The bimodule approach to HC suggests trying to link two Meq rings $A,B$ using tensor products rather than $\otimes$. To be specific, given $(A \otimes B)$, then this Meq result is by taking the direct sum of $(A,A,\mu)$ and $(Q,P,Q \otimes P \rightarrow A)$. But suppose instead we use the $A$-bimodule $(A \otimes A) \otimes (Q \otimes P)$ to link $A \otimes A$ and $Q \otimes P$:

\[
\begin{array}{ccc}
(A \otimes A) \otimes (Q \otimes P) & \xrightarrow{\mu \otimes 1} & Q \otimes P \\
\downarrow & & \downarrow \\
A \otimes Q \otimes P & \xrightarrow{a \otimes q \otimes p} & q \otimes p \\
\downarrow & & \downarrow \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}
\]

Thus we have $A \otimes Q \otimes P \rightarrow A$, whose cyclic $a \otimes q \otimes p \rightarrow q \otimes p$ module maps both to $(A \otimes)^{(x)}$ and $[B \otimes]^{(x)}$. The thing that doesn't work is that there is no obvious ring structure on $(A \otimes Q \otimes P) \otimes A$. We can write this bimodule either as $(A \otimes Q) \otimes P$ or $A \otimes (Q \otimes P)$. These lead respectively to the Morita contexts:

\[
\begin{pmatrix}
A & A \\
P & P \otimes (A \otimes Q)
\end{pmatrix}
\qquad \text{and} \qquad
\begin{pmatrix}
A & A \\
Q \otimes P & (Q \otimes P) \otimes A
\end{pmatrix}
\]
One calculates that the ring structures induced on \( P \otimes A \) and \( Q \otimes P \) are

\[
(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \otimes p_2 \otimes q_1 \otimes q_2
\]

\[
(q_1 \otimes p_1)(q_2 \otimes p_2) = q_1 \otimes q_2 \otimes p_1 \otimes p_2
\]

Another thing I can do is to consider the bimodule \((Q \otimes P) \otimes (A \otimes A) = Q \otimes P \otimes A\) which can be split either as \( Q \otimes (P \otimes A) \) or \((Q \otimes P) \otimes A\) leading to the Morita contexts

\[
\begin{pmatrix}
A & Q \\
(P \otimes A) & (Q \otimes P)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
A & Q \otimes P \\
A \otimes (Q \otimes P)
\end{pmatrix}
\]

The induced ring structures on \( P \otimes Q \) and \( Q \otimes P \) are resp.

\[
(p_1 \otimes q_1)(p_2 \otimes q_2) = p_1 \otimes p_2 \otimes q_1 \otimes q_2
\]

\[
(q_1 \otimes p_1)(q_2 \otimes p_2) = q_1 \otimes q_2 \otimes p_1 \otimes p_2
\]

These four products arise from the two associative products on the two dialgebras given by \( Q \otimes P \to A \)
and \( P \otimes Q \to B \).
Let $X_{nk}(P, Q)$ be the subset of $M_{nk}(P) \times M_{kn}(Q)$ consisting of $(p, q)$ such that $1 - pq$ is invertible. Fix $n$ and form the category with object set \( \bigcup_{k=0}^{\infty} X_{nk}(P, Q) \), where a map $(p, q) \to (p', q')$ is given by a matrix $a$ over $A$ such that $pa = p'$ and $qa = q'$. Notice that if there is a map $(p, q) \to (p', q')$, then $p'q' = pq = 1$. Thus $\pi_0$ of this category maps to $\text{GL}_n B$ by $(p, q) \mapsto 1 - p'q'$. Denote this category by $X_n$.

I claim that $\pi_0 X_n \to \text{GL}_n B$.

Thus $1 - p_1 q_1 = 1 - p_2 q_2 \iff$ there is a path in $X_n$ joining $(p_1, q_1)$ to $(p_2, q_2)$.

**Proof.** First we show $(p_1 q_1) \sim (p_1, p_2)j(8_1)$, where $\sim$ means in the same component of $X_n$.

We can write $q_1 = a q'$ since $Q = A Q$. Then

\[
\begin{pmatrix}
(p_1, p_2)j(8_1)
\end{pmatrix} = \begin{pmatrix}
(p_1, p_2)j(\begin{pmatrix}a & q' \\
0 & 0\end{pmatrix})
\end{pmatrix}
\]

\[
\begin{pmatrix}
(p_1 a & q')
\end{pmatrix} \to \begin{pmatrix}
(p_1, a q')
\end{pmatrix} = (p_1, q_1)
\]

Next given $(p_1 q_1)$ and $(p_2, q_2)$ such that $p_1 q_1 = p_2 q_2$, we know $q_1, q_2, a_3, p_3 q_1'$ such that (uses $p_3 q_1 \to B$)

\[
\begin{pmatrix}
p_1 & p_2 & p_3
\end{pmatrix} = \begin{pmatrix}
a_1 & a_2 & a_3
\end{pmatrix}
\]

Then $(p_2, q_2) \sim (p_2, p_3)j(\begin{pmatrix}8_2 \\
0 \end{pmatrix}) = (p_2, p_3)j(\begin{pmatrix}a_2 \\
0 \end{pmatrix}) q_2'
\]

\[
(p_1, q_1 q_2') \left\langle \begin{pmatrix} a_1 & q_2' \\
8_1 
\end{pmatrix}
\right\rangle = \begin{pmatrix} -p_2 q_2 - p_3 q_2' & g'
\end{pmatrix}
\]
Next note that \((p_1 g_1) \rightarrow (p_2 g_2)\).

Then the invertible matrices over \(A\) give

\[ 1 - g' p' = 1 - (g' p) a , \quad 1 - g p = 1 - a (g' p) \]

represent the same element of \(K_1 A\). Thus we have a well-defined map

\[ GL_n B \rightarrow K_1 A , \quad 1 - g p \mapsto [1 - g p] \]

Next we show this is a homomorphism.

\[
(1 - p_1 g_1)(1 - p_2 g_2) = 1 - p_1 g_1 - p_2 g_2 + p_1 g_1 p_2 g_2 \\
= 1 - (p_1, p_2) \begin{pmatrix} g_1 - g_1 p_2 g_2 \\ g_2 \end{pmatrix} \\
= 1 - (p_1, p_2) \begin{pmatrix} 1 - g_1 p_2 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}
\]

This product goes to the element of \(K_1 A\) represented by

\[
\star = 1 - \begin{pmatrix} 1 & -g_1 p_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} (p_1, p_2) \\
= 1 - \begin{pmatrix} g_1 (1 - p_2 g_2) \\ g_2 \end{pmatrix} (p_1, p_2) \\
= 1 - \begin{pmatrix} g_1 (1 - p_2 g_2) p_1 & g_1 (1 - p_2 g_2) p_2 \\ g_2 p_1 & g_2 p_2 \end{pmatrix} \\
= \begin{pmatrix} 1 - g_1 (1 - p_2 g_2) p_1 & -g_1 p_2 (1 - g_2 p_2) \\ -g_2 p_1 & 1 - g_2 p_2 \end{pmatrix}
\]

Recall

\[
\begin{pmatrix} 1 & -bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - bd^{-1} c & 0 \\ c & d \end{pmatrix} \sim \begin{pmatrix} a & bd^{-1} c \\ 0 & d \end{pmatrix}
\]
Thus \( \star \) is conjugate mod elements of 
\[
1 - g_2 \left( 1 - g_2 p_2 \right) p_1 \end{align*}
\]
\[
= 1 - g_1 \left( 1 - p_2 g_2 \right) p_1 - g_1 p_2 (1 - g_2 p_2) \left( 1 - g_1 p_1 \right) p_2 \end{align*}
\]
\[
= 1 - g_1 p_1.
\]

Oct 26, 1995: The above calculation and the one on p. 37 use the following identity:
\[
1 - \left( \frac{\delta_1}{\delta_2} \right) \left( p_1, p_2 \right) = \begin{pmatrix}
1 - \delta_1 p_1 & -\delta_1 p_2 \\
-\delta_2 p_1 & 1 - \delta_2 p_2
\end{pmatrix} \sim \begin{pmatrix}
1 - \delta_1 p_1 & 0 \\
0 & 1 - \delta_2 p_2
\end{pmatrix}
\]
\[
\star = d - c a^{-1} b = 1 - g_2 p_2 - g_2 p_1 (1 - g_1 p_1)^{-1} g_1 p_2 = 1 - g_2 \left( 1 - g_1 p_1 \right)^{-1} p_2
\]

In the composition situation above, \( p_2 \) is changed to
\[
(1 - p_1 g_1) p_2 = (1 - p_1 g_1) (1 - p_2 g_2) = 1 - p_1 g_1 - (p_2 - p_1 p_2) g_2 \quad \text{so} \quad \star = 1 - g_2 p_2.
\]

In the situation on p. 37: \( p_1 g_1 = p_2 g_2 \) so \( g_2 \) is changed to \(-g_2\)
and
\[
\star = 1 + g_2 \left( 1 - p_2 g_2 \right)^{-1} p_2 = \left( 1 - g_2 p_2 \right)^{-1}.
\]
Another calculation with adjoint functors, compare p.24. To show unicity of an adjoint:

\[ \text{Hom}(X, GY) = \text{Hom}(FX, Y) = \text{Hom}(X, G'Y) \]

Taking \( X = GY \), then \( 1_{GY} \) goes to \( GY \xrightarrow{\beta_GY} G'FGY \xrightarrow{G'\alpha_Y} G'Y \).

Similarly \( 1_{GY} \) goes back to \( G'Y \xrightarrow{\beta_GY} GFG'Y \xrightarrow{G\alpha_Y} GY \).

Thus to show the composition is the identity:

\[
\begin{array}{ccc}
GY & \xrightarrow{\beta_GY} & G'FGY \\
\downarrow & & \downarrow \\
GFGY & \xrightarrow{GF\beta_Y} & GFG'FGY \\
\downarrow & & \downarrow \\
GFG'Y & & GFG'Y \\
\end{array}
\]

Triangle commutes

by applying \( G \) to

\[
\begin{array}{ccc}
FX & \xrightarrow{F\beta_X} & FGFX \\
\downarrow & & \downarrow \\
FX & &FX \\
\end{array}
\]

when \( X = GY \).

Thus the composition \( G(\alpha_Y) \beta_GY \) equals

\[ G(\alpha_Y) \beta_GY = 1_{GY} \]
November 9, 1995

Given \((A \otimes B)\) completely firm, \(w: A \rightarrow B\) a ring homomorphism, and \(u': B \otimes_A A \rightarrow P\) a \((B,A)\)-bimodule isomorphism. To construct \(u: A \rightarrow P\), \(v: A \rightarrow Q\) such that

\[
\begin{align*}
  u(a_1 a_2) &= u(a_1) a_2 = w(a_1) u(a_2) \\
  v(a_1 a_2) &= a_1 v(a_2) = v(a_1) w(a_2) \\
  v(a_1) u(a_2) &= a_1 a_2 \\
  u(a_1) v(a_2) &= w(a_1) a_2
\end{align*}
\]

**First proof via adjunction:** \(u'\) gives \(B \otimes_A - \cong P \otimes_A -\), so

\[
\text{Hom}_A(M, Q \otimes_B N) = \text{Hom}_B(P \otimes_A M, N)
\]

\[
\cong \text{Hom}_B(B \otimes_A M, N)
\]

\[
= \text{Hom}_A(M, A \otimes_A N)
\]

whence \(Q \otimes_B - \cong A \otimes_A -\), an isom of the right adjoints. This gives an \((A,B)\)-bimodule isom \(v': A \otimes_A B \rightarrow Q\) such that \(u', v'\) respect the adjunction maps, i.e.

1. \(u' v': B \otimes_A A \otimes_A A \otimes_A B \rightarrow P \otimes A = B\)

   \(u' v' = \beta_{a_1 a_2 a_3 a_4} = a_1 \otimes w(a_1) \otimes w(a_1) \otimes a_4,\) and

2. \(v' u': A \otimes B \otimes B \otimes A \rightarrow Q \otimes P = A\)

   is the inverse of \(\beta_{a_1 a_2 a_3 a_4} = a_1 \otimes w(a_1) \otimes w(a_1) \otimes a_4.\)

Now define \(u, v\) to be the compositions

\[
\begin{align*}
  A = A \otimes A &\xrightarrow{w \otimes 1} B \otimes_A A \rightarrow P \\
  A = A \otimes A &\xrightarrow{1 \otimes w} A \otimes_A B \rightarrow Q
\end{align*}
\]

i.e.

\(u(a_1 a_2) = u'(w(a_1) \otimes a_2),\) \(v(a_1 a_2) = v'(a_1 \otimes w(a_2)).\)
Clearly, $v_r$ are $A$-bimodule maps, while

$u(q_1q_2) v(q_3q_4) = (u' \otimes v)(w(a_i) \otimes w(a_j))$

$= \alpha \left( \begin{array}{c}
w(a_1) \otimes w(a_3) \otimes w(a_4) \\
= w(q_1q_2q_3q_4)
\end{array} \right)$

and

$w(a_1) \otimes w(a_3) \otimes w(a_4) = \beta^{-1}(q_1q_2q_3q_4)$

Second proof: Define $u : A \to P$ as above, i.e.

