a

Try to do some work 15.05

I think I might review what needs to be done before you can finish. You can go back to Gelfand, but the next month should make mush of your brain. So where to start. Maybe review

What was Geigle-Lenzing existence of adjoints about? ~~[illegible]~~ Inverting S isomorphisms. You idea $\mathrm{Ext}^j(T, M)=0$ $j=0,1$ $\iff$ $\mathrm{Hom}(-,M)$ inverts maps with kernel + cokernel isom to T. Let's try to figure out what might have been meant. So consider the effect of h dim ≤ 1. Suppose we ~~[illegible]~~ go back to the ~~reg ex~~ example you know. Coherent sheaves on $\mathbb{P}^1$, $T=\mathcal{O}\oplus\mathcal{O}(-1)$ so that ? $\mathcal{A}=\mathrm{Coh}(\mathbb{P}^1)$. Then ~~I~~ have positive and negative sheaves. So what do you ~~[illegible]~~ know about the sheaf situation? If you take $T=\mathcal{O}\oplus\mathcal{O}(+1)$ then

$$\mathrm{Hom}(T, F) = H^0(F) \oplus H^0(F(-1))$$

$$\mathrm{Ext}^1(T, F) = H^1(F) \oplus H^1(F(-1))$$

Certainly ~~[illegible]~~ $H^0(F)=0 \Rightarrow H^0(F(-1))=0$

and $H^1(F(-1))=0 \Rightarrow H^1(F)=0$.

So what. There are sheaves F with $H^0(F)=H^1(F)=0$.

So what you want is to understand $F=F^+\oplus F^-$

~~[illegible]~~ reg sh closed under quotients extensions

They are like torsion modules, ~~reg~~ regular vbs closed under extns. + subobjects. They are like torsion-free modules.

money

what papers

What does this relate to $\perp$ cats. condition $\mathrm{Hom}(T, F) = 0$ closed under submod + extns. Condition $\mathrm{Ext}^1(T, F) = 0$ leads to a class closed under quotients and extns. assuming T has proj dim ≤ 1. Maybe the real issue is existence of adjoints. So suppose that $\mathcal{S}$ is a full subcategory of $\mathcal{A}$. What does it mean for the inclusion ${}^{\perp}\mathcal{S} \subset \mathcal{A}$ to have the appropriate adjoint? Let's ask for the following: $\forall M$ $\exists$ $M_{\#} \to M$ with kernel and cokernel in $\mathcal{S}$ and $M_{\#} \in {}^{\perp}\mathcal{S}$. At this point things would be much easier if I had a copy of [GL]. So suppose $\mathrm{Hom}(-, S)$ inverts nil isos? Consider $0 \to F^+ \to F \to F^- \to 0$ $\mathrm{Hom}(F^+, G^-) = 0$. ~~Keep up the~~ For coh sheaves on P^1 I don't see any interesting $\perp$ cat. If you take any line bundle $\mathcal{O}(n)$, ~~and ask for~~ you might as well ~~ask~~ take $\mathcal{O}$. Then for $\mathcal{S} = \{\mathcal{O}\}$

$$H^0(F) = H^1(F) = 0 \qquad \Rightarrow \qquad F = \mathcal{O}(-1) \otimes V$$

or

$$\mathrm{Hom}(F, \mathcal{O}) = 0 \qquad F = F_t \oplus F_{\geq \mathcal{O}(1)}$$

$$\mathrm{Ext}^1(F, \mathcal{O}) = 0 \quad \Rightarrow \quad F_t = 0$$

So its similar. But then maybe

$$F \longmapsto \mathcal{O} \otimes H^0(F)$$

is the appropriate adjoint.

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C

Anyway I didn't get much further. So let's think a bit. The question is whether $\mathcal{S}$ is always a Serre subcategory. What is ${}^{\perp}(\mathcal{S}^{\perp})$ all M such that $\mathrm{Ext}^*(M,N)=0$ for all N such that $\mathrm{Ext}^*(S,N)=0$ $S\in\mathcal{S}$. If $\mathcal{S} = {}^{\perp}(\mathcal{S}^{\perp})$? In general $\mathcal{S}^{\perp}$ and ${}^{\perp}\mathcal{S}$ are closed under extensions. $*=0,1$.

$$\mathcal{S}^{\perp} = \{N \mid \mathrm{Ext}^*(S,N)=0 \ \forall S\in\mathcal{S}\}.$$

$$0 \to N' \to N \to N'' \to 0$$

$$0 \to \mathrm{Hom}(S,N') \to \mathrm{Hom}(S,N) \to \mathrm{Hom}_S(N'') \to \mathrm{Ext}^1(S,N')$$
$$\to \mathrm{Ext}^1(S,N) \to \mathrm{Ext}^1(S,N'') \to 0$$

If proj dim $S \le 1$. then get ———

so you see nothing

General case you get $N', N'' \in \mathcal{S} \implies N \in \mathcal{S}$

$N, N'' \in \mathcal{S} \implies N' \in \mathcal{S}$.

Also $\mathcal{S} \cap \mathcal{S}^{\perp} = 0$. The good situation should be that $\mathcal{S} + \mathcal{S}^{\perp} = \mathcal{A}$ in some sense. So all this it seems that $\mathcal{S}$ is Serre. therefore you don't understand the relation with sheaves on P^1.

