Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories, i.e., a fully faithful and essentially surjective functor. Then there is a quasi-inverse $(G, \varepsilon, \eta)$ for $F$, which is unique up to canonical isomorphism and obtained as follows.

$F$ essentially surjective means we can choose for each $Y$ a $GY$ in $\mathcal{C}$ together with an isomorphism $\varepsilon_Y : FG(Y) \to Y$. Because $F$ is fully faithful, we can define $G$ uniquely on morphisms in $\mathcal{C}$ such that $G$ becomes a functor and $\varepsilon : FG \Rightarrow 1$ is an isomorphism. Then we have $\varepsilon.F : FGF \Rightarrow F$, so again as $F$ is fully faithful, there is a unique isomorphism $\eta : GF \Rightarrow 1$ such that $\eta.G = G.\varepsilon$ holds.

**Proof.** By definition, give $V : Y \to Y'$, then $G(V) : G(Y) \to G(Y')$ is the unique maps such that

\[
\begin{align*}
FG(Y) & \xrightarrow{F(G(V))} FG(Y') \\
\varepsilon_Y & \downarrow \quad \varepsilon_{Y'} \\
Y & \to \quad Y'
\end{align*}
\]

commutes. Thus $G(\varepsilon_Y)$ is unique such that

\[
\begin{align*}
FGFY & \xrightarrow{F(G(\varepsilon_Y))} FGY \\
\varepsilon_{FGY} & \downarrow \quad \varepsilon_Y \\
FGY & \to \quad Y
\end{align*}
\]

commutes and as $\varepsilon_Y$ is an isomorphism, this means $G(\varepsilon_Y)$ is unique such that $FG(\varepsilon_Y) = \varepsilon_{FGY}$.

But $\eta_X : GFX \to X$ by definition is unique such that $F(\eta_X) = \varepsilon_{FX}$. Taking $X = GY$, we find $\eta_{GY} = G(\varepsilon_Y)$ i.e. $\eta.G = G.\varepsilon$. 


Notation: Given \( \xi : F \to F' \), \( \xi' : G \to G' \) of functors which can be composed we write \( \xi \circ \xi' \) for the induced map on compositions:

\[
\begin{array}{ccc}
FG & \xrightarrow{\xi \circ \xi'} & FG' \\
\downarrow \xi & & \downarrow \xi' \\
F'G & \xrightarrow{F' \xi} & F'G'
\end{array}
\]

Also we write \( F \circ \xi \) instead of \( 1_F \circ \xi \). (Maybe + is a traditional notation).

Uniqueness of quasi-inverses. Let \((G, \varepsilon, \gamma), (G', \varepsilon', \gamma')\) be two quasi-inverses for \(F\). Then we have

\[
\begin{align*}
F \cdot \gamma &= \varepsilon \cdot F : FGFG \to F \\
F \cdot \gamma' &= \varepsilon' \cdot F : FG'F \to F \\
G \cdot \varepsilon &= \gamma G : GFG \to G \\
G \cdot \varepsilon' &= \gamma' G' : G'FG' \to G'
\end{align*}
\]

Now define \( \xi : G \to G' \) by either

1) \( G \xleftarrow{G \cdot \varepsilon'} GFG' \xrightarrow{\varepsilon' \cdot G} G' \)

1) \( G \xleftarrow{\gamma' \cdot G} G'FG \xrightarrow{G \cdot \varepsilon} G' \)

Let's check 1) is compatible with \( \varepsilon \)-maps:

\[
\begin{array}{ccc}
FG & \xleftarrow{\varepsilon \cdot \xi} & FGFG' \\
\downarrow \varepsilon & & \downarrow \varepsilon' \\
1 & = & 1
\end{array}
\]

and with \( \gamma \)-maps:

\[
\begin{array}{ccc}
GF & \xleftarrow{G \cdot \varepsilon' \cdot F} & GFG' \\
\downarrow \gamma & & \downarrow \gamma' \\
1 & = & 1
\end{array}
\]
Next show 1) = 1') by applying F to 1)
\[ e' : FG \xrightarrow{\eta : G} FG' \xrightarrow{F \cdot e} FG \xrightarrow{\epsilon : FG} 1 \]
Thus, F applied to 1) and 1') yield the same map, namely \((e')^* e\), so 1) = 1') as F is fully faithful.

Jan 4, 1996

Let \((G, \eta, \epsilon)\) and \((G', \epsilon', \eta')\) be quasi-inverses for F:
\[
\begin{align*}
\eta : GF & \xrightarrow{F} F, \quad \eta \circ \epsilon = 1, \quad \eta \circ F = F \circ \eta \\
\epsilon' : G'F & \xrightarrow{\eta'} GF', \quad \epsilon' \circ F = F \circ \epsilon', \quad \epsilon' \circ G'F = G'F \circ \epsilon' \\
\eta' : G'F & \xrightarrow{\eta'} GF', \quad (G \circ \epsilon)' \circ \eta' = \eta' \circ \epsilon' \\
\end{align*}
\]

Because F is fully faithful, \exists \phi : G \rightarrow G' such that F GF \xrightarrow{\phi} FG' \xrightarrow{\epsilon'} FG \xrightarrow{1} 1 \text{ commutes.}

Then
\[
\begin{align*}
FGF & \xrightarrow{F \cdot \phi \cdot F} FG'F \xrightarrow{\phi \cdot F} GF \xrightarrow{1} F \text{ comm.} \\
GF & \xrightarrow{\phi} G'F \xrightarrow{\phi} GF \xrightarrow{1} F \text{ comm.}
\end{align*}
\]

Hence we have \(\psi : (G, \epsilon, \eta) \xrightarrow{\psi} (G', \epsilon', \eta')\). To get a formula for \(\psi\), apply G:
\[
\begin{align*}
G & \xrightarrow{G \circ \epsilon} GFG \xrightarrow{G \circ \eta} GF \xrightarrow{1} G \text{ hence} \\
& \xrightarrow{G \circ \epsilon'} G' \xrightarrow{G \circ \eta'} G' \xrightarrow{1} G' \text{ hence}
\end{align*}
\]
which is 1) on p85. Similarly
\[
\begin{array}{c}
G \xrightarrow{\eta \cdot G} GFG \xrightarrow{\xi \cdot FG} GFG \xrightarrow{\eta' \cdot G'} G \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
G \xrightarrow{G \cdot \varepsilon} G' \xrightarrow{G' \cdot \varepsilon} G' \xrightarrow{\xi} G'
\end{array}
\]
which is 1) on p85.