$u(q_1q_2) = u'(wa_1 \otimes a_2)$

Clearly

$u(q_1q_2) = a(q_1)a_2 = wa_1u(a_2)$

Also

$u'(b \otimes a_2) = u'(bw(a_1) \otimes a_2) = bu'(wa_1 \otimes a_2) = bu(q_1q_2)$, hence

$u'(b \otimes a) = bu(a)$

Next from $u' : B \otimes_A A \to P$ we get an isomorphism

$Q \otimes_A A \to Q \otimes_B B A \xrightarrow{1 \otimes u'} Q \otimes P = A$

$gb \otimes a \to gb \otimes a \to gbu(a) \to gbu(a)$

so we have

$Q \otimes_A A \to \to A$

$g \otimes a \to gb(a)$

Define $v : A \to Q$ to be the composition
\[
A \xleftarrow{\sim} Q \otimes_A A \rightarrow Q \\
q_u(a) \rightarrow g \otimes a \rightarrow g \omega(a)
\]

Thus \( V(q_u(a)) = g \omega(a) \). It's clear that \( v \) is an \( A \)-bimodule map where \( A \) acts on the right of \( Q \) via \( \omega \):

\[
V(a_1 a_2) = a_1 V(a_2) = V(a_1) \omega(a_2)
\]

Better to write \( g u : a \rightarrow g \otimes b \otimes a \rightarrow g \otimes b \omega(a) \rightarrow g \omega(a) \):

\[
Q \otimes_A A = Q \otimes_B B \otimes A \sim Q \otimes_B A = A
\]

Thus we have:

\[
\begin{array}{c}
Q \otimes_A A \sim A \\
q_u(a) \otimes a_1 \rightarrow g \omega(a) \\
V(a_1) \otimes a_2 \leftarrow a_1 a_2
\end{array}
\]

and tensoring with \( P \) we have:

\[
\begin{array}{c}
B \otimes_A A \sim P \\
b \otimes a \rightarrow b \omega(a) \\
pv(a_1) \otimes a_2 \leftarrow p a_1 a_2
\end{array}
\]

Since \( a_1 a_2 \rightarrow V(a_1) \otimes a_2 \rightarrow V(a_1) \omega(a_2) \) is the identity, we get:

\[
V(a_1) \omega(a_2) = a_1 a_2
\]
Next \( g \otimes a \otimes a_2 \otimes a_3 \mapsto g \cdot u(a_1 \otimes a_2 \otimes a_3) \)

\[ = g \cdot u(a_1) \cdot a_2 \cdot a_3 \mapsto \nu(g \cdot u(a_1) \cdot a_2) \otimes a_3 \]

\[ = g \cdot u(a_1) \nu(a_2) \otimes a_3 \] is the identity, so we have \( g \cdot u(a_1) \nu(a_2) \otimes a_3 = g \otimes a_2 \otimes a_3 \), hence

\[ g \cdot u(a_1) \nu(a_2) = g \cdot \nu(a_2) \]

and so

\[ g \cdot u(a_1) \nu(a_2) = g \cdot \nu(a_1) \nu(a_2) \]

hence

\[ b \cdot u(a_1) \nu(a_2) = b \cdot \nu(a_1) \nu(a_2) \]

Thus \( \nu(a_3 \otimes a_4) \cdot \nu(a_1 \otimes a_2) = \nu(a_3 \otimes a_4) \cdot u(a_1) \cdot \nu(a_2) \)

\[ = u(a_3 \otimes a_4) \nu(a_1) \nu(a_2) = u(a_3 \otimes a_4) \cdot \nu(a_1) \cdot \nu(a_2) \]

\[ \nu(a_3 \otimes a_4) \cdot \nu(a_1 \otimes a_2) \]

\[ \nu(a_1 \otimes a_2) = u(a_1) \cdot \nu(a_2) \]

Given \( u : A \to P \), \( v : A \to Q \) usual props.

\[ (A) \otimes_A (A \otimes A) \to (A) \otimes_A (A \otimes Q) \]

\[ (u \otimes v) : M_2(A) \to (A \otimes Q) \]

get two homomorphisms of Morita contexts. Note 8 conditions

\[ a_1 \otimes a_2 = a_1 \otimes a_2 \]

\[ u(a_1) = u(a_2) \]

\[ v(a_1) = v(a_2) \]

\[ w(a_1 \otimes a_2) = w(a_1) \cdot w(a_2) \]

\[ v(a_1 \otimes a_2) = u(a_1) \cdot v(a_2) \]
We work over a commutative unital ground ring \( k \). Let \((A, B)\) be a completely prime Morita context (over \( k \)).

Prop. Assume \( A \) is \( h \)-unital and \( k \)-flat. Then \( B \) is \( h \)-unital iff \( P \overset{B}{\otimes} A \overset{A}{\otimes} Q \to B \) is a quasi-

(Previously I proved this ignoring the problem of flat bimodules being left and right flat, so the proof works over a field, probably also if we assume \( A, P, Q, B \) are all \( k \)-flat. I now want to check the above version carefully.)

Recall the previous argument:

\[
\begin{array}{c}
B \overset{B}{\otimes} P \overset{A}{\otimes} A \overset{Q}{\otimes} Q \to B \overset{B}{\otimes} B \\
\downarrow \\
P \overset{A}{\otimes} A \overset{Q}{\otimes} Q \to B 
\end{array}
\]

We know \( B \overset{B}{\otimes} P \to P \) is an \( A^\phi \)-nil-quis. Hence

\( A \) \( h \)-unital \( \Rightarrow \) \( B \overset{B}{\otimes} P \overset{A}{\otimes} A \Rightarrow P \overset{A}{\otimes} A \) \( \Rightarrow \) left vertical arrow is a quis.

Then if the bottom arrow is quis, so is the top arrow and we conclude that \( B \) is \( h \)-unital. This proves the direction \((\Leftarrow)\).

We know \( P \overset{A}{\otimes} A \to P \) and \( P \overset{B}{\otimes} Q \to B \) are \( B \)-nil-quis. If \( B \) is \( h \)-unital, then we have

quis \( B \overset{B}{\otimes} P \overset{A}{\otimes} A \Rightarrow B \overset{B}{\otimes} P \) and \( B \overset{B}{\otimes} P \overset{B}{\otimes} Q \Rightarrow B \overset{B}{\otimes} B \).

So the top arrow is a quis, as well as the left + right vertical arrows, and we conclude the bottom arrow is a quis, proving \((\Rightarrow)\).
To carry out this argument we need to make sense of the derived tensor products. Let \( E \to \tilde{A} \) be a flat \( A \)-bimodule resolution, let \( \hat{B} \to B \) and \( \hat{Q} \to Q \) be flat \( B^\text{op} \)-module and flat \( A \)-module resolutions respectively. Consider the square

\[
\begin{array}{ccc}
\hat{B} \otimes B & \otimes & E \otimes A \otimes \hat{Q} \\
\downarrow & & \downarrow \\
B \otimes B & \longrightarrow & \hat{Q}
\end{array}
\]

Now \( E \otimes A \) is a flat \( A \)-module resolution of \( A \).

In effect, \( E \) is a flat \( A^\text{op} \)-module complex (as \( \tilde{A} \otimes \tilde{A} \) is \( A^\text{op} \)-flat, \( \tilde{A} \) being \( k \)-flat) so \( E \otimes A = \tilde{A} \otimes \tilde{A} = A \). This \( E \otimes A \) is a resolution of \( A \).

Also \( E \otimes A \) is \( A \)-flat since \( (\tilde{A} \otimes \tilde{A}) \otimes A = \tilde{A} \otimes A \).

And \( A \) is \( k \)-flat.

So \( (\hat{B} \otimes B) \otimes A \otimes A = (\hat{B} \otimes B) \otimes A \) and similarly for \( P \) in place of \( \hat{B} \otimes B \). Thus we see that \( \hat{B} \otimes P \otimes A \otimes A \xrightarrow{\sim} P \otimes A \otimes A \), because we know \( \hat{B} \otimes P \to P \) is an \( A^\text{op} \)-nil-gens and \( A \) is \( k \)-unital. Then applying \( - \otimes A \) yields a quasi since \( \hat{Q} \) is \( A \)-flat. Thus the left vertical arrow is a quasi.

Something I should have mentioned earlier is that the condition \( P \otimes_A \hat{A} \otimes_A Q \to B \) can be interpreted as saying that \( P \otimes_A \hat{A} \otimes_A Q \to P \otimes_A A \otimes_A Q = B \) is a quasi, where \( P \to P \) is a flat \( A^\text{op} \)-module resolution.
We have quis

\[ \hat{P} \otimes_A A \otimes \hat{Q} \leftarrow \hat{P} \otimes_A E \otimes A \otimes \hat{Q} \rightarrow \hat{P} \otimes E \otimes A \otimes \hat{Q} \]

the first because \( \hat{P}, \hat{Q} \) are flat and \( E \otimes A \rightarrow A \) is a quis, the second because \( \hat{B} \rightarrow \hat{P} \) is a quis and \( E \otimes A \) and \( Q \) are \( A \)-flat.

At this point we can identify the bottom arrow in * with the map \( \hat{P} \otimes_A A \otimes \hat{Q} \rightarrow B \).

If this map is a quis, then so is the top arrow in the square *, whence \( \hat{B} \otimes_B B \rightarrow B \) is a quis, and \( B \) is \( h \)-universal. This proves (\( \Leftarrow \)).

Now \( \hat{P} \otimes E \otimes A \rightarrow P \) is \( \hat{P} \otimes_A A \rightarrow P \) which we know is a \( B \)-split quis. Assuming \( B \), we know \( B \rightarrow \hat{B} \) is \( h \)-universal we get a quis \( B \otimes_B \hat{P} \otimes E \otimes A \rightarrow \hat{B} \otimes_B P \), hence a quis \( \hat{B} \otimes \hat{P} \otimes E \otimes A \otimes \hat{Q} \rightarrow \hat{B} \otimes \hat{P} \otimes \hat{Q} \). Also \( \hat{P} \otimes A \hat{Q} = \hat{P} \otimes \hat{Q} \rightarrow B \) is a \( B \)-split quis, so we have a quis \( \hat{B} \otimes \hat{P} \otimes A \hat{Q} \rightarrow \hat{B} \otimes \hat{Q} \). Combining we see the top arrow in * is a quis, as well as the vertical arrows, so the bottom arrow is a quis, proving (\( \Rightarrow \)).
December 1, 1995

Multipliers again. Recall:

A left multipliers on \( B \) is an operator \( b \mapsto xb \)
that satisfies \( x(b_1b_2) = (xb_1)b_2 \). These form a ring
\[ M(B) = \text{Hom}_{B_{op}}(B, B) \text{ with product given by } (x_1x_2)b = x_1(x_2b). \]

A right multipliers on \( B \) is \( b \mapsto bx \) s.t. \( (b_1b_2)x = b_1(b_2x) \); we
get ring \[ M_r(B) = \text{Hom}_B(B, B)_{op} \text{ with product } b(x_1x_2) = (bx_1)x_2. \]

The ring of multipliers on \( B \) is the subring
\[ M(B) = \left\{(x^l, x^r) \in \text{Hom}_{B_{op}}(B, B) \times \text{Hom}_B(B, B)_{op} \mid b_1(x^l b_2) = (b_1 x^r)b_2 \right\}. \]

Suppose now \((A, B)\) is a completely finitelyMonte algebra.

We have
\[ M(B) = \text{Hom}_{B_{op}}(B, B) \cong \text{Hom}_{A_{op}}(A, A). \]

\[ x^l \mapsto \{y^l : bp \mapsto (xb)p \} \]
\[ x^r \mapsto \{y^r : gp \mapsto g(bxp) \} \]
\[ M_r(B) = \text{Hom}_B(B, B)_{op} \cong \text{Hom}_A(A, A)_{op}. \]
\[ x^r \mapsto \{y^r : gb \mapsto g(bx^r) \} \]
\[ \{x^r : p \mapsto p(gy^r) \} \]

Now check that if \( x^r \leftrightarrow y^r, \ x^l \leftrightarrow y^l \) in this way,
then the condition \((b_1 x^r)b_2 = b_1(x^r b_2)\) is equivalent
to the condition \((g_1 y^r)p = g_1 y^r p)\). Assume the first.
\[(g_{b_1}y^e)_{b_2}p = \frac{g(b, x^a)}{b_2}p = \frac{g(b, x^a)}{b_2}p \]
\[gb_1(y^e(b_2)p) = gb_1(x^eb_2)p = \frac{gb_1(x^eb_2)p}{n} \]

Conversely assume \((g_{y^e})p = g_{(y^ep)}\). Then
\[(q_{b_1}x^e)p_2y^e = (p_{1}(g_{y^e}))p_2y^e = \frac{p_1(g_{y^e})p_2y^e}{n} \]
\[p_1b_1(x^e(p_2y^e)) = p_1b_1(y^e(p_2)) = \frac{p_1b_1(y^e(p_2))}{n} \]

Conclude then that
\[
M(B) \cong \{ (y^e, y^e) \in \text{Ham}_A(p_2p) \times \text{Ham}_A(a, a) \mid g_{y^e}p = g_{y^ep} \}
\]

Recall that \(B = B^2 \implies \) any left multiplier commutes with any right multiplier:
\[(x^e(b, b_2))y^e = (x^eb_1)(b_2y^e) \]
\[x^e(b, b_2y^e) = x^e(b_1(b_2y^e)) = (x^eb_1)(b_2y^e).\]

Put another way, we have \(B \otimes B \to B\) with \(x^e \otimes 1 \) (resp. \(1 \otimes y^e\)) inducing \(x^e\) (resp. \(y^e\)) on \(B\); hence \(x^e\otimes 1, 1 \otimes y^e\) commute \(\implies x^e, y^e\) commute.

