Recall $U = \text{Cone}(k[\varepsilon] \otimes T \xrightarrow{1 - ze} k[\varepsilon] \otimes T)$.

Suppose we change $\varepsilon$ to $1 - z$, so that

$1 - ze$ becomes $1 - (1 - z)e = 1 - e + ze$. Then

$0$-cocycles on $U$ should be described by $(u_0, u_0', u_1, u_1', \ldots)$ satisfying

$[d, u_n] = 0, \quad [d, u_n'] = u_n(1 - e) + u_{n+1} e.$

Moreover, if $v : T \to U$ is a cocycle on $U$, then $v_i$ should be $(v_0, v_1, 0, 0, \ldots)$. Check:

$[d, v_0] = 0, \quad [d, v_1] = v_1 e - e^2 = (v_1)(1 - e) + 0 e.$

New description of $U$: $U = k[\varepsilon, \sigma] \otimes T$, the operators $\varepsilon, h$ on $T$ are extended to $\varepsilon = 1 \otimes e, h = 1 \otimes h$ (super sense) on $U$. The differential on $U$

is $d = d' + d''$, where $d'' = 1 \otimes d$ and $d' = (1 - e + ze) \partial$,

where $\partial = (\text{super})$ derivative with $\sigma = \text{degree} - 1$ derivation such that $\partial(1) = 1, \partial(\varepsilon) = 0$.

Then

$[d, \sigma] = [(1 - e + ze) \partial, \sigma] = 1 - e + ze$

$[d, h] = [(1 - e + ze) \partial, h] + [d'', h]

= - [1 - e + ze, h] \partial + e - e^2

= e - e^2 + (1 - z)[e, h] \partial$

The nice thing about the conditions

$[d, u_n] = 0, \quad [d, u_n'] = u_n(1 - e) + u_{n+1} e, \quad n > 0$

is that $u_n(1 - e) \sim u_n(1 - e)^2 \sim u_{n+1} e(1 - e) \sim 0 \quad \forall n > 0$

and also $u_{n+1} e \sim 0 \quad \forall n > 0$ so that we have

$u_0 = 0, \quad u_n \sim 0 \quad n > 1$. We proceed as in

the old notation to find $s_n$ satisfying $[d, s_n] = u_n$ for $n > 1$ and $[d, s_0] = u_0(1 - e)$. We have
\[ s_0 = u'_0 (1-e) + (u_0 - u_1) h \]
\[ s_n = u'_{n-1} e + u'_n (1-e) - (u_{n-1} - 2u_n + u_{n+1}) h \quad n \geq 1 \]

Next we compute the coboundary of \((s_0, 0, s_1, 0, \ldots)\).

For \( n \) large \((n \geq 1)\) should do) we find

\[ s_n (1-e) + s_{n+1} e = u'_n + [d, (-u'_{n-1} + 2u'_n - u'_{n+1}) h] \]
\[ + \frac{(-u'_{n-1} + 3u_n - 3u_{n+1} + u_{n+2}) [e, h]}{[d, (-s_{n-1} + 3s_n - 3s_{n+1} + s_{n+2}) [e, h]]} \]

At this point I want to shift from operations on cochains to operators in \(U\). If \( \phi: U \to Z \) corresponds to \((u_0, u'_0, u_1, u'_1, \ldots)\) then \( u_n = \phi z^n \), \( u'_n = \phi z^{n+1} \).

Thus the operator \( (u_0, u'_0, \ldots) \to s_n \) corresponds to the operator

\[ \sigma z^{n-1} e_j + \sigma^n (1-e) - (z^{n-1} - 2z^n + z^{n+1}) h_j. \]

This means I should examine the operator

\[ k = \sigma (z^{-1} e + 1 - e) - (z^{-1} - 2 + z) h \]

Then

\[ [d, k] = (1-e + \sigma e)(1-e + \sigma^{-1} e) - (z^{-1} - 2 + z)(e - e^2 + (1-z) [e, h] \sigma) \]
\[ = 1 - 2\sigma e + \sigma e (1-e) + \sigma^{-1} (1-e) e + e^2 - z^{-1} (e - e^2 + (1-z) [e, h] \sigma) \]
\[ = (z^{-1} - 2 + z)(1-z) [e, h] \sigma \]

i.e.

\[ [d, k] = 1 - z^{-1} (1-z)^3 [e, h] \sigma \]

So far I've done the stable (large \( n \)) calculation, and one can see how nicely it correlates to \( \bigotimes \). It appears that writing the last term in \( \bigotimes \) as \([d, -]\) amounts to the contraction:

\[ [d, k (1 + z^{-1} (1-z)^3 [e, h] \sigma)] = 1 \]

resulting from the fact that \((z^{-1} (1-z)^3 [e, h] \sigma)^2 = 0\).
Next examine the transient behavior.

Let $z^*$ be the toplyt operator corresponding to $z^{-1}$, so that $z^* z = 1$, $1 - z z^* = \text{projection onto } z^0 T \oplus \sigma z^0 T$. We have

$$[d, z^*] = [(1 - e + z e) \delta, z^*] = [e, z^*] e \delta$$

Put

$$k = \sigma (1 - e + z^* e) - (z^* - 2 + z) h$$

$$[d, k] = (1 - e + z e)(1 - e + z^* e) - \sigma [d, z^*] e - [d, z^*] h$$

$$- (z^* - 2 + z)(e - e^2 + (1 - z) [e, h] \delta)$$

$$= 1 - 2 e + e^2 + z (e - e^2) + z^* (e - e^2) + z z^* e^2 - e^2 + e^2$$

$$- \sigma [z, z^*] e \delta e - \delta [z, z^*] e \delta h - z^* (1 - z)^2$$

$$- z^* (e - e^2) + 2 (e - e^2) - z (e - e^2) - (z^* - 2 + z)(1 - z) [e, h] \delta$$

$$[d, k] = 1 + [z, z^*] (e^2 - e^2 \delta + eh \delta) - z^* (1 - z)^3 [e, h] \delta$$

Our choice for $h$ is probably not correct at the bottom, because $s_0 = u_0'(1 - e) + (u_0 - u_1) h$ not $u_0'(1 - e) + (2u_0 - u_1) h$. 
April 21, 1995

I propose to find a homotopy: \( \phi_i = [d, h] \)

at least in the case \([e, h] = 0\). Recall

\[ \phi_j = (u_0 e, u_0 h, 0, 0, \ldots) \]

when \( \phi = (u_0, u'_0, u_1, \ldots) \). Note that \([z^*_j, z] = 1 - z z^*\)

projects onto \((j \circ \sigma_j)(T)\). We have

\[
[z^*_j, z] ((1 - e) e + h^2) \begin{bmatrix} f & = & g e \\ \sigma_j & = & g h \\ z^*_j & = & 0 \\ z^*_j & = & 0 \end{bmatrix} \]

so that

\[ j_i = [z^*_j, z] (e (1 - e) + h^2) \]

Some formulas:

\[
[e, h] e + e[e, h] = [e^2, h] = [e - [d, h], h]
\]

\[
[e, h] e + e[e, h] = [e, h] - [d, h^2]
\]

so

\[
(1 - e)[e, h] = [e, h] e + [d, h^2]
\]

\[
[e, h] (1 - e) = e[e, h] + [d, h^2]
\]

\[
e[e, h] e = [e, h] (e - e^2) - [d, h^2 e]
\]

\[
= [e, h^2 h] - [d, h^2 e]
\]

\[
= [d, -[e, h] h - h^2 e] = [d, -e h^2 + h e h - h^2 e]
\]

\[
e[e, h] e = [d, -e h^2 + h e h - h^2 e]
\]
\[ s_0 = u_0'(1-e) + (u_0-u_1)h \]

\[ [d, s_0] = u_0'(1-e)^2 + u_1'(e-e^2) + (u_0-u_1)(e-e^2) \]

\[ [d, s_0] = u_0'(1-e) \]

\[ [d, u_n'(1-e) + (u_n-u_{n+1})h] = u_n'(1-e) \]

\[ [d, u_n'] = u_n'(1-e) + u_{n+1}e \]

\[ [d, u_{n+1}'e + (u_{n+1}-u_{n+2})h] = u_{n+1}e \]

\[ [d, u_{n+1}'e + (-u_{n+1}+u_n)h = u_n e] \quad n \geq 1 \]

\[ s_n = u_{n-1}'e + u_n'(1-e) + (-u_{n-1} + 2u_n - u_{n+1})h \quad n \geq 1 \]

\[ [d, s_n] = u_n \quad n \geq 1 \]

\[ s_n(1-e) + s_{n+1}e = u_{n-1}'(e-e^2) + u_n'(1-2e+e^2) + (-u_{n-1} + 2u_n - u_{n+1})h(e) \]

\[ + u_n' e^2 + u_{n+1}'(e-e^2) + (-u_{n+1} + 2u_{n+1} - u_{n+2})he \]

\[ = u_n' + (u_{n-1} - 2u_n + u_{n+1})[d, h] + (-u_{n-1} + 2u_n - u_{n+1})h \]

\[ + (u_{n-1} - 3u_n + 3u_{n+1} - u_{n+2})he \]

\[ = u_n' + [d, (-u_{n-1} + 2u_n - u_{n+1})h] \bar{o} \left( (-u_{n-1}(1-e)-u_n e + 2u_n (1-e) + 2u_{n+1}e \right) \]

\[ + (-u_{n-1} + 2u_n - u_{n+1})h + (u_{n-1} - 3u_n + 3u_{n+1} - u_{n+2})he \]

\[ = u_n' + [d, (-u_{n-1} + 2u_n - u_{n+1})h] + (-u_{n-1} + 3u_n - 3u_{n+1} + u_{n+2})h] \]

\[ + (-u_{n-1} + 3u_n - 3u_{n+1} + u_{n+2})he \]
\[ s_n (1-e) + s_{n+1} e = u_n' + [d_j (-u_{n-1} + 2u_n - u_{n+1})] e_j h_i. \]

\[ n \geq 1. \]

\[ s_0 (1-e) + s_1 e = u_0' (1-e e^2) + (u_0 - u_1) h (1-e) \]
\[ + u_0' e^2 + u_1' (e e^2) + (-u_0 + 2u_1 - u_2) h \]
\[ = u_0' + (-2u_0' + u_1') [d_j e_j h_i] + (u_0 - u_1) h \]
\[ + (-2u_0 + 3u_1 - u_2) h \]
\[ = u_0' + [d_j (2u_0' - u_1') h_j] - (2u_0' (1-e) + u_1' (1-e) - u_2') h \]
\[ + (u_0' - u_1') h + (-2u_0 + 3u_1 - u_2) h \]
\[ = u_0' + [d_j (2u_0' - u_1') h_j] + (2u_0 - 3u_1 + u_2) [e_j h_i] \]
\[ - u_0 h \]

\[ s_0 (1-e) + s_1 e = u_0' - u_0 h + [d_j (2u_0' - u_1') h_j] \]
\[ + (2u_0 - 3u_1 + u_2) [e_j h_i] \]

Put

\[ s'_n = (u_n' - 2u_n' + u_{n+1}') h_j + (s_{n-1} - 3s_n + 3s_{n+1} - s_{n+2}) [e_j h_i], \]

\[ n \geq 1. \]

Then for we have

\[ [d_j s_n'] + s_n (1-e) + s_{n+1} e = \]
\[ [d_j (u_{n-1}' - 2u_n' + u_{n+1}') h_j] + ([d_j s_{n-1}] - 3[d_j s_n] + 3[d_j s_{n+1}] - [d_j s_{n+2}]) [e_j h_i] \]
\[ + u_n' + [d_j (-u_{n-1}' + 2u_n' - u_{n+1}') h_j] + (-u_{n-1} + 3u_n - 3u_{n+1} + 3u_{n+2}) [e_j h_i] \]

\[ [d_j s_n'] + s_n (1-e) + s_{n+1} e = \]
\[ \left\{ \begin{array}{ll}
    u_n' & \text{if } n \geq 2 \\
    u_1' - u_0 e [e_j h_i] & \text{if } n = 1
  \end{array} \right. \]
At this point we have to adjust things to work well at $n=0,1$.

Put

\[ s_0' = (-2u_0'+u_1')h + (-2s_0 + 3s_1 + s_2)[e, h] \]

Then

\[ [d, s_0'] + s_0(1-e) + s_1 e = \]

\[ [d, (-2u_0'+u_1')h] + (-2[d, s_0] + 3[d, s_1] - [d, s_2])[e, h] \]

\[ + u_0' - u_0 h + [d, (2u_0'-u_1')h] + (2u_0 - 3u_1 + u_2)[e, h] \]

\[ = u_0' - u_0 h + 2u_0 e[e, h] \]

\[ [d, s_0] = u_0 - u_0 e \]

\[ [d, s_0'] + s_0(1-e) + s_1 e = u_0' - u_0 h + 2u_0 e[e, h] \]

\[ [d, s_1] = u_1 \]

\[ [d, s_1'] + s_1(1-e) + s_2 e = u_1' - u_0 e[e, h] \]

\[ [d, s_2] = u_2 \]

\[ [d, s_2'] + s_2(1-e) + s_3 e = u_2' \]

\[ \ldots \]

There are two ways to proceed.

First note that the cocycle $u_0 e[e, h]$ is reproduced by $1-e$.

\[ [d, -e^2 h + heh - he^2] = e[e, h] e \]

\[ [d, u_0 (-e^2 h + heh - he^2)] + u_0 e[e, h](1-e) = 0u_0 e[e, h] \]

Put

\[ \tilde{s}_0 = s_0 - 2u_0 e[e, h] \]

\[ \tilde{s}_0' = s_0' - 2u_0 (-e^2 h + heh - he^2) \]

\[ \tilde{s}_1 = s_1 + u_0 e[e, h] \]

\[ \tilde{s}_1' = s_1' + u_0 (-e^2 h + heh - he^2) \]
and \( \tilde{s}_n = s_n, \tilde{s}_n' = s_n' \) for \( n \geq 2 \).

Then we get

\[
[d, \tilde{s}_0] = u_0 - u_0 e
\]

\[
[d, \tilde{s}_1'] + \tilde{s}_1'(1-e) + \tilde{s}_1 e = u_1' - u_0 h
\]

\[
[d, \tilde{s}_1] = u_1
\]

\[
[d, \tilde{s}_1] + \tilde{s}_1'(1-e) + \tilde{s}_2 e = u_1'
\]

Now that we can contract cocycles, we get the required homotopy operator \( k \) such that \([d, k] = 1 - g_i\). Let's compute the modified \( h \) given by \( c_k \).

Think of \( 1: U \to T \) as the cocycle \((e, h, 0, 0, \ldots)\), then \( c_k \) should be the cochain \((\tilde{s}_0, \tilde{s}_1', \ldots)\) for \( u_0 = e, u_1' = h, u_n = u_n' = 0 \) for \( n > 1 \). Then \( c_k = \tilde{s}_0' \):

\[
\tilde{s}_0' = -2u_0 e [e, h] + u_1'(1-e) + (u_0 - u_1) h
\]

in general

\[
= -2e^2 [e, h] + h(1-e) + (e^0 - 0) h
\]

\[
= h + [e, h] - 2e^2 [e, h]
\]

\[
\therefore c_k = h + [e, h] - 2e^2 [e, h]
\]

\[
= h + [e, h] - e[e, h] - [e, h](1-e)
\]

\[
= h - [e, e^0] h
\]

The second way is to note that \((s_0, s_1', \ldots)\) gives a homotopy between \(1\) and \( y_i \) for a different \( i \), namely \((e, h - 2e[e, h], 0, e[e, h], 0, \ldots)\).
The corresponding modified $h$ is
\[ s_0 = u_0(1-e) + (u_0-u_1)h \]
\[ = (h - 2e[e, h])(1-e) + e h \]
\[ = h + [e, h] - 2e[e, h](1-e) \]
which is also equivalent to $h - [e, e, h]$. 

