January 3, 1996

Let $F : \mathcal{X} \longrightarrow \mathcal{Y}$ be an equivalence of categories, i.e. a fully faithful and essentially surjective functor. Then there is a quasi-inverse (G, ε, η) for F, which is unique up to canonical isomorphism and obtained as follows.

F essentially surjective means we can choose for each Y a GY in $\mathcal{X}$ together with an isomorphism $\varepsilon_Y : FG(Y) \xrightarrow{\sim} Y$. Because F is fully faithful we can define G uniquely on morphisms in $\mathcal{Y}$ such that G becomes a functor and $\boxed{\varepsilon : FG \xrightarrow{\sim} 1}$ is an isomorphism. Then we have $\varepsilon . F : FGF \xrightarrow{\sim} F$, so again as F is fully faithful, there is a unique isomorphism $\boxed{\eta : GF \xrightarrow{\sim} 1}$ such that $\boxed{F . \eta = \varepsilon . F}$. I claim that also $\boxed{\eta . G = G . \varepsilon}$ holds.

Proof. By definition given $v : Y \to Y'$, then $G(v) : G(Y) \to G(Y')$ is the unique map such that

$$\begin{tikzcd}
FG(Y) \arrow[r, "F(G(v))"] \arrow[d, "\varepsilon_Y"', "\wr"] & FG(Y') \arrow[d, "\varepsilon_{Y'}", "\simeq"'] \\
Y \arrow[r, "v"] & Y'
\end{tikzcd}$$

commutes. Thus $G(\varepsilon_Y)$ is unique such that

$$\begin{tikzcd}
FGFGY \arrow[r, "F(G(\varepsilon_Y))"] \arrow[d, "\varepsilon_{FGY}"'] & FGY \arrow[d, "\varepsilon_Y"] \\
FGY \arrow[r, "\varepsilon_Y"] & Y
\end{tikzcd}$$

commutes and as ε_Y is an isomorphism, this means $G(\varepsilon_Y)$ is unique such that $FG(\varepsilon_Y) = \varepsilon_{FGY}$. But $\eta_X : GFX \to X$ by definition is unique such that $F(\eta_X) = \varepsilon_{FX}$. Taking $X = GY$, we find $\eta_{GY} = G(\varepsilon_Y)$ i.e. $\eta . G = G . \varepsilon$.

Notation: Given $\xi : F \to F'$, $\zeta : G \to G'$ of functors which can be composed we write $\xi\circ\zeta$ for the induced map on compositions:

$$\begin{tikzcd} FG \arrow[r, "F\circ\zeta"] \arrow[d, "\xi\circ G"'] \arrow[dr, "\xi\circ\zeta"] & FG' \arrow[d, "\xi\circ G'"] \\ F'G \arrow[r, "F'\circ\zeta"'] & F'G' \end{tikzcd}$$

Also we ~~write~~ $F\circ\zeta$ instead of $1_F\circ\zeta$ [redacted]. (Maybe $*$ is a traditional notation).

Uniqueness of quasi-inverses. Let ([redacted] G, ε, η), ([redacted] G', ε', η') be two quasi-inverses for F. Then we have

$F\circ\eta = \varepsilon\circ F : FGF \to F$ $\qquad$ $F\circ\eta' = \varepsilon'\circ F : FG'F \to F$

$G\circ\varepsilon = \eta\circ G : GFG \to G$ $\qquad$ $G'\circ\varepsilon' = \eta'\circ G' : G'FG' \to G'$

Now define $\xi : G \simeq G'$ by either

1) $G \xleftarrow{G\circ\varepsilon'} GFG' \xrightarrow{\eta\circ G'} G'$

1') $G \xleftarrow{\eta'\circ G} G'FG \xrightarrow{G'\circ\varepsilon} G'$

Let's check 1) is compatible with ε-maps:

$$\begin{tikzcd} FG \arrow[d, "\varepsilon"'] & FGFG' \arrow[l, "FG\circ\varepsilon'"'] \arrow[r, "\overbrace{F\circ\eta}^{\varepsilon\circ F}\circ G'"] \arrow[d, "\varepsilon\circ\varepsilon'"'] & FG' \arrow[d, "\varepsilon'"] \\ 1 \arrow[r, phantom, "="] & 1 \arrow[r, phantom, "="] & 1 \end{tikzcd}$$

[redacted]

and with η-maps:

$$\begin{tikzcd} GF \arrow[d, "\eta"'] & GFG'F \arrow[l, "G\circ\varepsilon'\circ F"'] \arrow[r, "\eta\circ G'F"] \arrow[d, "\eta\circ\eta'"] & G'F \arrow[d, "\eta'"] \\ 1 \arrow[r, phantom, "="] & 1 \arrow[r, phantom, "="] & 1 \end{tikzcd}$$

$GF\circ\eta'$

Next show $1) = 1')$ by applying F to $1')$.

$$\begin{tikzcd}
FG \arrow[d, "\varepsilon"'] & \underset{\overset{\shortparallel}{\varepsilon'\cdot FG}}{} FG'FG \arrow[l, "F\cdot\eta'\cdot G"'] \arrow[r, "FG'\cdot\varepsilon"] \arrow[d, "\varepsilon'\cdot\varepsilon"'] & FG' \arrow[d, "\varepsilon'"] \\
1 \arrow[r, equal] & 1 \arrow[r, equal] & 1
\end{tikzcd}$$

Thus F applied to $1)$ and $1')$ yield the same map, namely $(\varepsilon')^{-1}\varepsilon$, so $1) = 1')$ as F is fully faithful.

Jan. 4, 1996 [redacted]

Let (G, ε, η) and $(G', \varepsilon', \eta')$ be quasi-inverses for F:

$$\begin{array}{ll}
\varepsilon : FG \xrightarrow{\sim} 1 & \varepsilon\cdot F = F\cdot\eta : FGF \to F \\
\eta : GF \xrightarrow{\sim} 1 & G\cdot\varepsilon = \eta\cdot G : GFG \to G \\
\varepsilon' : FG' \xrightarrow{\sim} 1 & \varepsilon'\cdot F = F\cdot\eta' : FG'F \to F \\
\eta' : G'F \xrightarrow{\sim} 1 & G'\cdot\varepsilon' = \eta'\cdot G' : G'FG' \to G'
\end{array}$$

Because F is fully faithful $\exists!\ \xi : G \to G'$ such that

$$\begin{tikzcd}
FG \arrow[r, "F\cdot\xi"] \arrow[d, "\varepsilon"'] & FG' \arrow[d, "\varepsilon'"] \\
1 \arrow[r, equal] & 1
\end{tikzcd}$$

commutes.