Next, I want to show using the uniqueness of a quasi-inverse that given \((F, G, \varepsilon, \eta)\) and \(\Theta: F \sim F'\), there is a corresponding \(\Xi: G \sim G'\) such that \(\Theta(\Xi)\) is an isomorphism \((F, G, \varepsilon, \eta) \sim (F', G', \varepsilon', \eta')\). i.e.
\[
\begin{array}{c}
FG \xrightarrow{\Theta \cdot \Xi} F'G' \\
\varepsilon \quad \eta \quad \varepsilon' \quad \eta'
\end{array}
\]
commute

In other words, we have \(\varepsilon = \varepsilon'(\Theta \cdot \Xi)\) and \(\eta = \eta'(\Xi \cdot \Theta)\).

The idea is that \(\Theta: F \sim F'\) makes \(G'\) into a quasi-inverse for \(F\). More precisely, we have an isomorphism
\[
\\Theta(1): (F, G', \varepsilon'(\Theta \cdot G'), \eta'(G \cdot \Theta)) \sim (F', G', \varepsilon', \eta')
\]
Then we know there is a \(\Xi: G \sim G'\) such that \(1, \Xi): (F, G, \varepsilon (G \cdot \Theta), \eta(G \cdot \Theta)) \sim (F, G', \varepsilon'(\Theta \cdot G'), \eta'(G \cdot \Theta))\)
i.e. \(\varepsilon = \varepsilon'(\Theta \cdot G')(F \cdot \Xi)\) and \(\eta = \eta'(G' \cdot \Theta)(\Xi \cdot F)\)
\(= \eta'(\Xi \cdot \Theta)\).

The first equation says:
\[
\begin{array}{c}
FG \xrightarrow{\mathcal{F} \cdot \Xi} FG' \xrightarrow{\Theta \cdot G'} FG' \\
\varepsilon \quad \varepsilon' \quad \varepsilon'
\end{array}
\]
\(= 1\) since the maps are isomorphisms and \(\mathcal{F}\) is fully-faithful.
March 12, 1996

I want to record formulas involved in the equivalences:

\[ \text{mod}(\mathbb{R}) \xrightarrow{\text{ext}_{\mathbb{R}}} \text{mod}(\#1) \]
\[ \downarrow \]
\[ \text{mod}(C[\mathbb{Z}]) \xleftarrow{\text{H}^0} \text{mod}(C[\mathbb{Q}]) \]

First consider the adjoint functor relations

(1)

\[ \text{Hom}(H \otimes V, W) = \text{Hom}(V, H^* \otimes W) \]
\[ \text{Hom}(V, H \otimes W) = \text{Hom}(H^* \otimes V, W) \]

where \( H \) is a finite-diml vector space. The adjunction maps in the former arise from the canonical maps

\[ \alpha : H \otimes H^* \rightarrow C \quad h \otimes h^* \rightarrow (h(h^*)) = (h^*h) \]
\[ \beta : C \rightarrow H^* \otimes H \quad 1 \rightarrow \sum c_i^* \otimes c_i \]

The adjunction maps in the latter arise from the canonical maps

\[ \alpha' : H^* \otimes H \rightarrow C \quad h^* \otimes h \rightarrow (h(h^*)) \]
\[ \beta' : C \rightarrow H \otimes H^* \quad 1 \rightarrow \sum c_i \otimes c_i^* \]

obtained from the preceding via the flips. Notice that \( \alpha' \beta = \alpha \beta' \) is \( \text{tr}(1) = \dim H \).

In the \( \mathbb{R}, H \) situation, \( H \otimes \mathbb{C}^2 \) comes equipped with a volume \( \Lambda^2 H = \mathbb{C} \) which we use to identify \( H^* \) and \( H \). We have a single adjoint functor relation

\[ \text{Hom}(H \otimes V, W) = \text{Hom}(V, H \otimes W) \]

arising from the canonical maps.
\[ \alpha : H \otimes H \rightarrow C \quad h_1 \otimes h_2 \mapsto h_1 h_2 \]

\[ \beta : C \rightarrow H \otimes H \quad 1 \mapsto e_2 \otimes e_1 - e_1 \otimes e_2 \]

**Check:** Let \( h = e_1 e_1 + e_2 e_2 \) so that \( e_2 = e_1 h, e_1 = e_2 h \)

\[ H \xrightarrow{\beta \circ \alpha} H \otimes H \otimes H \xrightarrow{\otimes \alpha} H \]

\[ h \mapsto (e_2 \otimes e_1 - e_1 \otimes e_2) \otimes h \mapsto e_2 (e_1 h) - e_1 (e_2 h) = h \]

\[ H \xrightarrow{\otimes \beta} H \otimes H \xrightarrow{\otimes \alpha} H \]

\[ h \mapsto h \otimes (e_2 \otimes e_1 - e_1 \otimes e_2) \mapsto (h e_2) e_1 - (h e_1) e_2 = h \]

Notice that \( C \xrightarrow{\beta} H \otimes H \xrightarrow{\alpha} C \) is

\[ 1 \mapsto e_2 e_1 - e_1 e_2 = -2(e_1 e_2) = -2. \]

I don't really understand this sign. Somehow it arises from the fact that the volume \( \Lambda^2 H = C \) determines two isos. of \( H \) with \( H^* \), i.e., \( \Lambda^2 H \subset H \otimes H \) and you can contract either factor of \( H \) with \( H^* \); these two isos. have opposite sign. Perhaps also this sign is related to what happens with the Fourier transform.

Next recall \( C[\sigma]_+ = C + C \sigma \) where \( \sigma = \bar{\sigma} \)

and \( \sigma^2 = \pm 1. \quad C[\sigma]_- = H \) where \( \sigma = \bar{\sigma} \) so \( \text{mod}(H) = \text{mod}(C[\sigma]_-) \) trivially. \( C[\sigma]_+ \cong M_2 \mathbb{R} \) so \( \text{mod}(\mathbb{R}) = \text{mod}(C[\sigma]_+) \) is a Morita equivalence, the functors being \( V \mapsto C \otimes V \), \( V \mapsto \Sigma V \).

