Goal: \( A \rightarrow A/\mathbb{I} = B \) \( IA = 0 \), \( A \) left flat \( B \) h-unital \( \Rightarrow K_A B = K_B \). \( B \) h-unital iff \( B \) it has a resolution by flat \( B \) module, same as a free flat \( \mathbb{B} \) of \( A \)-module. \( \Rightarrow B \) h-unital \( \Rightarrow \mathbb{Z} \otimes_A B = 0 \Leftrightarrow \mathbb{I} \otimes_A \mathbb{R} \). Concentrate.

\[ 0 \rightarrow \mathbb{I} \rightarrow A \rightarrow B \rightarrow 0 \]

exact sequence of \( B \)-module, where \( A \) is flat. Then \( B \) h-unital iff \( B \) is \( h \)-unital \( B \)-module iff \( I \) is an \( h \)-unital \( B \)-module.

Let's analyze this carefully

\[
\begin{pmatrix}
A \\ B
\end{pmatrix}
\]

Somehow I feel that the crux of the problem concerns the homology of \( GL(B) \) acting on \( A \otimes \mathbb{M}(I) \), where \( I \) is a \( B \)-module. What's our motivation?

At one point you studied cyclic homology of an algebra \( A/I = B \) using the DGA \( \mathbb{I} \otimes A \rightarrow A \) resolution of \( B \). And this led to a spectral sequence involving \( [I \otimes_A \mathbb{I}] \). In the present situation \( A \) acts as \( 0 \) on the right. So \( I \otimes_A \mathbb{Z} \otimes_A \mathbb{A} \) and

\[
[I \otimes_A \mathbb{I}] = (\mathbb{Z} \otimes_A I)^{\otimes n}.
\]

In the spectral sequence then you expect

Maybe you want to look at the relative \( HC(A \rightarrow B) \) which gives you

\[
\begin{array}{ccc}
B & A & I \\
\hline
\hline
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{I} & \mathbb{Z} & \mathbb{I} \\
\hline
\hline
\end{array}
\]

So it seems clear that the leading term is \( \mathbb{I} \otimes_A \mathbb{I} \).

So how do I proceed? Somehow the problem will be \( \mathbb{I} \) to invoke?

\[
0 \rightarrow \mathfrak{gl}(\mathbb{I}) \rightarrow \mathfrak{gl}(A) \rightarrow \mathfrak{gl}(B) \rightarrow 0
\]
Then you have all this invariant theory conn. with \( \Lambda \). Important is the grading, the degree in \( I \), so our problem is to understand something about \( H_x(\text{GL}(B)),\ H_x(\text{GL}(I)) \) when \( I \) is a \( B \)-module regarded as a bimodule with \( B \) acting trivially on the right. How now Brown cow.

Suppose \( I \) is a flat \( B \)-module. Should this homology vanish? What's important is the relative homology of \( \text{GL}(A) \to \text{GL}(B) \) whose leading term should be \( \mathbb{Z} \otimes_{\mathbb{B}} I \). Why. Naively, you read

\[
\text{Note: It seems the relative homology of } \\
\mathbb{Z} \otimes_{\mathbb{B}} \text{gl}(A) \to \text{gl}(B) \text{ is zero when } I \\
\text{is a flat } B \text{-module}. \text{ Does this use h-unitality of } B? \text{ Take } B \otimes I. \text{ Should it be true that } K_x(B \otimes I) = K_x(B) \text{ for } I \text{ form flat }\]

Q. Does Myun. of \( K \) for h-unital rings follow from Juslin's results? Some way of lifting excision?

03/05/17: Important I need to study how \( K_x \) behaves for extensions \( 0 \to I \to A \otimes B \to 0 \) such that \( IA = 0 \). This is the same as a \( B \)-mod map \( f: A \to B \). Have a \( \text{Mnd. cont.} \ (A \ B) \) \( \text{Have dual pair } (B, A, A \otimes B) \to B \) \

\[
0 \to I \to \tilde{A} \to \tilde{B} \to 0 
\]

so this is a square-zero extension where right mult by \( B \) on \( I \) is trivial. Thus \( I \) is a unitally bimodule
Classify extensions by $H^2(B, I)$. To how much is clear? I see an analogy with group extensions. It should be possible to see that $H^2(B, I) = \text{Ext}^1_B(B, I) \cong \text{Ext}^2_B(I, I)$

$f: B \otimes B \to I$

$x f(y, z) - f(xy, z) + f(x, yz) = 0$

$D: B \to I$

$D(xy) = Dxy + x D y$

$\text{Der}_B(B, I) = \text{Hom}_B(B, I)$

I want to understand $K_2$ for extensions

$0 \to I \to A \to B \to 0$ at. $IA = 0$

Same as $B$ module extensions of $B$ by $I$, same as square zero unital $B$ extensions of $\tilde{B}$ by the bimodule $I$.

We get $0 \to M(I) \to GL(A) \to GL(B) \to 0$

Note that $GL(2)$ operates here. This is a g/h ext.

I need to study how $K_2$ behaves for extensions $0 \to I \to A \to B \to 0$

such that $IA = 0$. Note $I^2 = 0$ so this is a square zero ext in a category $B$ by the $B$-bi-module $I$ where $IB = 0$. These extensions form a category equivalent to cat of $B$-modules $\mathbf{A}$ equipped with a map into $B$, $A, B$ are $\mathbf{M}$. e.g. $\begin{pmatrix} \tilde{A} & \tilde{B} \\ A & B \end{pmatrix}$$\tilde{B} \otimes A = A$

$A \otimes \tilde{B} = B$.
I don't understand what happens.

In any case you need to examine

\[ A \rightarrow B \implies K_x(A) \]

Review why.

\[ K_x(B) = K_x \left( \begin{array}{c|c} B & 0 \\ \hline 0 & B \end{array} \right) = K_x \left( \begin{array}{c} B \\ 0 \end{array} \right) \]

In general for a sum mant. \((A \oplus B)\) one has

- \(A\) is \( A\oplus B\) flat \(\iff P = P \otimes A \) is \(B\)-flat
- \(B\) is \(A\)-flat \(\iff B = P \otimes B \) is \(A\)-flat
- \(A\) is \(A\oplus B\) flat \(\iff Q = A \otimes Q\) is \(B\)-flat
- \(P\) is \(A\oplus B\) flat \(\iff B = P \otimes B\) is \(B\)-flat.

\[
A = \left( \begin{array}{c|c} 0 & 0 \\ \hline B & B \end{array} \right) = \left( \begin{array}{c} 0 \\ B \end{array} \right) \otimes \left( \begin{array}{c} B \\ 0 \end{array} \right)
\]

\[
\left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) = b_2 b
\]

\[
A \rightarrow A\oplus B\text{ flat } \iff P = \left( \begin{array}{c} 0 \\ B \end{array} \right) \text{ is } B\text{-flat} \iff B \text{ is } B\text{-flat} \iff Q = \left( \begin{array}{c} B \\ 0 \end{array} \right) \text{ is } A\text{-flat}
\]

\[
A \text{ is } A^{op}\text{-flat } \iff Q = \left( \begin{array}{c} B \\ 0 \end{array} \right) \text{ is } B\text{-flat} \iff B \text{ is } B\text{-flat.}
\]

Similarly for

\[
A = \left( \begin{array}{c|c} 0 & B \\ \hline 0 & B \end{array} \right) = \left( \begin{array}{c} 0 \\ B \end{array} \right) \otimes \left( \begin{array}{c} B \\ 0 \end{array} \right)
\]

\[
\left( \begin{array}{c} b_1 \\ b_2 \end{array} \right) = b_2 b
\]

- \(A\) is \(A\text{-flat} \iff P = \left( \begin{array}{c} 0 \\ B \end{array} \right) \text{ is } B\text{-flat} \iff B \text{ is } B\text{-flat}
- \(A\) is \(A^{op}\text{-flat} \iff Q = \left( \begin{array}{c} B \\ 0 \end{array} \right) \text{ is } B^{op}\text{-flat} \iff B \text{ is } B^{op}\text{-flat}

I know in this case that \(K_x(A) \cong K_x(B)\).

The interesting case is when \(A\) is \(A\)-flat?

Idea. Can look at \(C = \left( \begin{array}{c} A \\ B \end{array} \right)\)
Start again. Consider for any $B$-module surjection $A \xrightarrow{f} B$ the ring $A$ with $a_1 a_2 = f(a_1) a_2$, so we have a square zero extension

$$0 \rightarrow I \rightarrow A \xrightarrow{f} B \rightarrow 0$$

where $I$ is a $B$-bimodule such that $IB = 0$. Look at $K_*(A)$. Is $K_*(A)$ a right $\text{Hom}_B(B,B)$? A slight generalization: dual pair $(\tilde{B}, A)$, $A \otimes \tilde{B} \rightarrow B$, but this pairing is same as $B$-map $A \rightarrow \text{Hom}_B(B,B) = B$, so we replace surjectivity of $f$ by requiring $f(A)B = B$.

To simplify consider $B$-module extensions of $B$: $0 \rightarrow I \rightarrow A \xrightarrow{f} B \rightarrow 0$.

Consider a type of square zero extn of rings, and consider the relative $K_*$. Know $K_0(A) \times K_0(B)$. What kind of games can I play? Derived functor game. If $A$ is a lift flat $B$-module, then the result might be independent of $A$. $\begin{pmatrix} \tilde{B} & A \otimes \tilde{B} \rightarrow B \end{pmatrix}$ $\begin{pmatrix} A \otimes B \rightarrow B \\ A \otimes B \rightarrow B \end{pmatrix}$

Certainly we have $Q \otimes_P \tilde{B} = \tilde{B} \otimes A = A$ but $P \otimes_A Q = A \otimes \tilde{B} \rightarrow B$, $a \otimes b \rightarrow f(a) b$

$$f(a_1) b_1 f(a_2) b_2 = f(a_1 f(a_2) b_2)$$

$$f(a_1 b_1 f(a_2) b_2) = a_1 f(a_2) b_2$$
I am very confused.

For any $A \to B$ $B$-mod weij consider $K_\ast(A)$

If $A$ and $B$ are left flat, then $K_\ast(A) \to K_\ast(B)$.

This is OK if things are idempotent.

Why is true? $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ very things left flat

Point $B = A \otimes B$ and $A$ $A$-flat $\Rightarrow U_{A}A = A$

so $B$ acts on $B$

Assume $(A \otimes B)$ form and $A$ and $B$ are left flat rings. Then $P \otimes A = A \otimes A = A$ is $B$-flat or $A$ has right action on $A$ which is left $B$-flat, so get $K_\ast(A) = K_\ast(P \otimes B)$ $\to K_\ast(B)$. Now $B$ is $B$-flat $\Rightarrow Q \otimes B = B \otimes B = B$ is $A$-flat and we get $K_\ast(B) = K_\ast(P \otimes Q)$ $\to K_\ast(A)$ defined by the right action of $B$ on $B$

look at $K(A) = H_1(GL(A))/H_1(GL(Z))$

I want to find some sly methods for handling this stuff.

Look at $K_1(A) = 0$

I wonder if you're following a dead end. Should you be using finite sets and partial orderings. Volodin style.

Look at cyclic homology $\to I \to A \to B \to 0$

$DGA$ $C_\ast(I \to A)$ leads to $Fib (C_\ast(A) \to C_\ast(B))$

having a filt. with quotients $[I \otimes A]_n = (Z \otimes A)^n$
197 so if $I$ is a flat $A$-module, then it would seem that these cyclic tensor products reduce to $(I/\mathcal{A})^r$. Let's examine lower degrees.

$$\text{HC}_r(A) \to \text{HC}_r(B)$$

So the real puzzle. How to get a real understanding

$$0 \to I \to A \to B \to 0$$

$$0 \to \text{gl}(I) \to \text{gl}(A) \to \text{gl}(B) \to 0$$

$$\text{gl}(I) \to \text{gl}(A) \to \text{gl}(B)$$

$$S \Sigma \text{gl}(I) \to S \Sigma \text{gl}(B)$$

use invariant theory

$$\Lambda^p (\text{gl}(B) \otimes \mathbb{F})$$

Then take of coinvariants, of primitive

What about the representations of $\mathfrak{g}$?

Important seems to be first you embed $B$ in $\tilde{B}$ then take $\text{gl}(B)$ and then reduce with the reductive subalgebra $\mathfrak{g}$. This amounts to relative Lie alg homology $(\text{gl}(B), \mathfrak{g})$. One knows this is the same as the homology of $\text{gl}(B)$. So it seems that one is actually working with the squash new unital ring extension $0 \to I \to \tilde{A} \to \tilde{B} \to 0$. Question: When is the Lie alg homology of $\otimes \text{gl}(B)$ the same as the relative Lie alg hom of $(\text{gl}(B), \mathfrak{g})$? This should be the Lie version of excision, hyp. should be $B$ bi-unital. So there
some way I should be able to handle this?

So now there arises the question whether in Lie algebra homology you can see things like $h$-unital. You want $H_x(\mathfrak{gl}(B)) \to H_x(\mathfrak{gl}(\bar{B}), \mathfrak{gl}(k))$. This is what Suslin understands.

Problem. Defining $K_\mathfrak{h}(A)$ internally. The $K$-theoretica

relative

Working with $C_\mathfrak{h}(A)$ amounts to the Lie homology of $(\mathfrak{gl}(\bar{A}), \mathfrak{gl}(k))$. Hanlin said that if you want Lie homology of $\mathfrak{gl}(A)$ for $A$ non-unital, then it involves the bar homology of $A$. $\mathfrak{gl}(\bar{A}) = \mathfrak{gl}(k) \oplus \mathfrak{gl}(A)$

semi-direct product. An exact sequence

$$0 \to \mathfrak{gl}(A) \to \mathfrak{gl}(\bar{A}) \to \mathfrak{gl}(k) \to 0$$

$$H_\mathfrak{h}(\mathfrak{gl}(k), H_\mathfrak{h}(\mathfrak{gl}(A))) \to H_x(\mathfrak{gl}(\bar{A}))$$

$$H_p(\mathfrak{gl}(k)) \otimes H_\mathfrak{h}(\mathfrak{gl}(A)) = \mathfrak{gl}(k)$$

So you are not going to learn much about $H_\mathfrak{h}(\mathfrak{gl}(A))$.

What can we do?

$$\Lambda^p \mathfrak{gl}(A) = \Lambda^p (A \otimes A) = \left( \mathfrak{gl} \otimes \otimes A \otimes \otimes \right)$$

Is there a way to obtain the part corresponding to an irreducible rep $\rho$ of $\mathfrak{gl}(k)$. $\otimes \otimes = \vee \otimes \otimes \left( V^* \right) \otimes \otimes$

Take an irreducible $\rho$ of $\mathfrak{gl}(k)$. The idea is to tensor with $\mathfrak{gl}(k)$ and look at the result as a rep of $\Sigma_k$.

Look at $C = \Lambda \mathfrak{gl}(A)$ the complex of Lie chains on $\mathfrak{gl}(A)$. Conjugation action by $\mathfrak{gl}(\bar{k})$.
back to finite operations of reductive we have
\[ C = \bigoplus W_i \otimes \text{Hom}_\mathbb{C} (W_i, C) \]
where \( W_i \) ranges over the irreducible reps of \( g_f \). But now you want to use the fact that the irreps
\( g_f = V \otimes V^* \) are given somehow by reps.

of symmetric groups. \( g_f = \text{End}(V) \to \text{End}_{\Sigma_n} (V^{\otimes n}) \). What is the relation? Double comm. thm. says \( \text{End}(V^{\otimes n}) g_f = k[\Sigma_n] \)

\( V^{\otimes n} = \bigoplus W_i \otimes \mathbb{C} \).
There are some other things known. If \text{dim}(V) \geq 2 \), so that \( k[\Sigma_n] \to \text{End}(V^{\otimes n}) \), then each irreducible rep of \( \Sigma_n \) must occur in \( V^{\otimes n} \). So what do you want?? Go back to
\[ \Lambda^p (g_f \otimes A) = (g_f \otimes g_f \otimes \cdots) \Sigma_p \].
Now \( g_f \) acts on this

and we need somehow to describe non-trivial reps

of \( g_f \) occurring in this complex. You want to take \( W_i \) an irreducible rep of \( g_f \) and form \( \text{Hom}_{\mathbb{C}} (W_i, C) \). \( W_i \)

should occur in \( V \). This is nice

Yesterday I looked at Lie hom. of \( g_f(A) \). A moment later, a problem understood by Hanlon. \( g_f = \text{gl}(k) \) acts by adj.
on \( g_f(A) = g_f \otimes A \), and one can split the Lie chain ex

which is \( \Lambda^p (g_f \otimes A) \) into \( g_f \)-invariant subcomplexes according to the irreducible reps. of \( g_f \).

that the trival rep of \( g_f \) yields \( \Lambda^0 (g_f \otimes A) \), which reduces to the cyclic complex \( C_1(A) \). To handle a

general component, you need to use the description of irreducible \( g_f \)-modules.
If you do all this multilinear algebra, then you should find the result that the Lie homology \( gl(A) \) is given by the cyclic complex iff \( A \) is \( h \)-unital.

