September 5, 1995

I want to reduce Morita invariance to the simplest steps. Consider $M(A)$, $A = A^2$ a Roos category, and let $Q$ be a generator which is firm. Put $S = \text{Hom}_A(Q, Q)$. Then we have a functor

$$\text{mod}(S) \rightarrow M(A)$$

$$N \mapsto Q \otimes_S N$$

Roos' theorem should tell us that there an idempotent ideal $J$ in $S$ such that this functor induces an equivalence

$$M(S, J) \sim M(A)$$

One has the Morita context:

$$\begin{pmatrix}
A & Q \\
\text{Hom}_A(Q, A) & S
\end{pmatrix}$$

So it should be clear that $S$

$$J = \text{Im} \left\{ \text{Hom}_A(Q, A) \otimes_A A \rightarrow \text{Hom}_A(Q, Q) \right\}$$

In fact I might as well dispense with $S$ and consider the triple $(Q, \text{Hom}_A(Q, A) \otimes_A A, \psi)$ where $\psi$:

$$\begin{array}{c}
Q \otimes_A \text{Hom}_A(Q, A) \otimes_A A \\
\text{Hom}_A(Q, A) \otimes_A A \rightarrow A
\end{array}$$

This triple is the "maximum" one containing the $Q$ given at the outset. More precisely, given $(Q, P, Q \otimes_A P \rightarrow A)$, one has

$$\begin{array}{c}
P \rightarrow \text{Hom}_A(Q, A) \\
\psi^1
\end{array}$$

\begin{array}{c}
P \otimes_A A \\
\text{Hom}_A(Q, A) \otimes_A A
\end{array}$$
where a map $P \to \text{Hom}_A(A, A)$ inducing $\psi$ from $\psi$.

Let's fix

$(Q, P, Q \otimes_P A)$ and put $B = P \otimes_A Q$,

$C = \text{Hom}_A(A, A) \otimes_A A \otimes_A Q = \text{Hom}_A(A, A) \otimes_A Q$. We have Morita equivalences

$$M(B) = M(A) = M(C)$$

$P \otimes_A M \leftrightarrow M \rightarrow \text{Hom}_A(A, A) \otimes_A M$

$N \rightarrow Q \otimes_B N \rightarrow \text{Hom}_A(A, A) \otimes_B N$

The last functor is base extension with $B \to C$.

The relevant Morita contexts here are contained in

$$
\begin{pmatrix}
A & Q & Q \\
P & B & B \\
\text{Hom}_A(A, A) & C & C
\end{pmatrix}
$$

I want to understand $\begin{pmatrix} B & B \\ C & C \end{pmatrix}$ better.

We have a canonical map $\psi$

$$\psi : P \otimes_A Q \rightarrow \text{Hom}_A(A, A) \otimes_A Q$$

which is a ring homomorphism on one hand, and a right $C$-module map on the other.

Let's consider a ring $C$ and a right $C$-module map $\psi : B \to C$. Define a product on $B$ by

$b_1 \cdot b_2 = b_1 \psi(b_2)$

Then

$(b_1 \cdot b_2) \cdot b_3 = (b_1 \cdot b_2) \psi(b_3) = (b_1 \psi(b_2)) \psi(b_3) = b_1 (\psi(b_2) \psi(b_3))$. Thus $B$ is a ring. Also $\psi(b_1 \cdot b_2) = \psi(b_1 \psi(b_2)) = \psi(b_1) \psi(b_2)$ so
$f$ is a ring homomorphism.

An example of such a $f: B \to C$ is the inclusion of a right ideal in $C$.

September 6, 1995

Start with a firm and a generator $Q$ for $M(A)$. Take $P = \text{Hom}_A(Q, A) \otimes_A A$, let $B = P \otimes_A Q$.

Then the triple $(Q, P, Q \otimes P \to A)$ has the property that $P$ is the 'dual' of $Q$. Under the Morita equivalence $M(A) \cong M(B)$ associated to this triple equivalence $M(A) \cong M(B)$, one has $Q \mapsto P \otimes_A Q$, $P \mapsto P \otimes_A Q$, hence the triple goes into $(B, B, B \otimes B \to B)$. It should follow that $B$ as right module should be dual to $B$ as left module, i.e. $B \cong \text{Hom}_B(B, B) \otimes B$.

Let's check this. Consider more generally an arbitrary completely firm Morita context $(A, Q, P, B)$. I claim there is a canonical isomorphism

$$\text{Hom}_A(Q, A) \otimes_A Q \cong \text{Hom}_B(B, B) \otimes_B B$$

In other words the dual of $AQ$ under the Morita equivalence is the dual of $BB$. Pf. One has a comm. diagram

$$
\begin{array}{c}
\text{Hom}_A(Q, A) \otimes_B Q \otimes B \otimes A \\
\downarrow \\
\text{Hom}_A(Q, A) \otimes_B Q \\
\downarrow \\
\text{Hom}_A(Q, A) \otimes_A Q
\end{array}
\xrightarrow{\sim}
\begin{array}{c}
\text{Hom}_A(Q, A) \otimes_B Q \\
\downarrow \\
\text{Hom}_A(Q, A) \otimes_A Q
\end{array}
$$

\begin{align*}
\lambda \otimes p \otimes g &\mapsto \lambda \otimes p \otimes g \otimes g \\
\downarrow \\
(\lambda \otimes p) \otimes g \otimes g &\mapsto (\lambda \otimes p) \otimes g \\
\downarrow \\
\chi \otimes g \otimes g &\mapsto (\chi \otimes g) \otimes g
\end{align*}
On the other hand we have

$$\text{Hom}_A(Q,Q) = \text{Hom}_B(B,B)$$

by Morita equivalence, i.e. induced by
the functors $P \circ A \rightarrow Q \circ B$. Thus

$$\text{Hom}_A(Q,A) \otimes_A Q \cong \text{Hom}_A(Q,Q) \otimes_B P \circ A = \text{Hom}_B(B,B) \otimes_B B$$

---

September 8, 1995

For a Hopf category $M(A)$, a firm. For each "coordination" $(Q, P, \psi)$ we have a ring $P \circ A Q$, hence an abelian group $((P \circ A Q)_{ab})$. Can we take an appropriate inductive limit of these abelian groups?

To fix the ideas consider $M(k) = \text{mod}(k)$ where $k$ is a unital ring. Among all coordinations are those $(V, k, V \otimes_k U \rightarrow k)$, where $V \in P(k)$, $U = \text{Hom}(V, k) \in P(k^{op})$ and the pairing is the evident pairing. For such a triple $U \otimes_k V = \text{End}_k(V)$ and $(U \otimes_k V)^* = \text{Aut}_k(V)$. (I should have pointed out above that $(P \circ A Q)_{ab}$ is $\text{GL}(P \circ A Q) = \{ \text{invertible elts in } 1 + P \circ A Q \}$.

Given triples $(V, U, \psi)$, $(V', U', \psi')$ there is an obvious way of a homomorphism $U \otimes_k V \rightarrow U' \otimes_k V'$ arises, namely from a pair of maps $V \rightarrow V'$, $U \rightarrow U'$ such that $\psi$ is the restriction of $\psi'$.

Assume now that these triples are both $f_{\text{proj-reflexive}}$, i.e. $V \in P(k)$, $U$ = dual of $V$, $\psi = \text{canonical pairing}$. Then a map $(V, U, \psi) \rightarrow (V', U', \psi')$ arises when

$$(V, U, \psi) = (V, U, \psi) \oplus (V', U', \psi')$$

i.e. when we are given a retract situation $V \rightarrow V'$. 
The converse seems likely, namely a map \((V, V^*, \langle \cdot, \cdot \rangle) \rightarrow (V^*_1, V^*_1, \langle \cdot, \cdot \rangle)\) assuming the triples are f-proj reflexive. Proof.

Let \(a: V \rightarrow V^*_1\) and \(b: V^* \rightarrow V^*_1\) be compatible with the pairings: \(\langle v, \lambda \rangle = \langle a(v), b(\lambda) \rangle\) for all \(v \in V\), \(\lambda \in V^*\). Then \(\langle v, \lambda \rangle = \langle b \circ a(v), \lambda \rangle \Rightarrow b \circ a = 1_V\), so \(V\) is a retract of \(V^*_1\).
Let $B$ be an idempotent ring, let $f: P \to B$ be a surjection of left $B$-modules, where $P$ is firm. Then we get a coordinate system on $M(B)$ given by the triple:

\[
(P, B^{(2)}, P \otimes B^{(2)} \to B) \quad \text{with} \quad p \otimes b \otimes b' \mapsto f(p)b'b,
\]

Let $A = B^{(2)} \otimes_P B$ be the corresponding ring. Since $P$ is a firm $B$ module we have $A \to P$. To keep things simple, suppose $B$ firm. Let's calculate the product in $A = P$. By def. if $a_1 = b_1 \otimes p_1$, $a_2 = b_2 \otimes p_2$ in $A = B \otimes_B P$, then

\[
a_1 a_2 = b_1 \otimes f(p_1)b_2 p_2 \mapsto \frac{b_1 f(p_1)b_2 p_2}{f(b_1p)}
\]

Thus if we use $A \to P$ to identify $A$ and $P$ we have the product in $A$:

\[
a_1 a_2 = f(a_1) a_2
\]

and $f(a_1 a_2) = f(f(a_1) a_2) = f(a_1)f(a_2)$. So $f: A \to B$ is a surjective homomorphism. Let $K = \text{ker}(f)$. Then $K$ is an ideal in $A$ such that $KA = 0$ and $A$ is a $B=AK$ module.

This time start with a ring $B$, a $B$-module $A$ and a $B$-module map $f: A \to B$. Define

\[
a_1 a_2 = f(a_1) a_2
\]

This is an associate product:

\[
(a_1 a_2) a_3 = f(f(a_1) a_2) a_3
\]

\[
a_1 (a_2 a_3) = f(a_1) f(a_2) a_3
\]
and \( f(a_1 \cdot a_2) = f(f(a_1) \cdot a_2) = f(a_1) \cdot f(a_2) \)
so \( f: A \to B \) is a homomorphism. I have encountered this situation before; it generalizes the inclusion of a left ideal.

When \( f \) is surjective we have \( A/K = B \) where \( K = \ker(f) \) is an ideal in \( A \) such that \( KA = 0 \).

Let's start now with \( A \) a ring, \( K \) an ideal such that \( KA = 0 \), and put \( B = A/K \).

\[
\frac{A}{K} \otimes_{A/K} \frac{A}{K} = \frac{A}{K} \otimes_A \frac{A}{K} \cong \frac{A}{KA} + AK
\]

Thus \( B \) finitely \( \mathbb{Z} \) is an \( \mathbb{Z} \).