Then

$$\begin{tikzcd}
FGF \arrow[r, "F\cdot\xi\cdot F"] \arrow[d, "\varepsilon\cdot F = F\cdot\eta"'] & FG'F \arrow[d, "\varepsilon'\cdot F = F\cdot\eta'"] \\
F \arrow[r, equal] & F
\end{tikzcd}
\ \text{comm.} \ \overset{F \text{ f.f.}}{\Longrightarrow} \
\begin{tikzcd}
GF \arrow[r, "\xi\cdot F"] \arrow[d, "\eta"'] & G'F \arrow[d, "\eta'"] \\
1 \arrow[r, equal] & 1
\end{tikzcd}
\ \text{comm.}$$

Hence we have $\xi : (G, \varepsilon, \eta) \xrightarrow{\sim} (G', \varepsilon', \eta')$. To get a formula for ξ apply G

$$\begin{tikzcd}
G \arrow[d, equal] & GFG \arrow[l, "G\cdot\varepsilon"'] \arrow[r, "GF\cdot\xi"] \arrow[d, "\eta\cdot G"] \arrow[rr, bend right, "1"'] & GFG' \arrow[r, "G\cdot\varepsilon'"] \arrow[d, "\eta\cdot G'"] & G \\
G \arrow[rr, "\xi"] & & G' &
\end{tikzcd}$$

hence

$$\begin{tikzcd}
& GFG' \arrow[dl, "G\cdot\varepsilon'"'] \arrow[dr, "\eta\cdot G'"] & \\
G \arrow[rr, "\xi"] & & G'
\end{tikzcd}$$

which is 1) on p85. Similarly

$$\begin{tikzcd} G \arrow[rrr, bend left, "1"] & GFG \arrow[l, "\eta\cdot G"'] \arrow[r, "\xi\cdot FG"] \arrow[d, "G\cdot\varepsilon"] & G'FG \arrow[r, "\eta'\cdot G"] \arrow[d, "G'\cdot\varepsilon"] & G \\ & G \arrow[ul, equal] \arrow[r, "\xi"] & G' & \end{tikzcd}$$

yields

$$\begin{tikzcd} & G'FG \arrow[dl, "\eta'\cdot G"'] \arrow[dr, "G'\cdot\varepsilon"] & \\ G \arrow[rr, "\xi"] & & G' \end{tikzcd}$$

which is 1*) on p85.

Next I want to show using the essential uniqueness of a quasi-inverse that given (F,G,ε,η), $(F',G',\varepsilon',\eta')$ and $\theta : F \xrightarrow{\sim} F'$ there is a corresponding $\xi : G \xrightarrow{\sim} G'$ such that (θ, ξ) is an isomorphism $(F,G,\varepsilon,\eta) \xrightarrow{\sim} (F',G',\varepsilon',\eta')$

i.e.

$$\begin{tikzcd} FG \arrow[rr, "\theta\cdot\xi"] \arrow[dr, "\varepsilon"'] & & F'G' \arrow[dl, "\varepsilon'"] \\ & 1 & \end{tikzcd} \qquad \begin{tikzcd} GF \arrow[rr, "\xi\cdot\theta"] \arrow[dr, "\eta"'] & & G'F' \arrow[dl, "\eta'"] \\ & 1 & \end{tikzcd}$$

commute

in other words we have $\varepsilon = \varepsilon'(\theta\cdot\xi)$, $\eta = \eta'(\xi\cdot\theta)$.

The idea is that $\theta : F \xrightarrow{\sim} F'$ makes G' into a quasi-inverse for F. More precisely we have an isom.

$$(\theta, 1) : (F, G', \varepsilon'(\theta\cdot G'), \eta'(G'\theta)) \xrightarrow{\sim} (F', G', \varepsilon', \eta')$$

Then we know there is a $\xi : G \to G'$ such that

$$(1, \xi) : (F, G, \varepsilon, \eta) \xrightarrow{\sim} (F, G', \varepsilon'(\theta\cdot G'), \eta'(G'\cdot\theta))$$

i.e.
$$\varepsilon = \varepsilon'(\theta\cdot G')(F\cdot\xi) = \varepsilon'(\theta\cdot\xi) \quad \text{and} \quad \eta = \eta'(G'\cdot\theta)(\xi\cdot F) = \eta'(\xi\cdot\theta).$$

The first equation says:

$$\begin{tikzcd} FG \arrow[r, "F\cdot\xi"] \arrow[d, "\varepsilon"] & FG' \arrow[r, "\theta\cdot G'"] & F'G' \arrow[d, "\varepsilon'"] \\ 1 \arrow[rr, equal] & & 1 \end{tikzcd}$$

This determines ξ since the other maps are isomorphisms and F is fully-faithful

March 12, 1996

I want to record formulas involved in the equivalences:

$$\begin{tikzcd} \mathrm{mod}(\mathbb{R}) \arrow[r, shift left, "\mathrm{extn}"] \arrow[d, "\wr"] & \mathrm{mod}(\mathbb{H}) \arrow[l, shift left, "\mathrm{res}"] \arrow[d, equal] \\ \mathrm{mod}(\mathbb{C}[\sigma]_+) \arrow[r, shift left, "H\otimes -"] & \mathrm{mod}(\mathbb{C}[\sigma]_-) \arrow[l, shift left, "H\otimes -"] \end{tikzcd}$$

First consider the adjoint functor relations

$$(1) \qquad \begin{aligned} \mathrm{Hom}(H\otimes V, W) &= \mathrm{Hom}(V, H^*\otimes W) \\ \mathrm{Hom}(V, H\otimes W) &= \mathrm{Hom}(H^*\otimes V, W) \end{aligned}$$

where H is a finite-diml vector space. The adjunction maps in the former arise from the canonical maps

$$\alpha: H\otimes H^* \longrightarrow \mathbb{C} \qquad h\otimes h^* \longmapsto (h|h^*) = (h^*|h)$$

$$\beta: \mathbb{C} \longrightarrow H^*\otimes H \qquad 1 \longmapsto \sum e_i^*\otimes e_i$$

The adjunction maps in the latter arise from the canonical maps

$$\alpha': H^*\otimes H \longrightarrow \mathbb{C} \qquad h^*\otimes h \longmapsto (h^*|h)$$

$$\beta': \mathbb{C} \longrightarrow H\otimes H^* \qquad 1 \longmapsto \sum e_i\otimes e_i^*$$

obtained from the preceding via the flips. Notice that $\alpha'\beta = \alpha\beta'$ is $\mathrm{tr}(1) = \dim H$.

In the $\mathbb{R}, \mathbb{H}$ situation, $H \approx \mathbb{C}^2$ $\Lambda^2 H = \mathbb{C}$ which comes equipped with a volume we use to identify H^* and H. We have a single adjoint functor relation

$$\mathrm{Hom}(H\otimes V, W) = \mathrm{Hom}(V, H\otimes W)$$

arising from the canonical maps

$\alpha : H\otimes H \longrightarrow \mathbb{C}$ $\qquad h_1\otimes h_2 \longmapsto h_1\wedge h_2$

$\beta : \mathbb{C} \longrightarrow H\otimes H$ $\qquad 1 \longmapsto e_2\otimes e_1 - e_1\otimes e_2$

(here $e_1\wedge e_2 = 1$)

Check: Let $h = z_1e_1 + z_2e_2$ so that $z_2 = e_1\wedge h$, $z_1 = -e_2\wedge h$

$$H \xrightarrow{\beta\otimes 1} H\otimes H\otimes H \xrightarrow{1\otimes\alpha} H$$

$$h \longmapsto (e_2\otimes e_1 - e_1\otimes e_2)\otimes h \longmapsto e_2(e_1\wedge h) - e_1(e_2\wedge h) = h$$

$$H \xrightarrow{1\otimes\beta} H\otimes H\otimes H \xrightarrow{\alpha\otimes 1} H$$

$$h \longmapsto h\otimes(e_2\otimes e_1 - e_1\otimes e_2) \longmapsto (h\wedge e_2)e_1 - (h\wedge e_1)e_2 = h$$

Notice that $\mathbb{C} \xrightarrow{\beta} H\otimes H \xrightarrow{\alpha} \mathbb{C}$ is $1 \mapsto e_2\wedge e_1 - e_1\wedge e_2 = -2(e_1\wedge e_2) = -2$. (I don't really understand this sign. Somehow it arises from the fact that the volume $\Lambda^2 H = \mathbb{C}$ determines two isos. of H with H^*, i.e. $\Lambda^2 H \subset H\otimes H$ and you can contract either factor of H with H^*; these two isos. have opposite sign. Perhaps also this sign is related to what happens with the Fourier transform.)

