Let's go over the calculation which you didn't finish. Given $R, I, e, \bar{e}$ get $(eR, \bar{e}R, \alpha : eR/I \rightarrow \bar{e}R/I)$ and add $(1-e)R, (1-e)R, 1-e)R$ to get $eR \oplus (1-e)R, R, \bar{e} \oplus (1-e)$ mod I

$L \leq M \leq P \leftarrow L$

$eR \oplus eR \oplus eR$

$(1-e)R \oplus (1-e)R \oplus (1-e)R$

What you want to do is to show that $M = eR \oplus (1-e)R$ together with $M/MI \cong R/I$ yields an obj $M$ of $P(I)$. This means you need to produce an idempotent matrix over $\bar{I}$. Do this. Thus we want maps with composition identity:

$M \leftarrow \begin{bmatrix} \bar{p} & u \end{bmatrix} \begin{bmatrix} R & (e) \\ (\bar{e}) & M \end{bmatrix}$

$p^2 + uv = 1$

Important part of $M$:

$eR \leftarrow eR \oplus \bar{e}R \leftarrow eR$

$\begin{bmatrix} e \end{bmatrix} \leftarrow \begin{bmatrix} e-e\bar{e}e \end{bmatrix} \leftarrow \begin{bmatrix} 2 \bar{e}e-\bar{e}e \end{bmatrix}$

$\begin{bmatrix} 0 & 1-e & 0 & 0 \\ 0 & 1-e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} \bar{e} & 0 & 0 & 0 \\ 0 & 1-e & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$1 - p^2 = y^2$

$y = e-e\bar{e}e$?
\[
\begin{pmatrix}
\varepsilon_2 & 0 \\
0 & 1 - \varepsilon \\
\varepsilon & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\varepsilon \varepsilon & 0 & \varepsilon & 0 \\
0 & (\varepsilon + \varepsilon) & 0 & 0 \\
0 & 0 & 1 - \varepsilon & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
\varepsilon_2 \varepsilon \varepsilon & 0 & \varepsilon \varepsilon & 0 \\
0 & 1 - \varepsilon & 0 & 0 \\
\varepsilon \varepsilon & 0 & 1 - \varepsilon & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix} = \begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix}
\]

\[
\varepsilon - \varepsilon^2 = \varepsilon - (\varepsilon - \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon)^2
\]

how to do it simply: you first look at

\[
\begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix}
= \begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix}
\]

This leads to idempotent

\[
\begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix} + I
\]

on

\[
\begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix}
\]

Combine with

\[
\begin{pmatrix} 1 - \varepsilon & \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon & 1 - \varepsilon \end{pmatrix}
\]

and you get an idempotent on \( R^{\oplus 2} \).

\[
\begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon + 2 \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\
2 \varepsilon \varepsilon - 3 \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon + \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\
\varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\
(e-\varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon)^2
\]

\[
\begin{pmatrix} \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon \\ 1 - \varepsilon \end{pmatrix}
\]
Try writing this out always working inside of $R$. First however try $(E, R, E/EI = R/I)$

drift to

$R \leq E \leq P \leq R$

$x = 1 - gp : \ R \ominus I \leq \ R$

$y = 1 - p^g : \ E \ominus EI \leq \ E$

\[
\begin{pmatrix}
(8 \sqrt{x}) \\
(\sqrt{x})
\end{pmatrix}
\begin{pmatrix}
(p) \\
(\sqrt{x})
\end{pmatrix}
\begin{pmatrix}
(8 \sqrt{x}) \\
(\sqrt{x})
\end{pmatrix}
\begin{pmatrix}
(g \cdot x) \\
(g \cdot x)
\end{pmatrix}
= \begin{pmatrix}
(p^g) \\
(p^x)
\end{pmatrix}
\]

what's needed is an embedding of $E$ as a summand of $R \otimes N$, so as to get the idempotent over $\bar{E}$.

\[
\frac{E - \text{e-e-e}}{\text{e-e-e}} = E - \text{e-e-e} + \text{e-e-e}
\]

Check: $e - e(2e-e)^2$

$E \leftarrow E \leftarrow E \\
\uparrow_{K_2} \quad \uparrow_{K_{12}} \\
K \leftarrow K \leftarrow K$

\[
K(4A) \leftarrow K(4A) \leftarrow E, \quad K \leftarrow K \leftarrow E
\]

Theorem. Right $A$-flat.
Do on!!!! Let's try the following: Suppose you have a quasi-homomorphism \( A \to R \to \mathcal{I} \) and a Morita equivariant \( (I \to \mathcal{U}) \). Can you want to understand the map \( K_0(A) \to K_0(I) \to K_0(B) \)? You have your triple \( E \).

**Problem:** Given \( C \to R \to B = P \otimes_A Q \)

Better suppose \( R = \text{Mult}_A(P, Q) \) and

\[
C \cong R = \text{Mult} \text{ ring of } (P, Q) \text{ over } A
\]

\[
= \left\{ \text{Hom}_{A^{op}}(P, P) \times \text{Hom}_A(Q, Q)^{op} \mid \langle g, \rho(p) \rangle = \langle g, \rho, \rho(p) \rangle \right\}
\]

Then \( R \) contains an ideal \( I = \text{image of} \ P \otimes_A Q \)

\[ (p \otimes q)(p') = p \langle q, p' \rangle \]

\[ \lambda(p \otimes q) \cdot p' = (2p \otimes q)p' = p \otimes (2q)p' \]

\[ (\lambda(p \otimes q))(p') = \lambda((p \otimes q)p') = \lambda(p \langle q, p' \rangle) \]

\[ P \otimes_A Q \to I \ is \ a \ square \ 0 \ extension. \]

So you should get a map \( K_0(C) \to K_0(P \otimes_A Q) = K_0(A) \).

(necess to assume \( A = QP \) probably.)
So how does this subproblem proceed?

Suppose you have $L, M \in \mathcal{R}(R^o)\text{ and an isom module } I: L/LI \cong M/MI$.

Then this step seems tricky.

The real problem is whether how to go from $\mathcal{R}(I)$ to what you need. But here's a better idea: Consider getting.

Suppose you have this $K_0$ class given by $(L, M, \alpha)$ over $R$.

How are you going to represent it over $A$?

First look at $R, I$ and suppose you have a complex $U$. Attache $U$ complex of right modules to identity of $U$ defers to a

$U \otimes \text{Hom}_{R^o}(U, I) \to \text{Hom}(U, U)$

$U \otimes \text{Hom}_{R^o}(U, R)$

so I have to review my $K_0$ paper.

Basically you want $U, V, pairing V \otimes U \to \mathbb{C}, $ and $\langle \nu, \mu \rangle = \langle \nu, \mu \rangle$

get $U \otimes V \to \text{Hom}_{R^o}(U, U) \otimes \text{Hom}(V, V)^{op}$

try doing this with multiplier

$\text{Hom}_{R^o}(U \otimes V, U)$.
$U \otimes^C V \longrightarrow \text{Hom}_{C}(U, V)^{o^p}$

$\text{Hom}_{C}(U, U) \longrightarrow \text{Hom}_{C}(U, \text{Hom}(V, C))$

Multiplicative algebra is the fibre product. So then you deform $R$. Suppose you have then such a gadget over $C$ and a Morita eq. $(C \ Y)$

Then given $(U, V, < >)$ get $U \otimes^C Y, X \otimes^C V$ with pairing $(X \otimes^C V) \otimes_Z (U \otimes^C Y)$

$X \otimes^C C \otimes^C Y$

$A$

$(U \otimes^C Y) \otimes^A_{C} (X \otimes^C V) \longrightarrow \text{Hom}_{A}(X \otimes^C U, X \otimes^C V)$

$\text{Hom}_{A, \text{op}}(U \otimes^C Y, U \otimes^C Y) \longrightarrow \text{Hom}_{A, \text{app}}(X \otimes^C V \otimes U \otimes^C Y, A)$
Multipliers alg of \( U, V \) maps to that of \( U \otimes_C Y, X \otimes_C V \)

\[
\begin{align*}
\text{Hom}_{\text{cop}}(U, U) & \quad \text{Hom}_C(V, V) \\
\text{Hom}_{A^p}(U \otimes_C Y, U \otimes_C Y) & \quad \text{Hom}_A(X \otimes_C V, X \otimes_C V)
\end{align*}
\]

\[
\langle (x \otimes v), (y \otimes y) \rangle \quad \langle (x \otimes u \otimes v), u \otimes y \rangle
\]

\[
\langle u \rangle \langle v \rangle \mu \langle u \rangle \langle v \rangle \mu \langle u \rangle \langle v \rangle
\]

Also check that have hom.

\[
U \otimes_C V \longrightarrow U \otimes_C Y \otimes_C X \otimes_C V
\]

Also good condition is the possibility of deforming the identity map in the multi alg into \( U \otimes_C C^{(\infty)} \otimes V \)

So consider \( U, V, V \otimes U \longrightarrow C \) and pair

Assume \( U \otimes_C V \longrightarrow \text{Hom}_{\text{cop}}(U, U) \times \text{Hom}_{C \otimes \text{cop}}(V \otimes U, C) \)

Assume 1-def of the identity suppose working with complexes I think you know that \( C U \longrightarrow U \) is a homotopy equivalence. You have to go over these arguments.
B = U ⊗_C V

B ⊗_B B = (U ⊗_C V) ⊗_B (U ⊗_C V)

Let's understand. You need to begin with the class \( X \in U \otimes_C V \) whose action on \( U, V \) is homotopic to the identity.

\[
\begin{align*}
X & \in U \otimes_C V \quad X \in U \otimes_C V \\
U & \rightarrow (U \otimes_C V) \otimes U \\
& \rightarrow U \otimes_C U \\
& \rightarrow U \otimes V \\
& \rightarrow (U \otimes V) \otimes (U \otimes V) \\
& \rightarrow U \otimes C \otimes V \\
& \rightarrow U \otimes V
\end{align*}
\]

So you get an interesting structure

\[
B \otimes B \otimes B \rightarrow B \otimes B \rightarrow B
\]

So it seems that the arguments are formal

Homotopy equivalences, etc.

Maybe review why such a \((U, V)\) is hom to a finite proj complex. So suppose you have \((U, V)\) complexes of unitary \( R \)-modules such that \( I \in U \otimes_R V \) whose actions on \( U, V \) are homotopic to \( 1 \).

