Back to mathematics.

Review Nother invariance of $K_\ast$.

Alternative - stability for a field.

I want to review what I learned, and make a new attempt at Felsin's results via flatness.

$K_\ast(A) \overset{\text{def}}{=} \text{Ker} \{ K_\ast(A) \rightarrow K_\ast(A') \}$.

Main construction: $\to (P, Q, \otimes A)$ arbitrary dual $P$ over $A$ we associate a trace map

$\triangleright_{P} : K_\ast(\otimes A) \rightarrow K_\ast(A)$

by filtered colim, enough to do when $P \in \mathcal{P}(A)$ + naturality. Defin.

$A \otimes \hat{P}$

$K_\ast(\otimes A Q) \rightarrow K_\ast(\otimes A \text{Hom}_A(P, P))$

Point is that we have a rep of $\otimes A Q$ over $A$

$
\begin{array}{ccc}
K_\ast(\otimes A Q) & \rightarrow & K_\ast(\text{Hom}_A(P, P)) \\
\downarrow \text{can} & & \downarrow \text{can} \\
K_\ast(\hat{A}) & \rightarrow & K_\ast(\hat{A}')
\end{array}
$

Naturality, $(P, Q) \rightarrow (P', Q')$ factors

$K_\ast(\otimes A Q) \rightarrow K_\ast(\otimes A P)$

$\text{can}$

$\hat{P} \otimes \hat{Q}$

$P \otimes P'^* \rightarrow P \otimes P'^*$
So what am I doing? Arguing that for all \( P \in \mathcal{P}(\mathbb{A}^op) \) have canonical map

\[
\text{tr}^p : K_*(P \otimes_A P^*) \to K_*(\mathbb{A})
\]

such that \( \forall \) pair \( P, P' \) in \( \mathcal{P}(\mathbb{A}^op) \) you have commutativity or compatibility, you want

\[
K_*(P \otimes A P^*) \to K_*(P' \otimes A P'^*)
\]

For any \( u : P \to P' \) you want

\[
K_*(P \otimes A P^*) \xrightarrow{\text{can}} K_*(\mathbb{A})
\]

Hence

\[
P \xrightarrow{(1_0)} P \oplus P' \xrightarrow{(0_1)} P' \
\]

Consider (nonunital) category of \( P \to M \) where maps are \( P \xrightarrow{u} P' \) with a matrix over \( A \).

Is there a relation between Vaserstein's Whitehead lemma and writing \( E \) as a filtered colimit of f.g. free?
Problem. Central problem for me is to show that the two flat Monta rings have same $H_\pi(GL)$. Simp. gp argument reduces to lift Monta equivalence.

1600. Simplicial group argument. Given B idemp. say $B$-unit. Then 1 basic resolution of $B$ by flat from $B$-modules.

Interpret as s. rings.

Look at simp gp $\text{GL}(A)$ res by $\text{GL}(B)$. (loc.)

Left flat them. should say that $H_\pi(\text{GL}(A))$ constant functor, so you get $H_\pi(\text{GL}(A)) \to H_\pi(\text{GL}(B))$.

Is it possible to use this construction to say something about $H_\pi(\text{GL}(A)) \to H_\pi(\text{GL}(B))$, when $B$ is left flat? Independence of the flat resolution.

Start with the cat of $(A \to B)$. What do we know if $B$ is left flat. $(A \to B)$

$B$ is left flat $\Leftrightarrow$ $\mathbb{Q} \otimes_B B \otimes A$ is $A$-flat

$A$ is left flat $\Leftrightarrow$ $\mathbb{P} \otimes_A A = A \otimes_A A = A$ is $B$-flat,

Assuming the result of 2 often holds: $H_\pi(\text{GL}(B))$ should be a summand of $H_\pi(\text{GL}(A))$ in general, because can pick $\overline{A} \to A \to B$ with $\overline{A}$ flat.
A \otimes B = A \otimes B = B \otimes A

But then we have the map of B to A.

Go even it again. B flat to C sharp B = B in A flat.

Now we have A flat and B flat as A flat.

Thus the A flat and B flat act on B.

Why even it again? We have the map K(B) \rightarrow K(A).

So why if A is B flat and B is A flat, with A' flat we get

K(A) \rightarrow K(A)

This theorem is that A is B flat and B is A flat.

Point is that B flat is B flat.

In order to have A = B over A:

(A, B)

We want to have a map K(B) \rightarrow K(A).

So the conclusion is that A is B flat and B is A flat.
If I can't handle this case then I can't do the general case. Let us start with

There should be two ways to map $K_x(B)$ to $K_x(A)$

- $B$ acts on $B$ which is $A$-flat
- $B$ left acts on $A$ which is $A^p$-flat

First case: $A$ is a unitary $B$-mod mapping onto $B$

so $A = B \oplus I$ where $I \in \text{Mod}(B)$ and $IB = 0$.