It follows that \(B\) is naturally a bimodule over \(M(B) \times M(B)\) and that multiplication is a bimodule map from \(B \otimes B\) to \(B\). The multiplier condition \((b, x^e)b_2 = b_1(x^eb_2)\) means this multiplication map descends to \(B \otimes M(B)\). Thus \(B\) becomes an algebra over \(M(B)\) in some non-commutative sense.
Center of $\mathcal{M}(B)$.

First recall we have homomorphisms

$$\begin{align*}
B & \longrightarrow \text{Hom}_B(B, B) = \mathcal{M}_B(B) & b & \mapsto (b^\ell; b' \mapsto b b') \\
B & \longrightarrow \text{Hom}_B(B, B) = \mathcal{M}_B(B) & b & \mapsto (b^r; b' \mapsto b' b') \\
B & \longrightarrow \mathcal{M}(B) & b & \mapsto (b^\ell, b^r)
\end{align*}$$

Let $x = (x^\ell, x^r) \in \mathcal{M}(B)$, centralize the image of $B$, i.e.

$$b^\ell x^\ell = x^\ell b^\ell, \quad b^r x^r = x^r b^r \quad \forall b \in B. \quad \text{Thus}$$

$$b_1(x b_2) = x(b_1 b_2) \quad \quad (b_1 b_2) x = (b_1 x) b_2$$

Then $x(b_1 b_2) = b_1 x(b_2) = (b_1 x)b_2 = (b_1 b_2)x$, so that if $B = B^\ell$, then $b x = b x$, $\forall b \in B$.

In other words $x^\ell$ and $x^r$ are the same map from $B$ to $B$. Thus $x$ is a bimodule map $B \rightarrow B$.

Let's examine the unital ring $\text{Hom}_B(B, B)$ of $B$-bimodule maps $z : B \rightarrow B$. Then $(z, z) \in \mathcal{M}(B)$, since $z(b_1 b_2) = b_1 z(b_2) = b_1 (z b_2)$ and $z(b_1 b_2) = z(b_1) b_2 \Rightarrow (b_1 z) b_2 = b_1 (z b_2)$.

Here $z(b) = z b = b z$ in the left + right multiplier notation. If $x = (x^\ell, x^r)$ is any multiplier, then since any left + any right multiplier commute, we have $x^\ell z = z x^\ell$, $x^r z = z x^r$. So $(z, z)$ is in the center of $\mathcal{M}(B)$. In particular $\text{Hom}_B(B, B)$ is
a commutative unital ring.

Better version of preceding: Introduce the canonical homomorphism

\[ \mu : B \to M(B) \]

\[ \mu(b)b' = bb' \]

\[ b'\mu(b) = b'b \]

Then for any \( x \in M(B) \) we have

\[ x\mu(b) = \mu(xb) \]

\[ \mu(b)x = \mu(bx) \]

Check:

\[ (\mu(b)x)b' = \mu(b)(xb') = b(xb') = (bx)b' = \mu(bx)b' \]

\[ b'(\mu(b)x) = (b'\mu(b))x = (b'b)x = b'(bx) = b'\mu(bx). \]

and similarly for the other.

(*) shows that \( \mu \) is a bimodule map over \( M(B) \), in particular the image of \( \mu \) is an ideal in \( M(B) \).

Proof. The following unital rings are the same.

1) the center of \( M(B) \)
2) the centralizer in \( M(B) \) of \( \mu(B) \)
3) \( \text{Hom}(B, B) \), the ring of bimodule maps, \( B^{\text{Z}} \).

Proof. Notice that a b-bimodule map \( z : B \to B \), i.e., \( z(b, b') = z(b)b' = b, z(b') \) is the same thing as a multiplier \( x \) such that \( xb = bx' \), i.e., such that \( x \in \mathbb{Z} : B \to \mathbb{B} \). (Assuming \( B = B^{\text{Z}} \)) such a multiplier commutes with any other \( y = (y^x, y^y) \), since left and right multipliers commute:

\[ xy = (x^x, x^y)(y^x, y^y) = (x^x y^x, x^y y^y) = (x^x, x^y)(y^x, y^y) \]

\[ = (y^x x^x, x^y y^y) = (y^x y^y)(x^x, x^y) = yx \]

Thus 3) \( \subseteq \) 1) \( \subseteq \) 2).

Now let \( x \) commute with all \( \mu(b) \). Then
\[ x(bb') = (x\mu(b))b = (\mu(b)x)b' = b(xb') = b\mu(xb') = b\mu(\mu(b)x) = bb'x \]

showing \( xb = bx \) for all \( b \) since \( B = B^2 \).

Recall what we know about lift multipliers. The canonical map

\[ A \rightarrow \text{Han}_{A_{op}}(A, A) \quad a \mapsto (a' \mapsto a'a') \]

is an \( A_{op} \)-nil-isom, whence (assuming \( A \) firm)

\[ A \rightarrow \text{Han}_{A_{op}}(A, A) \otimes_A A \]

The other tensor product gives

\[ A \rightarrow A \otimes_A \text{Han}_{A_{op}}(A, A) = \mathcal{Q}_{\text{univ}} \]

where \( \mathcal{Q}_{\text{univ}} \) is universal wrt firm \( A \)-modules \( \mathcal{Q} \) equipped with a bimodule map \( \mathcal{Q} \otimes A \rightarrow A \). This leads to a Morita context studied by Steffen

\[
\begin{pmatrix}
A \rightarrow \mathcal{Q}_{\text{univ}} \\
A \rightarrow B
\end{pmatrix}
\]

Similarly we have canonical map

\[ A \rightarrow \text{Han}_A(A, A)^{op} \quad a \mapsto (a' \mapsto a'a) \]

which is an \( A \)-nil-isom, whence

\[ A \rightarrow A \otimes_A \text{Han}_A(A, A) \]

is universal wrt firm \( A_{op} \)-modules \( \mathcal{P} \) equipped with \( A \otimes \mathcal{P} \rightarrow A \).
The Morita contexts where

\[
\begin{pmatrix}
A & Q \\
P & B
\end{pmatrix} \quad \text{(also } \begin{pmatrix}
A = Q \\
P = B
\end{pmatrix})
\]

form an interesting class containing inclusions \( A \subseteq B \) such that \( BA = A, \ AB = B \), i.e., left ideal generating the ring. Also you get the situation where \( A \to B \) is a \( B \)-module map, and \( \frac{B}{\ker B} = A \upharpoonright T \) with \( IA = 0 \); \( (q, q') = f(q)q' \).

(*mips A, B with \( A \otimes_B B \to B, \ B \otimes_B A \to A \))

**Question:** Do there any interest in considering pairs of maps: \( u: P \to P' \) of \( A \otimes_B \)-modules and \( u^*: Q' \to Q \) of \( A \)-modules, which are adjoint:

\[
q'u(p) = u^*(q')p \quad \forall q' \in Q', \ p \in P.
\]

Such a pair \((u, u^*)\) should be analogous to a map of \( C^* \)-modules.

---

Rosen theorem link: Let \( Q \) be a generator of \( M(A) \),
\( R = \text{End}_A(Q)^{\text{op}}, \ B = \text{Im} \{ \text{Hom}_A(Q, A) \otimes_A Q \to R \} \). We have a Morita context \( (A, Q, R) \), yielding an equivalence of categories \( M(A) \sim \text{mod}(R)/\text{nil}(R,B) = M(R,B) \). The \( A^{\text{op}} \)-module corresponding to \( Q \) is \( P = \text{Hom}_A(Q, A) \otimes_A A \).

When \( Q = A \), \( P = B \) (the prime ring \( B^{(2)} \)) is Steffen's ring.

---
Consider the canonical maps

\[ B \xrightarrow{\mu^1} \text{Hom}_{B^{op}}(B, B) \]
\[ B \xrightarrow{\mu^2} \text{Hom}_B(B, B)^{op} \]
\[ B \xrightarrow{\mu} M(B) \]

Then \( \mu^1 \) is a \( B^{op} \)-morphism, so \( B \xrightarrow{\text{Hom}_{B^{op}}(B, B) \otimes B} \)
\( \mu^2 \) is a \( B \)-morphism, so \( B \xrightarrow{\text{Hom}_B(B, B)} \)
\( \mu \) is both so \( B \xrightarrow{M(B) \otimes B}_B, B \xrightarrow{B \otimes M(B)}_B \)

(To remember which side occurs, you look at the contravariant variable of the \( \text{Hom}(\_ , \_ ) \).)
December 7, 1995

I now want to record some observations I found when working on ring homomorphisms which induce $\text{Mod}_B$.

One idea was to consider a homomorphism of $\text{Mod}_A$:

$$(1, v) : (A \otimes_A A) \rightarrow (A \otimes_B A)$$

i.e.

$v(a_1)u(a_2) = a_1 a_2$

$u(a_1 a_2) = u(a_1) u(a_2) = u(a_1) v(a_2) = v(a_1) w(a_2)$

$w(a_1 a_2) = w(a_1) w(a_2) = u(a_1) v(a_2)$

( recalled there are 8 products to consider in a $\text{Mod}_A$.)

Now assume $QP = A$, $PQ = B$ (also $A, B$ idempotent).

Then

$$A \xrightarrow{u} P$$

is an $A$-nil-isomorphism

because $u(a) = 0 \Rightarrow a_1 q = v(a_1) w(a) = 0 \quad \forall a_1$

and $w(q a_2) P = u(a_1) v(a_2) P = u(a_1 v(a_2) P) \Rightarrow$

$w(a) \cdot \text{kills } \text{Coker}(u), \quad \forall a$.

Then

$$A^{(2)} \xrightarrow{\sim} A^{(2)} \otimes_A P$$

so for $M$ form over $A$ we have

$$M \xrightarrow{\sim} A^{(2)} \otimes_A P \otimes A^{(2)} \otimes_A M$$

$$a_0 q_1 q_2 m \mapsto a_0 \otimes a_1 \otimes u(a_2) \otimes m$$

Taking $M = Q \otimes_B N$, $N$ form over $B$ we get
\[ Q \otimes_B N \xrightarrow{\sim} A^{(2)} \otimes_A N \]
\[ a_0 a_1 a_2 \otimes n \longrightarrow a_0 \otimes a_1 \otimes u(a_2) g n \]

Now consider the maps in the opposite direction given by \( a_0 v(a_1) \otimes n \leftarrow\mathllap{1} a_0 \otimes a_1 \otimes n \). We have
\[
\begin{align*}
 a_0 v(a_1) u(a_2) g n & \leftarrow\mathllap{1} a_0 \otimes a_1 \otimes u(a_2) g n \\
 a_0 v(a_1) u(a_2) g n &= a_0 a_1 a_2 \otimes n
\end{align*}
\]
so this map is a one-sided, hence two-sided inverse. But note that \( a_0 v(a_1) \otimes n = v(a_0 a_1) \otimes n \) so that this inverse map factors through \( A^{(2)} \otimes_A N \). Thus it seems that \( A^{(2)} \otimes_A N = A \otimes_A N \) for \( N \) firm over \( B \). This can be checked much more simply as follows.

Observe that the maps
\[
\begin{align*}
 Q \otimes_B N & \hookrightarrow A \otimes_A N \\
 a_0 a_1 a_2 \otimes n & \longrightarrow a_0 \otimes u(a_2) g n \\
 v(a_1) \otimes n & \leftarrow\mathllap{1} a_0 \otimes n
\end{align*}
\]
are well-defined (use that \( Q \otimes_B N \) is \( A \)-firm), and are inverse to each other. For example
\[
\begin{align*}
 v(a_0 a_1 a_2) \otimes n & \leftarrow\mathllap{1} a_0 a_1 a_2 \otimes n \\
 a_0 a_1 v(a_2) \otimes n & \longrightarrow a_0 \otimes u(a_1) v(a_2) n = a_0 a_1 a_2 \otimes n
\end{align*}
\]

Similarly \( v: A \rightarrow A \) is an \( A \)-morphism so for \( M \) firm over \( A \) we have
\[
\begin{align*}
 M & = A \otimes_A M \xrightarrow{\sim} Q \otimes_A M \\
 a \otimes m & \longrightarrow v(a) \otimes m
\end{align*}
\]
whence \[ P \otimes_A M \xrightarrow{\sim} P \otimes_A Q \otimes_A M \xrightarrow{\sim} B \otimes_A M \]
\[ p \otimes am \xrightarrow{\sim} p v(a) \otimes m \]

Here are the formulas:

\[
\begin{array}{c|c}
A \otimes_A N & B \otimes_A M \\
\otimes & \otimes \\
a \otimes n & b \otimes m \\
v(a) \otimes n & v(b) \otimes m \\
q_1 \otimes w(a) \otimes q_3 & q_1 \otimes q_2 \otimes q_3 \\
q_1 \otimes w_2 \otimes q_3 & q_1 \otimes q_2 \otimes q_3 \\
\end{array}
\]

Here \( M \) is firm \( /A \) and \( N \) is firm \( /B \). We note that then the right sides \( Q \otimes_B N, P \otimes_A M \) are firm over \( A, B \) resp. Thus \( A \otimes_A N \xleftarrow{\sim} A^{(2)} \otimes_A N \).

Now \( B \otimes_A M \xleftarrow{\sim} B^{(2)} \otimes A \). (The former seems special, but the latter is true in general as \( B^{(2)} \otimes A \) is a \( B \)-bimodule, in particular an \( A \)-bimodule.

