module theory for idempotent rings
Morita equivalence, theory of
Morita invariance of cyclic homology for 2-unital
rings, also for $K_1$

how do I review this? you would need to
have an outline, introduction in your mind.
first an outline of the main ideas. This is
not important.
you would like an overview of the paper
the first part concerns $M(A)$

motivation:

to extend $R \rightarrow \text{mod}(R)$ for $R$ unital
to nonunital rings $A$.

$\text{mod}(\overline{A}) = \text{cat of } A$-modules is too big

If $A$ is unital with id elt $e$, then $\mathbb{Z}$

$\text{mod}(\overline{A}) = \text{mod}(A) \times \text{mod}(\overline{A}/A)$

$M = eM \oplus (1-e)M$

to cut $\overline{A}$ down

Def: from $A$-module $M$ by $A \otimes_A M \rightarrow M$
nil $A$-mod $M$ by $A^nM = 0$ for some $n$.

Thm: \{form $A$-mods\} $\rightarrow \text{mod}(\overline{A})/\bigoplus_{n>0} \text{mod}(\overline{A}/A^n)$

is fully faithful, is equiv when $A = A^2$.
go over proofs:

$a \sum_{g \in \text{mod}} : A \otimes_A M \rightarrow M$ is nil iso.

If $AX = M$, then $A \otimes_A A \otimes_A M \rightarrow 0$
2. \( M \) flat \( \iff \text{Hom}_A(M, -) \) is a right nil-injective.

\[ 0 \to N' \to N \to N'' \to 0 \]
\[ 0 \to \text{Hom}_A(M, N') \to \text{Hom}_A(M, N) \to \text{Hom}_A(M, N'') \to 0 \]

If \( AN' = 0 \) injective.

\[ A \otimes_A M \xrightarrow{\sim} M \]
\[ A \otimes_A N' \to A \otimes_A N'' \to 0 \]
\[ A \otimes_A N \to A \otimes_A N' \to 0 \]

\[ u : M \to N'' \text{ get } A \otimes_A M \xrightarrow{1 \otimes u} A \otimes_A N'' \xleftarrow{A \otimes f} A \otimes_A N \]

\[ M \to N'' \to N' \to N \]

Key props are:

1. \( M \) flat \( \iff \text{Hom}_A(M, -) \) is a right nil-injective.
2. \( M = A \otimes_A M \) flat \( \iff A \otimes_A A \otimes_A M \) is flat.

These imply equivalence:

\[ \text{flat}(A) \to \text{mod}(A)/\text{mod}(\tilde{A}/A). \]

Can I prove 1 in general.

\[ 0 \to N' \to N \to N'' \to 0 \]
\[ AN' = 0. \]
\[ A \otimes_A N' \to A \otimes_A N \to A \otimes_A N'' \to 0 \]
3. 

\[
\begin{align*}
N' & \longrightarrow N & \longrightarrow N'' & \longrightarrow 0 \\
\downarrow & \uparrow & \uparrow & \downarrow \\
A \otimes_A N' & \longrightarrow A \otimes_A N & \longrightarrow A \otimes_A N'' & \longrightarrow 0 \\
\end{align*}
\]

Assume now nil-iso.

Then 
\[ A \otimes_A M \cong M \]

Then 
\[ A \otimes_A M = \delta(M) \oplus K \]

where \( AK = 0 \).

but 
\[ A \otimes_A M = AM \Rightarrow \text{same for} \]

\[ A \otimes_A M. \]

\[ A \otimes_A M = A \otimes_A AM = A \otimes A M = A (A \otimes_A M). \]

You do this for \( \mathfrak{p} \) an ambient ring \( R \).

Independence of \( R \).

\[ \text{Prin}_R(A, A) \]

\[ \text{Prin}_R(R, A) \]

Why?

\[
\begin{align*}
0 & \longrightarrow A & \longrightarrow R & \longrightarrow R/A & \longrightarrow 0 \\
A \otimes_A M & \longrightarrow R \otimes_A M \\
\end{align*}
\]
4. Recall the independence of $R$ proof and also in the $h$-unital case. So what am I actually doing?

**Result:** $\text{Tor}_n^R(R/A, A) = 0 \iff \text{Tor}_n^\mathbb{A}(\mathbb{A}/A, A) = 0$.

**Why?** $L/L_A = L/L_R \otimes_R L/L_A$

Let $L/L_A = L/L_R \otimes_R L/L_A$

which derivation

Start with $A \otimes_A M \to M$. Then $M$

automatically is an $R$-module such that $AM = M$.

$A \otimes_A M \to A \otimes_R M$

$a \otimes a' m = a \cdot a' \otimes m = a \otimes (a' m)$

easy case is that $A \otimes_A N \to A \otimes_R N$ is $N$

is an $R$-module then $AN = N$, so $N \leq \text{prim}(R, A)$

$\iff N \in \text{prim}(\mathbb{A}, A)$. Conversely is that if

$M \leq \text{prim}(\mathbb{A}, A)$, then $M$ has unique $R$-mod. st.

let $A$-mod etc. + same as above.

Try your $h$-unital case now.

$L/L_A \otimes L/L_A = L/L_A \otimes_R L/L_A$

$\text{Tor}_n^\mathbb{A}(\mathbb{A}/A, A) \iff E^2_{1,0} = \text{Tor}_n^R(\mathbb{A}, \text{Tor}^\mathbb{A}(\mathbb{A}/A, R), A)$

$\iff R/A \otimes_R A = 0$

$\implies K^L/R_A = 0$ for all $A$-mod $R$-modules

next seems inductive.
5. A ideal in $R$, $M$ an $R$-module

$$\text{Tor}^R_p(R/A, M) = 0 \quad \forall p \leq n$$

$$\iff \text{Tor}^A_p(\mathbb{Z}, M) = 0 \quad \forall p \leq n.$$ 

**Lemma:** $\text{Tor}^R_p(R/A, M) = 0 \quad \forall p \leq n$

$$\iff \exists \text{ res. } F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

with $F_n$ $R$-flat and $AF_p = F_p \quad \forall p \leq n.$

**Proof:** ($\Leftarrow$) can use this res. to compute $\text{Tor}^p p \leq n$

($\Rightarrow$) induction on $n$.

If $n = 0,$ $\text{Tor}^R_0(R/A, M) = 0$ means $M = AM$

$\Rightarrow \exists F \to M$ with $F$ finitely flat.

$$0 \to M' \to F \to M \to 0$$

$$\to M'/AM' \to 0$$

Let $P$ be a proj. resol. of $M$. Then $p \leq n.$

$\text{Tor}^R_p(R/A, P) = 0$ for $p \leq n$ means $H_p'(P/\text{AP}) = 0.$

$P/\text{AP}$ projective over $R/A$ acyclic for $p \leq n.$

means $\exists h$ defined on $P/\text{AP}$ such that $[d, h] = 1$ in degrees $\leq n.$

lift $\overline{h}$ to $h$ on $P$

get $[d, h] = 1 - f$ where $f_p : P_p \to \text{AP}_p \quad p \leq n.$

Now take $\text{hom}(P \xrightarrow{F} P \xrightarrow{E} ) = F.$ flat complex

equipped with resolving $M.$
6.

Where are we??

So if $\text{Tor}^R_1(R/A, A) = 0$ p5, then

$\exists \ F_n \to \cdots \to F_0 \to A \to 0 \quad F_i$ form flat,

so it's also true over $\tilde{A}$.

why not review firm cover of an idempotent ring

A idempotent recall a hints. $B \to A$ a covering

when $B = B^2$, w. surj, $B \text{ker}(w) = \text{ker}(w) B = 0$.

$0 \to K \to B \to A \to 0$

B naturally an $A$-bimodule

$B \otimes B = B \otimes_A B \longrightarrow B \otimes_A A$

$\downarrow$

$B \otimes_B B \longrightarrow B \otimes_B A \cong B \otimes_A A \cong A \otimes_A A$

firm cover?

define covering of $B \to A = A^2$ to be $w: B \to A$ such that

$BK = KB = 0$ where $K = \text{ker}(w)$.

14. $A \otimes_A A \longrightarrow A$

$\sum a_i \otimes a_i'' = \sum (a_i \otimes a_i''')$

$q_1, q_2 \otimes \sum q_i q_j'' = q_1, q_2 \otimes \sum q_i q_j''$

etc.
Given $w: B \to A$ define $A \otimes_A B$ and $a_1 \otimes a_2 \mapsto b_1 b_2$ with $w(b_2) = a_2$. A well-defined surjective hom. $a_3 \otimes a_4 \mapsto b_3 b_4$.

$(a_1 \otimes a_2)(a_3 \otimes a_4) = a_1 a_3 \otimes a_2 a_4 \mapsto b_1 b_3 b_2 b_4$.

Composition coverings of $A$ are equiv. to subgroups of $Ker[\mathbb{Z}^2 \otimes_A A \to A^3 = H_2(A)]$. If both coverings $B \to A$, covering why is $M(B) = M(A)$.

More generally, we have equivalence if left modules when $B/I = A$ and $IB = 0$.

$B \otimes B \to N$.

Theory of Morita equivalence.

$w: A \to B$

$M(A) \xleftarrow{w^*} M(B)$

$B^{(2)} \to B \to B$

$M \mapsto B^{(2)} \otimes_A M \mapsto B \otimes_A M$

$A^{(2)} \otimes_A N \leftarrow N$

Adjointness: $\text{Hom}_A (M, N) \xrightarrow{\sim} \text{Hom}_B (B \otimes_A M, N)$

$x: B \otimes A^{(2)} \otimes_A N \to N$

$\beta: M = A^{(2)} \otimes_A M \to A^{(2)} \otimes_A B \otimes_A M$

$\delta: M \to A^{(2)} \otimes_A M$
8. bimodules are $\tilde{B} \otimes_A A^{(2)}$, $A^{(2)} \otimes \tilde{B}$

when $w$ induces a morphism fully faithful.

$A^{(2)} \cong A^{(2)} \otimes \tilde{B} \otimes A^{(2)}$

equivalent to $A \xrightarrow{w} \tilde{B}$ and some sort $A \otimes_A ^{\mathbb{R}}$

want $A \triangleleft B \triangleleft A$ and $K = \text{Ker}(w)$ to get $A \otimes_A ^{\mathbb{R}}$

You want $\alpha$ in $\text{Ker}(w)$.

$\tilde{B} \otimes A^{(2)} \otimes \tilde{B} \cong B^{(2)}$

$\tilde{B} \otimes \tilde{B} \cong \tilde{B}^{2}$

conversely if $\tilde{B} \otimes \tilde{B} = \tilde{B}$ then what method? $\text{YES}$.

$B \otimes A^{(2)} \otimes \tilde{B}$ onto $B$

$B \otimes \tilde{B}$ is a ring.

$(b_1 \otimes a_2 \otimes b_1)(b_3 \otimes \alpha \otimes b_4)$

too tricky

there's another idea.

$A \xrightarrow{w} \tilde{A} \subset B$

$\tilde{A} = A/I$ when $AIA = 0$.

$A \rightarrow A/IA: A/I$

$\uparrow$

kernel $I/IA$ killed by $\tilde{A}$

$AIA \rightarrow A/I$

left ideal $I$ kernel $I/IA$ killed by $A^{op}$

$\tilde{A} \subset \tilde{A}B$

$\uparrow$

$I$ right ideal

$\tilde{B} \subset \tilde{B}$
Left ideal case \( A \subseteq B \), \( BA = A, AB = B \)

\[
\begin{pmatrix}
A & B \\
A & B
\end{pmatrix}
\]

\[
M(A) \cong M(B)
\]

\[
M \mapsto A \otimes_A M = M
\]

\[
B \otimes N \leftarrow N
\]

So when \( \Theta \) is a left ideal in \( B \) gen. \( B \)

This identity between \( \text{frob} \) \( A \) + \( \text{frob} \) \( B \)-module

\[
A \otimes_A M = M
\]

acts on \( B \)-operators.

Can you prove the equivalence in this case?

\[
W(M) = B \otimes_A M \cong A \otimes_A M = M
\]

inc. \( \Theta \) is a \( \text{frob} \)-nil inc.