This should be the same as a ring with left identity $A = eA, e^2 = e, B = Ae$. Now you need to understand this case. What tools? You have times. $B \to A \to B$ composition 1. So what remains? What simplified possibilities are there?

Suppose we can vary $I$. Take $B$ to be a field

$$H_p^B(GL(B), H^B(M(I)))$$

Rationally $$H_p^B(M(I)) = \Lambda^B(M(I))$$
April 23, 97

Spend a few minutes on mathematics—how about stability for \( \text{GL}_n(F) \), \( F \) field

Symmetric groups \( S_n \) all this stuff about buildings.

Basic idea I think is about a group acting on a simplicial complex of high connectivity such as a building.

Some examples: \( V \) vector space

Take a simplicial complex of frames. \( \{ x_0, \ldots, x_r \} \)

Vertices = \( v \neq 0 \)

Over an infinite field you find that there are

\( \text{dim}(V) \)

This has no homology is a bouquet of spheres.

Now can I analyze this?

\( N = \text{dim}(V) \) \( G = \text{GL}(V) \)

Let's use semi-simplicial, I guess we get a complex of chains.

\[
0 \rightarrow M \rightarrow C_N \rightarrow \\
\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots 
\]

You've forgotten all of this. So there something you can do with the \( Q \) construction. Filter the cat of vector spaces. There is some sort of spectral sequence arising which involves \( H_*(\text{GL}_p, \mathbb{F}_p) \).

How can this work? Let me see what happens? Ideas. You have a filtration

\[
F_0 < F_1 < F_2 < \cdots
\]
of the \( Q \)-cat.

A cat cons. of \( V \)

\[
\text{v}
\]
There is a point here. Namely, \( P \rightarrow P' \) is a complex in which \( B \) acts. Somehow showing that it to exact sequences (A's) it's equivalent to Point. Each \( b \in B \) on this complex is \( 0 \).

So maybe it's a D6 module over \( B \to B \).

Define action of \( B[h] \) \( |h| = 1 \), \([d,h] = 1 \), \( h^2 = 0 \)

\[
h(p \otimes (p')) = p \cdot v(g) p'
\]

\[
h : \begin{array}{c}
H_B \rightarrow H_B(B') \\
p \cdot \rightarrow (p' \mapsto p \cdot v(g) p')
\end{array}
\]

\[
dh : p \cdot \rightarrow (p' \mapsto u(p \cdot v(g) p') = w(p,g) p')
\]

\[
h : p \cdot \rightarrow (p' \mapsto (p' \mapsto (p' \mapsto (p' \mapsto (p' \mapsto)
\]

Condition I used before, namely, U/AU acyclic.
Review: To set up equiv. between cat of f rings $B$ equipped with map to $A$ and the cat of $A$-modules $B$. First cat is obvious: object, $(\frac{p}{A}, Q, \langle \cdot \rangle : Q \otimes p \rightarrow A)$ maps $P, Q \xmapsto{w, v} P', Q' \rightarrow \langle w(p), v(p) \rangle = \langle v, p \rangle$. Equivalent category: from $\text{Mod}_A$ cat, $(A, Q)$ with $A$ fixed map $(w, v) : (A, Q) \rightarrow (A, Q')$.

2nd cat: Obj $B$ from $t$ with $F : m(A) \rightarrow m(B)$, map $(B, F) \xrightarrow{(w, \theta)} (B', F')$ cons of $w : B \rightarrow B'$ and $\theta : w^* F \sim F'$.

If $(B, F) \rightarrow (B', F') \rightarrow (B'', F'') \rightarrow \ldots$ comp. is $w_{i+1}^* \ldots w^*_1 F = w_1^* \ldots w_i^* F \circ w_{i+1}^* F \circ \ldots$

Equiv. cat Cons. of $B \otimes p$ where $p$ is a fixed invertible $B$-$A$-bimodule.

$B \otimes p \rightarrow B', p' \otimes B \rightarrow p'$

$\phi : B \otimes p \rightarrow B' \otimes p' \Rightarrow Q \otimes p \rightarrow Q'$

$B' \otimes p \rightarrow B' \otimes p' \Rightarrow Q' \otimes p \rightarrow Q' \otimes p'$

$B \otimes p \rightarrow B' \otimes p' \Rightarrow Q' \otimes p \rightarrow Q' \otimes p'$

$\phi : Q \rightarrow Q'$ $B$-$nil$-sem.