I can take \( V = \text{Hom}_R^\text{op}(U, R) \)

\[
U \otimes_R \text{Hom}_R^\text{op}(U, R) \rightarrow \text{Hom}_R^\text{op}(U, U)
\]

What does this mean \( 1 - f = [d, h] \) where \( f \) comes from \( X \in U \otimes_C V \).
Go over this point again.

Suppose you have a cycle in $U \otimes V$. What's the analogue of $X = \sum u_i \otimes v_i$? What do you hope for? dominated by a perfect complex. Thus you want a perfect complex $P$ and map $P \to U$ such that $\gamma \sim 1$. In this case you get $1 \in P \otimes R \to U \otimes_R V$.

Better would be to have $V$ replace $U$ by a complex of free modules, then $U$ is a filtered colimit of strictly perfect complexes.

$X \in U \otimes_R V = \lim_{\to R} P_i \otimes_R V$

Should work with $V$. Namely $X \in U \otimes_R V = \lim_{\to R} P_i \otimes_R V$ so you get $P_i \to U$ and $\gamma \in P_i \otimes_R V \to \text{Hom}(U_i, P_i)$

Therefore you find

**Argument.** Given $X \in U \otimes_R V$ write $X = \sum u_i \otimes v_i$ where $u_i, v_i$ are homog. Let $F$ be the free $R^\mathbb{P}$-module with $n$ generators graded appropriately, so that (ui) defines $F \to U$ and (vi) defines $V \to F$. Then $X_i$ is the comp. $U \to F \to U$
Now equip $F$ with diff $udv$:

$$udv u du = ud [X] d v = u X d^2 v = 0.$$ 

So if $X$ is kflp to $1_u U$ is dominated by $F$. 

Program. Take the thing you get from a quasi-hom $\tilde{B} \Rightarrow R \Rightarrow I \leftarrow \tilde{X}$, namely, $p, \tilde{p} \in \mathcal{P}(R^I)$ and an iso $\sim: \mathcal{P}/M \sim \tilde{P}/\tilde{M}$. Let $U: \tilde{p} \xrightarrow{p} \tilde{p}$

where $p, \tilde{p}$ lift $\prec$ and $\preceq$. Try thinking of $U$ as perfect complex with homotopy. The key idea is now to decide what to do.

I'm looking for a generalization of complexes and homotopy. In any case we have the odd operator $(p \tilde{p})$ and

$$X = 1 - (p \tilde{p})^2 = (1 - 8 p \tilde{p}) (1 - p \tilde{p}) : A \rightarrow \bigoplus_{A} I \otimes B.$$ 

Now apply Morita invariance $U$ becomes $U \otimes_Y : P \otimes Y \xrightarrow{P \otimes Y}$ $V$ becomes $X \otimes_Y V : \bigoplus_{R} X \otimes \tilde{R} \xrightarrow{X \otimes \tilde{P}}$ 

The point is maybe that this "dual pair" $(U \otimes_Y, X \otimes_V)$ over $A$, defines $\hat{h}$ with its $d, h$ has a $K_0$ class. Make life easy take $p = e R$, $\tilde{p} = \tilde{e} R$

Then $U \otimes_Y : e Y \xrightarrow{e Y}$ $X \otimes_Y U : X e \xrightarrow{X e}$

$U \otimes_Y \otimes_X \otimes_Y U : [e I \hat{e} e I \hat{e}]$ $I' = Y \otimes \hat{X}$

I get a "complex" over $A^p$ $e Y \xrightarrow{e Y}$ which because of the explicit dual defines an elt of $K_0(A)$.

I need the dual so as to "embed" it a perfect "complex".
\[1 - f = [d, h] \quad \xi \downarrow \frac{F}{U} \quad \text{id} = f\]

\[
(id \downarrow id) \cdot \text{id} = id \downarrow id \downarrow
\]

\[
(id \downarrow id) = (id \downarrow id) \downarrow
\]

\[
(id \downarrow id) \downarrow
\]

\[
F \quad \text{id} \downarrow
\]

\[
i \uparrow \downarrow j
\]

\[
U
\]

\[
(i_1 \downarrow i_1) \downarrow
\]

\[
(1-hd) \cdot df \downarrow
\]

\[
(1-hd) \cdot df \downarrow
\]

\[
(i_1 \downarrow i_1) \cdot (1-hd)
\]

\[
(1-hd) \cdot (1-hd)
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(1-hd) \cdot (1-hd)
\]

\[
(1-hd) \cdot (1-hd)
\]
\[ F \]

\[ i \uparrow i \downarrow j \]

\[ E \]

\[ K_1(R/I) \rightarrow K_0(R \times_{R/I} R) \rightarrow K_0 R \otimes K_0 R \rightarrow K_0(R/I) \]

\[ c' = i(1-dh) \]

\[ j'c' = (1-hd)i(1-dh) = f - hdf - fdh + hdfh \]

\[ hdf = h d(1-hd) \]

\[ fdh = (1-dh) dh \]

\[ hdf + fdh = hd + dh - hdhd - dhdh \]

\[ = [d, h] - [d, hdh] = [d, h'] \]

\[ h' = h - hdh \]

\[ [d, h'] = (1-f) - (1-f) dh - hd(1-f) \]

\[ = f dh + hdf \]

Is this a case of HPT?

\[ \text{Not obviously.} \]
So. \( U : \mathbb{P} \rightarrow \mathbb{P}^1 \)

\[
0 \rightarrow U \otimes I^\vee \rightarrow U \otimes U^\vee \rightarrow U/I \otimes U^\vee/Iu^\vee \rightarrow 0
\]

You lift \((0 \ g^{-1})\) to \( (0 \ g) \) : \( p \mapsto \begin{pmatrix} 0 & g \\ g & 0 \end{pmatrix} \)

Then \( 1 - \alpha^2 = \begin{pmatrix} 1 - gp & 0 \\ 0 & 1 - pg \end{pmatrix} \). Question: Can you refine \( \alpha \) so as to make \( 1 - \alpha^2 \equiv 0 \mod I^n \)?

Look for an polynomial \( p(x) \) such that \( p(1) = 1 \) and \( p(x) - 1 \) divisible by \((x-1)^n\). Then

\[
1 - p(x)^2 = (1 - p(x))(1 + p(x)) \equiv 0 \mod (x-1)^n \\
(x-1)^n(x+1)^n = (x^2-1)^n.
\]

Graph of \( p \)

It seems that \( p \) has degree \( \geq 2n+1 \). Not true for \( n=1 \).

If \( p(x) - 1 \equiv 0 \mod (x-1)^n \)
then \( p(x) \equiv 0 \mod (x-1)^{n-1} \)
so \( p'(x) \equiv 0 \mod (x^2-1)^{n-1} \)
and the smallest degree possible is \( p'(x) = c(x^2-1)^{n-1} \)
\( p(x) = c \int_0^x (x^2-1)^{n-1} dx \) where \( c = \frac{1}{\int_0^1 (x^2-1)^{n-1} dx} \)
\[ n = 1. \]
\[
\int_{0}^{x} (1-x^2) \, dx = \frac{x-x^3}{1-\frac{1}{3}} = \frac{3x-x^3}{2}
\]

\[ p(x) = \frac{1}{2} (3x-x^3) \quad \text{Check} \quad p(1) = 1 \quad \checkmark \]
\[
p'(x) = \frac{1}{2} (3-3x^2) \quad p'(1) = 0
\]

**Other method:** Starting from you want an involutive polar decomp.
\[
\alpha (\alpha^2)^{-\frac{1}{2}} = \alpha (1-(1-\alpha^2))^{-\frac{1}{2}}
\]
\[
= \alpha \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{2n-1}{2})}{n!} \frac{1}{(1+\alpha^2)^n}
\]
\[
= \alpha \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdots (2n-1)}{2^n \cdot n!} (\alpha^2 + 1)^n
\]
\[
= \alpha \left( 1 + \frac{1}{2} \frac{(\alpha^2)^1}{2} + \frac{1 \cdot 3}{2 \cdot 2!} \frac{(\alpha^2)^3}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 3 \cdot 4!} (\alpha^2)^7 \right)
\]
\[
= \alpha \left( 2 \cdot \frac{3}{8} + \frac{1}{4} \right)
\]
\[
= 1
\]
\[
2 \cdot \frac{5}{16} + 2 \cdot \frac{1}{2} \cdot \frac{3}{8} = 1
\]
\[
\text{does this have power of 2 in der.} \quad \text{YES,}
\]
\[
\frac{(2n)!}{2^n \cdot n! \cdot n!} \left( \frac{1-\alpha^2}{4} \right)^n
\]
\[
\frac{2n!}{n! \cdot n!} = \frac{2^n \cdot 1 \cdot 2 \cdot \cdots \cdot 2n-1}{n!} \in \mathbb{Z}
\]

\[
\left[ \frac{n}{2} \right] + \left[ \frac{n}{4} \right] + \cdots + \leq n, \quad n = 2^k
\]

\[
h = 4, \quad 4! = 24 \quad \oplus \quad 8
\]

\[
\frac{2n!}{n! \cdot (n-1)!} = \frac{(2n-1)!}{n! \cdot (n-1)!} = \frac{1 \cdot 3 \cdot (2n-1) \cdot 2^{n-1}}{n!}
\]

Approx. method: Let \( \alpha, \quad \alpha^2 - 1 \in I \)

I propose to change \( \alpha \) to \( \alpha(1 + \frac{1}{2}(1 - \alpha^2)) \)

\[
\left( \alpha \left(1 + \frac{1}{2}(1 - \alpha^2)\right) \right)^2 = \alpha^2 \left(1 + (1 - \alpha^2) + \left(\frac{1 - \alpha^2}{2}\right)^2\right)
\]

\[
\equiv 2\alpha^2 - \alpha^4 \mod I^2
\]

\[
1 - \left[ \alpha \left(1 + \frac{1}{2}(1 - \alpha^2)\right) \right]^2 \equiv 1 - 2\alpha^2 + \alpha^4 \mod I^2
\]

\[
\equiv (1 - \alpha^2)^2
\]

\[
\equiv 0
\]

So your approximation method takes \( \alpha = \left( \begin{array}{c} 0 \\ \frac{p}{8} \end{array} \right) \)

to \( \alpha \left( 1 + \frac{1}{2}(1 - \alpha^2) \right) = \left( \begin{array}{c} 0 \\ \frac{p}{8} \end{array} \right) \left( 1 + \frac{1 - \frac{8p}{2}}{2} \right) \)

\[
= \left( \begin{array}{c} 0 \\ \frac{p + \frac{p - 8p^2}{2}}{2} \end{array} \right)
\]

Replace \( p \) by \( \frac{3p - 8p^2}{2} \).
Recall last position. Given $(p, p, \theta)$ you let $U = p \oplus \hat{p}$, $V = U$

$0 \rightarrow U \otimes I \otimes U \rightarrow U \otimes U \rightarrow U/\text{im} \otimes U/\text{im} U \rightarrow 0$

$\text{Hom}_{K^p}(U, U)$

\[
\begin{pmatrix}
0 & 1 \\
p & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

I know that I can view $U$ as a complex in $L^1(R, I)$ with $p$ as affine and $\theta$ as a homotopy operator. If I regard $\theta p$ as fixed, then this gives more information than just $\theta$.