Now we can check compatibility isomorphisms with the adjunction maps in the case of \( B^{(2)} \otimes_A - = B \otimes_A - \)

and \( A^{(2)} \otimes - = A \otimes - \) and the canonical isos

\[
(P \otimes_A -) \circ (Q \otimes_B -) = 1, \quad (Q \otimes_B -) \circ (P \otimes_A -) = 1 .
\]

\[ a \otimes b \otimes a' m \xrightarrow{\sim} v(a) \otimes b u(a') \otimes m \]

\[ A \otimes B \otimes M \xleftarrow{\sim} Q \otimes B \otimes P \otimes M \]

\[ M \xleftarrow{\sim} M \]

\[ a_1 \otimes w(a_2) \otimes q_3 \otimes m \xrightarrow{\sim} v(a_1) \otimes w(a_2) \otimes u(a_2) \otimes q_3 \otimes m \]

\[ a_1 a_2 a_3 \]

\[ a_1 a_2 a_3 \]

\[ v(a_1) u(a_2) u(a_3) = a_1 a_2 a_3 \]
The preceding constructs a canonical isomorphism between the Morita equivalence given by the functors $P \otimes_A - \cong Q \otimes_B -$ and the pair $B^{(2)} \otimes_A - \cong A^{(2)} \otimes_A -$. Better to say we have constructed an isomorphism between the adjoint pair $(B^{(2)} \otimes_A - , A^{(2)} \otimes_A -)$ and the adjoint pair $(P \otimes_A - , Q \otimes_B -)$. As latter is an equivalence, so is the former.

Recall that when $(A, Q)$ is strictly idempotent:

$A = A^2 = Q P$, $P = PA = B P$, $Q = AQ = QB$, $B = B^2 = PQ$, then

we have $P \otimes_A A = B \otimes B P \otimes A = B \otimes B P$ is a finitely

$B A$ bimodule, and for $A \otimes_A Q = Q \otimes_B B$, and we have $Q \otimes_B P = A^{(2)}$, $P \otimes_A Q = B^{(2)}$. For the last point,
note that \( P_A \otimes Q = P_A \otimes A \otimes Q \) as \( P_A = P_A \otimes A = Q \)

and \( P_A \otimes A \) is \( A \)-firm.

Consider next a homomorphism \( A \rightarrow B \) which induces a \( \sum \), so that it factors \( A \rightarrow A/I = \overline{A} \subset B \) where \( AIA = 0 \) and \( \overline{AB} \overline{A} = \overline{A}, \overline{AB}B = B \). This \( \sum \) is the composition of ones belonging to the \( \sum \)-contexts

\[
\begin{pmatrix}
A & A/I_A \\
A/I_A & A/I_A
\end{pmatrix}
\begin{pmatrix}
\overline{A} & \overline{AB} \\
\overline{B} & B
\end{pmatrix}
\]

The functors are \( M \mapsto (B\overline{A} \otimes A/I_A) \otimes_A M \)

\[ N \mapsto (A/I_A \otimes \overline{AB}) \otimes_B N \]

Note that \( B\overline{A} \otimes A/I_A = \overline{A} \otimes A \otimes A/I_A \) is a \((B, A)\)-bimodule, \( \overline{A} \otimes A \)-firm, \( A/B \)-firm

Check:

\[
B\overline{A} \otimes A/I_A = B \otimes \overline{A} \otimes A/I_A
\]

\[
= B \otimes A \otimes A \left/ \text{Im} \left\{ B \otimes I \otimes A + B \otimes A \otimes I_A \right\} \right.
\]

\[
= B \otimes A \left( A/I_A \right) \left/ \text{Im} \left\{ A/I_A \otimes A + A/I_A \otimes B \right\} \right.
\]

\[
= A^{(a)} \otimes A^{(a)} = A^{(a)} \otimes A^{(a)}
\]

Sum \( A/I_A \otimes \overline{AB} = A/I_A \otimes \overline{A} \otimes B \)

\[
= A \otimes A \otimes B \left/ \text{Im} \left\{ A/I_A \otimes A + A/I_A \otimes B \right\} \right.
\]

\[
= A^{(a)} \otimes A^{(a)} = A^{(a)} \otimes B^{(a)}
\]
Here's a new idea for handling whether a ring hom \( w : A \rightarrow B \) induces an equivalence.

The condition that
\[
\beta : \begin{array}{c}
A^{(e)} \\
\downarrow
\end{array} \rightarrow \begin{array}{c}
A^{(e)} \otimes_{A} B^{(e)} \otimes_{A} A^{(e)}
\end{array}
\]
is an isomorphism is equivalent to \( A^{(e)} \rightarrow B^{(e)} \)
being an \( A \otimes A^{op} \)-nil-isom. This is equivalent
to \( A \rightarrow B \) being an \( A \otimes A^{op} \)-nil-isom, because
the kernels of \( A^{(e)} \rightarrow A \), \( B^{(e)} \rightarrow B \) are killed by
\( A \) (resp. \( B \) hence \( A \)) on both sides. Thus

1) \( \beta \) is an isomorphism

2) \( \omega_{i} : M(A) \rightarrow M(B) \) is fully faithful

3) \( A \xrightarrow{\omega} B \) is an \( A \otimes A^{op} \)-nil-isom

4) \[ AIA = 0 \quad \text{and} \quad \overline{ABA} \subset \overline{A} \quad \text{where} \ I = \ker(\omega) \]
and \( \overline{A} = \text{Im}(\omega) \)

are equivalent.

Now suppose this holds. Since \( F = \omega_{i} \), \( G = \omega^{*} \)
are adjoint functors with \( F \) fully faithful, equivalently \( \beta : \begin{array}{c}
I \\
\downarrow
\end{array} \rightarrow GF \), we can invert \( \beta \) and rewrite the adjunction conditions as
\[
\begin{array}{ccc}
FGF & \xrightarrow{\alpha} F \\
\downarrow \beta' & \downarrow \\
F \beta' & \end{array} \quad \text{and} \quad 
\begin{array}{ccc}
GF & \xrightarrow{\beta'} G \\
\downarrow \alpha & \downarrow \beta' \\
GF & \\
\end{array}
\]

This should tell us that \( \begin{pmatrix} 1 & G \\ F & 1 \end{pmatrix} \) is a
Morita context, more precisely that
\[
\begin{pmatrix}
\begin{array}{cc}
A^{(e)} & A^{(e)} \otimes_{A} B^{(e)} \\
\end{array} \\
\begin{array}{cc}
B^{(e)} \otimes_{A} A^{(e)} & B^{(e)} \\
\end{array}
\end{pmatrix}
\]
is a Morita context. All we need to obtain a Mor is the surjectivity
of \((B^2 \otimes A^2) \otimes (A^2 \otimes_A B^2) \to B\),
which amounts to the remaining condition
\(B \overline{A} B = B\).

I think I should have first done the preceding when \(A, B\) are firm rings.
Consider a map of Morita contexts
\[
(A \quad Q) \xrightarrow{(1 \quad v)} (A' \quad Q') \quad A = A^2 = Q^P \\
(P \quad B) \xrightarrow{(w \quad 1)} (P' \quad B') \quad B = B^2 = P^Q
\]
similarly with 's.

To prove that \( v \) is a Morita equivalence homomorphism and that the triangle of equivalences

\[
\begin{align*}
M(A) & \xrightarrow{Q \otimes B} M(B) \\
& \quad \xleftarrow{P \otimes A} \quad M(A) \\
& \quad \xrightarrow{M(B)} \quad M(B') \\
& \quad \xleftarrow{M(B')} \quad M(A)
\end{align*}
\]

commutes up to canonical isomorphism. One reason for your difficulties with this is the number of canonical isomorphisms you can write down. Going between two different categories yields 6 isomorphisms:

1) \( P' \otimes_A Q \otimes_B N \sim B' \otimes_B N \)
2) \( B' \otimes_B P \otimes_A M \sim P' \otimes_A M \)
3) \( Q' \otimes_B B' \otimes_B N \sim Q \otimes_B N \)
4) \( P \otimes_A Q' \otimes_B N' \sim B^{(2)} \otimes_B N' \)
5) \( B^{(2)} \otimes_B P \otimes_A M \sim P \otimes_A M \)
6) \( Q \otimes_B B^{(2)} \otimes_B N' \sim Q' \otimes_B N' \)

On the other hand going from a category to itself either clockwise or counterclockwise leads to 6 isomorphisms.
Another reason for difficulties is the fact
That you've relied on the naive transformations involving the Morita
equivalences with $P, Q$ and $P', Q'$. Thus
1) - 3) concerning $w_i = B'^{B^i}$ are equivalent
mainly, and similarly for 4) - 6) concerning $w^*$. But a priori $w_i$ and $w^*$ are only adjoint, not
inverse, so you must bring in adjunction
considerations to get between these groups, e.g. 1) is
equivalent to 4) by
\[
\text{Hom}_B (B' \otimes_B N, N') = \text{Hom}_B (N, B^{(2)} \otimes_B N')
\]
\[
\text{Hom}_B (P' \otimes_A Q \otimes_B N, N') = \text{Hom}_B (N, P \otimes_A Q' \otimes_B N')
\]
You might first prove 1) then deduce 4)
by adjunction, but if you give a formula for 1),
then it's probably simpler to also give a formula
for 4) and check compatibility with adjunction maps
than to compute the isomorphism 4) corresponding to 1).

Start this.
Claim \( v : Q \to Q' \) is a \( B^0 \)-nil-isom:

\( v(p) = 0 \implies g(p) = 0 \) \( g(p_i) = v(g) \) \( u(p_i) = 0 \)

\( g' w(p, g_i) = (g' u(p_i)) g_i = v((g' u(p_i)) g_i) \leq v(Q) \)

If \( N \) is \( B \)-form, then

\( Q \otimes_B N \xrightarrow{v} Q' \otimes_B N \)

\( g \otimes v \xrightarrow{\theta} v(g) \otimes n \)

\( (g' u(p) \theta \otimes n \xleftarrow{1} g' \otimes p q \theta n \)
Thus 

\[(P \otimes A Q) \otimes_B N \cong B \otimes_B N\]

\[p' \otimes g \otimes n \mapsto p' \circ (g \otimes n)\]

\[b_2 \circ (p) \otimes g \otimes n \mapsto b_2 \circ (p) g \otimes n\]

maps well-defined.

C: \[b_2 \circ (p) g \otimes n = b \circ (p g) \otimes n = b \circ (p g)\]

C: \[p' \circ (g \otimes n) \mapsto p' \circ (g \otimes n) \mapsto p' \circ (g \otimes n) = p' \circ (g \otimes n) = p' \circ (g \otimes n)\]

Next, \[\nu : P \rightarrow P'\] is \(B\)-nil-isomorphism.

\[B^{(2)} \otimes_B P \rightarrow B^{(2)} \otimes_B P'\]

If \(N'\) is \(B\)-form, then

\[B^{(2)} \otimes_B (P \otimes A Q) \otimes_B N' \rightarrow B^{(2)} \otimes_B (P' \otimes A Q) \otimes_B N'\]

maps well-defined.

Use \[B \otimes_B P \otimes_A Q \rightarrow B^{(2)} \otimes_B = B^{(2)}\]

C: \[b_2 \circ u(p) g \otimes n' \mapsto b_2 \circ u(p) g \otimes n' = b_2 \circ u(p) g \otimes n'\]

C: \[b_2 \circ p \otimes g \otimes n' \mapsto b_2 \circ p \otimes g \otimes n' = b_2 \circ p \otimes g \otimes n' = b_2 \circ p \otimes g \otimes n'\]

Next, check compatibility with the adjunction.
Record some additional formulas

\[ B' \odot_B P \odot_A M \sim p' \odot_A M \]
\[ b' \odot p \odot m \Rightarrow b' u(p) \odot m \]
\[ p' v(q) \odot p \odot m \Leftarrow p' \odot g p m \]

\[ Q \odot B^{(2)} \odot_B N' \sim Q' \odot_B N' \]
\[ q \odot b_1 \odot b_2 \odot n' \Rightarrow v(q b_1 b_2) \odot n' \]
\[ q \odot b_1 \odot b_2 \odot u(p) \odot n' \Leftarrow g b_1 b_2 p g' \odot n' \]

correspond under adjunction (i.e. are transpose)
Let's discuss the problem of defining iterated derived tensor products of bimodules. Let $A, B, C$ be (unital) rings, let $A \otimes_B C$ be (unital) $A$-bimodules. We wish to define $U \otimes_B V$ as a completion of $(A, C)$-bimodules determined up to quasi-isomorphism. It should also be functorial as $U, V$ range over the appropriate derived categories.

The obvious thing to do is to set $U \otimes_B V = \hat{U} \otimes_B \hat{V}$ where $\hat{U}$ (resp. $\hat{V}$) is a projective $(A, B)$-bimodule (resp. $(A, C)$-bimodule) resolution of $U$ (resp. $V$). These resolutions are defined and functorial up to homotopy, so you get a bifunctor on the derived categories. The same construction should be possible for iterated derived tensor products, even circular ones.

The problem with this is that the homology of this $U \otimes_B V$ might be different if the $A \otimes C$ structures were ignored. You would want to know that $\hat{U}$ is flat over $B$ or $\hat{V}$ is flat over $B$. You want there to be enough $B$-flat $A \otimes_B C$-bimodules which are flat over $B$ to be flat over $B$. Sufficient for this is for $A$ to be flat over the ground ring $k$. Similarly if $C$ is flat over $k$, then $B \otimes k C$ is $B$-flat.

Another thing you would like is associativity (which should be related to the composite functor condition). Suppose given $W$ and ask whether $U \otimes_B V \otimes_C W = U \otimes_B (V \otimes_C W) = (U \otimes_B V) \otimes_C W$.
You want \( \hat{\otimes}_B \) to respect guys, or a more sufficient condition would be for \( \hat{\otimes} \) to "good" for \( \hat{\otimes}_B \). We know \( \hat{\otimes} \) is made up of \( (B \otimes C) \otimes (C \otimes D) \).