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So where can you look next? What about the Gelfand stuff? Gelfand situation: You have a quiver and you reverse the arrows going into a sink. The indecomposables don't change. Ex.

$$\begin{tikzcd} \bullet \arrow[dr] & \\ & \bullet \\ \bullet \arrow[ur] & \end{tikzcd}$$

~~Here yo~~ Here you have

$$\begin{tikzcd} M_0 \arrow[dr] & \\ & M_{01} \\ M_1 \arrow[ur] & \end{tikzcd}$$

and you can split off the cokernel of $M_0 \oplus M_1 \to M_{01}$

Indecomposables are

$$\begin{tikzcd} 0 \arrow[dr] & \\ & k \\ 0 \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} k \arrow[dr] & \\ & 0 \\ 0 \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[dr] & \\ & 0 \\ k \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} k \arrow[dr, equal] & \\ & k \\ 0 \arrow[ur] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[dr] & \\ & k \\ k \arrow[ur, equal] & \end{tikzcd} \quad \begin{tikzcd} k \arrow[dr, equal] & \\ & k \\ k \arrow[ur, equal] & \end{tikzcd}$$

I should review all this stuff, all these examples. In general suppose you have a ~~vertex~~ sink x. This example is related to $\bullet \longrightarrow \bullet \longleftrightarrow \bullet$ The basic idea is that you can split off the cokernel of

$$\bigoplus_{y \to x} M_y \longrightarrow M_x$$

Then you are left with a surjection, so M_x is determined by the kernel, call it $K_x \subset \bigoplus_{y\to x} M_y$ ~~and~~ so you replace M by $\{M_y \mid y \neq x\}$ together with K_x at x and the maps $K_x \longrightarrow M_y$ for each $y \to x$. So you have a repn. of the new quiver where arrows going to x are reversed.

Let's see if we can recall how it works. You have an example. Let P_y be the projective module generated by k at y. $P_y(y') = \bigoplus_{y \to y'} k$ This should be paths from y to y'.

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So if x is a sink, then what? e

$a \to b \to e.$

$$k \to 0 \to 0$$
$$k = k \to 0, \quad k = k = k$$
$$0 \to k \to 0$$
$$0 \to k = k \qquad 0 \to 0 \to k$$

Let M be a module, i.e. rep. of quiver. first I want to get the cokernel $M_x / \bigoplus_{y \to x} M_y$. What's important is k_y k at y 0 elsewhere. Then k_y should be a quotient of P_y. In the example

$$a \to y \to x$$

k_x $\quad 0 \to 0 \to k$

P_x $\quad 0 \to k$

$$\begin{tikzcd} \arrow[dr] & \\ & \cdot x \\ \arrow[ur] & \end{tikzcd}$$

$k_x = P_x$ for a sink.

Instead look at
$$\begin{tikzcd} a \arrow[dr] & \\ & c \\ b \arrow[ur] & \end{tikzcd}$$

k_a :
$$\begin{tikzcd} k \arrow[dr] & \\ & 0 \\ 0 \arrow[ur] & \end{tikzcd}$$

P_a :
$$\begin{tikzcd} k \arrow[dr] & \\ & k \\ 0 \arrow[ur] & \end{tikzcd}$$

P_c,
$$\begin{tikzcd} 0 & \\ & k \\ 0 & \end{tikzcd}$$

exact. $\quad 0 \to P_c \to P_a \to k_a \to 0$

$$0 \to \mathrm{Hom}(k_a, M) \to M_a \to M_c \to \mathrm{Ext}^1(k_a, M) \to 0$$

So what happens? Basically you have a source a - it gives rise to

$$0 \longrightarrow \bigoplus_{a \to b} P_b \to P_a \to k_a \to 0$$

Now somehow I need to link this up

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with $\perp$ cats. ~~Th~~ Think of this as torsion f theory. $\mathrm{Hom}(k_a, M)$ – this give the part to split off. ~~The~~ So the Serre subcategory might be M supported at the source a. ~~[illegible]~~ Torsion and torsion free. Closed means what?

$$0 \to \mathrm{Hom}(k_a, M) \longrightarrow M_a \longrightarrow \prod_{a \to b} M_b \to \mathrm{Ext}^1(k_a, M) \to 0$$

$M \in \mathcal{S}^\perp$ means $M_a = \prod_{a\to b} M_b$. So it clear that $\mathcal{S}^\perp \xrightarrow{\sim} \mathcal{A}/\mathcal{S}$ I guess so. The only point here seems to be that $\mathcal{S}^\perp$ is the ~~repres.~~ source modules for the quiver with the ~~sink~~ source a. ~~[illegible]~~ So if true what about ~~[illegible]~~ changing the arrows. Start with M and split off the submodule supported at a. Then you end up with $0 \to M_a \longrightarrow \prod_{a\to b} M_b \longrightarrow \mathrm{Ext}^1(k_a, M) \to 0$.

So explain the relation. ~~so~~ You start with M a module for the quiver in which a is a source ~~[illegible]~~ You get an equivalence of cats between M ~~having~~ ~~[illegible]~~ ~~no~~ torsion free M for this quiver and ~~torsion~~ cotorsion-free N for the other quiver. Actually we are dealing with idempotent ideals in this case. What are the simple objects. ~~Simple [illegible]~~ Is it clear that the vertices correspond to simple ~~[illegible]~~ objects. Suppose given M. ~~[illegible]~~ No $\exists$ Interesting ~~[illegible]~~ simple representations of a tensor algebra. So if you stay away from loops then OKAY.

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This time I know about localizing subcats. How about derived categories? What next? What might happen to the derived categories? Take a source a. Can we find a functorial resolution? Try a sink maybe first. I think I need an examples.

$$\begin{tikzcd} a \arrow[dr] & \\ & c \\ b \arrow[ur] & \end{tikzcd}$$

Basic projectives:

P_a :
$$\begin{tikzcd} k \arrow[dr, equal] & \\ & k \\ 0 \arrow[ur] & \end{tikzcd}$$

P_b :
$$\begin{tikzcd} 0 \arrow[dr] & \\ & k \\ k \arrow[ur, equal] & \end{tikzcd}$$

P_c :
$$\begin{tikzcd} 0 \arrow[dr] & \\ & k \\ 0 \arrow[ur] & \end{tikzcd}$$

Q :
$$\begin{tikzcd} k \arrow[dr, equal] & \\ & k \\ k \arrow[ur, equal] & \end{tikzcd}$$

$$0 \to P_c \to P_a \oplus P_b \to Q \to 0$$

$$0 \to \mathrm{Hom}(Q,M) \to M_a \oplus M_b \to M_c \to \mathrm{Ext}^1(Q,M) \to 0$$