Now take \( H = H = C + C \sigma \) with \( C \) acting by left multiplication and \( \Lambda^2 H = C \) given by \( 1 \mapsto 1 \). \( \sigma \) on \( H \) is is left mult by \( \sigma \). (Reason for notation \( H \) is to avoid confusion arising from \( H \otimes V \) when \( C \) is left acting as \( H \).)

If \( V \in \text{mod}(C[\sigma]_+) \), then \( H \otimes V \) equipped with \( \sigma \otimes \) is in \( \text{mod}(C[\sigma]_-) \). Conversely \( W \in \text{mod}(C[\sigma]_-) \) implies \( H \otimes W \in \text{mod}(C[\sigma]_+) \).
Recall that restriction of scalars has both left and right adjoints. In the case of \( R = C \) these two adjoints are isomorphic:

\[
\text{Hom}_R(H, V^\otimes) = \text{Hom}_R(H, \mathbb{R}) \otimes_{\mathbb{R}} V^\otimes 
\]

where any nonzero element of \( H^* \) yields an isomorphism. In practice one takes a trace \( \tau : H^* \to \mathbb{R} \) which is unique up to a scalar, since \( H = \mathbb{R} \oplus [H, H] \).

We use the following isomorphism

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes v \mapsto 1 \otimes \varepsilon_1 v + 0 \otimes \varepsilon_2 v = \begin{pmatrix} \varepsilon_1 v \\ \varepsilon_2 v \end{pmatrix}
\]

to link the left adjoint (extension of scalars) from \( \text{mod}(R) \) to \( \text{mod}(H) \) with \( V \mapsto H \otimes V \).

Then we have isomorphisms (this uses that \( \otimes \) is compatible with \( \alpha, \beta \))

\[
\begin{align*}
\text{Hom}_R(H \otimes V, W) & \cong \text{Hom}_R(V, H \otimes W)^\otimes \\
\text{Hom}_H(H \otimes V, W) & \cong \text{Hom}_R(V^\otimes, (H \otimes W)^\otimes) \\
\text{Hom}_H(H \otimes V^\otimes, W) & \cong \text{Hom}_R(V^\otimes, W) \\
\text{Hom}_R(V^\otimes, W) & \cong \text{Hom}_R(H \otimes V^\otimes, W)
\end{align*}
\]

By Yoneda this yields a canonical isomorphism

\[
W \cong (H \otimes W)^\otimes \\
\omega \mapsto j \omega - 1 \otimes j \omega
\]

where \( j \) stands for \( \text{res}^H_R \). This identifies \( j \) with \( H \otimes - \) from \( \text{mod}(C\{0\}^-) \) to \( \text{mod}(C\{0\}^+) \).
Next we have iso

\[ \text{Hom}(\text{He} \circ W, V)^\sim \cong \text{Hom}(W, \text{He} \circ V)^\sim \]

\[ \text{Hom}_R((\text{He} \circ W), V^\sim) \cong \text{Hom}_H(W, \text{He} \circ V) \]

\[ \text{Hom}_R(W, V^\sim) \cong \text{Hom}_H(W, \text{He} \circ V^\sim) \]

\[ \text{Hom}_H(W, \text{Hom}_H(\text{He}, V^\sim)) \]

Thus we get a canonical isomorphism:

\[ \text{He} \circ V^\sim \cong \text{Hom}_H(\text{He}, V^\sim) \]

which amounts to an element \( \tau \) of \( H^* \), (take \( V^\sim = R \)).

Calculation gives \( \tau(1) = -2 \), \( \tau(i) = \tau(j) = \tau(k) = 0 \).
The six term exact sequence of kernels and cokernels becomes

\[ 0 \to \text{Ker}(ba) \to \text{Ker}(b) \to \text{Coker}(a) \to \text{Coker}(ba) \to 0 \]

so that it looks as if the complexes \( W \xrightarrow{ba} \ell_2 \otimes V \) and \( \ell_2' \otimes V \xrightarrow{b'a'} Q \) are quasi-isomorphic.

When you have more time, examine this carefully. Recall something similar appeared in connection with Vaserstein's lemma, more specifically, when you proved Morita invariance for \( K^! \).
Canonical resolutions over $\mathbb{P}^1$. If $F$ is a regular sheaf over $\mathbb{P}^1$ then it has a resolution of the form

$$0 \rightarrow \mathcal{O}(-1) \otimes W \rightarrow \mathcal{O} \otimes V \rightarrow F \rightarrow 0$$

Tensor this short exact sequence with

$$0 \rightarrow \Lambda^2 \mathcal{O}(-1) \rightarrow \mathcal{O} \otimes \mathcal{O}(1) \rightarrow 0$$

to get

$$0 \rightarrow \Lambda^2 \mathcal{O}(-2) \otimes W \rightarrow \Lambda^2 \mathcal{O}(-1) \otimes V \rightarrow \Lambda^2 \mathcal{O} \otimes F(-1) \rightarrow 0$$

$$0 \rightarrow \mathcal{O} \otimes \mathcal{O}(-1) \otimes W \rightarrow \mathcal{O} \otimes \mathcal{O} \otimes V \rightarrow \mathcal{O} \otimes F \rightarrow 0$$

$$0 \rightarrow \mathcal{O} \otimes W \rightarrow \mathcal{O}(1) \otimes V \rightarrow F(1) \rightarrow 0$$

$$0 \rightarrow 0 \rightarrow 0$$

whence

$$\Lambda^2 \mathcal{O} \otimes H^0(F(-1))$$

$$\mathcal{O} \otimes V \rightarrow H^0(F(1))$$

$$0 \rightarrow W \rightarrow \mathcal{O} \otimes V \rightarrow H^0(F(1)) \rightarrow 0$$

Which identifies $W \rightarrow \mathcal{O} \otimes V$ with the map

$$\Lambda^2 \mathcal{O} \otimes H^0(F(-1)) \rightarrow \mathcal{O} \otimes H^0(F)$$

induced by $\Lambda^2 \mathcal{O} \otimes F(-1) \rightarrow \mathcal{O} \otimes F$.