Let's see how hard it is to compute $1k^2j$. We need $s_0$ for the cocycle $ck$ which means we need $s_0, s'_0, s_1$ for $c$. Seems too hard.
New idea: Note that $U = k[z, \sigma] \otimes T$ with the differential
\[ d = (ze + 1-e)d + d \] (here $d$ is $1 \otimes d$) and I have changed notation to avoid the mistake of leaving out the second term in
\[
[x, h] = [d, h] + [(ze + 1-e)d, h] = e - e^2 + (1 - 2) [e, h] d)
\]
corresponds geometrically to the telescope

Let's consider the analogue of the infinite telescope in both directions. Put
\[
W = k[z] \otimes k[z, \varepsilon^{-1}] \otimes T = k[z, \varepsilon^{-1}] \otimes T \oplus \sigma k[z, \varepsilon^{-1}] \otimes T
\]
\[
W^+ = k[z] \otimes T \oplus \sigma k[z] \otimes T = U \text{ above}
\]
\[
W^- = k[z] \otimes T \oplus \sigma z^{-1} k[z] \otimes T
\]
Then $W^+$ and $W^-$ are subcomplexes of $W$ equipped with the differential $\tilde{d}:
\[
(ze + 1-e)d (k[z^{-1}] \otimes T) \subset \left( \begin{array}{c}
\sigma z^{-1} k[z^{-1}] \otimes T \\
(ze + 1-e) z^{-1} k[z^{-1}] \otimes T
\end{array} \right) \subset W^-
\]
We have $W^+ \cap W^- = T$, $W^+ + W^- = W$, whence an exact sequence of complexes
\[
0 \rightarrow T \rightarrow W^+ \otimes W^- \rightarrow W \rightarrow 0
\]
which is clearly locally split. We now show that $W$ is
contractible and it then follows that $T \sim W^+ \oplus W^-$. We have 

\[ [\tilde{z}, \sigma(z^{-1}e + 1 - e)] = (z^{-1}e + 1 - e)(z^{-1}e + 1 - e) \]

\[ = 1 - 2e + e^2 + (z^{-1} + z)(e - e^2) \]

\[ = 1 + (z^{-1} + 2 + z)(e - e^2) \]

Then 

\[ [\tilde{z}, \sigma(z^{-1}e + 1 - e) + (-z^{-1} + 2 - z)h] \]

\[ = 1 + (z^{-1} + 2 + z)(e - e^2) + (-z^{-1} + 2 - z)(e - e^2 + (1 - z)[e, h] \sigma) \]

\[ = 1 + (-z^{-1} + 2 - z)(1 - z)[e, h] \sigma = 1 - z^{-1}(1 - 2z + z^2)(1 - z)[e, h] \sigma \]

\[ = 1 - z^{-1}(1 - z)^3[e, h] \sigma \]

square zero

Thus 

\[ [\tilde{z}, h] = 1 \]

where

\[ k = (\sigma(z^{-1}e + 1 - e) - z^{-1}(1 - z)^2h)(1 + z^{-1}(1 - z)^3[e, h] \sigma) \]

I think this is the operator corresponding to the formulas on 269-270:

\[ s_n = u_{n-1}e + u_n(1 - e) + (-u_{n-1} + 2u_n - u_{n+1})h \]

\[ s_n' = (u_{n-1} - 2u_n + u_{n+1})h + [s_{n-1} - 3s_n + 3s_{n+1} - s_{n+2}][e, h] \]

Thus if $\phi \leftrightarrow (u_0, u_0', ..., )$, i.e., $\phi z^n = u_n$, $\phi \sigma z^n = u_n'$, then

\[ \phi k z^n = \phi(\sigma(z^{-1}e + 1 - e) - z^{-1}(1 - z)^2h)z^n \]

\[ = u_{n-1}e + u_n(1 - e) + (u_{n-1} + 2u_n - u_{n+1})h = s_n \]

\[ \phi k \sigma z^n = \phi(\sigma(z^{-1}e + 1 - e) - z^{-1}(1 - z)^2h)\sigma z^n \]

\[ + \phiren{\sigma e^{-1}z^{-1}(1 - z)^3[e, h] z^n} \]

\[ + \phi ( \text{Deer}. \]

\[ [\tilde{z}, h] = 1 \]

where
Let's now analyze the diagram

\[
0 \rightarrow T \xleftarrow{\alpha = \begin{pmatrix} 1 & \gamma \\ j^+ & 0 \end{pmatrix}} W^+ \oplus W^- \xrightarrow{\begin{pmatrix} \varepsilon & \eta \\ \delta & 0 \end{pmatrix}} W \rightarrow 0
\]

Here \( j^+, j^- \) are the obvious inclusions of \( T = \varepsilon^0 T \) in \( W^+, W^- \) (\( j^+: W^+ \rightarrow T \) are the maps corresponding to the cocycles \( 1, e, h, 0, \ldots \)) on \( W^+ \) and \( \{-1, 0, h, 1, e\} \) on \( W^- \). In general a cocycle on \( W^+ \) is a sequence \( (u_0^+, u_0^-, u_1, u_2, \ldots) \) with \( [d, u_0^+] = 0 \), \( [d, u_0^-] = u_0(1-e) + u_2 e \), and \( u_0^+ \) a cocycle on \( W^- \) is a sequence \( (u_2^-, u_1^-, u_0^-, u_0^-) \).

Check \( c^+ = (1, e, h, 0, \ldots) \), \( c^- = (-1, 0, h, 1, e) \) are cocycles:

\[
[d, e] = 0, \quad [d, h] = e(1-e) + oe; \quad [d, 1-e] = 0, \quad [d, h] = 0, \quad (1-e) + (1-e)e.
\]

Thus \( b = (c^+, c^-) \) is a map of complexes. One has \( ba = c^+j^+ + c^-j^- = e + 1-e = 1 \) so that the short exact exact sequence of complexes splits.

l is defined by \( lp = 1 - ab \).

Calculate \( b \). Given a cocycle on \( W^+ \oplus W^- \):

* \( (\ldots, u_1^- u_0^- | u_0^+, u_0^-, u_1, \ldots) \)

pull back via \( \alpha = \begin{pmatrix} 1 & \gamma \\ j^+ & 0 \end{pmatrix} \) to get \( u_0^+ - u_0^- = v \)

then pull back via \( b : (0, -h, -1+e | e, h, 0, 0, \ldots) \) to get \( (0, -vh, -v(1-e) | ve, vh, 0, \ldots) \).

Remove from * to get

\[
(\ldots, u_1^+, u_0^+, h^- u_0^-, u_0^+(1-e) + u_0^- e | u_0^+(1-e) + u_0^- e, u_0^- + u_0^+ h + u_0^- h, u_1, \ldots).
\]

(Note \( u_0^+ - (u_0^+ - u_0^-) e = u_0^+(1-e) + u_0^- e \), \( u_0^- + (u_0^+ - u_0^-)(1-e) = u_0^+(1-e) + u_0^- e \).
where we have

\((\ldots, u_{-1}, u_0| u_0^+, u_0^-, \ldots) \cdot l = \ldots\)

\((-\ldots, u_1, u_{-1}^+, u_{-1}^- h, u_0^+(1-e) + u_0^- e, u_0^- u_1^+ h + u_0^- h, u_1, \ldots\)\)

Return to

\[ T \xleftarrow{b} \xrightarrow{a} W^+ \oplus W^- \cong \text{lk} \quad \eta \mid \uparrow \xi \]

\[ U = W^+ \]

Let \(i = b \varepsilon, j = \eta \alpha, k^+ = \eta \text{lk} \varepsilon \). Then we have

\[ [a, k^+] = \eta [a, \text{lk} \varepsilon] \varepsilon = \eta (1 - ab) \varepsilon = 1 - g i. \]

We also have

\[ i = b \varepsilon = (i^+ - i)(1) = i^+ \]

\[ j = \eta \alpha = (1 \circ \alpha)(j^+_{-d}) = j^+ \]

and we know that \(i^+ j^+ = e\). Thus we get our \(A^\infty\) idempotent \(i^+ (k^+) j^+\) extending \(e\).

Calculate \(i^+ k^+ j^+ = b e \eta \text{lk} \varepsilon \eta \alpha\). Start with the identity \(\pi T\), pullback via \(b\) to get \((0, -h, -((e)|_e, h, 0, \ldots)\) then by \(e \eta\) getting \((C, 0, 0| e, h, 0, \ldots)\), then by \(k\) to get

\[ 0, e h, (e(1-e)) h - e h, 0 \]

\[ \ldots, u_{-1}, u_0, u_0', u_1, \ldots \]

Next we pull back via \(k\) and then

\[ p \varepsilon \eta \alpha = i^+ j^+ = j^+: T \to W^+ \]
This means all we have to do is to calculate the $S_0$ belonging to $u_1 = e_1 + e_2$, $u_0 = e - e_2$, $u_0 = h - e_1 h$ and the rest $0$.

\[ S_0 = u_1 e + u_0 (1 - e) + (-u_1 + 2u_0 - u_0) h \]
\[ = e e h + (h - e) (1 - e) + \frac{1}{2} h (e - e_2) h \]
\[ = e e h + h - h e - e h + e h + 2 e h - 2 e^2 h \]
\[ = h + [e, h] + 2 e h - 2 e^2 h \]
\[ = h + [e, h] - 2 e [e, h] \]
\[ = h + [e, h] - 2 e [e, h] - [e, h] (1 - e) \]
\[ = h - [e, [e, h]]. \]

Let’s straighten out the relation between solutions of

\[ [d, e_0] = 0 \]
\[ [d, e_1] = e_0 - e_0^k \]
\[ [d, e_2] = -e_0 e_1 + e_1 e_0 \]
\[ [d, e_3] = e_2 - e_0 e_2 + e_1^2 - e_2 e_0 \]
\[ [d, e_4] = -e_0 e_3 + e_1 e_2 - e_2 e_1 + e_3 e_0 \]
\[ [d, e_5] = e_4 - e_0 e_4 + e_1 e_3 - e_2 e_1 - e_3 e_1 - e_4 e_0 \]

and twisting cochains on the bar construction of the monomial algebra $ke$ with $e = e_2$, with values in a DG algebra $\Gamma$. The dual of $\text{Bar}(ke)$ is a poly ring $k[w]$ where $|w| = 1$ and $dw = -w^2$; note that $w \mapsto -w$ changes $dw = -w^2$ to $dw = w^2$. A twisting cochain from $\text{Bar}(ke)$ to $\Gamma$ is an
element \( \theta = \sum_{n \geq 0} \theta_n \omega^{n+1} \) of the tensor product DG algebra \( R \otimes k[[\omega]] \)
satisfying \( [d, \theta] + \theta^2 = 0 \). I should mention that \( \theta_n \in F_n = F^{-n} \) and that \( |\omega^1| = 1 \) for the upper indexing. One has
\[
[d, \theta] = \sum_{n \geq 0} [d, \theta_n] \omega^{n+1} + \sum_{n \geq 0} (-1)^n \theta_n [d, \omega^{n+1}] = \begin{cases} 0 & n \text{ odd} \\ -\omega^{n+2} & n \text{ even} \end{cases}
\]
\[
\theta^2 = \sum_{k, l \geq 0} (-1)^{k+1} \eta_k \theta_l \omega^{k+l+2}
\]

\[
[d, \theta] + \theta^2 = [d, \theta_0] \omega + ([d, \theta_1] - \theta_0 + \theta_0^2) \omega^2
\]
\[
+ ([d, \theta_2] - \theta_0 \theta_1 + \theta_1 \theta_0) \omega^3
\]
\[
+ ([d, \theta_3] - \theta_0 \theta_2 + \theta_1 \theta_1 + \theta_2 \theta_0) \omega^4
\]
\[
+ ([d, \theta_4] - \theta_0 \theta_3 + \theta_1 \theta_2 - \theta_2 \theta_1 + \theta_3 \theta_0) \omega^5
\]
\[
+ ([d, \theta_5] - \theta_1 \theta_4 + \theta_0 \theta_3 + \theta_1 \theta_3 + \theta_2 \theta_1 + \theta_3 \theta_0) \omega^6
\]
yielding
\[
[d, \theta_0] = 0 \quad \theta_0 = e_0
\]
\[
[d, \theta_1] = \theta_0 - \theta_0^2 \quad \theta_1 = e_1
\]
\[
[d, \theta_2] = \theta_0 \theta_1 - \theta_1 \theta_0 \quad \theta_2 = -e_2
\]
\[
[d, \theta_3] = -\theta_2 - \theta_0 \theta_2 - \theta_1 \theta_1 + \theta_2 \theta_0 \quad \theta_3 = -e_3
\]
\[
[d, \theta_4] = -\theta_0 \theta_3 - \theta_1 \theta_2 + \theta_2 \theta_1 - \theta_3 \theta_0 \quad \theta_4 = e_4
\]
\[
[d, \theta_5] = \theta_4 - \theta_0 \theta_4 - \theta_1 \theta_3 - \theta_2 \theta_2 + \theta_3 \theta_1 - \theta_4 \theta_0 \quad \theta_5 = e_5
\]
Thus it would appear that changing the signs of \( \Theta_n \) for \( n \equiv 2,3 \) (mod 4) converts the \( \Theta_n \) equations into the \( \alpha_n \) equations.

Consider now homotopy equivalences (earlier work: July 25, 1972 pp. 19-19), which yield a similar system of equations. The idea is that we have maps of complexes and homotopies

\[
\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\xrightarrow{f} & & \circ \circ \circ
\end{array}
\]

such that

\[
1 - gf = [d, h_X] \\
1 - fg = [d, h_Y]
\]

and compatibilities between these homotopies

\[
fh_X - hyf = [d, u] \quad gh_Y - h_Xg = [d, v]
\]

Introduce the operators on \( X \oplus Y \):

\[
\alpha_0 = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix} \quad \alpha_1 = \begin{pmatrix} h_X & 0 \\ 0 & h_Y \end{pmatrix} \quad \begin{pmatrix} o & v \\ u & 0 \end{pmatrix}
\]

Then

\[
[d, \alpha_0] = 0 \quad [d, \alpha_1] = \begin{pmatrix} 1 - gf & 0 \\ 0 & 1 - fg \end{pmatrix} = 1 - \alpha_0^2
\]

\[
[d, (u \quad v)] = \begin{pmatrix} 0 & gh_Y - h_Xg \\ fh_X - hyf & 0 \end{pmatrix} = \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix} \begin{pmatrix} h_X & 0 \\ 0 & h_Y \end{pmatrix} - \begin{pmatrix} h_X & 0 \\ 0 & h_Y \end{pmatrix} \begin{pmatrix} 0 & g \\ f & 0 \end{pmatrix}
\]

\[
= [\alpha_0, \alpha_1] = \alpha_0 \alpha_1 - \alpha_1 \alpha_0
\]

To find the higher equations, consider the case where \( X, Y \) are both SDR's of the same complex \( E \):
Let \( f = bi \), \( g = ja \). Then

\[
1 - gf = ji - jabi = j[d, c]i = [d, jci]h_x
\]
\[
1 - fg = ba - bya = b[d, k]i\alpha = [d, bkci] \frac{h_x}{h_y}
\]

\[
fh_x - hyf = bigci - bkabi = b(l - [d, k])ci - bk(l - [d, c])i
\]
\[
= b(- [d, k]c + k[d, c])i
= -b[d, k]c = [d, -bkci]
\]

\[
[d, jcka] = j(d, c)[k - c(d, k)]a = j((1-ab)k-c(k-uj))a = -(y\delta bka + jci)g(b)
= -gh_y + h_xg
\]

So put

\[
\alpha_0 = \begin{pmatrix} 0 & g(b) \\ b & 0 \end{pmatrix}, \quad \alpha_1 = \begin{pmatrix} jci \\ bka \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & jcka \\ bkc & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} jckci & 0 \\ 0 & bkc \end{pmatrix}
\]

Then

\[
[d, \alpha_0] = 0
\]
\[
[d, \alpha_1] = 1 - \alpha_0^2
\]
\[
[d, \alpha_2] = \begin{pmatrix} 0 & -(y\delta bka + jci)(b) \\ -(by + (c + bk)ka) & 0 \end{pmatrix} = -\alpha_0 \alpha_1 + \alpha_1 \alpha_0
\]
\[
\begin{bmatrix}
d, x_3
\end{bmatrix} =
\begin{bmatrix}
[d, j, ckei] & 0 \\
0 & [d, bkeka]
\end{bmatrix}
\begin{bmatrix}
\frac{j}{g}(-abke0 + cje - cka)i \\
0 & b(-jck + kabk - keij)a
\end{bmatrix}
= -x_0 x_2 + x_1^2 - x_2 x_0
\]

April 24, 1995

Analysis of the preceding formulas:

\[
\begin{bmatrix}
g & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
0 & g \\
b & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
0 & g \\
b & 0
\end{bmatrix}
= x_0
\]

\[
\begin{bmatrix}
g & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
k & 0 \\
0 & c
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
g & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
k & 0 \\
0 & c
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
g & 0 \\
0 & bka
\end{bmatrix}
= x_1
\]

Let us put \(x_n = \begin{bmatrix}
g & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
k & 0 \\
0 & c
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}.
Then
\[
x_2 = \begin{bmatrix}
g & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
0 & k & 0 \\
k & 0 & c
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
0 & gckak \\
bkeka & 0
\end{bmatrix}
\]

\[
x_3 = \begin{bmatrix}
g & 0 \\
0 & b
\end{bmatrix}
\begin{bmatrix}
0 & k & c \\
k & 0 & k
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & a
\end{bmatrix}
= \begin{bmatrix}
gckekai & 0 \\
0 & bkekka
\end{bmatrix}
\]

Here is the general pattern. We have an SDR situation:

\[
\begin{align*}
X &= \begin{bmatrix} 0 \\ b \end{bmatrix} \\
\oplus & \leftrightarrow \oplus \\
\bigcirc & \leftrightarrow \bigcirc \\
Y & \equiv \begin{bmatrix} 0 \\ a \end{bmatrix} \\
\oplus & \leftrightarrow \oplus
\end{align*}
\]
and an odd involution $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $E \oplus E$.

Then $\alpha_n = \eta(F^i \delta)^n E \delta$

satisfies

$[d, \alpha_n] = \eta \sum_{i=1}^{n-1} (F^i \delta)^{i-1} F[d, \delta] (F^{-i} \delta)^{n-i} F \delta$

$= \eta F(1 - \delta \eta)(F^i \delta)^{n-i} F \delta + \eta \sum_{i=2}^{n-1} (F^i \delta)^{i-1} (F^{-i} \delta)(F^{-i+1} \delta)(F^i \delta) F \delta$

$+ \eta (-1)^{n-i} (F^i \delta)^{n-i} F(1 - \delta \eta) F \delta$

Note $\eta F(1)(F^i \delta)^{i-1} = \eta F(F^i \delta)^{i-2} = 0$ as $\delta \eta = 0$.

$[d, \alpha_n] = -\eta \sum_{i=2}^{n-1} (F^i \delta)^{i-1} (F^{-i} \delta)(\eta(F^i \delta)^{n-i} F \delta) - (-1)^{n-1} (\eta(F^i \delta)^{n-i} F \delta)(\eta \delta F \delta)$

$= -\alpha_0 \alpha_{n-1} + \alpha_1 \alpha_{n-2} - \cdots - (-1)^{n-1} \alpha_n \alpha_0$

Suppose we start with the equations

$[d, \alpha_0] = 0$

$[d, \alpha_1] = 1 - \alpha_0^2$

$[d, \alpha_2] = -\alpha_0 \alpha_1 + \alpha_1 \alpha_0$

$[d, \alpha_3] = -\alpha_0 \alpha_2 + \alpha_1^2 - \alpha_2 \alpha_0$

and put $\alpha_0 = 2e_0 - 1$, $\alpha_1 = 4e_1$, $\alpha_2 = 8e_2$, $\alpha_3 = 16e_3$.