What I want is some module T probably $P_a \oplus P_b \oplus Q$ which will give via $\mathrm{Hom}(T,-)$ M_a, M_b and the submodule $\mathrm{Ker}(M_a \oplus M_b \to M_c)$,

Thus $\mathrm{Hom}(T,M) = M_a \oplus M_b \oplus \mathrm{Hom}(Q,M)$. What is $\mathrm{End}(T)$.

$$\mathrm{Hom}(T,T) = \begin{pmatrix} \mathrm{Hom}(P_a,P_a) & \mathrm{Hom}(P_a,P_b) & \mathrm{Hom}(P_a,Q) \\ \mathrm{Hom}(P_b,P_a) & \mathrm{Hom}(P_b,P_b) & \mathrm{Hom}(P_b,Q) \\ \mathrm{Hom}(Q,P_a) & \mathrm{Hom}(Q,P_b) & \mathrm{Hom}(Q,Q) \end{pmatrix}$$

$$= \begin{pmatrix} k & 0 & k \\ 0 & k & k \\ 0 & 0 & k \end{pmatrix}$$

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In general given a sink x what do h
[illegible] you do?

$$\begin{tikzcd} a \arrow[r, bend left] \arrow[r, bend right] & b \end{tikzcd}$$

$P_b : \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] & k \end{tikzcd}$

$P_a : \begin{tikzcd} k \arrow[r, bend left] \arrow[r, bend right] & k \oplus k \end{tikzcd}$

I want to control the cokernel of $M_a \oplus M_a \longrightarrow M_b$, so I take

$$\begin{tikzcd} 0 \arrow[r] & P_b \arrow[r] & P_a \oplus P_a \arrow[r] & Q \arrow[r] \arrow[d, equal] & 0 \\ & & & k \oplus k \to k & \end{tikzcd}$$

9-18th

Confusion:

$$\begin{tikzcd} & b \\ a \arrow[ur] \arrow[dr] & \\ & c \end{tikzcd}$$

$M_a \longrightarrow M_b \oplus M_c$

$$0 \longleftarrow k_a \longleftarrow P_a \longleftarrow P_b \oplus P_c \longleftarrow 0$$

But if you try to take $k_a \oplus P_b \oplus P_c$, then you get wrong endo ring.

So you need to understand the difference between

$$\begin{tikzcd} & b \\ a \arrow[ur] \arrow[dr] & \\ & c \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & b \arrow[dl] \\ a & \\ & c \arrow[ul] \end{tikzcd}$$

2nd:

$M_a \longleftarrow M_b \oplus M_c$

~~[illegible]~~ I have to understand the difference between [illegible] a source + [illegible] a sink

$$0 \longrightarrow P_a \longrightarrow P_b \oplus P_a \longrightarrow Q \longrightarrow 0$$

$$\left(\begin{tikzcd} & 0 \arrow[dl] \\ k & \\ & 0 \arrow[ul] \end{tikzcd}\right) \qquad \left(\begin{tikzcd} & k \arrow[dl, shift left] \arrow[dl, shift right] \\ k+k & \\ & k \arrow[ul] \end{tikzcd}\right)$$

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source versus sink.

source a, then the M with support a form a Serre subcat $\mathcal{S}$ of all M. $\mathcal{S}$ generated by k_a so $\mathcal{S}^\perp = M$ such that $\text{Ext}^1(k_a, M) = 0$. Here things are localizing - inclusion $i : \mathcal{S} \to \mathcal{A}$ has a right adjoint.

Case of a sink a. ~~Also~~ Again M with support a should form a Serre subcat. In general if I pick any vertex, then those M supported on that vertex form a Serre subcat closed under products. Why: [illegible] condition is $\text{Hom}(P_b, M) = 0$ $\forall\, b \neq a$. and P_b projective. ~~Thus assuming~~

What about $\cdot \rightrightarrows \cdot$? Two Serre subcats $W = 0$ get $\mathcal{O} \otimes V$

~~As well as~~ Other $V = 0$ get negative type

Yesterday I got one idea on the plane - namely to use Serre subcats in conn. with Gelfand business. Given a quiver, let $\mathcal{A}$ = abel. cat of its repns (modules). Then you have these proj. P^a_b for each vertex $\text{Hom}(P^a, M) = M_a$. P^a_b has basis the paths going from a to b. ~~to diff~~ Then $\{M \mid \underbrace{M_a}_{\text{Hom}(P^a, M)} = 0\}$ is a Serre subcat closed under prods so you ~~[illegible]~~ know $\mathcal{S}_a$ is bilocalizing. Functors $M \mapsto M^\#, M_\#$. You want

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to understand $M_\#$, $M^\#$ in the j case of a source and a sink. [redacted]

You believe that $\mathcal{A}/\mathcal{S}_a$ is the cat. of modules on the quiver with the vertex a deleted in general. $\mathcal{A}$ is an abel. cat with a small proj generator $\bigoplus_{\substack{\text{all}\\\text{vertices}}} P_*$. So we have all P_v $v \in Q$. $Q' = Q - \{a\}$. $\text{Hom}(\bigoplus_{v\in Q'} P_v, -)$ kills $\mathcal{S}_a$. ~~In~~ fact this is an exact functor kills the Serre subcat, so $\bigoplus_{v\neq a} P_v$ should be a proj gen. of $\mathcal{A}/\mathcal{S}$

Notice that $\bigoplus_{v\neq a} P_v \in {}^\perp\mathcal{S}$ since it inverts $\mathcal{S}$-isos.