Thus if \( C \) is \( k \)-flat, then \( \hat{\otimes} \) is a flat \((B, D)\)-bimodule complex.

**Summary:**
1. Given bimodule complexes \( A_B, B_C \), let \( \hat{U} \to U, \hat{V} \to V \) be flat bimodule resolutions. Then:
   \[ U \otimes_B V \equiv \hat{U} \otimes \hat{V} \]
   is a flat \((A, C)\)-bimodule complex if \( A \otimes B \otimes C \) is flat over \( A \otimes C \).

   *e.g.\( B \) is \( k \)-flat \( \Rightarrow \hat{U} \otimes \hat{V} \)

   *\( B \otimes C \) is \( B \)-flat, e.g. \( A \) is \( k \)-flat \( \Rightarrow \hat{U} \otimes \hat{V} \)

   *\( B \otimes C \) is \( B \)-flat, e.g. \( C \) is \( k \)-flat \( \Rightarrow \hat{U} \otimes \hat{V} \)

2. Given \( M_A, A_B, B_N \) complexes, then
   \[ M \otimes A_B \otimes N \equiv M \otimes A \otimes N \]
   provided \( \hat{M} \otimes \hat{N} \cong M \otimes N \), e.g. either \( A \otimes B \)

   *\( k \)-flat and \( M \otimes N \equiv M \otimes N \). *can be weakened to \( A \otimes B \equiv A \otimes B \)

3. \[ M \otimes_A \hat{\otimes}_A = \hat{M} \otimes_A \hat{\otimes}_A = \hat{M} \otimes_A \hat{\otimes}_A = \hat{M} \otimes_A \hat{\otimes}_A \]
   is true without flatness assumptions.
More adjoint functor stuff. Let
\((F, \alpha, \beta) \quad (F', \alpha', \beta')\)

be two pairs of adjoint functors between the same two categories. Consider \(\Theta : F \to F'.\)

\[
\begin{array}{c}
\text{Hom}(F'X, Y) \xrightarrow{\Theta^*} \text{Hom}(FX, Y) \\
\text{Hom}(X, G'Y) \xrightarrow{\Theta^*_X} \text{Hom}(X, GY)
\end{array}
\]

This diagram shows, thanks to Yoneda’s lemma, that there is a induced map \(\Theta^* : G' \to G\) which is called the transpose of \(\Theta\). Clearly we have
\[
\text{Hom}(F, F') = \text{Hom}(G', G) \quad \Theta \mapsto \Theta^*
\]
\(\Theta\) isom. \(\iff\) \(\Theta^*\) isom. and \((\Theta^*)^t = (\Theta^*)^{-1}\).

One has the following commutative squares:
\[
\begin{array}{ccc}
FG'Y & \xrightarrow{F \cdot \Theta^t} & FGY \\
\Theta \cdot G' & \downarrow & \downarrow G \cdot \Theta \\
F'GY & \xrightarrow{\alpha'} & Y
\end{array}
\]

\[
\begin{array}{ccc}
FX & \xrightarrow{\beta'} & GFX \\
\Theta^t \cdot F' & \downarrow & \downarrow \Theta^t \cdot F
\end{array}
\]

obtained from 1).

Consequently the isomorphisms \((\Theta, (\Theta^*)^t) : (F, \alpha) \to (F', \alpha')\)
(assuming \(\Theta\) is an isom.) are compatible with the adjunction maps, i.e.
\[
\begin{array}{ccc}
F'GY & \xrightarrow{\Theta \cdot (\Theta^*)^t} & FGY \\
\alpha' \downarrow & \downarrow & \downarrow \alpha
\end{array}
\]
\[
\begin{array}{ccc}
FX & \xrightarrow{\beta' \cdot (\Theta^*)^t} & G'FX \\
(\Theta^t)^* \cdot \Theta & \downarrow & \downarrow (\Theta^t)^* \cdot \Theta
\end{array}
\]

Commutes.
Let $F : X \to Y$ be an equivalence of categories, i.e., a fully faithful and essentially surjective functor. Then there is a quasi-inverse $(G, \varepsilon, \eta)$ for $F$, which is unique up to canonical isomorphism and obtained as follows.

$F$ essentially surjective means we can choose for each $Y$ a $G Y$ in $X$ together with an isomorphism $\varepsilon_Y : FG(Y) \to Y$. Because $F$ is fully faithful we can define $G$ uniquely on morphisms in $Y$ such that $G$ becomes a functor and $\varepsilon : FG \to 1$ is an isomorphism. Then we have $\varepsilon_F : FGF \to F$, so again as $F$ is fully faithful, there is a unique isomorphism $\eta : GF \to 1$ such that $\eta . G = G . \varepsilon$ holds.

Proof. By definition, given $v : Y \to Y'$, then $G(v) : G(Y) \to G(Y')$ is the unique map such that

$$FG(Y) \xrightarrow{F(G(v))} FG(Y')$$

$$\varepsilon_Y \downarrow \quad \xrightarrow{\eta_{G(v)}} \quad \varepsilon_{Y'}$$

$$Y \xrightarrow{v} Y'$$

commutes. Thus $G(\varepsilon_Y)$ is unique such that

$$FGFY \xrightarrow{FG(G(\varepsilon_Y))} FGY$$

$$\varepsilon_{FGY} \downarrow \quad \quad \varepsilon_Y$$

$$FGY \xrightarrow{\varepsilon_Y} Y$$

commutes and as $\varepsilon_Y$ is an isomorphism, this means $G(\varepsilon_Y)$ is unique such that $FG(\varepsilon_Y) = \varepsilon_{FGY}$. But $\eta_Y : GFX \to X$ by definition is unique such that $F(\eta_Y) = \varepsilon_{FX}$. Taking $X = GY$, we find $\eta_{GY} = G(\varepsilon_Y)$ i.e., $\eta . G = G . \varepsilon$. 


Notation: Given $\xi: F \to F'$, $\xi': G \to G'$ of functors which can be composed we write $\xi \circ \xi'$ for the induced map on compositions.

$$
\begin{array}{c}
FG \\ \xi \circ \xi'
\end{array} 
\xrightarrow{\xi \circ \xi'} 
\begin{array}{c}
FG' \\
\xi \circ \xi'
\end{array}
$$

Also we write $F \cdot \xi$ instead of $1_F \circ \xi$. (Maybe $+ \circ$ is a traditional notation).

Uniqueness of quasi-inverse. Let $(\eta, \varepsilon, \gamma), (\eta', \varepsilon', \gamma')$ be two quasi-inverses for $F$.

Then we have

$$
F \cdot \eta = \varepsilon \cdot F : FGFG \to F \\
G \cdot \varepsilon = \eta \cdot G : GFG \to G \\
F \cdot \gamma' = \varepsilon \cdot F : FG'G \to G \\
G \cdot \eta' = \gamma \cdot G' : G'FG \to G'
$$

Now define $\xi: G \to G'$ by either

1) $G \xleftarrow{\eta \cdot G} GFG' \xrightarrow{\eta' \cdot G'} G'$

1') $G \xleftarrow{\eta' \cdot G} G'FG \xrightarrow{\varepsilon' \cdot G'} G'$

Let's check 1') is compatible with $\varepsilon$-maps:

$$
\begin{array}{c}
FG \\ \varepsilon
\end{array} 
\xleftarrow{\varepsilon \cdot \varepsilon'} 
\begin{array}{c}
FGFG' \\
\varepsilon
\end{array} 
\xrightarrow{\varepsilon \cdot \varepsilon'} 
\begin{array}{c}
FG' \\
\varepsilon'
\end{array} 
\xrightarrow{1} 
\begin{array}{c}
1 \\
1
\end{array}
$$

and with $\gamma$-maps:

$$
\begin{array}{c}
GF \xleftarrow{\gamma \cdot \varepsilon' \cdot F} GFGF \xrightarrow{\gamma' \cdot G} G'F
\end{array} 
\xrightarrow{\gamma \cdot \gamma'} 
\begin{array}{c}
GF \xleftarrow{\gamma \cdot \gamma'} G'F \xrightarrow{\gamma' \cdot G} G'F
\end{array} 
\xrightarrow{1} 
\begin{array}{c}
1 \\
1
\end{array}
$$
Next show 1) = 1') by applying $F$ to 1'):

$$
\begin{align*}
  & FG \xrightarrow{\eta \cdot G} FGFG \xrightarrow{F \cdot \varepsilon} FG' \\
  & \varepsilon \downarrow \quad \varepsilon' \downarrow \quad \varepsilon' \downarrow \\
  & 1 = 1 = 1
\end{align*}
$$

Thus $F$ applied to 1) and 1') yield the same map, namely $(\varepsilon')^\dagger$, so 1) = 1') as $F$ is fully faithful.

Jan. 4, 1996

Let $(G, \varepsilon, \eta)$ and $(G', \varepsilon', \eta')$ be quasi-inverses for $F$:

$$
\begin{align*}
  & \varepsilon : FG \rightarrow 1 \quad \varepsilon : FG \rightarrow 1 \\
  & \eta : GF \rightarrow 1 \quad \eta : GF \rightarrow 1 \\
  & \varepsilon' : FG' \rightarrow 1 \quad \varepsilon' : FG' \rightarrow 1 \\
  & \eta' : G'F \rightarrow 1 \quad \eta' : G'F \rightarrow 1
\end{align*}
$$

$G \cdot \varepsilon = \eta \cdot G : GF \rightarrow G$

Because $F$ is fully faithful, $\exists \phi : G \rightarrow G'$ such that

$$
\begin{align*}
  & FG \xrightarrow{F \cdot \phi} FG' \\
  & \varepsilon \downarrow \quad \varepsilon' \downarrow \quad \varepsilon' \downarrow \\
  & 1 = 1
\end{align*}
$$

Then

$$
\begin{align*}
  & FGF \xrightarrow{F \cdot \phi \cdot F} FG'F \\
  & \varepsilon \cdot F = F \cdot \eta' \Rightarrow \varepsilon' \cdot F = F \cdot \eta' \\
  & F = F \text{ comm.}
\end{align*}
$$

Hence we have $\phi : (G, \varepsilon, \eta) \rightarrow (G', \varepsilon', \eta')$. To get a formula for $\phi$ apply $G$:

$$
\begin{align*}
  & G \xleftarrow{G \cdot \varepsilon} GFG \xrightarrow{GF \cdot \phi} GF \quad G \xrightarrow{G \cdot \varepsilon'} G' \\
  & \eta \cdot G \downarrow \quad \eta' \cdot G' \downarrow \\
  & G \rightarrow G' \quad G \rightarrow G'
\end{align*}
$$

$$
\begin{align*}
  & \text{hence} \\
  & G \cdot \varepsilon \xrightarrow{G \cdot \varepsilon} GFG \xrightarrow{GF \cdot \phi} GF \xrightarrow{G \cdot \varepsilon'} G
\end{align*}
$$
which is 1) on p85. Similarly

\[ \begin{array}{cccccc}
G & \xrightarrow{\eta} & GFG & \xrightarrow{\beta} & GFG & \xrightarrow{\gamma} & G \\
\downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \gamma \\
G & \xrightarrow{\varepsilon} & G' & & G' & & G'
\end{array} \]

which is 1) on p85.

Next I want to show the essential uniqueness of a quasi-inverse that given \((F, G, \varepsilon, \eta)\) and \(\Theta : F \rightarrow F'\) there is a corresponding \(\tilde{\eta} : G \rightarrow G'\) such that \(\tilde{\eta}(\Theta, \varepsilon')\) is an isomorphism \((F, G, \varepsilon, \eta) \sim (F', G', \varepsilon', \eta')\).

\[ \begin{array}{cccccc}
FG & \xrightarrow{\Theta} & F'G' & \xrightarrow{\tilde{\eta}} & G'F' \\
\varepsilon' \downarrow & & \varepsilon' \downarrow & & \varepsilon' \downarrow \\
1 & & 1 & & 1
\end{array} \]

commute

in other words we have \(\varepsilon = \varepsilon' (\Theta, \varepsilon)\) and \(\eta = \eta' (\tilde{\eta}, \Theta)\).

The idea is that \(\Theta : F \rightarrow F'\) makes \(G'\) into a quasi-inverse for \(F\). More precisely we have an iso:

\[ \begin{array}{cccccc}
(\Theta, 1) : (F, G, \varepsilon, \eta) & \sim & (F', G', \varepsilon', \eta') \\
\end{array} \]

Then we know there is a \(\tilde{\eta} : G \rightarrow G'\) such that

\[ \begin{array}{cccccc}
(1, \tilde{\eta}) : (F, G, \varepsilon, \eta) & \sim & (F', G', \varepsilon', \eta') \\
\end{array} \]

i.e. \(\varepsilon = \varepsilon' (\Theta, G')(\tilde{\eta}, \varepsilon)\) and \(\eta = \eta' (\tilde{\eta}, \Theta)(\tilde{\eta}, F)\)

The first equation says:

\[ \begin{array}{cccccc}
FG & \xrightarrow{\Theta} & F'G' & \xrightarrow{\tilde{\eta}} & G'F' \\
\varepsilon \downarrow & & \varepsilon' \downarrow & & \varepsilon' \downarrow \\
1 & = & 1 & & 1
\end{array} \]

This determines \(\tilde{\eta}\) since the other maps are isomorphisms and \(F\) is fully faithful.
March 12, 1996

I want to record formulas involved in the equivalences:

\[ \text{mod (R)} \xrightarrow{\text{extn}} \text{mod (H)} \]
\[ \downarrow \text{HS} \]
\[ \text{mod (C[S])} \xrightarrow{\text{HS}} \text{mod (C[S])} \]

First consider the adjoint functor relations

\[ H \otimes V, W = \text{Hom}(V, H^* \otimes W) \]
\[ \text{Hom}(V, H \otimes W) = \text{Hom}(H^* \otimes V, W) \]

where \( H \) is a finite-diml vector space. The adjunction maps in the former arise from the canonical maps

\[ \alpha : H \otimes H^* \rightarrow \mathbb{C} \]
\[ h \otimes h^* \mapsto (h(h^*)) = (h^*h) \]
\[ \beta : \mathbb{C} \rightarrow H^* \otimes H \]
\[ 1 \mapsto \sum e_i ^* \otimes e_i \]

The adjunction maps in the latter arise from the canonical maps

\[ \alpha' : H^* \otimes H \rightarrow \mathbb{C} \]
\[ h^* \otimes h \mapsto (h^*h) \]
\[ \beta' : \mathbb{C} \rightarrow H \otimes H^* \]
\[ 1 \mapsto \sum e_i \otimes e_i ^* \]

obtained from the preceding via the `flips`. Notice that \( \alpha' \beta = \alpha \beta' \) is \( \text{tr}(1) = \dim H \).