Let's go back and write things up. The main construction:

Given \((P, Q, \mathbb{Q}_n \to \mathbb{A})\) a dual pair over \(\mathbb{A} \) such that \(P\) is \(A^\text{op}\)-flat, one has a canonical map:

\[
K_* (P \otimes_A Q) \to K_* (\mathbb{A}).
\]

Properties:

\[
(\mathbb{P}, Q, < \cdot, \cdot>) \to (\mathbb{P}', Q', < \cdot, \cdot>)
\]

This naturality property means functional in \(P\) keeping \(Q\) fixed and functional in \(Q\) keeping \(P\) fixed.

Use \(K_*\) compat with filtered \(\lim\)'s. + fact that flat means \(P\) filtered limit of free \(A^\text{op}\)-modules.

If \(P\) free i.e. \(\mathbb{A}^n\), then have homom.:

\[
(P, Q) \to (P, P^\vee) \quad P^\vee = \text{Hom}_{A^\text{op}}(P, \mathbb{A})
\]

\[
\mathbb{P} \otimes_A Q \to \mathbb{P}^\vee \otimes_A Q = M_n A
\]

Put another way for \(P \in P(A^\text{op})\) one has an isom:

\[
K_* (\text{End}_{A^\text{op}}(P)) \to K_* (\mathbb{A}),
\]

\[
\mathbb{P} \otimes_A Q \to \text{End}_{A^\text{op}}(P)
\]

This makes naturality in \(\mathbb{Q}\) clear.
So what do we learn? If we have 
\((P, Q, Q \otimes P \to A)\), then we get 
\[ Q \to \text{Hom}_A(P, A) \text{ and } Q \otimes Q \to \text{Hom}_A(P, A). \]
When \(P\) is finite free, then \(Q \otimes \text{Hom}_A(P, A) = M_n(A)\).

So we get \(K_0(Q \otimes A) \to K_0(A)\), actually a little bit more. Namely a map into a possible \(K\)-theory defined using \(GL(A)\). What about a map \((P, Q) \to (P', Q')\)?

Again you factor \(P \to P'\) two cases are

\[ P \otimes Q \to P' \otimes Q \to \frac{P}{P'} \otimes Q \]

\[ K \otimes Q \to P' \otimes Q \to \frac{P}{P'} \otimes Q \]

Somehow these things are affine. Better is to compare the action of \(B = P' \otimes Q\) on \(P\) and \(P'\). In the first case you have \(0 \to P \to P_1 \to \frac{P}{P} \to 0\) and the \(B\) action on \(\frac{P}{P}\) is trivial \(\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\).

In second case action on subrep is trivial \(\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}\) which flips to \(\begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix}\).

Now is there something in some way when \(P\) not free. The problem here is that you could also take \(P\) pseudo free? \(P = \mathbb{A}^n\) instead of \(P = \mathbb{A}^n\).

So there some way to

Suppose the homology of \(A\)

Yet the proof in a good form.
Consider a simplicial scheme $(\Delta, q = A)$.

**Given** $P \to A$ an $A^\text{op}$ module map with $P = A^n$. To calculate $K_\ast(P \otimes A) \to K_\ast(A)$.

Really, $P \otimes A \to P \otimes \text{Hom}_{A^\text{op}}(P, A)$. Good problem.

Assume $A$ either left or right flat and idempotent, show that $\text{BGL}(A)^+ \to \text{fibre} (\text{BGL}(A)^+ \to \text{BGL}(I)^+)$ is a hnp. Keep on trying.

So back to $P \to A$ an $A^\text{op}$-map with $P$ free fin. Then get $P \otimes A$ acting on the $A^\text{op}$-module $P$, so

$$P \otimes A \to P \otimes_{A^\text{op}} \text{Hom}_{A^\text{op}}(P, A)$$

See I am somehow approximating $A$ by $P \otimes A \cong A^n$ with a funny mult. Then I have two representations of $P \otimes A$ in $P(A^\text{op})$ namely $\tilde{A}$ via the hom.

$P \otimes A \to A \subset \tilde{A}$ and $P$ via the action $\tilde{A}$ on $P = A^n$ which gives a hom $P \otimes A \to M_n(A)$. This argument shows that the trace map $K_\ast(A) \to K_\ast(A)$ assoc. to $(A \quad A)$ is the identity.

**A couple point you have to decide**

**Given** $A$ flat, say right flat. Use $(\Delta, q = A)$ get $K_\ast(P \otimes A, q) \to K_\ast(A)$ which should be the identity. So start by replacing $P$ by a free module $\tilde{A}$ approx. $A^n \to A$.

Then get $P \otimes A = A^n \otimes A = A^n$ with prod. $\tilde{a}_1 \otimes \tilde{a}_2 = \tilde{a}_1 \langle a_1, a_2 \rangle$ in general $(p_1 q_1)(p_2 q_2) = p_1 \langle q_1, p_2 \rangle q_2 = p_1 f(p_2 q_2) = p_1 f(p_2) q_2 = p_1 f(p_2 q_2)$.
So \( P = A^n \) is a ring and \( P \otimes_A P = P \otimes_A \mathcal{H} \). Also have homomorphisms:

\[
P = P \otimes_A A \rightarrow \text{End}_{\mathcal{H} P}(P, P)
\]

\[
p \otimes a \mapsto (p' \mapsto p \langle a, p' \rangle)
\]

Then you have two homs from \( P \) to matrices over \( A \), which you should be able to relate via exact sequences. Yes, the idea involves the map of dual pairs:

\[
(P, A, \langle a, p \rangle = \alpha f(p)) \rightarrow (\tilde{A}, A, A \otimes \tilde{A} \rightarrow A)
\]

\[
\tilde{a} \langle \tilde{a}, p \rangle = q \tilde{a}
\]

Actually these are really the affine groups it seems i.e. you add a line or column. And that's interesting because it's a consequence of taking \( Q \), the left module in the dual pair, to be \( A \).

Maybe you even learn something, namely, you can now take \( P \) to be \( A^n \).
Are there any implications of this argument when $A$ is not flat? Suppose we have a map $A^n \rightarrow A$ over $A^P$. Then $\text{P} \otimes_A A = \text{P}$ equipped with $p_1 p_2 = p_1 f(p_2)$. This is acted on the left by $\text{P}$ so we get $\text{P} \rightarrow M_\alpha(A)$?

$p \cdot p' = p f(p'), \quad p' \mapsto |p\rangle < f | p'\rangle$

is this a matrix, i.e. given $a \in A$ and $f \in \text{Hom}_A(A, A)$ is $af: a' \mapsto a f(a')$ null by an element of $a$.

I need insight, say from cyclic homology.

Critical case: $B$ h-unital, choose $A \rightarrow B$ any $B$-module map where $A$ is flat from over $B$.

$A$ is $B$-flat $\iff$ $Q_B A = B_B A = A$ is $A$-flat. It's in this situation that you must link $K_x(B)$ to $K_x(A)$. Here's an idea to try. $A$ is (constructed as an ind. limit of free $B$-modules). So can you construct $A$ start with?

$F \rightarrow A F \leftarrow F \rightarrow \cdots$

$M = \cdots$.

Do this starting with $B$ namely choose a free $B$-module $F$ with surj $F \rightarrow B$. Then you have the dual pair $B, F$, $F \otimes_B B$.

$\text{B} \otimes_F \text{B} \rightarrow \text{B}$

$\text{B} \otimes_F \text{B} = \text{BF}$

$A = \left\{ \begin{array}{l} \text{free} \ F_n \ \\
  \text{extremely concrete} \ A \otimes_A F_n \end{array} \right.$

semi-empirical approach. Start with $B$ h-unital then I have flat resolution of $B$ so that can be converted by Bock-Path to a semi-simp. mod over $B$

$A_0 \rightarrow A_1 \rightarrow \cdots$. 
Then get $s.s.$ exact sequence $B \text{s.s. ring } A$ of square-zero parts of $B$.

$$GL(A_n) \rightarrow GL(A) \rightarrow GL(\tilde{A}) \rightarrow GL(\tilde{A})$$

clearly exact because the kernels to $GL(\tilde{A})$ are just matrices. Then $\exists$ Spec sequence - apply $B = \mathbb{W}$ vertically get double s.s. set, apply $\mathbb{Z}$. Point is that $H_\ast(GL(A_n))$ mod. $f \leq 0$. Hence the $A_n$ are flat and $A$ may. So you get $H_\ast(Gl(A_n)) \rightarrow H_\ast(GL(\tilde{A}))$ so $H_\ast(A_n) = H_\ast(\tilde{A})$.

What happens when $B$ not h-unital.

03/07/99. Can you show that if $B$ is left flat, then the map $BGL(\tilde{B}) \rightarrow$ Fiber $\{BGL(\tilde{B}^+) \rightarrow BGL(\tilde{B})\}$ induces an isom in homology?

$$1 \rightarrow GL(B) \rightarrow GL(\tilde{B'}) \rightarrow GL(\tilde{B}) \rightarrow 1$$

$$BGL(B) \rightarrow BGL(\tilde{B}) \rightarrow BGL(\tilde{B}) \quad \text{fib}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$\text{Fib} \rightarrow BGL(\tilde{B})^+ \rightarrow BGL(\tilde{B})^+$$

$$E_2^{pq} = H_p(BGL(\tilde{B}), H_q(BGL(B))) \Rightarrow H_\ast(BGL(\tilde{B}))$$

$$H_p(BGL(\tilde{B})^+, H_q(BGL(\text{Fib})) \Rightarrow H_\ast(BGL(\tilde{B})^+)$$

Use comparison theorem. Induction will prob. yield

$$H_0(BGL(\tilde{B}), H_q(BGL(B))) \Rightarrow H_0(BGL(\tilde{B})^+, H_q(BGL(B)))$$

Thus you need $GL(\tilde{B})$ to act trivially on $H_\ast(BGL(B))$.

Continue with your flat res.

Suppose $Z^B_B \neq B$ begins in degree $n$.

e.g. $B$ Zam ideal.

$\text{Fib} \quad n = 2$

$\text{Fib} \quad n = 3$.

Same as $B$ being a fib.
$$M = A \otimes A \otimes M \otimes \otimes \otimes \otimes \otimes$$

$$F_0 \rightarrow M$$

$$F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \quad \text{for } p \gg p.$$
Start with $B$ idempotent and construct a flat resolution $A_n \to F_n \to F_{n-1} \to \cdots \to B \to 0$. Convert to a simplicial res.

$$
\begin{align*}
A_n & \equiv A_{n-1} \\
\downarrow & \downarrow \\
B & \equiv B = B
\end{align*}
$$

What exactly do we know?? For each $n$ one has $0 \to I_n \to A_n \to B \to 0$ an extension of $B$-modules with $A_n$ flat.

So $A_n$ is a ring $\mathfrak{m}a' = (\mathfrak{m}a)a'$. Let $G_n = GL(A_n)$. Then $\{G_n\}$ is a simplicial group.

$$
1 \to GL(I_n) \to G_n \to GL(B) \to 1
$$

First suppose $B$ is unitary, whence $F_\ast$ is a resolution of $B$. In general $GL(I_\ast)$ is the simp abelian group of matrices with entries in $I_\ast$. $F_\ast$ resolves $B \to I_\ast$ is acyclic $\iff$ $GL(I_\ast)$ acyclic $\iff$ $G_\ast$ resolves $GL(B)$.

Consider the double simp $\bigwedge^p G_p$.

$$
G_2 \times G_1 \times G_0 \to GL(B) \times GL(B)
$$

applying $Z[-]$ and taking long homology yields $H_q = \{0\}$ for $q > 0$. So $\bigwedge G_p$ degenerates yielding total homology is $H_x(GL(B))$ group homol.
In other direction we get

\[ H^\vee_0(G_0) \]

get spec. sequence \( E_2 = H_p(H^g_0(G_\ast)) \)

but we know that \( n \rightarrow H^g_0(G_n) \) is constant fun.

\[
\therefore \text{ find } A_0 \rightarrow B \text{ induces } H_0^g(GL(A)) \rightarrow H_0^g(GL(B)).
\]

Now suppose that \( F_\ast \) is a res. mod nil modules of \( B \). Then each \( A_n \) is flat so the second spectral sequence degenerates yielding \( H_\ast^g(GL(A_n)) \forall n \). Next what happens is that

the rows are no longer acyclic. In fact what do we know?

\[
\begin{array}{c}
\mathbb{Z}[G_2] \xrightarrow{\mathbb{Z}[G_1]} \mathbb{Z}[G_0] \twoheadrightarrow \mathbb{Z}[GL(B)^2] \\
\downarrow \\
\mathbb{Z}[G_2] \xrightarrow{\mathbb{Z}[G_1]} \mathbb{Z}[G_0] \twoheadrightarrow \mathbb{Z}[GL(B)]
\end{array}
\]

It seems that the \( p \)th row is the \( (p+1) \)th tensor product of the simp. ab. group \( \mathbb{Z}[G_\ast] \). Add the ring to \( \mathbb{Z}[GL(B)^{p+1}] \). So we have a complex \( \mathbb{Z}[G_\ast] \) with \( H_0(\mathbb{Z}[G_\ast]) = \mathbb{Z}[GL(B)] \). Look at the first non vanishing homology group. You have a simplicial group \( G_\ast \) with \( \pi_0(G_\ast) = GL(B) \) and

\[
\therefore \text{ with } \pi_1(G_\ast) = \pi_1(I_\ast) \text{ look at } \pi_1.
\]

\[
\begin{array}{c}
1 \rightarrow I_1(\pi_1) \rightarrow G_\ast \rightarrow GL(B) \rightarrow 0
\end{array}
\]

This is unnatural a bit. If \( B \) is prim., then \( \pi_0(I_\ast) = 0 \)
$0 \to I_0 \to F_0 \to B \to 0$

$\otimes B$ from iff $F_0 = B I_0$ in

which case $F$ maps onto $I_0$.

We have to be a little careful because of

the simplicial stuff. You start with $F_0 \to B$

then choose $F_1$

\[\begin{array}{ccc}
F_0 & \to & F_1 \\
\downarrow & & \downarrow \\
F_0 & \to & F_0
\end{array}\]

\[A_2 \to A_1 \to A_0 \to B \quad \text{the semi s. res.}\]

\[G_n = GL_\mathbb{Z}(\hat{A}_n)\]

\[0 \to I_n \to A_n \to B \to 0\]

\[\begin{array}{ccc}
I_0 & \to & I_0 \\
\downarrow & & \downarrow \\
F_0 & \to & F_0
\end{array}\]

\[F_0 \otimes F \quad F_0 \otimes F \to F_0 \]

\[B = B \quad \text{YES}\]

in any case $0 \to I_n \to A_n \to B \to 0$

Any we have this complex.

$\to F_n \to \cdots \to F_0 \to B \to 0$

\[\begin{array}{ccc}
I \to I \\
\downarrow & & \downarrow \\
F_0 & \to & F_0
\end{array}\]

\[\begin{array}{ccc}
A_0 & \to & A_0 \\
\downarrow & & \downarrow \\
B = B
\end{array}\]

\[\overset{\text{where homology starts in degree } n}{\text{One can think of } F_0 \text{ as a DG alg where right}}\]
\[\text{mult is nonzero only in } F_0 \text{. Construct simp. ring.}\]

$\to A_0 \to \cdots \to A_0 \to A_0 \to B \to 0$

$B = B$
$G_n = \text{GL}(\tilde{\mathcal{A}}_n)$ \quad m(I_n)

$G_2 \quad G_1 \quad G_0 \quad m(I)$

$G_2 \Rightarrow G_1 \Rightarrow G_0 \Rightarrow \text{GL}(\mathcal{B})$ = $\text{GL}(\mathcal{B})$)

So $M(I_n)$ is a simplicial abelian gp.

Now I need to calculate the homology of the simp. gp. $G$. Made of

\[ G^3 \Rightarrow G^2 \Rightarrow \text{GL}(\mathcal{B}) \Rightarrow, \]

Basic is you have something like a top gp with

$\pi_n = \text{GL}(\mathcal{B})$ and $\pi_n = M(I_n)$

$M(I_n I)$.

Enough to drive you nuts!

$M(I) \Rightarrow GL(\tilde{\mathcal{A}}) \Rightarrow GL(\mathcal{B})$.