Next suppose $G$ is a negative vector bundle. Then it has a dual canonical resolution of the form
Again, we get by tensoring

\[ 0 \rightarrow \mathcal{O}(1) \otimes W \rightarrow \mathcal{O} \otimes V \rightarrow 0 \]

whence

\[
\begin{align*}
H^0(\mathcal{O}(1)) \\
\downarrow \\
\Lambda^2 H \otimes H^1(\mathcal{O}(1)) & \cong \mathcal{O} H^1(\Lambda^2 H \otimes (\mathcal{O}(2) \otimes W)) \\
\downarrow \\
H \otimes V & \cong H \otimes H^1(G) \\
\downarrow \\
H^1(\mathcal{O}(1)) \\
\downarrow \\
0
\end{align*}
\]

The problem is now to identify the map \(W \rightarrow H \otimes V\) arising from this diagram with the map on \(H^0\) induced by \((\mathcal{O}(1) \otimes W \rightarrow \mathcal{O} \otimes V)\) tensored with \(\mathcal{O}(1)\).
We will construct various maps of complexes linked by \( R^I \)-maps. First map

\[
\Lambda^2 H \otimes G(-1) \quad [1] \quad \Lambda^2 H \otimes \Theta(-2) \otimes W \quad [1]
\]

\[
\downarrow \quad \longrightarrow \quad \downarrow
\]

\[
H \otimes G \quad [1] \quad (H \otimes \Theta(-1) \otimes W \to \Theta(1) \otimes V)
\]

where \( \longrightarrow \) is essentially obtained from the first two rows of the 3x3 diagram above. Second map

\[
\Lambda^2 H \otimes \Theta(-2) \otimes W \quad [1]
\]

\[
\downarrow \quad \longrightarrow \quad \downarrow
\]

\[
(H \otimes \Theta(-1) \otimes W \to \Theta(1) \otimes V) \quad (H \otimes \Theta(-1) \otimes W \to \Theta(1) \otimes V)
\]

Third map is inclusion

\[
(H \otimes \Theta(-1) \otimes W \to \Theta \otimes W)
\]

\[
\downarrow \quad \leftarrow \quad \downarrow
\]

\[
(H \otimes \Theta(-1) \otimes W \to \Theta(1) \otimes V) \quad \Theta \otimes V
\]

One can check that the dotted arrows induce isos on \( R^I \) for both source and target of the vertical arrows. So applying \( R^T \) we get a commutative square

\[
\Lambda^2 H \otimes H^I(G(-1)) \quad \sim \quad W
\]

\[
\downarrow \quad \downarrow
\]

\[
H \otimes H^I(G) \quad \sim \quad H \otimes V
\]

as desired.
Consider the problem of Morita invariance of K-theory for b-unital rings, but restrict one of the rings to be unital. Suppose then $A$ is unital and $(P,Q)$ is a form dual pair over $A$. $B = P \otimes_A Q$ is b-unital iff $P \otimes_A Q = P \otimes_A Q$, e.g. if either $P$ or $Q$ is flat over $A$. \( \text{If } Q \text{ is flat, then } A \text{ is an inductive limit of fg free modules, and similarly for } P. \) 

Note that surjectivity of $Q \otimes P \rightarrow A$ means $\exists p, q$ with $\sum_{a \in A} b_i p_i = 1$. In this case, replacing $(P,Q)$ by $(P,Q)^n$ and $B$ by $M_n B$, we reduce to the case where $\exists p \in P, q \in Q$ with $qp = 1$. Then $(P,Q) = (A,A) \oplus (X,Y), X = \{ p \in P \mid xq = 0 \}, Y = \{ y \in Q \mid yp = 0 \}$ so $B = \begin{pmatrix} A & Y \\ X & X \otimes_A Y \end{pmatrix}$ and the pairing $Y \otimes X \rightarrow A$ can be arbitrary. Also $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent in $B$ such that $A = eB, P = Be, Q = eB$, so we have the familiar Morita context

\[
\begin{pmatrix}
A = eBe & eB \\
Be & BeB = B
\end{pmatrix}
\]

We examined in connection with Dugu\'e\'s thesis.

Let's review this result. Start with $R, e = e^2 \in R, A = eRe, P = Re, Q = eR, B = ReR$. Hypotheses are:

- $\text{Re} \otimes eR \rightarrow B$ (i.e. $B$ fine) and $eR \in P(A)$. We have functors
  \[
  \begin{array}{ccc}
  \text{mod}(R) & \rightarrow & \text{mod}(A) \\
  L & \mapsto & eL \\
  M & \mapsto & \text{Re} \otimes eM
  \end{array}
  \]
  \[
  \begin{array}{ccc}
  \text{mod}(R) & \rightarrow & \text{mod}(A) \\
  N & \mapsto & N
  \end{array}
  \]
  \[
  \begin{array}{ccc}
  \text{mod}(R) & \rightarrow & \text{mod}(B) \\
  N & \mapsto & N
  \end{array}
  \]

\[
\left\{ \begin{array}{c}
\mu_{eR} \rightarrow R, \quad \text{if } e \in \text{reduced generators of } R \\
\mu_{eR} \rightarrow R, \quad \text{if } e \notin \text{reduced generators of } R
\end{array} \right\}
\]
which induce
\[ P(R) \to P(A) \to P(B) \subset P(R) \]

Here \( P(B) \subset P(R, B) \) is the full subcategory of small projectives in \( P(R, B) \), i.e., \( \bullet \in P(R) \) such that \( L = BL \).