Next recall $\mathbb{C}[\sigma]_\pm = \mathbb{C} + \mathbb{C}\sigma$ where $\sigma z = \bar{z}\sigma$ and $\sigma^2 = \pm 1$. $\mathbb{C}[\sigma]_- = \mathbb{H}$ where $\sigma = j$, so mod($\mathbb{H}$) = mod($\mathbb{C}[\sigma]_-$) trivially. $\mathbb{C}[\sigma]_+ \cong M_2\mathbb{R}$ so mod($\mathbb{R}$) = mod($\mathbb{C}[\sigma]_+$) is a Morita equivalence, the functors being $V_r \longmapsto \mathbb{C}\otimes_{\mathbb{R}} V_r$, $V \longmapsto V^\sigma$.

Now ~~[illegible]~~ take $H = \mathbb{H} = \mathbb{C} + \mathbb{C}j$ ~~[illegible]~~ with $\mathbb{C}$ acting by left mult-plication and $\Lambda^2 H = \mathbb{C}$ given by $1\wedge j = 1$. σ on H is is left mult by j. (Reason for notation H is to avoid confusion arising from $\mathbb{H}\otimes_{\mathbb{C}} V$ when $\mathbb{C}$ is left acting on $\mathbb{H}$.)

If $V \in$ mod($\mathbb{C}[\sigma]_+$), then $H\otimes V$ equipped with $\sigma\otimes\sigma$ is in mod($\mathbb{C}[\sigma_-]$). Conversely $W \in$ mod($\mathbb{C}[\sigma]_-$) $\Longrightarrow H\otimes W \in$ mod($\mathbb{C}[\sigma]_+$).

Recall that restriction of scalars has both left and right adjoints. In the case of $\mathbb{R}\subset\mathbb{C}$ these two adjoints are isomorphic:

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, V^{\sigma}) = \underbrace{\mathrm{Hom}_{\mathbb{R}}(\mathbb{H},\mathbb{R})}_{\mathbb{H}^*}\otimes_{\mathbb{R}} V^{\sigma} \xleftarrow{\sim} \mathbb{H}\otimes_{\mathbb{R}} V^{\sigma}$$

where any nonzero element of $\mathbb{H}^*$ yields an isom. In practice one takes a trace $\tau:\mathbb{H}\to\mathbb{R}$ which is unique up to a scalar, since $\mathbb{H}=\mathbb{R}\oplus[\mathbb{H},\mathbb{H}]$.

We use the following isomorphism

$$\boxed{\begin{array}{c} \mathbb{H}\otimes_{\mathbb{R}} V^{\sigma} \simeq H\otimes V \\ (z_1 1 + z_2 j)\otimes v \longmapsto 1\otimes z_1 v + j\otimes z_2 v = \begin{pmatrix} z_1 v \\ z_2 v\end{pmatrix}\end{array}}$$

to link the left adjoint (extension of scalars) from $\mathrm{mod}(\mathbb{R})$ to $\mathrm{mod}(\mathbb{H})$ with $V\mapsto H\otimes V$.

Then we have isos. (this uses that σ compatible with α,β)

$$\begin{array}{ccc} \mathrm{Hom}(H\otimes V, W)^{\sigma} & = & \mathrm{Hom}(V, H\otimes W)^{\sigma} \\ \| & & \| \\ \mathrm{Hom}_{\mathbb{R}}(H\otimes V, W) & & \mathrm{Hom}_{\mathbb{R}}(V^{\sigma}, (H\otimes W)^{\sigma}) \\ \| & & \\ \mathrm{Hom}_{\mathbb{H}}(\mathbb{H}\otimes_{\mathbb{R}} V^{\sigma}, W) & & \\ \| & & \\ \mathrm{Hom}_{\mathbb{R}}(V^{\sigma}, W) & & \end{array}$$

By Yoneda this yields a canonical isomorphism

$$\boxed{\begin{array}{c} \rho W \simeq (H\otimes W)^{\sigma} \\ w \longmapsto j\otimes w - 1\otimes jw \end{array}}$$

where ρ stands for $\mathrm{res}^{\mathbb{H}}_{\mathbb{R}}$. This identifies ρ with $H\otimes -$ from $\mathrm{mod}(\mathbb{C}[\sigma]_-)$ to $\mathrm{mod}(\mathbb{C}[\sigma]_+)$.

Next we have isos

$$
\begin{array}{ccc}
\mathrm{Hom}(H\otimes W, V)^\sigma & = & \mathrm{Hom}(W, H\otimes V)^\sigma \\
\| & & \| \\
\mathrm{Hom}_{\mathbb{R}}((H\otimes W)^\sigma, V^\sigma) & & \mathrm{Hom}_{\mathbb{H}}(W, H\otimes V) \\
\| & & \| \\
\mathrm{Hom}_{\mathbb{R}}(\rho W, V^\sigma) & & \mathrm{Hom}_{\mathbb{H}}(W, \mathbb{H}\otimes_{\mathbb{R}} V^\sigma) \\
\| & & \\
\mathrm{Hom}_{\mathbb{H}}(W, \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, V^\sigma)) & &
\end{array}
$$

Thus we get a canon. isom. $\boxed{\mathbb{H}\otimes_{\mathbb{R}} V^\sigma \simeq \mathrm{Hom}_{\mathbb{R}}(\mathbb{H}, V^\sigma)}$ which amounts to an element τ of $\mathbb{H}^*$, (take $V^\sigma = \mathbb{R}$).

Calculation gives $\tau(1) = -2$, $\tau(i) = \tau(j) = \tau(k) = 0$.

April 4, 1996

Observation

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \ell'_z \otimes V \arrow[d, "a'"] \arrow[dr, dotted, "b'a'"] & & \\
0 \arrow[r] & W \arrow[r, "a"] \arrow[dr, "ba"'] & H \otimes V \arrow[r, "b'"] \arrow[d, "b"] & Q \arrow[r] & 0 \\
 & & \ell_z \otimes V \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

The six term exact sequence of kernels and cokernels becomes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(ba) \arrow[r] & \mathrm{Ker}(b) \arrow[r] \arrow[d, equal] & \mathrm{Coker}(a) \arrow[r] \arrow[d, equal] & \mathrm{Coker}(ba) \arrow[r] & 0 \\
 & & \ell'_z \otimes Q \arrow[r, "b'a'"] & Q & &
\end{tikzcd}$$

so that it looks as if the complexes $W \xrightarrow{ba} \ell_z \otimes V$ and $\ell'_z \otimes V \xrightarrow{b'a'} Q$ are quasi-isomorphic.