Suppose \( A \) is h-unital. One has the context

\[
\begin{pmatrix} A & B \\ A & B \end{pmatrix} = \begin{pmatrix} A & A/K \\ A/K & A/K \end{pmatrix}
\]
diagram linking \( A \) and \( B \). Recall that \( B \) is h-unital \( \iff \) \( \quad \quad \quad \quad \begin{array}{c} P \otimes_A A \otimes_B Q \xrightarrow{\text{fin}} B \\ A \end{array} \)

This \( B \) is h-unital \( \iff \) \( A \otimes_A A \otimes_A B \xrightarrow{\text{fin}} B \)

\[ \iff \quad \quad \quad \quad \begin{array}{c} A \otimes_A A/K \xrightarrow{\text{fin}} A/K \end{array} \]

But one has \( A \) a map of \( A \)'

\[
A \otimes_A A \otimes_A A \otimes_A (A/K) \xrightarrow{\text{fin}} A \otimes_A A/K \xrightarrow{\text{fin}} K \quad K \quad A \quad A/K
\]

so we obtain

Claim: \( A \) is h-unital, \( K \subseteq A \) an ideal s.t. \( KA = 0 \).
Then \( B = A/K \) is h-unital \( \iff \) \( A \otimes_A K \xrightarrow{\text{fin}} K \) (i.e. \( K \) is an h-unital \( A \)-module).
Consider the map on $K_A$ induced by $A \rightarrow A/K = B$. Note that $K^2 \subseteq KA = 0$, so $B$ is a square zero extension of $B$ by the $B$-bimodule $K$ where the right multiplication is zero. We have then a group extension

$$1 \rightarrow M(K) \rightarrow GL(A) \rightarrow GL(B) \rightarrow 1$$

with abelian kernel. If $1 + \beta \in GL(B)$ ($\beta$ a matrix over $B$) and $K \in M(K)$, then

$$(1 + \beta)(1 + K)(1 + \beta)^{-1} = (1 + \beta)(1 + \beta)^{-1} + (1 + \beta)K(1 + \beta)^{-1} = 1 + (1 + \beta)K$$

Thus the action of $GL(B)$ on $M(K)$ defined by this group extension is given by left multiplication $(1 + \beta), K \mapsto (1 + \beta)K$.

I would like to understand when

$$K_1A = GL(A)_{ab} \rightarrow K_1B = GL(B)_{ab}$$

is an isomorphism. Conjecturally this happens if $A$ and $B$ are firm. Recall that assuming $A$ is firm, then $B = A/K$ is firm $\Rightarrow AK = K$.

Now we have an exact sequence

$$M(K)/[GL(A), M(K)] \rightarrow GL(A)_{ab} \rightarrow GL(B)_{ab} \rightarrow 0$$

so it would be nice to show that $AK = K$ implies $M(K) = [GL(A), M(K)] = \{ \alpha K \mid 1 + \alpha \in GL(A), K \in M(K) \}$.

I think we can take $(1 + \alpha)$ in $E(A)$. Thus

$$[e_{ij}, ke_{jk}] = ak e_{ik}$$

Here $j \neq i, h$. Thus the subgroup $[E(A), M(K)]$ contains $AK e_{ih}$ for all $i, h$, so contains $M(AK)$. In fact $[E(A), M(K)] = AN(AK)$. 
Claim: A idempotent, \( K \subset A \) ideal st. \( KA = 0 \), \( B = A/K \). If \( AK = K \), then \( K_1(A) \cong K_1(B) \).

Recall that starting with \( B \) idempotent and choosing a surjection \( P \rightarrow B \) of \( B \)-modules with \( P \) firm flat, we obtain a ring \( A \cong P \) which is left flat such that \( A/K \rightarrow B \). From the preceding we know \( B \) firm \( \iff \) \( AK = K \rightarrow K_1(A) \cong K_1(B) \).

On the other hand I think I've shown that for two left flat Morita equivalent rings \( A, A' \) one has a canonical iso \( K_1(A) \cong K_1(A') \). So it might be true that \( K_1(B) \rightarrow K_1(B') \) when \( B, B' \) are firm.

Let \( C = (A \oplus B) = (A) \oplus (A \oplus B) = (Q) \oplus (P \oplus B) \) be a completely-firm \( M \)-context. Then

\[
\begin{align*}
A \text{ is } A\text{-flat } & \iff P \otimes_A A = P \text{ is } B\text{-flat } \iff (A) \text{ is } C\text{-flat} \\
B \text{ is } B\text{-flat } & \iff Q \otimes_B B = Q \text{ is } A\text{-flat } \iff (Q) \text{ is } C\text{-flat} \\
(A \oplus B) \text{ is } A\text{-flat } & \iff (P \oplus B) \text{ is } B\text{-flat } \iff C \text{ is } C\text{-flat}
\end{align*}
\]

(the third is obtained by combining the first two).

\( C \text{ is } C\text{-flat } \iff A \text{ is } A\text{-flat and } B \text{ is } B\text{-flat}. \)
Recall equivalence between the data:
1) ring $B$, and $B$-module surjection $A \to B$.
2) ring $A$ and ideal $K \subseteq A$ such that $KA = 0$.

Claim: 3) If $A$ is a firm ring, then $B$ is firm ring $\Leftrightarrow AK = K$.
4) If $B$ is firm ring, then $A$ is firm ring $\Leftrightarrow A$ is firm $B$-module.

Pf. 3): $B \otimes_B B = A/K \otimes_A A/K = A \otimes_A A/\text{Im}(KA + A \otimes_A K) \xrightarrow{\sim} A/KA + AK = A/AK$ is iso $B \otimes_A AK = K$.

4) One has exact sequence
$$K \otimes_A A \to A \otimes_A A \to B \otimes_B A \to 0$$

Thus $A \otimes_A A \to B \otimes_B A$ (Proving 4).

When $A, B$ both firm, then we have the completely firm $M$ context.
$$\begin{pmatrix} A & B \\ A & B \end{pmatrix} = \begin{pmatrix} A \\ A \end{pmatrix} \otimes_A (A \otimes B) = \begin{pmatrix} B \\ B \end{pmatrix} \otimes_B (A \otimes B)$$

In this case this amounts to four cases:
$$A \otimes_A A \to A \quad A \otimes_B A \to B \quad B \otimes_B A \to B \quad B \otimes_B B \to B$$

Now let's look at $h$-unitality. One has
$$P \otimes_A A \otimes_A P = A \otimes_A A \otimes_A B$$
$$Q \otimes_B B \otimes_B P = B \otimes_B B \otimes_B A$$
Thus we get from our \( h \)-universal criterion:

5) If \( A \) is \( h \)-universal, then \( B \) is \( h \)-universal \( \iff \) \( A \overset{L}{\partial} B \to B \) quas.

6) If \( B \) is \( h \)-universal, then \( A \) is \( h \)-universal \( \iff \) \( B \overset{L}{\partial} A \to A \) quas.

From the maps of \( A \)'s:

\[
\begin{array}{ccc}
A \overset{L}{\partial} K & \longrightarrow & A \overset{L}{\partial} A \\
\downarrow & & \downarrow \\
K & \longrightarrow & A \\
\end{array}
\]

\[
\begin{array}{ccc}
B \overset{L}{\partial} K & \longrightarrow & B \overset{L}{\partial} A \\
\downarrow & & \downarrow \\
K & \longrightarrow & A \\
\end{array}
\]

we get:

5') If \( A \) is \( h \)-universal, then \( B \) is \( h \)-universal \( \iff \) \( A \overset{L}{\partial} K \to K \) quas.

6') If \( B \) is \( h \)-universal, then \( A \) is \( h \)-universal \( \iff \) \( B \overset{L}{\partial} K \to K \) quas.

In the situation \( A/K = B \), \( KA = 0 \), both \( A \), \( B \) \( h \)-universal, I want to show that \( K \cdot A \to K \cdot B \).

There is a group exact

\[
1 \to M(K) \to GL(A) \to GL(B) \to 1
\]

and Hochschild-Serre spectral sequence

\[
E^{2}_{pq} = H_{p}(GL(B), H_{q}(M(K))) \Rightarrow H_{*}(GL(A))
\]

so it's enough to have

\[
H_{*}(GL(B), H_{q}(M(K))) = 0 \quad q > 0.
\]
I think Juslin proves this with $\beta^{(n)}$ in place of $M(K)$ when he shows that
\[ H_x(GL(n) \times B^{(n)}) \to H_x(GL(n)) \]
for $B$ $h$-unital. In any case from
\[ H_x(GL(B), B^{(n)}) = 0 \]
one can deduce that
\[ H_x(GL(B), K^{(n)}) = 0 \]
for any $B$-module $K$ such that $B \otimes_B K \to K$ is a quasi-$H$-module by using a pseudo-free resolution of $K$.

Another point that gives some confidence in these ideas is the fact that the semi-direct product $C = B \ltimes K$, where right multiplication by $B$ in $K$ is trivial, is $h$-unital if $B$ is $h$-unital and $K$ is $h$-unital over $B$. In effect $Z \otimes_B B$ is a retract of $Z \otimes_C C$, so $C$ $h$-unital $\Rightarrow$ $B$ $h$-unital. The rest is clear from (6') above.

At some point I have to learn Juslin's methods, probably based on Vezzosi's model, and also the new ideas involving stable $K$-theory $\&$ $THH$.

Formulas:

\[ F \to BGL(A) \to BGL(A^+) \]

$P$ $A$-bimodules, $A$ unital

$K^S_x(A, P) = H_x(F, M(P))$

\[ K^S_x(A) \xrightarrow{P} K^S_x(A^+) \]

$P$ $A$-bimodules, $A$ unital

\[ K^S_x(A) \xrightarrow{\delta} H^*_x(ML) \]

\[ K^S_x(R) \xrightarrow{\delta} H^*_x(R) \]

$\delta$ $h$-unital

\[ H^*_x(R) = THH(R) \]

Thm. (D. & McCarthy. Annals '94) $K^S(R) = THH(R)$

Thm. (S. Wald. J.P.A. '92) $THH(R) = H^ML(R)$
Suppose \((A, Q)\) such that \(A\) is unital, \(P_A, AQ\) are finite \(\text{proj} A\)-modules, \(B = P \otimes_A Q\).

A perfect pairing \(Q \otimes_A P \rightarrow A\) is equivalent to a map \(P \rightarrow Q^* = \text{Hom}_A(Q, A)\) of \(\text{proj} A\)-modules. This can be factored \(P \rightarrow P_1 \rightarrow Q^*\) where the injection is a direct injection of \(\text{proj}\) right modules. For example, one can take \(P_1 = P \otimes Q^*\), \(f = p_2 \circ f\). Put \(Q_1 = P_1^* = \text{Hom}_A(P_1, A)\). Then we have direct injections \(P \rightarrow P_1\), \(Q \rightarrow Q_1\), such that \(f\) is the restriction of the canonical perfect pairing \(Q \otimes P_1 \rightarrow A\).