I want somehow to exploit the symmetry between $p$ and $\theta$ in order to obtain a motion of homotopy. Consider the determinant in the commutative case.

$E_1 \rightarrow E_0 \quad \text{yields} \quad \frac{\Lambda^p E_0 \otimes (\Lambda^p E_1)}{p}$

For a comm. unital domain $\text{map} E_i \in P(R)$, $p$ lifts $\theta: E_1 / IE_1 \rightarrow E_0 / IE_0$. Then $E_1$, $E_0$ have the same rank. This may be misleading.

What sort of homotopy possibilities? Replace

Start with $(U, V) \quad \text{a} \, \mathbb{Z}_2\text{-graded dual pair over} \, C$

$V \otimes U \rightarrow C$

Next idea. $U \otimes C \otimes V \rightarrow U \otimes V$

What is the sort of stuff you need.
84. How do I decide whether there is a future duality? My problem is this. Let \((U, V)\) be a super dual pair over \(C\), then I need to find the notion of a diagonal in \(U \otimes_c V\). It should be an element which when pulled into the multiplication algebra \(R = \text{Hom} \overset{\text{op}}{C}(U, U) \times \text{Hom}_C(V, V) \times \text{Hom}_C(U \otimes_U C, V \otimes U, C)\) has the form \(1 - \alpha^2\) where \(\alpha\) is odd.

Assume \(U \otimes_c V\) is unital \(1 = \sum u_i \otimes v_i\) and \(U \overset{(v_i)}{\longrightarrow} C^n \overset{(u_i)}{\longrightarrow} U\). Then what is \(R\)? \(R\) is \(\text{Hom} \overset{\text{op}}{C}(U, U) \cong \text{Hom}_C(V, V)^{op}\).

What is an element of \(R\)? \((\mu, \mu^*)\) such that \(\langle v, \mu, u \rangle = \langle v, \mu^* u \rangle\). Thus \(\mu^*\) and \(\mu\) are transposes. If the pairing is non-degenerate on one side:

\[ V \overset{\mu}{\longrightarrow} \text{Hom}_C(U, C) \]

then clearly \(\mu\) is determined by \(\mu^*\).
If \(V = \text{Hom} \overset{\text{op}}{C}(U, C)\) or \(C \otimes \text{Hom} \overset{\text{op}}{C}(U, C)\) etc., then \(R = \text{Hom} \overset{\text{op}}{C}(U, U)\).
Recall that \( \alpha' = \alpha (1 + \frac{1 - \alpha^2}{2}) \) satisfies

\[
1 - (\alpha')^2 = 1 - \alpha^2 \left( 1 + 1 - \alpha^2 + \frac{(1 - \alpha^2)^2}{4} \right) = 1 - 2 \alpha^2 + \alpha^4 + \frac{(1 - \alpha^2)^2}{4} = \alpha (1 + \frac{1}{4})(1 - \alpha^2)^2
\]

\[
\alpha' = \alpha (\alpha^2)^{-1/2} = \alpha (1 - (1 - \alpha^2))^{-1/2} = \alpha \sum_{n=0}^{\infty} \frac{1,3, \ldots, 2n-1}{n!} \left( \frac{1 - \alpha^2}{2} \right)^n
\]

I have to carefully review my K0 paper.

First I need to specify the kind of complexes. The example provided by Milnor's triples \( (\mathcal{P}, \mathcal{P}, P/\mathcal{P}) \)

Question: Multiplicities alg for \( A^\infty \) in the case \( T(V) \). Give a we have a dual pair \( (A^\infty, A^\infty) \) over \( A \) whence a multiplicative ring

\[
\text{Hom}_{A^\text{op}}(A^\infty, A^\infty) \times \text{Hom}_A(A^\infty, A^\infty)^{op}
\]

\[
\text{Hom}_{A^\text{op}}(A^\infty, A^\infty)
\]

If \( A \) is left or right flat then \( A^{(n)} = A^n \) is flat.

What happens? Can you describe the united ring \( \text{Hom}_{A^\text{op}}(A^\infty, A^\infty) \), forms \( M(A) \), etc. This seems to be constant.

\[
\text{Hom}_{A^\text{op}}(A^\infty, A^\infty) = \lim_{\leftarrow} \text{Hom}_{A^\text{op}}(A^\infty, A^k)
\]

\[
A = \tilde{T}(V) = V \otimes \tilde{A}
\]

\[
A^n = V \otimes^n \tilde{A}
\]
\[
\lim_{j} \text{Hom}_{A^{op}}(V^{\otimes j} \otimes \tilde{A}, A^k)
\]
\[
= \lim_{j} A^k \otimes V^{\otimes j} = A^k \otimes V^{\otimes \tilde{A}}
\]

First do \(\lim_{j} \text{Hom}_{A^{op}}(A^j, \tilde{A})\)
\[
= \lim_{j} \tilde{A} \otimes V^{\otimes j}
\]

Note that \(\lim_{j} \text{Hom}_{A^{op}}(A^j, X) = X \otimes V^{\otimes j}\)

Pretty clearly, this inductive system is what leads to the Torpshy algebra, i.e., used the canonical \(k \to V \otimes V^*: \)
\[
X \otimes V^{\otimes j} \to X \otimes V \otimes V^{\otimes j} \otimes V^{\otimes j+1}
\]

Thus \(\text{Hom}_{A^{op}}(A^\infty, \tilde{A}) = 0 \otimes V\)

Is it true that
\[
\text{Hom}_{A^{op}}(A^\infty, A^n) = A^n \otimes A \otimes V
\]

\[
0 \to A^n \to \tilde{A} \to \frac{\tilde{A}}{A^n} \to 0
\]

So the left multiplier algebra is \(\otimes V\) and similarly the right one.
Let's analyze following: \((U, V)\) super dual pair over \(B\), whence canon

\[
U \otimes V \rightarrow \text{Mult}(U, V) = \text{Hom}_B(U, \mathbb{C}) \times \text{Hom}_B(V, \mathbb{C})^{3}\]

Suppose given odd element \(x\) in the multiplier ring and an element \(f \in U \otimes V\) mapping to \(1 - x^2\).

Multplier rings are not functorial. But let's simplify things by assuming \(V \rightarrow \text{Hom}_B(U, B)\), in which case the multiplier ring is just the left multiplier ring \(\text{Hom}_B(U, U)\). So we ought to be able to picture everything using \(U\), e.g. \(x = (0 \, 0 \, 0)\) on \(U \oplus U_0\), which I like to view as \((0 \, 0 \, 0)\).

Choose \(F = \mathbb{C}^n\) and \(i = (v_i)\)

\[U \xrightarrow{i} \mathbb{C}^n\]

such that \(i \circ j = 0\).

I know that \(U\) becomes homotopy equivalent to a finite proj complex. I should work out why.

I recall the formulas were messy, but maybe manageable for length 1 complexes.

\[
\begin{align*}
F_1 & \xrightarrow{F_0} \\
U_1 & \xrightarrow{f_0} U_0
\end{align*}
\]

\[
\begin{align*}
& j_0 \circ i_0 = 0 \quad f_0 = 1 - dh \\
& j_1 \circ i_1 = 0 \quad f_1 = 1 - hd
\end{align*}
\]

What would you really like? is to modify \(i_0\) to \(i_0(1 - dh)\)

\(j_1\) to \(j_1(1 - hd)\)
An idea \( I = (V) \) \( J = (U) \)

\[
\begin{pmatrix}
B & V \\
U & U \otimes V
\end{pmatrix}
\]

replace \( B \) by \( M_B(B) \)
\( V \) by \( V^\otimes n \), \( U \) by \( U^\otimes n \)

Do you now work in this Morita context but entries are all super?

Let's now try to understand the calculations to be done. Two stages in my paper are involving my transitions to either a complex dominated by a fiber exp, the other being Ranicki's, which again was one-sided. Can I symmetrize somehow. At the moment we have the odd \( x \) in \( 
\begin{pmatrix}
0 & V \\
U & 0
\end{pmatrix}
\)

\[ 1 - x^2 = uu \]

\[
\begin{pmatrix}
f_0 \circ c (1 - dh) \\
(1 - dh)^2
\end{pmatrix}
\]

\[ F \rightarrow F_0 \]

\[ U \rightarrow U_0 \]

Anyway

\[ \frac{2h + dh}{1 - d} \]

\[ t \circ (1 - dh) \]

\[ \frac{1 - dh}{1 - d} \]

\[ \frac{2h + dh}{1 - dh} \]

\[ \frac{t \circ (1 - dh) d}{1 - dh} \]

\[ \frac{t \circ (1 - dh) d}{1 - dh} \]

\[ \frac{t \circ (1 - dh) d}{1 - dh} \]

\[ \frac{t \circ (1 - dh) d}{1 - dh} \]
Try a more straightforward idea. You work in the multiplier ring

\[(B \quad V) = \begin{pmatrix} B & V \\ U & \text{Mult}(U,V) \end{pmatrix}\]

What are trying to do? Life is difficult! You have \( \alpha \) odd \( \in \text{Mult}(U,V) \), \( U, V \)
even in \( U, V \). What sort of thing do you want to achieve? I think you want to allow \( \alpha \) to be perturbed modulo \( U \otimes B \).

Look at that question \( B \) separably \((U, V)\) super dual pair over \( B \). Then you Work in Multiplier ring. Is this a kind of dilation problem?