You have a top gp $G$ with $\pi_0 G = \text{GL}(\mathcal{B})$ and $\pi_i G = 0, \ 0 < i < n$

$\pi_n G = M(\pi_n I)$

$H_n(F_i) = \ker(F_i \rightarrow F_{i-1})/\text{im}(F_{i+1} \rightarrow F_i) = M_{n+1}/B_{n+1}$
It seems that

\[ 0 \to M_n \to F_{n-1} \to F_{n-2} \to \cdots \to F_1 \to F_0 \to B \to \mathcal{O} \]

\[ \text{Tor}_n^B(\mathbb{Z}, B) \xrightarrow{\sim} M_n / BM_n = H_{n-1}(F) \]

so if the bar homology \( \text{Tor}_x^B(\mathbb{Z}, B) \) begins in degree \( n \), this is \( H_{n-1}(F) \). e.g. if \( B \) is fibrant the bar homology begins in degree 2, and this is \( H_1(F) \) can be \( \neq 0 \).

Change \( n \) to \( n+1 \). Then bar homology begins in degree \( n+1 \), i.e. \( \text{Tor}^B_{n+1}(\mathbb{Z}, B) = 0 \iff H_k(F) = 0 \) for \( 0 \leq k < n \).

\[ 0 \to B \to F_0 \to B \to \mathcal{O} \]

Assume \( \text{Tor}_0^B(\mathbb{Z}, B) = 0 \)

\[ \text{Tor}_{n+1}^B(\mathbb{Z}, B) \implies M_{n+1} / BM_{n+1} = \frac{K_n(F_n \to F_{n+1})}{B} = H_n(F). \]

So going back to \( G = \text{GL}(\tilde{A_x}) \) top. gp.

with \( \pi_0(G) = \text{GL}(\tilde{B}) \) \( \pi_i(G) = 0 \) \( 0 < i < n \)

and \( \pi_n(G) = M(H_n(F)) = M(\text{Tor}_n^B(\mathbb{Z}, B)) \).

If this is true, then what do we find about \( H_*(\bar{W}(G)) \)?

The point I guess is that the simp. gp. \( G \) leads to a fibering

\[ B(G^{(e)}) \to BG \to B(\pi_0 G) \]

\[ \text{GL}(\tilde{B}) \]

starts in degree \( n \) with \( n \)th bar homology \( H_n(\text{GL}(\tilde{B})), M(\text{Tor}_n^B(\mathbb{Z}, B)) \) is some sort of obstruction.
Real problem: show that $A$ flat $\Rightarrow$ $GL(A)$ acts trivially on $H_*(BGL(A))$. Can show following:

\[ H_*(GL(A)) = \varinjlim H_*(GL(P \hat{\otimes} A)) \]

where $P \hat{\otimes} A$ is a filtered colimit of finite free $\hat{A}^p$-modules equipped with $P \rightarrow A$.

\[ P \hat{\otimes} A \rightarrow P \hat{\otimes} Hom_{\hat{A}^p}(P, A) = M_n(A) \text{ for } n \geq 0. \]

Suppose $A \in P(\hat{A}^p)$. Then we have a rep of $A$ by left mult., on $A \in P(\hat{A}^p)$. Get

\[ \hat{A} \rightarrow \text{End}_{\hat{A}^p}(P) \]

What is the point? The point may be that since $A \in P(\hat{A}^p)$, when one chooses $A \rightarrow \hat{A}^n$ to calculate

\[ K_*(\hat{A}) \rightarrow K_*(\text{End}_{\hat{A}^p}(A)) \rightarrow K_*(\hat{A}) \]

one actually factors thru $O$

\[ GL(\hat{A}) \rightarrow GL(\text{End}_{\hat{A}^p}(A)) \rightarrow GL(\hat{A}) \]

Problem: Assume $A \in P(\hat{A}^p) \quad \text{i.e. } f.g. \quad \text{free over } \hat{A}^p$

and $\hat{A}^p = A$. Recall $P \rightarrow P \hat{\otimes} A$, $P(\hat{A}^p) \rightarrow P(\hat{A}^p) \subset P(\hat{A}^p)$ induces a map $K_*(\hat{A}) \rightarrow K_*(\hat{A})$ which is idempotent.

Basic properties:

\[ U \in P(\hat{A}^p) \]

\[ 0 \rightarrow U \hat{\otimes} A \rightarrow U \rightarrow u/Au \rightarrow 0 \]

\[ 0 \rightarrow \bar{u} \hat{\otimes} \bar{A} \rightarrow \bar{u} \hat{\otimes} \bar{A} \rightarrow u/Au \rightarrow 0 \]
This tells me that

\[ 0 \rightarrow A \rightarrow \tilde{A} \rightarrow Z \rightarrow 0 \]

This business may be hard. I know that the triviality of affine group homology. I want to prove these results for $BGL(\tilde{A})^+$ when $A$ is flat.

Assume $A \in P(A^p)$. Suppose we start with $A \in P(A^p)$ i.e. $f$

\[ A \leftrightarrow \tilde{A} \rightarrow \tilde{A}^n \]

Then the functors defined on $P(A^p)$

\[ U \rightarrow U \otimes \tilde{A} \]

I get a homomorphism \[ \tilde{A} \rightarrow \text{End}_{A^{op}}(\tilde{A}^n) = M_n(\tilde{A}) \]

\[ \bar{a} \mapsto x \bar{a} y \]

This gives me a homomorphism and it apparently means that
Consider $P \to A$. Let $\tilde{A}^op$. Then $P \otimes_A A \to \tilde{A} \otimes_A A$.

What I am trying to calculate is the map $K_*(A) \to K_*(A)$ associated to

$$\begin{pmatrix} A & A \\ A & A \end{pmatrix}$$

You take as definition

$$K_*(A) = \ker \left[ K_*(\tilde{A}) \to K_*(\tilde{A}) \right]$$

so it's a functor of a non-unital ring. Then you assume

$$\lim_{\to} K_*(P \otimes_A A) = K_*(A)$$

Okay,

define $K_*(A) \leftarrow K_*(P \otimes_A A) \to K_*(A)$

Commutes

$$K_*(\tilde{A} \otimes \text{End}_{A^op}(P)) \cong K_*(\tilde{A})$$

What happens is that we seem to have a proof that $\text{ }$

 tomorrow you try to get $H_*(\text{GL}(A)) \to H_*(\text{GL}(A)^* \to \text{BGL}(B))$

03/08/97

Problem: Given two flat finitely ring $A, B$ which are $\mathbb{Z}$, is it true that $H_*(\text{GL}(A)) \sim H_*(\text{BGL}(B))$??

The case to look at carefully should be $A \otimes \text{BGL}(B)$?

In general I would expect that the arguments...
which I gave for \( K_x(A) \cong K_x(\tilde{A}) / K_x(\tilde{Z}) \)

should go through for \( H_\ast (BGL(A)) \) provided we have

\( A \to (\begin{pmatrix} A & A \\ 0 & 0 \end{pmatrix}) \) and \( A \to (\begin{pmatrix} A & 0 \\ A & 0 \end{pmatrix}) \) induce issues on \( H_\ast (BGL(-)) \).

Then I can use the simplicial group trick to handle the \( h \)-universal cases, and probably also

\( B \to A \otimes B \to B \)

for \( b \in B \).

see what happens for \( A \in \mathcal{P}(A^op) \),

\[
\begin{pmatrix} A & B \\ A & B \end{pmatrix}
\]

\( B = \text{Hom}_{\mathcal{A}^op}(A, A) = \mathbb{A} \otimes \text{Hom}_{\mathcal{A}^op}(A, A) = A \otimes B , \)

and \( B \otimes B A = A \) because \( B \) is universal. But in this case \( A \) amounts to a flat unitary \( B \)-module with \( B \)-map \( A \to B \) such that \( f(A) B = B \), so we can reduce by showing to the case of \( A = B^2 \), and the result is clear.

Next suppose \( A \) is flat and map a universal ring \( B \),

\[
\begin{pmatrix} A & Q \\ P & B \end{pmatrix}
\]

Then \( A \) arises from the closed

pair \( (Q, P, P \otimes Q \to B) \) over \( B \)

Wait: Go back to \( A \in \mathcal{P}(A^op) \),

\[
\begin{pmatrix} A & B \\ A & B \end{pmatrix}
\]

\( A \) is a \( B \)-module equipped with \( A \otimes B \to B \),

a \( B \)-bimodule map, i.e., \( A \to \text{Hom}_{\mathcal{B}^op}(B, B) = B \), i.e.,

\( A \otimes B \to B \) has the form \( a \otimes b \to f(a)b \), where \( f \)

is a \( B \)-module map. \( B \) is \( B \)-flat \( \Rightarrow B \otimes_B A \approx A \) \( B \)-flat

(in fact finite proj). Thus \( A \) can be any \( B \)-module equipped with \( B \)-map \( f : A \to B \) whose image (which is a left ideal)

(\( B \) as \( \mathcal{A} \) ideals).

Then the only filtration by filtered ind

limits can support \( A \) is finitely presented \( B \)-module.
Special case to be well understood.

\[ A \in \text{P}(A^p), \quad B = \text{Hom}_{A^p} (A, A) \]

Put another way, \( B \) is a unital ring, \( A \) is a unitary \( B \)-module equipped with a \( B \)-map \( f: A \to B \) such that \( f(A)B = B \). I now need to understand \( H_*(BGL(A)) \). Now I know that \( A \in \text{P}(A^p) \), in particular \( A \) is right flat. Consider

\[
\begin{pmatrix}
A & B \\
A & B
\end{pmatrix}
\begin{pmatrix}
B, A, A \otimes B & B \\
\otimes b & f(b)
\end{pmatrix}
\]

I know that

\[ A \text{ is } A \text{-flat} \iff \text{P}_A \otimes A = A \text{ is } B \text{-flat}. \]

Examine this case first.

But because \( B \) left acts on \( A \in \text{P}(A^p) \), we should get a \( \text{homom.} \quad B \to M_n(A) \). To simplify suppose \( A \subseteq B \) is a left ideal such that \( AB = B \), say \( y \cdot x = 1 \) \( \forall y \in A, \ x \in B \). Then we have

\[ \begin{array}{ccc}
A & \xrightarrow{y \cdot} & A \\
\downarrow \quad x & & \downarrow \quad x
\end{array} \]

Given \( b \in B \), then have \( b \) on \( A \), send \( b \) to \( x \cdot b \cdot y \). This is a homom. \( \phi: B \to A \).

Now ask about compositions

\[ A \to B \xrightarrow{\phi} A \xrightarrow{} B. \]

It's probably true for the old arg. that \( \phi \) induces the identity \( \text{H}_*(GL(B)) \), also that \( \phi f: A \to A \) induces the identity on \( \text{H}_*(GL(A)) \).
form dual pair over $B$ 
$B$ is unital 

$$\sum y \otimes x \in A \otimes B : \sum f(y) x = 1.$$  

To simplify suppose $f : A \to B$ is injective, whence $A$ is a left ideal in $B$ generating $B$ as an ideal. Also suppose $y \otimes x \in A \otimes B$, $yx = 1$. Then one has

$$A \xrightarrow{f} B \xrightarrow{\phi} A \xrightarrow{x} B$$

aim to show $\phi f : A \to A$ and $f \phi : B \to B$

$$a \mapsto xay \quad + \quad b \mapsto xby$$

induce maps on $\text{Hom}(BGL(-))$.

What is going on here? You have $A \in \mathcal{P}(A^\vee)$

so $A$ is a direct summand of a free module

$$0 \to A \xrightarrow{y} \tilde{A} \to \tilde{A}/xA \to 0$$

The limit $A \to B$ corresponds to $A$ acting on itself, $B \to \tilde{A}$ corresponds to $B$ acting on $A$ transported to the summand $yA$ of $\tilde{A}$. So what happens?

Thus we need to be able to compare the two $A$ actions on $\tilde{A}$, namely left mult. by $a$ and left mult. by $xay$.  
One repn. is $\tilde{A}$, the other is $\tilde{A} \otimes A$.

**Theorem**

Take $P = A$. Then you have $P \otimes_A A$ left acting on $P$ over $A^\text{op}$, and you have $P$ acting on $\tilde{A}$ through the homomorphism $P \otimes_A A \to \tilde{A}$. Draw a red arrow for action of $B = P \otimes_A A$.

$$P \xrightarrow{\text{red}} P \otimes \tilde{A} \xrightarrow{\text{red}} \tilde{A}$$

I'm thinking of functors from $P(\text{A}^\text{op})$ to $\text{B}?$$$

You have $P \otimes_A A$ acting on $P$ and on $\tilde{A}$.

Here $P$ and $\tilde{A}$ are in $P(\text{A}^\text{op})$. You represent $P$ and $\tilde{A}$ by $U$.

Start again, $(A, B)$. Draw an exact sequence of $U \in P(\text{A}^\text{op})$.

You assume $A \in P(\text{A}^\text{op}) \subset P(\tilde{A}^\text{op})$. You have functors $\otimes_A A$ of $U \in P(\text{A}^\text{op})$.

$$0 \to U \otimes_A A \to U \to U/UA \to 0$$

$$0 \to (U/UA)^2 \to (U/UA) \to (U/UA) \to 0$$
Which you can use to relate the functors $u \cdot A$, $u$, $u/Au \otimes A$, $u/Au \otimes \tilde{A}$ from $P(A^o \circ)$ to itself.

But now I really want to understand $H_x(BGL(A))$.

We have the functor $u \mapsto u \otimes A$. In the end I need to relate $H_x(BGL(A))$ with $H_x(BGL(A))$.

Take $(A \otimes A)$ Assume $A$ is right flat, $\infty$.

$A = \lim_i P_i \otimes A$ where $\lim_i P_i = A$

$P_i \in P(A^o \circ)$. Now $P \otimes A$ acts on $P \otimes \tilde{A}$

$P \otimes \tilde{A} \rightarrow P \otimes \text{Hom}^A_{A^o \circ}(P, A)$

$P \otimes \tilde{A} \rightarrow P \otimes \text{Hom}^A_{A^o \circ}(P, A)$

So we have $P \rightarrow \text{Hom}^A_{A^o \circ}(P, PA) = P \otimes A \otimes \text{Hom}^A_{A^o \circ}(P, \tilde{A})$

I also have $P \rightarrow A$

$P \otimes \tilde{A} \rightarrow P \otimes \text{Hom}^A_{A^o \circ}(P, \tilde{A})$

I also have $P \rightarrow A$

$P \otimes \tilde{A} \rightarrow P \otimes \text{Hom}^A_{A^o \circ}(P, \tilde{A})$
Suppose \( u : P \to P' \) can. with \( u \circ \theta : P \otimes_A Q \to P \otimes_A Q \).

How to think? You need to handle. You have \( \Phi \) given a refin of \( B \) in \( P \in \mathcal{P}(A^{op}) \) one has a map \( K_*(B) \to K_*(A) \).

Moreover an exact sequence of refinements \( 0 \to P_1 \to P_2 \to P_3 \to 0 \)

yields \( (\bar{[P_1]} + \bar{[P_3]}) = \bar{[P_2]} \). Also if \( B \) kills \( P/PA \)

then \( \bar{[P]} \to K_*(A) \).

So now give \( P \to P' \) your factor

\[ P \to \bar{[P]} \to P' \]

0 \to \bar{P} \to \bar{P}_1 \to \bar{P}_2 \to 0 \]

exact sequence of refinements of \( B \) in \( \mathcal{P}(A^{op}) \).

Now \( BP_2 = 0 \) so we have

\[ K_*(\bar{B}) \to K_*(\bar{A}) \]

\[ \bar{[P_2]} \text{ is zero in } K_*(\bar{A}) \].

\[ [P_2] \text{ is zero in } K_*(\bar{B}). \]
So review the steps.