What I want to do is to identify things properly. Main thm. Given

\[
\begin{array}{ccc}
P \otimes_A & \cong & M(A) \\
\Theta & \cong & P' \otimes_B \\
M(B) & \overset{\text{w}}{\longrightarrow} & M(B')
\end{array}
\]

Left ideal inc. \( A \subseteq B \), \( BA = A, BC = AB \)

In this case \( M(A) = M(B) \) in a def. sense and I need to check this agrees with \( w_i, w^* \).

\[
w(M) = B \otimes_A M \leftarrow A \otimes_A M
\]

No, just that \( w_i \) is an equiv.

General situation here is \( A \overset{f}{\rightarrow} B \) where \( f \) is a \( \text{left} \) \( B \)-module, \( f \) \( B \)-mod maps?
Adjunction of \( w_1, w^* \):
\[
\text{Ham}_A(M, A^{(2)} \otimes A N) \xrightarrow{\sim} \text{Ham}_A(M, N) \\
= \text{Ham}_B(\tilde{B} \otimes A M, N) \xrightarrow{\sim} \text{Ham}_B(\tilde{B} \otimes A M, N)
\]

---

go over main result: two categories \( \tilde{A} \), \( \tilde{B} \), dual pairs. \((P, Q, \phi : Q \otimes P \to A)\)

2. \( B, F : M(A) \to M(B) \)
   
   \[
   (B, F) \mapsto (B', F') \quad w : B \to B' \
   \theta : \omega F \to F'.
   \]

Anyway, we have you need a functor
\[
(P, Q, \phi) \mapsto \begin{cases} 
B = P \otimes_A Q \\ 
F = P \otimes_A -
\end{cases}
\]

\[
(1, \psi) : \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \mapsto \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix} \quad \text{obvious}
\]

but then you have to construct
\[
\delta : B' \otimes_B P \otimes_A M \xrightarrow{\sim} P' \otimes_A M
\]

where does this come from? obvious

Start with \( v : Q \to Q' \)

\( v \) is a \( B^{op} \)-null morph.

\[
v(g) = 0 \Rightarrow \delta \cdot g \cdot \delta = v(g) \cdot \delta \cdot g_1 = 0 \Rightarrow gB = 0
\]

\[
\delta \cdot w(pg) = \delta \cdot \begin{pmatrix} \psi \cdot \delta \cdot g \cdot \delta \cdot g_1 \
\end{pmatrix} = v(g) \cdot \delta \cdot g
\]

\[
\therefore \quad Q \otimes_B P \otimes_A M \xrightarrow{\sim} Q' \otimes_B P \otimes_A M
\]

\[
P' \otimes_A M = P' \otimes Q \otimes P \otimes M \xrightarrow{\sim} P' \otimes Q' \otimes P \otimes M = B' \otimes_B P \otimes_A M
\]
\[ b' \otimes u(p) \otimes m \rightarrow b' \otimes p \otimes m \]

other direction \[ Q \otimes B^{(2)} \otimes N' = Q \otimes B' \]
adjoint. maps. What is adjoint to \[ \Theta : B' \otimes p \otimes m \rightarrow p \otimes N' ? \]
meaning.

\[ \text{Ham}(F_x, Y) = \text{Ham}(x, G'Y) \]
\[ \text{Ham}(F'x, Y) = \text{Ham}(x, G') \]

\[ \Theta : F \rightarrow F' \] has \[ \Theta^t : G' \rightarrow G \]

\[ G'Y \rightarrow GF'G'Y \rightarrow GF'G'Y \rightarrow GY \]

\[ v(\theta b_2) b' u(p) \otimes n' \]
\[ (\theta b_2) b' \otimes n' \]
\[ \Theta b_1 \otimes b_2 \otimes w(b_3) u(p) \otimes n' \]
\[ \Theta b_1 \otimes b_2 \otimes u(p) b' \otimes n' \]

seems like \[ (\theta b_2) b' \otimes n' \rightarrow \Theta b_1 \otimes b_2 \otimes u(p) b' \otimes n' \]

\[ v(\theta b_2) u(p) b' \otimes n' \]
12. \( Q \xrightarrow{r} Q' \) is \( B^p \)-nil iso.

\( p \rightarrow p' \) is \( B \) nil comm.

\[ B^{(2)} \otimes B \rightarrow B \otimes B \]

\[ Q \otimes B^{(2)} \otimes P \]

\[ \phi \otimes M \xrightarrow{\gamma} B^{(2)} \otimes P \otimes M \]

\( B/I = A \)

\[ M = Q \otimes P \otimes M \xrightarrow{\phi} Q \otimes B^{(2)} \otimes P \otimes M \]

\[ Q \otimes B, N' = Q \otimes P \otimes Q \otimes B, N' \xrightarrow{\phi} Q \otimes B^{(2)} \otimes P \otimes \phi \otimes M \]

\[ B \otimes P \rightarrow P \]

\[ B \otimes P \rightarrow B \rightarrow p, p' \otimes p \]

\[ Q \otimes P \rightarrow A \]

\[ (b_1 p_1) b_2 \otimes p_2 = b_1 \otimes p_1 (b_2 p_2) \]
13. \[
\begin{align*}
\text{given } (A & Q) \quad \text{where } QP = A, \ PQ = B \\
M(A) & \approx m(B) \\
M & \approx \rho_{A} M \\
Q \approx \rho_{B} N & \approx N
\end{align*}
\]

\[
\begin{align*}
\beta \rho_{B} & \rightarrow P \\
(\sum b \cdot \rho_{B}) \cdot P = (\sum b \cdot \rho_{B}) \cdot \beta \rho_{B} \\
\beta \rho_{B} \cdot BP = PBP = PA \\
\Rightarrow (P/\beta P) A = 0.
\end{align*}
\]

\[
\begin{align*}
\rho_{B} \rho_{B} \rho_{A} & \rightarrow A \\
\sum (b \cdot \rho_{B}) \cdot P = (\sum b \cdot \rho_{B}) \cdot \rho_{B}
\end{align*}
\]

Module theory so what?? Given an A-faithful map \( Q \otimes P \rightarrow A \) the image is an ideal \( QP \) such that \( AQP = QPA = QP \). Then \( QPQP = Q \).

I have to work hard to get basic result. Should?

What are your main results??

- description of morphs: \( w:A \rightarrow B \) is a morph iff equivalence of categories

- vaguely: description of rings means map to \( A \)

- precisely two cats are equivalent

- first cat consists of \( (B, F: M(A) \approx M(B)) \)

- 2nd \( (P, Q, F) \) dual pairs.
14. Organize this stuff. Suppose you have
\[(A \quad Q) \rightarrow (A \quad Q^{'}) \]
you have to prove that \(B \rightarrow B^{' \prime} \) is a maehan.
Can prove \( w \) is a \( B \rightarrow B^{' \prime} \)-null map.

\[w(b) = 0 \Rightarrow \]

\[0 = v(b) w(b) u(p) = v(b) u(p) = B^{' \prime} \]
\[\Rightarrow p, \bar{q} \bar{b} \bar{p} b_1 = 0 \Rightarrow B^{' \prime} B = 0. \]

\[\frac{u(p) v(b) p^{' \prime} b^{' \prime} u(p)}{eA} \quad \frac{u(q) v(b)}{eA} = u(p, v(b, p^{' \prime} b^{' \prime} u(p)) v(q)} \]
\[\Rightarrow w(p, v(b, p^{' \prime} b^{' \prime} u(p)) v(q)) \]
\[= w(b, v(p, q)) \]
\[\Rightarrow B^{' \prime} w(b) B = w(b) \]

\[B^{' \prime} B = B^{' \prime} \quad \text{(OKAY)} \]

\[w(b) B^{' \prime} w(b) = u(p) v(q) p^{' \prime} q^{' \prime} u(p) v(q) \]
\[B^{' \prime} A = \frac{eA}{A} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \quad \frac{p^{' \prime} q^{' \prime}}{eA} \]

\[B^{' \prime} = B^{' \prime} B^{' \prime} B^{' \prime} = B^{' \prime} P^{' \prime} A Q^{' \prime} B^{' \prime} \]
\[\text{QBP} \]
\[B^{' \prime} = P^{' \prime} Q^{' \prime} P^{' \prime} Q^{' \prime} = P^{' \prime} A Q^{' \prime} = P^{' \prime} v(q) u(p) Q^{' \prime} \]
\[B^{' \prime} = P^{' \prime} Q^{' \prime} P^{' \prime} Q^{' \prime} = P^{' \prime} A Q^{' \prime} = P^{' \prime} A A Q^{' \prime} \]
\[\text{QBP} \]
\[= P^{' \prime} v(q) u(p) v(q) u(p) Q^{' \prime} \]
\[\leq B^{' \prime} w(b) B^{' \prime} \]
15. So what next?! 

\[ w(b) = 0 \quad \Rightarrow \quad \psi(Q) w(b) = \psi(P) \]

\[ \mathcal{B} \mathcal{B} B = \psi(Q) \psi(P) B \psi(Q) \psi(P) \]

\[ = \psi(P) \psi(Q) B \psi(P) \psi(Q) \]

\[ \mathcal{B} \mathcal{B} B = \mathcal{P} \mathcal{Q} b \mathcal{P} \mathcal{Q} \]

\[ w : B \rightarrow B' \quad \Rightarrow \quad w(b) = 0. \]

\[ p_{gb} p_{gb} \quad g_{gb} p_{gb} = \psi(g_{gb}) \psi(p_{gb}) \]

\[ = \psi(g) w(b) \psi(p_{gb}). \]

\[ \mathcal{B} \mathcal{B} B = \mathcal{P} \mathcal{Q} b \mathcal{P} \mathcal{Q} \quad \text{But} \quad Q_b P \subseteq Q_w b \quad \text{for} \quad w \in A \]

\[ \begin{pmatrix} A & Q \\ \psi & B \end{pmatrix} \quad \mathcal{B}' \otimes \mathcal{P} \otimes M \quad \mathcal{P}' \otimes \mathcal{M} \]

\[ \begin{array}{c}
\text{Given} \\
\psi_1 : \mathcal{P}_A \rightarrow \mathcal{P}_A \\
\mu_1 : \mathcal{P} \rightarrow \mathcal{P} \\
\end{array} \]

\[ \begin{array}{c}
\Theta : \mathcal{B}' \otimes \mathcal{P} \otimes M \rightarrow \mathcal{P} \otimes \mathcal{M} \\
\theta_1 : \mathcal{B}' \rightarrow \mathcal{B}' \otimes \mathcal{G} \rightarrow \mathcal{B}' \otimes \mathcal{G} \\
\end{array} \]

\[ \Theta \Theta : \psi_1 F \rightarrow F' \quad \text{given} \]

\[ \text{idea: Take} \\
\text{Hom}_B (\mathcal{B}' \otimes \mathcal{P}, \mathcal{P}') = \text{Hom}_B (\mathcal{P} \otimes \mathcal{B} \mathcal{Q}, \mathcal{P}') \]

\[ = \text{Hom}_B (\mathcal{P}, \mathcal{P}') \]
16. So what? \((A' B')\) describes eq. \(F'\)

\[ w_1 \text{ des. by } (B \quad B \quad B') \]

so \(w'_1\) des. by \((A \quad Q' \quad B')\)

now \(\Theta\) must set up an isom

\[
\begin{pmatrix}
A & Q \oplus B' \\
B' \oplus P & B'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
A & Q' \\
P' & B'
\end{pmatrix}
\]

But?

\[
\begin{pmatrix}
B & B \\
B & B
\end{pmatrix}
\rightarrow
\begin{pmatrix}
B & B \oplus B' \\
B' \oplus B & B'
\end{pmatrix}
\]

so you have a canonical

Suppose you have \(\Theta: w'_1 \rightarrow F'\), get

\[ w'_1 \rightarrow F'G \]

\[ \Theta \circ \Theta \circ \Theta \]

