{End}(Ve) \) module.

So here's an interesting question, namely, you have a quiver \( Q' = Q - a \) and its path alg \( R' \).

Where's the glue tying \( a \) to the rest.

Answer: you give two subsets of \( Q' \) representing

\[ x \rightarrow a \rightarrow y \]

might be empty.
Review: Starting with a quiver $Q$ we get the abelian category of its representations, $R$-modules where $R^Q = \text{End}(\bigoplus x \in Q \mathbb{V}_x)$. We get a Serre subcat of modules with support at the given vertex $a$, equiv. $\text{Hom}(\mathbb{V}_a, M) = M_a = 0$. The quotient cat $\mathcal{A}/\mathcal{S}$ is the modules for the quiver $Q' = Q - a$. $\text{Mod}(Q') R' = \text{End}(V')$, $V = V_a \oplus V'$. $\exists e \in R$ s.t. $V^2 = Ve$, $V' = Ve$ and then $R' = eRe$. Now we know $\mathcal{I} = \text{Mod}(R/\bigoplus A) \subset \text{Mod}(R)$ where $A$ generated by things factoring through $V^a$.

I forgot to say that $\mathcal{I} = \text{Mod}(k)$ provided there are no arrows $a$. Then we see that

$A = ReR$ \quad $Re = \text{Hom}(W, \text{Wait})$

$\text{Reps. of quiver} = R^Q$ modules.

Get this straight. $V$ is a generator of $A$, then let $R^Q = \text{End}(V)$, then have adjoint functors

$$\text{Mod}(R) \leftrightarrow \mathcal{A}$$

$$N \leftrightarrow \mathcal{V} \otimes_R N$$

$$\text{Hom}_R(V, M) \leftrightarrow M$$

$$\text{Mod}(R) \leftrightarrow \text{Mod}(S)$$

$$N \leftrightarrow V \otimes_R N$$

$$\text{Hom}_R(V \otimes_R N, M) = \text{Hom}_R(N, \text{Hom}_S(V, M))$$
So how do I want to think about representations of the quiver? As left modules, with the arrows acting as left multiplication. \( V = \bigoplus V^x \) should be \( R \) or the path algebra, when we convert the quiver module to an \( R \)-module, i.e.

\[
\bigoplus V^y = \bigoplus V^x = R
\]

and this acts on the left of \( \bigoplus M_x \) for any quiver module. \( V^x \) has basis the paths from \( x \) to \( y \).

Now return to the \( R \)-module description of quiver modules.

\[
V = \bigoplus V^x \quad \mapsto \quad R = \bigoplus R e^x
\]

\[
R e^x = \bigoplus V^x
\]

Now it should be possible to handle.

Above I succeeded in translating between a quiver rep. and an \( R \)-module. \( V = \bigoplus V^x \) is the smallest proj generator

\[
\text{Mod}(R) \xrightarrow{\sim} A
\]

\[
N \mapsto V \otimes_R N
\]

Hom \( (V, M) \xrightarrow{\sim} M \)

so the \( R \)-module corresponding to \( M \) is \( \bigoplus M^x \).
Review

\[ \text{Mod}(R) \rightarrow A \]
\[ V \rightarrow V \oplus R \]
\[ N \rightarrow V \otimes R \]
\[ \text{Hom}_R(V, M) \leftarrow M \]
\[ \text{Hom}_R(V, M) = M_x \]

Right \text{End}(V) \text{-module} \iff \text{left} \ R \text{-mod.}

\[ \text{Hom}_R(V, M) = \bigoplus \text{Hom}_R(V^x, M) = \bigoplus M_x \]
\[ \text{Hom}_R(V, V) = \bigoplus V^y = \bigoplus V^x \]
\[ \text{Hom}_R(V, V) = \text{Hom}_R(\bigoplus V^y, \bigoplus V^x) \]

\[ \oplus M_x \text{ is an } R \text{-module} \]

\[ V \text{ in } A \text{ corresps to } R \text{ as left } R \text{-module} \]
\[ \bigoplus V^y = \bigoplus V^x \]

\[ y \]

\[ y \]

\[ y \]

\[ y \]

\[ y \]

Confusion: Start again. \( V = \bigoplus V^x \) is a small proj. generator of \( A \).
\[ \text{Hom}_R(V^x, M) = M_x \text{, } M \]