**Consider dual pair** \((P, Q, A \otimes \mathbb{Z} \to A)\) over \(A\). If \(P \in \mathcal{P}(\mathbb{A})\),

then have \(P \otimes A \xrightarrow{\sim} \text{Hom}_{A^{op}}(P, P) \to \text{Hom}_{\mathbb{Z}}(P/P, P/P)\)

\[
K^*_X(P \otimes A) \to K^*_X(A) \quad \text{compatible with augmentation}
\]

\[
K^*_X(P \otimes Q) \to K^*_X(A)
\]

**Preliminaries:** \[\text{Rep}(\mathcal{B}, P(A^{op}))\]

\[
K_0(\text{Rep}(\mathcal{B}, P(A^{op}))) \to \text{Hom}_{\mathbb{Z}}(K^*_X(\mathcal{B}), K^*_X(\mathcal{A})).
\]

if \(P \in \mathcal{B}(A^{op})\) bimodule with \(P \in \mathcal{P}(A^{op})\)

then \(U \to U \otimes_{\mathcal{B}} P\), \(P(\mathcal{B}^{op}) \to P(\mathcal{A}^{op})\)

additive induces

\[
K^*_X(P(\mathcal{B}^{op})) \to K^*_X(P(\mathcal{A}^{op}))
\]

\[
\begin{pmatrix}
K^*_X(\mathcal{B}) \\
K^*_X(P(\mathcal{B}^{op}))
\end{pmatrix} \xrightarrow{\phi^P} \begin{pmatrix}
K^*_X(\mathcal{A}) \\
K^*_X(P(\mathcal{A}^{op}))
\end{pmatrix}
\]

\[
U \xrightarrow{U \otimes_{\mathcal{B}} P} U \otimes_{\mathcal{B}} P/P = (U/\mathbb{Z}) \otimes_{\mathbb{Z}} (P/P) \quad \text{multiplied by rank over } \mathbb{Z}
\]

\[
K^*_X(P) \xrightarrow{\phi^P} K^*_X(A)
\]

Claim this functional in \((P, Q) \quad (u, v); (P, Q) \to (P', Q')\)

\[
(P, Q) \to (P', Q') \to (P', Q')
\]
\[ B_1 \otimes A Q \rightarrow H_{A^{op}}(P', P') \]
\[ B_2 = P'^{A Q} \]
\[ P(B_2) \rightarrow \overline{P}(A^{op}) \]

\[ \overline{u} \otimes \overline{B_2} \otimes P' = \overline{u} \otimes P' \]

\[ u : P \rightarrow P' \quad \text{and} \quad B = P^{A Q} \]
\[ B' = P'^{A Q} \]
\[ \overline{P}(B_1) \rightarrow \overline{P}(A^{op}) \]
\[ \overline{P}(B_2) \rightarrow \overline{P}(A^{op}) \]

Fact: \( u : P \rightarrow P' \) is a \( B \)-nil isom.

Can factor: Two cases

\[ P \rightarrow P' \rightarrow P_0 \]
\[ P_0' \leftarrow P \rightarrow P' \]

\[ \phi = \phi + \phi_0 \quad \phi_0 = 0 \quad \text{in} \quad K_0(B) \]

\[ u \mapsto \overline{u} \otimes P_0 = (u / uB) \otimes P_0 \]
Use the fact that $P$ flat is a filtered ind limit of f.g. projectives.

$$K_*(\tilde{A}) = \pi_*(BGL(\tilde{A}^*))$$

Functional property: $P$ a $(B,A)$-bimodule such that $P \in P(A^{op})$, get $K_*(\tilde{B}) \xrightarrow{\phi_P} K_*(\tilde{A})$

e.g. a homomorphism $B \to A$

And now comes a big effort to show that $\pi_*(BGL(A)^*) = K_*(A)$ for $A$ flat and unital.

The idea I had is to adopt the unital ring proof of the affine group result. Let's begin.

Let's go over things until we understand them. The affine group $GL_n(A) \times A^n$, for $A$ unital, describes automorphisms of $0 \to A^n \to A^{n+1} \to A \to 0$ including the identity on $A$.

$$\begin{pmatrix} x & \beta \\ 0 & 1 \end{pmatrix}$$

So there is an affine group for each $n \geq 0$. Objects are exact sequences $0 \to V \to E \to V_0 \to 0$ where $V_0$ is fixed. Oriented operation $E \vee E$, $E \otimes \tilde{E}$.

The key point is that $E \vee E \cong E \otimes (V_0 \otimes \tilde{E})$, $E$ the split extension. So $\tilde{E}$ in the $K$-theory.

$E = \tilde{E}$. Now we have problems. What should be done?
A place to start might be the proof that $(A \overset{A}{A})$ and $A$ flat yields the identity map on $K^\wedge_*(A)$. **YES**.

Start with $P \to A$ then $P = \overset{\wedge}{A^n}$.

Start in general with $(A \overset{Q \otimes_A Q}{P \otimes_A Q})$. Let $H^*_c(GL(P \otimes_A Q)) \to H^*_c(GL(A))$.

Let $P \otimes_A Q \to \text{Hom}^A_{\text{op}}(P, P)$

$M_n(A) \subset M_n(\overset{\wedge}{A})$

So you see the problem. You will have $B \to M_n(A)$. Hence $GL_k(B) \to GL_{k^n}(A)$.

But things are defined at least, but there's an ordering problem, identifying $GL_k(M_n(A))$ with $GL_{k^n}(A)$.

There are delicate issues here.

Special case where $A \in P(A^\text{op})$, $B = \text{Hom}_{A^\text{op}}(A, A)$.

Such an $A$ arises from a unitary $B$-mod. map $f: A \to B$ such that $f(A)B = B$.

Dual pair $(B, A, A \otimes_A B \to B)$. For any such $A$ we know $A$ is right flat $\overset{\text{right flat}}{\to}$.

$B$ is not flat $\to B \otimes_B P = A$ is $A^\text{flat}$. 

$A \overset{B}{A}$
To now I would like to prove that \( \text{GL}(A) \to \text{GL}(B) \) induces an isomorphism on homology. First case would be \( A \to B \), then \( A = B \oplus N \) where it follows from the unital theory I think.

Let's examine this. \( A \xrightarrow{f} B \) given \( B \)-linear. \( A \) is ring. \( a_1 a_2 = f(a_1) a_2 \) \( f \) is a homom. If \( B \) unital, \( f \) into then \( A \) can lift \( B \) into \( A \), i.e. choose \( e \in A \) \( f(e) = 1 \). Then

\[
B \oplus N \xrightarrow{\sim} A \quad \text{N = Ker} (f).
\]

\[
b + n \xrightarrow{} be + n
\]

and \((b_1 n)(b_2 n_1) = b(b_1, n_1) = (bb_2, bn_1)\), so

I'm now interested in \( B \oplus N \) where \( N \) is an unitary \( B \)-module viewed as a \( B \)-bimodule with usual left action and \( 0 \) right action. Claim \( \text{GL}(B) \to \text{GL}(B \oplus N) \) homology too.

\[
\text{GL}(B) \times M(N)
\]

You should be able to prove this somehow. You expect to use a category of "Whitney sum" given by fibre product over \( N. \)

\[
\begin{array}{c}
B^r \to E \xrightarrow{\pi} E \to N \\
B^r \leftarrow \text{E} \to N
\end{array}
\]
Basic idea. For each \( n \leq N \) you have group \( \text{GL}_n(B) \times N^n \)

\[
\text{Hom}_B\left(B^n, B^n\right)^{op} = M_n(B)
\]

So you want to consider \( \mathfrak{fr} \) terms.

\[
0 \rightarrow N \rightarrow E \rightarrow B^n \rightarrow 0
\]

What I learn this way I think is that

\[
\lim_{\rightarrow n} H_x\left(\text{GL}_n(B) \times N^n\right) \leftarrow H_x(\text{GL}(B))
\]

Ultimately you will argue that the two homomorphisms

\[
\text{Aut}(E_n) \rightarrow \text{Aut}(E_n \oplus E_n)
\]

\[
\text{Aut}(E_n \oplus \overline{E_n})
\]

are conjugate?

Actually it might help to use the action of \( \text{GL} \) on \( \overline{\text{GL}} \)

There might be an alternative.

\[
M_n B \subset \begin{pmatrix} M_n B & B^n \\ 0 & 0 \end{pmatrix} \subset \begin{pmatrix} M_n B & M_n B \\ 0 & 0 \end{pmatrix}
\]

Another point. \( A \) is the same thing as a ring with left identity. \( A = Ae \oplus A(1-e) \)
I consider \((A, B)\) with \(B\) unital, i.e.

\[ f: A \to B \] a \(B\)-module map, \(A\) unitalary, such that \( f(A)B = B \). Thus \( f \) is a surjection onto a left ideal generating \( B \). Then \( f \) is a homomorphism when \( A \) has prod \( q_2 = f(q_1)q_2 \).

To show \( f: GL(A) \to GL(B) \) homology isom. We are considering the map of dual pairs \( \Phi \) over \( B \)

\[
(B, A, A \otimes B \to B) \xrightarrow{(f, 1)} (B, B, B \otimes B \to B). \]

It is natural to factor \( f: A \to B \) into \( A \xrightarrow{\pi} A \otimes B \xrightarrow{\pi_2} B \).

Then we are looking at

What does \( M \) now hold for the Volodin model?

I should carefully go over my arguments so as to see what I need to establish \( M \) in. Write them up!! Especially trans.

Basic construction. Given dual pair (\( P, Q \)) over \( A \) with \( P \in P(\tilde{A}^p) \).

Given dual pair \((\tilde{A}^n, Q)\) over \( A \) get homom.

\[
\tilde{A}^n \otimes Q_A \to A^n \otimes A^p. \] \( \text{Hom}(\tilde{A}^n, A) = M_n(A) \)

Now ask about maps, i.e. given \( u: \tilde{A}^p \to \tilde{A}^n \) have maps \((u, 1): (\tilde{A}^p, Q) \to (\tilde{A}^n, Q)\). Point: You factor \( u: \tilde{A}^p \to \tilde{A}^p \oplus \tilde{A}^n \to \tilde{A}^n \). The surjection case is simpler.
How do you propose to analyze this? You are constructing a system of matrix-like rings. For each $n$ you have $(\tilde{A}^n, A^n)$ dual pairs. And then for each $\tilde{A}^P \to \tilde{A}^n$ you have some link. So what happens? You eventually factor this map so really you are looking at $\tilde{A}^P \oplus \tilde{A}^n$.

\[
\begin{array}{ccc}
\tilde{A}^P & \xrightarrow{pr_1} & \tilde{A}^n \\
\downarrow & & \downarrow \\
\tilde{A}^P & \xrightarrow{pr_2} & A^n
\end{array}
\]

and all possible sections of $pr_1$. What sort of condition arises? Actually, what am I trying to do? You have a category somewhere—the objects are $\tilde{A}^n$ for $n \in \mathbb{N}$, and to each object $\tilde{A}^n$ you assign the ring $M_n(A) = \tilde{A}^n \otimes A^n$. Maybe the object is the dual pair $(\tilde{A}^n, A^n)$.

\[ P \mapsto P \otimes \text{Ham}_{A^P}(P, A) \]

You want a "space" $X$ which receives a map $BGL_n(A) \to X$ for each $n$. For each pair $p, l_n$ there is going to be some link between these maps. What is it that you want to know?

What do you need? There's a compatibility between. You need compat with maps of dual pairs. I think you can assume $Q$ doesn't change. The result...
The first condition

First, review factoring.

\[ U \xrightarrow{(f)} U \oplus V \xrightarrow{(0,1)} V \]

\[ U \xrightarrow{(1)} U \oplus V \xrightarrow{(f,1)} V \]

Simpler case \((P \oplus P \overset{p_0}{\leftarrow} P^*) \xrightarrow{p^*} (P, P^*)\)

Then we have homomorphisms.

\[(P \oplus P_0) \otimes_A P^* = P \otimes_A P^* \oplus P \otimes_A P^* \rightarrow P \otimes_A P^*\]

The condition is simply that if \(\Gamma = \text{endo ring}\) of \(0 \rightarrow P_0 \rightarrow P \oplus P_0 \rightarrow P \rightarrow 0\) induces \(0\) on \(P_0\), then the two maps \(\Gamma \rightarrow GL(A)\) have the same effect.

\[\phi : 0 \rightarrow P \overset{\psi}{\rightarrow} P \oplus P_0 \rightarrow P_0 \rightarrow 0\]

\[P \otimes_A (P^* \oplus P_0^*) \leftarrow (P \oplus P_0) \otimes_A (P \oplus P_0)^*\]
Problem: Find a good way to glue together the spaces $BGL_n, n > 0$ so as to incorporate these affine conditions. Is there an analogue of the $Q$ category?

Philosophy: I think you are trying to construct an analogue of a coproduct of $\mathcal{P}(\bar{A})$ by the action of $\mathcal{P}(\mathbb{Z})$.

Try for $Q$-category. Recall that a map in the $Q$ category has the form $W \xrightarrow{\phi} V_2 \xrightarrow{\psi} V_1$ where $\psi$ is adm. inj., $\phi$ adm. surj.

One has something similar, namely exact sequences of representations:

$0 \to P \overset{i}{\to} P' \to P \to 0$

$0 \to P \to P' \overset{\psi}{\to} P' \to 0$

where the action is trivial on $P$.

Get specific, injective $f$. 
Injective type \( 0 \to P \to P' \to P_0 \to 0 \)

\[ \text{Aut}(P) \times \text{Hom}(P_0, P) \hookrightarrow \text{Aut}(P') \]

\[ \begin{pmatrix} \text{P} \& \text{P}^* \\ 0 & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} \text{P} \& \text{P}^* \\ \text{P}_0 \& \text{P}_0^* \end{pmatrix} \]

Surjective type \( 0 \to P_0 \to P' \to P \to 0 \)

\[ \begin{pmatrix} \text{P} \& \text{P}^* \\ 0 & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} \text{P} \& \text{P}^* \\ \text{P}_0 \& \text{P}_0^* \end{pmatrix} \]

What might be a map??

What is the key idea? What should be an injective type map from \( P \) to \( P' \)?

Too hard. Given B choose a flat

What am I trying to do? I think it's true that
You need to get your arguments so clean that you know what should be true. You want to show that the two firm left flat rings which are to have the same $K_\ast(GL(-))$. Suppose this true in the strange form that when

Let's work out the details of trans.

Basic construction: Given $(P, Q, Q \otimes_A P \to A)$ $P$ flat $A^\mathbb{p}$-module, get canonical

$$T_p : K_\ast(P \otimes Q) \to K_\ast(A).$$

Clarify transitivity:

Given $(P, P^\mathbb{p})$ over $A$, $P$ $\mathbb{A}^p$-flat

$(Q, Q^\mathbb{p})$ over $B = P \otimes_A P^\mathbb{p}$ where $Q$ is $B^\mathbb{p}$-flat

get $(Q \otimes_B P^\mathbb{p}, P^\mathbb{p} \otimes Q^\mathbb{p})$ $(Q \otimes_B P^\mathbb{p}) \otimes (Q \otimes_B B) \to P^\mathbb{p} \otimes B \otimes P \to P \otimes B \to A$ with $Q \otimes_B P^\mathbb{p}$ $\mathbb{A}^p$-flat

Claim $K_\ast(Q \otimes_B P^\mathbb{p}) \otimes (Q^\mathbb{p} \otimes B)$ $\xrightarrow{T_{Q \otimes B}} K_\ast(A)$ $\xrightarrow{T_p} K_\ast(B)$

How to prove? Reduces to case $Q = B^\mathbb{p}$ and $Q^\mathbb{p} = Hom_{\mathbb{B}^\mathbb{p}}(Q, B)$

Suppose $Q$ is also left $A$ flat. Can suppose $P, Q$

f. free right $\mathbb{A}^p$-left $\mathbb{A}^p$ over $\mathbb{A}$. $\mathbb{A}^p$-flat

$P \otimes_A Q \to Q^\mathbb{p} \otimes_A Q$
Consider \((P, Q, Q \otimes A)\) where \(P \in P(A)\) and \(Q \in P(\overline{A})\). To show:

\[ T_P, T_Q : K_x(P \otimes A Q) \to K_x(A) \]

conincide. \(\text{Hom}_{A}^{\text{op}}(P, Q) \to P \otimes A Q \to \text{Hom}_{A}^{\text{op}}(Q, A) \) \[ M_n(A) \to M_n(A) \]

Roughly one should be the contragredient repn of the other.

Do the initial case first. \(P \in P(A)\), \(Q \in P(\overline{A})\) \(Q \otimes P \to A\) given. The point is that we have a map of dual pairs.

Consider the case where \(Q = P^\circ = \text{Hom}_{A}^{\text{op}}(P, A)\).

Then

\[ P \otimes A Q \to \text{Hom}_{A}^{\text{op}}(Q, A) \]

is

\[ \text{Hom}_{A}^{\text{op}}(P, P) \]

To now consider a non-mutual case. We have a map of dual pairs (\(P, Q) \to (P, P^*)\)) where \(P^* = \text{Hom}_{A}^{\text{op}}(P, A) = A \otimes P^\circ\)

\[ K_x(P \otimes A Q) \xrightarrow{T_Q} K_x(A) \]

\[ K_x(P \otimes_A P^*) \xrightarrow{T_{P^*}} K_x(A) \]

Point is that \(P \otimes_A P^* \to \)
What is so confusing.