So you're in situation of an isom of Ments

\[
\begin{pmatrix}
B & B \oplus B' \\
B' \oplus B & B'
\end{pmatrix}
\rightarrow
\begin{pmatrix}
B & P \oplus Q' \\
P \oplus Q & B'
\end{pmatrix}
\]

now you have an obvious hom. of \((B \quad B)\) into former. get

\[
\begin{pmatrix}
B & B \\
B & B
\end{pmatrix}
\rightarrow
\begin{pmatrix}
B & P \oplus Q' \\
P \oplus Q & B'
\end{pmatrix}
\]

\[ B' \oplus B \rightarrow P \oplus Q \]

yields \[ B' \oplus P \rightarrow P' \]
17. Alright, what happens? 
You seem to have a map of pairs
\((B, B', P) \rightarrow (P'_A, P'_B, Q)\)
but now translate back to A
\((B \otimes P, Q \otimes B) \rightarrow (P'_A Q \otimes B, Q \otimes P \otimes Q)\)
\((P', Q')\)

Thus, I am often is Example of cats. So you have

\[ C = (A, B) \rightarrow m(A) \xleftarrow{F} m(B) \]

\[ C' = (A, B') \rightarrow m(A) \xrightarrow{F'} m(B') \]

The only thing I should check is that, in the original case, there is no simple argument that constructs \(\theta: u \cdot F \sim F'\). Thus if you have \(u: P \rightarrow P'\), \(v: Q \rightarrow Q'\), how do you get \(B' \otimes P \rightarrow P'\)? Answer:

\[ p \otimes Q \otimes P \xrightarrow{Q \otimes P \otimes P} \otimes P' \]
\[ \iota' \otimes P \rightarrow \otimes A \]
\[ Q' \leftarrow Q \]

\(P \rightarrow P'\) is a \(B\)-nil ron. 
\[ \Rightarrow Q \otimes B P \rightarrow Q \otimes P' \]
18. \( p \rightarrow p' \) \( B \) nil-coin

\[ \Rightarrow p \rightarrow B \otimes_B p' \Rightarrow B \otimes_B p \rightarrow B \otimes_B B \otimes_B p' \rightarrow p' \]

\[ Q \rightarrow Q' \] \( B \) nil-coin.

\[ Q \otimes_B B \rightarrow Q' \otimes_B B \]

\[ p \rightarrow p' \] \( B \) nil-coin

\[ \Rightarrow B \otimes_B p \rightarrow B \otimes_B p' \]

\[ \Rightarrow B' \otimes_B (\otimes_B p) \rightarrow B' \otimes_B (\otimes_B p') \]

\[ \Rightarrow B' \otimes_B B \otimes_B p_M \rightarrow B' \otimes_B B \otimes_B p' \]

\[ B \otimes_B p_M \rightarrow B' \otimes_B B \otimes_B p' \]

\[ p' \otimes_A M \]

\[ \text{Assume} \ Q \rightarrow Q' \] \( B' \) nil-coin

\[ \Rightarrow q \otimes_B (\otimes_B p) \rightarrow q' \otimes_B (\otimes_B p') \]

\[ \Rightarrow q \otimes_B (\otimes_B p) \rightarrow q' \otimes_B (\otimes_B p') \]

\[ \Rightarrow q \otimes_B (\otimes_B p) \rightarrow q' \otimes_B (\otimes_B p') \]

\[ N' \subset M(B) \]

\[ q \otimes_B (\otimes_B p) \rightarrow q' \otimes_B (\otimes_B p') \]

\[ Q \otimes_B (\otimes_B p) \rightarrow Q' \otimes_B (\otimes_B p') \]

\[ M = Q \otimes_B p \otimes_A M \rightarrow Q' \otimes_B p \otimes_A M \]

\[ p' \otimes_A M \rightarrow p' \otimes_B (\otimes_B p) \]

\[ \text{Several ways to be} \]

\[ p \rightarrow p' \] \( B \) nil-coin.

\[ p_B g \rightarrow p' \]

\[ \phi(u(p)) = p_B g \rightarrow p' \]

\[ \phi(p) = p_B g \rightarrow p' \]

\[ \Rightarrow u(p) \rightarrow u(\phi(p)) \]

\[ \Rightarrow u(p) \rightarrow u(\phi(p)) \]

\[ \Rightarrow u(p) \rightarrow u(\phi(p)) \]

\[ p_B g \rightarrow p' \]

\[ \phi(u(p)) \]

\[ \phi(p) = p_B g \rightarrow p' \]
19. \[ B^a \otimes B \rightarrow B^a \otimes B \]

\[ B' \otimes B \otimes P \otimes M \rightarrow B' \otimes B \otimes P \otimes M \]

If an element \( s \) is a section, then \( s \) and \( w \cdot w' = 1 \).

\[ b \otimes b_1 \otimes p \otimes m \rightarrow b' \otimes v(b, b_1) \otimes v(p, m) \]

\[ b' \otimes p \otimes m \rightarrow b' \otimes v(p) \otimes m. \]

This is different from my argument:

\[ Q \rightarrow Q' \]

\[ B^a \text{-nil series.} \]

\[ P' \otimes Q \otimes P \otimes M \rightarrow P' \otimes Q' \otimes P \otimes M \]

\[ P' \otimes Q \otimes P \otimes M \rightarrow P' \otimes v(b) \otimes p \otimes m \]

other isomorphism between \( Q \otimes B, N' \) and \( Q \otimes B', N' \).
20. My old version uses $\Theta_{B}^{A} B \otimes B \otimes A \Rightarrow M$ for $M$ forms.

You construct steps: Start with $C \Rightarrow C'$

get $\Theta_{B}^{A} B \otimes A \Rightarrow P \otimes A M$ $w_{F} \Rightarrow F'$

and $\Theta_{B}^{A} B \otimes B \Rightarrow Q \otimes B N'$ $G_{w} \Rightarrow G'$

compatible with the pairing:

$w_{F} G w_{*} \Rightarrow 1$ $G w_{*} w_{F} \Rightarrow 1$

$F' G' \Rightarrow 1$ $G F \Rightarrow 1$

$B \otimes P \otimes Q \otimes B'$

$A B B$ $\Rightarrow p g b h b_{2} \Rightarrow p g b h b_{2} n'$

$\Theta_{B}^{A} B \Rightarrow \Theta_{B}^{A} B$ $b_{2} \Rightarrow p g b h b_{2} \Rightarrow \Theta_{B}^{A} B$ $b_{2} \Rightarrow p g b h b_{2} \Rightarrow \Theta_{B}^{A} B$

$P \otimes Q \otimes B' N'$ $b_{2} \Rightarrow p g b h b_{2} \Rightarrow \Theta_{B}^{A} B$

$A B B$ $\Rightarrow \Theta_{B}^{A} B$ $b_{2} \Rightarrow p g b h b_{2} \Rightarrow \Theta_{B}^{A} B$

$N'$ $b_{2} \Rightarrow p g b h b_{2} \Rightarrow \Theta_{B}^{A} B$

$Q' \otimes B' P \otimes A M$ $v(g b_{2}) \Rightarrow w(b_{2}) \Rightarrow \otimes$ $v(g b_{2}) \Rightarrow w(b_{2}) \Rightarrow \otimes$

next page
21. \( Q \otimes B^{(2)} \otimes B' \otimes P \otimes M \)

\[ s \uparrow \]

\[ Q \otimes B \otimes P \otimes A \otimes M \]

\[ s \downarrow \]

\[ M \]

\[ \otimes b_1 b_2 b_3 p \otimes m \]

\[ (b_1 b_2 b_3 p) \otimes m \]

---

**Formula for**

\[ M \xrightarrow{A^{(2)} \otimes B \otimes A} M \]

\[ q_1 \otimes q_{23} \]

**YES!**

---

**But you must inverse. The point is that**

\[ w(A)Bw(A) = w(A) \]

**so that**

\[ q_1 \otimes q_{23} \]

---

**It doesn’t change.** The only point is that one can calculate the transpose of \( \Theta \) and find it is \( \Theta^{-1} \).

\[ B : \tilde{F} \rightarrow F' \]

\[ \Theta : B' \otimes B \otimes P \otimes A \rightarrow P' \otimes P \]

\[ b' \otimes p \otimes m \rightarrow b' \otimes (p) \otimes m \]

\[ Q \otimes B^{(2)} \otimes B' \otimes P \otimes Q' \otimes N' \]

\[ Q \otimes B^{(2)} \otimes B' \otimes P \otimes Q' \otimes N' \]

\[ Q \otimes B^{(2)} \otimes P' \otimes Q' \otimes N' \]

\[ Q \otimes B^{(2)} \otimes P' \otimes Q' \otimes N' \]

\[ Q \otimes B^{(2)} \otimes P' \otimes Q' \otimes N' \]

\[ Q \otimes B^{(2)} \otimes P' \otimes Q' \otimes N' \]
22.

\[ Q \otimes B^{(2)} \otimes B \]

\[ g \otimes b_1 \otimes b_2 \otimes u(p) \otimes m' \]

So \( \theta : g b_1 b_2 p g' \otimes m' \rightarrow g b_1 b_2 \otimes u(p) g' \otimes m' \).

Similarly for \( \hat{\theta} : Q \otimes B^{(2)} \otimes B' \rightarrow Q' \otimes N' \)

\[ Gw \rightarrow G' \]

\[ F' \rightarrow F' Gw' \otimes F' \rightarrow F' G' w' F \rightarrow w' F. \]

\[ p' \otimes A M \]

\[ p' \otimes q b_1 b_2 b_3 p m \]

\[ p' \otimes g b_1 b_2 \otimes w(b_2) \otimes p \otimes m \]

\[ p' \otimes v(q b_1 b_2) \otimes w(b_2) \otimes p \otimes m \]

\[ p' \otimes v(q b_1 b_2) \otimes u(p) \otimes m \]

\[ p' \otimes q p m \rightarrow p' v(q) \otimes p \otimes m \]

This should take care of the functor from \( C \) to pairs \((B, F)\).
23. Now the converses. Suppose given $B$. Have you used surjective pairings:

where $B$ isom. here used $B = PQ$.

$$p : P \rightarrow P' \quad \text{B-nil isom.}$$

$$p, B(p') = P(v(B)p')$$

$$B^A \otimes P \otimes M \rightarrow B^A \otimes P' \otimes M$$

$$Q \rightarrow Q' \quad \text{B-nil iso.}$$

$$p' \otimes B \otimes P \otimes M \rightarrow p' \otimes B \otimes P \otimes M$$

Here have used $B' = P'Q'$.

$$w : B \rightarrow B' \quad \text{mrg home.}$$

$$w(b) \quad p_1 \otimes b \otimes p_2$$

$$w(b) = v(b, 6) \mu(b_2) = v'(g_1) w(6) \mu(b_2) = 0$$

$$\mu(b_1) v(b_2) \mu(b_2) \mu(b_2) = \mu(p_1 b_1 b_2) v(b_2) = w(p_1 b_1 b_2).$$

$$B' = P'Q' = P'Q'P'Q' = P'AAQ'$$

$$B' = P'Q'P'Q'P'Q' = P'AAQ'$$

$$= P' v(B) \mu(B) v(Q) \mu(P) Q'$$

$$w(b)$$
24. category. \( B, F : m(A) \to m(B) \)

\[
\text{Hom}((B, F), (B', F')) = \left\{ (\omega, \theta) \mid \omega : B \to B', \text{ homom.} \right\}
\]

\( B \xrightarrow{w_1} B' \xrightarrow{w_2} B' \)

\( \Theta_1 : w_1 F \to F' \quad \Theta_2 : w_2 F \to F'' \)

\[ w_2, F \]

\[ B \xrightarrow{w} B' \xrightarrow{w'} B'' \]

\( \Theta : w_1 F \sim F' \quad \Theta' : w_1' F \sim F'' \)

\[ w_1, w_1 F \xrightarrow{w_1(\theta)} w_1' F \xrightarrow{\theta'} F'' \]

\[ s \uparrow \]

\( (w, w)_! F \)

\[ w_1'' w_1' w_1 F \xrightarrow{w_1''(\theta)} w_1'' w_1' F' \xrightarrow{w_1''(\theta')} w_1'' F'' \xrightarrow{\theta''} F'' \]

\[ s \uparrow \]

\( (w'' w', w')_! F \xrightarrow{(w'' w', \theta)} (w'' w', F') \]

\[ s \uparrow \]

\( w'' w' \text{ tag with this is} \quad (w'' \theta)(w', \theta') \)

\( (\theta', w'')(\theta, w) = (\theta' \theta) \theta_! (w, w) \)

\( (\theta', w') (\theta, w) \)
So the only question is

\[ (w''w')w \rightarrow (w''w)w' \]

and this should be the usual cocycle condition.