Thus for a quiver Q, $\mathcal{A}(Q)/\mathcal{S}_a \simeq \mathcal{A}(Q-a]$, and we get ~~three~~ subcats realizing this quotient cat. ${}^\perp\mathcal{S}_a$ firm, $\mathcal{S}_a^\perp$ closed, reduced.

reduced means no submodule nor quotient module supported at a. Need examples.

$$x \longrightarrow a \longrightarrow y \qquad \underbrace{M_x \twoheadrightarrow M_a \hookrightarrow M_y}_{\text{reduced.}}$$

If $M_a \longrightarrow M_y$ not inj then

$0 \longrightarrow k \longrightarrow 0$ is a submodule

If $M_x \longrightarrow M_a$ not surj

$$\begin{tikzcd} M_x \arrow[r] & M_a \arrow[d] & \\ 0 \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

is a quotient module.

What might firm mean?? ~~Firm means~~ probably $M_x \xrightarrow{\sim} M_a$

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and closed prob. means $M_a \xrightarrow{\sim} M_y$.

Check this.

$$\begin{tikzcd} M_x \arrow[r] & M_a \arrow[r] & M_y \\ M_x \arrow[u, "\wr"] \arrow[r, shift left] \arrow[r, shift right] & M_x \arrow[u] \arrow[r] & M_y \arrow[u, "\wr"'] \end{tikzcd}$$

arb. maps.

$$\begin{tikzcd} M_x \arrow[r, "\sim"] & M_a \arrow[r] & M_y \\ N_x \arrow[u, "\wr"] \arrow[r] & N_a \arrow[u, twoheadrightarrow] \arrow[r] & N_y \arrow[u, "\wr"] \end{tikzcd}$$

this is an extn with kernel iso ℓ.

So much for this example. Next consider a source

$$\begin{tikzcd} a \arrow[r, bend left] \arrow[r, bend right] & x \end{tikzcd} \qquad M_a \rightrightarrows M_x$$

reduced $\begin{cases} M_a \hookrightarrow M_x \times M_x & \text{otherwise } k \rightrightarrows 0 \text{ is submod} \\ M_a = 0 & \text{otherwise } k \rightrightarrows 0 \text{ is quot. mod.} \end{cases}$

∴ reduced means just that $M_a = 0$.

firm is the same probably

closed should ~~mean~~ prob mean: $M_a \hookrightarrow M_x \times M_x$

check:

$$\begin{tikzcd} M_a \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M_x \arrow[d, equal] \\ M_x \times M_x \arrow[r, shift left] \arrow[r, shift right] & M_x \end{tikzcd}$$

so OKAY.

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This should also work with $$\begin{tikzcd} & x \\ a \arrow[ur] \arrow[dr] & \\ & y \end{tikzcd}$$ d

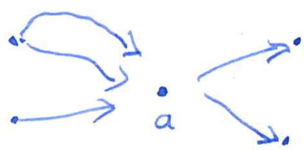
$$\begin{tikzcd} & M_x \\ M_a \arrow[ur] \arrow[dr] & \\ & M_y \end{tikzcd}$$

no quotient mod in S. means $M_a = 0$.

so reduced + firm same & mean $M_a = 0$.

closed:
$$\begin{tikzcd} M_a \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & M_x \times M_y \arrow[d, equal] \\ M_x \times M_y \arrow[r, shift left] \arrow[r, shift right] & M_x \times M_y \end{tikzcd}$$

What will the general picture be? You should have incoming and outgoing vertices.



So what does reduced means a submodule k^a means a non-zero element of M_a killed by outgoing arrows a quotient k^a means a linear fnl on M_a killing incoming arrows. Thus red means M_a gen by incoming arrows and cogen by outgoing arrows.

$$\bigoplus_{x\to a} M_x \twoheadrightarrow M_a^{red} \hookrightarrow \prod_{a\to y} M_y$$

What's ~~the~~ firm mean: probably that $\bigoplus_{x\to a} M_x \xrightarrow{\sim} M_a$ and closed means prob $M_a \xrightarrow{\sim} \prod_{a\to y} M_y$

---

my plans, Wed morning.
key.

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M

It might be pretty messy to remove an internal vertex. What can you do next? ~~Now go back to the tilting idea.~~

Wait: [GL] K-theory argument. What is $\mathcal{S}_a$? $\mathcal{S}_a$ = all module supported at a. There's a problem when there is a map from a to a.

So what happens. You have paths from a to a, but only to arrows from a to a matter. Thus if M has support at a, then only a loop $\circlearrowleft$ will act nontrivially - other paths gives 0. If we avoid loops then the Serre subcategories $\mathcal{S}_a$ are all ~~iso to~~ equiv. to $\mathbb{C}$.

How does this affect the preceding argument, namely the picture of a red. module. Basically any arrow follows into 4 types.

$a \to a$

$a \to b$ where $b \neq a$

$b \to a$ ______

$b \to c$ where $b \neq c$ and $b \neq a$ $c \neq a$

Then it should be clear. It seems OKAY.

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4.84	grapes + milk

Question is whether the condition that there is no arrow $a\circlearrowleft$ is the one Lusztig ~~was~~ imposes on his quivers.

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So there remains the question of the derived category. What exactly do I learn from this stuff? Now I have $\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$. I can ask the h-unital question. There's an idempotent ideal here. $R =$ algebra of the quiver. $\mathrm{End}(\bigoplus_x P^x)$. Remove the vertex a and see what arises. Have $P = \bigoplus_x P^x = P^a \oplus \underbrace{\bigoplus_{x \neq a} P^x}_{P'}$.

So it seems we have a Morita context. What is going on? Anyway think. You have

$$\mathcal{S} \to \mathcal{A} \to \mathcal{A}/\mathcal{S}$$

$$\begin{tikzcd} \mathrm{Mod}(R/I) \arrow[r] & \mathrm{Mod}(R) \arrow[r] & M(R,I) \arrow[d, phantom, "\wr"] \\ & & M(I) \cong M(R'). \end{tikzcd}$$

So what you need to find is an idempotent ideal in R Morita equiv. to R'. Expect maybe a Morita context $\begin{pmatrix} R' & \cdot \\ \cdot & R \end{pmatrix}$.