When you have more time examine this carefully. Recall something similar appeared in connection with Vaserstein's lemma, more specifically, when you proved Morita invariance for K^1.

May 24, 1996

Canonical resolutions over P^1. If F is a regular sheaf over P^1 then it has a resolution of the form

$$0 \to \mathcal{O}(-1)\otimes W \to \mathcal{O}\otimes V \to F \to 0$$

Tensor this short exact sequence with

$$0 \to \Lambda^2 H \otimes \mathcal{O}(-1) \to H\otimes\mathcal{O} \to \mathcal{O}(1) \to 0$$

to get

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda^2 H\otimes\mathcal{O}(-2)\otimes W \arrow[r]\arrow[d] & \Lambda^2 H\otimes\mathcal{O}(-1)\otimes V \arrow[r]\arrow[d] & \Lambda^2 H\otimes F(-1) \arrow[r]\arrow[d] & 0 \\
0 \arrow[r] & H\otimes\mathcal{O}(-1)\otimes W \arrow[r]\arrow[d] & H\otimes\mathcal{O}\otimes V \arrow[r]\arrow[d] & H\otimes F \arrow[r]\arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}\otimes W \arrow[r]\arrow[d] & \mathcal{O}(1)\otimes V \arrow[r]\arrow[d] & F(1) \arrow[r]\arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whence

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \Lambda^2 H\otimes H^0(F(-1)) \arrow[d] & \\
 & & H\otimes V \arrow[r, "\sim"] \arrow[d, equal] & H\otimes H^0(F) \arrow[d] & \\
0 \arrow[r] & W \arrow[r] & H\otimes V \arrow[r] & H^0(F(1)) \arrow[r]\arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

which identifies $W \to H\otimes V$ with the map $\Lambda^2 H\otimes H^0(F(-1)) \to H\otimes H^0(F)$ induced by $\Lambda^2 H\otimes F(-1) \to H\otimes F$.

Next suppose G is a negative vector bundle. Then it has a dual canonical resolution of the form

$$0 \to G \to \mathcal{O}(-1)\otimes W \to \mathcal{O}\otimes V \to 0$$

Again we get by tensoring

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Lambda^2 H\otimes G(-1) \arrow[r] \arrow[d] & \Lambda^2 H\otimes\mathcal{O}(-2)\otimes W \arrow[r] \arrow[d] & \Lambda^2 H\otimes\mathcal{O}(-1)\otimes V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H\otimes G \arrow[r] \arrow[d] & H\otimes\mathcal{O}(-1)\otimes W \arrow[r] \arrow[d] & H\otimes\mathcal{O}\otimes V \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G(1) \arrow[r] \arrow[d] & \mathcal{O}\otimes W \arrow[r] \arrow[d] & \mathcal{O}(1)\otimes V \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whence

$$\begin{tikzcd}
 & H^0(G(1)) \arrow[d] & \\
 & \Lambda^2 H\otimes H^1(G(-1)) \arrow[r, "\sim"] \arrow[d] & \underbrace{H^1(\Lambda^2 H\otimes\mathcal{O}(-2)\otimes W)}_{W} \\
H\otimes V \arrow[r, "\sim"] & H\otimes H^1(G) \arrow[d] & \\
 & H^1(G(1)) \arrow[d] & \\
 & 0 &
\end{tikzcd}$$

The problem is now to identify the map $W \to H\otimes V$ arising from this diagram with the map on H^0 induced by $(\mathcal{O}(-1)\otimes W \to \mathcal{O}\otimes V)$ tensored with $\mathcal{O}(1)$.

We will construct various maps of complexes linked by $R\Gamma$-isos. First map

$$\begin{tikzcd}
\Lambda^2 H \otimes G(-1)\,[1] \arrow[d] \arrow[dr, phantom, "\dashrightarrow"] & \Lambda^2 H \otimes \mathcal{O}(-2) \otimes W\,[1] \arrow[d] \\
H \otimes G\,[1] & \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O}(1) \otimes V\right)
\end{tikzcd}$$

where $\dashrightarrow$ is essentially obtained from the first two rows of the 3×3 diagram above. Second map

$$\begin{tikzcd}
\Lambda^2 H \otimes \mathcal{O}(-2) \otimes W\,[1] \arrow[d] \arrow[dr, phantom, "\dashrightarrow"] & \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O} \otimes W\right) \arrow[d] \\
\left(H \otimes \mathcal{O}(-1) \otimes W \to \mathcal{O}(1) \otimes V\right) & \left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O}(1) \otimes V\right)
\end{tikzcd}$$

Third map is inclusion

$$\begin{tikzcd}
\left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O} \otimes W\right) \arrow[d] \arrow[dr, phantom, "\dashleftarrow"] & \mathcal{O} \otimes W \arrow[d] \\
\left(H \otimes \mathcal{O}(-1) \otimes W \longrightarrow \mathcal{O}(1) \otimes V\right) & \mathcal{O}(1) \otimes V
\end{tikzcd}$$

One can check that the dotted arrows induce isos on $R\Gamma$ for both source & target of the vertical arrow. So applying $R^0\Gamma$ we get a commutative square

$$\begin{tikzcd}
\Lambda^2 H \otimes H^1(G(-1)) \arrow[r, phantom, "\simeq"] \arrow[d] & W \arrow[d] \\
H \otimes H^1(G) \arrow[r, phantom, "\simeq"] & H \otimes V
\end{tikzcd}$$

as desired.

August 1, 1996

Consider the problem of Morita invariance of K-theory for h-unital rings, but restrict one of the rings to be unital. Suppose then A is unital and (P,Q) is a firm dual pair over A. $B = P\otimes_A Q$ is h-unital iff $P\overset{L}{\otimes}_A Q \simeq P\otimes_A Q$, e.g. if either P or Q is flat over A. If Q is flat, then Q is an inductive limit of fg free modules, and similarly for P.

Note that surjectivity of $Q\otimes P \to A$ means $\exists\, p_i, q_i$ with $\sum_{i=1}^n q_i p_i = 1$. In this case, replacing (P,Q) by $(P,Q)^n$ and B by $M_n B$ we reduce to the case where $\exists\, p\in P,\ q\in Q$ with $qp = 1$. Then $(P,Q) = (A,A)\oplus(X,Y)$, $X = \{x\in P \mid xq = 0\}$, $Y = \{y\in Q \mid yp = 0\}$, so $B = \begin{pmatrix} A & Y \\ X & X\otimes_A Y\end{pmatrix}$ and the pairing $Y\otimes X \to A$ can be arbitrary. Also $e = \begin{pmatrix} 1 & 0 \\ 0 & 0\end{pmatrix}$ is an idempotent in B such that $A = eBe$, $P = Be$, $Q = eB$, so we have the familiar Morita context

$$\begin{pmatrix} A = eBe & eB \\ Be & BeB = B\end{pmatrix}$$

we examined in connection with Davydov's thm.