You assume \( P \simeq \bar{A} \). Right? Then you have the trace maps \( \tau_\mathfrak{p}, \tau_\mathfrak{q} \) are induced by

\[
\begin{align*}
\text{End}_{A_\mathfrak{p}}(P) & \quad \text{End}_A(\mathfrak{p}) \quad \text{End}_A(\mathfrak{q}) \\
\text{M}_n(A) & \quad \text{M}_p(A) \quad \text{M}_p(A)
\end{align*}
\]

But we know that

\[
\begin{align*}
\text{K}_x(\text{End}_{A_\mathfrak{p}}(P)) & \quad \text{K}_x(\text{End}_A(\mathfrak{p}) \circ \mathfrak{p}) \quad \text{K}_x(\text{End}_A(\mathfrak{q}) \circ \mathfrak{q}) \quad \text{K}_x(\bar{A}) \\
\text{K}_x(\text{End}_{A_\mathfrak{p}}(P)) & \quad \text{K}_x(\text{End}_A(\mathfrak{p}) \circ \mathfrak{p}) \quad \text{K}_x(\text{End}_A(\mathfrak{q}) \circ \mathfrak{q}) \quad \text{K}_x(\bar{A})
\end{align*}
\]

But suppose you want to prove this for \( H_x(BGL(-)) \).

\[
\begin{align*}
\text{P} & \quad \text{Hom}_A(\mathfrak{p}, A) \circ \mathfrak{p} = \text{M}_p(A) \\
\text{M}_n(A) & \quad \text{Hom}_A(\mathfrak{p}, A) \circ \mathfrak{p} = \text{M}_n(A)
\end{align*}
\]

It seems OK. You have \( (\mathfrak{p}, Q) \rightarrow (\mathfrak{p}, A \circ \mathfrak{p}) \).

This is too confusing.
2 hours from $P_A^A Q$ to matrices over $A$:

$$P_A^A Q \rightarrow P_A^A A_A^* P = M_n(A)$$

$$\begin{array}{c}
\uparrow \\
Q_A^A A^* Q = M_p(A)
\end{array}$$

03/13/97 Get details straight.

can be a pair $(P, Q)$ over $A$.

$$P^* = \text{Hom}_A(P, A) = A \otimes \text{Hom}_A(P, \tilde{A})$$

$$\begin{array}{c}
\text{can.} \ (P, Q) \rightarrow (P, P^*) \\
\text{and} \ P_A^A Q \rightarrow P_A^A P^* = M_n(A)
\end{array}$$

$$\text{GL}(P_A^A Q) \rightarrow \text{GL}(A)$$

### More

Let's discuss elementary considerations.

Idea: Use $\tilde{A}^n \rightarrow \tilde{A}^{m*}$ given by matrices over $A$. Thus when you construct a flat form over of $M=AM$ you do it as a $\lim F \rightarrow AP < P$ Can this help? Do try this. Have fun.

Start with $B=B^2$ let $F \rightarrow B$ be a free $B$-mod mapping onto $B$. Let $\begin{array}{c}
F \rightarrow F \rightarrow F \rightarrow \cdots
\end{array}$ $A_{\infty}$ be a fibre flat $B$-mod mapping onto $B$. $\begin{pmatrix} A & B \end{pmatrix}$
\[ A = \lim\limits_{\rightarrow} F \quad o \rightarrow I \rightarrow F \rightarrow B \rightarrow 0 \]

\[ GL(A) = \lim\limits_{\rightarrow} GL(F) \]

\[ o \rightarrow I \rightarrow F \rightarrow B \rightarrow 0 \]
\[ \forall \psi \quad \exists \phi \]

\[ o \rightarrow I \rightarrow F \rightarrow B \rightarrow 0 \]

\[ P = \tilde{A}^n \quad P^* = \text{Hom}_{A^{op}}(P, A) \quad P \otimes P^* = M_n(A) \]

given \((P, Q)\) a dual pair get \((P, Q) \rightarrow (P, P^*)\)

hence a homomorphism \(P \otimes Q \rightarrow M_n(A)\).

whence \(GL(P \otimes Q) \rightarrow GL(M_n(A)) = GL(A)\)

I now need to worry about a map of dual pairs \(P \rightarrow P'\). Factor \(P \xrightarrow{(1)} P \oplus P' \xrightarrow{(f, 1)} P'\)

\[
\begin{pmatrix} 1 & 0 \\
0 & 0 \end{pmatrix}
\]

This auto of \(P \oplus P'\) belongs to \(E(A)\) if \(f: P \rightarrow P'\)

If I restrict to such \(f\), then in homology the something won't matter

injective case \(P \rightarrow P \oplus P_0\) \((P, P^* \oplus P_0^*) \rightarrow (P \oplus P_0, P^* \oplus P_0^*)\)
It's not much cleaner except that I see that I can handle maps $f: P \to P'$ which are 0 modulo $A$.

Let's try to see what's true about $H_*(\text{GL}(A))$ when $A$ is flat.

Fix a ring $B$, say idempotent, and consider $A \to B$-module injections $f: A \to B$ such that $IA = 0$. I want to study the relation of $H_*(\text{GL}(-))$ for $A, B$. Such an extension is unique up to an $A$-mod injection $f: A \to B$, the product on $A$ being $a, q_2 = f(a)q_2$. Also $I^2 = f(I)I = 0$. 
You better go over what can be done with the functors $K_*(A) = \text{Ker}(K_0(A)) \to K_0(B)$.

Assuming $B = B^{\oplus r}$, can construct a free flat $B$-algebra $A$ over $B^{\oplus r}$.

\[\cdots \rightarrow P_i \rightarrow P_0 \rightarrow B \rightarrow 0\]

Convert to a $\mathbb{Q}$-linear complex by Dold-Kan:

\[\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0 \rightarrow B\]

Apply $\text{GL}$ to get a simplicial group:

\[\bigoplus G_i \rightarrow G_i \rightarrow G_0, \quad G_n = \text{GL}(A_n)\]

Because the $A_i$ are flat rings and $A_i \rightarrow B$ are all morphisms, all simplicial arrows are morphisms between left $\mathbb{Q}$-modules. I should instead use $\mathbb{Q}[\text{GL}(B)]$.

Consider the doubly complex. $\mathbb{Q}[\text{GL}(B)]$. Compute group homology:

\[\pi_0(\text{GL}(A)) = \text{GL}(\pi_0(A)) \cong \text{GL}(B^{(\geq 0)})\]

Spectral sequence abuts to $H_*(\text{GL}(A))$ for any $n$.

\[E_1^{p,q} = H_p^{\mathbb{Q}}(\text{GL}(A), \mathbb{Q}^{\otimes q})\]

Think of $\text{GL}(A)$ as a top group with $\pi_0(\text{GL}(A)) = \text{GL}(B^{(\geq 0)})$. If $B$ not prime you go no further. You are getting $H_1(B \otimes \text{GL}(B^{(\geq 0)})) = H_1(B \otimes \text{GL}(A))$ for the abutment in degree 1.
$A_2 \implies A_1 \implies A_0 \to B$

$G_n = GL(A_n)$

$G_0 \overset{c_1}{\to} G_1 \overset{c_2}{\to} G_2 \overset{b_1}{\to} F_1 \overset{f_1}{\to} F_0 \to B \to 0$

If $B$ is $h$-unital, then I know that $\{A_n\}$ resolves $B$.

so $BG \to BGL(B)$ is a h.eq. On the other hand, assuming $H^*(GL(-))$ agrees for flat $K$-rings, we should know that $H^*(BGL(B)) \to H^*(BG)$. Now weaken $B$ $h$-unital

to having a flat free resolution of length $n$.

$F_n \to F_{n-1} \to \cdots \to F_0 \to B \to 0$

exact.

$n=0$ $B$ idempotent.

$n=1$ $B$ finite.

So besides $\pi_0 G = GL(B)$ (assume $n\geq 1$)

the next ity gp in $\pi_n G = \tilde{M}(\tilde{B}(Z; B))$

yet integers straight

$F_n \to F_{n-1} \to \cdots \to F_0 \to B \to 0$

exact

with $F_i$ free flat for $0 \leq i \leq n$ \iff $\text{Tor}_i^B(Z; B) = 0$, $0 \leq i \leq n$.

Then $K_n = \ker(F_n \to F_{n-1})$, we have $K_n/BK_n = \text{Tor}_n^B(Z; B)$.

so if we choose $F_n$ to map into $BK_n$, we have the homology $H_{n+1}(F) = \text{Tor}_{n+1}^B(Z; B)$.

Replace $n+1$ by $n$.

Assume $\text{Tor}_i^B(Z; B) = 0$ $i < n$

$\neq 0$ $i = n$.

Then get $F_n \to F_{n-1} \to \cdots \to F_0 \to B \to 0$ exact in degrees $\leq n$.
Try again

\[ 0 \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow B \rightarrow 0 \]

\[ 0 \rightarrow Z_{n-1} \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow B \]

\[ Z_m/BZ_{n-1} = \text{Tor}_n^B(Z,B) \]

Consider a free flat complex \( F \) with \( H_i(F) = \begin{cases} B & i = 0 \\ 0 & 0 < i < n \\ Z & i = n \end{cases} \)

\[ 0 \rightarrow Z_n \rightarrow F_n \rightarrow F_{n-1} \rightarrow \cdots \rightarrow F_0 \rightarrow B \rightarrow 0 \]

\[ \text{Tor}_n^B(Z,B) \rightarrow \text{Tor}_{n+1}^B(Z,B) \rightarrow \text{Tor}_n^B(Z,Z_n) = Z_m/BZ_n \]

\[ X = H_n(F) = \text{Tor}_{n+1}^B(Z,B) \]

Conclusion: \( \text{Tor}_i^B(Z,B) = 0 \) for \( i < n \)

If free flat resolution \( F_n \rightarrow \cdots \rightarrow F_0 \rightarrow B \rightarrow 0 \),

then \( H_n(F) = \text{Tor}_{n+1}^B(Z,B) \).

Return to \( A_{n+1} \cong A_n \Rightarrow A_0 \rightarrow B \rightarrow 0 \)

\[ \text{here } T_n = \text{Tor}_{n+1}^B(Z,B) \]

\[ T_i(G) = \begin{cases} \text{GL}(B^{(n)}) & i = 0 \\ 0 & 0 < i < n \\ \text{Mat}_n(\text{Tor}_n^B(Z,B)) & i = n \end{cases} \]

So \( G \) an algebra of top gp with \( \hat{\lambda}_0 = \text{GL}(B) \), \( \hat{\lambda}_i = 0 \) for \( 0 < i < n \), and \( T_n \neq 0 \).
What is the homology of \( BG \)?

\[ 1 \to \widetilde{BG} \to BG \otimes \mathbb{P} \to B\pi_0 \to 1 \]

\[ E_{pq}^2 = H_p(B\pi_0, H_q(\widetilde{BG})) = H_{p+q}(BG) \]

and you get

\[
\begin{cases}
2 & \text{if } \delta = 0 \\
0 & 0 < \delta \leq n \\
\text{(trivial)} & \delta = n+1
\end{cases}
\]

This is a bit messy,

Let's go over it all again. \( B \) idempotent,

\( \mathbb{F} \) complex \( F_\bullet \) of flat \( \mathbb{F} \)-modules with aug to \( B \)

exact mod nil module

\[ \cdots \to F_i \to F_0 \to B \to 0 \]

\( B \)-module complex, make simplex

\[ \implies A_\ell \to A_0 \to B \to 0 \]

apply \( GL \) get s g p \( GL(A_\ell) \)

There is something going on here which linearizes in some way the functor \( H_\bullet(BGL(-)) \).

03/14/97 1410

Question: Do I have enough to handle \( BGL(A)^+ \sim \) fibre \( BGL(A)^+ \to BGL(2)^+ \) for \( A \) flat? The hope is to use some induction as follows. The inductive hypotheses would be that one has \( M_i \) of \( K_\bullet \) for flat rings in degrees \( < n \). Then via simplicial resolutions, you might be able to analyze an extension \( A_\ell \to B \) of flat rings.
What happens at the beginning? You should try to play the different themes together.

Begin with B flat and firm.

Let \( f: A \rightarrow B \) be a surjection of \( B \)-modules, with \( A \) flat \( B \)-mod. Then get \((A \hat{\otimes} B, A \otimes B \rightarrow B)\), actually from a dual pair \((B, A, A \otimes B \rightarrow B)\) and \(A \otimes B \rightarrow B, B \otimes A \rightarrow A\).

To show \( H_*(GL(A)) \rightarrow H_*(GL(B))\).

Now I have to use the techniques I've found. Take

\[ B^n \rightarrow A \]

\[ (b_1) \rightarrow \sum b_{a_i} \]

Then you are going to get

\[ \hat{B} \otimes P \rightarrow \hat{B} \otimes A \]

a ring homom. Actually what are you doing? You want to define a map \( H_*(GL(B)) \rightarrow H_*(GL(A))\).
Using the fact that $B$ is left $A$-flat, we have:

\[
\begin{pmatrix}
A & B \\
A & B
\end{pmatrix}
\]

is $A$-flat $\iff$ $A \otimes_A A = A$ is $B$ flat

$B$ flat $\iff$ $B \otimes_B B = B$ is $A$ flat.

Because $B$ is $A$-flat, and $B$ acts on the right, we expect to define this map by approx. So we approximate $B$ by $\hat{\Lambda}^n \to B$. So we have a map of dual pairs over $A$:

$B = A \otimes_A B \leftarrow B_\xi = A \otimes \hat{\Lambda}^n$

$\left( A, B, \frac{B \otimes A \to A}{b \otimes a \mapsto b \mathfrak{l} a} \right) \leftarrow \left( A, \hat{\Lambda}^n, \hat{\Lambda}^n \otimes A \to A, (\xi_i) \otimes a \mapsto (\sum i \xi_i \bar{b})_i \right)\left( f(\xi) \bar{b} \right)$

So you have an approximation $B \to$ mapping to matrices over $A$. Now you have this homomorphism $A \to B$. Look carefully at what you need. You have $H_\ast(GL(A)) \to H_\ast(GL(B))$.
Look carefully. You have $A \to B$ given, you need the inverse map in $K$-theory. This you get because $B$ is $B$-flat, hence $A$-flat. So if you approximate $Q \to A$ free, you get

$$(\tilde{B},Q,Q \otimes \tilde{B} \to B) \to Q^* \otimes Q$$

$$(\tilde{B},A,A \otimes \tilde{B} \to B) \to$$

$Q = \tilde{B} \otimes Q \to A = \tilde{B} \otimes A$

$$Q^* \otimes Q$$

Now suppose $Q \to A$ chosen so as to yield $H_x(GL(Q)) \to H_x(GL(A))$

We now apply the homom.

$A \to B$, and try to compare

$GL(Q) \to GL(A) \to GL(B)$
So they are obviously not the same because of \( u \). E.g., \( GL_1(\mathbb{R}) \to GL_1(A) \to GL_1(B) \).

And I already knew what happens I think. You have this map \( Q \to A \to B \).

Okay let's work this out carefully. You have a ring \( B \) a map \( Q=\tilde{B}^n \to B \), \( u(b_i) = \sum_i b_i x_i \) where \( x_i \in B \) fixed. Then you wish to compare the \( \mathcal{Q}_F \) action on \( \tilde{B}^n \) and on \( \tilde{B} \).

\[
q \tilde{g} q' = \sum_i (\sum_j g_i x_j) \tilde{g}'
\]

\[
u(qq') = \sum_{j=0}^{n} (\sum_i g_i x_j) \tilde{g}_j x_j' = u(q) u(q').
\]

\( u \) is a \( \mathcal{Q}_F \)-nil root.

\[
\begin{array}{ccc}
0 & \rightarrow & \tilde{B}^n \\
\downarrow & & \downarrow \nu \\
\tilde{B} & \rightarrow & B
\end{array}
\]

How does this help?

\[
\begin{array}{ccc}
\tilde{B}^n & \rightarrow & B^* \\
\downarrow & & \downarrow (0,1) \\
\tilde{B} & \rightarrow & \tilde{B}
\end{array}
\]

it seems that \( B \) acts here by

\[
\left( \begin{array}{c}
-u_1, \ldots, -u_n \\
1
\end{array} \right)
\]
I have to get control over this stuff. The problem is to show that if $0 \to I \to A \to B \to 0$
is an extension such that $IA = 0$, then $A, B$ are left flat, then $GL(A) \to GL(B)$ is a homotopy
ring.
I should be able to drop $A$ being left flat and still construct a map $H_\ast(GL(B)) \to H_\ast(GL(A))$. Why?
By choosing $A' \to A$ with $A'$ left flat.

The important thing here is that $B$ is $A$-flat
and this should follow from $B$ being left flat.