I need to analyze this situation carefully. Basically you have $R = \mathrm{End}(P)^{op}$ where $P = P^a \oplus P'$. P is the projective generator for $\mathcal{A}$

$P' = Pe$ $\qquad R' = \mathrm{End}(Pe)^{op} = \mathrm{Hom}_R(Pe, Pe)^{op}$
$\qquad = eRe$

Some ideal I. ReR. Can you make sense of h-unital? What I know about this situation is that there should be right + left R' modules — a dual pair generating R.

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So what do I find? ~~Not clear so far~~

You have ~~[illegible]~~ R'. Specify the ~~small~~ proj gen. $\oplus P^x = V$ of a. Split it into $V = V(1-e) \oplus Ve$ where $Ve^\perp = P^a$ $Ve = \bigoplus_{x \neq a} P^x$

$$R^{op} = \mathrm{End}(Ve^\perp \oplus Ve) = \begin{pmatrix} \mathrm{End}(Ve^\perp) & \mathrm{Hom}(Ve, Ve^\perp) \\ \mathrm{Hom}(Ve^\perp, Ve) & \underbrace{\mathrm{End}(Ve)}_{R'^{op}} \end{pmatrix}$$

Can look at $\mathrm{Hom}(P^a, Ve)$ ~~right~~ left $\mathrm{End}(Ve)$ module

$\mathrm{Hom}(Ve, P^a)$ ~~left~~ right $\mathrm{End}(Ve)$ module

What ~~[illegible]~~ should $\mathrm{Hom}(Ve, P^a) = \prod_{x \neq a} \mathrm{Hom}(P^x, P^a)$

$= \prod_{x \neq a} P^a_x$ basis paths from a to x

$\mathrm{Hom}(P^a, Ve) = \bigoplus_{x \neq a} \mathrm{Hom}(P^a, P^x) = \bigoplus_{x \neq a} P^x_a$ paths $x \rightsquigarrow a$

~~[illegible]~~ The pairings don't look too good, which is OK as what we should have is

$$\left(\begin{array}{c|c} \mathrm{End}(V) & \mathrm{Hom}(Ve, V) \\ \hline \mathrm{Hom}(V, Ve) & \mathrm{End}(Ve) \end{array}\right)$$

I don't see anything to learn.

~~[illegible]~~ Can I make any progress? The h-unital question. ~~[illegible]~~ I know that $R' = \mathrm{End}(Ve)^{op}$ is a

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path algebra. proj dim ≤ 1. Does this P
say anything about the R' modules
eR and Re. $R^{op}e = \text{Hom}(V, Ve)$
$eR^{op} = \text{Hom}(V, V)$

Your arrows again.

$V = \bigoplus_x P^x$ Put $R^{op} = \text{End}(V)$ to begin
so R acts on the right of V. Then

$\text{Hom}(Ve, V) = Re$

$\text{Hom}(V, Ve) =$

Set $R = \text{End}(V)^{op}$ so that R acts to the right on V. Then eR gives Ve.

$\text{Hom}(V, Ve) =$

$\text{Hom}(V, Ve)$ is naturally a right $\text{Hom}(V,V)$ module $\text{Hom}(V, Ve) = Re$ $\text{Hom}(Ve, V)$ left $\text{Hom}(V,V)$

$\shortparallel$

eR.

What can you say about these as eRe modules.

$\text{Hom}(Ve^{\perp}, Ve) = e^{\perp}Re$

$\text{Hom}(P^a, \bigoplus_{x \neq a} P^x)$ as a right ~~Hom~~ $\text{End}(Ve)$ module

So here's an interesting question, namely, You have a quiver $Q' = Q - a$ and its path alg R'. Where's the glue tying a to the rest.
Answer: you give two subsets of Q' representing

$x \longrightarrow a \longrightarrow y$ might be empty.

Review: ~~Starting~~ with a quiver we get the abelian categ of its representations. R-modules where $R^{op} = \mathrm{End}(\bigoplus_x V^x)$ $x \in$ quiv. $\mathcal{X}$

We get a Serre subcat $\mathcal{S}$ of modules M with support at the given vertex a, equiv. $\mathrm{Hom}(V^a, M) = M_a = 0$. The quotient cat ~~is~~ $\mathcal{A}/\mathcal{S}$ is the modules for the quiver $\mathcal{X}' = \mathcal{X} - a$. $\mathrm{Mod}(R')$ $R' \overset{op}{\cong} \mathrm{End}(V')$

$V = V^a \oplus V'$. $\exists\, e \in R$ s.t. $V^a = Ve^{\perp}$, $V' = Ve$ and then $R' = eRe$. Now we know $\mathcal{S} = \mathrm{Mod}(R/A) \subset \mathrm{Mod}(R)$ where A generated by things factoring through V^a.

I forgot to say that $\mathcal{S} = \mathrm{Mod}(k)$ provided there are no arrows $a \circlearrowleft$. Then we see that

$A = ReR$ $Re = \mathrm{Hom}$ Wait

~~mod~~ reps. of quiver $=$ R modules.

Get this straight. V ~~the~~ generator of $\mathcal{A}$, ~~then~~ let $R^{op} = \mathrm{End}(V)$, then have adjoint functors

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left] & \mathcal{A} \arrow[l, shift left] \\ N \arrow[r, mapsto] & V \otimes_R N \\ \mathrm{Hom}_{\mathcal{A}}(V, M) & M \arrow[l, mapsto] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left] & \mathrm{Mod}(S) \arrow[l, shift left] \\ N \arrow[r, mapsto] & V \otimes_R N \end{tikzcd}$$

$$\mathrm{Hom}_S(V \otimes_R N, M) = \mathrm{Hom}_R(N, \mathrm{Hom}_S(V, M))$$

So how do I want to think about representations of the quiver? As left modules, with the arrows acting as left multiplication. $V = \bigoplus_x V^x$ should be R or the path algebra when we convert the quiver module to an R-module. i.e.

$$\bigoplus_y V_y = \bigoplus_{y,x} V_y^x = R$$

and this acts on the left of $\bigoplus_x M_x$ for any quiver module. $V_y^x \simeq \mathrm{Hom}(V^y, V^x)$ has basis the paths from x to y.