$Q \otimes p = A$
Given \((u \ w) : (P, B) \to (P', B')\) form M and.

get iso \(\Theta : B' \otimes P \to P'\)

\[ \Theta(b' \otimes p) = b'u(p) \]

\[ \Theta^{-1}(p' \otimes q) = p'v(q) \otimes p \]

\(\Theta\) in \text{brims iso \ corresponding to} \ w_i F \to F'\)

can be identified with an isom.

hence \(w_i\) quasi- \text{invertible}, \(w_i\) is meghen.

\(\zeta\) is the \text{bi-module iso \ corresponding to} \ the \(\Theta\)-\text{ind iso} \(GW^* \rightarrow G'\)

can be indent with the iso of \text{gives} \text{funs} \(GW^* \to G'\) \text{corresponding iso of} \(\text{M and}\).

get iso of \text{M and}

\[
\begin{pmatrix}
1 & \zeta \\
\Theta & \omega
\end{pmatrix} : \begin{pmatrix}
A & Q \otimes B' \\
B' \otimes P & B'
\end{pmatrix} \to \begin{pmatrix}
A & Q' \\
P' & B'
\end{pmatrix}
\]

Converse: Given \(w : B \rightarrow B'\), \(\Theta : B' \otimes P \rightarrow P'\)

know \(w\) is meghen know \(\zeta : Q \otimes B' \rightarrow Q'\)

\[(\Theta, \zeta) : (w_i F, G \otimes \text{gists}) \rightarrow (F', G').\] \text{iso of pg fms}

Conversely,

\[
\begin{pmatrix}
A & Q \\
P & B
\end{pmatrix} \otimes \gamma \rightarrow \begin{pmatrix}
A & Q \otimes B' \\
B' \otimes P & B'
\end{pmatrix}
\]

\[\gamma(b) = w(b) \otimes p \]

\[\gamma'(b) = q \otimes w(b)\]

what sort of things
2. Given \((\rho B)\) form and \(w: B \rightarrow B'\) morphism \(w: B'\) form
get \(M(A) \sim M(B) \rightarrow m(B')\)

\[
\begin{pmatrix}
A & Q \otimes B' \\
B' \otimes P & B'
\end{pmatrix}
\]

\[
\begin{pmatrix}
b_i \otimes p \otimes b \otimes b' \otimes s_i
\end{pmatrix}
\]

\[
\begin{pmatrix}
\alpha(b_i \otimes p \otimes b \otimes b') = b_i' \otimes w(b_2) \otimes b_2' \\
\beta(b_3 \otimes b_4 \otimes b_5 \otimes b_6) = \beta_3 \otimes b_1 \otimes b_4 \otimes b_5 \otimes b_6 \otimes b_7
\end{pmatrix}
\]

\[
\begin{pmatrix}
q \otimes b_i \otimes b_i \otimes p
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sum b_i \otimes b_j \otimes p_j \otimes b_i' \otimes b_j' \otimes p_j'
\end{pmatrix}
\]

where \(w(b_i) = w(b_2) \otimes b_1' \otimes b_i' \otimes w(b_1)\)

Note that from \(\begin{pmatrix}
A & Q \\
B & P
\end{pmatrix} \xrightarrow{\mu_i \otimes w
\begin{pmatrix}
A & Q \otimes B' \\
B' \otimes P & B'
\end{pmatrix}
\]

\[
\begin{pmatrix}
q \otimes b_i \otimes b_i \otimes p
\end{pmatrix}
\]

\[
\begin{pmatrix}
pos \otimes b_i \otimes b_i \otimes p
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sum b_i \otimes b_j \otimes p_j \otimes b_i' \otimes b_j' \otimes p_j'
\end{pmatrix}
\]

where \(w(b_i') = w(b_i) \otimes b_1' \otimes b_i' \otimes w(b_1)\)
difficulty appears only in moving $B'$ to the right side. From $B' \circ B \rightarrow P'$ you can get

$$B' \circ B \Rightarrow P' \circ A$$

$$Q' \circ B \Rightarrow \Delta Q$$

but the prime is always on the left. So you need an adj. using that $B' \circ B, B \circ B'$ are "adjoint", $B' \circ B$ is invertible, hence the adjunction maps are isos.

$\alpha : FG \rightarrow L \quad \beta : L \rightarrow GF$

\[
\begin{align*}
F &\xrightarrow{F.\beta} FGF &\xrightarrow{\alpha.F} F \\
G &\xrightarrow{G.\beta} GFG &\xrightarrow{\alpha.G} G
\end{align*}
\]

is the identity is id.

\[
\begin{align*}
1 &\xrightarrow{\beta} GF &\xrightarrow{F'.\alpha.F} L \\
F &\xrightarrow{F'} G' &\xrightarrow{G'.\beta.G'} GFG' &\xrightarrow{G'.G.\alpha} G
\end{align*}
\]

The various publications mentioned above.