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Work a bit on the multiplier ring of \( A^\infty \), where \( A = T(E) \).

Let \( O_E \) be center alg of \( E \)

\[ O_E = T_E/(1 - \sum_k u_k^* u_k) \]

\[ A = T(E) \]

Review what I learned this morning about \( T_E = \mathbb{R} \)

We have the dual pair \( \kappa \rightarrow T_E \).

You have a dual pair over \( \mathbb{R} \)

\[ (T(E), T(E^*) \), direct sum \( (E^\otimes n, E^*\otimes n) \) for \( n \geq 0 \)

Then \( T(E) \otimes_k T(E^*) \) should be ideal of finite rank operators in \( \mathbb{R} \)

\[ R(1 - \sum \gamma_i s_i) R = T(E)(1 - \sum \gamma_i s_i) T(E^*) \]
Map $K_*(R) \to K_*(k)$ defined by a Kasparov module given by

$$T(E) \otimes E \xrightarrow{?} T(E)$$

These are finitely generated $R$-modules? In fact

$$0 \to T(E) \otimes E^* \to T_E \to T(E) \to 0$$

So $T(E) \in \mathcal{P}(R)$. Use basis $s_2, s_2^*$ for $T_E$ basis $s_1, 1$ for $T(E)$.

To compose $K_*(R) \to K_*(k) \to K_*(R)$, wait.

The first thing to understand is the map $K_0(R) \to K_0(k)$ defined by

$$T(E) \otimes E \xrightarrow{?} T(E)$$

So you start with $P \in \mathcal{P}(R^0)$ and tensor:

$$P \otimes_R T(E) \otimes E \xrightarrow{?} P \otimes_R T(E)$$

The construction of the operator $T_E$ is the subtle part of Kasparov's theory.

So how do we proceed? One idea is that you need a "connection" on $P$, and that such a thing comes from expressing $P$ as a summand of a free module. So what happens?

We have $U = U_1 \oplus U_0$, with the odd $U_0$.

In this example we have $U_1 \oplus U_0 = T(E) \otimes_R E \oplus T(E)$ as $R$-modules equipped by some odd $R$-operator of $\mathcal{P}(R)$ such that $-\alpha^2$ is in $U \otimes V$. 
Be specific. Follow Pimsner

Enlarge \( T(E) \otimes E = T(E) \otimes E \) to \( T(E) \) so

that \( \alpha \) can be chosen as \( \alpha : (0, 0) \).

Now you want to tensor with \( P \in T(R) \).

Say \( P = eR^\otimes n \). Then have operator \( e \)

acting on \( \otimes^n U \), say \( n = 1 \). What can you do?

What is the mechanism.

You have \( e \in R \) acting on \( U_1 \otimes U_0 \)

but not quite commuting with \( F \). Okay

\[ eU_1 \otimes uU_0 \quad eFe = \alpha \]

\[ 1 - \alpha^2 = e - eFeFe = eF_1 Fe - eFeFe \]


So you have \( R = T(E) \)

\[ U = T(E) \otimes T(E) \]

\[ F = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \]

\[ p_0 \text{ projection onto } E^\otimes \]

\[ p_0 = Z_{5,5} \]

I have difficulty analyzing what happens in this case.

However, there is another angle, namely to use a fijlyy
resolution of \( U \). Actually, you might first look

at the process \( P \mapsto P \otimes_R U : P \otimes_R U \mapsto P \otimes_R U \).

Bring \( I = U \otimes_R V \) in. What is \( U / IU \)? What is

\( T(E)/T(E) \)?

\[ 0 \rightarrow R \otimes E^* \rightarrow R \rightarrow T(E) \rightarrow 0 \]
\[ 0 \to (R/I) \otimes E^* \to R/I \to \otimes T(E)/IT(E) \to 0 \]

I generated by \( 1 - \sum s_is_i^* \). 

\[ (1 - \sum s_is_i^*) T(E) = k \quad \text{so} \quad T(E)/IT(E) = 0. \]

Very precise example. This is very nice. Start

You have a very specific situation, namely

\( R = F_E = T(E) \otimes T(E^*) \) in a funny way.

\( R \supset I = T(E) \otimes T(E^*) \) defined by a dual pair.

But in any case you have \( R \supset I \) in some equivalence.

But \( R \subset \text{Mult alg} (T(E), T(E^*)) = \text{Hom} (T(E), T(E)) \times \text{Hom} (T(E^*), T(E^*)) \).

Have \( R \) acting on \( T(E) \) and \( T(E) \otimes E = T(E)^{E_n} \).

You have two homoms. \( R \Rightarrow R \).

At the 2nd is non unital rather goes into \( \sum s_is_i^* = e^+ \).

\[ T(E) \otimes E \subset T(E) \otimes k \]

\[ R \rightarrow x \]

\[ x \rightarrow x + \]

look at the generators

\[ s_i \mapsto e_s^i e^+ = s_i^* \]

\[ s_i^* \mapsto e^- s_i e^+ = s_i^{-1} \]

\[ e = \sum \overline{s_i^*s_i^*} = e_s^i e^+ e^- s_i e^+ = e^+ s_i e^- \]

\[ e_s^i e^+ s_i^* e^- = e_s^i \]

These are the relations.
\[ s_i \rightarrow e^{+}\bar{s}_i e^+ \]

\[ s_i^* \rightarrow e^{-}\bar{s}_i^* e^- \]

\[ s_i = s_i^* e^+ \]

\[ s_i^* = e^{-}\bar{s}_i^* \]

\[ s_i - s_i' = (s_i e^+) e^- \]

\[ s_i^* - s_i^* = (e_s e^+) e^- \]

So we have a quasi-hom. \( R \xrightarrow{1} R \supset [I, 1] \).

So start with \( P \in \mathcal{P}(R^{op}) \). Then have \( (P, \theta, \mathcal{P}, \lambda) \). Now you want to convert this to

\[ \text{So start with } P \in \mathcal{P}(R^{op}), \text{ then have } (P, \theta, \mathcal{P}, \lambda). \text{ Now you want to convert this to} \]

\[ R = T_E \quad I = T_E (1 - \sum s_i s_i^*) \quad T_E \twoheadrightarrow T(E) \otimes T(E^*) \]

\[ e^+ = \text{proj onto } T(E^*) \rightarrow \]

\[ e^+ s_i = s_i \quad s_i^* = s_i^* e^+ \]

\[ s_i' = s_i e^+ \quad s_i^* s_i' = e^+ s_i^* e^+ = \delta_{ij} e^+ \]

\[ s_i^* = e^+ s_i^* \quad \text{unit hom} \]

\[ \text{Means you get } R \xrightarrow{\sigma} e^+ Re^+ \]

\[ s_i, s_i^*, s_e^+, e^+ s_i^* \]

This means you get \( \sigma \) is a homo. \( R \xrightarrow{1} R \)

Let \( P \in \mathcal{P}(R^{op}) \). What is \( \sigma^*_p(R) = P \otimes_R R = P \otimes e^+ R \) ?

Let this way. Look at the repn on \( T(E) \). Then \( \sigma \) is the rep. \( T(E) \otimes E \rightarrow \text{ rep on } R \). What do you need to get straight? Not much.

Lett go over the philosophy.
Philosophy. You have a quasi-

homomorphism. $B = R \times I$ and this induces $K_0(B) \to K_0(I)$.

You also have a map $\begin{pmatrix} I & Y \\ X & k \end{pmatrix}$, whence

an iso. $K_0(I) \to K_0(k)$. You need to get each of these in the best possible form.

Where to start? Universal case is $B = R \times R \times R$

know $\mathcal{P}(B^0)$ act of $(P_1, P_2, P_1/P_2 = P_2/P_1)$. $K_0(B)$ is $\mathcal{P}$-group of these triples, $K_0(I)$ is quotient of $K_0(B)$ by degenerate triples, really should write $K_0(R,I)$

and then prove exact $\xymatrix{ K_0(I,I) \ar[r] & K_0(R,I) }$.

Where to start? I think you want to take $(P_0, P_0, x)$. Wait. Try first to relate the two steps $K_0(I,I) \to K_0(R,I)$ and the Moi\-ta

invariance step $K_0(R,I) \to K_0(k)$. The problem is passing from $(P_0, P_0, x)$ over $R$ to similar data over $k$. You really have to get better control over your

$K_0$ paper.

What might work? You start with $(P_0, P_0, x)$.

and the dual triple $U = (P_0, P_1, z \tau)$. Then use Moi\-ta.

$\begin{pmatrix} R & Y \\ X & k \end{pmatrix}$

to get $(U \otimes_Y X \otimes_R U)$ over $k$. This is a super dual pair over $k$.

$0 \to U \otimes_I U \otimes_R R \to U \otimes_R U \to U \otimes_{II} R \to 0$

In this case if you have $\tilde{z}$ in $\tilde{y}$, can lift $\tilde{z}$.
To an odd $x \in U \otimes U^\vee$, then form $1-x^2$

$1-x^2 \in (U \otimes Y) \otimes (X \otimes U)$.

The problem is now to convert the latter into an element of $K_0(k)$.

So here is the crucial problem: to treat the dual pair $(U \otimes Y, X \otimes U)$ over $k$, together with the element $1-x^2$. This is probably not enough. You need a also, but you do have the multipliers $(x \otimes 1, 1 \otimes x^2)$ on $(U \otimes Y, X \otimes U)$. Yes!!!

So exactly what is at hand over $k$, you get a dual pair $(L, M) = (U \otimes Y, X \otimes U)$ over $k$ together with an $x_{1,1}$ in the multipliers.

Idea: $e, e = e \otimes 1, e = e \in I$, then $e \otimes U, e \otimes U$ are commensurable.

Start with $(Y, X)$

Somehow the basic construction amounts to the following. Let $(Y, X)$ be an odd-dual dual pair over the field, let $R = \text{mult. ring}$, $I = Y \otimes X \subset R$. We want to describe $K_0(R \otimes \mathbb{R}/I) \xrightarrow{\otimes} K_0(k)$. Canonical map. An object of $\mathcal{E} \mathcal{P}(R \otimes \mathbb{R}/I)_{X}$ is $(P, P, \phi)$ apply $- \otimes Y$ get $P \otimes X \xrightarrow{\otimes 1} P \otimes Y$ where $x$ is any lifting of $\phi$, $1-x^2 \equiv 0 \mod I$. 