Then $[d, 2e_0 - 1] = 0 \implies [d, e_0] = 0$.

$[d, 4e_1] = 1 - \alpha_0^2 = 1 - (2e_0 - 1)^2 = 4(e_0 - e_0^2) \implies [d, e_1] = e_0 - e_0^2$

$[d, 8e_2] = -(2e_0 - 1)4e_1 + 4e_1(2e_0 - 1) = -8e_0 e_1 + 4e_1 + 8e_1 e_0 - 4e_1$

$\implies [d, e_2] = -e_0 e_1 + e_1 e_0$
\[ [d, 16e_3] = -(2e_0-1)8e_2 + (4e_1)^2 - (8e_2)(2e_0-1) = 8e_2 - 16e_0e_2 + 16e_1^2 - 16e_2e_0 + 8e_2 \]

\[ \Rightarrow [d, e_3] = e_2 - e_0e_2 + e_1^2 - e_2e_0 \]

Another way to see these powers of 2 is from the formulas:

\[ \alpha_0 = \eta^F e, \quad \alpha_1 = \eta^F s^F e, \quad \alpha_2 = \eta^F s^F s^F e \]

First observe that since \( s^2 = \eta s = s \eta = \eta \) and \( \eta^2 = 1 \), we have on setting \( \rho = \frac{F+1}{2} \) or \( F = 2\rho - 1 \):

\[ \alpha_0 = 2\eta^F e - 1, \quad \alpha_1 = 4\eta^F s^F e, \quad \alpha_2 = 8\eta^F s^F s^F e \]

Moreover, \( e_0 = \eta^F e, \quad e_1 = \eta^F s^F e, \quad e_2 = \eta^F s^F s^F e \), can be seen to satisfy the e-equations. In fact, the e-equations hold for \( e_n = \eta^F e \), where \( 1-\eta = [d, h] \). So in the case of interest one has taken \( 1-\eta = [d, s] \) and applied \( p \) on both sides.

\[ [d, pe^F p] = p(1-\eta)p = p(\eta s \eta p) \]

Visually, one has

\[ T \xrightarrow{\eta} U \xrightarrow{p} pU \]

so \( U \) is a homotopy a direct summand of \( p \) and \( pU \) is a direct summand of \( U \).
Let $X \rightarrow Y \rightarrow Z$ be maps of complexes.

Let $M(f)_n = \begin{bmatrix} x_n \\ \oplus \\ x_{n-1} \\ \oplus \\ Y_n \\ \oplus \\ Z_n \end{bmatrix}$ be

the mapping cylinder of $f$; picture $X \rightarrow Y$. Recall the maps $X \rightarrow M(f)$ $i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $j = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $p = (f \circ o \circ i)$ $k = (0 \circ 0 \circ 0)$ $o \circ 0 \circ 0 \circ 0 \circ 0 \circ 0$ satisfying: $[d, e] = [d, p] = [d, j] = 0$ $p i = f$ and the SDR relations: $p j = 1$ $1 - j p = [d, h]$ $h^2 = p h = b f = 0$.

Let $N = M(f) \times M(g); $ picture $N_n = X_n \oplus X_{n-1} \oplus Y_n \oplus Y_{n-1} \oplus Z_n$.

$b = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & g & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $k = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $d = \begin{pmatrix} d & -1 \\ -d & f \\ d & -1 \\ g & d \end{pmatrix}$ $N^g \rightarrow$ is an SDR.
Miscellaneous comments.

1. Double mapping cylinder for the maps \( T \leftarrow T \xrightarrow{f} T \) is the homotopy equivalence to \( T \), in fact one has an SDR of \( M \) onto \( T \) given by \( M \xrightarrow{h} T \)

\[ M_n = T_n \oplus T_{n-1} \oplus T_n \] \[ d = \begin{pmatrix} d & f \cdot i \\ -d & f \end{pmatrix} \]

\[ i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad p = \begin{pmatrix} f & 0 & 1 - f \end{pmatrix} \quad h = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

The embeddings correspond to \( f \) and \( 1 - f \) respectively. The doubly infinite iteration of \( W \):

\[ T \xrightarrow{f} T \xrightarrow{1-f} T \]

is the sum of subcomplexes \( \hat{W} - \hat{W}^+ \), such that \( \hat{W} - \hat{W}^+ = T \).
One gets then an exact sequence
\[ 0 \rightarrow T \rightarrow W^* \oplus W^+ \rightarrow W \rightarrow 0 \]
which realizes a \( \Delta \) of "descent."

\[ T \rightarrow \text{holin}(T, \eta_1 f) \oplus \text{holin}(T, f_1) \rightarrow \text{holin}(T, f_1(\eta_1 f)) \quad \]
\[ \text{holin}(T_1, \eta_1 T_1 f_1) \quad \text{holin}(T_1, T_1 f_1) \ldots \]

Suppose \( f = e \) where \( e - e^2 = [d, h] \). In
note in general that
\[ W = k[z, z^{-1}] \otimes T \oplus \sigma k[z, z^{-1}] T \]
\[ \frac{\beta}{\alpha} = 1 \otimes d + (\sigma^{-1} + \sigma^{-1} f) \frac{\beta}{\alpha} \]
\[ \sigma^2 q \approx \sigma \]
\[ [\beta, \sigma] = 1. \]

When \( f = e \) we have the near homotopy
\[ h = \sigma(z^{-1} e + e^{-1}) + (z^{-1} + 2 + z) h \]
\[ [\beta, h] = [1 \otimes d, \sigma] (e^{-1} + e) d, \sigma] (z^{-1} e + e^{-1}) \]
\[ = (z^{-1} + 2 + z) [\beta, h] + (e^{-1} + e z) [\beta, \sigma] (z^{-1} e + e^{-1}) \]
\[ -(z^{-1} + 2 + z) [e^{-1} + e z, h] d \]
\[ = z^{-1} (e^{-1} + e z) + 2 (e e^{-2}) + z (e e^{-2}) + (e^{-1}) e^{2} + e^{2} + z (e^{-2} + e^{2} + 1 e e^{-2}) \]
\[ - (z^{-1} + 2 + z) (1 + z) [e, h] d. \]
\[ [\beta, h] = 1 - z^{-1} (1 + z)^{3} [e, h] d. \]

There are fewer signs than before. The matrix
of \( h \) relative to \( z^{-1} \sigma, \sigma, \sigma, z f, z f, z f, z f, z f, z f \ldots \) is
Another point is that the term $z^2(1+z)^3[e, h^2]$ is symmetric in the sense that if you think of $|e|=2$ and $|h|=1$, then $|e|=1$ so $|2e|=-3$ and $|2e^2|=3$.

It seems to be a waste of time to continue with these calculations as things get much too complicated.

Note the map

\[
\begin{pmatrix}
  h \\
  e \\
  2h \\
  e-1 \\
  h
\end{pmatrix}
\]

\[
\begin{pmatrix}
  h \\
  2h \\
  e-1 \\
  h
\end{pmatrix}
\]

is $\sim 0$, a

homotopy being

2. Homology of the DGA $\Gamma = k(e, h)$ where $|e|=0$

$|h|=1$, $[d, e]=0$, $[d, h]=e-e^2$. See p.263 for background.

$\Gamma: \quad \text{AhA} \xrightarrow{d} \text{AhA} \xrightarrow{d} A \quad A = k[e, h, z]$

$A \otimes A = k[x, y, z] \otimes k[x, y, z]$

$Z_1 = (x-y)k[x, y]$, $B_i$ is ideal in $k[x, y]$ generated by $g(x)-g(y)$ with $g \in (x^2-x)k[x]$. Put $g = (x-x^3)f(x)$, then
\[ g(x)-g(y) = (f(x)-f(y)) (x-x^2) + f(y) (x-x^2-y+y^2) \]
\[ \in k[x,y] (x-y)(x-x^2) + k[x,y] (x-y)(1-x-y) \]
\[ \hbox{in } B_1 \]

Why? \((x-y)(x-x^2) \leftrightarrow (e-e^2)[e,h] = [d,h][e,h] = [d,h][e,h]\]
\((x-y)(1-x-y) \leftrightarrow [e,h] - e[e,h] - [e,h] e = [e-e^2,h] \]
\[ = [d,h], h] = [d,h] \]

Thus \(B_1 \leftarrow \begin{pmatrix} ((x-y)(x-x^2), (x-y)(1-x-y)) \end{pmatrix} \)
\[ H_1 = \frac{Z}{B_1} \cong \frac{k[x,y]}{(x-x^2) - (1-x-y)} = \frac{k[x]}{(x-x^2)} = \frac{k[e,k]}{e} \]

Conclusion is that \(H_1\) is 2-dimensional, really free of rank 2 over \(k\). It is generated as a bimodule over \(H_0 = k[x]/(x-x^2) = k \otimes k e\) by \([e,h] \leftrightarrow x-y\).

Now we know that \(H_1\) is spanned by the four elements \(e[e,h]e, e^2[e,h]e, e[e,h]e^2, e^2[e,h]e^2\).
Also \(e[e,h] e, e[e,h] e^2\) are boundaries, so we get a canonical isomorphism
\[ \Lambda^1(k \otimes k e) \cong H_1 \]
\[ de \mapsto \text{class of } [e,h] \]

Question: Does the isomorphism \(\Lambda^1(k \otimes k e) \cong H_4\)
for \(n=0,1\) extend to an isomorphism of graded algebras
\[ \Lambda^1(k \otimes k e) \cong H_4(F) ? \]

Change notation: Let \(d\) be replaced by \(\partial\).
Then \(\partial\) is the degree -1 derivation of \(\Gamma = A \langle h^\perp \rangle, A = k[e]\)
such that \(\partial(h) = e-e^2, \partial(a) = 0\). Let \(d\) be the degree +1 derivation such that \(d(a) = [h,a], d(h) = h^2\).
Then \[ d^2(a) = d[h, a] = [h^2, a] - [h, [h, a]] = 0 \]
\[ d^2(h) = d(h^2) = h^2 h - h h^2 = 0 \]
so \[ d^2 = 0. \]
Also \[ [d, \partial](a) = 0 + \partial[h, a] = [e - e^2, a] = 0 \]
\[ [d, \partial](h) = d(e - e^2) + \partial[e, h] = [h, e - e^2] + (e - e^2) h - h (e - e^2) = 0. \]
Thus \[ [d, \partial] = 0. \] Thus \( d \) induces a degree +1 derivation on \( H_*(\Gamma) \) such that \( d^2 = 0 \). This then induces a map of DGA's \( \Omega(k \oplus ke) \to H_*(\Gamma) \) extending the identity in degree 0.

April 30, 1995

Notation: \( \Gamma = k \langle e, h \rangle = A \langle h \rangle \), \( A = k[e] \), \( B = k[e]/(e - e^2) = k + k e \). One has a homomorphism

(1) \[ A \langle h \rangle \to B \langle h \rangle \]

compatible with the differentials \( \partial, d \) defined by

\[ \partial(e) = 0, \quad \partial(h) = e - e^2 \quad \text{on } A \langle h \rangle \]
\[ d(e) = [h, e], \quad d(h) = h^2 \]

resp. \( \partial = 0, \quad d(e) = [h, e], \quad d(h) = h^2 \quad \text{on } B \langle h \rangle \).

Thus we get a homomorphism on homology and \( \partial \)

(2) \[ H_*(\Gamma) \to B \langle h \rangle \]

In degree zero this is the obvious iso \( H_0(\Gamma) = B \).
In degree one, since \( H_1(\Gamma) \) is generated by the class of \( [h, e] \) as \( B \)-bimodule the image is contained in \( B(h e - e h) = \Omega^1 B \subset B \otimes B = B h B \). From our computation of \( H_1(\Gamma) \) we see that (2) is surjective.

\[ \square \]

It's now clear that \( H_*(\Gamma) \) is at least as big as \( \Omega \), namely our homomorphism \( \Omega B \to H_*(\Gamma) \) is onto.
Further evidence for $H_*(\Gamma) \cong \mathcal{O}B$.

Consider the $\mathcal{O}$-adic filtration of $\Gamma = A\langle h \rangle$ with respect to the ideal $J$ generated by $h$ and $\Delta(h) = e-e^2$. Then

$$\Gamma/J = A(e, h)/(h, e-e^2) = B$$

and because $\Gamma$ is a tensor algebra it should be true that

$$\bigoplus_{n \geq 0} J^n/J^{n+1} = T_B(J/J^2).$$

Picture:

\[
\begin{array}{cccc}
A & \rightarrow & \Gamma \\
\rightarrow & & \\
A_hA & \rightarrow & I \\
\rightarrow & \hspace{1cm} J \\
A_hA_hA & \rightarrow & I\cdot A + A\cdot I \\
\rightarrow & \hspace{1cm} J \\
\end{array}
\]

$J/J^2$ should be the complex

\[
\begin{array}{cccc}
BhB & \rightarrow & I/I^2 \\
\rightarrow & \hspace{1cm} (e-e^2) \\
B \otimes B & \rightarrow & B \\
\end{array}
\]

Thus $J/J^2$ should be quasi-regular $\mathcal{O}B[1]$ and so $\text{gr}^J(\Gamma)$ quasi-regular $\bigoplus_{n \geq 0} \mathcal{O}B[n]$.

Conclude that if the spectral sequence for this filtration converges, then $H_*(\Gamma) \cong \mathcal{O}B$. 
Let's start with a htepy retract situation

\[ U \xrightarrow{g} T \xrightarrow{i} T \]

Let \[ e_n = i \circ j \circ f \hspace{1cm} e_n \in \text{Ham}(T, T) \]

1. \[ [d, e_o] = 0 \]
2. \[ [d, e_1] = e_0 - e_o^2 \]
3. \[ [d, e_2] = -e_0 e_1 + e_1 e_0 \]

\[ \cdots \]

\[ [d, e_{n+1}] = \begin{cases} e_n & \text{even} \\ 0 & \text{odd} \end{cases} - \sum_{j=0}^{n} (-1)^j e_j e_{n-j} \]

Let

\[ d = \begin{pmatrix} d & 1 & -e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ -d & e_0 & e_1 & e_2 & \cdots \\ d & 1 & -e_0 & -e_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

Now, \[ T = T \oplus T[1] \oplus T[2] \oplus \cdots \]

The identity (2) are equivalent to \( d^2 = 0 \).

Let

\[ k = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

operate on \( \vec{T} = T \oplus T[1] \oplus T[2] \oplus \cdots \), \( |d| = -1 \).

The identities (2) are equivalent to \( d^2 = 0 \).

Let

\[ k = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]

operate on \( \vec{T} \), \( |k| = +1 \).
Then

\[ \dd t_k = \begin{pmatrix} 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ -d & e_0 & e_1 & e_2 & \cdots \\ d & 1-e_0 & -e_1 & \cdots \\ -d & e_0 & \cdots \\ d & \cdots \end{pmatrix} \]

\[ \dd h_d = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ d & 1-e_0 & -e_2 & -e_3 & \cdots \\ -d & e_0 & e_1 & e_2 \\ d & 1-e_0 & -e_1 \\ -d & e_0 \end{pmatrix} \]

\[ [\dd h_d] = \begin{pmatrix} 1-e_0 & -e_1 & -e_2 & -e_3 & \cdots \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \\ & & & & 1 \end{pmatrix} \]

= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} e_0 & e_1 & e_2 \cdots \end{pmatrix}

Let \( a = \dd \alpha : T \to \dd T \) and \( b = \dd \beta : \dd T \to T \). Note \( \dd a = ad \) and that \( a \) is injective. Since \( ab \) commutes with \( \dd \alpha \) \( ([\dd \alpha, [\dd \alpha, k]] = [\dd \alpha^2, k] = 0) \), we have \( a(b \dd a - \dd a b) = ab \dd a - \dd a ab = 0 \) \( \Rightarrow b \dd a = db \) by the injectivity.
Actually the last step should have been done differently, namely:

\[
[A, h] = I - \begin{pmatrix} e_0 & e_1 & e_2 & \cdots \\
0 & 0 & \ddots & \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

where

\[
\tilde{d} \tilde{c} = \tilde{c} d 
\]

Now clearly \( \tilde{d} \tilde{c} = \tilde{c} d \) so \( \tilde{c} : U \to \tilde{T} \) is a map of complexes. I claim \( \tilde{d} : \tilde{T} \to U \) is also a map of complexes.

\[
\tilde{d} \tilde{c} = (s, h_j, h_j^2, \ldots) \begin{pmatrix} d & -c h_j & -i h_j^2 \\
0 & y & c h_j \\
0 & 0 & 1-c y \\
0 & 0 & 0 \\
\end{pmatrix} = d \tilde{c}.
\]

\[
\tilde{d} \tilde{c} = (1-i) c h - i c h
\]

\[
\tilde{d} \tilde{c} = ((1-g) h - h (1-j)) d + h^2 j d
\]

\[
d h^2 j = (1-g) h^2 - h (1-j) h + h^2 (1-j) d - h^3 j d
\]
Thus we have maps

\[ \tilde{\gamma} : U \leftrightarrow \tilde{T} \]

and a map \( \tilde{\gamma} \) such that \( \tilde{\gamma} \tilde{\gamma} = [\tilde{\gamma}, k] \).