Cancel: Consider B. Things + have a M \rightarrow B. Fibres are M(b).

\[ A \xrightarrow{w} B \xrightarrow{\text{Hom}_B (M_j, N)} \]

\[ \text{Hom}_A (M, A^n(\theta) \otimes A) \xrightarrow{w \otimes A} \]

\[ \text{Hom}_B (B \otimes M_j, N) \]

So what? Assoc. \( (\theta, w') (\theta, w) = (\theta, w' \circ (\theta, w) \circ (\theta, w)) \)

\( = (\theta, w_1 \circ (\theta, w') \circ (\theta, w_2) \circ (\theta, w) \circ (\theta, w) \circ (\theta, w)) \)

\( = (\theta, w) \circ (\theta, w') \circ (\theta, w_1) \circ (\theta, w_2) \circ (\theta, w) \circ (\theta, w) \circ (\theta, w) \)

Try to go back over category stuff. Structure on the cat of B, F.

Question. Consider the cat of Abelian groups + morphisms. Take a component. What is the homotopy type??

2-groupoid assoc. to A, B Equiv. (M(A), M(B)).

get fibre cat in groupoids. To each A you have M(A).

First the base cat = component of Abelian groups + morphisms.

fibre cat fibre M(B) over B.

What does this "fibre" bundle tell you about B? Actually it's a fibration. Fix a fibre say A look at self-equiv of M(A). Anyways what's this?

Pair B, F: M(A) |-> M(B) equiv. pairs B, P where P is a firm (B, A)-bimodule which is invertible - not a cat?

w: B -> B(\omega) pairs B, F
maps (B, F) -> (B', F') -> (B'', F'')

(\omega', \omega)(\omega, \omega) = (\omega', \omega') (\omega, \omega)
27. 

\[(B, P) \xrightarrow{(\theta, w)} (B', P') \xrightarrow{(\theta', w')} (B'', P'')\]

\[\theta: B' \otimes_b P \rightarrow P' \quad \theta': B'' \otimes_{b'} P' \rightarrow P''\]

**Given**

\[Q = A \otimes_A \text{Ham}_B (P, B) \quad \text{where } P \text{ invertible}\]

\[
\text{Ham}_B (P \otimes_A M, N) = \text{Ham}_A (M, \text{Ham}_B (P, N))
\]

\[
= \text{Ham}_A (M, B \otimes_A A \otimes_A \text{Ham}_B (P, N))
\]

\[Q \otimes P = \]

**Start with** \[\theta: B' \otimes_b P \rightarrow P'\]

\[(\theta')^{-1}: G \rightarrow G'\] Another way: Uniqueness of \(g\)-inv. says \(J'\) \[\delta: Q \otimes B' \rightarrow B' \otimes Q\] compatible with pairings i.e.

\[(Q \otimes B') \otimes (B' \otimes P) \sim Q \otimes B P = A\]

\[Q' \otimes B' P \sim A\]


$$(B^c_B B^c_B) \otimes (Q \otimes B^c_B) \Rightarrow B^c_B \otimes B^c_B \Rightarrow B^c_B$$

Define $S_{B^c_B B^c_B}$. $\Theta : w : F \Rightarrow F'$, $\Theta_t : G \Rightarrow G^w$

$$Q_{B^c_B B^c_B} \otimes N'$$

$$s$$

$$Q_{B^c_B B^c_B} \otimes B^c_B \otimes P \otimes Q_{B^c_B} \otimes N'$$

$$s$$

$$Q_{B^c_B B^c_B} \otimes B^c_B \otimes P' \otimes Q'_{B^c_B} \otimes N'$$

$$s$$

$$Q_{B^c_B B^c_B} \otimes B^c_B \otimes N'$$

$$(g b p) B^c_B$$

$$8 p B^c_B$$

$$s$$

$$Q_{B^c_B B^c_B} \otimes B^c_B \otimes P \otimes Q'$$

$$s$$

$$Q_{B^c_B B^c_B} \otimes B'$$

$$8 \otimes (bp) B^c_B$$

$$8 p B^c_B$$

$$s$$

$$Q_{B^c_B B^c_B} \otimes B^c_B$$

$$8 \otimes (bp) B^c_B$$

$$8 p B^c_B$$
29. This is okay.

What should be the conceptual proof? You have

$$F \not\triangleright F'$$

What should be the conceptual proof? You have

$$\Theta : \psi \cdot F \rightarrow F'$$

eyou compute the $M$ contents belonging to $\psi \cdot F$

$$C_F = \begin{pmatrix} A & \Theta \circ B' \\ B \circ P & B' \end{pmatrix}$$

pairings

$$(\Theta \circ B') \circ (B \circ P) = \Theta \circ P \Rightarrow A$$

$$(B \circ P) \circ (\Theta \circ B') = B' \circ B \circ B' = B'.$$

Get an isom then

$$\begin{pmatrix} A & \Theta \circ B' \\ B' \circ P & B' \end{pmatrix} \rightarrow \begin{pmatrix} A & Q' \\ P' & B' \end{pmatrix}$$

So what else. Want yelch.
30. \( \Theta : w_1 F \rightarrow F' \)
\[ \tilde{u} : B' \otimes_B P \rightarrow P' \]
\[ \tilde{u} : b' \otimes p \rightarrow b' \circ (p) \]
\[ P = B \otimes_B P \xrightarrow{\text{well}} B \otimes_B P \xrightarrow{\tilde{u}} P' \]
\[ \tilde{u} : P \rightarrow P' \]
\[ u : P \rightarrow P' \]
\[ n(b_p) = \tilde{u}(b \otimes p) \]

\[ G' \]
\[ G' \]
\[ S' \]
\[ Gw \ast w_1 F G' \]
\[ Gw \ast F' G' \]
\[ Gw \ast \]

So you get \( Q' \rightarrow Q \otimes B' \)
\[ (g \otimes p) g' \rightarrow g \otimes u(p) g' \]
Inverse has form \( g \otimes b' \rightarrow v(g) b' \). Conclude then
\[ v(g) u(p) g' = (g \otimes p) g' \]
\[ v(g) u(p) b' = \text{G} b' \]
\[ v(g) u(p) = g \text{G} \]

\[ b' = w^{-1}(b) \]
\[ u(p) v(g) b' = w(p \circ g) b' \]
\[ u(p) v(g) b' = w(p \circ g) b' \]
\[ u(p) v(g) b' = w(p \circ g) b' \]
\[ u(p) v(g) b' = w(p \circ g) b' \]
and a morphism \( \sigma : B \to B' \) of rings. Then get \( (Q \otimes B', B' \otimes B P) \) with

\[
Q \otimes B' \otimes B' \otimes B P = Q \otimes B' \otimes B' \otimes B \otimes B P \]

\[
\tilde{\sigma} : B' \otimes P \to P' \quad \text{and} \quad \tilde{\sigma}^* \to \tilde{\sigma}^* \\
G' \to G' \otimes B' \\
Q' \to Q \otimes B' \\
g \circ h \to g \circ u(p)h' \\
g \circ b_1 \otimes b_2 \circ p \\
B \otimes B' \otimes B' \otimes B
\]

Direct fiber version: \( P \to P' \to Q \to Q' \)

\( v(g)u(p) = g \cdot p \). Then construct the isomorphism

\[
B' \otimes P \to P' \\
Q \otimes B' \to Q'
\]

Proof: \( u : P \to P' \) is B-nil iso means that

\[
\Rightarrow P = B \otimes B P \to B \otimes B P' \\
\Rightarrow B' \otimes B P \to P'
\]

\( M(B) = M(B') \)

By equivalence.
Similarly:
\[ Q \rightarrow Q' \]

\[ \Rightarrow Q = Q \otimes_B B \rightarrow Q \otimes_B B' \]

\[ \rightarrow Q \otimes_B B' \cong Q \otimes_B B \otimes_B B' = Q' \]

The final step can either be done by:
\[ \tilde{u} : B \otimes_B p \rightarrow p' \]
\[ \tilde{u}(b \otimes p) = b' u(p) \]
\[ u(bp) = \tilde{u}(w(b) \otimes p) \]

\[ \tilde{v} : Q \otimes_B B' \rightarrow Q' \]

satisfying:
\[ (B' \otimes p) \otimes_A (Q \otimes_B B') \rightarrow B' \otimes_B B' \otimes_B B' \rightarrow B' \]

\[ b'_1 u(p) \cdot v(b) \cdot b'_2 = b'_1 w(p) \cdot b'_2 \]

\[ b'_1 = w(b'_1) \]

\[ (Q \otimes_B B') \otimes_B (B' \otimes_B p) \rightarrow Q \otimes_B B' \]

\[ \downarrow \tilde{u} \otimes \tilde{v} \]

\[ Q' \otimes_B p' \rightarrow A \]

\[ v(b_1) \cdot w(b_2) \cdot p = g_b \cdot b_2 \cdot p \]

\[ v(b_1) \cdot u(b_2) = g'_b \cdot b_2 \cdot p \]
\[
\text{Suppose } C = (A, Q) \text{, then.}
\]
\[
\begin{pmatrix} A \\ P \end{pmatrix} \Theta_A (A, Q) = \begin{pmatrix} A \Theta_A A \\ P \Theta_A A \end{pmatrix}
\]
should be strictly firm.

\[
\Theta_A A \Theta_A Q \rightarrow P \Theta_A A \Theta_A Q
\]

\[
Q = A Q
\]

\[
0 \rightarrow K \rightarrow A \Theta_A Q \rightarrow Q \rightarrow 0
\]

\[
P \Theta_A K \rightarrow P \Theta_A A \Theta_A Q \rightarrow P \Theta_A \rightarrow 0
\]

\[
P \Theta_A K = 0.
\]
Useful result seems to be that to convert a spinor \((A\otimes Q)\) to a form \(\otimes\) it suffices to use tensor rep.

\[
\begin{align*}
A' &= A\otimes_A A = Q\otimes_B B \\
P' &= P\otimes_A A \\
Q' &= A\otimes_A Q = B\otimes_B B
\end{align*}
\]
Given \( (\begin{array}{cc} A & AB \\ BA & B \end{array}) \) with \( A = A^2 = QP \) and \( B = B^2 = PQ \):

\[ A^4 = AB^2 A = ABBA \leq A \]

\[ A^2 = ABA \leq A \]

\[ A^4 = AB^2 A = ABBA \leq A \]

\[ A = A^2 = QP \]

\[ B = B^2 = PQ \]

\[ w(b) = 0 \]

\[ g_1 B \cdot p_2 = w(g_1) w(b) u(p_2) = v(g_1) w(b) u(p_2) = 0 \]

Okay, getting clearer.

\[ w \cdot \text{morphism tells us that} \quad m(B) \xrightarrow{w_k} m(b) \]

Composite again:

\[ m(A) \rightarrow m(B) \rightarrow m(b') \]

described by:

\[
\begin{pmatrix}
A & B \\
B' & B'
\end{pmatrix}
\]

Proofs:

\[
\begin{pmatrix}
B \otimes P \\
B
\end{pmatrix} = \begin{pmatrix}
\omega \otimes B \\
B
\end{pmatrix} \rightarrow \begin{pmatrix}
B' \\
B'
\end{pmatrix}
\]

\[ b_1 \otimes p \otimes g \otimes b_2 \rightarrow b_1 \omega(p_{b_1}) b_2 \]

\[
\begin{pmatrix}
\omega \otimes B \\
B
\end{pmatrix} \begin{pmatrix}
\omega(B) \\
B
\end{pmatrix} \rightarrow A
\]

\[ \omega(g_{b_1}) \otimes \omega(b_2) \otimes p \rightarrow g_{b_1 b_2} p \]
87. Given \((P, Q)\) and \(w : B \rightarrow B'\), you get a dual pair \(B' \otimes_B P, Q \otimes_B B'\) with pairing:

\[
\left( Q \otimes_B B' \right) \otimes_B \left( B' \otimes_B P \right) \twoheadrightarrow 0
\]

\[Q \otimes_B B' \otimes_B P \leftarrow Q \otimes_B B' \otimes_B P = A\]

You can consider \((P, Q)\) as a dual pair over \(B\).

\[\text{Have } P \otimes Q \text{ over } B \quad \text{YES}\]

Monotone equiv. etc.