Now return to the R-module description of quiver modules.

$$V = \bigoplus_x V^x \longmapsto R = \bigoplus_x \underbrace{Re^x}_{\text{columns of } R.}$$

$Re^x = \bigoplus_y V_y^x$. Now it should be possible to handle

Above I succeeded in translating between a quiver rep. and an R-module. $V \cong \bigoplus_x V^x$ is the dist. small proj generator $R = \mathrm{End}(V)^{op}$

$$\begin{tikzcd} \mathrm{Mod}(R) \arrow[r, shift left] & \mathcal{A} \arrow[l, shift left] \end{tikzcd}$$

$$N \longmapsto V \otimes_R N$$

$$\mathrm{Hom}_{\mathcal{A}}(V, M) \longleftarrow\!\shortmid M$$

So the R-module corresp to M is $\bigoplus_x M_x$

Review $\quad \mathrm{Mod}(R) \simeq \mathcal{A}$ $\qquad$ S

$$N \longmapsto V \otimes_R N$$

$$\underbrace{\mathrm{Hom}_{\mathcal{A}}(V, M)}_{\text{right End}(V)\text{-module } \therefore \text{ left } R\text{-mod.}} \longleftarrow M$$

$R = \mathrm{End}(V)^{op}$

$V = \bigoplus_x V^x$

$\mathrm{Hom}_{\mathcal{A}}(V^x, M) = M_x$

$$\mathrm{Hom}_{\mathcal{A}}(V, M) = \bigoplus_x \mathrm{Hom}_{\mathcal{A}}(V^x, M) = \bigoplus_x M_x$$

$$\mathrm{Hom}_{\mathcal{A}}(V, V) = \bigoplus_y V_y = \bigoplus_{y,x} V^x_y$$

$$\underline{\mathrm{Hom}_{\mathcal{A}}(V, V)} = \mathrm{Hom}_{\mathcal{A}}\left(\oplus V^y, \oplus V^x\right) \ ?$$

$\bigoplus_x M_x$ is an R-module

V in $\mathcal{A}$ corresp to R as left R-module

$$\therefore \bigoplus_y V_y = \bigoplus_{x,y} V^x_y$$

---

confusion. Start again $V = \bigoplus_x V^x$ is a small proj. generator of $\mathcal{A}$. $\mathrm{Hom}(V^x, M) = M_x$. M corresp to R-module $\mathrm{Hom}(V, M) = \bigoplus_x \mathrm{Hom}(V^x, M) = \bigoplus_x M_x$

This is a right $\mathrm{End}(V)$ module, hence a left R-module.

$$\begin{tikzcd}
\mathrm{Hom}(V^y, V^x) \otimes \mathrm{Hom}(V^x, M) \arrow[r] & \mathrm{Hom}(V^y, M) \\
\underbrace{V^x_y} \otimes M_x \arrow[r] & M_y
\end{tikzcd}$$

(with $\mathrm{Hom}(V^y, V^x) = V^x_y$ and $\mathrm{Hom}(V^x, M) = M_x$)

has basis consisting of paths $x \rightsquigarrow y$.

Now look at a vertex a which we want to remove

$$V = V^a \oplus \overbrace{\bigoplus_{x \neq a} V^x}^{V'} \qquad V^x = Ve^x = Re^x$$

probably $\mathrm{Hom}(V^x, M) = \mathrm{Hom}_R(Re^x, M) = e^x M = M_x$

Life is hard to do

What is the your aim . I have this idempotent $e = (e^a)^\perp$ in R, hence the usual $\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$. The question is whether ReR is h-unital. I think this means that

$$Re \overset{L}{\otimes}_{eRe} eR \xrightarrow{\sim} ReR$$

So it ends up with examining Re and eR. Now

$$Re = eRe \oplus e^\perp Re$$
$$eR = eRe \oplus eRe^\perp$$

so the issue probably amounts to whether

$$e^\perp Re \overset{L}{\otimes}_{eRe} eRe^\perp \xrightarrow{\sim} e^\perp ReRe^\perp$$

$e^\perp Re = e^a R \sum_{x \neq a} e^x$ has basis of paths from a vertex $x \neq a$ to a

$eRe^\perp = \sum_{x \neq a} e^x Re^a$ paths from a to outside.

It seems that $eRe^\perp$ is a free eRe-module with basis the edges $a \longrightarrow x$ outside. If so the higher Tors vanishes. I know that $Re^\perp$ is a proj (in fact free) R-module, so what about $eR \otimes_R Re^\perp$. You would need eR ~~to~~ projective over eRe.

So ~~this~~ let's check out this pen?

There's a problem remaining from yesterday

Go back over the data from the beginning.

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Sept 4. 0515

Review: You have a quiver, $\mathcal{A}$ is the abelian cat of its reps over k. $\mathcal{A} = Mod(R)$ where R = path algebra of quivers. Specifically you have projectives V^x in $\mathcal{A}$ such that $Hom(V^x, M) = M_x$ and for each arrow $x \to y$ you get $M_x \to M_y$ i.e. $V^y \to V^x$. $V = \bigoplus_x V^x$ is a small proj. generator, so if $R = End(V)^{op}$ then we have $\mathcal{A} \simeq Mod(R)$ given by $M \mapsto Hom(V, M)$, $N \mapsto V \otimes_R N$. Note R in $Mod(R)$ corresp to V. We have projections $e^x \in R$ corresp to proj of V onto its summand V^x. $\therefore V^x \Leftrightarrow Re^x$