We have some obvious maps
\[ K_\ast(P(B)) \to K_\ast(P(R)) \to K_\ast(P(R/B)) \]
\[ \downarrow \]
\[ K_\ast(P(A)) \]

Now \( B = \text{Re} \otimes e R, e R \subset P(A) \Rightarrow B \subset P(B) \), so we have a resolution of \( f_i \) by projective modules
\[ 0 \to B \to R \to R/B \to 0. \]

This should imply that any object in \( P(B) \) has a
\[ 0 \to p \to p_0 \to V \to 0 \]
which with \( p \in P(R) \). Hence by resolution we get a map
\[ K_\ast(P(R/B)) \to K_\ast(P(R)) \]
Claim \( K_\ast(P(R/B)) \to K_\ast(P(R)) \to K_\ast(P(R/B)) \) is the identity. Given
\[ 0 \to p_1 \to p_0 \to V \to 0 \]
for proj \( h \)-res.

of \( V \in P(R/B) \) one has
\[ 0 \to \text{Tor}_1^R(R/B, V) \to p_1/Bp_1 \to p_0/Bp_0 \to V \to 0 \]
This \( \text{Tor} = 0 \) since \( V \) is a summand of \( (R/B)^{\ast} \) and
\[ \text{Tor}_1^R(R/B, R/B) = B/B^{\ast} = 0. \]

At this point we know \( K_\ast(P/B) \) and \( K_\ast(P(R/B)) \) are direct summands of \( K_\ast(P(R)) \). Consider the exact sequence of \( \otimes \) functors.
0 \to B \otimes_B L \to L \to L/BL \to 0

from \( P(R) \) to \( P'(R) \) (= modules admitting length \( \leq 1 \) resolutions from \( P(R) \)). By additivity and \( K_*(P(R)) \to K_*(P'(R)) \), we get that

\[ K_*(R) \to K_*(P(B)) \oplus K_*(R/B) \to K_*(R) \]

is the identity. It follows that

\[ K_*(P(B)) \oplus K_*(R/B) \to K_*(R) \]

\[ \cong K_*(A) \]

I think this is correct. When \( R = B \) we then get \( K_*(A) = K_*(P(B)) \to K_*(B) \) (by \( K_*(B) \to K_*(B/B) \)), which is the Morita invariance result I am after.

Let's now return to the original setting \((A, Q)\) with \( A \) unital, \( P \otimes B \to A \), \( P \otimes A = B \), and suppose \( Q \in P(A) \). Now \( Q \) is a generator for \( \text{mod}(A) \) since we have \( P \otimes B \to A \), so without affecting the Morita invariance question we should be able to replace \( A \) by the new unital ring \( A' = \text{Hom}_A(Q, Q)^1 \). We have to compose the maps given by

\[
\begin{pmatrix}
A' & Q^* \\
Q & A
\end{pmatrix}
\begin{pmatrix}
A & Q \\
P & B
\end{pmatrix}
\]

where \( Q^* = \text{Hom}_A(Q, A) \).

\[
\begin{pmatrix}
A' & Q^* & Q \otimes_A Q \\
Q & A & Q \\
P & Q & P & B
\end{pmatrix}
\begin{pmatrix}
A' & Q^* & Q = A' \\
P & Q = B & B
\end{pmatrix}
\]
This transformation reduces us to a Morita context (put A for A') of the form

\[
\begin{pmatrix}
A & Q = A \\
P & B
\end{pmatrix}
\]

where \( P \) can be any \( A^\oplus \)-module. The pairing \( A \otimes P \to A \) which must be surjective is given by an \( A^\oplus \)-module map \( f: P \to A \), namely \( a \otimes p \mapsto af(p) \). Surjectivity means the right ideal \( f(P) \) in \( A \) generates \( A \) in the sense that \( Af(P) = A \). Let's try to understand the case where \( B \) is a right ideal in \( A \) such that \( AB = A \).

Let's try to understand the case where \( B \) is a right ideal in \( A \) unital and \( \exists y \in A, x \in B \) such that \( xy = 1 \).
Recall setup: \((A, B)\) \(A\) unital, \(B\) right ideal in \(A\) satisfying \(AB = A\).

We know the following.

- This Morita context is split, being associated to the firm dual pair \((B^*, A)\) over \(A\) where the pairing is \(A \otimes B^* \longrightarrow A, a \otimes b \mapsto ab\). Hence \(A \otimes B^* \cong A\).

- Since \(A\) is unital we know \(B \in \mathcal{P}(B), A \in \mathcal{P}(B^*)\) are dual to each other and \(A \cong \text{Hom} (A, A), A \cong \text{Hom}_B (B^*, B)^*\).

- functors on modules,

\[
\begin{align*}
M(B) \subset \text{mod}(B) & \longrightarrow \text{mod}(A) \longrightarrow M(B) \subset \text{mod}(\tilde{B}) \\
P(B) \subset P(\tilde{B}) & \longrightarrow P(A) \longrightarrow P(B) \subset P(\tilde{B})
\end{align*}
\]

\[
L \longrightarrow A \otimes_B L \mapsto B \otimes_A A \otimes_B L = B \otimes_B L
\]

So just from \(P(A) \cong P(B) \subset P(\tilde{B}) \longrightarrow P(A)\)

\[
V \mapsto B \otimes_A V \mapsto A \otimes_B B \otimes_A V = V
\]

we find \(K\_\* (A) \longrightarrow K\_\* (\tilde{B}) \longrightarrow K\_\* (A)\) is the identity.

\(f\) is induced by \(L \mapsto A \otimes_B L\), i.e. extension of scalars with \(\tilde{B} \longrightarrow A\), so \(f\) is induced by this homomorphism. Now \(i\) is induced by \(V \mapsto B \otimes_A V\), where \(B\) is regarded as a representation of \(A\) in \(P(B)\); in fact we have \(A = \text{Hom}_B (\tilde{B}, B)^*\). If we choose an embedding of \(\tilde{B}\) as a direct summand of \(\tilde{B}\), then we get a homomorphism \(A \longrightarrow M_n (\tilde{B})\). This homomorphism induces \(i\).
For example suppose $f_j \in A$, $x \in B$ satisfying $y x = 1$. Then we have

\[ \overline{B} = \overline{B}y \oplus \overline{B}(1-xy) \]

Why? $1-xy$ is idempotent and $(1-xy)$ kills $B_x$. Also we have $\overline{B} \xrightarrow{\cdot y} \overline{B}y \xrightarrow{x} \overline{B}$ is the identity so $\cdot y : \overline{B} \xrightarrow{x} \overline{B}y$.

It's better to give the pair of maps of $B$-modules $\overline{B} \xrightarrow{\cdot y} \overline{B} \xrightarrow{x} \overline{B}$ with composition $1$. The corresponding homomorphism $A \xrightarrow{\cdot x} \overline{B}$ is then $a \mapsto xay$. Check: $(xq_1y)q_2y = xq_1q_2y$.