In this way, we embed \(B = P \otimes_A Q\) into the "matrix" ring \(P_1 \otimes A Q_1 = \text{Hom}_A(Q_1, Q)\).

\[
\begin{array}{ccc}
P \otimes_A Q & \rightarrow & P_1 \otimes_A Q \\
\downarrow & & \downarrow \\
P \otimes_A Q_1 & \rightarrow & P_1 \otimes_A Q_1
\end{array}
\]

\(P \otimes A P^*\)

The square is part of a \(\otimes\) commutative diagram.

Consider \(A\) a field. \(P_1 \otimes_A Q_1\) is a matrix algebra, \(P \otimes_A Q\) is a right ideal which can be roughly viewed as made of "rows" \(p \otimes Q\), while \(P_1 \otimes A\) is a left ideal made of "columns". Their intersection \(P \otimes A\) is a subring which has zero multiplication when \(\langle a, p \rangle = 0\). This
situation is ruled out in the case of a $M$ equivalence.

(In fact the situation $A = P \otimes A \subset P\otimes A, Q = B$
 satisfies $B = B^2 = BAB$ (assuming $P, Q \neq 0$)
 and $ABA = A$; but not $A = A^2$ where $\langle q, p \rangle = 0$.)

Next I would like to extend the field situation
in a geometric direction, i.e. take $P, Q$ to
 correspond to vector bundles over $X$ and $A = C(X)$.
Then $\psi: Q \otimes_A P \rightarrow A$ is onto
iff $f: P \rightarrow Q^*$ is nonzero at each $x \in X$.
(Actually since $A$ is commutative, the fact that $\psi$ is
an $A$-bimodule map implies that $\psi$ descends to a
pairing $Q \otimes_A P \rightarrow A$:
$\psi((p, q), p') = \psi(q, p') = \psi(p, q, p')$,
$\psi(q, p) = \psi(q, p) a_i = \psi(q, p a_i) = \psi(q, a_i p).$

Suppose $P = A^{k}$, $Q = A^{l}$, let $p_i, q_j$ be
bases for $P, Q$. Then $\psi: Q \otimes_A P \rightarrow A$
 is given by a $k \times l$ matrix over $A$:
$b_{ij} = \psi(q_j, p_i)$. We can identify $B = P \otimes_A Q$
with $k \times l$ matrices $B = \sum_{i,j} p_i \otimes q_j b_{ij}$. Then

$$\left(\sum_{i,j} p_i \otimes q_j b_{ij}\right) \left(\sum_{i',j'} p_{i'} \otimes q_{j'} b_{i'j'}\right)$$

$$= \sum_{i, j, i', j'} p_i \otimes a^{i}_{j} b_{ij} a^{i'}_{j'} b_{i'j'}$$

Thus $B = \text{M}_{kl}(A)$ with product
$\alpha^1 \cdot \alpha^2 = \alpha^1 \beta \alpha^2$. 

September 22, 1995

I want to prove Morita invariance for cyclic homology of b-unital rings. The key idea is to make use of ring homomorphisms which are Morita equivalences.

Let $A$ be a b-unital ring. $HC_*(A)$ is defined to be the homology of the Connes–Tsygan bicomplex of $A$, equivalently the homology of the pre-cyclic module $\mathbb{M}_1 \to A^{\otimes n+1}$. The mixed complex module $\mathbb{M}_1 \to A^{\otimes n+1}$. The mixed complex behind $HC_*(A)$ is the cone on $1 - 2 : (A^{\otimes n+1}, b) \to (A^{\otimes *}, b)$:

\[
\begin{align*}
& b' \\
& A \otimes A^2 \xrightarrow{1 \cdot \lambda} A^2 \\
& b' \quad -b' \\
& A \xrightarrow{1 \cdot \lambda} A
\end{align*}
\]

The Hochschild homology corresponding to this cyclic homology is the homology of this mixed complex, which can also be described as $(\Omega \tilde{A}, b)$. Thus $HH_*(\tilde{A})$ is what I called the reduced Hochschild homology $\tilde{HH}_*(\tilde{A})$.

Now $\tilde{A}^{\otimes *}$ can be calculated as follows. Start with the standard $A$-bimodule resolution of $\tilde{A}$:

\[
\cdots \xrightarrow{b'} \tilde{A} \otimes A^2 \otimes \tilde{A} \xrightarrow{b'} \tilde{A} \otimes A \otimes \tilde{A} \xrightarrow{b'} \tilde{A} \otimes \tilde{A}
\]

I will assume from now on that $A$ is flat over the groundring (default groundring is $\mathbb{Z}$). Then the above complex is a flat $A$-bimodule res. of $\tilde{A}$. Now $M \otimes_A \tilde{A}$ is a flat $A$-bimodule res. of $\tilde{A}$. Now $M$ as $A$-bimodule

\[M \otimes_A \tilde{A} = (M \otimes A^{\otimes *}, b') = (M \otimes A^{\otimes *}, b)
\]

so that $M \otimes_A$ is given by the complex

\[M \otimes_A \tilde{A} \otimes (\tilde{A} \otimes A^{\otimes *}, b') = (M \otimes A^{\otimes *}, b)
\]

In particular $A \otimes_A = (A^{\otimes *}, b)$, which does not give the reduced Hochschild homology $\tilde{HH}_*(\tilde{A})$ in general.
However for $h$-unital rings

the $b'$ complex is acyclic, so the Hochschild homology $\text{HH}_*(A)$ belonging to $\text{HC}_*(A)$ is

the homology of $A\otimes_A^L$. In fact there's a $\Delta$:

$$(A^{\otimes_{n+1}}, b) \rightarrow (\tilde{\Delta} A, b) \rightarrow (\tilde{\Delta} A, b)[1]$$

showing that $\text{HH}_*(A\otimes_A^L) \rightarrow \text{HH}_*(A) \iff A$ is unital

If $\varphi: A \rightarrow B$ is a homomorphism of rings, then $\varphi$ induces a map of mixed complexes $\tilde{\Delta} A \rightarrow \tilde{\Delta} B$, and formally one has that $\text{HH}_*(A) \rightarrow \text{HH}_*(B)$

$\iff \text{HC}_*(A) \rightarrow \text{HC}_*(B)$. For $h$-unital rings we have

$\text{HH}_*(A) = \text{HC}_*(A\otimes_A^L)$ and so we have $A\otimes_A^L B \rightarrow B\otimes_B^L B$ is a quasi $\iff \text{HC}_*(A) \rightarrow \text{HC}_*(B)$. This is what I want to use to prove Morita invariance for $\text{HC}_*$ of $h$-unital rings.

Let $\varphi: A \rightarrow B$ be a homomorphism between $h$-unital (hence prime) rings which is a Morita equivalence. The corresponding Morita context is

$$(A \otimes_A^L B) = \begin{pmatrix} A & A\otimes_A B \\ B\otimes_A^L A & B \end{pmatrix}$$

Notice that there are homomorphisms

$$M_2 A \rightarrow (A \otimes_A^L B) \rightarrow M_2 B$$

The idea now is to use say the latter map of Morita contexts to produce

$A\otimes_A^L \rightarrow B\otimes_B^L$ from our proof of

Minv. of $\text{HH}$.
More precisely we have

\[
A_\text{\textz}^B_A \leftarrow \mathcal{Z} \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A = \mathcal{Z} \mathcal{A}_\text{\textz} A \mathcal{B}_\text{\textz} A = \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A \leftarrow A \mathcal{B}_\text{\textz} B \quad \text{and so we can conclude that } \mu_x : \text{HH}_x(A) \cong \text{HH}_x(B)
\]

hence \( \text{HC}_x(A) \rightarrow \text{HC}_x(B) \). This proves

\[\text{If } \mu : A \rightarrow B \text{ is a homomorphism of } \text{h}-\text{unital rings which is also a Morita equivalence, then } \mu_x \text{ reduces } \text{HH}_x \text{ and } \text{HC}_x.\]

The next step will be to try to handle a Morita equivalence with context \( C = (A, Q) \) by means of the evident homomorphism \( A \leftarrow C \leftarrow B \). The problem is that \( C \) need not be \( \text{h}-\text{unital} \) even if \( A, B \) are.

Recall that if \( A \) is \( \text{h}-\text{unital} \), then

1) \( B \) \( \text{h}-\text{unital} \) \( \iff \) \( \mathcal{Z} \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} Q \rightarrow B \)

2) \( C \) \( \text{h}-\text{unital} \) \( \iff \) \( (\mathcal{A} \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A (A, Q)) \rightarrow C \)

\[\leftarrow (\mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A (A, Q)) \rightarrow \left( \begin{array}{c} A \\ Q \end{array} \right) \]

Here are cases where \( C \) is \( \text{h}-\text{unital} \)

1) \( A \) \( \text{h}-\text{unital} \), \( B \) left + right flat (equiv \( \mathcal{B}_\text{\textz} A \mathcal{B}_\text{\textz} A \) flat)

2) \( A \) left + right flat (hence \( \text{h}-\text{unital} \)) and \( B \) \( \text{h}-\text{unital} \) \( \text{Note } 1) \) \( \text{and } 1) \) are symmetric.

2) \( A, Q \) \( A \)-flat (this \( \Rightarrow \) \( P, Q \) \( B \)-flat and \( C \) is \( C \)-flat i.e. we have a completely left-flat situation)
The next discussion is perhaps of no real 

interest. I am interested in 

situations where \( P \odot_A Q \Rightarrow B \), \( Q \odot_B P \Rightarrow A \), so that 

the simpler proof works:

\[
A \odot_A Q \Rightarrow Q \odot_B P \Rightarrow A \\
B \odot_B Q \Rightarrow Q \odot_B P \Rightarrow B 
\]

The question is whether this situation occurs for \((A, C)\) 

and \((B, C)\) in the cases 1), 2) above.

Consider 2) the completely left flat situation. Then 

\( C = (A) \odot_A (A, Q) \) is left \( C \)-flat, so \((A), (B)\) are \( C \)-flat.

Then \( C = (A) \odot_A (A, Q) \) and from \( C = C \odot_C C \)

we get \( A = (A, Q) \odot_C (A) \). Thus the short proof

(probably the one used by Block + Steghen) works for \( A \subset C \)

and also for \( B \subset C \) by symmetry.

Next take 1) \( A \)-unital, \( B \subset C \)-flat \( \Leftrightarrow B \)-left-right-flat.

Then \( C = C \odot_C C \Rightarrow A = (A, Q) \odot_C (A) \), \( B = (B) \odot_C (B) \).