[Insert diagram or figure here if necessary]
Let $f: P \to PI \subseteq P$. Then what about $f \circ 1: P \otimes Y \to PI \otimes Y \subseteq P \otimes Y$.

What do I know about $P \otimes Y$? If you write $P$ as a summand of $R^n$, typically saying that then $f$ is given by $M_n(R)$, you have $P \otimes Y \subseteq Y \otimes M_n(I) \otimes \otimes Y \subseteq P \otimes Y$.

You know nothing about $P \otimes Y$ other than the fact it is a subset of $Y \otimes Y$ and complemented subspace of $Y \otimes Y$. But no other structure I can see. The point is that this $f \circ 1$ is nuclear.

What picture my might be obtained, arise.

For any object of $F(R \times R, R)$, you choose lifting $x$ of $o$ to get $P \Rightarrow \tilde{P}$, then you get $P \otimes Y \subseteq \tilde{P} \otimes Y$ such that $1 - (\xi o)^*$ is nuclear.

Next take $R = T(E) \otimes (X \otimes k)$ such that $Y = T(\xi)\otimes X = T(\xi)\otimes X \subseteq R$.

We look at two actions of $R$ on $T(\xi)$, the first is the obvious one, the second is the action on $T(\xi) \otimes E = T(\xi)$, extended by $\otimes 0$. $x(s_i^*) = s_i^* e^+$ $\sigma(s_i) = e^+ s_i^*$ $\sigma(1) = e^+$. This gadget should yield a map $K^*(\xi) \to K^*(\xi)$. What is it on $P \otimes F(R^0)$.
I should focus on the idea that $T(E) \otimes_k E$, $T(E)$ are almost isomorphic representations of $R$ over $k$. From this viewpoint, it is clear that $P \otimes T(E) \otimes E$, $P \otimes T(E)$ are almost isomorphic $k$-modules for any $P \in \mathcal{P}(R^p)$.

Idea: Is there a way to dilate almost isomorphic reps?

Question: Graph of a unitary

What does to be done. We have two involutive reps of $R = \mathcal{P}(E)$ namely $T(E) \otimes E$ and $T(E)$ which are almost isomorphic, i.e., odd $\alpha$ in $T(E) \otimes E$ such that $[R, \alpha]$ and $1 - \alpha^2$ finite rank. I think it is always possible to dilate $\alpha$. $\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ $1 - \alpha^2 = \begin{pmatrix} 1 - \delta P & 0 \\ 0 & 1 - \delta \end{pmatrix}$

Unitarily, given a contraction $\alpha = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$, then

$$\begin{pmatrix} V^* - cc^* & c^* \\ c & -V^* - cc^* \end{pmatrix} \begin{pmatrix} 1 & c^* \\ c & -V^* - cc^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$c(1 - cc^*) - \sqrt{1 - cc^*}c$$

\[11/25/97\] Problem:

\[
\begin{pmatrix} R = \mathcal{P}(E) & Y = T(E) \\ X = T(E^*), \delta \end{pmatrix} \quad \text{I} = Y \otimes X
\]

We have reps of $R$ on $T(E) \otimes E$ and $T(E)$ which are almost isomorphic, specifically, there are maps $T(E) \otimes E \xrightarrow{\iota} T(E)$ such that $\alpha = \begin{pmatrix} 0 & c^* \\ c & 0 \end{pmatrix}$ satisfies $[R, \alpha] \subseteq I$, $1 - \alpha^2 \in I$. Alternatively, one can replace $T(E) \otimes E$ by $T(E)$ which is $T(E)$ with $R$ acting through the group hom $\varphi: R \to R$ then $\alpha^2 = 1$. This gives a map $K_0(R) \to K_0(k)$, $P \in \mathcal{P}(R^p) \mapsto P \otimes T(E)$ $P \otimes T(E)$. Set the seed.
do you have \( \frac{\langle \text{reg. action} \rangle}{T(E) \otimes E} = \frac{T(E)}{T(E)} \)

\[ P \otimes R T(E) \otimes E = P \otimes T(E) \]

How do we produce the maps between these? One way is to write \( P = \frac{R^0}{R} \), then have

\[ (T(E) \otimes E) \longrightarrow T(E) \]

Clearer,

\[ P \otimes R T(E) = \frac{R^0}{R} \otimes R T(E) \]

\[ = (\otimes) \left( T(E) \otimes E \right) \]

simply you have \( T(E) \otimes E \longrightarrow T(E) \)

Take the project \( \oplus \) idempotent matrix \( e \) over \( R \), and act on both sides

\[ \oplus e (T(E) \otimes E) \]

\[ e T(E) \otimes E \]

My viewpoint. You have two \( R \)-modules \( T(E) \otimes E \) and \( T(E) \) so you get \( P \in P(R \otimes P) \)

\[ P \otimes T(E) \otimes E \rightarrow P \otimes T(E) \]

Two vector spaces. But the important part is that these are in isomorphism. The two \( R \)-modules are almost isomorphic. Notice we know \( T(E) \in P(R) \)

(this is probably irrelevant) because you do not use the map \( P(R) \rightarrow P(R/I) \) as \( IT(E) = T(E) \)
Keep on trying. The fundamental problem will be to clean up the idea that elements of $K_0(A)$ are represented by super dual pairs $(U,V)$ over $A$ together with an odd $x \in \text{Mult}(U,V)$ and even $y \in U \otimes_A V$ satisfying

$$f \mapsto 1 - x^2.$$ I already know this is true because such data yield a complex $U$ of $A^{op}$-modules with homotopy operator:

$$\partial = \begin{pmatrix} \partial_{U} & \partial_{V} \\ \partial_{V}^{op} & \partial_{U}^{op} \end{pmatrix}$$

and then $f = 1 - x^2 = \begin{pmatrix} 1 - \partial_{U} & \partial_{V} \\ \partial_{V}^{op} & 1 - \partial_{U}^{op} \end{pmatrix}$ is nuclear.

But I don't understand the equivalence relation.

\[\begin{array}{ccc}
\text{Given } (U,V) \text{ super dual pair over } A \\
& \alpha \in \text{Mult}(U,V) \land \text{odd} \\
f \in U \otimes_A V & f \text{ even} & \partial(f) = 1 - \alpha^2
\end{array}\]

\[\begin{array}{ccc}
0 & U \otimes_A V \xrightarrow{\partial} \text{Mult}(U,V) & 0 \otimes_A 0
\end{array}\]

Suppose $\alpha' = \alpha + \partial g$. Then

$$1 - \alpha'^2 = 1 - \left( \alpha^2 + \alpha \partial g + \partial \alpha + \partial^2 g \right)$$

$$= 1 - \alpha^2 - \partial (\alpha g) - \partial (\partial g)$$

$$= \partial (f - \alpha g - g \alpha + \partial g)$$

in the sense of the product in $U \otimes_A V$.

Important is $\partial xy = \partial x y$ for $x, y \in U \otimes_A V$

$$\partial(xy) = \partial x y - x \partial y.$$
Next thing is that given \( x, g \), \( \partial f = 1 - x^2 \)

one can improve \( g \). Let

\[
\begin{align*}
\alpha' &= \alpha + \frac{1}{2} \alpha (1 - \alpha^2) \\
\frac{1}{4} \partial (x f + f a)
\end{align*}
\]

Wait

\[
\begin{align*}
\alpha' &= \alpha + \frac{1}{2} \alpha (1 - \alpha^2) \\
1 - \alpha^2 &= 1 - \alpha^2 - \alpha^2 (1 - \alpha^2) + \frac{1}{4} \alpha^2 (1 - \alpha^2)^2 \\
1 - 2\alpha^2 + \alpha^4 &
\end{align*}
\]

First point is that \( \alpha f - f a \Rightarrow \alpha (1 - \alpha^2) - (1 - \alpha^2) \alpha = 0 \)

Try changing \( \alpha \) to \( \alpha' = \alpha + \frac{1}{2} \alpha (1 - \alpha^2) \)

\[
\begin{align*}
g &= \alpha f \\
&(f - \alpha g - g \alpha - g \partial g) \\
&= f - \alpha f - \alpha f a - \alpha f^2 (1 - \alpha^2) \\
&= f - \alpha f f - 2\alpha f a + \alpha f a^2
\end{align*}
\]

\[
\partial (f - \alpha g - g \alpha - g \partial g) = 1 - \alpha^2 - \alpha \partial g - \partial g \alpha - \partial g \partial g
\]

\[
= 1 - (\alpha + \partial g)^2
\]
\[ \alpha' = \alpha + \frac{1}{2} \alpha (1 - \alpha^2) = \alpha + \alpha \frac{1 - \alpha^2}{2} = \alpha + \alpha \delta(\frac{1}{2}) \]

Want \( \partial \hat{g} = \frac{1}{2} \alpha (1 - \alpha^2) \)

If so then

First assume \( \delta U \otimes V \) by range \( UV \subset \text{Mult} \).

Then \( \partial : UV \rightarrow \text{Mult} \) is injective. So

\[ \alpha(f \hat{g}) = (1 - \alpha^2) \alpha = \alpha(1 - \alpha^2) = \partial(\alpha f) \Rightarrow \alpha f = f \alpha. \]

But I assume if commutes with \( \alpha \)? I know that \( \delta(\alpha f \hat{g}) = 0 \) so that \( \alpha f = f \alpha \) is killed by any \( f \in \mathfrak{g} \).

In particular \( 1 - \alpha^2 \)

\[ \partial(\hat{f}^2) = -f \hat{f} + \alpha f f \]

i.e. \( f(1 - \alpha^2) = (1 - \alpha^2) f \quad f \alpha^2 = \alpha^2 f \)

The first case to understand is when \( I \subset \text{Mult} \).

\[ \alpha(\alpha \hat{f} + f \hat{g}) - (\alpha f + \alpha \hat{g}) \alpha = \alpha^2 f + \alpha f \alpha - \alpha f \alpha - f \alpha^2 \]

\[ = \alpha^2 f - f \alpha^2 = 0 \]

\[ \hat{g} = \frac{1}{\alpha} (\alpha \hat{f} + f \hat{g}) \]

\[ \partial(\hat{g}) = \frac{\alpha(1 - \alpha^2) + (1 - \alpha^2) \alpha}{4} \]

\[ = \frac{1}{2} \alpha (1 - \alpha^2). \]

and we know \( \alpha \hat{g} = \hat{g} \alpha \)

\( f \) gets changed to \( f - \alpha \hat{g} - \alpha \hat{g} + \frac{1}{2} \alpha (1 - \alpha^2) \)

\[ f - \frac{\alpha(\alpha \hat{f} + f \hat{g}) + (\alpha f + \alpha \hat{g}) \alpha}{4} + \frac{1}{2} \alpha (1 - \alpha^2) \]

comes from \( I \otimes \mathfrak{g} \).
Begin with \( \Delta(x) = 1 - x^2 \).