Let's review this result. Start with R, $e = e^2 \in R$, $A = eRe$, $P = Re$, $Q = eR$, $B = ReR$. Hypotheses are: $Re\otimes_A eR \xrightarrow{\sim} B$ (i.e. B firm) and $eR \in \mathcal{P}(A)$. We have functors

$$\begin{array}{ccccccc} \mathrm{mod}(R) & \longrightarrow & \mathrm{mod}(A) & \xrightarrow{\sim} & \underset{\mathcal{M}(R,B)}{\overset{\|}{\mathcal{M}(B)}} & \subset & \mathrm{mod}(R) \\ L & \longmapsto & eL & & N & \mapsto & N \\ & & M & \longmapsto & Re\otimes_A M & & \end{array}$$

which induce

$$\mathcal{P}(R) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(R)$$

Here $\mathcal{P}(B) \simeq \mathcal{P}(R, B)$ is the full subcategory of small projectives in $\mathcal{M}(R, B)$, i.e. $L \in \mathcal{P}(R)$ such that $L = BL$.

We have some obvious maps

$$\begin{tikzcd} K_*(\mathcal{P}(B)) \arrow[r, hook] \arrow[dr, "\sim"] & K_*(\mathcal{P}(R)) \arrow[r] \arrow[d] & K_*(\mathcal{P}(R/B)) \\ & K_*(\mathcal{P}(A)) & \end{tikzcd}$$

Now $B = Re \otimes_A eR$, $eR \in \mathcal{P}(A) \Longrightarrow B \in \mathcal{P}(B)$, so we have a resolution by f.g projective modules

$$0 \longrightarrow B \longrightarrow R \longrightarrow R/B \longrightarrow 0.$$

This should imply that any object V in $\mathcal{P}(B)$ has a $0 \to P_1 \to P_0 \to V \to 0$ with $P_i \in \mathcal{P}(R)$. Hence by resolution we get a map $K_*(\mathcal{P}(R/B)) \to K_*(\mathcal{P}(R))$.

Claim $K_*(\mathcal{P}(R/B)) \to K_*(\mathcal{P}(R)) \to K_*(\mathcal{P}(R/B))$ is the identity. Given $0 \to P_1 \to P_0 \to V \to 0$ ~~f~~proj R-res. of $V \in \mathcal{P}(R/B)$ one has

$$0 \longrightarrow \mathrm{Tor}_1^R(R/B, V) \longrightarrow P_1/BP_1 \longrightarrow P_0/BP_0 \longrightarrow V \longrightarrow 0$$

This Tor $= 0$ since V is a summand of $(R/B)^n$ and $\mathrm{Tor}_1^R(R/B, R/B) = B/B^2 = 0$.

At this point we know $K_*(\mathcal{P}(B))$ and $K_*(\mathcal{P}(R/B))$ are direct summands of $K_*(\mathcal{P}(R))$. Consider the exact sequence of exact functors

$$0 \longrightarrow B \otimes_B L \longrightarrow L \longrightarrow L/BL \longrightarrow 0$$

from $\mathcal{P}(R)$ to $\mathcal{P}'(R)$ (= modules admitting length ≤ 1 resolutions from $\mathcal{P}(R)$). [redacted] By additivity and $K_*(\mathcal{P}(R)) \xrightarrow{\sim} K_*(\mathcal{P}'(R))$, we get that

$$K_*(R) \longrightarrow K_*(\mathcal{P}(B)) \oplus K_*(R/B) \xrightarrow{\text{[redacted]}} K_*(R)$$

is the identity. It follows that

$$\underset{\substack{\| \\ K_*(A)}}{K_*(\mathcal{P}(B))} \oplus K_*(R/B) \xrightarrow{\sim} K_*(R)$$

I think this is correct. When $R = \tilde{B}$ we then get $K_*(A) = K_*(\mathcal{P}(B)) \xrightarrow{\sim} K_*(B)$ $\left(\overset{\text{def}}{=} K_*(\tilde{B})/K_*(\mathbb{Z})\right)$, which is the Morita invariance [redacted] result I am after.

Let's now return to the original setting $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ with A unital, [redacted] $Q \otimes P \twoheadrightarrow A$, $P \otimes_A Q = B$, and suppose $Q \in \mathcal{P}(A)$. Now Q is a generator for mod(A) since we have $Q \otimes P \twoheadrightarrow A$, so without affecting the Morita invariance question we should be able to replace A by the nice unital ring $A' = \mathrm{Hom}_A(Q,Q)^{op}$. We have to compose the maps given by

$$\begin{pmatrix} A' & Q^* \\ Q & A \end{pmatrix} \qquad \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

where $Q^* = \mathrm{Hom}_A(Q, A)$.

$$\begin{pmatrix} A' & Q^* & Q^* \otimes_A Q \\ Q & A & Q \\ P \otimes_A Q & P & B \end{pmatrix} \qquad \therefore \begin{pmatrix} A' & Q^* \otimes_A Q = A' \\ P \otimes_A Q = B & B \end{pmatrix}$$

This transformation reduces us to a Morita context (put A for A') of the form

$$\begin{pmatrix} A & Q=A \\ P=B & B \end{pmatrix}$$

where P can be any A^{op}-modules. The pairing $A \otimes P \longrightarrow A$ which must be surjective is given by an A^{op}-module map $f : P \longrightarrow A$, namely $a \otimes p \longmapsto af(p)$. Surjectivity means the right ideal $f(P)$ in A generates A in the sense that $Af(P) = A$. Suslin's excision theory should take care of the surjection $P \otimes_A Q \twoheadrightarrow f(P) \otimes_A Q$, so the important case to consider is when $P = B$ is a right ideal in A such that $AB = A$.

Let's try to understand the case where B is a right ideal in A unital and $\exists\, y \in A$, $x \in B$ such that $yx = 1$.

<u>August 2, 1996</u>

Recall setup: $\begin{pmatrix} A & A \\ B & B \end{pmatrix}$ A unital, B right ideal in A satisfying $AB = A$.

We know the following.

- This Morita context is firm, being associated to the firm dual pair (B, A) over A where the pairing is $A \otimes B \longrightarrow A$, $a \otimes b \mapsto ab$. Hence $A \otimes_B B \xrightarrow{\sim} A$.

- Since A is unital we know $B \in \mathcal{P}(B)$, $A \in \mathcal{P}(B^{op})$ are dual to each other and $A \xrightarrow{\sim} \operatorname{Hom}_{B^{op}}(A, A)$, $A \xrightarrow{\sim} \operatorname{Hom}_B(B^{\bullet}, B)^{op}$.

- functors on modules

$$\begin{tikzcd} \mathcal{M}(B) \arrow[rrr, bend left, "1"] \subset \operatorname{mod}(\tilde{B}) \arrow[r] & \operatorname{mod}(A) \arrow[r, "\sim"] & \mathcal{M}(B) \subset \operatorname{mod}(\tilde{B}) \end{tikzcd}$$

$$\mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(\tilde{B})$$

$$L \longmapsto A \otimes_B L \mapsto B \otimes_A A \otimes_B L = B \otimes_B L$$

So just from
$$\mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(A)$$
$$V \mapsto B \otimes_A V \longmapsto A \otimes_B B \otimes_A V = V$$

we find $\boxed{K_*(A) \xrightarrow{i} K_*(\tilde{B}) \xrightarrow{j} K_*(A) \text{ is the identity.}}$

j is induced by $L \mapsto A \otimes_B L$, i.e. extension of scalars wrt $\tilde{B} \to A$, so j is induced by this homomorphism. Now i is induced by $V \longmapsto B \otimes_A V$, where B is regarded as a representation of A in $\mathcal{P}(B)$; in fact we have $A = \operatorname{Hom}_B(B, B)^{op}$. If we choose an embedding of B as a direct summand of $\tilde{B}^n$, then we get a homomorphism $A \longrightarrow M_n(\tilde{B})$. This homom. induces i.