Also \( \tilde{\gamma} = (g, b, \ldots)(i) = i = 1 - [d, h] \).

Thus \( \tilde{\gamma}, \tilde{\gamma} \) give a homotopy equivalence of \( U \) and \( \tilde{T} \).

**Misc.**

1. If \( e \) is an operator on a module \( M \), then the two canonical dilations of it to an idempotent are \( (e, e - e^2) \) and \( (e, 1) \) on \( M \oplus M \).

2. If \( e - e^2 \in I \), then \( I^2 = 0 \), then

\[ (e + \delta e)^2 = e + \delta e \quad \text{where} \quad \delta e = (2e - 1)(e - e^2). \]

3. Given \( e_0 = e, h \) with \( [d, e] = 0 \) \( \quad [d, h] = e - e^2 \),

we know that \( e_1 = h - ad(e)^2 h = h - e^2 h + 2he - he^2 \)

satisfies \( [d, e_1] = e_0 - e_0^2 \) and \( [d, e_2] = -e_0 e_1 + e_1 e_0 \)

for some \( e_2 \). I have calculated that

\[ e_2 = -h^2 + 3e h^2 - 4he + 3h^2 e \]

works.
The problem is to show how a homotopy idempotent can be refined to an $A_\infty$ idempotent. I hope to do this by studying the complexes $T^{(n)} = T \oplus T \oplus \cdots \oplus T$

with differential

$$
\begin{pmatrix}
    \text{d} & -e_0 & -e_1 & \cdots & -e_n \\
    -\text{d} & e_0 & & \cdots & +e_{n-1} \\
    & -\text{d} & e_0 & & \cdots & -e_{n-2} \\
    & & & \ddots & \ddots & \ddots \\
    & & & & \text{d} & -e_0 \\
\end{pmatrix}
= I - \begin{pmatrix}
    e_0 & +e_1 & +e_2 & \cdots & +e_n & 0 \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    -e_{n-1} & \cdots & \cdots & \cdots & \cdots & \cdots \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
= I - \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots \\
    0 & \cdots & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
e_0 & e_1 & \cdots & e_{n+1} \\
\end{pmatrix}
- \begin{pmatrix}
    e_{n+1} \\
    \vdots \\
    0 \\
\end{pmatrix}
\begin{pmatrix}
    0 & 0 & \cdots & 1 \\
\end{pmatrix}
$$

Examples

$$
\begin{pmatrix}
    \text{d} & -e_0 \\
    -\text{d} & 1 \\
\end{pmatrix}
\begin{pmatrix}
    0 & 0 \\
    1 & 0 \\
\end{pmatrix}
= \begin{pmatrix}
    1-e_0 & \text{e_0} \\
    \text{e_0} & 1-e_0 \\
\end{pmatrix}
= \begin{pmatrix}
    1 & 0 \\
    0 & 1 \\
\end{pmatrix}
- \begin{pmatrix}
    0 & \text{e_0} \\
    \text{e_0} & 0 \\
\end{pmatrix}
\begin{pmatrix}
    0 & 0 \\
    1 & 0 \\
\end{pmatrix}
$$

\[\text{this ends with}\]
\[1-e_0 \text{ for n+1 even, e_0 for n+1 odd}\]
\[
\begin{pmatrix}
1 - e_0 & -e_1 \\
-d & e_0 \\
-d & e_0
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} = 
\begin{pmatrix}
1-e_0 & -e_1 & 0 \\
\alpha & 1 \beta & -e_1 \\
\delta & e_0 & e_0
\end{pmatrix}
\]

\[
= 
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
- 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} 
\begin{pmatrix}
e_0 & e_1 & e_2
\end{pmatrix}
= 
\begin{pmatrix}
-f_2 \\
-te_1 \\
-1-e_0
\end{pmatrix}
\]

Let's examine the case of \( T^{(1)} \). Note that we have maps of complexes

\[
T \xrightarrow{(e_0 \ e_1)} T^{(1)} \xrightarrow{(-e_1 \ e_0)} T[1]
\]

because

\[
1 \sim \underbrace{\begin{pmatrix} 1 \\
0 \end{pmatrix}}_{\alpha} (e_0 \ e_1) + \underbrace{\begin{pmatrix} -e_1 \\
e_0 \end{pmatrix}}_{\beta} (0 \ 1)
\]

and \( \beta \alpha = 0 \), it follows that \( \alpha, \beta \) are orthogonal idempotents up to homotopy.

Note that \( T^{(1)} \) depends only on \( e_0 \), but the upper operators in (x) depend on \( e_1 \). The hope I have is that by constructing a suitable homotopy splitting of \( T^{(1)} \) I am forced to modify \( e_1 \) so that \( e_2 \) exists. In (x) the composition of the top arrows \( -e_1 + e_1 e_0 \) is \( 0 \) iff \( e_2 \in \mathcal{F} \).
Abstract question. Suppose $X, Y$ objects in a category having the same object $Z$ as retract

\[
\begin{array}{c}
X \xleftarrow{i} \overset{a}{\rightarrow} Y \xrightarrow{b} Z \xrightarrow{z} Y \\
\end{array}
\]

Let $u = a_j$, $v = b_j$. Then

\[
\begin{align*}
uv &= a_j b = ab = \text{the projector on } Y \\
vu &= b_j a = i_j = \text{the projector on } X
\end{align*}
\]

and

\[
\begin{align*}
uvu &= ab a_j = ay = u \\
vuv &= v b = cb = v
\end{align*}
\]

I think the way to summarize the preceding is to say that an isomorphism between two objects $(X, c_0)$, $(Y, c_1)$ in the Karoubian envelope of a category is specified by a pair of maps $X \xleftarrow{u} Y$ such that $uvu = u$, $vuv = v$, $vu = e$, $uv = e'$.

Let's apply this to

\[
T \xleftarrow{v = (e_0 \ e_1)} \overset{T(1)}{\rightarrow} T(c_1)
\]

$u = (1)$

Then

\[
\begin{align*}
uvu &= (1 \ 0) (e_0 \ e_1) (1) = (e_0) \quad \text{so} \\
\mu - uvu &= (1 - e_0)
\end{align*}
\]
\[
\begin{pmatrix} 1 - e_0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - e_0 \\ \mathbf{0} \\
1 - e_0 \end{pmatrix}(0)
\]
\[
= \begin{pmatrix} d & 1 - e_0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix}(1)
\]
\[
= \begin{pmatrix} d & 1 - e_0 \\ -d & 1 \end{pmatrix}(0) + (0) d
\]

showing that \( u \sim \nu \nu u \).

Next \( \nu \nu \nu = (e_0 e_1)(1)(e_0 e_1) = (e_0 e_0 e_0 e_1) \)

so \( \nu - \nu \nu \nu = ((1 - e_0)e_0, (1 - e_0)e_1) \). Now

\[
(e_0 e_1)\begin{pmatrix} d & 1 - e_0 \\ -d & 1 \end{pmatrix} = (e_0 e_1)(1 - e_0)
\]
\[
= (e_0 (1 - e_0), e_1 (1 - e_0))
\]
is \( \sim 0 \). Subtracting this from \( \nu - \nu \nu \nu \) we find

\( \nu - \nu \nu \nu \sim (0, -e_0 e_1 + e_1 e_0) \)

We want this to be \( \sim 0 \) i.e. of the form

\[
(\alpha \beta) \begin{pmatrix} d & 1 - e_0 \\ -d & 1 \end{pmatrix} - d(\alpha \beta)
\]
\[
= (-[d, \alpha] \quad \alpha (1 - e_0) - [d, \beta])
\]

Thus we want to find \( \alpha, \beta \) such that

\[
[d, \alpha] = 0
\]
\[
\alpha (1 - e_0) - [d, \beta] = -[e_0, e_1]
\]
I believe we know this is possible iff \([e_0, e_1] = 0\).
Similarly consider
\[ T(1) \quad u' = \begin{pmatrix} 0 & 1 \end{pmatrix} \quad T[1] \]
\[ v' = \begin{pmatrix} -e_1 \\ e_0 \end{pmatrix} \]

Then I've calculated that \( u' - u'v'u' \sim 0 \)
but that \( v' - v'u'v' \sim 0 \) iff \(-e_0[e_0, e_1]\)
is a boundary.

Thus I want \( e_0[e_0, e_1] \sim 0 \) and
\( [e_0, e_1]e_0 \sim 0 \), which I believe happens iff
\( [e_0, e_1] \sim 0 \).
May 4, 1995

I can now give the inductive construction which refines an homotopy idempotent to an $A_{\infty}$-idempotent.

We start with $e_0$ in $T$ such that $[d, e_0] = 0$ and $e_0 \sim e_0^2$. Let $e_1$ be such that $[d, e_1] = e_0 - e_0^2$. Form $T^{(1)} = T \oplus oT$ with differential $(d, 1 - e_0)$. We have maps of complexes

\[
\begin{align*}
T^0 & \xrightarrow{v = (e_0, e_1)} T^{(1)} \xrightarrow{v' = (-e_1, e_0)} T^{[1]} \\
u & = (1, 0) \quad u' = (0, 1)
\end{align*}
\]

The arrows at the bottom are part of a complex. One has

\[
I = 
\begin{bmatrix}
(d, 1 - e_0) & (0, 0) \\
(-d, 1) & (1, 0)
\end{bmatrix} + 
\begin{bmatrix}
(1) & (0, e_1) \\
(0, e_0) & (0, 1)
\end{bmatrix} = I
\]

\[
u' u = 0, \quad v u = e_0, \quad u' v' = e_0, \quad v v' = -e_0 e_1 + e_1 e_0
\]

\begin{itemize}
\item Let $z = v v' = -e_0 e_1 + e_1 e_0$. The homology class of $z$ is an obstruction to finding $e_2$ such that $[d, e_2] = -e_0 e_1 + e_1 e_0$.
\item We have
\[
v v' = v\left([d, h] + u v + v' u'\right) v' = v u v' v + v v' u' v' + [d, v h v'] \]
\item i.e.,
\[
z = e_0 z + z e_0 + [d, v h v']
\end{itemize}

This implies $(1 - e) z \sim z e_0$, $z(1 - e) \sim e_0 z$ hence
\((-e)z(1-e) \sim 0\) and \(e_0 \varepsilon e_0 \sim 0\), so

\[ z \approx e_0 \varepsilon (1-e_0) + (1-e_0)ze_0. \]

Let's consider now a change from \(e_0\) to \(\varepsilon_1 = e_1 + \delta e_1\), where \([d, \delta e_1] = 0\) so that \([d, \varepsilon_1] = e_0 - e_2\). Then \(\delta z = -e_0 \delta e_1 + \delta e_1 e_0\) and we would like to arrange \(z + \delta z \sim 0\) i.e.

\[ z = e_0 \delta e_1 - \delta e_1 e_0. \]

This implies \(e_0 \varepsilon (1-e_0) \approx e_0 \delta e_1 (1-e_0)\)

\[(1-e_0)z e_0 \sim -(1-e_0) \delta e_1 e_0\]

so if we put \(\delta e_1 = e_0 \varepsilon (1-e_0) - (1-e_0)ze_0\) we have

\[ e_0 \delta e_1 - \delta e_1 e_0 \equiv e_0^2 \varepsilon (1-e_0) - e_0 (1-e_0)ze_0 \]

\[ -e_0 \varepsilon (1-e_0) e_0 + (1-e_0)ze_0^2 \]

\[ \sim e_0 \varepsilon (1-e_0) + (1-e_0)ze_0 \sim z\]

as desired. Actually a simpler choice is

\[ \delta e_1 = z(1-e_0) - (1-e_0)z = [e_0, z] \]

since \(e_0 \delta e_1 = e_0 \varepsilon (1-e_0) - e_0 (1-e_0)z \sim e_0 \varepsilon (1-e_0) \sim z,

-\delta e_1 e_0 \sim z(1-e_0) e_0 + (1-e_0)ze_0\]

For this choice of \(\delta e_1\), we have the choice

\[ \varepsilon_1 = e_1 + \delta e_1 = e_1 + [e_0, z] = e_1 - [e_0, [e_0, e_1]] \]

which was found before.

Next stage. Suppose \(e_0, e_1, e_2\) given satisfying the first 3 \(A_\infty\)-identity equations we have maps.
\[ u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ T \leftarrow T^{(2)} \rightarrow T^{(2)} \]

\[ \mathbf{v} = (e_0, e_1, e_2) \quad \mathbf{v}' = (e_0, e_1, e_2) \]

\[ 1 = \begin{pmatrix} d & -e_0 & -e_1 \\ -d & e_0 & e_1 \\ d & e_0 & e_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (e_0, e_1, e_2) + \begin{pmatrix} -e_2 \\ -e_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \]

\[ 1 = u\mathbf{v} + \mathbf{v}'u' + [d, h] \]

\[ u'u = 0, \quad vv = e_0, \quad u'v' = 1 - e_0, \quad vv' = z \quad \text{where} \]

\[ z = -e_0e_2 + e_1^2 + e_2(1 - e_0) \]

One then has

\[ z = vv' = vv' + vv'u'v' + [d, h]v' \]

\[ z = e_0z + z(1 - e_0) + [d, v hv'] \]

This implies

\[ (1 - e_0)z \sim z(1 - e_0) \quad \text{and} \]

\[ z \sim e_0ze_0 + (1 - e_0)z(1 - e_0) \]

Let's now consider a change \( \delta e_2 \) such that

\[ [d, \delta e_2] = 0. \]

Then

\[ \delta z = -e_0\delta e_2 + \delta e_2(1 - e_0), \]

and we would like

\[ z + \delta z \sim 0 \quad \text{i.e.} \]

\[ z = e_0\delta e_2 - \delta e_2(1 - e_0) \]

The simplest choice appears to be

\[ \delta e_2 = z e_0 - (1 - e_0)z \]

for

\[ e_0\delta e_2 = e_0ze_0 - e_0(1 - e_0)z + z(1 - e_0)z(1 - e_0) \]

\[ -\delta e_2(1 - e_0) = -ze_0(1 - e_0) + (1 - e_0)z(1 - e_0) \]
Next consider $T^{(3)}$ whose diff is
\[
\begin{pmatrix}
  d & 1-e_0 & e_1 & e_2 \\
  -d & e_0 & e_1 \\
  d & 1-e_0 \\
  -d
\end{pmatrix},
\]
(We recognize this as the cone on the maps
\[
\begin{pmatrix}
  -e_2 \\
  e_1 \\
  1-e_0
\end{pmatrix}: T[2] \to T^{(2)},
\]
but this doesn't seem useful.)

We assume $e_2$ has been modified so that there exists an $e_3$ such that $[d, e_3] = -e_0 e_2 + e_1^2 + e_2 (1-e_0)$. Then we have the identity
\[
l = \begin{pmatrix}
  d & 1-e_0 & e_1 & e_2 \\
  -d & e_0 & e_1 \\
  d & 1-e_0 \\
  -d
\end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix}
  1 \\
  e_0 e_1 e_2 e_3 \\
  0 \\
  e_0
\end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ u & v & v' & u' \end{pmatrix}
\]
leading to maps
\[
T \xrightarrow{\nu} T^{(3)} \xleftarrow{\nu'} T[3]
\]
such that
\[
l = \nu \nu' + \nu' \nu' + [d, h]
\]

$\nu' \nu = 0$, $\nu \nu = e_0$, $\nu' \nu' = e_0$, $\nu \nu' = e_0$, $\nu' = z$ where
\[
z = -e_0 e_2 + e_1 e_2 - e_2 e_1 + e_3 e_0
\]

One has
\[
\nu \nu' = \nu \nu \nu' + \nu \nu' \nu' \nu' + [d, \nu \nu' \nu']
\]
or
\[
z = e_0 z + z e_0.
\]
As before for $T^{(2)}$
\[
\delta z = -e_0 \delta e_3 + \delta e_3 e_0,
\]
so we can take
\[
\delta e_3 = [e_0, z] \quad \tilde{e}_3 = e_3 + [e_0, z]
\]
to kill the class of $z$, and then $e_0 \notin \mathbb{F}$.
Construction: Let $\tilde{T} = T \oplus \sigma T \oplus \sigma^2 T \oplus \ldots$
equipped with the twisted differential given by the $A_\infty$-idempotent $e_0, e_1, \ldots$. We have the identity
\[
I = \begin{bmatrix}
(d & 1-e_0 & \cdots \\
-1 & \ddots & \vdots \\
\vdots & \ddots & 1
\end{bmatrix} \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} + \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
e_0 & e_1 & e_2 & \cdots
\end{pmatrix}
\]

We can view $\tilde{T}$ as a kind of double complex with the columns all equal to $T$

\[
\begin{array}{c}
T_0 \leftarrow e_0 \leftarrow T_1 \leftarrow e_0 T_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
T_1 \leftarrow e_0 \leftarrow T_2 \leftarrow e_0 T_3 \\
\downarrow \quad \downarrow \quad \downarrow \\
T_2 \leftarrow e_0 \leftarrow T_3 \leftarrow e_0 T_4 \\
\downarrow \quad \downarrow \quad \downarrow \\
T_3 \leftarrow e_0 \leftarrow T_4 \leftarrow e_0 T_5 \\
\downarrow \quad \downarrow \quad \downarrow \\
\vdots \quad \vdots \quad \vdots \\
\end{array}
\]

except that the total differential has higher components like the differentials in a spectral sequence.

Now suppose $T$ supported in $[0, m]$. Then $e_{m+1} = e_{m+2} = \ldots = 0$ for obvious reasons. I claim that $(e_0, e_1, \ldots); \tilde{T} \rightarrow T$ is zero in degrees $> m$.