Then
\[
\Delta(-\Delta) = f \Delta f = (\partial f) f (-\Delta) f
\]

i.e. \( f \) commutes with \(-\Delta\). Replace \( f \) by
\[
\frac{f + \partial f}{2}
\]

\[
\mathbb{R}^2 \cong \mathbb{R}^2
\]

\[
\kappa[x] \otimes \kappa[y] \longrightarrow \kappa[x]
\]

Form DGA: gen. by \( x \) in deg \( 0 \), \( y \) in deg \( 1 \).

\[ d(y) = 1 - x^2. \]

Truncated
\[
\kappa[x] y \kappa[x] y \kappa[x] \rightarrow \kappa[x] y \kappa[x]
\]

\[
d(y f(x) y) = (1 - x^2)f(x) y - y f(x)(1 - x^2)
\]

\[
\kappa[x] \otimes \kappa[y] \otimes \kappa[x]
\]

Form DGA: \( \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R} \rightarrow \mathbb{R} \)

where \( dl = 1 - x^2 \). What is homology?

\[
d(fyg f_1 y f_2) = f_0 (1 - x^2) f_1 y f_2 - f_0 y f_1 (1 - x^2) f_2
\]

so we obtain
\[
\kappa[x] \otimes \kappa[x] / f_0 g \otimes f_1 - f_0 \otimes f_1 g f_1
\]

for all \( g \in (1 - x^2) \kappa[x] \).
\[ \mathbb{k}[x] \otimes \mathbb{k}[x] \]

What is this rank 2 free right \( \mathbb{k}[x] \)-module with basis \( \{1, x\} \)?

\[
0 \longrightarrow \mathbb{k} \longrightarrow \mathbb{k}[x] \otimes \mathbb{k}[x] \xrightarrow{\cdot x} \mathbb{k}[x] \xrightarrow{\cdot (1-x)} \mathbb{k}[x] \longrightarrow 0
\]

Is \( \mathbb{k}[x] \) as \( \mathbb{k}[x] \)-bimodule.

\[ \mathbb{k}[x] \otimes (1-x^2) \mathbb{k}[x] \]

no non-units

\[ \mathbb{k} \] generated by \( x \otimes (1-1 \otimes x) \) bimodule.

Look for \( D: \mathbb{k}[x] \to M \) \( \mathbb{k}[x] \)-derivation with \( M \)
a bimodule such that \( D \) kills \( (1-x^2) \mathbb{k}[x] \). \( D \) completely determined by \( D \times (1-x^2) \mathbb{k}[x] \)

\[
(1-x^2) D x + (D(1-x^2)) x = D((1-x^2)x)
\]

similarly \( (Dx)(1-x^2) = 0 \). So \( M \)

is spanned by \( Dx, xDx, Dx \)

\( D(1-x^2) = 0 = D(x^2-1) = (Dx)x + xDx \). So it seems that

\( M \)

is spanned by \( Dx, xDx = -(Dx)x \). It's apparently related to \( \mathbb{Q}(\mathbb{k}[x]) \). Clearly.

So where are we now?

Look at \( \mathbb{k}[x] \otimes (1-x^2) \mathbb{k}[x] \). The product

\[
(f y g_1)(f_2 y g_2) = f y g_1 f_2 (1-x^2) g_2
\]

so this is associated the dual pair \( (\mathbb{k}[x], \mathbb{k}[x]) \) over \( (1-x^2) \mathbb{k}[x] \)

where \( \langle g, f \rangle = g(1-x^2)f \). What is the Hilbert-Mumford ring? Hom \( \mathbb{k}[x] \) \( (1-x^2) \mathbb{k}[x] \).
10. Start again. To understand the algebra where you have an odd operator \( \cdot \) equipped with an odd operator \( \times \) such that \( y = 1 - x^2 \) is "compact", better consider a dual pair \((U,V)\) over \(A\) and form the DGA
\[
U \otimes A V \xrightarrow{\ d \ } \text{Mult}(U,V)
\]

Study the algebra arising from elements \( x \in \text{Mult} U \otimes_A V \) sat \( d(y) = 1 - x^2 \).

Universal case: Look at DGA gen. by \( x,y \) of degree 0,1 resp. satisfying \( d(y) = 1 - x^2 \). This is
\[
\rightarrow M \otimes_R M \rightarrow M \rightarrow R
\]
where \( R = k[x] \), \( M = R \otimes_R M \) and \( d(y) = 1 - x^2 \). We want to truncated this to
\[
M / d(M \otimes M) \rightarrow R
\]
\[
d(f \otimes g, f \otimes f_2) = f_0(1-x^2)f_1 \otimes g_2 f_2 - f_0 g_1 f_1(1-x^2)f_2
\]

So you have \( R \otimes R / \{ f_0 g \otimes f_2 - f_0 \otimes g f_2 \} \) where \( g \in (1-x^2)k[x] \).

Thus our bimodule is \( R \otimes_R (1-x^2)R \).

Actually this arises from the dual pair \((k[x], k[x])\) over \( (1-x^2)R \) where the pairing is \( \langle g, f \rangle = g(1-x^2)f \).

Check:
\[
(f_1 g_0) (f_1 g_1) = f_1 g_0 f_1 (1-x^2) g_1
\]
\[
\langle g_0, f_1 \rangle
\]
\[ \text{Hom} \left( k[x^2], k[x] \right) \rightarrow \begin{array}{c}
\begin{array}{c}
(1-x^2)k[x] \\
u \\
\end{array} \\
\begin{array}{c}
\rightarrow \\
u(1) \\
\end{array} \\
(\tilde{g} \rightarrow \tilde{g}f) \\
\end{array} \]

This makes \( k[x] \) a summand of the right mult. ring. So look at \( u \) such that \( u(1) = 0 \). Then \( u \) kills \( (1-x^2)k[x] \) so \( u \in \text{Hom} \left( (1-x^2)k[x] \middle/ (1-x^2)k[x] \right) = 0 \).

Next point: call \( R = R \left( \left( \begin{array}{c}
\end{array} \right) \right) \).

\[ 0 \rightarrow K \rightarrow R \otimes R \rightarrow R \rightarrow 0 \]

\[ D(\tilde{f}) = f \circ \text{id} \otimes f = f y - y f \]

\[ D : R \rightarrow K \text{ universal deriv. killing } (1-x^2)R. \]

So only \( D x \) can be \( \neq 0 \). We find

\[ (x^2 - 1) D x + D (x^2 - 1) D x = D (x^2 - 1) x = 0 \]

\[ \Rightarrow (x^2 - 1) D x = 0 \]

\[ (D) (1-x^2) = 0. \]

At most 4 of \( x \) alts. \( x_0 D x, x D x, (D x) x \), \( x D x \)

\[ 0 = D (x^2 - 1) = D x x + x D x, \quad D x + x D x x = 0 \]

And we do have such \( x = 0 \) because

\[ R \rightarrow R / (1-x^2)R \rightarrow \Omega^1 \left( k[F] \right). \]

Most element in \( R \) commute with \( y \). So \( xy - yx \) and \( xy - yx \) are the two elements spanning the kernel.
Let's see if you can get anywhere.

You have $x \in \mathbb{R}$

You can refine $x$, $x' = x + \frac{1}{2}x(1-x^2)$

$$1-(x')^2 = 1-x^2(1+1-x^2+\frac{1}{4}(1-x^2)^2)$$

$$= (1-x^2)^2 - \frac{x^2}{4}(1-x^2)^2$$

$$= (1-x^2)^2 \left(1-\frac{x^2}{4}\right)$$

We know the kernel of $\phi$ contains $xy-yx$ and $xy-yx \mapsto (xy-yx) = x^2y-xyx$. Kernel.

So in the end you have this bimodule of operators $k[x] \otimes_{k[x]} k[x]$ generated by $y$.

$x(xy+yx) = x^2y + xyx$

$(xy+yx)x = xyx + yx^2$

Look at the center of this bimodule all element $\overline{y}$ such that $\overline{xy} = \overline{yx}$.

$f(x)g + yf(x)$

Example $f(x)(x^2-1)g$

Because $xf(x)(x^2-1)g = yf(x)(x^2-1) = f(x)(x^2-1)g$.

So anything $f(xy)+g(f(x)$ $f(0) = 0$

$x^3y = xyx^2$ $x^3y = x(x^2-1)y + xy$
when we write elements down \( hf(x)g(x) \)
you first divide by either \( f \) or \( g \) by \( x^2 - 1 \).

Anything \( (x^2 - 1) f(x) g(x) \) is in the center

\[
x (xy - yx) = x^2 y - xy x
\]

\[
\begin{align*}
(x^2 - 1) \frac{z}{xy - yx} &= \frac{x^2 y - xy x}{-xy x + y x^2} = 2 (x^2 y - xy x) \\
x z &= 2xz
\end{align*}
\]

\[
\begin{align*}
[x, z] &= 2xz \\
[x, xz] &= x [x, z] = x 2xz = 2x^2 z = 2z
\end{align*}
\]

\[
[x, y] = z
\]

\[
[x, 2y - xz] = 2z - 2z = 0
\]

\[
2y - x(xy - yx) \in \text{ center}
\]

The center maps isom. to

\[
(1 - x^2) k[x] z
\]

basis \( \{ 1, x, y, xy, yx, xz \} \)

\[
(1 - x^2) k[x] y = z
\]

\[
\begin{align*}
(1 - x^2) x^n y &= x y + y x \\
2y - x(xy - yx)
\end{align*}
\]

Thus it seems I can arrange \( y \) to commute with \( x \), replace \( y \) by \( y - \frac{1}{2} x [x y - y x] \)
\[ y - \frac{1}{2} x (x y - y x) \] should commute with \( x \).
\[
\begin{align*}
x(xz) &= x(x^2 y - x y x) \\
&= x(y x^2 - y x) \\
&= x y x^2 - y x^3
\end{align*}
\]

\[
[x, z] = x z - z x
\]

\[
[x, x y - y x] = x z
\]

\[
[x, -\frac{1}{2} x z] = x z - z x = 2 x z
\]

\[
(x z) = x z - 2 x = 2 x z
\]

\[
(l - x^2) z = d y z = + y d z = 0.
\]

\[
[x, y - \frac{1}{2} x z] = z - z = 0
\]

Anyway what happens. Stop.

\[
x \quad d y = 1 - x^2
\]

\[
x' = x + \frac{1}{2} x (1 - x^2)
\]

\[
l - x'^2 = l - x^2 \left(1 + (1 - x^2 + \frac{(1 - x^2)^2}{4})\right)
\]

\[
= l - 2 x^2 + x^4 - \frac{x^2 (1 - x^2)^2}{4}
\]

\[
= (1 - x^2)^2 \left(1 - \frac{x^2}{4}\right)
\]

\[
\text{What is } y' \quad \text{dy} = 1 - x^2 \text{dy} \quad \text{where } [x, y] = 0.
\]

\[
\begin{align*}
l - x'^2 &= d y d y \left(1 - \frac{x^2}{4}\right) \\
y' &= y \left(1 - x^2 \right) \left(1 - \frac{x^2}{4}\right)
\end{align*}
\]

Basically we need equivariant relation.