But also

\[
(A, Q) \odot_A (A, Q) = (A \odot_A A, A \odot_A Q) = (A, Q) \\
B = (B) \odot_B (B) = (B, B, B) = (B, B) 
\]

finally

\[
C \odot_B C \Rightarrow C \odot_B B \Rightarrow B 
\]

so in case 1) the short proof works for both \( A \subset C \)

and \( B \subset C \). Same for 1) by symmetry.

At this point I understand somewhat when

\( C \) is \( h \)-unital, and I should be able to prove
Morita invariance of $\mathcal{H}C$ for $h$-unital rings

Let's proceed by considering a Hoop category $M$ and all its coordinate systems. Suppose to fix the ideas that $M = M(D)$ with $D$ some idempotent ring. Then we can choose a new coordinate system $(D, V, W_D, V \otimes W \to D)$ where $V, W$ are flat firm over $D$. Then $A = W \otimes_D V$ is both left and right flat.

Now let $B_1, B_2$ be rings module to $D$, hence to $A$. Then we get an isomorphism $\mathcal{H}C(B_1) \cong \mathcal{H}C(B_2)$ from the homomorphisms:

$$B_1 \cong (A \ast B_1) \cong A \cong (A \ast B_2) \cong B_2$$

This is OK because $A$ both left and right flat, $B_i$ $h$-unital $\Rightarrow (A \ast B_i)$ is $h$-unital.

Next check the independence of the choice of $A$.

Diagram of inclusions:

\[ \begin{array}{ccc}
B_1 & \to & (A \ast B_1) \\
\uparrow & & \downarrow \\
A' & \to & (A' \ast B_1)
\end{array} \]

\[ \begin{array}{ccc}
A & \to & (A \ast B_2) \\
\downarrow & & \downarrow \\
A' & \to & (A' \ast B_2)
\end{array} \]

\[ \begin{array}{ccc}
B_1 & \to & (A' \ast B_1) \\
\uparrow & & \downarrow \\
A' & \to & (A' \ast B_2)
\end{array} \]
This shows the isom $HC(B_1) \cong HC(B_2)$ is independent of the choice of $A$.

Here's another way to see that $B_1, B_2$ $h$-unital. Let $A$ be left $+$ right flat, e.g., $A$ unital. Then we know given a $\text{M} \in (P \otimes B)$ that

$$B \text{ left unital } \Rightarrow P \otimes_A Q = P \otimes_A A \otimes_A Q \cong B.$$ 

Consider

$$\left( \begin{array}{c}
A \\
P_1 \\
P_2
\end{array} \right) \otimes_A (A, Q_1, Q_2) = \left( \begin{array}{c}
\begin{array}{c}
A \\
Q_1 \\
Q_2
\end{array} \\
P_1 \\
P_2
\end{array} \right)$$

Then $B_1 \text{ left unital } \Rightarrow P_1 \otimes_A Q_1 \cong P_1 \otimes_A Q_1$

$B_2 \text{ left unital } \Rightarrow P_2 \otimes_A Q_2 \cong P_2 \otimes_A Q_2$

$\left( \begin{array}{c}
B_1 \\
P_1 \otimes_A Q_1 \\
P_2 \otimes_A Q_2
\end{array} \right) \text{ left unital } \Rightarrow P_i \otimes_A Q_j = P_i \otimes_A Q_j \forall i, j$

So you get a non-$h$-unital example (as before) by arranging $\text{Tor}_i(P_i, Q_j) = 0$ for $i = j$ and $\neq 0$ for some $i \neq j$.

It seems there is a natural category structure on coordinate systems given by homomorphisms of the corresponding rings which induce the $M$-equivalence.
If \( A \) is a monoidal ring, its multipliers ring \( \text{Mult}(A) \) consists of pairs \( x = (x^a, x^b) \) of operators on \( A \) which we write \( a \mapsto ax^a \) and \( a \mapsto x^a a \) satisfying

\[
(\alpha_1, \alpha_2) x^a = \alpha_1 (\alpha_2 x^a) \\
(\alpha_1 x^a) \alpha_2 = \alpha_1 (x^a \alpha_2) \\
x^b (\alpha_1, \alpha_2) = (x^a \alpha_1) \alpha_2
\]

The first condition says \( x^a \in \text{End}_A(A) \), the third that \( x^b \in \text{End}_A(A) \), and the second says that \( x^a \) and \( x^b \) are adjoint with respect to the pairing \( \mu : A \otimes A \rightarrow A \). This obviously generalizes to a triple \((Q, P, \psi)\), namely let \( \text{Mult}(Q, P, \psi) \) be the set of pairs \((x^a, x^b) \in \text{End}_A(Q) \times \text{End}_A(P)\) such that \( \langle x^a q, p \rangle = \langle q, x^b p \rangle \), i.e.

\[
Q \otimes P \xrightarrow{x^a \otimes 1 \otimes x^b} Q \otimes P \xrightarrow{\psi} A
\]

has composition zero, equivalently

\[
P \xrightarrow{x^a} \text{Hom}_A(Q, A) \\
\xrightarrow{x^b} \text{Hom}_A(Q, A)
\]

commutes. \( M = \text{Mult}(Q, P, \psi) \) is clearly a subring of \( \text{End}_A(Q) \times \text{End}_A(P) \).

Assume the triple \((Q, P, \psi)\) such that \( Q, P \) are firm and \( \psi \) is surjective. Then I claim...
that
\[ M(Q, P, \psi) \cong \text{Mult}(\rho_A) \]

because \( \text{End}_A(Q) = \text{End}_B(B) \), \( \text{End}_{A^e}(P) = \text{End}_{B^e}(B) \)

and adjointness condition is preserved under the 

Morita equivalence. Here I use that

\[ Q \otimes P \xrightarrow{x^2-10y^6} Q \otimes P \xrightarrow{\psi} A \]

\[ \text{B} \otimes \text{B} \xrightarrow{x^2-10y^6} \text{B} \otimes \text{B} \xrightarrow{\mu} \text{B} \]

go into each other via \( P \otimes_A Q \) and \( Q \otimes_B P \).

(Strictly speaking \( Q \otimes B \otimes P = A^{(2)} \), but then one can follow with \( A^{(1)} \rightarrow A \). Note that because \( Q, P \) are finitely-

generated \( A \)-modules \( Q \otimes P \rightarrow A \) lifts uniquely to \( A^{(1)} \).)

We can use this to compute multiplier rings.

For example let \( A \) be a field and let \( Q, P \) be finite dimensional. If the pairing \( Q \otimes P 

\rightarrow A \) is non-degenerate then \( B = \text{Mult}(B) \) because \( P \cong \text{Hom}_A(Q, A) \otimes \) so \( x^2 = (x^2) \).

In general one has
\[ 0 \rightarrow P_0 \rightarrow P \rightarrow P/P_0 \rightarrow 0 \]

\[ 0 \rightarrow Q_0 \rightarrow Q \rightarrow Q/Q_0 \rightarrow 0 \]

where \( P_0 = Q^\perp, \ Q_0 = P^\perp \) for the pairing

\[ P \rightarrow Q^* \]

\[ \downarrow \]

\[ P/P_0 \rightarrow (Q/Q_0)^* \]

so the multiplier ring is a fibre product consisting of
pairs $\psi(x^k, x^l)$ with $x^k$ an endo of $Q_0$, respecting $Q_0$, and $x^l$ an endo of $P$ respecting $P_0$ such that on \( P/P_0 = (Q/Q_0)^\times \), we have $x^l = (x^k)^\ast$.

If $Q, P$ are infinite dimensional and the pairing is non-degenerate so that $P \subset Q^\ast$ and $Q \hookrightarrow P^\ast$, then the multiplier ring is the ring of endomorphisms of $P$ having transposes defined on $Q$. This might be equivalent to some sort of continuity for the weak topology on $P$ coming from $Q$.

Another comment: If $A$ is a ring such that $A = A^2$, then any left multiplier $x^l \in \text{Han}_L(A, A)$ commutes with any right multiplier $y^r \in \text{Han}_R(A, A)$:

$$(x^l(a_1 a_2)) y^r = (x^l a_1 a_2) y^r = x^l a_1 (a_2 y^r) = x^l (a_1 a_2 y^r) = x^l (a_1 y a_2 y^r)$$

In diagrams:

$$\begin{array}{ccc}
A \otimes A & \longrightarrow & A \\
\downarrow x \otimes 1 & & \downarrow x^l \\
A \otimes A & \longrightarrow & A
\end{array}$$

Thus $A$ is a bimodule over $M = \text{Mult}(A)$ when $A = A^2$.

(Also it seems when $A$ has trivial left annihilator and right annihilator in general.)

Also you have $A \longrightarrow M$ satisfying $x \mu(xa) = \mu(xa)$, $\mu(aya) = \mu(aya)$ so that $\mu(A)$ is an ideal in $M$. Thus $\mu(aya) = (x^a y a)$, so if it is injective (same as trivial left and right annihilator) we see $A$ is a bimodule over $M$. To this step that you need either $A = A^2$ or $\mu$ injective.
Here's an exercise I found difficult. Let $F, G$ be adjoint functors, $\alpha : FG \to 1$, $\beta : 1 \to GF$ the adjunction maps. One knows that $F$ is fully faithful $\iff$ $\beta$ is an isom. In effect there is a commutative triangle
\[
\text{Hom}(F(X), F(X')) = \text{Hom}(X, GF(X'))
\]
\[
\text{Hom}(X, X')
\]
\[
f
\]
\[
(\beta_{X'})_*
\]
\[
\text{etc.}
\]

The exercise is to give a proof at least of $\Rightarrow$ using properties of $\alpha, \beta$ and avoiding Yoneda's lemma. (I wanted this for bimodule arguments in connection with Morita equivalence.)