So over the basic construction: $(P, Q)$ over $A$ $P$ is $A^p$-flat. Claim have canonical map

$$K_\ast(P \otimes_A Q) \to K_\ast(A)$$

Can suppose $P$ is free $A^p$-module $\mathbb{A}^n$.

$$P \otimes_A Q \to P \otimes P^* = M_n(A)$$

induces

$$K_\ast(M_n(A)) = K_\ast(A).$$

naturality.

$$(w, 1) : (P, Q) \to (P, Q)$$

can assume $w(P) \subset P A$.

$$\begin{array}{ccc}
P & \xrightarrow{(u)} & P \\
\downarrow{(0)} & & \downarrow{(0)} \\
P \oplus P' & \xrightarrow{(w, 1)} & P' \\
\downarrow{(0, 1)} & & \\
P' & \xrightarrow{(u)} & \\
\end{array}$$
I need to ask what I must know.

Let's begin with $A$ idempotent, and let's try to understand well what we need. To do what? To define a trace map

$$K_x(P \otimes_A Q) \to K_x(A)$$

with the properties. I think the important thing is to define trace maps

$$K_x(P \otimes_A Q) \to K_x'(A)$$

when $(P, Q)$ is a dual pair over $A$ and $P = \tilde{A}^n$ for some $n$.

We need to a trace map

$$K_x(\tilde{A}^n \otimes_A (\tilde{A}^n)^*) \to K_x(A)$$

for each $n \geq 0$. $n = 0$ is trivial. Exactly what do I need to make them consistent?

For each $n \geq 0$, I need to analyze a map $u: P \to P'$.

I need to analyze a map $u: P \to P'$. You need

$$P \otimes_A P' \xrightarrow{u \otimes 1} P \otimes_A P'$$

and

$$P \otimes_A P' \xrightarrow{1 \otimes u^*} P \otimes_A P'$$

Your method consists of factorization $P \xrightarrow{u} P \otimes_A P' \to P'$.

The effect of $u$ is confounding me.

The important point is that $u$ is a nil isomorphism.

I have to think carefully about the kind of $u$'s, I think I can restrict to $u: P \to P'$ which maps $P$ into $P'$, because these are the kinds of
in Glue, \( A \).

Consider the diagram. The important point should be this:

May be you need

\( (1, 0) \), \( (\sigma) \), etc. (\( \sigma \))

Alternatively:

\( \Phi_0 \) (\( \Phi \))

(\( \Phi_0 \))

\( \Phi \)
Try again. You have $u: P \to P'$, a map and you want to compare

$$P \otimes P'^* \xrightarrow{u \otimes 1} P' \otimes P'^*$$

$$\otimes \downarrow \omega u^*$$

$$P \otimes P'^* \xrightarrow{\omega u^*} M_n(A)$$

$$(p_1 \otimes \lambda_1')(p_2 \otimes \lambda_2') = p_1 <\lambda_1', u(p_2)> \otimes \lambda_2'$$

$$(p_1 \otimes \lambda_1'u)(p_2 \otimes \lambda_2' u) = p_1 <\lambda_1'u, p_2> \otimes \lambda_2'u$$

You have some kind of correspondence between $M_n(A)$ and $M_{n'}(A)$ given by $M_{nn'}(A)$. In fact it's very clear, namely $u \in M_{nn'}(A)$ and the correspondence is our friend $v \mapsto u v \in M_{nn'}$.

$$\oplus \ M$$
03/12/97

Yesterday you reached something like the Hochschild complex. You have objects \( n \in \mathbb{N} \) and for each \( i \). You have the sequence of objects

First for a ring \( A \) you can form

\[
\{(x_0, x_1, \ldots, x_n) \mid 1 - x_0 x_i \text{ invertible}\}, \quad \{x_0 \mid 1 - x_0 \text{ invertible}\}
\]

This appears to be a cyclic set

\[
A \otimes A \otimes A \rightarrow A \otimes A \rightarrow A
\]

\[
(x_0, x_1, x_2) \mapsto x_0 x_1 x_2, \quad x_0, x_1, x_2, \quad x_2 x_0, x_1
\]

Better check well-defined.

\[
P \otimes P^* \xrightarrow{u \otimes 1} P \otimes P^*
\]

\[
\downarrow \otimes u^* \quad \text{Once } u \text{ is fixed, you get a ring structure on } P \otimes P^*, \text{ namely (p} \otimes \lambda')(p_2 \otimes \lambda_2') = p_1 \otimes \lambda_1'(u(p_2) \otimes \lambda_2')
\]

and you can consider invertible elements \( 1 - \nu \in P \otimes P^* \).

Now these are elements of the form \( 1 - u \nu \)

\[
(u \otimes 1)(1 - \nu) = 1 - (u \otimes 1) \nu = 1 - u \nu; \quad \rho' \mapsto \rho
\]

\[
(u \otimes 1)(\rho \otimes \lambda') = u(\rho) \otimes \lambda' = u \cdot (\rho \otimes \lambda')
\]

Similarly

\[
(1 \otimes u^*)(\rho \otimes \lambda') = \rho \otimes \lambda' u = (\rho \otimes \lambda) \cdot u
\]

\[
(p_1 \otimes \lambda_1')(p_2 \otimes \lambda_2' u) = p_1 \lambda_1'(u(p_2) \otimes \lambda_2') = (p_1 \otimes \lambda_1')(u(p_2) \otimes \lambda_2') \cdot u
\]

Seems clear. At level 0 you have
You don't have much time to get this into shape. You have The structure should come from the Hochschild complex of a ring with many objects. Let $X$ be the set of objects. For each $x \in X$ you have $A_{xy}$ and for each $x, y, z$ you have $A_{yz} \odot A_{yx} \rightarrow A_{xz}$ associative. The thing you are after is a complex of the form 

$$\bigoplus_{x, y, z} A_{xy} \otimes A_{yz} \otimes A_{zx} \rightarrow \bigoplus_{x, y} A_{xy} \otimes A_{yx} \rightarrow \bigoplus_{x} A_{xx}$$

so basically you have a ring $A = \bigoplus_{x, y} A_{xy}$ with a matrix decomposition relative to set $X$. There's an obvious trace map in the present case from the Hochschild complex of the ring with matrix decomposition to the Hochschild complex of the ring $A$.

Where does $K$-theory enter? One actually produces an invertible subring. Inside $A_{xy} \otimes A_{yz} \otimes A_{zx}$ you consider $x_0 \otimes x_1 \otimes x_2$ or should I look at tuples $(x_0, x_1, x_2)$ such that $1 - x_0 x_1 x_2 \in A_{xx}$ is invertible? An interesting question:

How to obtain $K$-theory?

Give $x \otimes y \in A_{xy}$ look at $x \in A_{xx}$ such that $1 - x$ invertible. Then give $x, y$ consider $(x_0, x_1) \in A_{xy} \otimes A_{yx}$ such that $1 - x_0 x_1$ invertible in $A_{xx}$, whence...
What can I say about the structure? You have \((x_0, x_1)\). YES.

What sort of structure? Each \(x\) gives \(G_{xx}\). A pair \(x, y\) gives a

But what about bar homology?

\[
\rightarrow \quad A_{x_0} \oplus A_{x_1} \rightarrow A_{xy}
\]

Let's try to write up something that will clarify the situation.

Let's begin with the basic case. You have a dual pair \(P, Q\) over \(A\) such that \(P\) is \(A^0\) flat. To construct a canonical map

\[
K_*(P \otimes_A Q) \rightarrow K_*(A)
\]

Use the fact that \(P\) is a filtered colimit of finite free \(A_{op}\) modules. This result admits strengthening.

If \(P\) is a filtered flat you have Wodzicki's refinement

\[
\tilde{A}^n \xrightarrow{a^n} \tilde{A}^p \xrightarrow{f_{a^p}} \tilde{A}^q
\]

Hence

Thus given \(xa = 0\) in \(A\), \(\sum_{i=1}^n x_i a_i = 0\), \(i \leq n\)

\(f_{a^p}\) such that \(x = ya\) and \(\Rightarrow a^p a = 0\).

I think this means that \(\Rightarrow P\) can be written as a filtered lim of a system \(\otimes\) in \(P(A_{op})\) such that the transition maps are 0 modulo \(A\).
This fact should be useful.

So we restrict to $P$ of the form $A^n$ and we want naturality w.r.t. maps $P \to P'$ of such modules which are zero modulo $A$. i.e. $u \in P' \otimes_A P^*$

Basic Construction: Have

$$P \otimes_A Q \to P \otimes_A P^* = M_n(A)$$

which induces

$$K_*(P \otimes_A Q) \to K_*(M_n(A)) \to K_*(A)$$

Want naturality in the dual pair $P \to P'$, $Q \to Q'$

$$P \otimes_A Q \to P' \otimes_A Q' \to P' \otimes_A P'^*$$

can take $Q' = P'^*$. Then must check

$$P \otimes_A Q \to P' \otimes_A P'^*$$

I think you can take both $Q = Q' = P'^*$

$$P \otimes_A Q \to P' \otimes_A P'^*$$

Good diagram
So the critical case you need to treat is for any $u: P \to P'$ in $P(A^*p)$ and the map $(p, p^*) \to (P', p'^*)$. You then need an isomorphism:

$$
P \otimes_A p^* \xrightarrow{u \otimes 1} P' \otimes_A p'^*
$$

Notice that you haven't used $u: P \to P'$ has image in $P^A$.

Review: You have for $(p, q, q \otimes p \to A)$ $p \in P(A^*p)$ a homomorphism $P \otimes_A Q \to P \otimes_A p^* \hookrightarrow M(A)$. What I need is some feeling for why

The basic construction takes $(p, q, q \otimes p \to A)$ $p \in P(A^*p)$ and assigns the homomorphism $P \otimes_A Q \to P \otimes_A p^*$.

The gadget I'm after should be given by the groups $GL(\otimes p^*)$ and I need to put in relations to get naturality. Give $\psi(\lambda, \nu): (p, q) \to (p', q')$ case ass. $\nu = 1$ for $p^*$. 

\[ \begin{array}{c}
P \otimes_A Q \xrightarrow{\psi} P' \otimes_A Q' \\
\downarrow \\
P \otimes_A p^* \xrightarrow{u \otimes 1} P' \otimes_A p'^* \\
\downarrow \\
P \otimes_A p^*
\end{array} \]
You have the idea to factor \( u \).

What are you doing? Instead of just \( \mathbb{P}_A \mathbb{P}^* \) and \( \mathbb{P}_A \mathbb{P}'^* \), we now have to consider \( (\mathbb{P} \mathbb{P}')_A \otimes (\mathbb{P}' \mathbb{P}^*) \). So how does this affect things? Before you had any \( u \in \mathbb{P}_A \mathbb{P}^* \) and you used \( u \) to make \( \mathbb{P}_A \mathbb{P}^* \) into a ring.
Review end of yesterday's work.

The basic construction assigns to a dual pair \((P, A, Q_2 : P \to A)\) over \(A\) with \(P \in \text{P}(A^{op})\) the homomorphism \(P \circ A \to P \circ A \circ \text{P}^*\). The gadget \(X\) I'm looking for is generated by the groups \(\text{GL}(P \circ A \circ \text{P}^*)\) and I need to put in the relations necessary for naturality.

For a map \((P, A) \to (P', A')\)

\[
\begin{array}{ccc}
P \circ A & \xrightarrow{u}
& P' \circ A' \\
\downarrow & & \downarrow \\
P \circ A \circ \text{P}^* & \xrightarrow{\text{GL}(P \circ A \circ \text{P}^*)} & P' \circ A' \circ \text{P}^*
\end{array}
\]

Naturality reduces to the case \(Q = Q' = \text{P}^*\).

We want

\[
\begin{array}{ccc}
\text{GL}(P \circ A \circ \text{P}^*) & \rightarrow & \text{GL}(P' \circ A' \circ \text{P}^*) \\
\downarrow & & \downarrow \\
\text{GL}(P \circ A \circ \text{P}^*) & \rightarrow & \text{GL}(P' \circ A' \circ \text{P}^*)
\end{array}
\]

\(\xrightarrow{X}\)

to commute up to homotopy for each \(u : P \to P'\).

Factoring \(u\)

\[
\begin{array}{ccc}
P & \xrightarrow{(u)} & P \oplus P' \xrightarrow{(0, 1)} P' \\
\downarrow & & \downarrow \\
P \circ A \circ \text{P}^* & \xrightarrow{(P \circ A \circ \text{P}^*) \circ (0, 1)} & P' \circ A \circ \text{P}^*
\end{array}
\]

leads to

\[
\begin{array}{ccc}
(P \circ A \circ \text{P}^*) \circ (0, 1) & \rightarrow & P' \circ A \circ \text{P}^* \\
\downarrow & & \downarrow \\
(P \circ A \circ \text{P}^*) \circ (0, 1) & \rightarrow & P' \circ A \circ \text{P}^*
\end{array}
\]

\[
\begin{array}{ccc}
P \circ A \circ \text{P}^* & \xrightarrow{\text{GL}(P \circ A \circ \text{P}^*)} & P' \circ A \circ \text{P}^* \\
\downarrow & & \downarrow \\
P \circ A \circ \text{P}^* & \xrightarrow{\text{GL}(P \circ A \circ \text{P}^*)} & P' \circ A \circ \text{P}^*
\end{array}
\]

\[
\begin{array}{ccc}
(P \circ A \circ \text{P}^*) \circ (0, 1) & \rightarrow & P' \circ A \circ \text{P}^* \\
\downarrow & & \downarrow \\
(P \circ A \circ \text{P}^*) \circ (0, 1) & \rightarrow & P' \circ A \circ \text{P}^*
\end{array}
\]

\[
\begin{array}{ccc}
(P \circ A \circ \text{P}^*) \circ (0, 1) & \rightarrow & P' \circ A \circ \text{P}^* \\
\downarrow & & \downarrow \\
(P \circ A \circ \text{P}^*) \circ (0, 1) & \rightarrow & P' \circ A \circ \text{P}^*
\end{array}
\]
Trying to probe the cyclic homology type of \( A \) using group rings. Ideally (unital) \( k[G] \to A \) is equivalent to a gp hom \( G \to A^\times \).

More generally you consider unital rings \( \mathfrak{m} \to A \), i.e. \( \mathfrak{m} \in \mathcal{P}(A^\times) \) and a hom. \( G \to \text{End}_{\mathcal{P}(A)}(\mathfrak{m}) \).

Then you get \( k[G] \to P \otimes_{A} P^\vee \) and hence a map from the cyclic homology type of \( k[G] \) to the cyclic homology type of \( A \). These must be interesting possibilities in the non-unital context.

At some point use the Burghelea result on cyclic homology of group rings, namely, the identity conjugacy class leads to the group homology tensored with \( k[\mathfrak{m}] \) sitting as a summand of the cyclic homology.

\[
\begin{array}{ccc}
HC & \longrightarrow & HP \\
\downarrow & & \downarrow \\
HH & \longrightarrow & HC
\end{array}
\]

\[
k[G] \longrightarrow M_*(A)
\]

\[
H_*(G) \to \text{Tor}_* k[G] \otimes k[G] (k[G], k[G]) \to \text{Tor}_* A \otimes A (A, A)
\]

mechanism of Dennis trace
cyclic set giving \( EG \times G \to G \times G \Rightarrow G \)

b complex
To get the homology of the group ring, you fix the product to be 1. So if you want the Dennis trace it is very simple. But now how do I correlate this construction with what I am doing now?

I take a group homomorphism $G \to \text{GL}_n(A)$ and consider $k[G] \to M_n(A)$. Then I can make $BG$ into the Hochschild cycle of $M_n(A)$.

$BG$ appears as the cyclic subset of $EG \times EG$

Consider \( g_0, g_1, \ldots, g_n \) $\Rightarrow g_0 g_1 \cdots g_n = 1$.

So what am I doing? Replacing $M_n(A)$ by $A$.

So inside the Hochschild complex of $A$ I have a cyclic subset consisting of $(g_0, \ldots, g_n)$ such that $g_0 \cdots g_n = 1$. So how does this relate to the usual case?

I can then for $A$ unital you can look for finite “cycles” $(g_0, \ldots, g_n)$ in $A$ s.t. $g_0 \cdots g_n = 1$.