I propose that the homepage will be edited by a group of topologists representing

How will the homepage be edited?

I propose adjusting our policies to the changes in the environment.
Notation. \((P, Q)\) from dual pair over \(A\), \(B = P \otimes_A Q\) corresponds from \(\Sigma_I\)

We have \(\alpha\) map \(m(A) \simeq m(B)\) given by

\[F = P \otimes_A - , \quad G = Q \otimes_B - \]

\((P', Q')\) another fdp, \(B', F', G'\) same def...

\((u, v): (P, Q) \to (P', Q')\) a map of dual pairs

\(u: B \to B', \quad v(pq) = u(p) v(q)\) corresponds.

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Real Logic

1) Given \((P, Q)\) \(u: B \to B'\) get \((B' \otimes_B P, Q \otimes_B B')\)

describe \(m(A) \simeq m(B) \simeq m(B')\)

2) Given \((P, Q) \to (P', Q')\) get \((u, v): (B' \otimes_B P, Q \otimes_B B') \to (P' \otimes_A Q')\)

int. a map of fdp's yields a map of f rings equipped

with maps to \(A\).

3) Conversely, given \((u, v): (P, Q) \to (P', Q')\) \(u: B \to B'\) and

\(v\) from canonical \(P \to B' \otimes_B P\), \(Q \to Q \otimes_B B'\)

\(b \mapsto u(b) \circ P, \quad g \mapsto g \otimes g \circ w(b)\).
So what happens?

04/24 06:23. I’d like to reconstruct the stability arguments I found years ago. These were based on Stiefel manifolds, analogs of made out of Grassmannian sequences. Review $E$ vector bundles over $X$, when can you split off a trivial line bundle? Method: Choose $G \otimes V \to E$. Use Schur’s things. Mainly

Suppose you want to show $BG_n \to BG_{n+1}$ is a hom. dim. in a certain range. Put $Y = BG_{n+1}$, $X = EG_{n+1} \times G_n (G_{n+1}/G_n)$, so you have a fibration $X \to Y$.

Then analyze à la Groth. $\exists x \times X \to X \to Y$, esp. eq.

Pattern here: $G_{n+1}$ set $\exists (G_{n+1}/G_n)^2 \to (G_{n+1}/G_n)$ and $\to$ what’s going on is you have $G = G_{n+1}$ acting on a simplicial set which is acyclic such that the $\Delta$-set $G_n$ set of vertices is $G_n$.

Example: $G_n = Aut(V)$ unimodular

First work over a field. $G$ acts in $V=0$ trans. with stabilizer $G_{n-1}$ at least nilp. There’s a simplicial set of consisting of frames $\Delta_0$, $

\begin{align*}
\left|GL_2(F_2)\right| &= 3 \cdot 2 = 6 \\
\left|GL_3(F_2)\right| &= 7 \cdot 6 \cdot 4 = 168 \\
\left|GL_2(F_3)\right| &= 8 \cdot 6 \\
\left|GL_2(F_5)\right| &= \frac{24 \cdot 20}{4} = 120 \\
\left|GL_2(F_7)\right| &= 15 \cdot 12
\end{align*}$
April 27, 1977 1546

I have done little mathematics since March 23. Only a few pages on April 5. Tomorrow I think I start some lectures, talk on June 20 looms ahead.

How to get started? Lecture?

Instead look at stability for a field and see if you can work out your old result as well as Huskin’s. What should be the basic idea? First handle mod p homology. p invertible in F.

The key is to consider the simplicial set consisting of indep. sequences $\left( x_0, \ldots, x_p \right)$

0-simple $X_0 = V - \{0\}$
1-simple $X_1$ pairs of mid. vectors. $X_1 \subset X_0 \times X_0$

In general $X_n \subset X_{n+1}$. We get an s.c.

\[ \begin{array}{c}
\exists \quad \mathbb{Z}[X_0] \to \mathbb{Z}[X_1] \to \mathbb{Z} \to 0 \\
\exists \quad \mathbb{Z}[X_1] \to \mathbb{Z}[X_0] \to \mathbb{Z} \to 0
\end{array} \]

Notice no degeneracies. Can make a simplicial set by allowing repetitions. What sort of relations arise? Inside $V$, say $V = F^3$

\[ \begin{array}{c}
\mathbb{Z}[x_3] \to \mathbb{Z}[x_2] \to \mathbb{Z}[x_1] \to \mathbb{Z}[x_0] \to \mathbb{Z} \to 0
\end{array} \]

This complex should be acyclic by general position arguments in degrees < 3. If true, then what sort of result do we get? $G$ acts trans. on $X_p$

$p \leq 3$ say $X_p = G/G_p$