Assuming $F$ fully faithful we know that
\[
\alpha, F \circ \beta : FG \to F \text{ has the form } F \xi \text{ where } \xi : GF \to I \text{ is unique. Since }
\]
\[
\begin{array}{ccc}
F & \xrightarrow{\beta} & FGF \\
\downarrow & & \downarrow \alpha F \\
F & \xrightarrow{\alpha F} & F
\end{array}
\]
is the identity, we have $({\xi F}_{F})F_{\beta} = 1_F$, so $F(\beta F) = 1_F$ and then $\beta F = 1$. Next by naturality of $\beta$

\[
\begin{array}{ccc}
GF & \xrightarrow{\beta} & I \\
\downarrow & & \downarrow \beta \\
GFGF & \xrightarrow{\xi F \beta} & GF
\end{array}
\]

commutes. On the other hand $GF \xi = G \times F$ and we know that $G \beta G_{\xi} : GFG \to G$ is $1_G$, so
\[
\beta F = (GF \xi)(\beta GF) = 1GF, \text{ showing that } \beta F = 1_{GF}.
\]

above square
Higgins thesis on Lie-adgiers.
A dialgebra D is a bimodule with two assoc. operations $d_1 \cdot d_2$ and $d_1 \times d_2$ satisfying
\[
d_1 \cdot (d_2 \cdot d_3 - d_2 \times d_3) = 0 \\
(d_1 \cdot d_2 - d_1 \times d_2) \times d_3 = 0 \\
d_1 \times (d_2 \cdot d_3) = (d_1 \times d_2) \cdot d_3
\]

Example. Let A be an assoc. algebra, M an A-bimodule, and $f: M \to A$ a bimodule map.
Then $m_1 \cdot m_2 = m_1 f(m_2)$, $m_1 \times m_2 = f(m_1) m_2$
are associative operations on M (recall the former makes sense when $f: M \to A$ is only a right A-module map, and the latter requires only that $f$ be a left A-module map). Then
\[
m_1 \cdot (m_2 \cdot m_3 - m_2 \times m_3) = m_1 \frac{f(m_2 f(m_3) - f(m_2)m_3)}{f(m_1) f(m_3) - f(m_1) f(m_3)} = 0 \\
(m_1 \cdot m_2 - m_1 \times m_2) \times m_3 = f(m_1 + m_2) - f(m_1)m_2)m_3 = 0 \\
m_1 \times (m_2 \cdot m_3) = (m_1 \times m_2) \cdot m_3 = f(m_1)(m_2 f(m_3) - f(m_1)m_3) = 0
\]
so M is a dialgebra.
Let $(D_3, \cdot, \times)$ be a dialgebra, let
\[
N = \text{Im} \{ D \otimes D \to D \}
\]
Then $D \cdot N = N \times D = 0$.
$N \cdot N \subseteq D \cdot N = 0$ and $N \times N < N \times D = 0$. In particular
Also \( d_1 \cdot d_2 \equiv d_1 \otimes d_2 \) modulo the

subspace \( N \). Thus

\[
\begin{align*}
D \times N + N &= D \times N + N = N \\
N \times D + N &= N \times D + N = N
\end{align*}
\]

so \( N \) is an ideal for both \( \cdot \) and \( \ast \)

of square zero. \( \ast \) On \( D/N \), \( \cdot \ast \ast \ast \)

\( D/N \) is an assoc. algebra regarded as a dialgebra

in a trivial way. \( N \) is a \( D/N \)-bimodule

with left action given by \( \ast \) and right action

by \( \cdot \); these commute by the third axiom.

\( \square \square \square \square \square \square \square \square \square \square \square \)

So \( D \) is a dialgebra extension

\[
\begin{tikzcd}
0 \rar & N \lar & D \lar & D/N \lar & 0
\end{tikzcd}
\]

of the associative algebra \( D/N \) by the \( D/N \)-bimodul

of \( N \). Presumably there is some sort of analogue

of Hochschild cohomology connected with these

extensions.

\( D \) is a unital dialgebra where \( 1 \in D \) such

that \( d \cdot 1 = 1 \otimes d = d \otimes 1 \). In this case \( D/N \)

is unital and \( N \) is a unital \( D/N \)-bimodule.

Let's split \( \ast \) linearly (possible if \( D/N \) proj.

as \( k \)-module). Then \( \cdot \ast \) are given by appropriate

2-cycles in \( A = D/N \) for the two \( A \)-bimodule structures

in \( N \) having zero on one side. It looks like the

Hochschild cohomology \( H^*(\tilde{A}, N) \) vanishes in degrees > 0

and is \( \tilde{A} \otimes N \) in degree 0 when the right multiplication

of \( A \) on \( N \) is zero. In effect

\[
H^*(\tilde{A}, N) = H^* \{ \text{Hom} \; \tilde{A} \otimes \tilde{A}^\ast \otimes \tilde{A}, N \} \]
\[ H^* \left( \Lambda^0 \Lambda^0 \otimes, N \right) \cong 0 \]

because the right mult. of \( \Lambda \otimes N \) factors through \( \Lambda / \Lambda = k \), and because

\[ \Lambda \otimes \Lambda \to \Lambda \otimes \Lambda \to \Lambda \]

should be a proj. resolution of \( k \).

If so, then we can assume \( D = N_0 \times D / N \)

for the \( \times \) product, arbitrariness is a
derivation \( D / N \to N_0 \), which should be linear.
Then the \( \times \) product should be given by a
2-cocycle \( A^{\times 2} \to \Omega \Lambda A \), which should be a
coboundary.

A Leibniz algebra \( L \) is a \( k \)-module equipped
with bilinear operation \( l \cdot l' \) satisfying

\[
(l \cdot m) \cdot n = (l \cdot n) \cdot m + l \cdot (m \cdot n).
\]

In other words right mult by any \( a \in L \) is a
derivation of \( (L, \cdot) \). Thus we have a map

\[ L \to \text{Der}(L, \cdot) \quad m \mapsto a \cdot m \]

such that \( R(m \cdot n) = R_n R_m - R_n R_m \), so \( R \) is
a homomorphism of Leibniz algebras (maybe \( -R \) ?).

Note that \( l \cdot (m \cdot n) = 0 \); alt: \( R(a \cdot m) = -[R_n, R_m] = 0 \).

Let \( K = \text{Im} \left( L \otimes L \to L \otimes L \right) = \text{span} \left\{ m \cdot n \mid m, n \in L \right\} \)

\( \otimes L' \to L \otimes L' \otimes L' \).

(\text{char } k \neq 2). Then \( L \cdot K = 0 \) and \( K \cdot L \subseteq L \)
because right mult. \( -R(l) \) is a derivation of \( L \) hence
preserves \( K \). Thus we have an extension.
of Leibniz algebras, where $K/K = 0$ and $L/K$ is a Lie algebra (since modulo $K$ we have $d.K = 0$, which is the antisymmetry condition). Right multiplication makes $K$ a Lie module over $L/K$.

A functor from dialgebras to Leibniz algebras. Given $(D, \circ, *)$ a dialgebra, then $(D, d.d' - d.x.d)$ is a Leibniz algebra.

If $M \rightarrow A$, $m.m' = m.f(m')$, $m.x.m' = f(m).m'$, then the assoc. Liebniz alg is $M$ with operation $m \circ m' \rightarrow m.f(m') - f(m).m$.

Important example: $M = A \oplus A$, $f(a, b) = b$. Then we get the Leibniz algebra

$$(a, b) \circ (a', b') = (a, b) b' - b'(a, b) = ([a, b], [b, b'])$$

denoted $L \oplus A$ by Higgins. His universal enveloping algebra $U_L(A)$ is left-adjoint to this. Apparently there is a more interesting universal enveloping dialgebra of a Leibniz algebra.
Let \((D, \cdot, \ast)\) be a dialgebra: \(\cdot, \ast \) associative +
\[
\begin{aligned}
d_1 \cdot (d_2 \cdot d_3 - d_2 \ast d_3) &= 0 \\
(d_1 \cdot d_2 - d_1 \ast d_2) \ast d_3 &= 0 \\
d_1 \ast (d_2 \cdot d_3) &= (d_1 \ast d_2) \cdot d_3
\end{aligned}
\]

Example: Let \(A\) be an associative alg, \(M\) an \(A\)-bimodule, \(f: M \rightarrow A\) an \(A\)-bimodule map. Then \(m_1 \ast m_2 = m_1 f(m_2), m_1 \ast m_2 = f(m_1)m_2\) makes \(M\) into a dialgebra.

Conversely, given a dialgebra \(D\), let \(N\) be the image of \(D \otimes D \rightarrow D, d \otimes d' \mapsto d \cdot d' - d \ast d'\). Then \(A = D/N\) is an associative algebra with product \(\cdot\) and \(\ast\). \(D\) is an \(A\)-bimodule induced by \(\cdot\) and \(\ast\). \(D\) is an \(A\)-bimodule induced by \(\ast\) with left \(A\)-mult (resp. right \(A\)-mult) induced by \(\cdot\) (resp. \(\ast\)). The canonical surjection \(f: D \rightarrow A\) is an \(A\)-bimodule map such that \(d_1 \cdot d_2 = d_1 f(d_2), d_1 \ast d_2 = f(d_1)d_2\).

Check this: The first and second identity above give \(D \cdot N = N \ast D = 0\). Now
\[
\begin{aligned}
d \ast n &= d \ast n - d \cdot n \in N \Rightarrow D \cdot N \subset N \\
n \cdot d &= n \ast d - n \ast d \in N \Rightarrow N \cdot D \subset N
\end{aligned}
\]
so both \(\cdot\), \(\ast\) on \(D\) descend to \(A = D/N\) making \(A\) an assoc. algebra. Next \(D \cdot N = 0 \Rightarrow \ast\) on \(D\) descends to a right mult. \(D \otimes A \rightarrow D\) making \(D\) a right module over \(A\), since \(\cdot\) is associative. \(N \otimes D \rightarrow A\otimes 0 \rightarrow D\) making \(D\) a right module over \(A\), since \(\cdot\) is associative. The third identity above implies \(D\) is left \(A\)-module. The third identity above implies \(D\) is left \(A\)-module. Similarly \(\ast\) descends to \(A \otimes D \rightarrow D\) making \(D\) a right \(A\)-module. The third identity above implies \(D\) is left \(A\)-module. Finally, I should have said that the left \(A\)-module structure is defined by
\[
f(d_1)d_2 = d_1 \ast d_2, d_1 f(d_2) = d_1 \ast d_2.
\]
Suppose now that \( M \xrightarrow{f} A \) is given as above, with \( f \) surjective, and assume that \( A \) is unital and that \( M \) is a unitary \( A \)-bimodule. Choose \( \xi \in M \) such that \( f(\xi) = 1 \in A \).

Let \( N = \text{span of } m_1m_2 - m_1f(m_2), m_1 = f(m_1)m_2 - f(m_2)m_1 \).

Then \( N \subseteq \ker(f) \) and conversely, given \( n \in \ker(f) \) we have
\[
\xi \ast n - \xi \circ n = f(\xi)n - \xi f(n) = 1n = n.
\]

This shows that the dialgebra \( M \) determines \( A \).

So start with a unital dialgebra \( D \). This means we are given \( 1_D \) such that \( 1_D \ast d = d \ast 1_D = d, \forall d \).

We've seen that if \( N = \text{span of } d_1, d_2, \text{ then } A = D/N \text{ is an assoc. algebra.} \)

\( D \) is an \( A \)-bimodule, \( f: D \rightarrow A \) is an \( A \)-bimodule map such that \( d_1 \times d_2 = f(d_1)d_2, d_1d_2 = d_1, f(d_2) \).