Now jump to the problem of $\mathcal{S}$ = Serre subcat of $M \ni M_a = Hom(V^a, M) = 0$, where a is a fixed vertex. $\mathcal{S} \simeq Mod(k)$ if $\exists$ no arrows $a \circlearrowright$. ~~Next~~

$$\begin{tikzcd}
\mathcal{S} \arrow[r] \arrow[d, equal] & \mathcal{A} \arrow[r] \arrow[d, equal] & \mathcal{A}/\mathcal{S} \arrow[d, equal] \\
Mod(k) \arrow[d, equal] & Mod(R) & M(R,A) \arrow[d, "\wr"] \\
R/A & & Mod(eRe)
\end{tikzcd}$$

$e = \sum_{x \neq a} e^x = (e^a)^\perp$

A is the ideal ReR. The problem is where A is h-unital, which I think is equiv. to $Re \overset{\mathbb{L}}{\otimes}_{eRe} eR \xrightarrow{\sim} ReR$ being a qis.

Now $eRe = End(Ve)^{op} = Hom_R(Re, Re)^{op}$ is the path alg for the quiver with a deleted.

$Re = eRe \oplus e^a Re$. Re is projective over R. You need to understand it over eRe. Shift to eR.

$$R = Hom(\oplus V, \oplus V) = \bigoplus_{x,y} e^x R e^y$$

You really need better control over R. Anyway start with $M = \bigoplus M^x$ and for each arrow $x \xrightarrow{u} y$ in the quiver you have $M_x \to M_y$. If $x \xrightarrow{u} y \xrightarrow{v} z$, then in R you have the path (u, v) acting as

$vu : M_x \to M_y \to M_z$. Stupid point you waste much time at. Let's make paths run the way you compose. Thus $\xrightarrow{u} \xrightarrow{v}$ is vu, so that the product of ring elts corresp. to comp.

Look at $R = \bigoplus e^x R e^y$ $\quad e^x R e^y$

$M_x = \operatorname{Hom}_R(Re^x, M) = e^x M$. So $e^x R e^y : e^y M \to e^x M$.

$e^x R e^y$ has basis all paths from y to x

Look at $e^a R e = \bigoplus_{x \neq a} e^a R e^x$. This has basis all paths going from a vertex $x \neq a$ to a. Take one path:

$$x_n \xrightarrow{u_n} y_{n-1} \to \cdot\, y_2 \xrightarrow{u_2} y_1 \xrightarrow{u_1} a \qquad u_1 \cdots u_n$$

examine this as a $(eRe)^{op}$ module. Claim? might be that any such path has a first point $= a$.

Let's write the path

$$y_0 \to y_1 \to \cdots \to y_{n-1} \to y_n = a \qquad \text{length } n.$$

~~We know $y_0 \neq a$~~. There is a least p such that $y_p = a$.

Look just at Re as eRe module. $Re = \bigoplus_{x \neq a} e^y R e^x$

paths starting at $x \neq 0$ going anywhere. Point is that any such path has a first point ~~not~~ ? One possibility. Looks like you ~~made~~ a mistake. You assume that eRe is the path alg of the quiver with a removed. So what seems to be happening? You have a problem. $eRe = \bigoplus_{\substack{x \neq a \\ y \neq a}} e^y R e^x$ involves paths which might contain a. So there's an idempotent idea

W

So I seem to end up with a difficulty.

I have the M context. You were working with

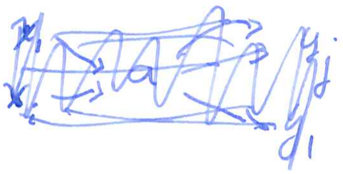
$\begin{pmatrix} R & Re \\ eR & eRe \end{pmatrix}$ gives the ideal ReR in R?

Go back to

$$\begin{tikzcd} \mathcal{S} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{A} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathcal{A}/\mathcal{S}. \arrow[d, phantom, "\shortparallel"] \\ \mathrm{Mod}(k) \arrow[d, phantom, "\shortparallel"] & \mathrm{Mod}(R) & \mathrm{Mod}(eRe) \\ R/ReR & & \end{tikzcd}$$

The mistake seems to be to identify eRe with the path alg of the quiver with the vertex a removed. I think what happens is that you have to add arrows, namely for each pair $x \to a$ and $a \to y$ you add an arrow $x \to y$. Why? Go back to what you found about $\mathcal{S}^\perp$ and ${}^\perp\mathcal{S}$ and the reduced modules.

reduced



$$\begin{tikzcd} x \arrow[rr, phantom, "="] \arrow[dr] & & y \\ & a \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} x \arrow[rr, phantom, "="] \arrow[dr] & & y \\ & a \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} M_x \arrow[r] \arrow[rr, bend left, "1", "\text{ism.}"'] & M_a \arrow[r] & M_x \\ 0 \arrow[u, "0"] \arrow[r] & k \arrow[r] \arrow[u] & 0 \arrow[u, "0"'] \end{tikzcd}$$

shows that $M_a \hookrightarrow M_x$

dually $M_x \twoheadrightarrow M_a$ surjective. ~~As a K-module what happens?~~ ~~So it's not this case~~.

so reduced amounts to

$$\bigoplus_{x \to a} M_x \twoheadrightarrow M_a \hookrightarrow \bigoplus_{a \to y} M_y \qquad \text{reduced}$$

$$\xrightarrow{\sim} \qquad \text{firm}$$

$$\hookrightarrow \qquad \text{closed}$$

What's important here is the map $\bigoplus_{x \to a} M_x \longrightarrow \bigoplus_{a \to y} M_y$

X

for all M. Again $\bigoplus_{x\to a} M_x \xrightarrow{\phi} \bigoplus_{a\to y} M_y$. ~~[illegible]~~ ϕ Map of functors, first rep. $M_x = \mathrm{Hom}(P^x, M)$. Clearly ϕ is a matrix ϕ_{xy}, more precisely $\phi_{x\to a, a\to y}$, ~~and~~ Same as the operators ~~[illegible]~~ assoc. to the path $x\to a\to y$ for each $x\to a$, $a\to y$. So it's clear. Anyway let's check this out. Consider

$$\begin{tikzcd} a \arrow[r, shift right, "u"'] & b \arrow[l, shift right, "v"'] \end{tikzcd}$$