Kasparov module - generalization of an object of \( \mathcal{D}(A^{op}) \).

You want \( (U, V) \) super dual pair over \( A \) together with \( x \in \text{Mult}(U, V) \), \( y \in U \otimes_A V \) with \( d y = 1 - x^2 \).
We have two dual bases in $\mathbf{R} = (U_i, V_i)_{i=0,1}$.

For $i = 0$, \( V_i = T(E^*) \), \( U_i = T(E) \), the direct sum of \( E \otimes \mathbb{C} \).

For all $n > 0$, we have \( U_i = T(E) \otimes E \), \( V_i = E \otimes T(E^*) \).

We have this basis for $T(E) \otimes E \rightarrow T(E)$.

Also, we have a compact $R \rightarrow \mathbb{R}$.

You want to compute the composition $K_0(R) \rightarrow K_0(k) \rightarrow \mathbb{R}$.

This composition is given by $\mathcal{S}$, a Kasparov product. Basically, $U_0 \otimes R$, $U_1 \otimes R$. This is a pair of $R$-bimodules, $U_0 \otimes R$ and $U_1 \otimes R$.

In principle, this consists of $T(E) \otimes E \otimes R \oplus T(E) \otimes R$ and a subtle "operator" between them. What you want to do is to take $P \in P(R^2)$, then form

\[ P \otimes T(E) \otimes E \otimes R \oplus P \otimes T(E) \otimes R \]

with some $\otimes R$. Then tensor with $R$.

Then someone you homotoped the differential to get $P \otimes T(E) \otimes (E \otimes R \rightarrow R)$.

The probably not true because it would give $P \otimes T(E) \otimes T(E^*)$ which is $P \otimes (T(E) \otimes T(E^*))$.
Is there a way to see what to do.

Somehow you will apply $P \otimes R$ to a bimodule $(T(E) \otimes E \otimes T(E)) \otimes R$. There should be some twisting.

$P \otimes (T(E) \otimes E \otimes T(E)) \otimes R$

Intuitively this is $P \otimes k \otimes R$, which should give $P$ by the homotopy $\epsilon : R \to R$ joining the identity to the "augmentation".

If $P = R$ I must get $R$.

What method? Can you construct a bimodule resolution

$\rightarrow R \otimes E \otimes R \rightarrow R \otimes (E \otimes E^*) \otimes R \rightarrow R \otimes R \rightarrow R \rightarrow 0$

Let $D : R \to M$ be a derivation determined by $D(s_i), D(s_i^*), D(s_j)T(s_i^*) + s_jD(s_i^*) = 0$

$R = T/J$

$0 \rightarrow T/J^2 \rightarrow R \otimes \mathcal{O}(T) \otimes R \rightarrow \Omega(T/J) \rightarrow 0$

12/01/97 Let $R = T(E)$, have $R \rightarrow k \rightarrow R$ homos.

And you want to understand $K_0(R) \rightarrow K_0(k) \rightarrow K_0(R)$.

The idea will be to compare with the bimodule resolution

$0 \rightarrow R \otimes E \otimes R \rightarrow R \otimes R \rightarrow R \rightarrow 0$

The problem is to link $P \otimes k \otimes R$ with $P$.

$0 \rightarrow P \otimes E \rightarrow P \rightarrow P \otimes k \rightarrow 0$
12/01/97 cont. \( R = T(E) \), have hom R \rightarrow R.

To understand \( K_0(R) \rightarrow K_0(k) \rightarrow K_0(R) \), breakely to see if it's L. Use somehow the bimodule resolution

\[
0 \rightarrow B \otimes E \otimes R \rightarrow R \otimes R \rightarrow R \rightarrow 0
\]

which gives a \( \Delta \) functorial resolution

\[
0 \rightarrow P \otimes E \otimes R \rightarrow P \otimes R \rightarrow P \rightarrow 0
\]

Easier to look at. Willemsen free product paper.

\[
0 \rightarrow R \otimes E \otimes M \rightarrow R \otimes M \rightarrow M \rightarrow 0
\]

What techniques do you know? filtered rings, projective line, Hasse-Knörrer papers? Bass FT.

Let's revert to old notation. A unital \( T \) variable, consider \( R = A[T] \), filtered ring \( F_p A[T] = A + AT + \cdots + AT^p \).

Graded ring \( \bigoplus_{p \geq 0} F_p A[T] = A[T_0, T_1] \). Surface basis \( h^p T^i \) \( 0 \leq i \leq p \). Also \( \bigoplus h^p F_p R = A[h, hT] \). Use graded \( A[T_0, T_1] \) modules, mixed \( \otimes \) to to get

\[
\left( \frac{U}{A[T_0, T_1] / (T_0^n) \modu} \right) \quad \text{graded} \quad A[T_0, T_1] \quad \text{modules} \quad \text{graded} \quad A[T] \quad \text{modules}
\]

Try again.
I have to understand $K_0(\mathcal{A}[T])$. Basic idea goes back to Grothendieck at least in the regular case. Basic idea - take $P \in P(A[T])$ extend to something over $P^A$ and then? Take $P \otimes A[T^{-1}]$ and extend to $A[T^{-1}]$. This can do somewhat by embedding $P$ as a summand of $A^\oplus r$. 

Given $P$ you consider the appropriate building of $A[T^{-1}]$ lattices $L$ in $P[T^{-1}] = P \otimes A[T^{-1}]$. $P, L$ together define a "module" over $P^A$.

Let's recall some of the ideas. Start with

$$GL_n(A[T,T^{-1}]) = \bigotimes_{\mathbb{Z}} GL_n(A[T,T^{-1}]) = \bigotimes_{\mathbb{Z}} (M_n A)[T,T^{-1}] \times$$

What's important is the order in $T$. There's a linearization procedure to reduce to an invertible matrix of the form $aT + b$. This defines a bundle over $P^A$ of the form $\mathcal{O} \otimes P_0 \oplus \mathcal{O}(-1) \otimes P_1$ roughly where $P_0 \oplus P_1 = A^N$. Why? Suppose $\mathcal{O} \otimes b \in A[T^d,T^{-1}]$ is invertible

$$A[T^{-1}](aT + b), A[T] \cup \mathcal{T} \mathcal{A}[T^{-1}], (aT + b)^{-1} A[T]$$
So you find that

$$T^\top A[T^{-1}] \oplus (aT+b)^{-1}A[T] = A[T, T^{-1}]$$

Look at $K = T^\top A[T^{-1}] \oplus (aT+b)^{-1}A[T]$. It is projective over $\mathbb{P}$. 

$$K = (aT+b)^{-1}A[T] / A[T]$$

You can also look at

$$(aT+b)^{-1}A[T^{-1}] , \ A[T]$$

$$(a+bT^{-1})^{-1}A[T^{-1}] T^{-1} , \ A[T]$$

$$(a+bT^{-1})^{-1}A[T^{-1}] T^{-1} , \ A[T]$$

Basically you are computing

$$\mathbb{P}(\mathbb{H}(F))$$

$$(a+bT^{-1})^{-1}A[T^{-1}] / A[T^{-1}] = A[T^{-1}] / (a+bT^{-1})A[T^{-1}]$$
I don't understand the geometry, but basically you are examining a clutching function \( a + b \), i.e., a \( \text{Kronecker module} \) such that \( a + b \) is invertible for \( a \neq 0, \infty \), hence you have a torsion sheaf \( F \) over \( \mathbb{P}^1 \) with support at \( 0, \infty \). Then \( A = H^0(F) \) and the splitting of \( A \) results from the splitting of the support.

10/21/97 So what next? Hjelmslev linearization.

IDEA to be explored later: For example, maps of \( R \)-modules \( M \to N \) is nuclear when it is in the image of

\[
\text{Hom}_R(M, R) \otimes_R N \to \text{Hom}_R(M, N)
\]

equivalently, if it factors \( M \to R^m \to N \) for some. In representation theory one introduces a category of \( G \)-modules, a triangulated category (I think), in which maps factoring through a projective \( G \)-module are put equal to zero. Analogous to null homotopic maps.