For a non-unital what happens? Is there an analogue? What would you like? You need different. It seems completely different. Instead of $(g_0, \ldots, g_n)$ such that $g_0 \cdots g_n = 1$, you consider $(x_0, \ldots, x_n)$ s.t. $(1-x_0 \cdots x_n)^{-1}$ exists.
Dennis trace review: Suppose $A$ unital.
Consider representations of groups over $A$, i.e. group homs $G \to \text{GL}_n(A)$, equivalently ring homs $k[G] \to M_n(A)$. Such a $f$ induces a map from cyclic hom types $k[G]$ to $A$. Apply Buehler. The former splits according to the conjugacy classes of $G$. Focus on identity pest. Key result is that you get a divisible $S$ module for the cyclic homology, whose Hochschild homology is $H_c(BG)$.

$$
\begin{align*}
\mathcal{H}^c_k(k[G]) & \longrightarrow \mathcal{H}^c_k(k[G]) \\
\downarrow & \downarrow \\
\mathcal{H}^c_k(k[G]) & \longrightarrow \mathcal{H}^c_k(k[G]) \\
HH_k(k[G]) & \longrightarrow \mathcal{H}^c_k(k[G]) \\
\mathcal{H}^c_k(k[G]) & \longrightarrow \mathcal{H}^c_k(k[G])
\end{align*}
$$

In fact there is probably a canonical map $H_c(BG) \longrightarrow H^c_k(k[G])$, which extends to an isomorphism $k[G] \otimes H_c(BG) \longrightarrow H^c_k(k[G])$.

Easy to understand the Dennis trace $H_c(BG) \longrightarrow HH(A)$ cyclic set $EG \times G^c$ realized by $G^3 \cong G \times G \cong G$ with usual faces $(g_0, g_1, g_2) \mapsto (g_0 g_1, g_2)$, $(g_0, g_1 g_2)$, $(g_2 g_0, g_1)$.

Induces map $k[EG \times G^c] \longrightarrow $ Hochschild of $A = k[G]$. Put another way, given $A$ look at the cyclic set of sequences of...
invertibles \( g_0, \ldots, g_n \) in \( A \) with these faces.

\[ B \text{ is the sub cyclic set of } (g_0, \ldots, g_n) + g_0 \cdots g_n = 1. \]

So the Dennis trace map is the map \( H_x(BG) \to HH(A) \)
induced by the map of cyclic sets
\[
\{(g_0, \ldots, g_n) \in G(A)^{n+1} \mid g_0 \cdots g_n = 1\}
\]
\[
\mapsto (g_0 \otimes \cdots \otimes g_n) \in M_r(A)^{n+1}
\]
\[
\tr
\]
\[
\tr (g_0 \otimes \cdots \otimes g_n) \in A^{n+1}
\]

What about the non-unital D.T. map?

\[ k[G] \longrightarrow P \otimes_A Q \]

\[ k[G] \longrightarrow \widehat{P \otimes_A Q} \]

\[ \text{unital homom.} \]

A good question may be what are needed to handle the non-unital case.

A good question may be what are needed to handle the non-unital case. A Hoch complex has bar complex included.

Proposal: Find characteristic classes of representations of groups on dual pairs \((P, Q)\) over \( A \) with \( P \) flat, eventually \( P \in \mathcal{P}(\beta^0 A) \). This means assembling \( H_x(G) \), \( G = (P \otimes_A Q)^* \) for all these pairs.

First construct Dennis trace.
214 1140 Do what's going on?

renormalized Dennis trace map. Given dual pair \((P, Q, < > : Q \otimes P \rightarrow A)\). We have ring \(P \otimes Q\) and can consider \((P \otimes Q)^{\times} = \left\{ \frac{1 - \sum p \otimes q}{1 - \sum q \otimes p} \right\}\) invertible.

As before have map of cyclic sets from \(k[BG]\) to, better

\[
\begin{align*}
(G^3) & \Rightarrow (G^2) \Rightarrow (G) \\
\vdots & \vdots & \vdots \\
\hat{\mathcal{E}} \hat{\mathcal{E}} \hat{\mathcal{E}} & \Rightarrow \hat{\mathcal{E}} \hat{\mathcal{E}} \Rightarrow \hat{\mathcal{E}}
\end{align*}
\]

Do I have a trace map to Hochschild \(x \otimes A\)?

Guess? True Hoch \(cx\) has bar complex as quotient. Actually what do you do for \(A = \hat{\mathcal{E}} \otimes A\). I guess what works is to go into \(\hat{B}\) then normalize. Actually the true Hoch \(cx\) is produced by the cyclic traces of Connes - Tsygan. So what? Godg, \(\cdots \cdots dB < \Omega(B)\). So what about from \(B = P \otimes Q\) to \(A\)?

It seems that we have a trace map \(\text{No}\)

\[
B^\otimes n = (P \otimes Q)^\otimes (\cdots) \otimes \cdots \otimes (\cdots)
\]

Did I even get the trace map straight for matrices? With \(A \otimes \cdots \otimes A \rightarrow \otimes_{s} A \otimes \cdots \otimes_{s} A\).

This seems okay.

\[
\begin{align*}
& (P \otimes A \otimes P) \otimes (P \otimes A \otimes P) \\
& (V \otimes A \otimes V) \otimes
\end{align*}
\]
You have to understand $\tilde{M} \tilde{B}$ for a $B$-bimodule $M$. Let $E \to M$ be a flat $B$-bimodule resolution. Need to assume $\tilde{B} \otimes \tilde{B}$ is left and right flat, e.g., if $B$ flat over $k$. Let $F \to \tilde{B}$ be a flat bimodule res. of $\tilde{B}$. Then

$$M \otimes_{\tilde{B}} F \otimes_{\tilde{B}} \to E \otimes_{\tilde{B}} F \otimes_{\tilde{B}} \to E \otimes_{\tilde{B}} \tilde{B} \otimes_{\tilde{B}} = E \otimes_{\tilde{B}}$$

quis because can suppose $F = \tilde{B} \otimes \tilde{B}$

so now take $\tilde{B} = k[G]$, $M = \mathbb{Z}^h$

Use $F \to \tilde{B}$, $E \to \mathbb{Z}$. Then

$$k \otimes_{\tilde{B}} F \otimes_{\tilde{B}} \cong E \otimes_{\tilde{B}}$$

Take $F$:

$$k \otimes_{\tilde{B}} F \otimes_{\tilde{B}} k \otimes_{\tilde{B}} B \otimes_{\tilde{B}} B \to \tilde{B} \otimes_{\tilde{B}} \tilde{B} \otimes_{\tilde{B}} \tilde{B} \otimes_{\tilde{B}} \tilde{B} \to \tilde{B} \otimes_{\tilde{B}}$$

Then $k \otimes_{\tilde{B}} = k \otimes_{\tilde{B}} F \otimes_{\tilde{B}}: B^2 \to B \to k$

so there should be little problem!

Anyway we agree that it is clear that

$$k \otimes_{\tilde{B}} \to \tilde{B} \otimes_{\tilde{B}} \to k \otimes_{\tilde{B}}$$

amounts to the 2 columns of the double ex.

Now what about Dennis trace
It seems you are using $P$ free so as to get a left action on $P$, effectively writing $P = V \otimes \tilde{A}$.

13:52. We’ve been through this before— if you want to compare Hochschild for $A$ and $B = P \otimes_R Q$ you use a bicomplex

$$B \otimes B \xleftarrow{\sim} P \otimes_A Q \otimes_B A = Q \otimes_B P \otimes_A \rightarrow A \otimes_A A$$

if $P$ is $A$-flat
and $P \otimes_A Q = B$
so you use something like

$$P \otimes_A Q \otimes_B B$$

now ask whether there’s a good way to handle the Dennis trace map. Where does $h$-unital enter? What sort of things happen. Here $A, B$ are not unital. Let’s understand the nature of the argument. You have $k[G] \rightarrow \tilde{P} \otimes_A Q = \tilde{B}$.

You get $k[G] \otimes k[G] \rightarrow \tilde{B} \otimes_B \rightarrow B \otimes_B \rightarrow 0$ because $k[G]$ unital.

It seems like there is always a map from 15:26. Treat these problems. How to handle? Start again. You have $k[G] \rightarrow \tilde{P} \otimes_A Q = \tilde{B}$.

Wait: Take $\tilde{B} = k[G]$. Have $\Delta$

$$B \otimes_B \rightarrow \tilde{B} \otimes_B \rightarrow \tilde{Z} \otimes_B$$
But the point somehow is that you have $\mathbb{Z}[G] \rightarrow \tilde{B}$, a map of augmented rings. What am I going to do.

You have $0 \rightarrow B \rightarrow \tilde{B} \rightarrow \mathbb{Z} \rightarrow 0$

Exact seq of $B$-bimodules, hence

$0 \rightarrow B \otimes B \rightarrow \tilde{B} \otimes B \rightarrow \mathbb{Z} \otimes B \rightarrow 0$

Visualize using columns of matrices.

You now take $B = \mathbb{Z}[G] = I[G]$. So life goes on slowly. But there is a Dennis trace map which should go from $H_x(BG) \rightarrow HH_x(Z[G])$. This I sort of understand, namely $BG$ is simp. set of $(g_0, \ldots, g_n) \mapsto g_0 \cdot g_n = 1$, and this simplex goes to $g_0 \otimes \cdots \otimes g_n \in \tilde{B} \otimes B^{\otimes n} = \text{quotient of } B^{\otimes n+1}$ by degenerate

This might be very interesting.

Summary: We have this simplicial (cycle) model for $BG$ namely consisting of $(g_0, \ldots, g_n) \mapsto g_0 \cdot g_n = 1$

Standard model $(g_1, \ldots, g_n)$

$C^2 \cong Z G$

$d_0(g_0g_1) = g_2 \quad d_1(g_0g_1) = g_1g_2$

$d_2(g_1g_2) = g_1$
Now use \((g_0, g_1, g_2)\), \(g_0 g_1 g_2 = 1\).

\[
\begin{align*}
(g_2^{\dagger}, g_1^{\dagger}, g_3, g_2) & \xrightarrow{d_0} (g_2^{\dagger}, g_2, g_3) \\
& \xrightarrow{d_1} (g_2^{\dagger}, g_2, g_3) \\
& \xrightarrow{d_2} (g_2^{\dagger}, g_2, g_3) \\
& \xrightarrow{d_3} (g_2^{\dagger}, g_2, g_3)
\end{align*}
\]

Deleting commas and crossover.

So a simplex in \(BG\) \((g_1, g_2, \ldots, g_n)\) goes to \((g_1, g_2, \ldots, g_n)^{\dagger} \otimes g_1 \otimes \ldots \otimes g_n\) in \(BG \otimes \mathbb{Z}^n\).

The image of this element in the 0' complex is \(g_1, \otimes g_2, \otimes \ldots \otimes g_n\). Does this sound reasonable? Is it a map of complexes?

Go back to \(BG\)

\[
\mathbb{Z}[G^3] \equiv \mathbb{Z}[G^2] \equiv \mathbb{Z}[G] = pt
\]

It seems likely that the normalisation is

\[
\rightarrow I[G] \otimes I[G] \otimes I[G]
\]

here \(I(G) = \mathbb{Z}[G]/\mathbb{Z}\). Keep on trying!!

\[
d(g_1, g_2, g_3) = (g_2, g_3) - (g_1g_2, g_3) + (g_1, g_2g_3) - (g_1, g_2)
\]

\[
d(g_1, g_2) = (g_2) - (g_1g_2) + (g_1)
\]

But what you want is \(I[G]^{\otimes n}\) inside \(\mathbb{Z}[G]^{\otimes n}\) somehow.
Look back for a more intelligent method.

Why do you care?

What do I want to do so to ended [G[6]]

\[ Z[G[6]] \]

\[ \text{becomes} \]

\[ G \]

\[ \text{becomes} \]

\[ Z[G] \]

\[ \text{becomes} \]

\[ Z[G[6]] \]
Actually what you care about is the composition
\[ Z[BG] \longrightarrow (B^{\otimes n+1}, b) \longrightarrow (B^{\otimes b}, b') \longrightarrow (B^{\otimes b'}, b) \]

and this should be the normalization of the simplicial abelian group $Z[BG]$. The first point is to identify $BG$ (in degree $n$ is $G^n$ with faces deleting commas, degeneracies inserting 1's, and
c. \( (g_1, \ldots, g_n) = \begin{pmatrix} g_1 & \cdots & g_n \\ g_2 & \cdots & g_n \\ \vdots & \ddots & \vdots \\ g_n & \cdots & g_n \end{pmatrix} \end{pmatrix}

with the subsemiplexial set of $BG \times \mathbb{Z}$ consisting of $(g_1, \ldots, g_n)$ in deg. $n + g_0 \cdots g_n = 1$, faces delete commas and crossover degeneracies insert 1's. After $g_0$ until after $g_n$, in degree $n$ $Z[BG] = Z[G^n]$

\[ (g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_n)^\circ \otimes g_1 \otimes \cdots \otimes g_n \]

\[ \mapsto (g_1, \ldots, g_n)^\circ \otimes g_1 \otimes \cdots \otimes g_n \]

\[ -b'(g_1 \otimes g_2) = -(g_1-1) \cdot (g_2-1) \cdot g_1 \cdot g_2 + \delta(g_1) \cdot (g_2-1) \cdot g_1 + \delta(g_2) \cdot g_1 
\]

\[ (g_1, g_2) \mapsto g_2 - g_1 g_2 + g_1 \mapsto \delta(g_2) - \delta(g_1 g_2) + \delta(g_1) = \delta(g_2) - g_1 \cdot g_2 + \delta(g_1) = -(\delta(g_1))(\delta(g_2)) \]
It seems as if we have lifted the complex \((B^\otimes 1_B)\) yielding \(H_*(BG)\).

You have this map of complexes:

\[
\begin{align*}
\mathbb{Z}[BG] & \longrightarrow (B^\otimes 1_B) \longrightarrow (B^\otimes 1_B) \\
(g_1, \ldots, g_n) & \longrightarrow (g_1, \ldots, g_n) \otimes g_1 \otimes \cdots \otimes g_n
\end{align*}
\]

This composition is probably the projection on the normalized quotient complex. This is clear if the degeneracies in \(BG\) insert 1's. So what it means is that the above composition is a homotopy equivalence. So you find that the homology \(H_*(BG)\) is a summand of the Hochschild homology \(HH_*(\mathbb{Z}[G])\) — this is something you know!

This is confusing because we have linked bar homology of \(\mathbb{Z}[G]\) with something. I want mainly to go from \(\mathbb{Z}[G] \longrightarrow \text{PQ}\) to a map \(H_*(BG) \longrightarrow H_*(A)\). So how do I get to the bottom of this??? Try

--

03/19/97

Go over what you learned yesterday. Dennis trace. Consider a unital ring homom. \(k[G] \longrightarrow A\), equiv. a group hom \(G \longrightarrow \text{Aut}(A)\).

Map of cyclic sets

\[
\begin{align*}
E_6 \times E_6 G^c & = \{(g_0, \ldots, g_n)\} \\
A^\otimes 1_B & \downarrow \\
g_0 \otimes \cdots \otimes g_n
\end{align*}
\]

\(B_6 \subset E_6 \times E_6 G^c = \{(g_0, \ldots, g_n) | g_0 \cdot g_n = 1\}\).

\(B_6 = G^n, \quad d_i (g_1, \ldots, g_n) = (g_{i+1}, \ldots, g_n)\quad 0 \leq i < n, \quad \delta_i (g_1, \ldots, g_n) = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} g_1, \ldots, g_n \end{pmatrix} = \begin{pmatrix} g_1, \ldots, g_n \end{pmatrix} \).

\[d_i (g_1, \ldots, g_n) = (g_{i+1}, \ldots, g_n) \quad 0 \leq i < n, \quad \delta_i (g_1, \ldots, g_n) = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} g_1, \ldots, g_n \end{pmatrix} = \begin{pmatrix} g_1, \ldots, g_n \end{pmatrix} \)
Suppose $A = \tilde{B}$

$\mathbb{Z}[BG] \longrightarrow (\tilde{B} \otimes \tilde{b}) \longrightarrow (\tilde{B} \otimes \tilde{B} \otimes \tilde{b}) \longrightarrow (\tilde{B} \otimes b', -b')$

$\mathbb{Z}[G^\times] \longrightarrow \mathbb{Z}[G^\times']$

$(g_1 \cdots g_n) \quad (g_1 \cdots g_n)^{-1} \quad g_1 \cdots g_n \longrightarrow \tilde{g}_1 \otimes \cdots \otimes \tilde{g}_n$

What does this mean? Have

$0 \longrightarrow (B^{\otimes\times' \tilde{b}}) \longrightarrow (\tilde{B} \otimes B^{\otimes \tilde{b}}) \longrightarrow (B^{\otimes \tilde{b}}, -b') \longrightarrow 0$

so it seems that this splits canonically. But this should mean that

$H_*(\mathbb{Z}[G], \mathbb{Z}[G]) \cong \bigoplus_{i+1} H_*(BG)$

This is no surprise, but it would seem that the $\otimes$ $(B^{\otimes\times' \tilde{b}})$ which computes $B^{\otimes \tilde{b}}$ is slightly removed from the group homology. Puzzle,

Anyway consider next $\mathbb{Z}[G] \longrightarrow P \otimes_A Q$.

Idea: get map on Hoch homology, then you want to use $M$ inv. of Hochschild to get $H_*(BG) \longrightarrow HN(A)$.