In the \( \mathbb{R}, H \) situation, \( H \otimes \mathbb{R}^2 \) comes equipped with a volume \( \Lambda^2 H = \mathbb{C} \) which we use to identify \( H^* \) and \( H \). We have a single adjoint functor relation

\[ \text{Hom}(H \otimes V, W) = \text{Hom}(V, H \otimes W) \]

arising from the canonical maps.
\[ \alpha : H \otimes H \rightarrow C \quad \iota \otimes \iota \rightarrow h_1 h_2 \]

\[ \beta : C \rightarrow H \otimes H \quad 1 \rightarrow e_2 \otimes e_1 - e_1 \otimes e_2 \quad \text{(here } e_1 e_2 = 1) \]

Check: Let \( h = e_1 e_2 \) so that \( e_2 = e_1 h, e_1 = -e_2 h \)

\[ H \xrightarrow{\beta} H \otimes H \xrightarrow{\iota \otimes \iota} H \]

\[ h \mapsto (e_2 \otimes e_1 - e_1 \otimes e_2) \otimes h \mapsto e_2 (e_1 h) - e_1 (e_2 h) = h \]

\[ H \xrightarrow{\iota \otimes \iota} H \otimes H \xrightarrow{\alpha} H \]

\[ h \mapsto h \otimes (e_2 \otimes e_1 - e_1 \otimes e_2) \mapsto (h e_2) e_1 - (h e_1) e_2 = h \]

Notice that \( C \xrightarrow{\beta} H \otimes H \xrightarrow{\iota \otimes \iota} C \) is

\[ 1 \mapsto e_2 \otimes e_1 - e_1 \otimes e_2 = -2(e_1 e_2) = -2. \]

I don't really understand this sign. Somehow it arises from the fact that the volume \( \Lambda^2 H = C \) determines two isos of \( H \) with \( H^* \), i.e. \( \Lambda^2 H \subset H \otimes H \) and you can contract either factor of \( H \) with \( H^* \); these two isos have opposite sign. Perhaps also this sign is related to what happens with the Fourier transform.

Next recall \( C[\sigma]_+^+ = C + C_0 \) where \( \sigma z = z \sigma \)

and \( \sigma^2 = \pm 1 \). \( C[\sigma]_- = H \) where \( \sigma = j \), so \( \sigma \)

\[ \text{mod}(H) = \text{mod}(C[\sigma]_-) \text{ trivially.} \]

\( C[\sigma]_+ = M_2 \mathbb{R} \) so \( \text{mod}( \mathbb{R} ) = \text{mod}( C[\sigma]_+ ) \) is a Morita equivalence, the functors being \( V \mapsto C \otimes V \), \( V \mapsto V^\sigma \).

Now take \( H = H = C + C_j \)

with \( C \) acting by left multiplication and \( \Lambda^2 H = C \) given by \( 1 \otimes j = 1 \). \( \sigma \) in \( H \) is

is left mult by \( j \). (Reason for notation \( H \) is

to avoid confusion arising from \( H \otimes V \) when \( C \)
is left acting in \( H \).)

If \( V \in \text{mod}(C[\sigma]_-) \), then \( H \otimes V \) equipped

with \( \sigma \otimes \sigma \) is in \( \text{mod}(C[\sigma]_+) \). Conversely \( W \in \text{mod}(C[\sigma]_+) \)

\[ \Rightarrow H \otimes W \in \text{mod}(C[\sigma]_-). \]
Recall that restriction of scalars has both left and right adjoints. In the case of $RCC$ these two adjoints are isomorphic:

$$\text{Hom}_R(H, V^\sigma) \cong \text{Hom}_R(H, R) \otimes_{R^*} V^\sigma \cong H \otimes V^\sigma$$

where any nonzero element of $H^*$ yields an isomorphism. In practice one takes a trace $\tau: H \to R$ which is unique up to a scalar, since $H = R \oplus [H, H]$.

We use the following isomorphism:

$$H \otimes V^\sigma \cong H \otimes V$$

$$\left(\sum_i z_i \otimes \sigma_i\right) \otimes \sigma \mapsto \sum_i \sigma_i \otimes z_i \sigma + g \otimes \sigma = \left(\sum_i \sigma_i \otimes z_i \sigma\right)$$

to link the left adjoint (extension of scalars) from $\text{mod}(R)$ to $\text{mod}(H)$ with $V \mapsto H \otimes V$.

Then we have isos:

$$\text{Hom}_R(H \otimes V, W^\sigma) \cong \text{Hom}_R(V, H \otimes W)^\sigma$$

$$\text{Hom}_R(H \otimes V, W) \cong \text{Hom}_R(V, H \otimes W)^\sigma$$

$$\text{Hom}_R(H \otimes V^\sigma, W) \cong \text{Hom}_R(V^\sigma, H \otimes W)^\sigma$$

By Yoneda this yields a canonical isomorphism:

$$W \cong (H \otimes W)^\sigma$$

$$w \mapsto j \otimes w - 1 \otimes jw$$

where $j$ stands for res$^H_R$. This identifies $j$ with $H \otimes -$ from $\text{mod}(C[0,1])$ to $\text{mod}(C[0,1]^+)$. 
Next we have

\[ \text{Hom}(H \otimes W, V) = \text{Hom}(W, H \otimes V) \]

\[ \text{Hom}_{\mathbb{R}}((H \otimes W)^{\sigma}, V) = \text{Hom}_{\mathbb{H}}(W, \mathbb{H} \otimes V) \]

\[ \text{Hom}_{\mathbb{R}}(\mathfrak{p} W, V^{\sigma}) = \text{Hom}_{\mathbb{H}}(W, \mathbb{H} \otimes V^{\sigma}) \]

\[ \text{Hom}_{\mathbb{H}}(W, \text{Hom}_{\mathbb{R}}(H, V^{\sigma})) \]

Thus we get a canonical isomorphism:

\[ \text{Hom}_{\mathbb{R}}(\mathfrak{p} W, V^{\sigma}) \xrightarrow{\sim} \text{Hom}_{\mathbb{R}}((H \otimes V)^{\sigma}) \]

which amounts to an element \( \tau \) of \( H^1 \), (take \( V = \mathbb{R} \)).

Calculation gives \( \tau(1) = -2 \), \( \tau(i) = \tau(j) = \tau(k) = 0 \).
The six term exact sequence of kernels and cokernels becomes

\[ 0 \rightarrow \text{Ker}(ba) \rightarrow \text{Ker}(b) \rightarrow \text{Coker}(a) \rightarrow \text{Coker}(ba) \rightarrow 0 \]

so that it looks as if the complexes \( W \xrightarrow{b} \ell_2 \otimes V \) and \( \ell_2 \otimes V \xrightarrow{ba'} Q \) are quasi-isomorphic.

When you have more time examine this carefully. Recall something similar appeared in connection with Vaserstein's lemma, more specifically, when you proved Morita invariance for \( K^1 \).
Canonical resolutions over $\mathbb{P}^1$. If $\mathcal{F}$ is a regular sheaf over $\mathbb{P}^1$ then it has a resolution of the form

$$0 \to \mathcal{O}(1) \otimes W \to \mathcal{O} \otimes V \to \mathcal{F} \to 0$$

Tensor this short exact sequence with

$$0 \to \Lambda^2 \mathcal{O}(1) \otimes \mathcal{O}(1) \to \mathcal{O} \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

to get

$$0 \to \Lambda^2 \mathcal{O}(2) \otimes W \to \Lambda^2 \mathcal{O}(1) \otimes V \to \Lambda^2 \mathcal{O}(1) \otimes \mathcal{F} \to 0$$

$$0 \to \mathcal{O} \otimes \mathcal{O}(1) \otimes W \to \mathcal{O} \otimes \mathcal{O} \otimes V \to \mathcal{O} \otimes \mathcal{F} \to 0$$

$$0 \to \mathcal{O} \otimes W \to \mathcal{O}(1) \otimes V \to \mathcal{F}(1) \to 0$$

$$0 \to 0 \to 0$$

where

$$\Lambda^2 \mathcal{O} \otimes H^0(\mathcal{F}(1))$$

$$H \otimes V \twoheadrightarrow H \otimes H^0(\mathcal{F})$$

$$0 \to W \to H \otimes V \to H^0(\mathcal{F}(1)) \to 0$$

which identifies $W \to H \otimes V$ with the map

$$\Lambda^2 \mathcal{O} \otimes H^0(\mathcal{F}(1)) \to H \otimes H^0(\mathcal{F})$$

induced by $\Lambda^2 \mathcal{O}(1) \otimes \mathcal{O}(1)$. Next suppose $\mathcal{G}$ is a negative vector bundle. Then it has a dual canonical resolution of the form...
Again we get by tensoring

\[
0 \rightarrow G \rightarrow H_{(-1)} \otimes W \rightarrow O \otimes V \rightarrow 0
\]

where

\[
\begin{align*}
H^0(G(1)) & \rightarrow \\
\Lambda^2 H \otimes H^1(G(1)) & \rightarrow \Lambda^2 H \otimes H^0(G(-1)) \otimes W \rightarrow H^0(\Lambda^2 H \otimes G(-2) \otimes W) \rightarrow 0
\end{align*}
\]

\[
\begin{align*}
H \otimes V & \rightarrow \Lambda^2 H \otimes O \otimes V \rightarrow O \otimes V \rightarrow 0
\end{align*}
\]

\[
\begin{align*}
H^0(G(1)) & \rightarrow \\
\Lambda^2 H \otimes H^1(G(1)) & \rightarrow \Lambda^2 H \otimes H^0(G) \rightarrow H^1(H \otimes V) \rightarrow 0
\end{align*}
\]

The problem is now to identify the map \(W \rightarrow H \otimes V\) arising from this diagram with the map \(H^0\) induced by \((O(-1)) \otimes W \rightarrow O \otimes V\) tensored with \(O(1)\).
We will construct various maps of complexes linked by $R^\ast$-use. First map

$$
\Lambda^2 H \otimes G(-1) \otimes W \overset{[1]}{\longrightarrow} \Lambda^2 H \otimes \Omega(-2) \otimes W \overset{[1]}{\longrightarrow} H \otimes G \overset{[1]}{\longrightarrow} (H \otimes \Omega(-1) \otimes W \longrightarrow \Omega(1) \otimes V)
$$

where $\longrightarrow$ is essentially obtained from the first two rows of the $3 \times 3$ diagram above. Second map

$$
\Lambda^2 H \otimes \Omega(-2) \otimes W \overset{[1]}{\longrightarrow} (H \otimes \Omega(-1) \otimes W \longrightarrow \Omega(1) \otimes V)
$$

Third map is inclusion

$$
(H \otimes \Omega(-1) \otimes W \longrightarrow \Omega(1) \otimes V) \quad \Omega(1) \otimes V
$$

One can check that the dotted arrows induce isos on $R^\ast$ for both source and target of the vertical arrow. So applying $R^\ast$ we get a commutative square

$$
\Lambda^2 H \otimes H^1(G(-1)) \overset{\sim}{\longrightarrow} W \overset{\sim}{\longrightarrow} H \otimes H^1(G) \overset{\sim}{\longrightarrow} H \otimes V
$$

as desired.
Consider the problem of Morita invariance of K-theory for b-unital rings, but restrict one of the rings to be unital. Suppose then A is unital and (P, Q) is a form dual pair over A. B = P ⊗_A Q is b-unital iff \( P^\perp \otimes_A Q = P \otimes_A Q \), e.g., if either P or Q is flat over A. Then Q is an inductive limit of fg free modules, and similarly for P.

Note that surjectivity of \( Q \otimes R \to A \) means there exist \( \sum_i b_i p_i = 1 \). In this case, replacing (P, Q) by \((P, Q)^n\) and B by \( M_n B \), we reduce to the case where \( \exists p \in P, q \in Q \) with \( qp = 1 \). Then \( (P, Q) = (A, A) \oplus (X, Y) \), where \( X = \{ p \in P \mid xp = 0 \} \), \( Y = \{ q \in Q \mid qp = 0 \} \). So \( B = (A \oplus Y, X) \) and the pairing \( Y \otimes X \to A \) can be arbitrary. Also \( e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) is an idempotent in \( B \) such that \( A = eBe \), \( P = Be \), \( Q = eB \), so we have the familiar Morita context

\[
\begin{pmatrix}
A = eBe & eB \\
Be & BeB = B
\end{pmatrix}
\]

as examined in connection with Dadelev's thesis.