In effect this shows that $(e_0, \ldots)$ does to $(\tilde{T})_m$ and it's evident that $(\tilde{T})_{m+1} \rightarrow T_{m+1}$ is zero.

In fact this picture is unnecessary at
\((e_0, e_1, \ldots) : \tilde{T} \rightarrow T\) is of degree zero and hence zero on \((\tilde{T})_n\) for \(n > m\) since \(T_n = 0\) there.

Thus on \(\tilde{T}\) we have \([d, h] = 1\) in degrees \(> m\).

Now consider more generally a complex \(E\) with a homotopy \(h\) satisfying \([d, h] = 1\) in degrees \(> m\):

\[
\begin{align*}
E_{m+2} & \xrightarrow{d} E_{m+1} & \xrightarrow{h_{m+1}} & E_m \\
\end{align*}
\]

Let \(h' = h\) on \(E_n\) for \(n \geq m+1\) and \(h' = 0\) on \(E_n\) for \(n < m\). Then

\[
[d, h'] = \begin{cases} 
1 & \text{on } E_n \quad n \geq m+1 \\
0 & \text{on } E_n \quad n < m
\end{cases}
\]

Note that \(dh'_m\) on \(E_m\) is a projector, since

\[
d_{m+1}h'_m d_{m+1} = d_{m+1}(h_m d_{m+1} + d_{m+2} h_{m+1}) = d_{m+1}.
\]

Thus \([d, h']\) is a projector on the complex \(E\), and splits \(E\) into \([d, h'] E \oplus (1 - [d, h']) E\). The former is contractible, so the latter is \(h.e.q\) to \(E\). The latter is

\[
0 \rightarrow 0 \rightarrow (1-d_{m+1}h'_m)E_m \rightarrow E_{m-1} \rightarrow \ldots
\]

This argument applied to \(\tilde{T}\) shows that \(\tilde{T}\) is \(h.e.q\) to a subcomplex \(\tilde{T}\) in degrees \(< m\) to a direct summand of \((\tilde{T})_m\) in degree \(m\) and to zero in degrees \(> m\).
Consider a Morita context \( (R, P, Q) \) and let \( A = QP < R \), \( B = PA < S \). Let \( U \) be a \( P \)-perfect complex over \( R \) such that \( U/AU \) is contractible as a complex of \( R/A \) modules, i.e. \( \exists \overline{h} \in U/AU \) such that \([d, \overline{h}] = 1\), and \( \overline{h} \) commutes with \( R \)-multiplication. Since \( U \) is projective in each degree \( \exists h \in U \) compatible with \( R \)-multiplication which lifts \( \overline{h} \):

\[
\begin{array}{ccc}
U & \xrightarrow{h} & U \\
\downarrow & & \downarrow \\
U/AU & \xrightarrow{\overline{h}} & U/AU
\end{array}
\]

Put \( f = 1 - [d, h] \) in \( U \). One has

\[
\begin{array}{cccccc}
& & & 1 & & \\
& & & \downarrow & & \\
0 & \rightarrow & A_\otimes R U & \xrightarrow{\mu} & U & \rightarrow & U/AU & \rightarrow & 0 \\
& & & \downarrow{1} & & \downarrow{1} & \downarrow{1 - [d, h]} = 0 \\
& & & \downarrow{f} & \downarrow{f} & \downarrow{f} & \\
0 & \rightarrow & A_\otimes R U & \xrightarrow{\mu} & U & \rightarrow & U/AU & \rightarrow & 0
\end{array}
\]

so there is a unique \( \varphi : U \rightarrow A_\otimes R U \) such that \( f = \mu \varphi \). Moreover \( \varphi \mu = 1_\otimes f \). Here we have used flatness of \( U \) for the exactness of the rows above.

We then have

\[
\begin{array}{ccc}
0 & \rightarrow & \otimes h \\
& & \downarrow{\overline{h}} \\
A_\otimes R U & \xrightarrow{\mu} & U
\end{array}
\]

satisfying

\[
[d, 1_{\otimes h}] = 1_{\otimes [d, h]} = 1_{\otimes 1} - 1_{\otimes f} = 1_{\otimes A_\otimes R U - \varphi \mu} \\
[d, h] = 1 - f = 1 - \mu \varphi
\]
so that $\eta$ is a homotopy inverse for $\mu : A \otimes_R U \to U$. (Notice also the compatibility of the homotopies with $\mu$:

$\mu(1 \otimes h) = h \mu$, which means we have a contraction on Cone($\mu$):

$$
\begin{pmatrix}
  d & \mu \\
  -d & -\mu(1 \otimes h) - h \mu
\end{pmatrix}
\begin{pmatrix}
  d h + d d + \mu \eta \\
  -d \eta + \mu \eta
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 \\
  0 & 1 \otimes 1
\end{pmatrix}
$$

We can iterate this homotopy equivalence to higher order:

$$
\begin{array}{c}
\longrightarrow A^{(2)} \otimes_R U \\ \eta \mu = \mu \eta
\end{array}
\begin{array}{c}
\longleftarrow A \otimes_R U \\ \eta \mu = \mu \eta
\end{array}
$$

Let's call a complex of $R$-modules $E$-formal (wrt the ideal $A$) when $\mu : A \otimes_R E \to E$ is a homotopy equivalence. Formally it then follows that $M \longrightarrow M \otimes_R E$ from right modules to complexes carries a nil-adjunction into a $b$-equivalence.

For example consider $Q \otimes S P \longrightarrow A$, whose kernel $K$ is killed by $A$:

$$
\begin{array}{c}
K \otimes_R A \\ Q \otimes S P \otimes A \\ Q \otimes_S P \\
\end{array}
\begin{array}{c}
\longrightarrow A \otimes A \\
\longrightarrow A \otimes A \\
\longrightarrow A
\end{array}
$$

because $\left( \sum_{e \in K} \phi_e \otimes p_e \right) \otimes (s \otimes p) \longrightarrow \sum_{e \in K} \phi_e \otimes p_e \otimes p = 0$
Better method is to observe that
the two maps \( Q \otimes P \otimes Q \otimes P \rightarrow Q \otimes P \)
sending \( q_1 \otimes q_2 \otimes q_3 \otimes q_4 \) to \( q_1 p_1 q_2 p_2 \) and
\( q_1 \otimes p_1 q_2 \otimes p_2 \) resp. coincide. The former
factors through \( A \otimes R, Q \otimes P = Q \otimes P \otimes Q \otimes P / K \otimes Q \otimes P, \)
the latter factors through \( Q \otimes P \otimes A = \chi / Q \otimes P \otimes K, \)
therefore we get a well-defined map
\[
A \otimes A \rightarrow Q \otimes S
\]
such that \( q_1 p_1 \otimes q_2 p_2 \mapsto q_1 p_1 q_2 p_2 = q_1 \otimes p_1 q_2 \otimes p_2. \)

Anyway from the commutative diagram of \( R \)-modules
\[
\begin{array}{ccc}
Q \otimes P \otimes A & \rightarrow & A \otimes A \\
\downarrow \otimes \mu & & \downarrow \mu \\
Q \otimes S \otimes P & \rightarrow & A
\end{array}
\]
we get a comm. diagram of complexes
\[
\begin{array}{ccc}
Q \otimes P \otimes A \otimes U & \rightarrow & A \otimes R \otimes A \otimes U \\
\downarrow & & \downarrow \\
Q \otimes S \otimes P \otimes U & \rightarrow & A \otimes R \otimes U
\end{array}
\]
where the vertical maps are bi-\( q_1 \)'s since \( U \) is
\( h \)-firm. Conclude \( Q \otimes S \otimes P \otimes U \rightarrow A \otimes R \otimes U \) is a
\( h \)-equiv.

Next consider \( P \otimes R \). We know this is
\( h \)-firm, since \( - \otimes P \) carries \( B \)-nil rios to \( A \)-nil rios.
and $- \otimes_R U$ carries $A$-nil iso to h.equiv. We want to show that $P \otimes_R U$ is h.equiv. to a s.perfect complex.

We know the map $Q \otimes_R P \otimes_R U \rightarrow U$ is a homotopy equivalence. Let $q : U \rightarrow Q \otimes_R P \otimes_R U$ be a homotopy inverse so that in particular $vp = 1_U$. Because $U$ is strictly perfect, $q$ is equivalent to a $0$-cycle in

$$\left( U^* \otimes_R Q \right) \otimes_R (P \otimes_R U) \simeq \text{Hom}_R(U, Q \otimes_R P \otimes_R U)$$

Thus for each degree $n$ we can write

$$q_n = \sum_{j=1}^n \xi_j \otimes v_j \quad \xi_j \in \text{Hom}_R(U_n, Q) \quad v_j \in P \otimes_R U_n$$

so $q_n$ factors

$$U_n \xrightarrow{q_n} Q \otimes_R P \otimes_R U_n$$

and $1 \otimes q_n$ factors

$$P \otimes_R U_n \xrightarrow{1 \otimes q_n} P \otimes_R Q \otimes_R P \otimes_R U_n \xrightarrow{1 \otimes v_j} P \otimes_R U_n$$

Thus $1 \otimes q_n$ is nuclear. This shows that the identity map of $P \otimes_R U$ has a deformation $1 \otimes vp$ which is nuclear. (I should have mentioned that $U_n$ being $A$-nil, there are only finitely many $q_n \neq 0$.)
So we see that $P \otimes R U$ is h.e.g. to a s-perfect complex.

---

We have now shown that if $U$ is s-perfect + h-firm, then $P \otimes R U$ is h-perfect + h-firm.

It follows that $U$ is h-perfect + h-firm $\iff P \otimes R U$ is h-perfect and h-firm.

Additional comments.

1. Suppose we only assume that $U$ is an h-firm complex of projective modules. Then we should still be able to factor $\phi_n$.

\[
\begin{array}{ccc}
U_n & \overset{\phi_n}{\longrightarrow} & Q \otimes P \otimes R U_n \\
(\otimes) & \downarrow & \oplus Q \\
\bigoplus_{j} & & (1 \otimes 0_j)
\end{array}
\]

In effect, we can add $\bigoplus U_n^t$ to $U_n$ to make it free, and replace $(U_n, \phi_n)$ by $(U_n + U_n^t, \phi_n + 0)$.

If $U_n = \bigoplus_{i} R$, then factor each summand $R_i$ as above and take the direct sum. $J$ is then a disjoint union of finite sets indexed by $I$.

Hence it's clear that $1 \otimes \psi_{R_n} : P \otimes R U_n \to P \otimes R U_n$ factors through a free $S$-module for each $n$.

This means that $P \otimes R U$ is an h-retreat of a complex of free $S$-modules, hence h.e.g. to a complex of free $S$-modules which is right bold if $U$ is.
This checks the previous result that $\mathcal{P} \otimes U$ is deg to a complex of projectives when $U$ is a h-form complex of projectives. (Things appear slightly better since the complexes need not be rigid bdd)

2. Suppose $\mathcal{U}$ has a deformation $\psi$ which factors degewise as follows:

\[
\begin{array}{ccc}
\mathcal{U}_n & \longrightarrow & \mathcal{U}_n \\
\downarrow \psi_n & & \downarrow \psi_n \\
\mathcal{A}^n & \subseteq & \mathcal{R}^n
\end{array}
\]

Then we know that $\mathcal{U}$ is an h-retract of $T = \bigoplus T_n$, $T_n = R^n$, with differential $id$. Moreover $\exists : \mathcal{U} \rightarrow T$ has image contained in $AT$ since $\exists = \iota (1 - dh)$. Thus $1 - \psi \iota = [d, \exists]$ on $\mathcal{U}$. redudes to $1 = [d, \exists]$ on $U/\mathcal{U}$, so that $U/\mathcal{U}$ is contractible. But $\exists$ has the lifting $\exists$, so it's clear that $U$ is h-form. NO, we don't have $\mathcal{A} \otimes U = U$. 
It seems to be worthwhile to understand length one complexes better. Reasons: Any perfect pure complex can be split into length one perfect pure complexes in a certain sense. Also the Atiyah-Bott-Shapiro treatment of relative $K$.

I recall that a map of length one complexes

$$
\begin{array}{c}
X_1 \xrightarrow{df} X_0 \\
\downarrow f_1 \downarrow \downarrow f_0 \\
Y_1 \xrightarrow{dy} Y_0
\end{array}
$$

is a quasi iff

$$(\ast) \quad 0 \rightarrow X_1 \xrightarrow{\cdot (-dx)} X_0 \oplus Y_1 \xrightarrow{(t, dy)} Y_0 \rightarrow 0$$

is exact, and it is a homotopy equivalence iff this sequence is split exact. In fact a splitting of the sequence is equivalent to a homotopy inverse data for $f$, that is $g: Y \rightarrow X$ and homotopy operators $h_x, h_y$ such that $1 - gf = [d, h_x], 1 - fg = [d, h_y]$. (more needed; see below)

The correspondence is given by

$$
\begin{array}{c}
\Rightarrow
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}
\begin{array}{c}
\Rightarrow
\end{array}
$$

Let’s check this statement. It seems you have to assume the homotopies $h_x, h_y$ are compatible with
Respect to either $f$ or $g$, i.e.
either $-f_1 h_x + h y f_0 \rightarrow 0$ or
$-h_x g_0 + g_1 h y \rightarrow 0$

Note that \( f \) for degree reasons
\( \alpha_0 \Rightarrow = 0 \) in these cases, i.e. \( [f, h] : X \to Y \)
has degree $+1$, so \( [f, h] = [d, k] \) means \( [f, h] = 0 \)
since $k$ has degree 2.

Observe that

\[ r_i = h_x d + g_1 f_1 = 1 \]
\[ g_s = f_0 g_0 + d h_y = 1 \]

\[ (r_i s) j = \begin{pmatrix} -d \\ f_1 \end{pmatrix} (h_x g_1) + \begin{pmatrix} g_0 \\ h_y \end{pmatrix} (f_0 d y) \]
\[ = \begin{pmatrix} d h_x + g_0 f_0 \\ -d g_1 + g_0 d \\ -f_1 h_x + h y f_0 \\ f g_1 + h y d y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -f_1 h_x + h y f_0 \\ 1 \end{pmatrix} \]

\[ r s = -h_x g_0 + g_1 h y \]

Now

\[ (r_i s) j = 1 \iff -f_1 h_x + h y f_0 = 0 \]
\[ \downarrow \]

\[ r s + s j = s \Rightarrow \alpha r s = 0 \iff r s = r i r s = 0 \]

and \( r s = 0 \iff -h_x g_0 + g_1 h y = 0 \).

Conversely, \( r s = 0 \), and note that \( (r_i s j) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is invertible. But
\[(1 - ir - sj)\iota = i - cri = i - i = 0\]
\[(1 - ir - sj)s = s - cAs - sy = s - s = 0\]

So \((1 - ir - sj)(ir + sj) = 0 \implies 1 = ir + sj\).

Now recall what a contraction for \(\text{Cone}(f)\) looks like:

\[
\begin{bmatrix}
(-dx, dy) \\
(f, dy)
\end{bmatrix}
\begin{bmatrix}
(hx, gx) \\
hx
\end{bmatrix}
= \begin{bmatrix}
[d, h_x] + gf & -[d, g] \\
- [f, h_x] + [d, y] & [d, h_y] + fg
\end{bmatrix}
= \begin{bmatrix} 1 & 0 \\
0 & 1 \end{bmatrix}
\]

So a contraction for \(\text{Cone}(f)\) satisfies \([f, h] = 0\).

Similarly a contraction for \(\text{Cone}(g)\) satisfies \([g, h] = 0\).

It would have been simpler to point out that \((x)\) is \(\subseteq \text{Cone}(f)\). Then a contraction for \(\text{Cone}(f)\) satisfies \([f, h] = 0\), so one doesn't have to compute \(ir + sj\).

The problem with the cone is that it is a complex of length 2. Here's a criterion inside length one complexes.

\[\text{Prop. } X \text{ and } Y \text{ are homotopy equivalent } \iff \exists \text{ an isomorphism } X \oplus (Y_0 \xrightarrow{1} Y_0) \cong \left( X_0 \xrightarrow{1} X_0 \right) \oplus Y\]

\(\Leftarrow\) is obvious

\(\Rightarrow\) we prove by a formula for the rimi.
\[
\begin{pmatrix}
-d_x & g_0 \\
-f_1 & h \\
\end{pmatrix}
\begin{pmatrix}
-d_x & g_1 \\
-f_0 & d_y \\
\end{pmatrix}
\begin{pmatrix}
-1 & g_0 \\
-f_0 & d_y \\
\end{pmatrix}
= \begin{pmatrix}
-d_x h_x + g_0 f_0 \\
-f_1 h_x + h_y f_0 \\
\end{pmatrix}
\begin{pmatrix}
-d_x g_1 + g_0 d_y \\
-f_1 g_1 + h_y d_y \\
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
-d_x h_x \\
-f_0 \\
\end{pmatrix}
\begin{pmatrix}
-d_x & g_0 \\
-f_0 & d_y \\
\end{pmatrix}
\begin{pmatrix}
-1 & g_0 \\
-f_0 & d_y \\
\end{pmatrix}
= \begin{pmatrix}
-d_x h_x g_0 + g_0 f_0 \\
-f_0 g_0 + d_y h_y \\
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
0 & 1 \\
\end{pmatrix}
\]

\[\text{etc.}\]
A monounital ring, $\text{spht}(A) =$ category of strictly perfect complexes $\sim$ over $A$ which a homotopy form wrt $A$, i.e. $U/AU \simeq 0$. Define $K_0(\text{spht}(A))$ to be the abelian group generated by $[U]$ depending on the isom. class of $U$ subject to the relations:

i) $[U \oplus U'] = [U] + [U']$

ii) $U \sim 0 \Rightarrow [U] = 0$

iii) $0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0$ exact $\Rightarrow [U] = [U'] + [U''].$

Define $K_0'(A)$ to be the abelian group with generators $[U]$, for each $\text{spht}$ complex of length $1$ $U_1 \xrightarrow{d} U_0$ s.t. $U_2/\text{AU}_1 \simeq U_2/\text{AU}_0$, where $[U]$ depends only on $U$ up to isomorphism, with the same relations:

1) $[U \oplus U'] = [U] + [U']$

2) $U \sim 0$ (i.e. $U_1 \xrightarrow{d} U_0$) $\Rightarrow [U] = 0$.