For example suppose $\exists\, y \in A,\ x \in B$ satisfying $yx = 1$. Then we have

$$\tilde{B} = By \oplus \tilde{B}(1-xy)$$

Why? $1-xy$ is idempotent and $\cdot(1-xy)$ kills By.
$\tilde{b} = \underbrace{\tilde{b}xy}_{\in B} + \tilde{b}(1-xy) \in By + \tilde{B}(1-xy)$.

Also we have $B \xrightarrow{\cdot y} By \xrightarrow{\cdot x} B$ is the identity so $\cdot y : B \xrightarrow{\sim} By$.

It's better to give the pair of maps of B-modules

$$B \xrightarrow{\cdot y} \tilde{B} \xrightarrow{\cdot x} B$$

with composition 1. The corresponding homomorphism $A \to \tilde{B}$ is then $a \mapsto xay$. Check: $(xa_1y)(xa_2y) = xa_1a_2y$.

Now our problem becomes showing that

$$K_*(\tilde{B}) \xrightarrow{f} K_*(A) \xrightarrow{i} K_*(\tilde{B})$$

is projection onto $K_*(B)$. Look at this from the viewpoint of $H_*(GL(-))$. Use Suslin's result that because B is h-unital $H_*(GL(B))$ is the homology of the fibre of $BGL(\tilde{B})^+ \to BGL(\mathbb{Z})^+$. Then it seems we want to know that the homomorphism

$$B \hookrightarrow A \longrightarrow B$$
$$a \longmapsto xay$$

induces the identity on $H_*(GL(B))$.

Another way to say this might be the obvious representations of $GL_n(B)$ on B^n and $\tilde{B}^n$ in $P(\tilde{B})$ have the same stable characteristic classes. Somehow you want to deduce this from the exact sequence

$$0 \to B^n \to \tilde{B}^n \to \mathbb{Z}^n \to 0$$

Consider the chain of homoms.

$$\hookrightarrow A \longrightarrow B \hookrightarrow A \longrightarrow B \hookrightarrow A \rightarrow$$
$$a \longmapsto xay$$

Notice that $A \longrightarrow A$, $a \mapsto xay$ is a nonunital ring homomorphism between unital rings. Is it a morphomorphism? We have M. equiv. given by
$$\begin{pmatrix} A & Ay \\ xA & xAy \end{pmatrix}$$

Setting: $B < A$ unital, $BA = B$, $\exists y \in A$, $x \in B$ s.t. $yx = 1$.

We have homomorphisms: $A \to B$, $a \mapsto xay$ and the inclusion$_A^{B<A}$. These induce maps $BGL(A)^+ \to BGL(B)^+$ and $BGL(B)^+ \longrightarrow BGL(A)^+$. The question is whether they are inverse up to homotopy. Looks at the compositions.

Consider $A \xrightarrow{\phi} A$, $a \longmapsto xay$. This is a non unity preserving homomorphism, but it still induces group homomorphisms $GL_n(A) \longrightarrow GL_n(A)$ for all n. How? If $e = \phi(1) = xy$, then one has a homom. of unital rings $A \longrightarrow eAe$ followed by the inclusion $eAe < A$. The idea is that $Ae \in \mathcal{P}(A)$ has $Hom_A(Ae, Ae) = eAe$, so $\mathcal{P}(eAe)$ is equivalent to the full Karoubian subcat of $\mathcal{P}(A)$ which is generated by Ae. We get the functor

$$\mathcal{P}(A) \longrightarrow \mathcal{P}(eAe) \subset \mathcal{P}(A)$$

$$V \mapsto Ae_\phi \otimes_A V \longmapsto Ae_\phi \otimes_A V$$

Here Ae_ϕ means Ae with A acting on the right via ϕ.

Let's calculate this for $\phi(a) = xay$. Note that

$Ae = Axy \subseteq Ay$ and $Ay \subset Ayxy \subset Axy$. $\therefore Ae = Ay$

Take $V = A^n$. Then $Ay_\phi \otimes_A A^n \xrightarrow{\sim} Ay_\phi^n$. Now you choose a split embedding of Ay into a free A^m in order to get a representation of $G = \mathrm{Aut}(V)$ by matrices. In this case

$$\begin{aligned} Ay \oplus A(1-xy) &\xrightarrow{\sim} A \\ (axy \quad a(1-xy)) &\longleftarrow\!\shortmid\ a \\ (a_1 y \quad a_2(1-xy)) &\longmapsto a_1 y + a_2(1-xy) \end{aligned}$$

We have isom.

$$\begin{aligned} Ay_\phi \otimes_A M &\xrightarrow{\sim} M \\ a'y \otimes m &\longmapsto a'yxm = a'm \\ y \otimes m &\longleftarrow\!\shortmid\ m \end{aligned}$$

Check: $a'y \otimes m \mapsto a'm \mapsto y \otimes a'm$ $= y x a' y \otimes m$,

~~[illegible]~~ $m \mapsto y \otimes m \mapsto m$,

Take $M = A$ get isom

$$\boxed{\begin{aligned} Ay_\phi &\xrightarrow{\sim} A \\ a'y &\longmapsto a'yx = a' \\ a'y &\longleftarrow\!\shortmid\ a' \end{aligned}}$$

Better $A \longrightarrow Ay$, $a' \mapsto a'y$ so right mult by a on A

note that $a'y\,\phi(a) = a'y\,xay = a'ay$,

corresp. to right mult by xay on Ay.

If $g \in \mathrm{Aut}_A(A^n)$, ~~then~~ you get induced autom on

$Ay \otimes_A (A)^n \simeq (Ay)^n \underset{\cdot y}{\overset{\cdot x}{\rightleftarrows}} A^n$: $\quad g = 1+\alpha$ on A^n becomes

$1 \otimes (1+\alpha)$ on $Ay \otimes_A A^n$, i.e. $\underset{x(1+\alpha)y}{\phi(1+\alpha)}$ on Ay^n, to which

you add $1-xy$ on $A(1-xy)$. Thus we get $1-xy + x(1+\alpha)y$

$= 1 + x\alpha y$.