Let's review this result. Start with \( R, e = e^2 \in R \), \( A = eRe \), \( P = Re \), \( Q = eR \), \( B = ReR \). Hypothetical are:

\( \text{Re} \otimes eR \to B \) (i.e., B form) and \( eR \in \text{F}(A) \). We have functors

\[
\begin{align*}
\text{mod}(R) & \to \text{mod}(A) & \to M(B) & \subset \text{mod}(R) \\
L & \mapsto eL & N & \mapsto N & M & \mapsto Re \otimes_A M
\end{align*}
\]
which induce

\[ P(R) \rightarrow P(A) \rightarrow P(B) \subset P(R) \]

Here \( P(B) \cong P(R, B) \) is the full subcategory of small projectives in \( M(R, B) \), i.e. \( L \in P(R) \) such that \( L = BL \).

We have some obvious maps

\[
\begin{align*}
K_\ast(P(B)) & \rightarrow K_\ast(P(R)) \rightarrow K_\ast(P(R/B)) \\
\downarrow & \downarrow \\
K_\ast(P(A)) & 
\end{align*}
\]

Now \( B = R e \otimes_A e R \), \( e R \in P(A) \rightarrow B \in P(B) \), so we have a resolution by fg projective modules

\[ 0 \rightarrow B \rightarrow R \rightarrow R/B \rightarrow 0. \]

This should imply that any object in \( P(B) \) has a \( 0 \rightarrow P_1 \rightarrow P_0 \rightarrow V \rightarrow 0 \) with \( P_i \in P(R) \). Hence by resolution we get a map \( K_\ast(P(R/B)) \rightarrow K_\ast(P(R)) \).

Claim \( K_\ast(P(R/B)) \rightarrow K_\ast(P(R)) \rightarrow K_\ast(P(R/B)) \) is the identity. Given \( 0 \rightarrow P_i \rightarrow P_0 \rightarrow V \rightarrow 0 \) for proj h-res. of \( V \in P(R/B) \) one has

\[ 0 \rightarrow \text{Tor}_1^R(R/B, V) \rightarrow P_1/BP_1 \rightarrow P_0/BP_0 \rightarrow V \rightarrow 0 \]

This \( \text{Tor} = 0 \) since \( V \) is a summand of \((R/B)^n\) and \( \text{Tor}_1^R(R/B, R/B) = B/B^2 = 0 \).

At this point we know \( K_\ast(P/B) \) and \( K_\ast(P(R/B)) \) are direct summands of \( K_\ast(P(R)) \). Consider the exact sequence of functors
$0 \rightarrow B \otimes L \rightarrow L \rightarrow L/B \rightarrow 0$

defines $\mathcal{P}(R)$ to $\mathcal{P}'(R)$ ($=$ modules admitting length 1 resolutions from $\mathcal{P}(R)$). By additivity and $K_\ast(\mathcal{P}(R)) \rightarrow K_\ast(\mathcal{P}'(R))$, we get that

$K_\ast(R) \rightarrow K_\ast(\mathcal{P}(B)) \oplus K_\ast(R/B) \rightarrow K_\ast(R)$

is the identity. It follows that

$K_\ast(\mathcal{P}(B)) \oplus K_\ast(R/B) \rightarrow K_\ast(R)$

"$K_\ast(A)$"

I think this is correct. When $R = \mathfrak{B}$ we then get $K_\ast(A) = K_\ast(\mathcal{P}(B)) \rightarrow K_\ast(B)$ ($\Rightarrow K_\ast(B)/K_\ast(\mathfrak{Z})$), which is the Morita invariance result I am after.

Let's now return to the original setting $(\begin{array}{c} A \\ P \\ Q \\ B \end{array})$ with $A$ central, $\mathbb{Q} \otimes P \rightarrow A$, $P \otimes Q = B$, and suppose $Q \in \mathcal{P}(A)$. Now $Q$ is a generator for $\text{mod}(A)$ since we have $Q \otimes P \rightarrow A$, so without affecting the Morita invariance question we should be able to replace $A$ by the map central ring $A' = \text{Hom}_A(Q, Q)$. We have to compose the maps given by

$$\begin{pmatrix} A' & Q^* \\ Q & A \end{pmatrix} \begin{pmatrix} A & Q \\ P & B \end{pmatrix}$$

where $Q^* = \text{Hom}_A(Q, A)$.

$$\begin{pmatrix} A' & Q^* & G \otimes Q \\ Q & A & Q \\ P \otimes Q & P & B \end{pmatrix} \rightarrow \begin{pmatrix} A' & Q^* \otimes Q = A' \\ P \otimes Q = B & B \end{pmatrix}$$
This transformation reduces us to a Monticelli context (put A for A') of the form

\[
\begin{pmatrix}
A & Q=A \\
B & \quad P=B
\end{pmatrix}
\]

where P can be any \( A^f \)-module. The pairing \( A \otimes P \rightarrow A \) which must be surjective is given by an \( A^f \)-module map \( f: P \rightarrow A \), namely \( a \otimes p \mapsto af(p) \). Surjectivity means the right ideal \( A f(P) = A \), \( f(P) \) in \( A \) generates \( A \) in the sense that \( Af(P) = A \). Let's try to understand the case where \( B \) is a right ideal in \( A \) unital and \( \exists y \in A, x \in B \) such that \( yx = 1 \).
Recall setup: \((A, A)\) a unital, \(B\) right ideal in \(A\) satisfying \(AB = A\).

We know the following.

- This Morita context is split, being associated to the fiber dual pair \((B, A)\) over \(A\) where the pairing is 
  \[ A \otimes B \to A, \quad a \otimes b \mapsto ab. \]
  Hence 
  \[ A \otimes_B B \to A. \]

- Since \(A\) is unital we know \(B \in \text{C}(B), A \in \text{C}(B)\) are dual to each other and 
  \[ A \to \text{Hom}_B(B, B)^\circ, \]
  \[ A \to \text{Hom}_B(B, B)^\circ. \]

- functors on modules

\[ \begin{align*}
  M(B) \subset \text{mod}(B) & \to \text{mod}(A) \xrightarrow{\iota} M(B) \subset \text{mod}(B) \\
  P(B) \subset P(B) & \to P(A) \xrightarrow{\sim} P(B) \subset P(B)
\end{align*} \]

\[ L \to A \otimes_B L \to B \otimes_A L = B \otimes_B B \]

So just from 
\[ \begin{align*}
  P(A) & \sim P(B) \subset P(B) \to P(A) \\
  V & \to B \otimes_A V \to A \otimes_B B \otimes_A V = V
\end{align*} \]

we find 
\[ K_*(A) \xrightarrow{\iota} K_*(B) \xrightarrow{j} K_*(A) \] is the identity.

\(J\) is induced by 
\[ L \to A \otimes_B L, \quad \text{i.e.} \] extension of scalars w.r.t. \(\tilde{B} \to A\), so \(j\) is induced by this homomorphism. Now \(\iota\) is induced by 
\[ V \to B \otimes_A V, \] where \(B\) is regarded as a representation of \(A\) in \(P(B)\); in fact we have 
\(A = \text{Hom}_B(B, B)^\circ\).

If we choose an embedding of \(B\) as a direct summand of \(\tilde{B}\), then we get a homomorphism 
\[ A \to M_n(B). \] This homomorphism induces \(\iota\).
For example suppose \( j \in A \), \( x \in B \) satisfying \( yx = 1 \). Then we have

\[
\tilde{B} = By \oplus \tilde{B}(1-xy)
\]

Why? \( 1-xy \) is idempotent and \( \cdot (1-xy) \) kills \( By \).

\[
\tilde{B} = By + \tilde{B}(1-xy) \in By + \tilde{B}(1-xy).
\]

Also we have \( B \xrightarrow{y} By \xrightarrow{x} B \) is the identity so \( y : B \xrightarrow{\cdot y} By \).

It's better to give the pair of maps of \( B \)-modules

\[
B \xrightarrow{y} \tilde{B} \xrightarrow{x} B
\]

with composition \( 1 \). The corresponding homomorphism \( A \rightarrow \tilde{B} \) is then \( a \mapsto xay \).

Check: \( (x_{a_1}y)(x_{a_2}y) = x_{a_1a_2}y \).

Now our problem becomes showing that

\[
K_x(\tilde{B}) \xrightarrow{i} K_x(A) \xrightarrow{i} K_x(\tilde{B})
\]

is projection onto \( K_x(B) \).

Look at this from the viewpoint of \( H_x(GL(B)) \). Use Suslin's result that because \( B \) is \( h \)-universal \( H_x(GL(B)) \) is the homology of the fibre of \( BGL(B)^+ \rightarrow BGL(\mathbb{Z})^+ \). Then it seems we want to know that the homomorphism

\[
B \rightarrow A \rightarrow B
\]

\( a \mapsto xay \)

induces the identity on \( H_x(GL(B)) \).

Another way to say this might be the obvious representations of \( GL_n(B) \) on \( B^n \) and \( \tilde{B}^n \) in \( P(B) \) have the same stable characteristic classes. Somehow you want to deduce this from the exact sequence

\[
0 \rightarrow B^n \rightarrow \tilde{B}^n \rightarrow \mathbb{Z}^n \rightarrow 0
\]
Consider the chain of homoms:

\[ \cdots \to A \to B \to A \to B \to A \to \cdots \]

\[ a \to xy \]

Notice that \( A \to A, \quad a \mapsto xy \) is a non-unital ring homomorphism between unital rings. Is it a morphism? We have \( M \), given by:

\[
\begin{pmatrix}
A & Ay \\
xA & xAy
\end{pmatrix}
\]

Setting: \( B \subseteq A \) unital, \( BA = B \), \( \forall y \in A, x \in B \) s.t. \( yx = 1 \).

We have homomorphisms: \( A \to B, \quad a \mapsto xy \) and \( B \subseteq A \). These induce maps \( BGL(A)^+ \to BGL(B)^+ \) and \( BGL(B)^+ \to BGL(A)^+ \). The question is whether they are inverse up to homotopy. Look at the compositions.

Consider \( A \to A, \quad a \mapsto xy \). This is a non-unity preserving homomorphism, but it still induces group homomorphisms \( GL_n(A) \to GL_n(A) \) for all \( n \). How?

If \( e = \phi(1) = xy \), then one has a homomorphism of unital rings \( A \to eAe \) followed by the inclusion \( eAe \subseteq A \). The idea is that \( Ae \in \phi(A) \) has \( \text{Hom}_A(eAe, eAe) = eAe \), so \( P(eAe) \) is equivalent to the full Karoubian sub-cat of \( PA \) which is generated by \( Ae \).

We get the functor

\[
P(A) \to P(eAe) \subseteq P(A)
\]

\[ V \mapsto Ae \otimes V \to A \otimes V
\]

Here \( Ae \) means \( A \) with \( A \) acting on the right via \( \phi \).

Let's calculate this for \( \phi(a) = xy \). Note that

\[ Ae = Axy \subseteq Ay \quad \text{and} \quad Ay \subseteq Axy \subseteq Axy. \]

\[ \therefore \quad Ae = Ay \]

Take \( V = A^n \). Then \( A^n \otimes A^n \to A^n \). Now
you choose a split embedding of $A_g$ into a free $A^m$ in order to get a representation of $\text{Aut}(V)$ by matrices. In this case

\[
A_g \oplus A(1-xy) \xrightarrow{\sim} A
\]

\[
(a_{xy}, a(1-xy)) \mapsto 1 \alpha
\]

\[
(a_{y}, a_2(1-xy)) \mapsto a_1y + a_2(1-xy)
\]

We have isomorphism.

\[
A_g \otimes_{A} M \xrightarrow{\sim} M
\]

\[
a_g \otimes m \mapsto a_g' \otimes m = a'_g \otimes m
\]

\[
y \otimes m \mapsto y \otimes m \mapsto m
\]

\[
A_g \otimes_{A} M \xrightarrow{\sim} A
\]

\[
a_g \mapsto a_g' \mapsto a'_g
\]

\[
y \mapsto 1 \otimes y
\]

Take $M = A$ get isomorphism. Better $A \mapsto A_g$, $a \mapsto a_g$.

Note that $a_g \otimes (a_2(1-xy)) = a_g'(1-xy) = a'_g$. Let $\alpha \in A$.

Consequently, right mult by $x$. Let $\phi : A \to A_g$.

If $g \in \text{Aut}(A^n)$, then $\phi$ gets induced automorphism on $A_g \otimes (A_g)^n \xrightarrow{\alpha} (A_g)^n$.

\[
g = 1 + \alpha \text{ on } A^n \text{ becomes}
\]

\[
x(1+\alpha) \otimes y \approx \phi((1+\alpha) \otimes y)
\]

You add $1-xy$ on $A(1-xy)$. Thus we get

\[
1-xy + x(1+\alpha)y = 1 + x\alpha y.
\]

This calculation identifies the effect of the homomorphism $a \mapsto x\alpha y$ on $GL_n(A)$ with what we get from the functor $P(A) \to P(eA) \subset P(A)$.

\[
V \mapsto A_g \otimes_{A} V
\]
Assume $B = B^2$ such that $B \in P(B)$, i.e. $B$ is a finitely generated $\widetilde{B}$-module which is firm (since $B = \widetilde{B}$). We have functors

$$P(B) \xrightarrow{\sim} P(\widetilde{B}) \xrightarrow{L} B \otimes \widetilde{B} = BL$$

whose composition is $\sim L$. On the other hand $B$ is a generator from $P(B)$, so one has an equivalence $P(A) \sim P(B)$. $V \mapsto B \otimes A V$ where $A = \text{Hom}_B(B, B)^\circ$.