Now, map such that \( f(d_1d_2) = f(d_1)f(d_2), f(d_1d_2) = f(d_1)d_2, f(d_1) = f(d_1d_2) = f(d_1f(d_2)), \) which implies that \( A \) is unital with \( f(1_D) = 1_A \).

Clearly \( D \) is a unital bimodule over \( A \). In fact, clearly \( D \) is not uniquely determined; it can be any \( \xi \in D \) such that \( f(\xi) = 1_A \).

The conclusion of the above discussion is that a unital dialgebra \( D \) is equivalent to a quadruple \( (A, M, f, \xi) \), where \( A \) is a unital assoc. alg, \( f: M \rightarrow A \) an \( A \)-bimodule map, and \( \xi \in M \) satisfies \( f(\xi) = 1_A \).

Let's now work out the Lie analogue. Again, start with a Lie algebra \( g \) and a \( g \)-module map \( f: M \rightarrow g \). Then define \( m \circ m' = f(m)m' - f(m')m \). This gives an opposite Lie algebra, i.e. where \( L_m \) is a derivation.
of \((M, \ast)\) for each \(m \in M\). Check
\[
m_1 \cdot m_2 = f(m_1) m_2 - f(m_2) m_1
\]
\[
f(m_1 \cdot m_2) = \left[ f(m_1), f(m_2) \right]
\]
\[
m_1 \cdot (m_2 \cdot m_3) = f(m_1) \left( f(m_2) m_3 - f(m_3) m_2 \right) - f(m_2) f(m_3) m_1
\]
\[
(m_1 m_2) \cdot m_3 = \left[ f(m_1), f(m_2) \right] m_3 - f(m_2) \left( f(m_1) m_3 - f(m_3) m_1 \right)
\]
\[
m_1 \cdot (m_2 m_3) = \frac{f(m_1)(f(m_2) m_3 - f(m_3) m_2)}{f(m_2) m_3 - f(m_3) m_2} - \left[ f(m_1), f(m_2) \right] m_3
\]

Conversely given a Lie algebra \(J\):
\[
l : (m, n) = (l, m) \ast n + m \ast (l, n)
\]
we have a map \(J \rightarrow \text{Der}(J, \ast)\), \(l \rightarrow L^l = l \cdot\cdot\cdot\)
which is a homomorphism \(L : m = \left[ L, L_m \right]\). Let \(N = \text{span of} \\{m \ast m \mid m \in J\}\). Then \(L_{m, m} = \left[ L_m, L_m \right] = 0 \Rightarrow \left[ L_m, L_m \right] = 0 \Rightarrow \left[ L, L_m \right] = 0 \Rightarrow \)
descends to \(J/N \rightarrow \text{Der}(J, \ast)\), making \(J\) a Lie module over \(J/N\). I should have noted earlier that \(N : J = 0\), \(J : N \supset N\), so that \(J/N\) is a quotient algebra of \(J\) and that \(J/N\) is a Lie algebra of \(J\). Moreover \(f : J/N \rightarrow \mathfrak{g}\) is a \(g\)-module map. But \(j \cdot k = f(j) k - f(k) j\) ?

Start again with a Lie module map \(M \rightarrow \mathfrak{g}\)
and define \(m \ast m' = f(m) m'\). Then
\[
m_1 \cdot (m_2 \cdot m_3) = f(m_1) (f(m_2) m_3 - f(m_3) m_2)
\]
\[
(m_1 m_2) \cdot m_3 = \left[ f(m_1), f(m_2) \right] m_3 - f(m_2) \left( f(m_1) m_3 - f(m_3) m_1 \right)
\]
\[
m_1 \cdot (m_2 m_3) = \frac{f(m_1)(f(m_2) m_3 - f(m_3) m_2)}{f(m_2) m_3 - f(m_3) m_2} - \left[ f(m_1), f(m_2) \right] m_3
\]
\[
\left[ m_1, m_2 \ast m_3 \right] = (m_1, m_2) \ast m_3 + m_3 \ast (m_1, m_2)
\]

Leibniz alg.
Let \( J \) be a Lie algebra:

\[
j(k, l) = (j, k) \cdot l + k \cdot (j, l)
\]

i.e.

\[
J \xrightarrow{f} \text{Der}(J)
\]

\[
f_j = f^j.
\]

and also \( L \cdot J = \left\langle L \right\rangle \cdot J \). Let \( N = \text{span of } f_j \). Then \( J \cdot N \subseteq N \), \( N \cdot J = 0 \Rightarrow \)

\[
L \otimes L \xrightarrow{\otimes} L, \quad j \cdot k \mapsto j \cdot k \text{ descends to } L \otimes N \otimes L \rightarrow L.
\]

\( L \otimes L \rightarrow L, \quad j \cdot k \mapsto f(j)k \). So we learn that any Lie algebra \( J \)

gives from a triple \((g, M, f : M \rightarrow g)\) where \( f \)

is surjective. Then can I conclude that \( g \) is determined by \((M, \cdot)\)?

Look at the above analogy. Given \( M \rightarrow A \)

and \( A \)-bimodule maps, we want to know when

\[
M \otimes_A M \xrightarrow{f \cdot f} M \rightarrow A \rightarrow 0
\]

is exact. Better: Start with \( M \rightarrow A \), then divide \( M \) by span of \( \left\{ f(m) \cdot m' - m \cdot f(m') \right\} \) to get an associative algebra extension \( B \otimes A \rightarrow A \rightarrow 0 \) such that

\[
B \cdot K = K, B = 0.
\]

The natural condition for sort of extension not to exist is for \( A \) to be firm: \( A \otimes A \twoheadrightarrow A \),

i.e.

\[
\text{H}_1(A) = \text{H}_2(A) = 0.
\]

The analogue of this in the Lie case would be \( H_2(j) = \text{H}_2(g) = 0 \). Consider \( M \rightarrow g \) and divide \( M \) by the span of \( m \cdot m = f(m) \cdot m \). Then we should have a Lie algebra extension

\[
0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{h} \rightarrow g \rightarrow 0
\]
such that \( k \) is a trivial \( q \)-module, 

i.e. \( k \) is a central extension of \( q \). So we want \( H_{2}(\mathfrak{g}) = 0 \), possibly also \( H_{1}(\mathfrak{g}) = 0 \), to recover \( q \) from the Leibniz algebra \( M \).

Next consider homology. If \((D, \cdot, *)\) is a dialgebra

then the analogue of \( \text{HH}_{0} \) for \( D \) is

\[
D/\text{span} \{ d \cdot d' - d' \cdot d \}.
\]

As \( m \cdot m' - m' \cdot m = m \cdot f(m') - f(m') \cdot m \), it is clear that, when \( \partial \) arises from \( M \to A \), one has \( D/[D, D] \cdot * = M/[A, M] \).

The following assertions seem correct.

1) A dialgebra \( D \) is equivalent to a triple \((A, M, f)\), where \( A \) is an \( \text{assoc. alg.} \), \( M \) an \( A \)-bi-module, and \( f: M \to A \) is an \( A \)-bi-module map, such that \( f \) is surjective and the kernel of \( f \) is spanned by \( m \cdot f(m') - f(m') \cdot m \), \( \forall m, m' \in M \).

2) A unital dialgebra (without a specific choice of \( 1 \)) \( D \) is equivalent to a triple \((A, M, f)\) where \( A \) is unital, \( M \) is \( A \)-bi-module over \( A \), and \( f \) is surjective.

2) A Lie algebra \( \mathfrak{g} \) is equivalent to a triple \((g, J, f)\), where \( g \) is a Lie alg., \( J \) a \( g \)-module, \( f: J \to g \) \((g, J, f)\), where \( g \) is a Lie alg., \( J \) a \( g \)-module, \( f: J \to g \) is surjective and the kernel of \( f \) is spanned by \( f(\xi) - f(\eta) \), \( \forall \xi, \eta \in J \).

The above should be equivalences of categories.

Let's assume this and try to calculate free objects.

Let \( V \) be a \( k \)-module and look for the free non-unital dialgebra generated by \( V \):

\[
\xymatrix{ M \ar[r]^{f} & A \ar[d] \ar@{-->}[l]_V \ar[r] & \bar{T}(V) \\
 T(V) \oplus \bar{T}(V) \ar[r] & \bar{T}(V) }
\]
I don't understand the initial case.

Next the free Leib algebra.

\[ \begin{array}{c}
\text{free } U(V) \text{ module} \\
generated \text{ by } V
\end{array} \quad \xrightarrow{f} \quad L(V) \]

\[ U(L(V)) \otimes V \xrightarrow{f} L(V) \]

\[ f(V) \] is the free Leib algebra generated by \( V \)

Next there is a functor \( \text{DiAlg} \to \text{Leib} \) which takes \( M \xrightarrow{f} A \) to \( \text{of} = (A, [, ]) \) and \( J = M \).

Considered as \( g \)-module via \( a \cdot m = [a, m] \). This functor has an left adjoint - the universal dialg generated by a Leib algebra. It takes \( J \xrightarrow{f} \text{of} \)

to \( A = U(g) \) and \( M = (U(g) \otimes J \otimes U(g)) \),

Here \( g \) acts on \( U(g) \otimes J \otimes U(g) \) via

\[ g(X)(\alpha \otimes f \otimes \beta) = -\alpha X \otimes f \otimes \beta + \alpha \otimes Xf \otimes \beta + \alpha g \otimes X \beta. \]

Put another way \( g \) acts internally on the \( U(g) \) bimodule \( U(g) \otimes U(g) \), and we couple this to the action on \( J \).

\( M \) is the universal \( g \)-module generated by the \( g \)-module \( J \). As is a vector space, it should be true that \( M \cong U(g) \otimes J \).

Higgins' \( \text{A}_{\text{Leib}} : \text{Leib} \to \text{A}_{\text{Alg}} \).

Given \( B \) assoc alg, consider the dialgebra \( B \otimes B \xrightarrow{\mu} B \).