Now our problem becomes showing that $K_\ast(\overline{B}) \xrightarrow{\cdot x} K_\ast(A) \xrightarrow{\cdot y} K_\ast(\overline{B})$ is projection onto $K_\ast(B)$. Look at this from the viewpoint of $H_\ast(GL(\overline{B}))$. Use Suslin's result that because $B$ is h-unital $H_\ast(GL(B))$ is the homology of the fibre of $BGL(\overline{B}) \xrightarrow{x} BGL(\mathbb{C})$. Then it seems we want to know that the homomorphism

\[ \overline{B} \xrightarrow{\cdot x} A \xrightarrow{\cdot y} B \]

induces the identity on $H_\ast(GL(B))$.

Another way to say this might be the obvious representations of $GL_n(B)$ on $B^n$ and $\overline{B}^n$ in $P(\overline{B})$ have the same stable characteristic classes. Somehow you want to deduce this from the exact sequence

\[ 0 \rightarrow B^n \xrightarrow{\cdot x} \overline{B}^n \xrightarrow{\cdot y} \mathbb{Z}^n \rightarrow 0 \]
Consider the chain of homomorphisms.

\[ \xymatrix{ A & B & A & B & A \\
\phi \ar[r] & \psi \ar[r] & \chi \ar[r] & \delta \ar[r] & \varepsilon } \]

Notice that \( A \to A, \phi \to \varepsilon \) is a non-unital ring homomorphism between unital rings. Is it a unital homomorphism? We have a group given by

\[
\left( \begin{array}{cc}
A & Ay \\
xA & xAy
\end{array} \right)
\]

Setting: \( B \leq A \) unital, \( BA = B \), \( \forall y \in A, \forall x \in B \), \( yx = 1 \).

We have homomorphisms: \( A \to B, \phi \to \psi \) and the inclusion. These induce maps \( \text{BGL}(A)^+ \to \text{BGL}(B)^+ \) and \( \text{BGL}(B)^+ \to \text{BGL}(A)^+ \). The question is whether they are inverse up to homotopy. Looks at the compositions.

Consider \( A \to A, \phi \to \varepsilon \). This is a non-unital preserving homomorphism, but it still induces group homomorphisms \( \text{GL}_n(A) \to \text{GL}_n(A) \) for all \( n \). Has \( e = \phi(1) = xy \), then one has a homomorphism of unital rings \( A \to eAe \) followed by the inclusion, \( eAe \leq A \). The idea is that \( A \to eAe \) is equivalent to the full Karoubian subcat of \( \text{P}(A) \) which is generated by \( A \).

We get the functor

\[
\text{P}(A) \to \text{P}(eAe) \subseteq \text{P}(A)
\]

\[
V \to A \otimes A V \to A \otimes V
\]

Here \( A \otimes V \) means \( A \otimes V A \) with \( A \) acting on the right via \( \phi \).

Let's calculate this for \( \phi(1) = xAy \). Note that \( A \phi \otimes A V \to A \otimes V \).

\[
Ae = Axy \leq Ay \quad \text{and} \quad Ay \leq Ayx y \leq Axy. \quad \therefore \quad Ae = Ay
\]

Take \( V = A^n \). Then \( A \phi \otimes A^\phi A^n \to A \phi A^n \).

Now
you choose a split embedding of $A_y$ into a free $A^m$ in order to get a representation of $\text{Aut}(V)$ by matrices. In this case

\[
A_y \oplus A(1-xy) \cong A
\]

\[
(a \cdot xy, a(1-xy)) \mapsto -1 \cdot a
\]

\[
(a_1, y, a_2(1-xy)) \mapsto a_1y + a_2(1-xy)
\]

We have isomorphism $A_y \otimes A \cong M$.

Check: $a' \otimes m \mapsto a'y \otimes m$,

$\lambda m \mapsto \lambda \otimes m \mapsto m$, $\lambda \cdot y \otimes m \mapsto m$.

Take $M = A$, get isomorphism $A_y \otimes A \cong A$.

Better $A \to A_y, a' \to a'y$

note that $a'y \otimes m = a'y \cdot xy = a'y$,

so right mult by $a'$ in $A$

corresponds to right mult by $xy$ in $A_y$.

If $g \in \text{Aut}(A_y^n)$, then you get induced automorphism $A_y \\otimes A^n \cong (A_y)^n \to A^n$.

If $g = 1 + x$ on $A$, it becomes

$A_y \otimes A^n \cong (A_y)^n \to A^n$, i.e., $\phi(1 + x)$ on $A_y^n$, to which

$x(1 + x)y$ adds.

Thus we get

\[1 - xy + x(1 + x)y = 1 + xxy.\]

This calculation identifies the effect of the homomorphism $a \mapsto xy$ on $\text{GL}_n(A)$ with what one gets from the functor $A_y \mapsto \text{P}(eAe) \subset \text{P}(A)$.

$V \mapsto A_y \otimes_A V$
August 9, 1996

Assume $B = B^2$ such that $B \in \mathcal{P}(B)$, i.e., $B$ is a $fg$ proj $\bar{B}$-module which is firm (since $\bar{B} = BB$).

We have functors

$$\mathcal{P}(B) \xrightarrow{\sim} \mathcal{P}(\bar{B}) \xrightarrow{L} B \otimes_{\bar{B}} \bar{L} = BL$$

whose composition is $\sim 1$. On the other hand $B$ is a generator from $\mathcal{P}(B)$, so one has an equivalence $\mathcal{P}(A) \sim \mathcal{P}(B)$, $V \mapsto B \otimes_{\bar{A}} V$ where $A = \text{Hom}_B(B, B)^{op}$.

Consequently $K_{\sim}(\mathcal{P}(B)) = K_{\sim} A$. The above functors give maps $K_{\sim} A \mapsto K_{\sim} \bar{B} \mapsto K_{\sim} A$ with composition the identity. Consider next the other composition $K_{\sim} \bar{B} \mapsto K_{\sim} A \mapsto K_{\sim} \bar{B}$ induced by $L \mapsto B \otimes_{\bar{B}} L = BL$.