Work out formulas again: Let \( U = \{ U_i \to U_0 \} \) be a complex of \( R \)-modules, such that \( U_0 \) is homotopic to an \( A \)-nuclear map. Find a sheaf with a f. proj. complex \( T_0 \to T_0 \) which is acyclic modulo \( A \). We can choose \( h: U_0 \to U_1 \) such that \( 1 - h \) on \( U_0 \) and \( 1 - dh \) on \( U_1 \) are \( A \)-nuclear. The main point is the case \( A = R \). Let

\[
1 - dh = 0: U_0 \to T_0 \to U_0 \\
1 - hd = 0: U_1 \to T_1 \to U_1
\]

where \( T_0, T_1 \) are finite free modules. Actually
Let $T_1$ be the pull-back $V$ in

\[
\begin{array}{ccc}
T_1 & \xrightarrow{d} & T_0 \\
\downarrow{d_1} & & \downarrow{d_0} \\
U_1 & \xrightarrow{d} & U_0
\end{array}
\]

then $(T \to T_0) + (U_1 \to U_0)$ is a quasi. In fact $j$ is a hsq because its core is contractible.

\[
0 \to T_1 \xrightarrow{\begin{pmatrix} (h_1^u) & (h_0) \\ (d) & 0 \end{pmatrix}} U_1 \xrightarrow{j_1} U_0 \to 0
\]

1 on $T_1 = c_1 f_1 + dh^T$

\[
\begin{pmatrix} -d_1 \\ d \end{pmatrix} \begin{pmatrix} -c_1 & h^T \end{pmatrix} + \begin{pmatrix} h_1^u \\ c_0 \end{pmatrix} (d^T j_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

\[
j_1 = 1 - \frac{dh}{c_1}
\]

**Argument:**

\[
1 - T_1 = (c_1 h^T) (-d_1) = (-c_1 h^T) \begin{pmatrix} 1 - h_1 u_d & h_1 u_0 \\ -c_0 & 1 - c_0 s_0 \end{pmatrix} (-d_1)
\]

**Okay. Yes**
What do I do next? \( K_0(A[T]) = K_0(A) \oplus \text{Nil}_0(A) \)

Let \( P \in \mathcal{P}(A) \) be an \( A \)-module. \( A[T] \otimes_A P \to A[T] \otimes_A P \)

Let \( P \in \mathcal{P}(A[T]) \), let \( \bar{P} = \otimes \left( \frac{P}{TP} \in \mathcal{P}(A) \right) \).

You want to compare \( P \) with \( A[T] \otimes_A \bar{P} \). Have a surj \( P \to \bar{P} \) and since \( P \) proj over \( A \), can lift to \( \bar{P} \to P \) and then you get \( A[T] \otimes_A \bar{P} \to P \). What to hope for. Over a field you have a vector bundle over the affine line.

I need to understand \( A[T] \)!

Let's begin with \( M \) a f.g. \( A[T] \)-module. Want to extend to a module over \( P \). Use \( A[T^{-1}] \)? Take \( \bar{P} \) over \( A \) look at building at \( \infty \).

\[ M \otimes_{A[T]} A[T^{-1}] \]

To get started we might use graded module approach.

Begin with \( M \) a f.g.

\[ L \in \text{Hom}(M, A[T]) \otimes A[T] \]

Later? Let's make a systematic attempt to calculate \( K_0(A[T]) \). List ideas to pursue.

graded modules over \( S = A[T^{-1}] \)
modules over \( X = \mathbb{P}_A^1 \)

diagonal class in \( K^0(X \times X) \) - review
modules over \( \mathbb{P}_A^1 \) and \( \text{K}-\text{modules} \)
Here's something worth pursuing. A bi-module resolution like the Koszul complex

\[ \cdots \to \mathbb{R}^2 \mathbb{V} \otimes \mathbb{R} \to \mathbb{R} \mathbb{V} \otimes \mathbb{R} \to \mathbb{R} \mathbb{R} \to \mathbb{R} \to 0 \]

in the case \( R = S(V) \) yields a functorsial projective resolution for any module:

\[ \cdots \to \mathbb{R}^2 \mathbb{V} \otimes \mathbb{M} \to \mathbb{R} \mathbb{V} \otimes \mathbb{M} \to \mathbb{M} \to 0 \]

Idea of diagonal approximation, Thom form for the tangent bundle, etc.

What is the analog for modules over \( X = \mathbb{P}^1 \)

Let \( \mathbb{F} \) module over \( \mathbb{P}^1 \)

\[ H^1(\mathbb{F}(-1)) = 0 \quad \Rightarrow \quad H^1(\mathbb{F}(-1)) = H^1(\mathbb{F}(0)) = H^1(\mathbb{F}(1)) = \cdots \]

\[ 0 \to \mathbb{F}(n-1) \to \mathbb{F}(n) \oplus^2 \to \mathbb{F}(n+1) \to 0 \]

\[ 0 \to H^0(\mathbb{F}(-1)) \to H^0(\mathbb{F}) \oplus^2 \to H^0(\mathbb{F})(1) \to 0 \]

\[ 0 \to H^0(\mathbb{F}) \to H^0(\mathbb{F})(1) \oplus^2 \to H^0(\mathbb{F})(2) \to 0 \]

\[ 0 \to \mathbb{F} \otimes \mathbb{H}^0(\mathbb{F}) \to \mathbb{F} \to 0 \]

\[ H^0(\mathbb{F}) = H^1(\mathbb{F}) = 0. \]

\[ \mathbb{F} \otimes \mathbb{H}^0(\mathbb{F}) \to \mathbb{F} \to 0 \]

\[ H^1(\mathbb{F}) = 0 \quad \Rightarrow \quad \mathbb{O} \otimes H^0(\mathbb{F}(1)) \to F'(1) \]

\[ L_0 \quad H^1(\mathbb{F}(-1)) = 0 \quad \Rightarrow \quad \mathbb{O} \to \mathbb{O} \otimes H^0(\mathbb{F}(-1)) \to \mathbb{O} \otimes H^0(\mathbb{F}) \to \mathbb{F} \to 0 \]

\[ F = \left\{ \mathbb{O} \otimes R[1](\mathbb{F}(-1)) \to \mathbb{O} \otimes R[1](\mathbb{F}) \right\} \]

Vaguely really modules \( \Lambda^2 V \).
Tilting object $O \oplus O(-1) = T$

has $R^i \text{Hom}(T, T) = 0$ \For $i \neq 0$

$\text{End}_O(T) = \begin{pmatrix} O(-1) & 0 \\ 0 & k \end{pmatrix}$ \newline $V = \Gamma(O(0))$

Yet equivalence of derived cats.

Idea: Start with a \textit{web} over $A[T]$

regard as quasi-coherent sheaf on $\mathbb{P}^1_A$

It is flat so it is a filtered ind. limit of f.g. free modules.

Let's be naive. Suppose we have $M$ an $A[T]$-mod

and \[
M \rightarrow A[T]^{ \oplus n} \rightarrow \ast \rightarrow M
\]

\[
M^v \leftarrow t_i A[T]^{ \oplus n} \leftarrow t_i M^v
\]


Let $F_0 M$ be a f.g. $A$-submodule of $M$ generating $M$.

set $F_p M = F_p^0 + TF_0 + \cdots + T^p F_0$

\[
0 \rightarrow N \rightarrow A[T]^{n} \rightarrow M \rightarrow 0
\]

\[
U \quad U \quad U
\]

\[
0 \rightarrow N \wedge F_p A[T]^{n} \rightarrow F_p A[T]^{n} \rightarrow F_p M \rightarrow 0
\]

Want finitely presented

\[
0 \rightarrow M \rightarrow A[T]^{n} \rightarrow A[T]^{n} \rightarrow M \rightarrow 0
\]

Filtered mess.
Problem: To understand $K_*(A[t])$.

I think you want to start with the FT of Bass about $K_1(A[t,t^{-1}])$, as this is closer to the geometry. The point is to work with modules over $P_A$ where you have canonical resolutions and finiteness. An element $g \in GL_n(A[t,t^{-1}])$ is a clutching function for a VB, over $P_A$. On the other hand, twisting $E$ with $E(n)$ yields a VB with canonical resolution

$$0 \to \mathcal{O}(1) \to \mathcal{O} \to \mathcal{O}(E_n) \to E_n \to 0$$

where $\mathcal{O} \in \mathcal{P}(A)$. Cech

Now translate into matrix calculations.

$$A[t] \mathcal{O} A[t,t^{-1}] \mathcal{O} A[t^{-1}]$$

Let $E = (E_+, E_-, \psi_+ E_+ = \psi_- E_-)$

```
\Gamma' = \begin{cases} 
\tau_+ & \text{if } g(t) = f_+ \\
\tau_- & \text{if } g(t) = f_-
\end{cases}
```

It is possible to build
Let's go over Davydov again.

R until e is idempotent in R

\[ 0 \to \ker R \to R \to R/\ker R \to 0 \]

\[ e^1 \ker R \to e^1 \ker R \]

Assumptions:

\[ R \ker e \to e^1 \ker R \]

Now \( e^1 \ker R \to e^1 \ker R \to e^1 \ker R \to e^1 \ker R \)

\[ e = 1 - z z^* \]

\[ R/\ker R = 0 \to k[z, z^{-1}] \]

Why is this true?

But wait: Take \( R = \mathcal{F}_E, \text{deg}(E) = 1 \)

\[ = k[z, z^*]/z z^* = 1 \]

Thus you get

\[ \mathcal{K}_1(k) \to K_1(R) \to K_1(k[z, z^{-1}]) \]

\[ \to K_0(k) \to K_0(R) \to K_0(k[z, z^{-1}]) \]

Here \( k \) can be any unital ring.

Use that \( \mathcal{K}_1(k[z, z^{-1}]) = k \mathcal{K}_1(k) \oplus N_1(k) \oplus N_1(k) \oplus K_0(k) \)

Thus \( \mathcal{K}_1(R) = \mathcal{K}_1(k[z]) \oplus \mathcal{K}_1(k[z^{-1}]) \)

What is \( e^1 \ker R \to e^1 \mathcal{F}(E) \oplus \mathcal{F}(E^*) e^1 = \mathcal{F}(E) \oplus \mathcal{F}(E^*) \)

\[ e^1 = \sum z_i z_i^* \text{ kills } 1 \]

Reproduces rest.

So look at basis \( s_\alpha s_\beta^* e^1 = s_\alpha s_\beta^* \) for \( \beta \).
$$e^{-s_x}s_eta^e = s_x$$ if \( |x| > 0 \) 

$$e^{-s_x}s^*_eta e^i = s_x s^*_eta$$ if \( |x|, |\beta| \) both \( > 1 \).

$$e^{-s_x}e^i = s_x e^i$$ if \( |x| > 1 \)

$$s_x e^i = \bigoplus s_x s_i s^*_i$$

Consider

$$s_x e^i e^i s^*_\beta = s_x \bigoplus s_i s^*_i$$

satisfy same relations

$$e^{-s^*_i} s_i e^i = s_i e^i$$

$$s_x e^i s^*_\beta e^i = s_x s_i s^*_i$$

$$e_i = ss^*_i$$

$$e^i Re^i$$

$$ss^*_i ss^*_i$$

$$ss^*_i ss^*_i$$

$$ss^*_i ss^*_i$$

$$ss^*_i ss^*_i$$

$$ss^*_i ss^*_i$$