The idea is to take \(P \in M(B), \quad Q \in M(B')\) together with \(P \otimes Q \rightarrow B\).

Interesting question: You know that \(P \otimes_A -\) has right adjoint \(N \mapsto \text{Ham}_{B}^{(2)}(P, N)\).

\[\text{Ham}_{B}(P \otimes_A M, N) = \text{Ham}_{A}(M, \text{Ham}_{B}^{(2)}(P, N))\]

\[\quad = \text{Hom}_{A}(M, A \otimes_{A} \text{Ham}_{B}(P, N))\]

What does it mean that this functor is?

What would you like to know? Basic idea is to consider \(N \mapsto \text{Ham}_{B}(P, N)\). This is a nasty functor of \(N\) but the image of \(\mathcal{M}_{B}\) by \(A \otimes_{A}\) might be nicer. This should relate to Roster theorem proof.
Anyway, \((B, B) \to (B', B')\) 

Still after the logic of cats.

Given \(B, F : m(A) \to m(B)\)

Complete \(F\) to \(F_G, \varepsilon : F_G \to 1, \eta : GF \to 1\)

What we do have is an obvious functor from

\[ \text{Idea: Stick to Meets with A fixed.} \]

Consider the next result!!

\[
\begin{array}{c}
B, F : m(A) \to m(B) \\
M(A) \xrightarrow{F} M(B) \xrightarrow{w} M(B')
\end{array}
\]

\(B, F\) have filtered + cofiltered cat. over prim rings + morphisms.

\(B, F\) a map of \(w : B \to B'\)

is an isom. \(\theta : w^* F \to F'\)

\(\text{i.e. } \theta : \text{VA}\)

so we have over the cat of idempotent rings \(B\)

this cof cat of prim modules.

\[
\text{Hom}(N, N')_w = \text{Hom}_B(N, N') = \text{Hom}_B((B \otimes B N, N'))
\]

\[
\text{Hom}_B((B \otimes B N, N')) = \text{Hom}_B(N, B \otimes B N')
\]

\(\text{a map } w : N \to N' \text{ same as a } B \text{-mod.}\)
\( \theta : w_1 N \to N' \quad \theta' : w_1 N' \)

\[
\begin{align*}
M & \xrightarrow{f} m' \xrightarrow{g} m'' \\
B & \xrightarrow{f} B' \xrightarrow{g} B''
\end{align*}
\]

\[
(M, B) \xrightarrow{\theta, \xi} (M', B') \xrightarrow{\xi, \eta} (M'', B)
\]

\[
(g \xi)(f \theta) = \overline{g f} \\
(\theta, f)(\xi, g) = (\theta f(\xi), f g) \\
(\xi, g)(\theta, f) = (\xi g f(\theta) \circ g f, g f)
\]

\[
(g f)! \eta \xrightarrow{c_{g f}} g f, \epsilon M \xrightarrow{g(\theta)} g M' \xrightarrow{\xi} M''
\]

\[
M \xrightarrow{f_1 M} M' \\
M \xrightarrow{g_1 M} M' \xrightarrow{\xi} M''
\]

\[
B \xrightarrow{f} B' \xrightarrow{g} B''
\]
A cofiber cat of modules.

$B \mapsto \text{M}(B)$ functor an object of next want a functor $F: \text{M}(A) \to \text{M}(B)$ i.e. a family of objects $F(M) = M \in \text{M}(A)$. Leads to replacing $\text{M}(B)$ by $\text{Hom}(\text{M}(A), \text{M}(B))$. Leads to replacing fibre over $\text{M}(A)$ fibre over $B$ is $\text{M}(B)$.

$\text{steinther} (\text{M}(A), \text{M}(B)) = \text{form} (B \otimes A^{\text{op}})$

then have $B \mapsto B \otimes A^{\text{op}}$

$\underbrace{(B, P)} \xrightarrow{(w, \alpha)} \underbrace{(B', P')} \quad \text{OKAY.}$

it means a hom. $B \to B'$ and $B \otimes A^{\text{op}} \to B' \otimes P'$. criterion that $B \otimes P$ has an adjoint bi-module i.e. $Q$

$A \xrightarrow{Q} \otimes B P$

$P \xrightarrow{Q} \otimes B P$

Return! Very basic idea: $\text{Eq}(\text{M}(A), \text{M}(B))$ "functor" of $B$ get a fibre category of $(B, F)$. Behind:

at $B, P$

\[
\begin{align*}
\text{M}(A) & \xrightarrow{w} \text{M}(B) \\
\text{E} & \xrightarrow{F} \text{F}' \\
\text{M}(B) & \xrightarrow{w^*} \text{M}(B')
\end{align*}
\]
41. Given \((B; F, G) \xrightarrow{(\omega, \delta, \chi)} (B'; F', G') \rightarrow (B'', F'', G'')\)

\[ \Theta: \omega \in F \rightarrow F' \]

point in \((\omega_1, \omega_2)^*(F, G)\) yields Matrix \[(A \ Q \ B) \rightarrow (A \ Q') \]
\[(B \ Q \ B') \rightarrow (P \ Q' \ B') \]

Composite map

**Question:** Given \(B^\text{PA}\) can you tell when \(P\) is invertible. You can forget \(A\) first, and then \(P\) should generate \(M(B)\). Then go through Rosen analysis. Put \(R = \text{Hom}_B(P, P)\) and you get \(\text{mod}(R) \rightarrow M(B)\)

\[ M \rightarrow \text{P} \otimes_B M \]

It seems you plan to remove the old \(C \rightarrow C'\) version.

\[ \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \quad \text{PA} \leftarrow P \]

\[ C = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} \]

Assume \(A = A^2 = QP\), \(B = B^2 = PQ\)

\[ C^2 = \begin{pmatrix} A & AQ + QB \\ PA + PB & B \end{pmatrix} \]

But \(AQ = QPQ = QB\)

And \(PA = PQP = BP\).

So \(C^2\) is strictly idempotent i.e.

\[ C \otimes C^2 \]

\[ C = C^2 \leftrightarrow P \leq 0, PA \text{ and } Q = Q \]
Start with $C \Rightarrow A = A^2 = AP$, $B = B^2 = P'Q'$.

Replace $C$ by $C^2 = \overline{A'A''A''A''Q} = (\overline{A} \overline{A''} P A'' P) Q''$.

Then $AQ' = AAQ = AQ = Q'$, $Q'B = Q'$.

Since $P'A = P'P''$,

$P'Q' = BPQ'B = B^3 = B$.

$Q'P' = AQPA = A^3 = A$.

Suppose $C$ strictly idempotent.

$(A) \otimes (A \otimes Q) \text{ is a spinor.}$

$P \otimes A M$

$P \otimes A A$ = $BP \otimes P \otimes A A$

Correct from brane side as $B \otimes P \otimes A A$

$P \otimes A A = B(P \otimes A A)$

$P \otimes \eta \eta = P \otimes (P \otimes \eta)$

$P \otimes A A = B \otimes P \otimes A = B ^{2} \otimes B$.

Neg hom.

$B \otimes A A$ = $A A$ = $A^{2} \otimes B$

how do I do this.
43. Once \( PA = P \) and \( AQ = Q \), then the dual pair is \((A^{(2)}_A, A^w_A Q)\). So the matrix must be:

\[
\begin{pmatrix}
A^{(2)}_A & A^w_A Q \\
P \otimes A & P \otimes A
\end{pmatrix}
\]

Example:

\[
\begin{pmatrix}
A & AB \\
BA & B
\end{pmatrix}
\]

Assume \( A = A^2 \), \( AB \prec A \), \( B = B \prec A \).

Do what else is there?

Assume \( C \) is idempotent.

Then:

\[
\begin{pmatrix}
P \otimes A & B \otimes B \\
Q \otimes A & A \otimes A
\end{pmatrix}
\]

It seems I felt that I needed to know that \( A, B, P, Q \) form + any pairings \( \Rightarrow \) form.

Yes because given \((P, Q)\) form, dual pair so that \( P \otimes P \rightarrow A \) is surjective; how do you get that \( P, Q \) B form; and \( Q \otimes P \rightarrow A \)?

Write stuff out & maybe some theory of meg.

\[
\text{Megism:}
\]

\[
\text{H}_{A}(M \otimes A) \rightarrow \text{H}_{A}(N \otimes A)
\]

\[
\alpha : \mathcal{B} \otimes A \rightarrow \mathcal{M} \rightarrow \mathcal{M}
\]

\[
\beta : \mathcal{M} \rightarrow A^{(2)}_A \otimes A
\]

\[
\omega(M) = B \otimes A \quad \omega(N) = A \otimes A
\]

\[
\begin{array}{cccc}
\mathcal{A} & \mathcal{B} & \mathcal{C} & \mathcal{D} \\
6 & q_1 & q_2 & m \\
6 & 0 & 1 & (m, n) \\
6 & 0 & 0 & (m, n) \\
\end{array}
\]
4. if $H \cong \beta$ then for $M = A^{(e)}$

$$A^{(e)} \sim A^{(1)} \otimes B \otimes A^{(2)} \Rightarrow A \xrightarrow{w} B \quad \text{with } A \otimes_B A \cong A$$

$A \text{ Ker}(w) A = 0$ and $\omega(A) B \omega(A) \leq \omega(A)$

Next, $w$ equiv $\otimes B$ also sin.

$$B \otimes A^{(1)} \otimes B^{(2)} \sim \otimes A^{(2)}$$

$$\Rightarrow B \omega(A)^2 B^2 = B^2 \quad \text{i.e. } B \omega(A) B = B$$

cor: assume this holds. Factor $\omega$ into $A \xrightarrow{\omega(A)} B$.

can suppose 1) $\omega$ surj. $B = A/I$ where $AIA = 0$.

2) $A \subseteq B$, $ABA < A$, $BAB = B$.

$$A \rightarrow A/ \quad A \otimes_A N \rightarrow N$$

four cases.

1) $w: A \rightarrow A/I$ canon. surj., $I$ ideal $\Rightarrow IA = 0$.

2) $A/I = 0$.

3) $w: A \rightarrow B$ inclusion of subalg. st. $\forall A \subseteq B$. $BA < A$, $AB = B$.

4) $AB < A$, $BA = B$, $IM = 0 \Rightarrow M$ artinian. an $A/I$-module.

Case 1)

$$A \otimes_A M \sim M$$

$$A/I \otimes_A M = A/I \otimes_A I M$$

$\Rightarrow M(\alpha) = M(\alpha I)$

$$\Rightarrow M = M(\alpha I)$$

2) $m(\alpha I) = m(\alpha I)$

3) $A \otimes_A M \sim M$ needs to be worked out - variant of the may arg.
$\omega: A \to B$ factors

$A \to A/AK \to \overline{A} \subset \overline{AB} \subset$

$A \xrightarrow{\omega} A/AK \xrightarrow{\omega'} A/K = \overline{A} \xrightarrow{\omega_2} \overline{AB} \xrightarrow{\omega_4} B\overline{B} = B$

$A \otimes_A M \to M \Rightarrow M$ has unique $B$-mod. st.

such that $b(\omega m) = (b \omega) m$

$\mathbb{N} \otimes_B N \to \mathbb{N}$

$A \otimes_B B \to B \to 0$

$a \otimes b \rightarrow ab$

$A \otimes_B B \to B \to 0$

$a \otimes b = (a \otimes b, a \otimes b)$

$A \otimes_B A \to A \to 0$

$(b \otimes a) \alpha' = b \otimes a'$

Take $(p_A, A, Q, Q \otimes P \to A)$ twin dual pair

put $B = P \otimes A$ etc. whence get $M$-context.

\[
\begin{pmatrix}
A & Q \\
p & B
\end{pmatrix} =
\begin{pmatrix}
p \\
A
\end{pmatrix} \otimes_A \begin{pmatrix}
A & Q
\end{pmatrix}
\]

need to see that $B, P, Q$ B. prime $Q \otimes_B P \to A$. 