Philosophy is simple, but there seem to be technical problems. Start with $A$ non unital, consider a dual pair $(P, Q)$ over $A$ with $P \in \mathcal{D}(A^{op})$ and a rep $\mathbb{Z}[G] \longrightarrow \tilde{P} \otimes_A Q$. I can suppose $Q = P^{\otimes A_0} \otimes_{A_0} (P, A)$.

Put $B = P \otimes_A Q$. Have map

$B \otimes_B = P \otimes_A Q \otimes_B = Q \otimes_B P \otimes_A \longrightarrow A \otimes_A$

and map $\mathbb{Z}[G] \otimes_B \mathbb{Z}[G] \longrightarrow B \otimes_B$.
So how do I proceed next?

Try to handle it: given \( \overline{\mathbb{Z}[G]} \to \mathbb{P} \otimes \mathbb{A} \)

to map \( \mathbb{Z}[G] \) to \( \mathbb{A}^{\otimes} \).

Take \( \mathbb{P} \in \mathcal{P}(A^\otimes) \), but can suppose

suppose \( \mathbb{P} \in \mathcal{P}(A^\otimes) \), \( \mathbb{Q} = \mathbb{P} \), choose non-unital embedding \( \mathbb{P} \otimes \mathbb{A} \to \mathcal{M}_n(A) \) - usual business of embedding in a trivial bundle.

These are problems continuing. What should I do? Can

I have two approaches to Morita invariance of \( \mathbb{H} \).

given \( (\mathbb{A}, \mathbb{Q}) \) pair right flat

then \( \mathbb{A} \otimes A \leftarrow \mathbb{Q} \otimes \mathbb{P} \otimes \mathbb{A} \leftarrow \mathbb{P} \otimes \mathbb{Q} \otimes \mathbb{B} \leftarrow \mathbb{B} \otimes \mathbb{B} \)

double \( \mathbb{Q} \otimes \mathbb{B}^\otimes \otimes \mathbb{P} \otimes \mathbb{A}^\otimes \) gluing the diagram.

There are things here I find puzzling. K-theory viewpoint. Maybe your approach to K-theory is biased, namely gluing \( (\mathbb{P} \otimes \mathbb{Q})^\times \) for appropriate dual pairs over \( \mathbb{A} \).

Hence \( \text{Hom}_{\text{gps}} (\mathbb{G}, \mathbb{B}^\times) = \text{Hom}_{\text{gpr}, \text{rig}} (\mathbb{Z}[G], \mathbb{B}) \)

\[ = \text{Hom}_{\text{gpr}, \text{rig}} (\mathbb{Z}[G], \mathbb{B}) \]
latter can fractionalize, e.g., to $\mathbb{Q}$, as well as $\mathbb{Q}$ with $p$-adic $\mathbb{Q}$.

Exercise. Show that $\mathbb{Q}(x)$ is a transcendental extension of $\mathbb{Q}$.

Note that $\mathbb{Q}(x)$ is the field of rational functions in $x$ with coefficients in $\mathbb{Q}$.

Induces $K^p(\mathbb{Q}) \rightarrow \mathbb{Q}(A)$

$K^p(\mathbb{Q}) \rightarrow \mathbb{Q}(A)$

$P \mathcal{O} \rightarrow P \mathcal{O}^*$

Here $\mathcal{O}^* = A^* \mathcal{O}^*$

The canonical map $\mathcal{O} \rightarrow (P^*, P^*)$ where $P^* = \mathcal{O} \mathcal{P}^*$

affines to define in a natural way for affine pairs with $P^* \mathcal{P}(\mathcal{P})$.

Review basic construction of base map $K^p(\mathbb{Q}) \rightarrow \mathbb{Q}(A)$.

I am slowly getting the terminology. Here we should work more on the details, using $K^p(A) = K_0(K^p(A))$. I should review for a few hours with details.

Do you have a tutorial? I don't know what is happening. I think when you are practicing group rings, you have to get the terminology. Here we should work more on the details.
reduce to

\[ \mathbf{P} \rightarrow \mathbf{P} \oplus \mathbf{P'} \rightarrow \mathbf{P'} \]

I must do this carefully.

General case: \( u : \mathbf{P} \rightarrow \mathbf{P'} \quad u \in P \otimes A \)

\[ P = P \otimes A P' \]

\( (p_1 \otimes \lambda_1)(p_2 \otimes \lambda_2) = p_1 \lambda_1, u(p_2) > \otimes \lambda_2 \)

\[ P \otimes A P' \xrightarrow{u \otimes 1} P' \otimes A P' \]

\[ 1 \otimes u^* \rightarrow P \otimes A P' \]

Note \( \nu \in P \otimes A P' \) such that \( 1 + \nu \) invertible

\[ \Rightarrow (u \otimes 1) \nu = \nu : \mathbf{P'} \rightarrow \mathbf{P'} \quad \text{and} \quad 1 + \nu \text{ invertible in } \mathbf{P'} \]

But suppose you factor \( u \)

Two cases: \( \quad \mathbf{P} \rightarrow \mathbf{P} \oplus \mathbf{P'} \)

\( (\mathbf{P}) \otimes A \stackrel{(0,1) \otimes 1}{\longrightarrow} (\mathbf{P}) \otimes A \)

\[ \begin{array}{c}
\downarrow \\
\downarrow \\
\mathbf{P} \otimes (p' \otimes p') \end{array} \quad \stackrel{(u) \otimes 1}{\longrightarrow} \quad \begin{array}{c}
\downarrow \\
\downarrow \\
\mathbf{P} \otimes (p' \otimes p') \end{array} \]

The real puzzle is whether one can assume \( u : \mathbf{P} \rightarrow \mathbf{P'} \text{ is zero modulo } A \).

It seems that the general case is to construct the trace for a general flat fibring \( \mathbf{P} \); you can write \( \mathbf{P} \) as ind. limit of free \( f' \) modules over matrices \( f, A \).
Let's try to do a little. Take $B$ flat ring, $A = B \oplus I$, $I$ is a $B$-module regarded as $B$-bimodule with $IB = 0$. $(B \oplus I, B)$

Now assume $I = B$ then we have $A = 1$. NO.

Better $A$ is a $B$-module equipped with $B$-map $f : A \to B$.

$a \otimes b \mapsto f(a)b$

Now suppose $A = B \oplus I$, $f = p_1 : A \to B$. Then have homs. $A \to B < A$, actually maps of dual pairs $(\tilde{B}, A) \to (\tilde{B}, B) \to (\tilde{B}, A)$

$\langle a, b \rangle = f(a)\tilde{b}$, \hspace{1cm} $\langle b, \tilde{b} \rangle = \tilde{b}b$ \hspace{1cm} $\langle b, \tilde{b} \rangle = f(b)\tilde{b}$

I want to show that $\Delta : A \to B \otimes A$ induces the identity on $\text{Hom}_A (\text{GL}(A))$. I think I know that this is true for $K_A(A) = \text{Ker} (K_A(A) \to K_A(\mathbb{Z}))$.

Digress to ask whether, instead of looking at $(B \oplus B_0)$ with $B$ left flat, it might be better to work with $B \oplus B_1$ and $B$ left flat. Or bypass symmetry $B \oplus B_0$ with $B$ right flat. Show $(A \oplus B)$ $B$ is $B^k$ flat $\iff B \otimes \mathbb{P} = A \otimes \mathbb{P}$. $\Delta$ is $A^k$
\[
\begin{align*}
\text{B is } B^{op} \text{-flat } & \iff B \otimes_B P = A \text{ is } A^{op} \text{-flat} \\
B \text{ is } B \text{-flat } & \iff Q \otimes B = B \text{ is } A \text{-flat} \\
A \text{ is } A \text{-flat } & \iff P \otimes_A A = A \text{ is } A \text{-flat} \\
A \text{ is } A^{op} \text{-flat } & \iff A \otimes Q = A \otimes B = B \text{ is } B^{op} \text{-flat}
\end{align*}
\]

I want to take \( A = B \)

I persist stupidly in trying to show flat rings have Morita inv. for \( \text{Hom}(B \otimes (-)) \). What you should be doing is everything you can concerning the construction you know works. For example, what about \( B \) and \( B^{op} \) rings which are both left and right flat, what about dual pairs \( (P, Q) \) over \( A \) where \( P \) is \( A^{op} \)-flat and \( Q \) is \( A \)-flat. Show the two trace maps. Can assume \( P \in P(A^{op}) \) and \( Q \in P(A) \). Then you have \( Q \rightarrow \text{Hom}_{A^{op}}(P, A) = A \otimes_A P \) and \( P \rightarrow \text{Hom}_A(Q, A) = A \otimes_A A \).

Then we can look at

\[
\[
\]

So you need to know whether the representations of \( B = P \otimes_A Q \) on \( P \) and on \( Q \) This ought to follow from functoriality with \( P \rightarrow Q^* \subset Q \).
Let's review the implications that $\mathbf{K}$ is $\mathbf{E}$.

Monte invariance $\Rightarrow K$ triviality of affine duals.

Assume flat left flat, $f: A \rightarrow A/I = B$. Two cases $IA = 0$. Then $A$ is $B$-module mapping onto $B$ and $a_1a_2 = f(a_1)a_2$. Here $(A \ B)$ and $A$ is $A$ flat $\Leftrightarrow P \otimes_A A = A \otimes_A A = A$ is $B$ flat.

Note $A$ is $A^{op}$ flat $\Rightarrow A \otimes A = A \otimes B = B$ is $B^{op}$ flat.

True for any $f: A \rightarrow B$. This is in $\mathbf{K}$.

So you want to prove that for $B$ a right flat ring, and any $B$-module map $f: A \rightarrow B$ s.t. $f(A)B = B$ that $GL(A) \rightarrow GL(B)$ is a homology iso. This seems much simpler than what I was trying to do.
Consider a left Morita equiv. from \((A \times B, A \otimes B \rightarrow B)\), say given by a \(A\)-dual pair \((B, A, A \otimes B \rightarrow B)\). Then \(A \cong B \otimes B\) is a ring with \((a_1)(b_2 a_2) = \delta \langle a_1, b_2 \rangle a_2\). To simplify suppose \(\langle a, b \rangle = f(a)b\) where \(f: A \rightarrow B\) is a \(B\)-bimod. map. Then \(a \langle b_2 a_2 \rangle = f(a)b_2 a_2\), i.e. \(a_1 a_2 = f(a_1)a_2\).

Suppose given \((A \times B, A \otimes B \rightarrow B)\) left Morita equiv. \((B, A, A \otimes B \rightarrow B)\) right Morita equiv. Then \(A\) is \(A^{op}\)-flat \(\iff A \otimes A^\vee = A \otimes B = B\) is \(B^{op}\)-flat. So therefore, once \(B\) is \(B^{op}\)-flat then so any \(A\) left Morita equiv. to \(B\). Now let's try to really understand this, especially in terms of matrix equations.

Assume \(f: A \rightarrow B\) gives the pairing. Can you show \(A\) is \(A^{op}\)-flat if \(B\) is \(B^{op}\)-flat by equations?

\[ A \rightarrow A^{op} \rightarrow A \]

\[ 0 = u a \quad u a' \]

\[ a' a = 0 \quad a = u' a' \]
Thus if you have
\[ A \to A^{\infty} \to A \]

apply the hom \( f: A \to B \)
you get \( f(u_0) f(a) = 0 \)
so can factor \( a \)
\[ f(u_0) = 0 \]

The idea should be that if \( A \) is a filtered
ind lim \( \text{ind lim} \)
to we have \( \text{ind lim} \) \( (B,A) \to (B,B) \) and
we assume \( B \) is right flat, \( \text{ind lim} \)
can write \( B \)
only lim \( \text{ind lim} \) of \( P_i \to B \)
to as filtered ind lim of \( P_i \to B \).

\[ M(P_i \otimes B) \to GL(P_i \otimes A) \to GL(P_i \otimes B) \]
But OK we have \( B \) \( \text{B is right flat}/B \)
\[ \text{B is right flat}/B \]
\[ \text{A is right B/A} \]
\[ \text{B is right flat}/B \]
\[ \text{A is right B/A} \]

The fact that \( A \) is flat and \( B \) left acts
on \( A \) might lead to a K map \( B \to A \)
Suppose \( B \) is flat. Then \( 0 \to I \to A \to B \to 0 \) is exact in \( \text{Mod}(B) \). General case: dual pair \((B, A)\).

Then \( A = \text{Hom}_B(B \otimes A) \) is right flat. \((A \to B)\)
f: \( A \to B \) is a map hom of right flat maps.

To prove \( \text{GL}(A) \to \text{GL}(B) \) induces isom. on \( H^*(B(-)) \)

\[ 0 \to \text{M}(I) \to \text{GL}(A) \to \text{GL}(B) \to 0 \]
group extension. \( E^2_{1,0} = H_0(\text{GL}(B), H_0(\text{M}(I))) \Rightarrow H^0(\text{GL}(A)) \)

Observe that \( A \) can be \( B \oplus I \) with any \( B \)-module, e.g. \( B \). You have a lot of freedom here, namely, any \( B \)-ring left \( I \) Morita equivalent to \( B \), any \( B \)-mod \( A \) equipped with \( f: A \to B \), e.g. \( B \).

Do I understand the semisimple case, \( B \) semisimple, \( A \) any \( B \)-module? This is the case \( A \in P(A^{op}) \), \( B = \text{End}_{A^{op}}(A, A) \), that I studied — Duskin.

Have \( A \to B \) hom. \( A_B \) \( B \)-module.

We have \( B \otimes A \) inducing \( B \to K_x(B) \to K_x(A) \).

\( U \in P(A^{op}) \)

\[ U \to U \otimes_B A \]

\( V \in P(B) \)

\[ V \to V \otimes_B A \]

\[ B^\prime \to A^\prime \]

\[ U \to U \otimes_A B \to U \otimes_A B \otimes_A A = U \otimes_A A \]
\[ A \in \mathcal{P}(A^\text{op}) \quad B = \text{Hom}_{A^\text{op}}(A, A) \]
\[ A \rightarrow B \quad f(A) B = B. \]

For example, if \( A = B \oplus B \) and \( f = \text{pr}_1 \).

\[ \begin{array}{c}
\begin{bmatrix} A & B \\ A & B \end{bmatrix} \\
\begin{bmatrix} U \rightarrow U \otimes_A B \\ V \otimes_B A \leftarrow V \end{bmatrix}
\end{array} \]

One composition is the identity, the other is \( U \rightarrow U \otimes_A A \) which is \( A^\text{op} \)-nil semi-direct to the identity. And I guess we know how to handle this. If I am comparing \( U \rightarrow U \otimes_A P = U \otimes_A A \) with \( U \rightarrow U \otimes_A P' = U \otimes_A A = U \). We have the:

\[ P \otimes_A Q \rightarrow P' \otimes_A Q \]

You asked.

\[ A \in \mathcal{P}(A^\text{op}) \subset \mathcal{P}(\tilde{A}^\text{op}) \]
\[ A \xrightarrow{y} \tilde{A}^n \quad y \times 1 \]

\( P(A^\text{op}) \), \( U \rightarrow U \otimes_A A \) - I want to see the effect on objects. So we get \( \tilde{A} \rightarrow M_n(A) \subset M_n(\tilde{A}) \)
\[ \tilde{a} \rightarrow x \tilde{a} y \]

and I need to compare this homomorphism with the identity. So how to proceed? Factor \( A \subset \tilde{A} \) into:
\[ A \rightarrow A \oplus \tilde{A} \rightarrow \tilde{A} \]
Suppose $A = B \oplus I$, $I$ a unitary $B$-module, $B$ unital. Note that $A$ has a left identity: $A = eA$, $B = eA$.

This implies that $A \cong \tilde{A}$ is a summand of the right $A$-module $\tilde{A}$. We need to consider $B \otimes \tilde{A}$. In this situation besides the homomorphism $A \to B$, we also have a homomorphism $B \to A$. The second is in general replaced by the bimodule $B \otimes \tilde{A}$. Wait: A homomorphism $B \to A$ yields $V \mapsto V \otimes B \otimes \tilde{A}$. But here we have $V \mapsto V \otimes A$ which is defined.

Let's take this more generally. Point is that $B$ is the multiplier algebra.

So what? You have...

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03/23/97 Consider $A = B \oplus I$, $I$ a unitary $B$-module. Can I find a simple proof that $GL(A) \to GL(B)$ induces $\sigma_{im}$ on $H_A$. This is the simplest case, but more generally we can look at any dual pair $(B, A)$ over $B$, equiv. a $B$-module $A \to B \to f(A)B = B$. Such on $A$ is right flat, in fact, in $\mathcal{P}(A^\circ)$.