This gives a functor \( \text{Alg} \to \text{DiAlg} \) and we look for the lift adjoint

\[ B \otimes B \xrightarrow{\mu} B \]

\[ \xrightarrow{(x, y) \mapsto f} \]

\[ M \xrightarrow{f} A \]

Thus the universal \( B \) seems to be \( T_\mu (M) = A \oplus M \oplus M \otimes A \).
Now take $M \rightarrow A$ to be
\[(U(g) \otimes T \otimes U(g))_q^g \rightarrow U(g)\] and we get $T_q(U(g) \otimes T)$ which is approx $U(g) \otimes T(J)$.
This agrees with Higgins’ PBW theorem saying that $q_0 U_q \otimes J = S(g) \otimes T(J)$. 
October 15, 1975

Moore invariance of $K_1$ for form rings, a direct approach. Consider $A, P, Q,$
$\phi: Q \otimes P \rightarrow A$ arbitrary $A$-bimodule map, and
$B = P \otimes Q$ the associated ring. I propose to
define a map $GL(B) \rightarrow K_1[A]$. Suppose given
$1-b \in GL(B)$. We can choose
$p_i = (p_{ji}) \in P^s, \quad q_j = (q_{ij}) \in Q^s, \quad 1 \leq i \leq n, 1 \leq j \leq s$
such that $b = p_i q_i = (p_{ji} q_{ik})$ using summation
convention. Let $p = (p_1 \cdots p_n) \in M_{s \times n}(P)$, $q = (q_1 \cdots q_n) \in M_{n \times s}(Q)$:

$$
\begin{pmatrix}
M_{s \times n}(A) \\
M_{n \times s}(Q)
\end{pmatrix}
\begin{pmatrix}
M_{s \times n}(P) \\
M_{n \times s}(B)
\end{pmatrix}
$$

Because $1-b = 1-pq$ is invertible, we know
that $1-bp \in GL_{n \times n}(A)$. Our task is to show
that the class $[1-qp] \in K_1[A]$ is independent of
the choice of $p, q$.

By replacing $B$ by $M_{s \times B}$, $P$ by $P^s = M_{s^2}(P)$,
$Q$ by $Q^s = M_{s^2}(Q)$, we should be able to reduce
to $s = 1$. To $b = p_i q_i = (p_1 \cdots p_n)(q_1 \cdots q_n)$.

Suppose now that we have two choices:

$b = p_i' q_i' = p_j' q_j' \quad 1 \leq i \leq k, 1 \leq j \leq l$

Consider $p = (p', p'')$, $q = (q', -q'')$. Then $pq = 0$. 
\[ 1 - \delta p = \begin{pmatrix} 1 - \delta \rho' & -\delta \rho'' \\ \delta \rho' & 1 + \delta \rho'' \end{pmatrix} \]

Claim: this is congruent mod \( E(A) \) to
\[
\begin{pmatrix} 1 - \delta \rho' & 0 \\ 0 & (1 - \delta \rho'')^{-1} \end{pmatrix}
\]

In effect,
\[
\begin{pmatrix} 1 & 0 \\ 0 & (1 - \delta \rho'')^{-1} \end{pmatrix} \begin{pmatrix} 1 - \delta \rho' & -\delta \rho'' \\ \delta \rho' & 1 + \delta \rho'' \end{pmatrix} = \begin{pmatrix} 1 - \delta \rho' & -\delta \rho'' \\ 0 & * \end{pmatrix}
\]

where \( * \) is
\[
= 1 + \delta \rho'' + \delta \rho' (1 - \delta \rho'')^{-1} \delta \rho'' = 1 + \delta \rho' \left( 1 + \left( (1 - \delta \rho'')^{-1} \delta \rho'' \right) \right)
\]

\[
= 1 + \delta \rho' (1 - \rho'' \delta')^{-1} \rho'' = 1 + \delta \rho' \left( (1 - \rho'' \delta')^{-1} \rho'' \right)
\]

\[
= (1 - \delta \rho')^{-1}
\]

Now we are reduced to showing that \( \rho \rho = 0 \)

\[ \implies 1 - \delta p \in E(A), \text{ because then it follows} \]

\[ \begin{pmatrix} 1 - \delta \rho' & 0 \\ 0 & (1 - \delta \rho'')^{-1} \end{pmatrix} = \begin{pmatrix} 1 - \delta \rho' & -\delta \rho'' \\ 0 & (1 - \delta \rho'')^{-1} \end{pmatrix} \]

is zero in \( K \cdot A \).

So we have to understand the $\mathbf{\text{condition}}$ of the

\[ \sum_{i=1}^{n} p_i \otimes q_i = 0 \text{ in a tensor product } P \otimes_A Q. \]  \( \text{(Here is where we will use the fact that } B = P \otimes_A Q \text{ and not just } PQ. \) \) One way this condition arises is when \( q = (q_i) \) can be factored \( q = a \rho' (g_i = a_{ij} q_j) \) such that \( p a = 0. \)
For then \( p \otimes q = p \otimes q' = p a \otimes q' = 0. \)

(With indices \( p_i \otimes q_i = p_i \otimes q_j' = p_i q_j' \otimes q_j = 0. \))

It seems that the converse, i.e. \( p \otimes q = 0 \Rightarrow \exists \ q = a q' \) such that \( pa = 0 \), is not true. But we have

**Lemma:** If \( p \otimes q = 0 \) in \( \tilde{A} \otimes A Q \), then \( \exists \ p', \ q', \ a, \ a' \) such that

\[(p \ p')(a') = 0, \quad (q') = (a) q'. \]

In other words, if we enlarge \( p \) to \( (p \ p') \) and \( q \) to \( (q') \), then we get the desired factorization. In the above \( a, a' \) are matrices over \( \tilde{A} \), but when \( Q = A Q \) we can further factor \( q' = a'' q'' \) with \( a'' \) a matrix over \( A \) and so assume \( a, a' \) are over \( A \).

Proof. We can suppose \( P \) is finitely generated. Let \( p' \) be a finite set of generators. Consider the exact sequence

\[0 \rightarrow K \rightarrow T \oplus T' \xrightarrow{f} P \rightarrow 0 \]

where \( T, T' \) are free \( \tilde{A} \)-modules with bases \( x, x' \), and \( f(x) = p, \ f(x') = p' \). We have an exact sequence

\[K \otimes_A Q \rightarrow T \otimes_A Q \oplus T' \otimes_A Q \rightarrow P \otimes_A Q \rightarrow 0\]

so \( \exists \ k \in K, \ g \in Q \) such that

\[k \otimes q' \mapsto x \otimes g\]
We have \( k = xa + xa' \) for unique \( a, a' \) since \( (x x') \) is a basis for \( \mathbb{T} \oplus \mathbb{T}' \). Then

\[
x \otimes q = (xa + xa') \otimes q' = x \otimes aq' + xa' \otimes q'
\]

where \( q = aq' \), \( a'q' = 0 \). Also \( k = xa + xa' \rightarrow 0 \) in \( P \) implies that \( pa + p'a' = 0 \). \( \ldots \) \( (p, p')(\langle a \rangle) = 0 \)

and \( \langle b \rangle = \langle a' \rangle q' \).

---

Here's the problem you ran into when you don't allow the extra elements \( p' \). Suppose \( p \in P \), \( q \in Q \) such that \( p \otimes q = 0 \) in \( P \otimes A \). Consider the exact sequence

\[
0 \longrightarrow A/\alpha \longrightarrow P \longrightarrow P/pA \longrightarrow 0
\]

where \( \alpha = \{ p \in A \mid \alpha(p) = 0 \} \). Then

\[
\text{Tor}_i^A(P, Q) \longrightarrow \text{Tor}_i^A(P/pA, Q) \longrightarrow A/\alpha \otimes Q \longrightarrow P \otimes Q
\]

is not exact. 

I went \( q \in \alpha \otimes Q \), for then \( q = aq' \) with \( pa = 0 \).
So if I take \( P \) projective, \( P/pA \) not right flat, then I can find \( Q \) such that \( \text{Tor}_i^A(P/pA, Q) \otimes \neq Q/\alpha \otimes Q = 0 \), and I get a counterexample.
So back now to \( p \circ g = 0 \) and the problem of showing that \( 1 - gp \in E(A) \).

Use the lemma to get \( (p, p')(\alpha') = 0 \), \( (g') = (\alpha')g' \).

Consider

\[
1 - (\alpha')(p, p') = \begin{pmatrix}
1 - gp & -gp' \\
0 & 1
\end{pmatrix}
\]

This is equivalent to \( 1 - gp \) modulo \( E(A) \).

Next

\[
1 - (\alpha')(p, p') = 1 - (\alpha')(g'(p, p')) = 1 - (\alpha')(g'p, gp')
\]

\[= 1 - \alpha' \alpha' \quad \text{where} \ \alpha' \alpha = 0.
\]

But Vaserstein's identity tells us that
\[
\begin{pmatrix}
1 - \alpha' & 0 \\
0 & 1 - (\alpha' \alpha')
\end{pmatrix}
\]
is in \( E(A) \) in general, so we conclude that
\( 1 - \alpha' \alpha' \) and \( 1 - gp \) are in \( E(A) \) as desired.

I forgot to give the simpler example, namely if \( pa = 0 \), \( g = ag' \), then

\[
1 - gp = 1 - a(g'p) \quad \text{where} \ a' \alpha = 0
\]

so \( (1 - gp, 0) \) is a product of elementaries.

At this point I should know that given \( 1 - b = 1 - pq \in GL(P(G_\alpha)) \), that \([1 - gp] \in K_\alpha A\) depends only on \( 1 - b\) and not on the choice of \( p, g \).
Basic forms of the Vaserstein identity:

\[
\begin{pmatrix} 1 & -g \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

Thus,

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
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\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
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\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
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\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
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\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \circ & 1 \end{pmatrix}
\]
October 19, 1995

Attempt to define $HH$ intrinsically for a Ross category $M$.

The first idea is to pick $P \in M$, $Q \in M$ and a surjection $f: Q \otimes P \rightarrow A$, where $P$ and $Q$ are flat. Suppose I choose a coordinate system $M \cong M(A)$, $A$ a field. Then $P$ becomes a field flat $A$-module, $Q$ becomes a field flat $A$-module, and $f: Q \otimes P \rightarrow A$ is an $A$-bimodule surjection. In general, if $M \rightarrow A$ is an $A$-bimodule map, one gets a presheaf of abelian groups $[M \otimes A]^*$. In fact before this one has a presimplicial $A$-bimodule with augmentation to $A$:

\[
\begin{array}{ccc}
M \otimes M \otimes M & \xrightarrow{\Delta} & M \otimes A \times M \\
A & & A \\
\end{array}
\]

(this is $T_A(M[1]) = R_A(M \rightarrow A)$ which is a DG algebra after making it a complex)

In the case of $Q \otimes P \rightarrow A$ we get the augmented complex

\[
\begin{array}{ccc}
Q \otimes B \otimes B \otimes P & \rightarrow & Q \otimes B \otimes P \\
Q & \rightarrow & Q \otimes P \\
& & A \\
\end{array}
\]

where $B = P \otimes A Q$, and the presheaf object

\[
\left[ (Q \otimes P) \otimes A \right]^* = [P \otimes A Q \otimes A]^* = [B \otimes A]^*
\]

which gives the Hochschild homology

\[
H_*(\tilde{B}, B) = H_*(B \otimes_B L)
\]

of $\tilde{B}$ with coefficients equal to the bimodule $B$.