Corresp to \( R \)-module \( \text{Hom}_R(V, M) = \bigoplus \text{Hom}_R(V^x, M) = \bigoplus M_x \)

\[ \text{Hom}(V^y, V^x) \otimes \text{Hom}(V^x, M) \rightarrow \text{Hom}(V, M) \]

\[ V^x \otimes M_x \rightarrow M_y \]

\[ V^x \]

\[ V^y \]

\[ V^x \]

\[ V^x \]

\[ V^x = Ve^x = R e^x \]

\[ \text{Hom}(V^x, M) = \text{Hom}_R(R e^x, M) = e^x M = M_x \]

Now look at a vertex \( a \) which we want to remove.

\[ V = V^a \oplus V^x \]

\[ V^x = Ve^x = R e^x \]

Life is hard to do.
What is the aim. I have this idempotent \( e = (e^2) \) in \( R \), hence the usual \( (R \oplus eR \oplus eRe) \). The question is whether \( ReR \) is \( h \)-unital. I think this means that \( Re \oplus eR \to ReR \) so it ends up with examining \( Re \) and \( eR \). Now

\[
 Re = eRe \oplus eRe^t \quad eR = eRe \oplus eRe^t
\]

so the issue probably amounts to whether

\[
 e^t Re \oplus eRe \to e^t ReRe^t
\]

\[
 e^t Re = e^t R \sum_{x \neq a} e^x \quad \text{basis of paths from a vertex } \mathcal{X} \text{ to a}
\]

\[
 eRe^t = \sum_{x \neq a} e^x Re^t \quad \text{paths from a to outside}
\]

It seems that \( eRe^t \) is a free \( eRe \)-module with \( \mathcal{X} \) basis the edges \( a \to x \) outside. I know that \( Re^t \) is a proj (in fact free) \( R \)-module, so what about \( eR \oplus Re^t \). You would need \( eR \oplus \) projection over \( eRe \).

So this lets check out this pen?

There's a problem remaining, from yesterday.

So back over the data from the beginning.
Reviewer: You have a quiver, $A$ is the abelian set of its reps over $k$. $\mathcal{A} = \text{Mod}(R)$ where $R = \text{path algebra of quiver}$. Specifically, you have projectives $V \cong \bigoplus V^x$ in $\mathcal{A}$ such that $\text{Hom}(V^x, M) = M^x$ and for each arrow $x \to y$ you get $M_x \to M_y$ and $V^x \to V^y$. $V = \bigoplus V^x$ is a small proj. generator, so if $R = \text{End}(V)^\oplus$ then we have $\mathcal{A} = \text{Mod}(R)$ given by $M \mapsto \text{Hom}(V, M)$, $N \mapsto V \otimes_R N$. Note $R$ is $\text{Mod}(R)$ corresponding to $V$. We have projections $e^x \in R$ coming from proj of $V$ onto its summand $V^x$. Now jumps to the problem of $\mathcal{S} = \text{barrel subset of } M \mapsto M_a = \text{Hom}(V_a, M) = 0$, where $a$ is a fixed vertex. $\mathcal{S} \subseteq \text{Mod}(k)$ if I no arrows $a \to a$. Let $\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$ $\text{Mod}(k) \to \text{Mod}(k) \to M(k, a)$ is $\text{Mod}(k, Re)$. $A$ is the ideal $ReR$. The problem is where $A$ is $k$-unital, which I think is equivalent to $Re \otimes eRe \subseteq Re$ being a quiver.

Now $eRe = \text{End}(Ve)^\oplus = \text{Hom}(Re, Re)^\oplus$ in the path alg for the quiver with $a$ deleted. $Re = I + eRe \oplus e^2Re$. $Re$ is projective over $R$. You need to understand it over $eRe$. Shift to $eR$.

$$ R = \text{Hom}(\bigoplus V, \bigoplus V) = \bigoplus eRe \otimes x, y $$

You really need better control over $R$.

Anyway, start with $M = \bigoplus M^x$ and for each...
Arrow \[ x \to y \] in the quiver you have \[ M_x \to M_y \]. If \[ x \to y \to z \], then in \[ R \] you have the path \((u, v)\) acting as \[ vu : M_x \to M_y \to M_z \]. Stupid point you wrote much time at. Let's make paths run the way you compose. Thus \( x \to y \) is \( vu \), so that the product of my alts comes to comp.