This calculation identifies the effect of the homom. $a \mapsto xay$ on $GL_n(A)$ with what one gets from the functor

$$\begin{aligned} P(A) &\longrightarrow P(eAe) \subset P(A) \\ V &\longmapsto Ae_\phi \otimes_A V \end{aligned}$$

August 4, 1996

Assume $B = B^2$ such that $B \in \mathcal{P}(B)$, i.e. B is a fg proj $\tilde{B}$-module which is firm (since $B = BB$). We have functors

$$\mathcal{P}(B) \subset \mathcal{P}(\tilde{B}) \longrightarrow \mathcal{P}(B)$$

$$L \longmapsto B \otimes_B L = BL$$

whose composition is $\simeq 1$. On the other hand B is a generator from $\mathcal{P}(B)$, so one has an equivalence $\mathcal{P}(A) \xrightarrow{\sim} \mathcal{P}(B)$, $V \mapsto B \otimes_A V$ where $A = \mathrm{Hom}_B(B,B)^{op}$. Consequently $K_*(\mathcal{P}(B)) = K_* A$. The above functors give maps $K_* A \xrightarrow{\iota} K_* \tilde{B} \xrightarrow{f} K_* A$ with composition the identity. Consider next the other composition $K_* \tilde{B} \xrightarrow{f} K_* A \xrightarrow{i} K_* \tilde{B}$ induced by $L \mapsto B \otimes_B L = BL$. One has functorial exact sequences from $\mathcal{P}(\tilde{B})$ to $\mathrm{mod}(\tilde{B})$

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & B \otimes_{\mathbb{Z}} \bar{L} \arrow[r, equal] \arrow[d] & \tilde{B} \otimes_{\mathbb{Z}} \bar{L} \arrow[d] & \\
0 \arrow[r] & BL \arrow[r] \arrow[d, equal] & F(L) \arrow[r] \arrow[d] & \tilde{B} \otimes_{\mathbb{Z}} \bar{L} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & BL \arrow[r] & L \arrow[r] \arrow[d] & \bar{L} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

where $F(L) = L \times_{\bar{L}} (\tilde{B} \otimes_{\mathbb{Z}} \bar{L})$. In $K_0(\tilde{B})$ we have from the two exact sequences involving $F(L)$.

$$[F(L)] = [B \otimes_B L] + [\tilde{B}]\, r(L) = [B]\, r(L) + [L]$$

where $r(L) = \text{rank}_{\mathbb{Z}}(\bar{L})$. Write this

$$[L] = [B\otimes_B L] + ([\tilde{B}]-[B])\, r(L)$$

This yields a direct sum decomposition

$$K_*\ \text{[redacted]}\ A \underset{j}{\overset{\iota}{\rightleftarrows}} K_*\tilde{B} \underset{[\tilde{B}/B\otimes_B -]}{\overset{(\tilde{B}\otimes_{\mathbb{Z}}-)-(B\otimes_{\mathbb{Z}}-)}{\leftrightarrows}} K_*\mathbb{Z}$$

in degree 0 at least. But it should hold for all degrees, since functorial exact sequences are additive. $\therefore\ K_*\mathcal{P}(B) = K_*B \overset{\text{df}}{=} K_*(\tilde{B})/K_*(\mathbb{Z})$.

We want to understand the above arguments better. We have $F(L) = F\otimes_B L$, where F ~~is~~ the B bimodule $F = \tilde{B}\times_{\mathbb{Z}}\tilde{B}$, $b(x,y)=(bx,by)$, $(x,y)b=(xb,0)$.
We have B-bimodule exact sequences

$$0 \longrightarrow B_\varepsilon \xrightarrow{b\mapsto(0,b)} F \xrightarrow{pr_1} \tilde{B} \longrightarrow 0$$

$$0 \longrightarrow B \xrightarrow{b\mapsto(b,0)} F \xrightarrow{pr_2} \tilde{B}_\varepsilon \longrightarrow 0$$

where $\tilde{B}_\varepsilon$, B_ε mean the right [redacted] action of B is [redacted] via the augmentation $\varepsilon:\tilde{B}\to\mathbb{Z}$. We can split these exact sequences compatibly with left B-action using $\Delta:\tilde{B}\to F$. Thus

$$F = (B,0)\oplus\Delta\tilde{B} = (0,B)\oplus\Delta\tilde{B}$$

giving two isomorphisms of F with $B\oplus$ [redacted] $\tilde{B}$ in $\mathcal{P}(\tilde{B})$.
Take the former.

$$\begin{array}{ccc} F & \simeq & B\oplus\tilde{B} \\ (u+v,v) & \longleftarrow & (u, v) \\ (x,y) & \longmapsto & (x-y\ \ y) \end{array}$$

Then right mult by b is

$$(u\ \ v) \longmapsto (u+v,v)b = (ub+vb,0) \longmapsto (ub+vb\ \ 0) = (u\ \ v)\begin{pmatrix} b & 0 \\ b & 0\end{pmatrix}$$

Take the latter isom.

$$F \xrightarrow{\sim} B \oplus \tilde{B}$$

$$(v', u'+v') \longleftarrow (u' \quad v')$$

$$(x, y) \longmapsto (y-x \quad x)$$

and right mult by b is

$$(u' \; v') \mapsto (v', u'+v') \xrightarrow{b\cdot} (v'b, 0) \mapsto (-v'b \;\; v'b) = (u' \; v')\begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$$

Observe $(u', v') \mapsto (v', u'+v') \longmapsto (-u' \;\; u'+v') = (u', v')\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}.$$

What does this mean? The first homomorphism $b \longmapsto \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix}$ from B to $\mathrm{Aut}_B(B \oplus \tilde{B}) = \begin{pmatrix} A & A \\ B & \tilde{B} \end{pmatrix}$ arises from the exact sequence $0 \longrightarrow B \longrightarrow F \longrightarrow \tilde{B}_\varepsilon \longrightarrow 0$. It extends to $\begin{pmatrix} B & 0 \\ B & 0 \end{pmatrix}$ which by Suslin should be K-equivalent to $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. The second homomorphism $b \mapsto \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}$ extends to $\begin{pmatrix} 0 & 0 \\ B & B \end{pmatrix}$ which should be K-equiv. to $\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$. Since these are conjugate this should mean that the representations $B \longrightarrow A = \mathrm{Aut}_B(B)$ and $B \longrightarrow \tilde{B} = \mathrm{Aut}_B(\tilde{B})$ are somehow equivalent

Let A be a left ideal in R unital. Recall that a) R/A is projective $\iff$ A has a right identity: $a = ae$ $\forall a$.

b) R/A is flat $\iff$ A has local right identities: $\forall a_1, \dots, a_n$ $\exists a$ $a_j(1-a) = 0$ (resp. this holds for $n=1$.)

Suppose A is an ideal in R such that R/A is right flat, so that $\forall$ modules M

$$A \otimes_R M \xrightarrow{\sim} AM$$

Then taking $M = R/A$ we get $A \otimes_R R/A = A/A^2 = 0$. Also we have $M = AM \implies M$ is firm.

Conversely assume these two conditions, and let M be any module. Since $A = A^2$, $AM = A(AM)$ so AM is firm. Also $A \otimes_R (M/AM) = 0$. So we have a diagram with exact rows

$$\begin{tikzcd}
 & A \otimes_R AM \arrow[r] \arrow[d, "\cong"] & A \otimes_R M \arrow[r] \arrow[d] & A \otimes_R (M/AM) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & AM \arrow[r] & M \arrow[r] & M/AM \arrow[r] & 0
\end{tikzcd}$$

with $A \otimes_R (M/AM) = 0$,

showing that $A \otimes_R M \xrightarrow{\sim} AM$ for all M. $\therefore$ R/A is right flat. $\therefore$

Prop. R/A is right flat for an ideal A iff $A = A^2$ and $M = AM \implies M$ is firm.

It would be better to formulate this independently of R as follows

Prop. A has local left identities $\iff$ $A = A^2$ and $M = AM \implies M$ is firm for all modules M.