3) $0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0$ exact $\Rightarrow [U] = [U'] + [U''].$

restricted to length one complexes.

I would like to check that in the presence of 1) + 2), the relation 3) is equivalent to

3') if $\xymatrix{U_1 \xrightarrow{f} U_0}$, $U_0 \xrightarrow{g} V_0$ are length one $\text{spht}$ complexes, then $[U_1 \xrightarrow{f} U_0] + [U_0 \xrightarrow{g} V_0] = [U_1 \xrightarrow{f+g} V_0]$.
3) $\Rightarrow$ 3') follows from the general result relating $\text{Cone}(gf)$ to $\text{Cone}(f), \text{Cone}(g)$ in the case of maps of complexes $X \to Y \to Z$.

\[ \text{Cone}(f) \quad \text{Cone}(f) \cup \text{Cyl}(g) \quad \text{Cone}(g) \]

namely we have an exact sequence

\[ 0 \to \text{Cone}(f) \to \text{Cone}(f) \cup \text{Cyl}(g) \to \text{Cone}(g) \to 0 \]

together with a SDR of $\gamma$ onto $\text{Cone}(gf)$. 1)+2) together with this SDR gives $[\text{Cone}(f) \cup \text{Cyl}(g)] = [\text{Cone}(gf)]$.

3') $\Rightarrow$ 3'. Suppose given an exact sequence of length 1 split complexes

\[ 0 \to U_i \to V_i \to W_i \to 0 \]

\[ \begin{array}{ccc}
0 & \to & U_0 \\
\downarrow f' & & \downarrow f'' \\
0 & \to & V_0 & \to & W_0 & \to & 0
\end{array} \]

because $W_n$ is projective this sequence splits locally, so we can assume $V = U \oplus W_i$ for $i = 0, 1$ and $f = \left( \begin{array}{c} f' \\ u \end{array} \right): U_i \oplus W_i \to U_0 \oplus W_0$.

But $f$ factors

\[ \left( \begin{array}{c} f' \\ u \\ f'' \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\
0 & f'' \end{array} \right) \left( \begin{array}{c} 1 \\ u \\ 1 \end{array} \right) \left( \begin{array}{c} f' \\ 0 \\
0 & 1 \end{array} \right) \]

i.e.
by 3') we have \([f] = \begin{pmatrix} (f'0) & (1\,0) \\ (0\,1) & (0\,f'') \end{pmatrix} = [f'] + [f'']\]

There's an obvious map
\[

K_0(A) \longrightarrow K_0(\text{spf}(A))

\]

induced by the inclusion of length 1 complexes.

We next define a map in the opposite direction.

Given \(U\) in \(\text{spf}(A)\), let \(\bar{U} = U/AU\) and \(U^\# = \bar{\Lambda} \otimes \bar{U}\). \(\bar{U}\) is contractible over \(\bar{\Lambda}\), hence \(U^\#\) is contractible over \(\bar{\Lambda}\). One has a canonical iso \(U^\# = \bar{U}\). This isomorphism can be lifted to a map \(U^\# \xrightarrow{f} U\) unique up to homotopy. In effect one has

\[

\begin{array}{ccc}
U^\# & \xrightarrow{f} & U \\
\otimes & \downarrow & \downarrow \\
\bar{U}^\# & \xrightarrow{f} & \bar{U} \\
\otimes & \downarrow & \downarrow \\
0 & \xrightarrow{0} & 0 \\
\end{array}

\]

and because \(U^\#\) is projective, the obstruction to the instance of uniqueness lie in \(H_x \text{Hom}_{A}(U^\#, AU)\), which is zero as \(U^\#\) is contractible.
On the other hand there is a lifting $g: U \to U^\#$ unique up to homotopy $\bar{u}^\#$.

\[ \begin{array}{c}
U \xrightarrow{g} U^\# \\
\downarrow \\
U^\#
\end{array} \]

for the obstructions lie in $H_k Hom^A(U, A\bar{u}^\#)$, which is zero as $A\bar{u}^\# = A \otimes_\mathbb{Z} \bar{u}$ is contractible.

If $U, V$ are two strictly perfect cys of $\mathbb{A}$ modules and $f: U \to V$ is a map such that $\bar{f}: \bar{U} \xrightarrow{\sim} \bar{V}$, then the cone on $f$ is the total complex of

\[ \begin{array}{c}
\cdots \xrightarrow{d} U_2 \xrightarrow{d} U_1 \xrightarrow{d} U_0 \\
\downarrow \quad \downarrow \quad \downarrow \\
V_2 \xrightarrow{d} V_1 \xrightarrow{d} V_0
\end{array} \]

The columns are length one sp hf complexes, so

\[ X(U \xrightarrow{f} V) = \sum (-1)^0 \begin{bmatrix} f_0: U_0 \to V_0 \end{bmatrix} \in K_0^A(\mathbb{A}) \]

is defined.

Properties: a) $X(U \xrightarrow{f} V) + X(V \xrightarrow{g} W) = X(U \xrightarrow{g \circ f} W)$

b) if $U, V$ are contractible and $f: U \to V \xrightarrow{\sim} \bar{f}$, then $X(U \xrightarrow{f} V) = 0$.

a) is clear from b) property above.
For b) use the short exact sequences

\[ 0 \rightarrow \mathbb{Z}_g u \rightarrow U \rightarrow \mathbb{Z}_g u' \rightarrow 0 \]
\[ 0 \rightarrow \mathbb{Z}_g V \rightarrow V \rightarrow \mathbb{Z}_g V' \rightarrow 0 \]

to get \( V_g \). 

\[ [f_g^*] = [Z_g f] + [Z_g f'] \in K'_0(A). \]

where \( \sum (i) \delta [f_g^*] = 0 \).

Now restrict to \( U \in \text{sphf}(A) \), and choose liftings \( f: U' \rightarrow U \), \( g: U \rightarrow U' \) as above.

We have

\[ X(f: U' \rightarrow U) + X(g: U \rightarrow U') = X(gf: U' \rightarrow U') \]

This shows \( X(f: U' \rightarrow U) \) is independent of the choice of \( f \).

We now show there is a well-defined map

\[ K_0(\text{sphf}(A)) \rightarrow K'_0(A) \]

sending \([U] \) to \( X(f: U' \rightarrow U) \).

We have to check the relations are satisfied.

i) \[ X(f: U' \rightarrow U) + X(f': U' \rightarrow U') = X(fgf: U'' \rightarrow U'') \]

ii) \( U \circ U \circ v \circ 0 \) both \( U, U' \rightarrow U \) so \[ X(f: U \rightarrow U) = 0. \]

iii) Suppose \( 0 \rightarrow U' \rightarrow U \rightarrow U'' \rightarrow 0 \) a given exact sequence in \( \text{sphf}(A) \). Then \( 0 \rightarrow \bar{U} \rightarrow \bar{U} \rightarrow \bar{U}'' \rightarrow 0 \) is an exact sequence of contractible complexes over \( Z \), so it splits. Thus one has
0 \to U^\# \overset{r}{\to} U^\# \overset{r}{\to} U''^\# \to 0

\phi' \downarrow \quad \phi \downarrow \quad \phi'' \downarrow \quad \phi''

0 \to U' \overset{i}{\to} U \overset{\phi}{\to} U'' \to 0

\begin{align*}
\text{where we've chosen } f', f'' \text{ lifting the canonical iso modulo } A. & \quad \text{Define } f := \phi f' + \phi f''.
\text{Because } U''^\# \to 0 \text{ the lifting } \phi \text{ of } f'' \exists. & \quad \text{Define } f \text{ to be } \phi f' + \phi f''.
\text{Then we have a map of exact sequences of complexes} & \quad \text{We have a map of exact sequences of complexes}
0 \to U'^{\#} \to U^\# \to U''^\# \to 0

0 \to U' \to U \to U'' \to 0

\text{where the vertical arrows are iso. modulo } A, & \quad \text{where the vertical arrows are iso. modulo } A,
\text{hence } [f_0] = [f'_0] + [f''_0] \text{ in } K^A_0(A). & \quad \text{hence } [f_0] = [f'_0] + [f''_0] \text{ in } K^A_0(A).

\begin{align*}
\chi(f : U^\# \to U) = \chi(f' : U'^\# \to U') + \chi(f'' : U''^\# \to U'')
\end{align*}

\text{proving iii).} \quad \text{proving iii).}

\text{Final step is to check the maps} \quad \text{Final step is to check the maps}
K^A_0(A) \to K^A_0(\text{split}(A)) \to K^A_0(A)

\text{are inverse. The first map is onto since given} \quad \text{are inverse. The first map is onto since given}
U \text{ we know } [U] = [U^\#] + [\text{Cone}(f : U^\# \to U^\#)] \quad U \text{ we know } [U] = [U^\#] + [\text{Cone}(f : U^\# \to U^\#)]

\begin{align*}
\ii_{0} \sum \phi_0 [U^\#_{g_0} \to U_{g_0}] & \in \text{Image of } K^A_0(A).
\end{align*}

\text{So it remains to show the composition is } 1.
Take a length one split complex $U; P \xrightarrow{d} Q$. Then we have

$$p^\# \sim q^\#.$$  
$fi \uparrow \downarrow f_0$  
$P \xrightarrow{d} Q$  

so $\chi(f; U^\# \to U) = [f_0: Q^\# \to \mathbb{R}] - [f_1: P^\# \to P]$ in $K^e(A)$. But

$$[p^\# \xrightarrow{d} Q] = [f_0] + \underbrace{[d: P^\# \to Q^\#]}_{0}$$

$$= [d: P \to Q] + [f_1]$$

$\therefore [u] = [f_0] - [f_1] = \chi(f; U^\# \to U)$ as desired.
July 1, 1995

There is a problem linking \( K_1 A \), for \( A \) commutative \( A = A^2 \), to \( K_0 A \) defined via perfect fun complex. For example when \( A \) is a radical ring \( (A = J(A)) \), there are no such complexes except contractible ones while \( K_1 A \) can be non-trivial.

A better way to say this is that the idea of obtaining the higher \( K_n A \) from the category of perfect fun complexes up to homotopy, say via Waldhausen theory, seems not to work for a radical ring. One might still hope to use perfect fun complexes, but some new ideas are needed.

It seems \( K_0 R \) can be obtained as \( K_1 \) of the suspension of \( R \) for \( R \) unital. Also, \( K_0 R \) sits naturally as a summand of \( K_1 (R[z, z^{-1}]) \). Note that \( A = A^2 \implies A[z, z^{-1}] \) has the same property. So perhaps (under some extra h-unital conditions on \( A \)) one can find the "good" \( K_0 A \) sitting as a summand of the Vaserstein \( K_1 (A[z, z^{-1}]) \).

So it seems worthwhile to understand Bass' theory for \( K_1 (A[z, z^{-1}]) \). There are canonical maps

\[
K_0 R \rightarrow K_1 (R[z, z^{-1}]) \rightarrow K_0 R
\]

with composition equal to the identity. These maps come from the Atiyah-Bott elementary proof of the periodicity theorem. The former assigns
to an idempotent matrix $e$ over $R$ the
invertible matrix $\varepsilon p + 1 - p$ over $R[\varepsilon, \varepsilon^{-1}]$.
The latter "constant, for $g \in GL(R[\varepsilon, \varepsilon^{-1}])"$ the "rank 1
vector bundle" $E_g$ over $R P^1$ obtained by
planching and assigns to $[g] \in K_1(R[\varepsilon, \varepsilon^{-1}])$ the
class $\left[ R \Gamma [E_g(-1)] \right]$

An important step in the Atiyah-Bott proof
is studying families of maps
$$az + b : V \to W \quad z \in C$$
($V, W$ two finite-dimensional vector spaces or bundles) such that
$az + b$ is invertible for $z \in S^1$. Such a family
splits canonically into two pieces where $az + b$
is invertible inside $S^1$ and on the second it is
invertible outside (including $z = \infty$). The projection
operator is
$$\frac{1}{2\pi i} \oint (az + b)^{-1} \overline{adz} \quad \text{on } V$$
(and probably $\frac{1}{2\pi i} \oint \overline{adz} \frac{1}{az + b} \quad \text{on } W$).

Let's discuss this splitting result algebraically.
Suppose given $az + b \in M_n(R[\varepsilon]) \cap GL_n(R[\varepsilon, \varepsilon^{-1}])$.
Replace $R$ by $M_n R$ to reduce to the case $az + b \in GL(R[\varepsilon, \varepsilon^{-1}])$
with $a, b \in R$.

Consider the diagram
Work with right module structure so that left multiplication \( az+b \cdot \) is a module map.

The above diagram leads to a canonical isomorphism

\[
\begin{align*}
R & \xrightarrow{\cong} R[z]/(az+b)R[z] \oplus R[z^{-1}]/(az+bz)R[z^{-1}]
\end{align*}
\]

of right \( R \)-modules. This means we have a decomposition \( R = eR \oplus e^+R \) where \( e \) is an idempotent in \( R \). Now we find \( e \). This corresponds to \((1,0)\) on the right above.

Left to \((1,0)\) in \( R[z] \oplus R[z^{-1}] \), which goes to \( 1 \in R[z,z^{-1}] \) and corresponds under the isomorphism to

\[
\begin{align*}
\left( (az+b)^+ , (az+b)^-_z \right) \in R[z] \oplus R[z^{-1}]
\end{align*}
\]

Let \( (az+b)^-_z = \sum c_k z^k \) so that

\[
\begin{align*}
(az+b) \sum c_k z^k = \sum c_k z^k (az+b) = 1 \quad \text{c.o.}
\end{align*}
\]

\[
\begin{align*}
ac_{k-1} + bc_k = c_{k-1}a + c_kb = \delta_{k0}
\end{align*}
\]
\[
(a+b)^+ = \sum_{k>0} c_k z^k, \quad (a+b)^- = \sum_{k<0} c_k z^k
\]

\[
(a+b)^+ (a+b)^- = \sum_{k>0} a c_k z^{k+1} + b c_k z^k
\]

\[
= b c_0 + \sum_{k>1} (a c_{k-1} + b c_k) z^k = b c_0
\]

\[
(a+b)^- (a+b)^+ = \sum_{k<0} a c_k z^{k+1} + b c_k z^k
\]

\[
= a c_0 + \sum_{k<-1} (a c_{k-1} + b c_k) z^k = a c_{-1}
\]

Then \((1,0) - (b c_0, a c_{-1}) = (1-b c_0, a c_{-1}) = \Delta(a c_{-1}).\)

Thus \[e = a c_{-1} = \text{Res} (a (a+b)^- dz)\]

Let's interpret the preceding argument. Go back to a family \(a+b: V \to W\) with \(V, W\) f.d. vector spaces over \(\mathbb{C}\), and assume \((a+b)^+ F\) for all \(z \neq 0, \infty\). Then we obtain a coherent sheaf \(F\) defined by

\[
0 \to \mathcal{O}(-1) \otimes V \xrightarrow{a+b} \mathcal{O} \otimes W \xrightarrow{\mathcal{O}} F \xrightarrow{\mathcal{O}} 0
\]

where \(F\) has support \(\subset E_0, \times 3\). \(F\) is regular (Mumford) and we have canonical isomorphism

\[
W \xrightarrow{\sim} \Gamma(F) \quad \Gamma(F(-1)) \xrightarrow{\sim} H^1(\mathcal{O}(-2) \otimes V) = V.
\]

The obvious decomposition of \(F\) into \(F^+ \oplus F^-\) then induces the corresponding splitting of the family \(a+b\).
Interesting point: One has managed to extend the notions of spectrum and characteristic polynomial from operators on a f.d. vector space \( V \rightarrow W \), namely those which are transverse to the graph of multiplication by \( z \) for generic \( z \). Thus \( a z + b : V \rightarrow W \Leftrightarrow (a, b) V \subseteq W \) is transversal to \((1, -z) W = \ker \) of \((w_1, w_2) \rightarrow z w_1 + w_2\). The spectral sheaf is \( F \) as above (which generalizes \( V \) as a \( \mathbb{C}[z] \)-module with \( z \) acting as the operator). The characteristic polynomial of the correspondence is the divisor of \( F \), which should give a homogeneous poly of degree \( d \) in the homogeneous cords \((z, z)\) defined up to a scalar factor.

\[
\Lambda^\text{max}(\mathcal{O}(-1) \otimes V) \rightarrow \Lambda^\text{max}(\mathcal{O} \otimes W)
\]

\[
\mathcal{O}(-d) \otimes \Lambda^d V \rightarrow \mathcal{O} \otimes \Lambda^d W
\]

This gives a homogeneous poly of degree \( d \) in the homogeneous cords \((z, z)\), defined up to a scalar factor.

\( z = \infty \) is not an eigenvalue \( \Leftrightarrow a^{-1} \) exists in which case we have the char poly of \( -a^{-1}b \) (or \(-ba^{-1}\) (example)).