Look at \( R = \bigoplus e^x Re^x e^x Re^x \). \( M_x = \text{Hom}_R (Re^x, M) = e^x M \). Do \( e^x Re^x : e^x M \to e^x M \). \( e^x Re^x \) has basis all paths from \( y \) to \( x \). Look at \( e^x Re = \bigoplus e^x Re^x \). This has basis all paths going from a vertex \( y \to x \) to \( a \). Take a path:

\[ x \to y_1 \to y_2 \to \cdots \to y_n \to a \]

examine this as a \((eRe)^{op}\) module. Claim? might be that any such path has a first point \( = a \).

Let's write the path

\[ y_0 \to y_1 \to \cdots \to y_{n-1} \to y_n = a \]

We know \( y_0 \to a \). There is a least \( p \) such that \( y_p = a \).

Look just at \( Re \) as \( eRe \) module. \( Re = \bigoplus e^x Re^x \). Paths starting at \( x \to 0 \) going anywhere. Point is that any such path has a first point \( = a \)? One possibility. Looks like you made a mistake. You assume that \( eRe \) is the path alg of the quiver with \( a \) removed. So what seems to be happening? You have a problem. \( eRe = \bigoplus e^x Re^x \) involves paths which might contain \( a \).

So there's an inessential idea.
So I seem to end up with a difficulty. I have the M content. You were working with 
\( (R \otimes R) \) gives the ideal \( R \otimes R \) in \( R^2 \).

Go back to \[ \mathbb{L} \rightarrow A \rightarrow A/[. \]

\[ \text{Mod}(k) \quad \text{Mod}(R) \quad \text{Mod}(eR) \]

\[ R/\text{ker} \]

The mistake seems to be to identify \( eR \) with the path alg of the quiver with the vertex a removed. I think what happens is that you have to add arrows, namely for each \( x \rightarrow a \) and \( a \rightarrow y \) you add an arrow \( x \rightarrow y \). Why? Go back to figure out what you found about \( J^+ \) and \( J^- \) and the reduced modules.

\[ x = y \]

\[ \text{dual to } M_a \rightarrow M_x \]

\[ M_z \rightarrow M_a \rightarrow M_z \]

\[ 0 \rightarrow k \rightarrow 0 \]

\[ x \rightarrow a \]

\[ a \rightarrow y \]

What's important here is the map \[ \oplus M_x \rightarrow \oplus M_y \]
for all $M$. Again $\otimes M_x \xrightarrow{\phi} \otimes M_y$. 

$\phi$ map of functors, first rep. $M_x = \text{Hom}(P^x, M)$. Clearly $\phi$ is a matrix $\phi_{xy}$, more precisely $\phi_{x \rightarrow a, a \rightarrow y}$, same as the operators assoc. to the path $x \rightarrow a \rightarrow y$ for each $x \rightarrow a, a \rightarrow y$. So it's clear. Anyway let's check this out. Consider $\xrightarrow{\epsilon} \bigtriangleup$. Then $x$

$\mathcal{R}$ is generated by $e_a, e_b, a, b$ satisfying $e_a^2 = e_a$, $e_b^2 = e_b$, $e_a e_b = e_b e_a = 1$, $a e_a = b e_b$. The kernel category is such that $\nu: M_b \rightarrow M_a$. Thus you just get $G_b$.

So this is pretty interesting because now you can see some measure of the complexity of a quiver. Each time you remove a vertex you acquire more arrows. This gives a quotient abelian, so if you progressively delete all but one vertex you end up with a quotient abelian cat with one vertex. This has the form $\xrightarrow{\epsilon}$, and the ring is a tensor algebra so you get wild ramification type. Need example. Look at quivers with two vertices, all arrows from one to the other $\xrightarrow{\epsilon} \bigtriangleup$. If we delete a then end up with $2 \times 3 = 6$ loops at $b$. $\bigtriangleup$
Question: Can you understand what happens if you delete two vertices first case: not joined by any arrow.

Three vertices $a \xrightarrow{p} c \xrightarrow{r} b$ arrows from $c$ to itself.

If you delete a source or sink then you do not add any arrows. Maybe finite rank type means that you can delete without producing loops.

Okay. I hours work. Observe two ex:

1. $a \rightarrow b \xrightarrow{c} c \rightarrow c$

2. $a \rightarrow b \xrightarrow{c} c \rightarrow c$

In example 1. the $K_0$-theory should be $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}$.