$\alpha$
96. $A \otimes B \otimes N \xrightarrow{1_{\otimes}} A \otimes N$

\[ \begin{array}{c}
A \otimes B \otimes N \xrightarrow{1_{\otimes}} A \otimes N \\
B \otimes N \xrightarrow{\mu} N
\end{array} \]

Yes!! Last again.

$A = B \quad BA = A$  
$AB = B$

So what???

$M \xrightarrow{1} A \otimes A = M$

$N = B \otimes B \leftrightarrow N$

$M$ $A$-form from $A \otimes A M = M$ get

$B$ action $b(a m) = (ba) m$.

Then

$B \otimes A \xrightarrow{B} A$

$B \otimes P \xrightarrow{1_{\otimes}} P$

$b \otimes p \otimes p' = b \otimes p' \otimes p'$

$b \otimes a \otimes b_1 = b a \otimes b_1$  
$b \otimes a \otimes a_1 = b a \otimes a_1$

$B \otimes M = B \otimes B A \otimes A M$
47. \( A \subset R \)

\[
\begin{align*}
M(A, A) & \quad M(R, A) \\
M & \quad A \otimes_A M \\
\otimes_R N & \quad N
\end{align*}
\]

Start with \( A \otimes_A M \rightarrow M \)

get \( R \) action on \( M \) \( r(am) = (ra)m \)

is a form \( A \otimes M \rightarrow A \otimes_R M \)

\[ ar \otimes a'm = ara' \otimes_M = a \otimes ra'm \]

Suppose then that \( A \otimes_R N \rightarrow N \).

\[
\begin{align*}
A \otimes_A M & \rightarrow M \\
b(am) & = (ba)m
\end{align*}
\]

\[
\begin{align*}
A \otimes_A M & \rightarrow B \otimes_B M \rightarrow M \\
b \otimes am & \\
ba \otimes m
\end{align*}
\]

\[
\begin{align*}
B \otimes_B M & \rightarrow M \\
B \otimes_B A \otimes_A M & \rightarrow B \otimes_B M
\end{align*}
\]
48. \[ A \otimes_A M = M \]
\[ B \otimes_B N = N \]

and then have

M \rightarrow A \otimes_A M = M

B \otimes_B N = N \leftarrow N

\[ \text{why form } B \otimes_A A \rightarrow BA = A \]

because \[ B \otimes_B P \rightarrow P \] is \( A \text{ op-nil} \) with.

need \[ B \otimes_A A \rightarrow A \] is \( A \text{ op-nil} \) isomorphism.

easier \[ A \otimes_A M \rightarrow B \otimes_B M \rightarrow M \] easy case.

other \[ N \in M(B) \] want \( N \) to be \( A \)-fim.

\[ A \otimes_A N = A \otimes_B B \otimes_B N \rightarrow B \otimes_B N \]

need that \[ A \otimes_B B \rightarrow B \] is \( B \text{ op-nil} \) with.

\[ \text{Contrfim. } F: M(A) \rightarrow M(B) \]
\[ \text{mod}(A) \rightarrow \text{mod}(B) \]

\[ M \rightarrow F(A \otimes_A M) = P \otimes_A M \]
\[ P = F(A^{(a)}) \]

\( P \) - \( A \)-nil with \( P \otimes_A A \rightarrow \)

\[ m(A \text{ op}) = \text{restrfim} (m(A), A \text{ op}) \]

\[ P \rightarrow P \otimes_A - \]
A map \( \varphi : M' \to M \) is inj.

When kernel is nil.

\[
B \otimes P \otimes A \to P
\]

\[
\Rightarrow B(\otimes) P \otimes A(\otimes) \to B \otimes P \otimes A \to P
\]

\[
\Rightarrow B(\otimes) P \to P \quad \text{and} \quad B \otimes P \otimes A \to P.
\]

\[
\text{Prim } B, A \text{-bimodule } = \text{prim } (B \otimes A, B \otimes A)
\]

\[P \text{ flat.} \quad \varnothing M' \to M \text{ inj in } M(A)\]

i.e. \[
0 \to K \to M' \to M \quad \text{ex. } AK = 0
\]

\[
\Rightarrow 0 \to P \otimes A K \to P \otimes A M' \to P \otimes A M
\]

\[P \text{ flat } A^{op} \quad \varnothing P \otimes A K = P \otimes A K = 0
\]

\[
P \otimes A \text{ exact } M(A) \to \varnothing B.
\]

\[
\text{hence. } 0 \to M' \to M \text{ in } \]

\[P \text{ } A^{op}-\text{flin } \Rightarrow P \otimes A \text{ - awerts } A \text{-module.}
\]

\[
0 \to M' \to M \to M'' \to 0
\]

\[
A M' = 0 \quad P \otimes A M' \to P \otimes A M \to P \otimes A M'' \to 0
\]

\[
A \otimes M' \to A \otimes M \to A \otimes M'' \to 0
\]

\[
0 \to M' \to M \to M'' \to 0
\]
\[ P \otimes A \otimes M' \rightarrow P \otimes A \otimes M \]
\[ \downarrow s \quad \downarrow s \]
\[ P \otimes M' \rightarrow P \otimes M \]

Sections.

Please modules \rightleftharpoons modules/nil-modules

bílot, Serre subcats + Roos theorem.

independence of \( R \).

Please bimodules + stone functors.

adj functors assoc. to a ring homom.

examples.

Rings

Claim: If \( C \) 0 idem then the normalized is

\[
\begin{pmatrix}
A_A \otimes A & A_Q \otimes A \\
A_A & A_Q \otimes A
\end{pmatrix}
\]

\[
\begin{pmatrix}
A_A \otimes A & A_Q \otimes A \\
A_A & A_Q \otimes A
\end{pmatrix}
\]

\[
B \otimes B = g \otimes B
\]

\[
B \otimes B = g \otimes B
\]
51. \((A) \otimes_A (A)\) 

\(P' \otimes A Q'\) where \(P' = P' A\), \(Q' = A Q'\) and \(Q' P' = A\).

Thus \(P' \otimes A Q'\) is a fibre ring.

\[(A) \otimes_A (B)\] \(P' = P' \otimes A (2)\) 
\(Q' = Q' \otimes A (2)\) 

\(Q' \otimes P' \rightarrow A (2) \otimes_A Q \otimes P \otimes A (2) \rightarrow A (5) \rightarrow A (3)\)

\(P' \otimes Q' \rightarrow P \otimes A (2) \otimes A (2) \otimes_Q Q \rightarrow P \otimes Q \rightarrow B\)

\(P \otimes A (2) \otimes Q \otimes B (2) \rightarrow B \otimes B (2) \rightarrow B (2)\)

\(Q \otimes B (3) \otimes P \otimes A (2) \rightarrow A \otimes A (2) \rightarrow A (2)\)

Example. \(Q \otimes P \rightarrow A\) unital
\(p_0 \otimes p_0 \rightarrow 1\)

\((P, Q) = (A, A) \otimes\)

\(P = p_0 A \otimes P'\)
\(Q = A q_0 \otimes Q'\)

\(\sum \xi_i : p_i = 1\).
$52. \quad \text{dual } (A \ Q) \ A$

$A \text{ f.g. } P_A, \ Q \text{ f.g. } Q_B$

$P = P \otimes_A A$ is $B$-f.g. and $Q_B$ f.g.

$B = P \otimes_A Q$ \quad \therefore \quad B$-f.g. ring.

$Q \otimes_B P \longrightarrow A \quad \text{Ker} \quad Q \otimes_B P \longrightarrow A$

**Idea:** \quad $F: \text{mod}(R) \longrightarrow A B$

Right exact.

$F(R) \otimes_R M \longrightarrow F(M)$

$\text{f.g. } (R^0, A^0) = \text{exterior } (m(R^0), A^0)$

$A \subset R \quad RA \subset A$

**Remark:** \quad A ideal in $R$

$F^a$, flat, $FA = A$.

$F \otimes_A N \longrightarrow F \otimes_R N$

$\Rightarrow$ to see if $M$ flat

$F' \otimes_A M$

$N \longrightarrow F \otimes_A N \longrightarrow F \otimes_R N$

$F \quad \text{A}^0$ flat f.g.

$F \otimes_A N \longrightarrow F \otimes_R N$

$\therefore \quad F$ flat

$F \otimes_A M \longrightarrow F \otimes_R M$

Assume $F$ is flat
\[ \sum_{r \in \text{m}_i} \rightarrow \sum_{r \in \text{m}'_i} = 0. \]
\[ a \sum_{r \in \text{m}_i} = \bigotimes_{r \in \text{m}_i} a_{r,0} \sum_{r \in \text{m}'_i} = 0. \]
\[ \sum_{r \in \text{m}_i} \]
\[ a \sum_{r \in \text{m}'_i} = \sum_{r \in \text{m}'_i} = 0 \]

\[ 0 \rightarrow F \otimes K \rightarrow F \otimes M' \rightarrow F \otimes M \]

\[ \text{exact start = flat form at modules} \]

\[ P \rightarrow P \otimes_R \left( \text{frin}(R, A) \rightarrow \text{Ab} \right) \]

\[ \text{is exact} \quad \text{obvious from view of} \quad \text{frin}(R, A) \text{as mod}(R)/\text{nil} \]

\[ \text{Other direction:} \quad \text{Given} \quad F \quad \text{functor} : \quad \text{frin}(R, A) \rightarrow \text{Ab} \]
\[ \text{compose with} \quad A^{(2)} \otimes_R - \quad \text{to get} \quad M \rightarrow F(A^{(2)} \otimes_R M) \]
\[ F(A^{(2)} \otimes_R M) = P \otimes_R M \quad P = F(A^{(2)}) \]

\[ P \text{ must be form.} \quad \text{Why?} \quad \text{That?} \quad 0 \rightarrow M' \rightarrow M \]

\[ \text{for modules} \quad P \otimes_R A \rightarrow P \Rightarrow P \otimes_R - \quad \text{rives in} \quad \text{nil incs.} \quad 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \]

\[ AN' = 0 \Rightarrow P \otimes_R N' = P A \otimes_R N' = P A \otimes_R N = 0 \]

\[ P \otimes_R N = P \otimes_R N'' \quad \text{(a)} \]

\[ AN'' = 0 \Rightarrow K \rightarrow A \otimes_R N' \rightarrow A \otimes_R N \rightarrow 0 \]

\[ AK = 0 \]
char. of finite module

\[ \text{Tor}^0_R (M, N) = 0 \quad p = 0, 1. \]

\[ \text{Ext}^p_R (M, M) = 0 \quad p = 0, 1. \]

\[ \text{Hom}_R (M, -) \text{ isos. nonc.} \]

\[ \otimes_R M \]

**Proof:**

\[ 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \]

\[ \text{Hom}_R (P, N) = \text{Hom}_{R/I} (R/I \otimes_R P, N) \]

\[ E^0_I = \text{Ext}^p_R (\text{Tor}^0_R (R/I, M), N) = \text{Ext}^{p+1}_R (M, N) \]

\[ \Rightarrow \text{Tor}^0_R (R/I, M) = 0 \quad g = 0, 1 \]

Conversely, take \( N \) injective \( R/I \)-module

\[ \text{Hom}_R (\text{Tor}^0_R (R/I, M), N) = \text{Ext}^0_R (M, N) \]

Consider objects of all \( M \rightarrow \text{Ext}^1_R (M, N) = 0 \quad i = 0, 1 \).

Closed under extension.

\[ \text{Tor}^0 \rightarrow \text{Tor}^1 \rightarrow \cdots \]
55. If $Q$ is a left $R$-module, then

\[ R/A \otimes_R Q = 0 \quad Q = AQ \]

\[ \Rightarrow N \otimes_R Q = 0 \quad \text{all nil modules} \]

\[ \text{Tor}_1^R(N, Q) \] is right exact and vanishes for $N = R/A \Rightarrow$ vanishes for all nil modules.

To see that $N \to N \otimes_R Q$ inverts nil isom.