One has functorial exact sequences from $\mathcal{P}(\bar{B})$ to $\mathcal{P}(B)$

$$0 \rightarrow B_L \rightarrow F(L) \rightarrow \bar{B} \otimes_{\bar{Z}} \bar{L} \rightarrow 0$$

$$0 \rightarrow B_L \rightarrow L \rightarrow \bar{L} \rightarrow 0$$

where $F(L) = L \times (\bar{B} \otimes \bar{L})$. In $K_0(\bar{B})$ we have from the two exact sequences involving $F(L)$:

$$[F(L)] = [\bar{B} \otimes \bar{L}] + [\bar{L}] = [B] r(L) + [L]$$
where \( r(L) = \text{rank}_\mathbb{Z}(L) \). Write this

\[
L = [B \otimes \mathbb{Z}] + ([B] - [B]) r(L)
\]

This yields a direct sum decomposition

\[
K_x \otimes A \xrightarrow{\text{L}} K_x \otimes \mathbb{Z} \xrightarrow{\text{J}} (B \otimes \mathbb{Z}) - (B \otimes \mathbb{Z})
\]

in degree 0 at least. But it should hold for all degrees, since functional exact sequences are additive: \( K_x P(B) = K_x B \cong K_x (\mathbb{Z})/K_x (\mathbb{Z}) \).

We want to understand the above arguments better. We have \( F(L) = F \otimes L \), where \( F \) is the \( B \)-brinodule \( F = \mathbb{B} \times \mathbb{Z} \mathbb{B} \), \( b(x,y) = (bx, by) \), \( (x,y)b = (xb, y) \).

We have \( B \)-brinodule exact sequences

\[
0 \rightarrow B \xrightarrow{b \mapsto (0, b)} F \xrightarrow{p_{21}} \mathbb{B} \xrightarrow{\epsilon} 0
\]

\[
0 \rightarrow B \xrightarrow{b \mapsto (b, 0)} F \xrightarrow{p_{22}} \mathbb{B} \xrightarrow{\epsilon} 0
\]

where \( \mathbb{B} \) is the natural \( B \)-action of \( B \) via the augmentation \( \epsilon: \mathbb{B} \rightarrow \mathbb{Z} \). We can split these exact sequences compatibly with left \( B \)-action using \( \Delta: \mathbb{B} \rightarrow F \). Thus

\[
F = (B, 0) \oplus \Delta \mathbb{B} = (0, B) \oplus \Delta \mathbb{B}
\]

giving two isomorphisms of \( F \) with \( B \oplus \mathbb{B} \) in \( P(B) \).

Take the former. \( F \xrightarrow{\cong} B \oplus \mathbb{B} \)

\[
(u + v, v) \leftarrow (u \ast v)
\]

\[
(x, y) \xrightarrow{\cong} (x - y, y)
\]

Then right mult by \( b \) is

\[
(u, v) \mapsto (u + v, v)b = (ub + vb, 0) \mapsto (ub + vb, 0) = (u, v)
\]

The matrix for right multiplication by \( b \) is

\[
\begin{pmatrix}
0 & b \\
0 & 0
\end{pmatrix}
\]
Take the latter isom. 
\[ F \cong B \oplus \tilde{B} \]
\[
\begin{align*}
(u', v') & \mapsto (u', v') \\
(x, y) & \mapsto (y-x, x)
\end{align*}
\]
and right mult by \( b \) is
\[
\begin{aligned}
(u', v') & \mapsto (v', u'+v') \xrightarrow{b} (vb, o) \mapsto (-vb, vb) = (u', v')( \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix} )
\end{aligned}
\]
Observe \( (u', v') \mapsto (v', u'+v') \mapsto (-u', u'+v') = (u', v')( \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} )\]
\[
\begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix}
\]

What does this mean? The first homomorphism \( b \mapsto \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \) from \( B \) to \( \text{Aut}_B(B \oplus \tilde{B}) = \begin{pmatrix} A & A \\ B & \tilde{B} \end{pmatrix} \)

arises from the exact sequence \( 0 \to B \to F \to \tilde{B}_e \to 0 \).

It extends to \( \begin{pmatrix} B & 0 \\ B & 0 \end{pmatrix} \) which by Fushin should be \( K \)-equivalent to \( \begin{pmatrix} B & 0 \\ B & 0 \end{pmatrix} \). The second homomorphism \( b \mapsto \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \) extends to \( \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \) which should be \( K \)-equivalent to \( \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \). Since these are conjugate this should mean that the representations \( B \to A = \text{Aut}_B(B) \) and \( B \to \tilde{B} = \text{Aut}_B(B) \) are somehow equivalent.
Let $A$ be a left ideal in $R$ with $1$. Recall that

a) $R/A$ is projective $\iff$ $A$ has a right identity: $a = ae \forall a$.

b) $R/A$ is flat $\iff$ $A$ has local right identities: $\forall a_1, \ldots, a_n$, $\exists a_j(-a) = 0$ (resp. this holds for $n = 1$.)

Suppose $A$ is an ideal in $R$ such that $R/A$ is flat, so that $A$ modules $M$

$A \otimes_R M \xrightarrow{\sim} AM$

Then taking $M = R/A$ we get $A \otimes_R R/A = A/A^2 = 0$.

Also we have $M = AM \Rightarrow M$ is firm.

Conversely assume these two conditions, and let $M$ be any module. Since $A = A^2$, $AM = A(AM)$ so $AM$ is firm. Also $A \otimes_R (M/AM) = 0$. So we get a diagram

\[
\begin{array}{cccccc}
A \otimes_R AM & \rightarrow & A \otimes_R M & \rightarrow & A \otimes_R (M/AM) & \rightarrow & 0 \\
\downarrow & & & & & & \\
0 & \rightarrow & AM & \rightarrow & M & \rightarrow & M/AM & \rightarrow & 0
\end{array}
\]

showing that $A \otimes_R M \xrightarrow{\sim} AM$ for all $M$. \therefore R/A is right flat.

**Prop.** $R/A$ is right flat for an ideal $A$ iff $A = A^2$ and $M = AM \Rightarrow M$ is firm.

**Prop.** $A$ has local left identities $\iff$ $A = A^2$ and $M = AM \Rightarrow M$ is firm for all modules $M$. It would be better to formulate this independently of $R$ as follows.
Assume $A$ is such a ring. Then

$$M = AM \implies \text{Hom}_R(R/A, M) = 0.$$ 

In effect if $K = \text{Hom}_R(R/A, M)$, then $AK = 0$ and

$$0 \to A_K \to A_M \to A_{M/K} \to 0$$

$$0 \to K \to M \to M/K \to 0$$

using $M = AM$ and $M/K = A(M/K)$. \therefore $K = 0$.