\( z = 0 \) is not an eigenvalue \( \Leftrightarrow b^{-1} \exists \) exists in which case the char poly = \(|z - b^{-1}a|\).
Changes of coordinates: assume \( a \neq b \) is invertible at \( z = 1 \). Then

\[(a+b)^{-1}(a+b) \circ 1\]

has the same spectrum, which reduces to the case of \( z \alpha + (1-a) \). Now put \( z = 1 - \frac{1}{\omega} \) so that \( z = 0, \infty \) correspond to \( \omega = 1, 0 \) resp. Then

\[z \alpha + (1-a) = 1 - \frac{1}{\omega} \alpha = \frac{1}{\omega}(\omega - \alpha)\]

so that we're looking for the spectrum of \( \alpha \).

Note that \( z \alpha + (1-a) \in GL(\mathbb{R}[z, z^{-1}]) \iff \alpha(1-a) \) is nilpotent.

**Prof.** \((\Rightarrow)\) \((z \alpha + (1-a))(z^{-1} \alpha + (1-a)) = \alpha^2 + (1-a)^2 + (z + z^{-1})a(1-a) = 1 + \frac{1}{z-2+z^{-1}}\alpha(1-a) \) is invertible showing \( z \alpha + (1-a) \) is invertible over \( \mathbb{R}[z, z^{-1}] \).

\((\Rightarrow)\) If \((z \alpha + (1-a))h(z) = 1\) with \( h \) invertible over \( \mathbb{R}[z, z^{-1}] \), then \( h(z)h(z^{-1}) \) is an inverse for \( 1 + (z-2+z^{-1})a(1-a) \). Now \( h(z)h(z^{-1}) \) is a matrix over the subring of \( \mathbb{R}[z, z^{-1}] \) consisting of Laurent series invariant under \( z \mapsto z^{-1} \). This is a polynomial ring \( \mathbb{R}[x] \), where \( x = z-2+z^{-1} \) (I should have pointed out that \( z \alpha + (1-a) \) and \( z^{-1}a+(1-a) \) commute, so that their inverses \( h(z) \) and \( h(z^{-1}) \) also commute). Thus \( 1 + x \alpha(1-a) \) is invertible in \( \mathbb{R}[x] \), which implies \( \alpha(1-a) \) is nilpotent.

**Question:** Can any of this \( P \) stuff shed light on homotopy idempotents?
Note that

\[ T[z] \oplus z^{-1}T[z^{-1}] = T[z, z^{-1}] \]

\[ b = 1 - a \]

\[ (az + b, az + b) \sim az + b \]

\[ T \rightarrow T[z] \oplus T[z^{-1}] \rightarrow T[z, z^{-1}] \]

\[ \text{Cone}(T[z]) \oplus \text{Cone}(T[z^{-1}]) \]

is closely related to things we examined before: telescopes made of

\[ \text{At this point I want to leave } az + b \]

and look at a more general \( g \in GL(R[z, z^{-1}]) \).

Again take \( g \in GL_1(R[z, z^{-1}]) \). Associate to \( g \) the length 1 complex

\[ R[z] \oplus z^\ast R[z^{-1}] \stackrel{(in, -in)}{\longrightarrow} R[z, z^{-1}] \]

Note that if \( g = 1 \) this complex is acyclic and if \( g = z \) it has homology \( R \) on the left only.

This complex should be the Čech complex for \( E_g(-1) \) over \( P^1 \). Observe it isomorphic to

\[ z R[z] \oplus z^{-1} R[z^{-1}] \]

\[ \text{(in, -in)} \rightarrow R[z, z^{-1}] \]
If \( g \in R[z^{-1}] \), then \( gR[z^{-1}] \subset R[z^{-1}] \)
so that the above complex is equivalent to
\[
gR[z^{-1}] \xrightarrow{\cdot g} R[z,z^{-1}] / zR[z] \\
\cong R[z^{-1}]
\]

The homology on the right is
\[ R[z^{-1}] / gR[z^{-1}] \]

Actually, \( g \in R[z^{-1}] \) is unnecessary. One has
\[ zR[z] \xrightarrow{1} zR[z] \] as a contractible subcomplex so that \( \bigodot \) is quasi to
\[
g \cdot R[z^{-1}] \xrightarrow{\cdot g} R[z,z^{-1}] / zR[z] \\
\cong R[z^{-1}]
\]

which is a (kind of) Toeplitz operator on \( R[z^{-1}] \)
associated to \( g \).

Consider
\[
g^{-1}R[z] \oplus z^{-1}R[z^{-1}] \xrightarrow{(g^{-1},z^{-1})} R[z,z^{-1}]
\]
which is isomorphic to \( \bigodot \). This is quasi to
\[
R[z] \xrightarrow{g^{-1}} g^{-1}R[z] \subset R[z,z^{-1}] \longrightarrow R[z]
\]
where the map is the Toeplitz operator on \( R[z] \)
associated to \( g^{-1} \). To get the correct sign I should replace \( g \) by \( g^{-1} \) and consider the above complex as a chain complex with the shift of degree +1.
So our $\Gamma(E_g(-1))$ becomes (essentially) the Toeplitz operator

$$ R[z] \xrightarrow{f(g)} R[z] $$

Different versions introduce the Toeplitz algebra $R<[z,z^*]/(1-z^2z) \sim R[z] \otimes_R R[z^*]$, and the Toeplitz extension

$$ 0 \to J \to T \to K \to R[z; z^{-1}] \to 0 $$

This gives $K_1(R[z; z^{-1}]) \to K_0(J) \cong K_0(R)$

where the latter comes from Morita invariance.

Concretely given $g \in GL_1(R[z; z^{-1}])$ we lift $g, g^{-1}$ to $f(g)$ and $f(g^{-1})$ over $T \to K$, then form the complex

$$ T \xrightarrow{f(g)} T $$

Put $T = T \to K$, $e = \begin{bmatrix} 1 & -zz^* \end{bmatrix}$. We have a Morita context.

$$ \begin{pmatrix} T & Te \\ eT & eTe \end{pmatrix} = \begin{pmatrix} T & R[z] \\ R[z^*] & R \end{pmatrix} $$

The mirage under the Morita context of the complex $1)$ is

$$ R[z] \xrightarrow{f(g)} R[z] $$

since $T \otimes Q = Q = R[z]$. The other point is that if $g \in R[z]$, then
\[ \phi(g) = g \quad \text{so that the only} \]

homology is \( \mathbb{R}[z]/\mathbb{R}[z] \) in degree 0.

Now we also know that \( \phi(g^{-1}) \) gives a parameter for \( 2 \). Observe that

\[ \phi(z^k) \phi(z^l) = \phi(z^{k+l}) \quad \text{if } l \geq 0 \]

This is obvious for \( k \geq 0 \), so suppose \( k \leq 0 \) and put \( \varepsilon = -k \geq 0 \). Then

\[ \phi(z^k) \phi(z^l) = (z^*)^\varepsilon z^l = \begin{cases} (z^*)^k z^l & \text{if } \varepsilon \geq k \\ z^l & \text{if } \varepsilon \leq k \end{cases} \]

Thus if \( g \in \mathbb{R}[z] \) we have \( \phi(g^{-1}) \phi(g) = \phi(g^{-1}g) = 1 \).

So we have

\[ \begin{array}{ccc} \mathbb{R}[z] & \xrightarrow{\phi(g)} & \mathbb{R}[z] \\ \phi(g^{-1}) & \xleftarrow{\phi(g)} & \end{array} \]

and \( 1 - \phi(g) \phi(g^{-1}) \) projects onto \( H_0 \).

Unfortunately, when \( g = az + b \), this projector \( 1 - \phi(g) \phi(g^{-1}) \) does not seem to be the one studied on p. 326-7.
July 18, 1975

Program: To understand the following and the relations between them.


2. Cone and suspension of a ring (used by Karoubi, Wagoner to deloop).

3. John Roe’s finite propagation C^* algebras.

Rough background: Roe told me at Lancaster that his stuff and Pedersen’s ideas are closely related. Ranicki in his book on lower K+L theory discusses Pedersen–Weibel, at least a metric space version using open cones instead of the integer lattices in the original PW paper. I assumed 1+2 were very similar, but this now seems naive.

Related ideas: Negative K groups via (Lawuent) polynomial extensions (Bass–Heller–Swan). Toeplitz algebras, periodicity proofs. Controlled K-theory

Let’s begin with the cone on a unital ring A.

The key idea here is to embed P(A) in a P(R) having trivial K-theory, because of an infinite direct sum argument.

First examine when K_0(R) = 0.

Claim: K_0(R) = 0 ⇐⇒ there is an N such that P ⊕ R^k ∼= R^k for all P in P(R).

Proof: (⇐) obvious.

(⇒) I know that K_0(R) = \text{Iso}(P(R)) \times N/\sim

where \((L^n_m, n) \sim (L^{'n}_m, n')\) \implies \exists k \; P \oplus R^k \oplus R^k = P' \oplus R^k \oplus R^k.

Hence K_0(R) = 0 \implies \forall P \; \exists k \; \text{st.} \; P \oplus R^k \sim R^k.
In particular, find $k$ such that $R \oplus R^k \cong R^k$, where $R^k \cong R^{k+1} \cong R^{k+2} \cong \ldots$

Now given $P$, we know $\exists k$ such that $P \oplus R^k \cong R^k$ and then $P \oplus R^k \cong R^k$ follows because either $l < k$ and you can add $R^{k-l}$ or $l \geq k$ and you have $R^l \cong R^k$.

Recall the Eilenberg swindle. Assume $P \oplus Q = F$ free, then

$$P \oplus Q \oplus P \oplus Q \oplus \ldots = (F \oplus F \oplus \ldots)$$

is

$$P \oplus Q \oplus P \oplus Q \oplus P \oplus \ldots = P \oplus (F \oplus F \oplus \ldots)$$

(This might be useful in connection with $\mathcal{F} + 1 \cdot e$ and Bott maps.)

Here's a simpler argument. Given $P \in \mathcal{P}(R)$ suppose that $\Sigma P = \bigoplus_{n \in \mathbb{N}} P$ is in $\mathcal{P}(R)$.

Since there's a canonical isomorphism $P \oplus \Sigma P \cong \Sigma P$ it follows that $[P] = 0$ in $K_0(R)$.

Thus we have $K_0(A) = 0$, a additive category if there exists a functor $\Sigma : A \to A$ together with an isomorphism $id \oplus \Sigma = \Sigma$.

Question: Suppose $\exists \Sigma R$ in $\mathcal{P}(R)$ such that $R \oplus \Sigma R \cong \Sigma R$. Does it follow that $K_0(R) = 0$?

Note that $\Sigma R$ is a summand of $R^k$ for some $k$, hence $R \oplus R^k \cong R^k$. Thus the question becomes whether $R$ stably trivial $\Rightarrow K_0(R) = 0$. 

Perhaps Cuntz’s algebra \( \mathbb{O}_2 \) should be examined.

Let’s focus attention on the situation where the K-theory is trivial because of an infinite direct sum functor \( \oplus \).

First example: Consider the additive category \( A \) of vector spaces of countable dimension. This is of the form \( \mathcal{P}(R) \) where \( R \) is the ring of endomorphisms of \( k(\infty) = k(\mathbb{N}) \). \( R \) is the ring of matrices \( (a_{ij}), \sum_0 \) with finite columns. (Finite means \( a, e, 0 \).

Because the countable direct sum is defined in this category the K-theory of \( R \) is trivial.

I think the case \( C(A) \) on a ring is the ring of matrices \( (a_{ij}), \sum_0 \) with both rows and columns finite. Here’s an interesting way to get this ring: (Recall \( A \) is a unital ring).

Consider the category. The objects are triples \( (\mathbb{Q}, \mathcal{P}, <, >) : \mathbb{Q} \oplus \mathbb{P} \rightarrow A \), where \( <, > \) is an \( A \)-bimodule map. A map

\[
(\mathbb{Q}, \mathcal{P}, <, >) \rightarrow (\mathbb{Q}', \mathcal{P}', <, >)
\]

is a pair of maps \( \mathbb{Q} \rightarrow \mathbb{Q}', \mathcal{P} \rightarrow \mathcal{P}' \) satisfying \( <u(\mathbb{Q}), \mathcal{P}'> = <\mathbb{Q}, u'(\mathcal{P'})> \).

I think this is an additive category which is Karoubian. There’s an obvious direct sum operation for families of triples, although whether it gives the categorical direct sum is not clear.
Now consider the triples \((A_\infty^\infty, A_\infty A_\infty, \langle \rangle)\)
where \(A_\infty^\infty\) is the left \(A\)-module of \(\infty\) finite row vectors, \(A_\infty A_\infty\) is the right \(A\)-module of \(\infty\) finite column vectors, and \(\langle \rangle\) is dot product. Here \(A_\infty^\infty = \bigoplus A\), etc. Then the ring of endos of this triple is the ring of row + column finite matrices, i.e. the cone \(C(A)\).
Let's discuss Pedersen–Weibel. Given an additive category $C$, their first delooping is defined as follows. The objects are family $(P_n)_{n \in \mathbb{Z}}$ of objects in $C$, and a map $(P_n) \rightarrow (Q_n)$ in $C'$ is a matrix $(\eta_{mn})$ with $\eta_{mn} \in \text{Hom}_C(P_n, Q_m)$, such that the support of $(\eta_{mn})$ is contained in $\{(m, n) \mid |m-n| \leq r\}$ for some $r$. The $k$-th delooping $C^{k}$ is defined similarly using $\mathbb{Z}^k$ instead of $\mathbb{C}$.

Note that $\text{Hom}_{C^1}(P, Q)$ has a natural filtration indexed by $\mathbb{N}$ which is exhaustive and compatible with composition. One can extend the definition of $C^1$ to similarly filtered additive categories $C$. One then has $C^k = (C^{k-1})^!$, allowing inductive proofs.

A key point in the proofs I think is to break $\mathbb{Z}$ up into $(-\infty) \cup \mathbb{N}$, then argue that objects supported over $\mathbb{N}$ form a subcategory whose $K$-theory is trivial, because of an infinite direct sum argument. Thus given $\mathcal{P} = (P_n)_{n \in \mathbb{Z}}$ we can shift: $\sigma \mathcal{P} = (P_{n-k})_{n \in \mathbb{Z}}$ and form

$$\sum \mathcal{P} = \bigoplus_{k \geq 0} \sigma^k \mathcal{P} = P_0 \oplus P_1 \oplus P_2 \oplus \ldots \oplus P_0 \oplus P_1 \oplus \ldots \oplus P_0 \oplus \ldots$$

The canonical isomorphism $P \oplus \sum \mathcal{P} \cong \sum \mathcal{P}$ holds in $C^1$. More precisely one has a decomposition

$$\sum \mathcal{P} = P \oplus s \sum \mathcal{P}$$
and an isomorphism $\sigma \Sigma P \cong \Sigma P$

of finite propagation.

Suppose now that $C = P(A)$. It's clear that $C$ is not of the form $P(R)$. In effect you would need a generator $P^k(P_n)$ for $C$, and you can always manufacture a $(Q_n)$ which grows too fast to be a summand of some $P^k$.

This means the Pedersen-Weibel construction leaves the $K$-theory of rings (maybe just unital rings). I find this surprising in view of what we told me and also the theory of $C(A)$.

One thing we can do is specify a growth rate. To fix the ideas, work over $\mathbb{N}$ and consider those objects $P = (P_n)_{n \in \mathbb{N}}$ such that $P_n$ is a summand of $A^\otimes n$, where $\tau(n) \leq C f(n)$, $f$ being the growth function. Then $(A^\otimes n)$ should be a projective generator for this subcategory.

If we take $f(n) = 2^n$, then this subcategory is closed under the functor $\Sigma^i$. Since

$$\text{rk}(P_0 \oplus \cdots \oplus P_n) = \sum_{k=0}^{n} \text{rk}(P_k) \leq C \sum_{k=0}^{n} 2^k \leq C 2^{n+1}$$
July 28, 1995

Let $L_n(R, A)$ be the category of fg proj $R$-module complexes which become contractible modules $A$ and which are $n$-dual chain complexes (supported in degrees $0 \leq k \leq n$).

Form the Grothendieck group $L_n(R, A)$ generated by objects of $L_n(R, A)$ where the relations are additivity for locally split (short) exact sequences and equality for homotopy equivalent complexes.

We should have have Morita invariance for $L_n(R, A)$, in particular $L_n(\tilde{A}, A) \cong L_n(R, A)$. In any case suppose from now on that $R = \tilde{A}$.

We have obvious maps

$$L_1 \to L_2 \to \cdots \to \varinjlim_n L_n$$

$$\cong K_0(A) \quad \cong K_0(\mathcal{Pf}(A))$$

and propose to show these are isomorphisms. I know $K_0(A) \to K_0(\mathcal{Pf}(A))$ is an isomorphism, so it's enough to show $L_n \to L_n$ is surjective.

Given $U : U_n \to U_{n-1} \to \cdots \to U_0$ in $L_n(\tilde{A}, A)$, we choose $f : U^* \to U$ inducing the identity on $A$. 
We know $U^\#$ splits into elementary complexes, hence we can restrict $f$ to the "bottom" summand to get a map $f^*: C(U_0^\#) \to U_0$ reducing to the identity mod $A$ in degree 0.