It is sufficient to show for inj cats killed by $A$

\[ N \to N 
\]

\[ \text{Tor}_1^R(N'', Q) \to \text{Tor}_0^R(N', Q) \to \text{Tor}_0^R(N, Q) \to N' \otimes_R Q \]

Converse

\[ A \to R \]

\[ 0 \to \text{Tor}_1^R(A, N) \to A \otimes_R N \to N \to \text{Tor}_0^R(R/A, N) \to 0 \]

vanishing $\text{Tor}_i^R = 0 \Rightarrow$ nil, nil isom.

\[ 0 \to L \to L' \to N \to 0 \]

\[ 0 \to \text{Tor}_1^R \to L_1 \otimes_R Q \to L_0 \otimes_R Q \to N \otimes_R Q \to 0 \]

\[ A \otimes_R M \to M \]

\[ 0 \to S' \to S \to S'' \to 0 \]

take $\text{Tor}$

\[ \text{where } S \to S' \to \cdots \]

\[ S \to f^{(n)} \]

A is a Macaulay graded ring of some sort. $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded ring? NO. Think of it as the dual pair $A @ P$, $A @ Q$ $(A, A) \oplus (P, Q)$.

Idea: Suppose $A$ has a left ann. ideal? Idea: This time you notice things such that $M(A)$, $M(B)$ are isom. cats. Two cases $A \rightarrow A/I$ where $IA = 0$ $A < B$ where $BA = A$, $AB = B$ $w: A \rightarrow B$

Question: Is there a maximal such $B$? Real question: Can you eliminate the nil ideals somehow?

First question clear, you take $P = A$ and look at possible $Q$. Then $Q = A^\times \text{Hom}_{A@B}(P, A)$ $B = P @ A$

Life is hard. Think a little $M(A)$ generator $A$.

You will work things in such a way that $M(A) \subseteq M(B)$ via a homomorphism $A \rightarrow B$. In other words, $B$ naturally acts on any $A$-module $M = A @ M$. Obvious first candidate is $M$?
57. \( \text{Hom}_{A^{\text{op}}}(A, A) \) acts on \( M \).

But now this \( B \) is and

\[ A \to \text{Hom}_{A^{\text{op}}}(A, A) \]

kernel \( K \) such that \( KA = 0 \).

\[ A \to AK \to \text{Hom}_{A^{\text{op}}}(A, A) \]

\( f \in \text{Hom}_{A^{\text{op}}}(A, A) \) such that \( f(a, a') = f(a) a' \)

shows the image \( AK \otimes \text{Hom}_{A^{\text{op}}}(A, A) \) is a left ideal.

\[ 0 \to K \to A \to \text{Hom}_{A^{\text{op}}}(A, A) \to \otimes C \to 0 \]

\( K, C \) should be \( A \)-nil, which agrees with \( Q = A \otimes A \text{Hom}_{A^{\text{op}}}(A, A) \).

Other case: \( \text{Take largest start with } A \)

look for \( J \to Z \to A \) \( JZ = 0 \) so that \( Z \) is a left \( A \)-module.

So you just take \((A, Q) \to (A, A)\).

\[ \text{M}(Q) \quad \text{M}(A) \]

\( A \) a ferm \( Z \) a ferm \( A \)-module \( f: Z \to A \) surj

\[ z_1 z_2 = f(z_1) z_2 \]

By this construction you increase the left arm of \( A \). What about right arm? \( \text{Hom}_{A}(Z, Z) \)
Proper picture: Look at the R Mod Cat $M(A)$ and the generator $A$. You want all $A \oplus_A M$ modules of the form 

\[(A \oplus_A Q) \quad \text{i.e. } P = A \oplus_A Q = \frac{\text{Hom}_A(A, A)}{\text{Hom}_A(A, A)}\]

So you are choosing $Q$ and $Q \oplus A \rightarrow A$. You want all $Q \rightarrow \text{Hom}_A(A, A)$ which lead to $Q \oplus A \rightarrow A$. Among the choices are quotients of $A$ \text{i.e.} $Q = A/I$ where $IA = 0$. And left ideals $\mathfrak{a} \leq A$ such that $\mathfrak{a} \oplus A \rightarrow A$ \text{i.e.} $\mathfrak{a}A = A$. YES!!!

Back to writing!

Take $A \leq B$ right ideal: $\begin{cases} AB \leq A \\ BA = B \end{cases}$

Then $M(A^\text{op}) = M(B^\text{op})$

$V \oplus_A A = V \Rightarrow V$ has unique $B^\text{op}$-module struc.
\[ M(A^{op}) \quad m(B^{op}) \quad (A \quad B) \]

\[ V \quad \quad \quad V \otimes_A B \]

\[ W \otimes_B A \leftarrow \quad W \quad \quad \quad \]

Idea is that

\[ \begin{array}{c}
m(A) \quad m(B) \\
\downarrow \quad \downarrow \\
\text{rundfun}(m(A), AB) = \text{ref}(m(B), AB)
\end{array} \]

\[ V \quad \quad \quad (M \quad \rightarrow \quad V \otimes_A M) \quad \quad \quad \]

\[ (N \quad \rightarrow \quad V \otimes_A N) \quad \rightarrow \quad V \otimes_B A^{(2)} \]

\[ W \quad \quad \quad (N \quad \rightarrow \quad W \otimes_B N) \quad \quad \quad \]

\[ (M \quad \rightarrow \quad W \otimes_B V \otimes_A M) \quad \rightarrow \quad W \otimes_B A^{(2)} \]

\[ \begin{array}{c}
M(A^{op}) = M(B^{op}) \\
\text{(w-m-t)} \\
\text{rundfun}(m(A), AB) \rightarrow \text{ref}(m(B), AB)
\end{array} \]

\[ (M \rightarrow V \otimes_A M) \quad \rightarrow \quad (N \rightarrow V \otimes_A A^{(2)} \otimes A^{(2)} N) \]

\[ W \otimes_A A^{(2)} \]

\[ (M \rightarrow W \otimes_B V \otimes_A M) \quad \leftarrow \quad (N \rightarrow W \otimes_B N) \]

\[ \text{WT}(V). \]
\[ A \xrightarrow{w} B \]

\[
\text{Hom}_{m(A)}(M, w^*N) = \text{Hom}_{m}(M, N) = \text{Hom}_{m(B)}(w, M, N)
\]

\[ B^{(2)} \rightarrow B, \quad B < B^{(2)} \quad \text{\text{\[ B^{(2)} \otimes_A M \rightarrow B \otimes_A M \rightarrow B \otimes_A M \text{.} \]}} \quad \text{hence \quad A^{(2)} \text{nil}} \quad \text{is,} \quad \text{hence \quad A^{(2)} \text{nil}} \quad \text{is,}
\]

\[
\text{Hom}_{A}(M, B \otimes_A M) \rightarrow \text{Hom}_{A}(M, N)
\]

\[
= \text{Hom}_{B}(B \otimes_A M, N)
\]

\[ \text{A} \rightarrow \text{A / I} = \tilde{A} \subset B \]

\( A^{(2)} \otimes_A B \rightarrow B \otimes_A A^{(2)} \)

\[ \text{Assume} \quad IA = 0, \quad BA \subset B, \quad \tilde{AB} = B. \]

Then have maps \( w: A \rightarrow B \), a \( B \)-module structure on \( A \) such that \( w \) is a \( B \)-mod. map.

So we have a \( B \)-mod. structure on \( A \), a \( B \)-module maps \( w: A \rightarrow B \) such that \( \tilde{AB} = B \).

Also we have \[
\begin{array}{ccc}
q_1 & q_2 \\
\hline
\end{array}
\]

Let's go back to see if you can rig up some non-degenerate. Yesterday I analyzed \( M \)-contents \( (A \otimes B) \quad (A \otimes B) \quad B \otimes_A A \rightarrow A \quad A \otimes A \rightarrow B \)

Point \( (A \otimes B = Q) \quad \phi : Q \otimes A \rightarrow A \quad Q \rightarrow A \otimes \text{Hom}_{A^f}(A, A) \)
61. Ultimately what's going on is that I have $M(A)$ and the generator $A$ chosen. A choice of $Q$ somewhere amounts to choosing enough maps $g_i : A \to A$? Just what does this mean?

Think abstractly. You have $M$ a $K$-algebra and some generator $P$.

Think abstractly. You are given $M$ and $M' = \text{rextfun}(M, qv)$. Given $Q \in M$ $P \in M'$ you can form $Q \otimes \mathbb{Z} P \in \text{rextfun}(M, M')$.

That you want a surj $Q \otimes \mathbb{Z} P \to 1$

$$0 \to \text{Tor}_i (A, k) \to A \otimes_A A \to A \to A/A^2 \to 0$$

ind of $k$

NO because $k$ enters into what happens? A idempotent $k$-algebra $M(A)$ is ind of $k$, so

$$A \otimes_{\mathbb{Z} A} A \to A \otimes_{\mathbb{Z} A} A$$

must be an isomorphism

$$\text{form} (\mathbb{Z} \otimes_A A) \leftarrow \text{form} (\mathbb{Z} \otimes_A A)$$

Anyway form covering

$$\begin{array}{ccc}
A \otimes_A A & \to & B \otimes_B B \\
\downarrow & & \downarrow \\
A & \to & B
\end{array}$$
62. Propose exploring the idea of making a firm ring non-degenerate. Start with $A$.

The left-right arm.

$0 \rightarrow I \rightarrow A \rightarrow \text{Hom}_A(A, A) \rightarrow \text{Hom}_A(A, A)$

General case: $P \rightarrow P \otimes_A Q = B$.

$P \otimes_A Q \rightarrow M(B) \rightarrow \text{Hom}_A(B, B) \times \text{Hom}_A(Q, Q)$

$\text{Hom}_B(B, B) \text{ Hom}_B(B, B)$.

Suppose we keep $P, Q$ to avoid confusion. Let's say we have the firm dual pair $(P, Q)$ over $A$ given. What do we know about $B \rightarrow M(B)$?

Homotopy of things are nice. I know that the cotorsal analogue of Calzetti's alg is $B$-nil because $B$ is an ideal in $M(B)$.

Idea. You want to go from $P, Q$ to another firm dual pair $(P', Q') \rightarrow (P, Q')$ so that $B \rightarrow B'$ detect pairs of the left-right arm ideal. 

Thus if you have $b \neq 0$, $Bb = bB = 0$, you would like to contract see if you can arrange that $b$ is detected in some $P'$ or $Q'$.

Try enlarging $P$,

$\text{Hom}(fp, flat)$
Go back over nuclear maps. What

\[ \text{Hom}_R(M, R) \otimes_R N \to \text{Hom}_R(M, \Omega) \]

this is an ism. if \( N \) flat and \( M \) f.p.