Then ~~the~~ $R =$ generated by e_a, e_b, u, v satisfying $e_a^2 = e_a$, $e_b^2 = e_b$, $e_a v = v e_b = v$, $u e_a = e_b u = u$. The firm category is such that $v : M_b \xrightarrow{\sim} M_a$. Thus you just get

$$\begin{tikzcd} b \arrow[loop left, "uv"] \end{tikzcd}.$$

~~[illegible]~~

So this is pretty interesting because now you can see some measure of the complexity of a quiver. Each time you remove a vertex you acquire new arrows. This gives a quotient abelian, so if you progressively ~~deleting~~ delete all but one vertex you end up with a quotient abelian cat with one vertex. This has the form

, and the ring is a tensor algebra so you get wild ramification type. Need examples. Look at quivers ~~the~~ with two vertices all arrows from one to the other

$$\begin{tikzcd} a \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & b \arrow[l, shift right=5] \arrow[l, shift right=7] \end{tikzcd}$$

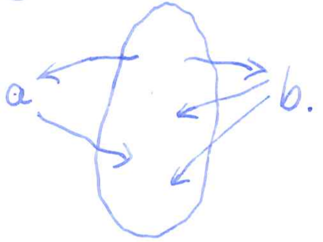
. If we delete a then end up with $2 \times 3 = 6$ loops at b.

~~[illegible]~~

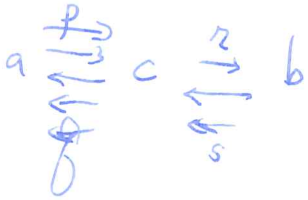
$$\begin{tikzcd} a \arrow[loop left] \arrow[r, "1"] \arrow[r, shift right=3, "2"'] & b \arrow[l, bend right] \end{tikzcd}$$

y

Question: Can you understand what happens if you delete two vertices, first case: not joined by any arrow.



[redacted] Three vertices



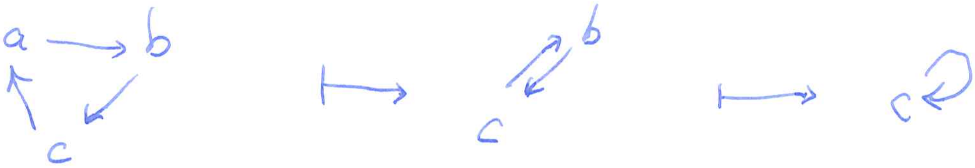
pq + rs arrows from c to itself

If you delete a source or sink you do not add any arrows. Maybe finite repn type means that you can delete without producing loops.

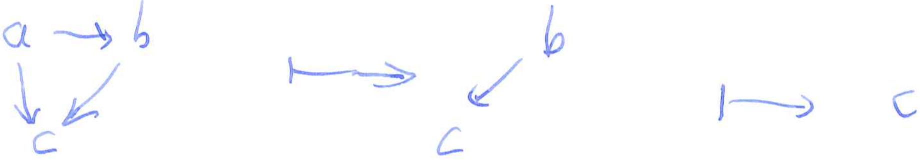
[redacted]

OKAY 1 hours work. Observe two ex:

1.



2.



In example 1. the K_0-theory should be $\mathbb{Z} + \mathbb{Z} + \mathbb{Z}$

K-theory is a bad measure since the tensor alg has the same K-theory as k. The examples are

Kronecker quiver

$$\begin{tikzcd} a \arrow[r, shift left] & b \arrow[l, shift left] \end{tikzcd} \quad \text{and} \quad \underline{\begin{tikzcd} a \arrow[r, shift left] \arrow[r, shift right] & b \end{tikzcd}}$$

$$\begin{tikzcd} M_a \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & M_b \\ k \arrow[u] & 0 \arrow[u] \end{tikzcd}$$

$\therefore M_a \hookrightarrow M_b \times M_b$

$$\begin{tikzcd} M_a \arrow[r, shift left] \arrow[r, shift right] & M_b \arrow[d] \\ 0 \arrow[r] & k \end{tikzcd}$$

compare the representations of the quivers

$$\begin{tikzcd} a \arrow[r, shift right, "g"'] & b \arrow[l, shift right, "f"'] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, shift left] \arrow[r, shift right] & b \end{tikzcd}$$

the second is the Kronecker quiver whose representations you know ~~in terms~~ via coherent sheaves on $\mathbb{P}^1$. The indecomposables are $\xi^+(\mathcal{O}/m_\lambda^{n+1})$, $\lambda \in \mathbb{P}^1$, $n \geq 0$ and $\xi^+(\mathcal{O}(n))$, $n \geqslant 0$ and $\xi^-(\mathcal{O}(-n))$ $-n \leq -1$. Let's describe the situation for the first quiver. Suppose given $V \underset{g}{\overset{f}{\rightleftarrows}} W$. $\lambda \in \mathbb{C}$

$V = \bigoplus V_\lambda \qquad V_\lambda = \{ v \in V \mid \exists n \ (fg - \lambda)^n v = 0 \}$

$W = \bigoplus W_\lambda \qquad W_\lambda = \{ w \mid \exists n \ (gf - \lambda)^n w = 0 \}$

$g(fg)^n = (gf)(gf)(gf)g = (gf)^n g$

$g(fg - \lambda)^n = (gf - \lambda)^n g \implies g(V_\lambda) \subset W_\lambda$

$f(gf - \lambda)^n = (fg - \lambda)^n f \implies f(W_\lambda) \subset V_\lambda$

Now $fg - \lambda$ is locally nilpotent on V_λ, so if $\lambda \neq 0$ fg is invertible on V_λ. sim gf