For the case $C(f)$:

\[
\begin{array}{cccccccc}
U_0^\# & \to & U_0^\# \\
\downarrow & & \downarrow \\
U_0 & \to & U_0 \\
\end{array}
\]

If $n \geq 2$, this is in $L_n$, and it is h.eq. to $U_0$. On the other hand, $U_0^\# \to U_0 \in L_1$ is a subcomplex and the quotient is the suspension of a complex $V$ in $L_{n-1}$. So we have

\[
[U] = [U_0^\# \to U_0] - [V] \in \text{Im} \{L_{n-1} \to L_n\}
\]

Another point is that for $L_n$ one obtains the same Grothendieck group if one weakens

\[
U \cap V \Rightarrow [U] = [V] \to U \cap 0 \Rightarrow [U] = 0.
\]

This is not obvious. The argument when the dimension of the chain complexes increases is as follows. Given a h.eq. $f: U \to V$ one forms the mapping cylinder $M(f)$ which satisfies

\[
M(f) = V \oplus C(U) \quad \text{in general}
\]

\[
= U \oplus C(f) \quad \text{if } f \text{ a h.eq.}
\]

contract.
The problem is that $M(f)$ is upper-dimensional in general. However one can modify things as follows. We have

$$
\begin{align*}
M(f)_k &= U_k \oplus V_{k-1} \oplus V_k \\
C(U) &
\end{align*}
\begin{pmatrix}
 d & -1 \\
-1 & d
\end{pmatrix}

\xymatrix{
0 \ar[r] & U \ar[r] & M(f) \ar[r] & C(f) \ar[r] & 0
}

Picture of $M(f)$ at the top:

$$
\begin{align*}
U_n \ar[r]^d & U_{n+1} \\
\ar[d]^{(-1,+)} & \ar[d]^{(-1,+)}
\end{align*}
\begin{align*}
U_n \oplus V_n \ar[r]_{d \oplus 0} & U_{n+1} \oplus V_{n+1}
\end{align*}

Replace $M(f)$ by $M(f)/d_{M}(U_n)$. Now $d_{M}(U_n)$ is an elementary complex, hence contractible. We want to see that it maps isomorphically onto a direct summand of both $C(U)$ and $C(f)$ (ignoring differentials). It will then follow that we have locally split exact sequences.
\[
\frac{d_m(u_n)}{d_m'(u_n)}
\]
\[
0 \rightarrow u \rightarrow M(f) \rightarrow C(f) \rightarrow 0
\]
\[
M'(f) \rightarrow C'(f)
\]
\[
0 \rightarrow 0
\]

whence 
\[
M'(f) = U \oplus C'(f)
\]

with \(C'(f) \cong 0\). Similarly 
\[
M'(f) = V \oplus C'(u)
\]

with \(C'(u) \cong 0\).

Now \(d_m'(u_n)\) is obviously contained as a summand in \(C(f)\):
\[
U_n \rightarrow U_{n-1} \rightarrow \ldots
\]

Now 
\[
U_n \rightarrow U_{n-1} \rightarrow C(f) \rightarrow U_n \rightarrow U_{n-1}
\]

\[
V_n \rightarrow V_{n-1}
\]
is contractible so 
\[
0 \rightarrow U_n \rightarrow V_n \oplus U_{n-1}
\]
is a split injection and we win.
I want to derive a formula for the inverse of \( \text{Ker} L(R, A) \rightarrow \text{Ker} L(R, A) \), that is, to express \([U] \in \text{Ker} L(R, A)\) in terms of classes of length one complexes.

The initial idea goes as follows. Suppose \( U \) is a chain complex, and let \( h \) be a bipy operator such that \( dh + hd = 1 \mod A \). We can attach an elementary complex using the map

\[
\xrightarrow{d_1} U_2 \xrightarrow{d_0} U_1 \xrightarrow{d_0} U_0
\]

This gives the complex \( U \)

\[
\xrightarrow{(d_3)} U_3 \oplus U_2 \oplus U_1 \oplus U_0 \xrightarrow{(d_3, d_2, h_0)} U_0
\]

and the de Rham subcomplex such that the quotient is \( U' \):

\[
\xrightarrow{(d_3)} U_3 \oplus U_2 \oplus U_1 \oplus U_0 \xrightarrow{(d_3, d_2, h_0)} U_0
\]

Thus in the Grothendieck group we have

\[\[ [U] = [d_3, h_0: U_0 \rightarrow U_0] - [U'] \]

Now notice the transition \( U \rightarrow U' \); \( U_0 \) has been shifted to be with \( U_2 \). Repeating this process
should lead to a din 1 complex made of $\tilde{U}$ even, $\tilde{U}$ odd with differential constructed from $\alpha, \beta =$ hopefully $\alpha + \beta$.

Consider $\tilde{U} = U \oplus U[1] \oplus U[2] \oplus \ldots$

With the differential and homotopy

$$d = \begin{pmatrix} d & 0 \\ -d & -1 \\ d & 0 \\ -d & -1 \\ d & -1 \\ d & -1 \end{pmatrix}$$

$$k = \begin{pmatrix} 0 & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ -1 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since

$$\begin{pmatrix} -d & -1 \\ d & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

it's clear that we have the complex

$$\tilde{U} = U \oplus C(U[1]) \oplus C(U[2]) \oplus \ldots$$

Now conjugate by the invertible operator

$$\begin{pmatrix} 1 & h \\ 1 & \cdot \end{pmatrix} \oplus \begin{pmatrix} 1 & h \\ 1 & \cdot \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 1 & h \\ 1 & \cdot \end{pmatrix}$$

on $(U \oplus U[1] \oplus U[2] \oplus U[3]) \oplus \ldots$

$$\begin{pmatrix} 1 & -h \\ 1 & \cdot \end{pmatrix} \begin{pmatrix} d & 0 \\ -d & -1 \\ d & 0 \\ -d & -1 \end{pmatrix} = \begin{pmatrix} d & 0 \\ -d & -1 \\ d & 0 \\ -d & -1 \end{pmatrix}$$
Conjugation doesn't change the sign, since
\[
\begin{pmatrix}
1 & h \\
0 & 1 + h
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & h \\
0 & 1 + h
\end{pmatrix}
\text{commutes.}
\]

You should be more careful but the point is
that
\[
\begin{pmatrix}
1 + h & 0 \\
0 & 1
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & -1 \\
0 & 0
\end{pmatrix}
\text{ commute if } a \text{ and } b \text{ do, and this holds for}
\]
a = \begin{pmatrix} 1 + h \\ 1 \end{pmatrix} \quad b = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},
\]
Here's a picture of \(\tilde{A}, \tilde{d}\):
Suppose \( U \) is 3-dimensional and consider the resulting \( \partial U \).

One can see this complex is \( \partial U \) since it's closed under the homotopy operator \( \tau \). The circled \( U \)
Claim: the subcomplexes make up the steps of a filtration.

From this complex we should get the formula

\[
[u] = 2 [u_0 \to^f u_0] - [u_1 \to^f u_1] + [u_2 \to^f u_2]
- [d + h: U_2 \oplus U_0 \to U_3 \oplus U_1]
\]

Actually before this complex I could have done the simpler

\[
\begin{array}{c}
U_3 \\
\overrightarrow{U_2} \\
\overrightarrow{U_1} \\
\overrightarrow{U_0} \\
\end{array}
\]

which gives

\[
[u] = [u_0 \to^f_0 u_0] - [u_1 \to^f_1 u_1]
+ [d + h: \frac{U_3}{U_1} \to \frac{U_2}{U_0}]
\]

Observe that

\[
[d + h: U^+ \to U^-] + [d + h: U^- \to U^-]
= [\frac{(d + h)^2}{U^+} \to U^-] = [\frac{(d + h)^2}{U^-} \to U^-]
\]

but $1 - f + h^2$ is triangular:
so we get

\[ \sum_{i} \left[ 1 - f_{2i} \right] = \sum_{i} \left[ 1 - f_{2i+1} \right] \]

or \[ \sum (w)^n \left[ 1 - f_{n} \right] = 0 \]

General formula:

\[
\begin{array}{c}
U_0 \rightarrow U_2 \\
U_1 \rightarrow U_2 \\
U_2 \rightarrow U_2 \\
\end{array}
\]

\[ [u] = \left[ d + h : U^+ \rightarrow U^+ \right] \]

\[ + \sum_{i \geq 0} (p-i)[1-f_{2i}] - \sum_{i \geq 0} (p-i)[1-f_{2i+1}] \]

\[ [u] = \left[ d + h : U^+ \rightarrow U^+ \right] \]

\[ - \sum_{i \geq 0} i[1-f_{2i}] + \sum_{i \geq 0} i[1-f_{2i+1}] \]

Note that under the map \( K_0 A \rightarrow K_0 A \), the class \([1-f_n]\) go to zero so we get

\[ \text{Im} [u] \text{ in } K_0 A = \left[ d + h : U^+ \rightarrow U^+ \right] \]

which is a Whitehead type formula.
In trying to establish Morita invariance for Hochschild homology of b-antiyal rings in general I seem to encounter the problem that a free bimodule over \( R \) need not be flat as left or right \( R \)-module. Let's discuss aspects of this problem.

Let's start with the formula

\[
\mathcal{X} \otimes_R \mathcal{M} = (\mathcal{X} \otimes \mathcal{M}) \otimes_{\mathcal{R} \otimes \mathcal{R}^{op}} \mathcal{R} = (\mathcal{X} \otimes \mathcal{M})/\alpha(\mathcal{X} \otimes \mathcal{M})
\]

where \( \alpha \) is the left ideal in \( \mathcal{R} \otimes \mathcal{R}^{op} \) generated by \( a \otimes 1 - 1 \otimes a \), \( a \in \mathcal{R} \). Let \( \mathcal{E} \rightarrow \mathcal{R} \) be a free (flat should be enough) \( \mathcal{R} \)-bimodule resolution of \( \mathcal{R} \). Suppose \( \mathcal{X}, \mathcal{M} \) are projective resolutions of \( \mathcal{X} \) and \( \mathcal{M} \) over \( \mathcal{R}^{op} \) and \( \mathcal{R} \otimes \mathcal{R}^{op} \). Then

\[
\mathcal{X} \otimes \mathcal{M} \hookrightarrow (\mathcal{X} \otimes \mathcal{M}) \otimes_{\mathcal{R} \otimes \mathcal{R}^{op}} \mathcal{E} = \mathcal{X} \otimes \mathcal{E} \otimes \mathcal{M}
\]

should be a quasi. Why:

\[
\mathcal{X} \otimes \mathcal{M} = (\mathcal{X} \otimes \mathcal{M}) \otimes_{\mathcal{R} \otimes \mathcal{R}^{op}} \mathcal{R} \leftarrow (\mathcal{X} \otimes \mathcal{M}) \otimes_{\mathcal{R} \otimes \mathcal{R}^{op}} \mathcal{E}
\]

complex of \( \text{proj} \) \( \mathcal{E} \) modules

because tensoring with a \( \text{proj} \) \( \mathcal{E} \) respects quasi-
The point however is that $\hat{X} \otimes \hat{M} \rightarrow X \otimes Z$ is not necessarily a quasi. Thus we have

$$X \otimes^L_R M \rightarrow (X \otimes \hat{M}) \otimes_{Z \otimes R^g R} R$$

but it's not clear what the homology of $X \otimes \hat{M}$ is.

Consider a Morita context $(A, Q, \hat{A})$ with everything finite and flat on both sides. Let $E \rightarrow \hat{A}$ be a projective $\hat{A}$-bimodule resolution and $F \rightarrow \hat{B}$.

Consider

$$A \otimes A E \otimes A \overset{1}{\leftarrow} Q \otimes F \otimes A P \otimes F \otimes A$$

$$\Rightarrow$$

$$P \otimes E \otimes A Q \otimes F \otimes A \overset{2}{\rightarrow} B \otimes F \otimes B$$

Claim 1 is a quasi. Since $E$ consists of proj $\hat{A}$ bimodules we know $\otimes A E \otimes A$ respects quasi. Check: $T \otimes A (\hat{A} \otimes \hat{A}) \otimes A = \hat{A} \otimes A T \otimes A \hat{A} = T$

Now $F \rightarrow \hat{B}$ quasi and $Q$ flat over $B^g R_1$. $P$ flat over $B$ simply $Q \otimes B F \otimes B P \rightarrow Q \otimes B \beta \otimes B P = Q \otimes B P = A$ is a quasi. Thus 1 is a quasi.

A similar argument holds in the case
of which uses $P, Q$ flat over $A^{op}$ and $A$ resp.

Now I think we know that in an everything firm Morita context that

$P$ firm flat over $A^{op} \iff P \otimes_A Q = B$ is firm flat over $B^{op}$

$B$ firm flat over $A \iff B \otimes A^{op} = B$ is firm flat over $A^{op}$

So in an everything firm situation, if $B$ is flat on both sides then we have a canonical map (after inverting quas)

\[
\begin{array}{ccc}
B & \otimes & F \\
\otimes_B & \otimes & \otimes_B \\
A & \otimes & E \\
\otimes_A & \otimes & \otimes_A
\end{array}
\]

Apparently I can define $HH$ for a Root category using biflat coordinates. But I don't see how to go further. For example, I suppose I assume $P, Q, \tilde{A}$ flat (equiv. $B$ is $\tilde{B}$ biflat), this is a biflat coordinatization. Further assume $A, \tilde{A}$ flat (equiv. $P$ is $B^{op}$ flat), I want $\otimes$ to be a quas, and it suffices that

\[
Q \otimes_{\tilde{B}} F \otimes_{\tilde{B}} P \rightarrow Q \otimes_{\tilde{B}} P = A
\]

because $P$ is biflat be a quas., and it further suffices that $Q \otimes_{\tilde{B}} F \otimes_{\tilde{B}} \tilde{B} = 0$ be a quas. Can this be done by a leg of $F \rightarrow \tilde{B}$ as a map over $\tilde{B}$? Certainly there's a section $F \rightarrow \tilde{B}$. 


Let \( A \to B \) be a homomorphism of nonunital rings. We have the restriction of scalars functor

\[
\text{mod}(B) \to \text{mod}(A)
\]

which is exact and carries \( B \)-nil modules into \( A \)-nil modules, hence it induces an exact functor

\[
\text{M}(B) \to \text{M}(A)
\]

Suppose \( A, B \) idempotent. Then for \( N \) a \( B \)-module, \( M \) an \( A \)-module we have

\[
\begin{align*}
\text{Hom}_{\text{M}(A)}(M, N) &= \text{Hom}_{\text{M}(B)}(M, \text{Hom}_B(B^{(2)}, N)) \\
&= \text{Hom}_A(A^{(2)} \otimes_A M, \text{Hom}_B(B^{(2)}, N)) \\
&= \text{Hom}_B(B \otimes_A A^{(2)} \otimes_A M, \text{Hom}_B(B^{(2)}, N)) \\
&= \text{Hom}_B(B \otimes_A A^{(2)} \otimes_A M, N) \\
&= \text{Hom}_{\text{M}(B)}(B^{(2)} \otimes_A A^{(2)} \otimes_A M, N)
\end{align*}
\]

In other words there's an extension of scalars functors \( \text{M}(A) \to \text{M}(B) \) given by

\[
\begin{align*}
M &\to A^{(2)} \otimes_A M \\
&\uparrow \text{M}
\end{align*}
\]

\[
\begin{align*}
&\to B \otimes_A A^{(2)} \otimes_A M \\
&\uparrow \text{basechange}
\end{align*}
\]

\[
\begin{align*}
&\to B^{(2)} \otimes_A A^{(2)} \otimes_A M \\
&\uparrow \text{if you want}
\end{align*}
\]

\[
\begin{align*}
&\text{M} \text{ firm}
\end{align*}
\]
For example, let \( R \rightarrow T \) be a commutative ring homomorphism where \( R, T \) happen to be unital. Then we get the adjoint functors
\[
\begin{align*}
M(R) & \longrightarrow M(T) \\
\text{mod}(R) & \longleftarrow \text{mod}(T)
\end{align*}
\]

\[
M \mapsto T \otimes_R M = (T \otimes T^e)^e \otimes_R M
\]

since
\[
Te \otimes e^e M = Te \otimes Te^e e^e M = 0
\]

rest. of scalars
they make:
form over \( R \)

\[
eN = R \otimes N \longleftarrow R
\]

where \( e \) is the image of \( 1_R \) in \( T \).

Consider now a given ring \( A \) and a triple \( (\mathbb{Q}, P, Q \otimes P \rightarrow A) \) with \( P, Q \) fixed and \( \psi \) arbitrary. We have the Morita context
\[
\begin{pmatrix}
A & A \\
P & Q \otimes P
\end{pmatrix}
\]

with ideals
\[
\begin{pmatrix}
A & Q \\
P & (A, Q) = A
\end{pmatrix}
\]

\[
\begin{pmatrix}
A \\
P
\end{pmatrix} \otimes (A, Q) = C.
\]

So this context gives a Morita equivalence
\[
\begin{align*}
M(A) & \longrightarrow M(C) \\
M & \longleftarrow (A) \otimes (A)
\end{align*}
\]
On the other hand we have a monomodal ring homomorphism \( A \xrightarrow{(A \otimes B)} C \)
which induces a functor \( M(A) \rightarrow M(C) \)
as above. It sends \( M \) to the limit version \( A \otimes_A M \) followed by extension of scalars

\[
C \otimes_A A \otimes_A M = (A \otimes_A (A \otimes Q) \otimes_A A \otimes M
\]

\[
= (A \otimes Q) \otimes_A M
\]

since \( Q \otimes_A A = Q \otimes_A A^2 = QA \otimes_A A = 0 \).

Thus we see that the Morita equivalence associated to the context \((*)\) coincides with
extension and restriction of scalars \( \text{wrt } A \subset (A \otimes_Q) \).

Let's check the other functor

\[
N \mapsto (A \otimes Q) \otimes_C N
\]

\[
m(C) \rightarrow m(A)
\]

\[
N \mapsto A \otimes_A N = A \otimes_A C \otimes_C N
\]

\[
= A \otimes_A (A \otimes Q) \otimes_C N
\]

\[
= (A \otimes_Q) \otimes_C N
\]