I seem to recall using this to decide

when an element of \( \mathbb{P} \otimes R \mathbb{N} \) is zero. What you want is to take \( \mathfrak{e} \in \mathbb{P} \otimes R \mathbb{A}, \mathfrak{e} \neq 0 \), and to find \( \mathfrak{p}' \)

and to find \( \mathfrak{q} \sim \mathfrak{p}' \rightarrow R \mathbb{A} \) so that

\[ \mathbb{P} \otimes R \mathbb{A} \to \text{Hom}_R(\mathfrak{p}', R) \otimes_R \mathbb{P} \otimes R \text{Hom}_R(\mathfrak{p}', R) \]

\[ \text{Hom}_R(\mathfrak{p}, \mathbb{P}) \]

takes \( \mathfrak{e} \) somewhere non-zero.

\[ \mathbb{P} \otimes R \mathbb{A} \]

keeping \( \mathbb{A} \) fixed but

embiggening \( \mathfrak{p} \), i.e., universal case is

\[ \mathbb{Q} \rightarrow \text{Hom}_R(\mathfrak{q}, R) \otimes_R \mathbb{A} \]

\[ \mathbb{P} \rightarrow \text{Hom}_R(\mathfrak{p}, R) \otimes_R \mathbb{A} \]

\[ \mathbb{A} \subset \mathbb{B} \]

\[ \mathbb{B} \mathbb{A} \subset \mathbb{A} \]

\[ \mathbb{A} \mathbb{B} = \mathbb{B} \]

\[ \mathbb{A} \rightarrow \mathbb{B} \]

\[ \mathbb{M} \]

\( \mathbb{A} \)-flat \( \mathbb{A} \mathbb{M} = \mathbb{M} \).

\[ \mathbb{B} \otimes_R \mathbb{M} \]

\( \mathbb{B} \)-flat \( \mathbb{C} \mathbb{M} \).

\[ \mathbb{B} \otimes_R \mathbb{M} = \mathbb{B} \otimes_R \mathbb{A} \mathbb{M} = \mathbb{B} \mathbb{w}(\mathbb{A}) \otimes_R \mathbb{M} \]
\[ A \to R \]
\[ M \to A \otimes_A M \]
\[ N = R \otimes_R M \to N \]

\text{M finite}

\[ \text{Hom}_A(M, -) \text{ invertible and isom.} \]
\[ \text{Ext}^i_A(M, N) = 0 \quad \text{N nil and } i = 0, 1 \]

\[ \otimes_R M \text{ nil and isos.} \]
\[ \text{Tor}^i_R(B/A, M) = 0 \quad \text{for } i = 0, 1 \]

\[ M \text{ is a coherent of } \text{a map of finite flat modules.} \]

\[ 0 \to N' \to N \to N'' \to 0 \]
\[ \text{AN}'' \to 0 \to \text{Hom}(M, N') \to \text{Hom}(M, N) \to \text{Hom}(M, N'') \to \text{Ext}^1(M, N) \to 0 \]

\[ \text{since } AM = M \]
\[ \text{and } AN'' = 0. \]

\[ 0 \to N \to M \to M \to 0 \quad \text{N-nil} \]
\[ \text{Hom}(M, N) \to \to \text{Ext}^1(M, N) \]

\[ \text{Ext}^i_R(M, E) = H^i(\text{Hom}_R(P, E)) \]
\[ = H^i(\text{Hom}_{R/A}(R/A \otimes_R P, E)) \]
\[ = \text{Hom}_R(\text{Tor}^i_{R/A}(R/A, M), E). \]

So this Ext vanishes \[ \Rightarrow \text{Tor}^i = 0 \quad \text{for } i = 2, 3. \]
\[ E_{2}^{p} = \mathcal{E}xt_{R/A}^{p}(\text{Tor}_b^{R/A}(R/A, M), N) \]

\[ \Rightarrow \mathcal{E}xt_{R}^{*}(M, N). \]

If the Tor's vanish for \( q = 0, 1 \), then \( \mathcal{E}xt_{R}^{i}(M, N) = 0 \) \( \forall i = 0, 1 \) all \( R/A \)-mod.

Conversely if these \( N \) is \( R/A \)-mod.

\[ \text{Hom}_{R/A}(\text{Tor}_b^{R/A}(R/A, M), N) = \mathcal{E}xt_{R}^{1}(M, N) \]

Let \( A \) be unital to fix the ideas.

Consider a dual pair \((P, Q)\) over \( A \) where \( P, Q \) are f.g. free.

You want to understand the ring \( \text{End}_{P \otimes Q} \).

In terms of matrix rings over \( A \).

A basic trick you have is to replace \( B \) by \( A \rightarrow B \) where \( A \) is flat in both roles and the ideal \( I \) such that \( AIA = 0 \). The typical thing you have is degenerate. You hope relate the degenerate thing to a non-degenerate one.

Start with \( A \) embed into \( B = (A, P, Q) \)

1/25 09:50 stop wasting time

Start with \( A \) unital, \((P, Q)\) f.g. dual pair where \( P, Q \) are f.g. projective. Then we can choose to simplify suppose \( (P, Q) = (A, A) \oplus (P_0, Q_0) \) where the pairing \( Q_0 \otimes P_0 \rightarrow A \) is arbitrary. You should be able to embed \((P_0, Q_0)\) into a non-degenerate pair. Method \( Q_0 \rightarrow \text{Hom}_{A \otimes B}(P_0, A) = P_0^* \)
You have
\[ Q_0 \oplus (0 \ 1) \oplus P_0^* \rightarrow P_0^* \rightarrow (0 \ 1) \]

\[ (0 \ 1) \oplus (1 \ 0) \oplus (0 \ 1) = P \]

Take \[ Q = Q_0 \oplus P_0^* \]
\[ P = Q_0^* \oplus P_0 \]

usual pairings
\[ Q_0 \otimes Q_0^* \rightarrow A \]
\[ P_0^* \otimes P_0 \rightarrow A \]

pairing \[ Q_0 \otimes P_0 \rightarrow A \]

any other pairing \[ P_0^* \otimes Q_0^* \rightarrow A \] such that \[ (0 \ 1) \] remains non-degenerate.

So, if I start with \[ (A \ Q_0 \ P_0^*) \]
I can embed it in \[ (A \ Q_0^* \ P_0) \]

So what is the problem now? You've managed to embed \[ (A \ Q_0 \ P_0^*) \] inside the non-deg. thing. But the question is whether you can embed the non-deg. thing inside a matrix ring over
the deg. alg. The A part! Then what happens??

A \in \text{M}_nA

\cap \cup \cap

B \longrightarrow \text{M}_nB

e.g. Whatever the theory of the Grassmannians you want to be able to link different homomorphisms \( C \rightarrow \text{M}_nC \).

A homomorphism \( C \rightarrow \text{M}_nC = \text{Hom}(C; C) \) is an idempotent matrix. These form a theory I am interested in; morphisms - means rank 1 projections. this space has homotopy type \( \text{P}(C) \). Homotopy type

Algebraically what happens? Higher rank maps of dual pairs

\[ \text{M}C \rightarrow \text{M}_nC \]

\[ (C, C^\circ) \rightarrow (C', C'^\circ) \]

\[ (W, W^*) \rightarrow (V, V^*) \]

amounts to

\[ W \rightarrow V \]

\[ W^* \rightarrow V^* \]

\[ \alpha, \beta \] commute with duality

so that

\[ (\alpha(w), \beta(\mu)) = (w, \mu) \]

\[ (\beta, \alpha)(w, \mu) = \beta \alpha = 1. \]

\[ W \rightarrow V. \] So what I learn is that the full subcat of monoidal categories is the category of dual pairs over \( \text{A} \) and split injectors. You know that Volodin's "topology" must be applied to get the good K-theory thing.
First part.

Properties of f-injective modules.

\[ M \text{ f-inj} \iff \text{Hom}(M, -) \text{ mnr. and cso.} \]

=> suff to consider any m kernel killed by a inj cokernel.

\[ 0 \to N' \to N \to N'' \to 0 \]

\[ 0 \to \text{Hom}(M, N) \to \text{Hom}(M, N) \to \text{Hom}(M, N') \to \text{Ext}^1(M, N) \]

\[ 0 \text{ if } M = 0 \]

\[ AN'' = 0. \]

If \( N' \) nil then inj, so you need to lift

\[ 0 \to W' \to N \to N'' \to 0 \]

can assume \( f = 1_M \)

\[ \begin{array}{c}
\text{A} \otimes_R N' \\
\text{A} \otimes_R N \\
\text{A} \otimes_R M
\end{array} \to 0 \]

\[ 0 \to N' \to N \to M \to 0 \]

\[ \leq \]

\[ \text{A} \otimes_R M \to M \text{ nil and s.} \]

\[ 0 \to \text{Ext}^1(M, N) \to \text{Hom}(M, N) \to \text{Hom}(M, N') \to \text{Ext}^2(M, N) \]

\[ 0 \to N \to E \to M \to 0 \]
69. Even if (b) + (c) is formal

\[ (c) \Rightarrow (b) \] clear: \[ 0 \to N' \to N \to N'' \to 0 \]

\[ \text{N}' \text{ nil } \Rightarrow Y \text{ (can be Hanin)} \]

\[ N'' \text{ nil } \]

(a) \Rightarrow (c) \quad \text{M finitely torsion} \quad \text{Tor}^R_j(R/A, M) = 0 \quad j = 0, 1.

\[ E^p_{1} = \text{Ext}^p_R(\text{Tor}^R_0(R/A, M), N) \Rightarrow E^p_{1} \] (M, N)

\[ = 0 \quad p = 2, \quad g = 0, 1 \]

(a) \quad \text{M finite}

(b) \quad \text{Hanin}^R(M, -) \text{ is nont-nilpotent}

(c) \quad \text{Ext}^R_R(M, N) = 0 \quad \forall N \text{ nil } \quad (\text{resp. AN} = 0)

\[ j = 0, 1. \]

(d) \quad \text{Tor}^R_j(W, M) = 0 \quad \forall j = 0, 1.

(e) \quad \text{Tor}^R_j(W, M) = 0 \quad \forall j = 0, 1.

(f) \quad \text{M kernel of a map between finite flat modules.}

\[ \text{An} \to \text{Ar} \to \text{Ar} \to \text{Ar} \to 0 \]

\[ 0 \to N \to E \to M \to 0 \quad \text{AN} = 0 \]

(g) \quad \text{M finite}

(h) \quad \text{Ext}^R_j(M, N) = 0 \quad \forall j = 0, 1 \text{ and all } N \text{ sat AN} = 0.

(i) \quad \text{same as (b) but for all nil modules.}

(j) \quad \text{Hanin}^R(M, -) \text{ is nont-nilpotence.}

(k) \quad \Rightarrow (b) \text{ M = AM } \Rightarrow \text{Hanin}^R(M, N) = 0 \quad \text{if AN} = 0.

\[ \text{given } 0 \to A \to E \to M \to 0. \]

Need \( \otimes \) conditions.

M finitely flat modules and \( \otimes \).
Back to a unital. I think I ought to be able to handle left + right flat rings meg to a unital ring.

Take \( B = P \otimes_A Q \) \( P \leq A \) \( \text{f.g. proj.} \). To simplify assume \( p_0 \in P \), \( q_0 \in Q \), \( p_0 q_0 = 1 \in A \).

Then get \( A \to B \) actually \( (A, A) \to (P, Q) \)

I think I can embed \( (P, Q) \to (A^\oplus, A)^n \)

\[ \begin{array}{cccc}
A & \to & M_n(A) & \to & M_3(B) & \to & M_9(A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \to & M_n(B) & \to & (P, Q) & \to & (P, Q)^n
\end{array} \]

First thing to understand is the Buskin case.

Rough idea is that we have an admissible move \( P \subset P \).

Try something else

Start with \( P, Q \). Have \( P \leq P \).

Can enlarge \( Q \subset Q \).

We have \( B \subset B_1 \), \( P \otimes A Q = P \otimes A Q_1 \).

Actually this looks pretty nice!

Start with any pairing \( Q \to \text{Hom}_{A^\oplus}(P, A) \)

Enlarge \( Q \).
What to do?

\[ P \otimes A \cong P \otimes A \]

\[ P \otimes A \rightarrow Q \]

\[ P \otimes A \rightarrow Q \]

\[ Q \otimes Q \]

\[ (P, Q) \subset (P, Q) \subset (Q, Q) \]

Can assume \( Q \rightarrow Q \) nice. Then \( Q \rightarrow P^* \) says \( Q \rightarrow P \) nice. But natural complements.

tensor product \[ A \otimes_A M \]

\[ 0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0 \]

if \( A = A^* \) then

1) \( AN' = 0 \) \( \Rightarrow \) \( A \otimes_R N \)
2) \( AN'' = 0 \) \( \Rightarrow \) \( A \otimes_R N'' \)

In particular

1) \( A \otimes_R M \rightarrow M \)
2) \( A \otimes_R M \rightarrow M \) 

Criteria:

- \( M \) flat, \( \text{T}_{R}^{R} (R/A, M) = 0 \) for \( i = 0, 1 \).
- \( A \otimes_R M \) is flat, \( A \otimes_R M \) is \( R \)-module.
- \( M \) is kernel of maps of \( R \)-module.

Suppose \( M \) flat and \( A \otimes M = M \). Then

\[ 0 \rightarrow A \rightarrow R \rightarrow R/A \rightarrow 0 \]

\[ 0 \rightarrow A \otimes_R M \rightarrow M \rightarrow M/AM \rightarrow 0 \]