Alternate proof using local left identities: Let $Am = 0$, write $m = \sum a_i m_i$ and choose $a \in A$ such that $(1-a) a_i = 0$. Then $m = am = 0$.

Prop. Let $A$ be a ring satisfying $A = A^2$. TFAE

1) $A$ has local left identities
2) $AM = M$ is firm for all modules $M$
3) $M = \{m \mid Am = 0\}$ is zero for all firm modules $M$

It remains to check $3) \implies 2$.

Take a module $M$ s.t. $AM = M$. Then $A \otimes A M$ is firm and the kernel of $A \otimes A M \to AM$ is killed by $A$. By 3) the kernel is zero, and so $M$ is firm.

Question: Is any idempotent ring Morita equivalent to a ring with local left identities?

Suppose $(A \otimes B)$ strictly firm such that $B$ has local left identities. Then $P$ as a $B$-module satisfying $P = BP$ has local identities in the sense that $\forall p_i \exists p_i'$. \exists $b$ such that $(1-b)p_i = 0$. Conversely, if this condition holds then as $B = PQ$, the ring $B$ has local left identities. In this situation we also know that $B$ is $B^\mathbb{Q}$ flat, hence $P$ is $A^\mathbb{Q}$-flat. Now, starting with $A$ idempotent we have a sequential way to construct firm flat right modules $P$. Can
This be modified to yield local left identities or is there an obstruction?

I think we can arrange $Q$ to be essentially free in the following sense. We want, starting from a finite set of $p_j$, to construct $b = \Sigma p_j g_j$ satisfying $p'_j = \Sigma p_j (g_j p'_j)$ for all $p_j$. Here $p_j g_j$ can be added to what we already have. The function of $g_j$ is to provide an $A^Q$-linear map $P \rightarrow A$ (or maybe $\tilde{A}$).

Imagine constructing $P Q$ inductively adding at each stage the necessary $p_j g_j$. Then $P$ is a flat free module over $A$ and the $g_j$ give linear functionals on $P$. So we can replace the $Q$ linear functionals on $P$. So we can replace the $Q$ we might have with $AF$, where $F$ is a free $A$-module whose basis elements map to the $g_j$. In other words we have $F \otimes P \rightarrow A$ hence $AF \otimes P \rightarrow A$.

Consider $A$ maximal ideal in a valuation ring $R$ such that the principal ideals are $R \varepsilon$, $\varepsilon \in U \cup -1, 2^n \mathbb{Z}$. A firm flat $A^Q$-module $P$ is a torsion free $R$-module such that for any $p \in P$ $\exists \varepsilon > 0, p \in P$ such that $p = p \varepsilon$. Suppose we have an affine Morita context $(A Q \; \; \; B)$ where $B$ has local left identities. Then we know $P$ is $A^Q$ flat firm and for every finite set $p_j \exists b = \Sigma p_j g_j$ such that $p'_j = \Sigma p_j g_j p'_j$.

Take a single $p'$. We have $p' \in \Sigma p_i R$ which is a torsion free finitely generated $R$-module.
Replacing \( p_i \) by suitable linear combinations over \( R \), we can assume they form an \( R \)-basis, and also that \( p' \in p_i R \).

Then \( p' = \sum p_i \delta_i p' \implies \delta_i p' = 0 \) for \( i \neq 1 \). If \( p' = p_1 \delta_1 u \), then \( p' = p_1 \delta_1 p' \implies p_1 \delta_1 u = p_1 \delta_1 p_1 \delta_1 u \), so \( p_1 = p_1 \delta_1 p_1 \), and so \( \delta_1 p_1 = 1 \in R \). This contradicts the facts that \( \delta_1 p_1 \in A \).

Try for a computational proof as follows. The condition \( p'_j = \sum p_i \delta_i p'_j \) says the \( B \)-module \( W = \sum Bp'_j \) satisfies \( W \subseteq BW \). So \( W \) is finitely generated and \( W = BW \), so there should exist a simple object in \( M(B) \). Strictly speaking, there's a non-nil simple \( B \)-module. But \( M(B) \cong M(A) \) and \( A \) is a radical ring so \( M(A) \) has no simple objects.
Continue with the nondegeneracy question—whether any idempotent \( A \) is Morita equivalent to \( A \oplus 1 \).

I consider a special \( A \) where factoring: \( a = \sum a_i e_i \),
can be done explicitly and simply.

Let \( R \) be a valuation ring with value group \( \mathbb{Z} \). Let \( \mathbb{Z} = R_{\mathbb{Z}} \), say there are powers \( z^e \) for \( z \in U \mathbb{Z}_R \) so the principal ideals are \( \{ Rz^e \} \). Let \( m = U Rz^e \) be the maximal ideal of \( R \).

Take \( A = m/mz \) and note that \( A = m/mz = R_{zR} m \) is flat over \( R \).

It should be clear from \( 0 \to k \overset{z}{\to} A \to A \overset{1}{\to} 0 \)
that \( A = \overline{A}^{(2)} \). So we have a firm flat commutative ring with \( z \) a nonzero element \( z \) killed by \( A \).

When I consider a Morita context \( (R, A, B) \)
with \( A \) a firm flat \( R \)-module \( B \), such that

Claim there are firm flat \( A \)-modules \( \mathcal{M} \) such that

\[
A^M = 0. \text{ Let } F = \bigcup_{e > t} Rz^e \text{ where } t \text{ is a real no.}
\]

\( M = F/Fz^e \) is a firm flat \( A \)-module. Let \( x \in F \),

satisfy \( mx \in Fz^e \). Up to units, I can suppose \( x = z^e \).

Then \( z^{2^k e} \in Fz^e \) with \( e > t \).

Thus in this case \( m = F/Fz^e \) has \( m \) nonzero element killed by \( A \).
Let's take $Q = F_t / F_{t^2}$, $F_t = U \mathbb{R} e^\xi$. I'd like to find an appropriate $P$. The obvious candidate which pairs nicely with $Q$ is $P = F_t / F_{t^2}$. The pairing $Q \otimes P \to A$ is surjective and in fact it looks like $Q \otimes_A P \to A$, whence $B = P \otimes_A Q$ is also $A$. So $Q \otimes_A P \to A$, although I've managed to make $A$, $B$, zero, $B$ is still degenerate.