Assume A is such a ring. Then

$$M = AM \implies \operatorname{Hom}_R(R/A, M) = 0.$$

In effect if $K = \operatorname{Hom}_R(R/A, M)$, then $AK = 0$ and

$$\begin{tikzcd}
\cancel{A \otimes_A K} \arrow[r] \arrow[d] & A \otimes_A M \arrow[r, two heads] \arrow[d, "\wr"] & A \otimes_A (M/K) \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & K \arrow[r] & M \arrow[r] & M/K \arrow[r] & 0
\end{tikzcd}$$

using $M = AM$ and $M/K = A(M/K)$. $\therefore K = 0$.

Alternate proof using local left identities: Let $Am = 0$, write $m = \sum a_i m_i$ and choose $a \in A$ such that $(1-a)a_i = 0$. Then $m = am = 0$.

Prop. Let A be a ring satisfying $A = A^2$. TFAE

1) A has local left identities
2) $AM = M \implies M$ firm for all modules M
3) ${}_AM = \{m \mid Am = 0\}$ is zero for all firm modules

It remains to check $3) \Rightarrow 2)$. Take a module M s.t. $AM = M$. Then $A \otimes_A M$ is firm and the kernel of $A \otimes_A M \twoheadrightarrow AM$ is killed by A. By 3) the kernel is zero, and so M is firm.

Question: Is any idempotent ring Morita equivalent to a ring with local left identities?

Suppose $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ strictly firm such that B has local left identities. Then P as a B-module satisfying $P = BP$ has local identities in the sense that $\forall p_1, \ldots, p_n$ $\exists b \in B$ such that $(1-b)p_i = 0$. Conversely if this condition holds then as $B = PQ$, the ring B has local left identities. In this situation we also know that B is B^{op} flat, hence P is A^{op}-flat. Now, starting with A idempotent we have a sequential way to construct firm flat right modules P. Can this be modified to yield local left identities or is there an obstruction?

I think we can arrange Q to be essentially free in the following sense. We want, starting from a finite set of p'_μ, to construct $b = \sum_i p_i q_i$ satisfying $p'_\mu = \sum_i p_i (q_i p'_\mu)$ for all μ. Here p_i, q_i can be added to what we already have. The function of q_i is to provide an A^{op}-linear map $P \longrightarrow A$ (or maybe $\tilde{A}$).

Imagine constructing P, Q inductively adding at each stage the necessary p_i, q_i. Then P is a flat firm module over A and the q_i give linear functionals on P. So we can replace the Q we might have with AF, where F is a free $\tilde{A}$-module whose basis elements map to the q_i. In other words we have $F \otimes_{\mathbb{Z}} P \twoheadrightarrow A$ hence $AF \otimes P \twoheadrightarrow A$.

Consider A = maximal ideal in a valuation ring R such that the principal ideals are Rz^ε, $\varepsilon \in \bigcup \frac{1}{2^n}\mathbb{Z}$. A firm flat A^{op}-module P is a torsion free R^{op}-module such that for any $p \in P$ $\exists\, \varepsilon > 0, p_1 \in P$ such that $p = p_1 z^\varepsilon$. Suppose we have an sfirm Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ where B has local left identities. Then we know P is A^{op} flat firm and for every finite set p'_j $\exists\, b = \sum p_i q_i$ such that

$$p'_j = \sum_i p_i q_i p'_j \qquad \forall j$$

Take a single p'. We have $p' \in \sum p_i R$ which is a torsion free finitely generated R-module.

Replacing the p_i by suitable linear combinations over R, we can assume they form an R-basis, and also that $p' \in p_1 R$.

Then $p' = \sum p_i q_i p' \implies q_i p' = 0$ for $i \neq 1$. If $p' = p_1 z^\varepsilon u$, then $p' = p_1 q_1 p'$, so $p_1 z^\varepsilon u = \underline{p_1 q_1 p_1} z^\varepsilon u$, so $p_1 = p_1 q_1 p_1$, and so $q_1 p_1 = 1 \in R$. This contradicts the facts that $q_1 p_1 \in A$.

Try for a less computational proof as follows. The condition $p_j' = \sum_i p_i q_i p_j'$ says the B-module $W = \sum \tilde{B} j_j'$ satisfies $W \subset BW$. So W is finitely generated and $W = BW$, so there should exist a simple object in $\mathcal{M}(B)$. Strictly speaking theres a non nil simple B-module. But $\mathcal{M}(B) \simeq \mathcal{M}(A)$ and A is a radical ring so $\mathcal{M}(A)$ has no simple objects.

August 16, 1996

Continue with the nondegeneracy question – whether any idempotent A is Morita equiv. to a B which injects into its multiplier ring. I consider a special A where factoring: $a = \sum a_i' a_i''$ can be done explicitly and simply.

Let R be a valuation ring with value group $\bigcup_{n \geq 0} \mathbb{Z}\frac{1}{2^n}$, say there are powers z^{ε} for $\varepsilon \in \bigcup \mathbb{Z}\frac{1}{2^n}$ so the principal ideals are $\{Rz^{\varepsilon}\}$. Let $m = \bigcup_{\varepsilon > 0} Rz^{\varepsilon}$ be the maximal ideal of R. Take $A = m/mz$ and let $\bar{A} = m/Rz$. Actually we start with $\bar{A} \subset \bar{R} = R/Rz$, and note that $A = m/mz = \bar{R} \otimes_R m$ is flat over $\bar{R}$ and satisfies $\bar{A}A = A$ so that A is firm flat over $\bar{A}$. It should be clear from $0 \to k \xrightarrow{z} A \to \bar{A} \to 0$ that $A = \bar{A}^{(2)}$. So we have a firm flat commutative ring with a nonzero element z killed by A. When I consider a firm Morita context $\begin{pmatrix} A & Q \\ P & B \end{pmatrix}$ this element z kills everything, creating difficulties with making things nondegenerate.

Claim there are firm flat A-modules M such that ${}_A M = 0$. Let $F = \bigcup_{\varepsilon > t} Rz^{\varepsilon}$ where t is a real no. F is a flat R-module such that $mF = F$ so $M = F/Fz$ is a firm flat A-module. Let $x \in F$ satisfy $mx \in Fz$. Up to units I can suppose $x = z^{\varepsilon}$ with $\varepsilon > t$. Then $z^{2^{-k}} z^{\varepsilon} \in Fz \ (\forall k) \Longrightarrow 2^{-k} + \varepsilon > t+1 \ (\forall k)$ $\Longrightarrow \varepsilon \geq t+1$. If $t \notin \bigcup 2^{-n}\mathbb{Z}$ then $\varepsilon > t+1$, so $x \in Fz$. Thus in this case $M = F/Fz$ has no nonzero element killed by A.

Let's take $Q = F_t / F_t z$, $F_t = \bigcup_{\varepsilon > t} R z^{\varepsilon}$.

I'd like to find an appropriate P. The obvious candidate which pairs nicely with Q is $P = F_{-t} / F_{-t} z$. The pairing $Q \otimes P \longrightarrow A$ is surjective and in fact it looks like $Q \otimes_A P \xrightarrow{\sim} A$, whence $B = P \otimes_A Q$ is also A. So although I've managed to make ${}_A Q$, $P_{(A)}$ zero, B is still degenerate.