Recall that $HH_*(B) = HH_*(\tilde{B})/\mathbb{Z}$ where

\[
HH_*(\tilde{B}) = H_*(\tilde{B} \otimes_{\mathbb{Z}} L)
\]

and the exact sequence $0 \rightarrow B \rightarrow \tilde{B} \rightarrow \mathbb{Z} \rightarrow 0$.
of $B$-bimodules yield a Δ:

$$\mathcal{B} \longrightarrow \tilde{\mathcal{B}} \longrightarrow \mathcal{Z} \longrightarrow$$

hence a long exact sequence

$$H_*(\tilde{\mathcal{B}}, \mathcal{B}) \longrightarrow HH_*(\tilde{\mathcal{B}}) \longrightarrow H_*(\mathcal{Z}, \mathcal{B}) \longrightarrow$$

base homology $H_B^*(B)$

except for $\mathcal{Z}$ in degree 0.

Thus one has

$$H_*(\tilde{\mathcal{B}}, \mathcal{B}) \longrightarrow HH_*(\tilde{\mathcal{B}}) \longrightarrow H_B^*(\mathcal{B}) \longrightarrow$$

which one can also see using the s.e.s. of complexes

$$0 \longrightarrow (\mathcal{B} \otimes (x+1), b) \longrightarrow \mathcal{C} \longrightarrow (\mathcal{B} \otimes (x+1), b') \longrightarrow 0.$$}

The point is that there is a canonical map

$$H_*(\tilde{\mathcal{B}}, \mathcal{B}) \longrightarrow HH_*(\tilde{\mathcal{B}})$$

which is an isomorphism $\iff$ $H_B^*(\mathcal{B}) = 0$, i.e., $B$ is $h$-unital.

When $P_A, Q$ are flat, $B$ is left and right flat, and conversely. In particular $B$ is $h$-unital so we see that $[(Q \otimes P) \otimes A]^k$ gives the Hochschild homology.

It seems that we have proved

Profl. Given $Q_A, P_A, \varphi: Q \otimes P \to A$, then as usual $[(Q \otimes P) \otimes A]^k$ gives the Hochschild homology iff $P_A$ is $h$-unital.
Intrinsic construction of $HH$ and $HC$ for a Poisson category $M$.

Let's choose a coordinate system $M = M(A)$ to do our calculations. Let $M \rightarrow A$ be an $A$-bimodule, where $M$ is a form flat $A$-bimodule. Then we have a cyclic module $[M \otimes_A]^\otimes$ which we claim computes the Hochschild and cyclic homology of $M$. (The latter may be defined as $HH_*(B)$ and $HC_*(B)$ for any coordinate system $M = M(B)$ such that $B$ is $h$-unital.)

The first case to consider is when $M = Q \otimes P$ where $A$, $P$ are form flat over $A$. Then

$$[M \otimes_A]^\otimes = [(Q \otimes P) \otimes_A]^\otimes = [P \otimes Q]^\otimes = B^\otimes$$

is the cyclic module associated to the ring $B$. Now $B$ is left and right flat in this case, in particular $h$-unital. Thus we know $B^\otimes$ yields $HH_*(B)$ and $HC_*(B)$, which are the $HH_*$ and $HC_*$ associated to $M$ as $B$ is $h$-unital. (I recall it is not a complete triviality that $H_*(B^\otimes, B) = HH_*(B)$, but that this uses the $h$-unitality of $B$.)

Next case: $A$ $h$-unital, $M$ form flat $A$-bimodule. I propose to identify $H_*([M \otimes_A]^\otimes, b)$ with $H_*(A \otimes_A)$. Consider the DG alg:

$$\cdots \rightarrow M \otimes_A M \rightarrow M \rightarrow A$$
where the differential $d$ is the unique degree $-1$ derivation on $T_A(M)$ which is zero on $A$ and $f:M \rightarrow A$ on $M$.

Since $M$ is a flat $A$-bimodule so is $T^n_A M$ for all $n \geq 1$. Indeed, $M$ is a filtered inductive limit of fg free bimodules, $(\tilde{A} \otimes \tilde{A})^n$, so we restrict to seeing that $(\tilde{A} \otimes \tilde{A})^n \otimes (\tilde{A} \otimes \tilde{A}) = \tilde{A} \otimes \tilde{A} \otimes \tilde{A}$ is a flat $A$-bimodule, which results from $A$ being flat over the ground ring $Z$.

Also, since $M$ is a firm $A$-bimodule so is $T^n_A M$, $\forall n \geq 1$. This is clear since firm for a bimodule means firm on both sides.

Thus

$$\rightarrow M \otimes_A M \rightarrow M \rightarrow A \rightarrow 0$$

is a complex of firm flat $A$-bimodules with augmentation to $A$.

Next we show this is a resolution module and left $A$-modules. Look at the homology of this DG ring: $H_*(T_A M, d)$. Left and right multiplication by $A$ on this homology factor through left + right mult by $H_0(T_A M, d) = 0$. Thus $H_*(T_A M, d)$ is killed by $\tilde{A} \otimes \tilde{A} \otimes \tilde{A}$. (More concretely given $a \in A$ choose $\xi \in M$ such that $d(\xi) = a$, then $h = \xi$ satisfies $[d, h] \alpha = d(\xi \alpha) + \xi d(\alpha) = a \alpha$, showing that $A \cdot H_*(T_A M) = 0$.

Now $A \cdot h$-unital $\iff A \otimes_A -$ kills complexes with nil-homology. Apply this functor to the complex

$$\rightarrow M \otimes_A M \rightarrow M \rightarrow \tilde{A}$$

because all modules are flat over $A$ we know that
\[ A \otimes_A T^\infty_A M = A \otimes_A T^\infty_A M = T^\infty_A M \]

On the other hand since \( T^\infty_A M \) has nil-homology, and \( A \) is \( h \)-unital we know this complex is acyclic.

At this point we have a flat bimodule resolution of \( A \)

\[ \cdots \rightarrow M \otimes_A M \rightarrow M \rightarrow A \]

so applying \(- \otimes_A \) gives

\[ ([M \otimes_A]^\otimes A, b) \sim A \otimes_A A \]

Thus for \( A \) \( h \)-unital, \( M \) flat \( \text{bimod} \rightarrow A \), we have \( \text{HH}_k^* ([M \otimes_A]^\otimes A, b) \sim \text{HH}_k^* A \), a canonical isomorphism.

A remaining point is that given two such \( M \)'s, say \( M_1 \) and \( M_2 \), we can form either \( M_1 \otimes_A M_2 \) or \( M_1 \oplus M_2 \) and then get

\[ ([M_1 \otimes_A M_2]^\otimes A, b) \sim ([M_1 \oplus M_2]^\otimes A, b) \]

and thus get not only a canonical isomorphism of \( \text{HH}_k \), but also \( \text{HC}_k \), for the cyclic modules \([M_1 \otimes_A]^\otimes A\) and \([M_2 \otimes_A]^\otimes A\).

Then taking \( M_2 = Q \otimes P \), we identify these with \( \text{HH}_k^* (B), \text{HC}_k^* (B) \).

Finally we want to get beyond assuming \( A \) is \( h \)-unital. The point will be that if we have \( Q \otimes P \rightarrow A \) with \( Q \) flat over \( A \), then the transport of \( M \) to the \( \text{bimodule} \ P \otimes_A M \otimes_A Q \) over \( P \otimes_A Q = B \) is a flat \( \text{bimodule} \) over \( B \).
Consider the rings \( A \otimes B \), \( B \otimes B^\# \). Then

\[ P \otimes Q \] is a left-\( B \otimes B^\# \), right-\( A \otimes A^\# \) bimodule, and

\[ Q \otimes P \] is a left-\( A \otimes A^\# \), right-\( B \otimes B^\# \) bimodule.

Moreover, these rings are firm, and the bimodules are firm on both sides. Also,

\[
(P \otimes Q) \otimes_{A \otimes A^\#} (Q \otimes P) = (P \otimes Q) \otimes_{A^\#} (Q \otimes P) = B \otimes B
\]

\[
(Q \otimes P) \otimes_{B \otimes B^\#} (P \otimes Q) = (Q \otimes P) \otimes_{B^\#} (P \otimes Q) = A \otimes A
\]

Thus, we should have a completely firm Monta context:

\[
\begin{pmatrix}
A \otimes A^\# & Q \otimes P \\
P \otimes Q & B \otimes B^\#
\end{pmatrix}
\]

We can then conclude that

\[
M \longrightarrow (P \otimes Q) \otimes_{A \otimes A^\#} M = P \otimes_A M \otimes_A Q
\]

carries firm flat \( A \)-bimodules to firm flat \( B \)-bimodules.

I use here that firm \( A \)-bimodules, i.e., \( A \)-bimodules

\( M \) which are firm on either side, are the same

as \( \otimes \) \( A \otimes A^\# \)-modules which are firm

wrt the ideal \( A \otimes A^\# \), equivalently firm modules

for the ring \( A \otimes A^\# \). Check:

\[
(A \otimes A) \otimes_{A \otimes A^\#} M = A \otimes_A M \otimes_A A
\]

and

\[
A \otimes_A M \otimes_A A \longrightarrow M \implies M = AM A \subset AM, MA \subset M
\]

Also,

\[
A \otimes_A M \otimes_A A \longrightarrow A \otimes M \otimes_A A
\]

\( \text{since } AM = M \)

\( \Rightarrow \) \( A \otimes_A M \otimes_A A \longrightarrow M \) \( \text{is left and right firm.} \)
Theorem: Let $A$ be an ideal in $B$ and $M$ a $B$-module. Then $A \otimes_B M \simto M \Rightarrow B \otimes_B M \simto M$.

Proof: The hypothesis implies $- \otimes_B M$ inverts $A$-nil-ideal, in particular $A \subset B \subset B$. 

Application: If $M$ is a trinodule over $A$, then $A \otimes_A M \otimes_A A \simto M \iff A \otimes_A M \simto M$ and $M \otimes_A A = M$.

Pf. ($\Leftarrow$) clear. ($\Rightarrow$): The hypothesis says that the $A \otimes A$ module $M$ is flat over $A \otimes A$. The lemma says $M$ is also flat with the ideal $A \otimes A$, i.e. $A \otimes A \otimes_A A \simto M$, but $M \otimes_A A \simto M$, so $A \otimes_A M \simto M$. 

added

Directed proof for a $(B,A)$-trinodule $P$ that $B \otimes_B P \otimes_A A \simto P \iff B \otimes_B P \simto P$ and $P \otimes_A A \simto P$.

$\Leftarrow$ clear. $\Rightarrow$: $B^{(u)} \otimes_B P \otimes_A A^{(u)} \simto B \otimes_B P \otimes_A A \simto P$ and clear $B^{(u)} \otimes_B P \otimes_A A^{(u)}$ is flat on both sides.