Consequently $K_x(P(B)) = K_xA$. The above functors give maps $K_x A \mapsto K_x \widetilde{B} \mapsto K_x A$ with composition the identity. Consider next the other composition $K_x \widetilde{B} \mapsto K_x A$ induced by $L \mapsto B \otimes \widetilde{B} = BL$.

One has functorial exact sequences from $P(B)$ to $\text{mod}(B)$:

$$0 \to BL \to F(L) \to \frac{B \otimes \widetilde{B}}{Z} \to 0$$

$$0 \to BL \to L \to \frac{\widetilde{B}}{Z} \to 0$$

where $F(L) = L \times (B \otimes \widetilde{B})$. In $K_0(B)$ we have from the two exact sequences involving $F(L)$:

$$[F(L)] = [B \otimes \widetilde{B}] + [\widetilde{B}] r(L) = [B] r(L) + [L]$$
where \( r(L) = \text{rank}_B(L) \). Write this

\[
[L] = [\mathcal{O}_B L] + ([B]-[B]) r(L)
\]

This yields a direct sum decomposition

\[
\begin{array}{ccc}
\mathcal{K}_* A & \underset{f}{\longrightarrow} & \mathcal{K}_* \mathcal{B} & \underset{g}{\longrightarrow} & \mathcal{K}_* \mathcal{Z} \\
\end{array}
\]

in degree 0, at least. But it should hold for all degrees, since functorial exact sequences are additive. i.e. \( \mathcal{K}_* P(B) = \mathcal{K}_* B \cong \mathcal{K}_*(\mathcal{O})/\mathcal{K}_*(\mathcal{Z}) \).

We want to understand the above arguments better. We have \( F(L) = F \otimes L \), where \( F \) is the \( B \)-bimodule \( F = \tilde{B} \times \mathbb{Z} \tilde{B} \), \( b(x, y) = (bx, by) \), \( (x, y)b = (xb, y) \).

We have \( B \)-bimodule exact sequences

\[
\begin{array}{ccc}
0 & \longrightarrow & B \overset{b}{\longrightarrow} F \overset{pr_1}{\longrightarrow} \tilde{B} & \longrightarrow & 0 \\
0 & \longrightarrow & B \overset{b}{\longrightarrow} F \overset{pr_2}{\longrightarrow} \tilde{B}_\varepsilon & \longrightarrow & 0 \\
\end{array}
\]

where \( \tilde{B}_\varepsilon \) means the right action of \( B \) is via the augmentation \( \varepsilon: \tilde{B} \rightarrow \mathbb{Z} \). We can split these exact sequences compatibly with left \( B \)-action using \( \Delta: \tilde{B} \rightarrow F \). Thus

\[
F = (B, 0) \oplus \Delta \tilde{B} = (0, B) \oplus \Delta \tilde{B}
\]

giving two isomorphisms of \( F \) with \( B \oplus \tilde{B} \) in \( P(B) \).

Take the former. \( F \cong B \oplus \tilde{B} \)

\[
(\sigma + v, v) \leftarrow (u \circ v)
\]

\[
(x, y) \longrightarrow (x-y, y)
\]

Then right mult by \( b \) is

\[
(\sigma, v) \rightarrow (\sigma + v, v)b = (ab + vb, 0) \rightarrow (ab + vb, 0) = (u, v)(b, 0)
\]
Take the latter isom.

\[ F \cong B \oplus \tilde{B} \]

\[(v', u+v') \leq (u', v') \]

\[(x, y) \mapsto (y-x, x) \]

and right mult by \(b\) is

\[(u', v') \mapsto (V', u'+v') \mapsto (Vb, 0) \mapsto (-Vb, v'b) = (u', v') \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix} \]

Observe \((u', v') \mapsto (V', u'+v') \mapsto (-u', u'+v') = (u', v') \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix} \)

What does this mean? The first homomorphism

\[ b \mapsto \begin{pmatrix} b & 0 \\ b & 0 \end{pmatrix} \]

from \(B\) to \(\text{Aut}_B(B \oplus \tilde{B}) = \begin{pmatrix} A & \tilde{A} \\ \tilde{B} & B \end{pmatrix} \)

arises from the exact sequence \(0 \to B \to F \to \tilde{B}_e \to 0\).

It extends to \( \begin{pmatrix} B & 0 \\ B & 0 \end{pmatrix} \) which by functor should be \(K\)-equivalent to \( \begin{pmatrix} B & 0 \\ \tilde{B} & B \end{pmatrix} \).

The second homomorphism

\[ b \mapsto \begin{pmatrix} 0 & 0 \\ -b & b \end{pmatrix} \]

extends to \( \begin{pmatrix} 0 & 0 \\ \tilde{B} & B \end{pmatrix} \) which should be \(K\)-equiv.

\( \begin{pmatrix} 0 & 0 \\ \tilde{B} & B \end{pmatrix} \), since these are conjugate this should mean that the representations \(B \to \tilde{B} = \text{Aut}_B(B)\) and \(B \to \tilde{B} = \text{Aut}_B(\tilde{B})\) are somehow equivalent.
Let $A$ be a left ideal in $R$ minimal. Recall that

a) $R/A$ is projective $\iff$ $A$ has a right identity: $a = ae$, $\forall a$.

b) $R/A$ is flat $\iff$ $A$ has local right identities: $\forall a_1, \ldots, a_n. \forall a \forall j (a_j)(\sum a_i) = 0$ (resp. this holds for $n=1$.)

Suppose $A$ is an ideal in $R$ such that $R/A$ is flat, so that $A$ modules $M$.

\[ A \otimes_R M \rightarrow AM \]

Then taking $M = R/A$ we get $A \otimes_R R/A = A/A^2 = 0$.

Also we have $M = AM \Rightarrow M$ is firm.

Conversely assume these two conditions, and let $M$ be any module. Since $A = A^2$, $AM = A(AM) \Rightarrow AM$ is firm. Also $A \otimes_R (M/AM) = 0$. So we have a diagram with exact rows:

\[ \begin{array}{ccccccc}
0 & & & & & & 0 \\
A \otimes_R AM & \rightarrow & A \otimes_R M & \rightarrow & A \otimes_R (M/AM) & \rightarrow & 0 \\
\downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \rightarrow & AM & \rightarrow & M & \rightarrow & M/AM \rightarrow 0
\end{array} \]

showing that $A \otimes_R M \sim AM$ for all $M$. $\therefore$ $R/A$ is right flat.

Prop. $R/A$ is right flat for an ideal $A$ iff $A = A^2$ and $M = AM \Rightarrow M$ is firm.

It would be better to formulate this independently of $R$ as follows:

Prop. $A$ has local left identities $\iff$ $A = A^2$ and $M = AM \Rightarrow M$ is firm for all modules $M$. 

Prop.
Assume $A$ is such a ring. Then

$$M = AM \implies \text{Hom}_R(R/A, M) = 0.$$ 

In effect if $K = \text{Hom}_R(R/A, M)$, then $AK = 0$ and

$$0 \to K \to M \to M/K \to 0$$

using $M = AM$ and $M/K = A(M/K)$. \text{Then} $K = 0$.

Alternate proof using local left identities: Let $Am = 0$, write $m = \sum a_i m_i$ and choose $a \in A$ such that $(1-a) a_i = 0$. \text{Then} $m = am = 0$.

Prop. Let $A$ be a ring satisfying $A = A^2$. Then

1) $A$ has local left identities
2) $AM = M \implies M$ firm for all modules $M$
3) $M = \{m | Am = 0\}$ is zero for all firm modules

It remains to check $3) \implies 2$). \text{Take} a module $M$ s.t. $AM = M$. Then $A \otimes_A M$ is firm and the kernel of $A \otimes_A M \to AM$ is killed by $A$. By $3$) the kernel is zero, and so $M$ is firm.

\text{Question: Is any idempotent ring Morita equivalent to a ring with local left identities?}

Suppose $(A, Q)$ strictly firm such that $B$ has local left identities. Then $P$ as a $B$-module satisfying $P = BP$ has local identities in the sense that $\forall p_i \in B$ there is $B$ such that $(1-b)p_i = 0$. Conversely, if this condition holds then as $B = PQ$, the ring $B$ has local left identities. In this situation we also know that $B$ is $B^p$ flat, hence $P$ is $A^p$-flat. Now, starting with $A$ idempotent we have a sequential way to construct firm flat right modules $P$. Can
Can this be modified to yield local left identities or is there an obstruction?

I think we can arrange $Q$ to be essentially free in the following sense. We want, starting from a finite set of $\tilde{p_j}$, to construct $b = \sum_i p_i \tilde{z}_i$ satisfying $p_j' = \sum_i p_i (\tilde{z}_i p_j)$ for all $p_j$. Here $p_i \tilde{z}_i$ can be added to what we already have. The function of $\tilde{z}_i$ is to provide an $A^\# \text{-linear}$ map $P \to A$ (or maybe $A^\#$).

Imagine constructing $PQ$ inductively adding at each stage the necessary $p_i \tilde{z}_i$. Then $P$ is a flat firm module over $A$ and the $\tilde{z}_i$ give linear functionals on $P$. So we can replace the $Q$ we might have with $AF$, where $F$ is a free $A^\#$-module whose basis elements map to the $\tilde{z}_i$. In other words we have $F \otimes P \to A$ hence $AF \otimes P \to A$.

Consider $A$ = maximal ideal in a valuation ring $R$ such that the principal ideals are $R \varepsilon$, $\varepsilon \in \mathbb{U}$ $2^n \mathbb{Z}$. A firm flat $A^\#$-module $P$ is a torsion free $R^\#$-module such that for any $p \in P$ there exists $\varepsilon > 0, p_1 \in P$ such that $p = p_1 \varepsilon$. Suppose we have an $A$-firm Morita context $(A, Q, B, p, B)$ where $B$ has local left identities. Then we know $P$ is $A^\#$ flat firm and for every finite set $p_j' \exists b = \sum_i p_i \tilde{z}_i$ such that

$$p_j' = \sum_i p_i \tilde{z}_i p_j'$$

Take a single $p'$. We have $p' \in \sum p_i R$ which is a torsion free finitely generated $R$-module.
Replacing a $p_i$ by suitable linear combinations over $R$, we can assume they form an $R$-basis, and also that $p_i \in p_iR$.

Then $p' = \sum p_i \delta_i p' \implies \delta_i p' = 0$ for $i \neq 1$. If $p' = p_1 \delta u$, then $p' = p_1 \delta p_i$ so $p_i \delta u = p_i \delta p_i \delta u$. So $p_i = p_i \delta p_i$, and so $p_i \delta p_i = 1 \in R$. This contradicts the facts that $\delta_i p_i \in A$.

Try for a computational proof as follows. The condition $p_j = \sum p_i \delta_i p_j$ says the $B$-module $W = \sum b_j$ satisfies $W = BW$. So $W$ is finitely generated and, so there should exist a simple object in $M(B)$. Strictly speaking, there's a non-nil simple $B$-module. But $M(B) \simeq M(A)$ and $A$ is a radical ring so $M(A)$ has no simple objects.
Continue with the nondegeneracy question: whether any idempotent $A$ is Morita equivalent to a $B$ which injects into its multiplier ring. I consider a special $A$ where factoring $a = \sum q_i a_i$ can be done explicitly and simplify.

Let $R$ be a valuation ring with value group $\mathbb{Z}$, say there are powers $\zeta^e$ for $\zeta \in \mathbb{Z}/\mathbb{Z}$ so the principal ideals are $\{\mathbb{Z}^e\}$. Let $m = \bigcup \mathbb{Z}^e$ be the maximal ideal of $R$. Take $A = m/m\mathbb{Z}$ and let $\bar{A} = m/R\mathbb{Z}$. Actually we start with $A \subset \bar{A} \subset R/\mathbb{Z}$ and note that $A = m/m\mathbb{Z} = \bar{A} \otimes \mathbb{Z}$ is flat over $R$ and satisfies $\bar{A}^2 = A$ so that $A$ is firm, flat over $\bar{A}$.

It should be clear from that $A = A^{(2)}$. So we have a firm flat commutative ring with a nonzero element $\zeta$ killed by $A$. When I consider a Morita context $(A, Q)$ this element $\zeta$ kills everything, and makes things nondegenerate. Claim there are firm flat $A$-modules $M$ such that $M = 0$. Let $F = \bigcup \mathbb{Z}^e\mathbb{Z}$ where $t$ is a real no.

$F$ is a flat $R$-module such that $mF = F$ for $M = F/F\mathbb{Z}$ is a firm flat $A$-module. Let $x \in F\mathbb{Z}$ satisfy $mx \in F\mathbb{Z}$. Up to units I can suppose $x = \zeta^e$ with $\zeta > t$. Then $\zeta^{2-k}\zeta^e \in F\mathbb{Z} \Rightarrow 2^{-k+1} \geq t+1$ if $t \in \bigcup \mathbb{Z}^e\mathbb{Z}$ then $\zeta > t+1$, so $\zeta > t+1$. Thus in this case $M = F/F\mathbb{Z}$ has no nonzero element killed by $A$. 
Let's take $Q = F_t(F_t^2)$, $F_t = U_{\mathbb{R}^\mathbb{Z}}$. I'd like to find an appropriate $P$. The obvious candidate which pairs nicely with $Q$ is $P = F_t^2/F_t^2$. The pairing $Q \otimes P \to A$ is surjective and in fact it looks like $Q \otimes A \to A$, whence $B = P \otimes A$ is also $A$. So $Q \otimes A \to A$, although I've managed to make $A$, $P$, zero, $B$ is still degenerate.