Let $A$ be a small abelian category and consider

$$\text{Lex}(A, \text{Ab}) \longrightarrow \text{Add}(A, \text{Ab})$$

left exact

functors

additive

functors

According to Gabriel, $\text{Lex}(A, \text{Ab})$ is an abelian category, and $F \rightarrow R^0F$ is left adjoint to the above inclusion.

The observation is that this fits into the framework of localizing subcategories of Grothendieck categories. Call $F : A \rightarrow \text{Ab}$ effaceable if given $\xi \in F(M)$, $\exists M \xrightarrow{i} N$ st $\xi$ is killed by $i$.

The effaceable functors form a Serre subcategory of $\text{Add}(A, \text{Ab})$ closed under direct sums.

Let $M \hookrightarrow N$ in $A$. Then we have in $\text{Add}(A, \text{Ab})$ an exact sequence

$$0 \longrightarrow \frac{h^\bullet M}{h^\bullet M} \longrightarrow h^\bullet N \longrightarrow h^\bullet M \longrightarrow \frac{h^\bullet M}{\text{Im} h^\bullet N} \longrightarrow 0$$

where $\frac{h^\bullet M}{\text{Im} h^\bullet N}$ is effaceable. Indeed, let $\xi \in (\frac{h^\bullet M}{\text{Im} h^\bullet N})(X) = \text{Hom}(M,X) / \text{Im} \text{Hom}(N,X)$. Define $X \rightarrow Y$ by pushout

$$\begin{array}{ccc}
M & \rightarrow & N \\
\Downarrow & & \Downarrow \\
X & \rightarrow & Y
\end{array}$$

Then $\xi$ goes to zero in $(\frac{h^\bullet M}{\text{Im} h^\bullet N})(Y)$.

Let $E$ be an injective functor, which is
affaceable-free. Then applying \( \text{Hom}(-, E) \) to \( \mathfrak{X} \) we get

\[
0 \rightarrow E(M) \rightarrow E(N) \rightarrow E(N/M) \rightarrow 0
\]

showing that \( E \) is exact.

It then follows that the kernel of any map between injective affaceable-free functor is left exact.

Conversely given \( F \) left exact, it's affaceable-free (this is equivalent to preserving mono's), so there is a mono \( F \rightarrow E \) with \( E \) injective affaceable-free. Then one checks \( E/F \) is affaceable-free, so it follows that \( F \) is the kernel of a map between injective affaceable-free functors.

Some remaining points to be clarified:

Why is the localization functor given by

\[
R^0F(M) = \lim_{\leftarrow M \in N} \ker \left( F(N) \rightarrow F(N/M) \right)
\]

Is there any link of this to the fact that

\[
\text{Lex}(A, \mathfrak{A}) = \text{Pro} A
\]

i.e. the left exact functors are of the form

\[
F(M) = \lim_{\rightarrow} \text{Hom}(X_\alpha, M)
\]

for some filtered inverse system \( \{X_\alpha\} \) in \( A \).

Consider now \( R, I \) with \( R \) morthorsian commutative. Then we have

\[
\]
coherent sheaves on $U = \mathbf{Sp}(R)-\mathbf{Sp}(R/I)$

\[
\begin{align*}
\text{fg mod}(R) & \xrightarrow{\cong} \text{mod}(R) \\
\text{fg null}(R,I) & \xrightarrow{\text{null}(R,I)} \text{tors}(R,I)
\end{align*}
\]

call this $A$

then $A$ is the ind category $\text{Lex}(\alpha, ab)$ quasi-coherent sheaves on $U$

some natural questions

1. exactness of embedding

\[
\begin{align*}
\text{mod}(R) & \xrightarrow{\sim} \text{Coform}(R,I) \\
\text{tors}(R,I) & \xrightarrow{\text{null}(R,I)} \text{mod}(R)
\end{align*}
\]

also the restriction to $\frac{\text{fg mod}(R)}{\text{fg null}(R,I)}$

2. we have an embedding

\[
\begin{align*}
\text{form}(R,I) & \hookrightarrow \text{Lex}(\frac{\text{mod}(R)}{\text{null}(R,I)}, ab) \\
M & \mapsto \text{Hom}(M,-)
\end{align*}
\]

Does the fact that $\text{Lex}$ is abelian (mod set theory) help?

Observation: If $M$ form, or more generally $M=\text{IM}$, then any quotient $N$ of $M$ which is fin gen.
also satisfies $N=\text{IN}$ and this means $\exists a A$ such that $(1-a)N=0$. The support of $N$ is closed in $\mathbf{Sp}(R)$ and disjoint from $\mathbf{Sp}(R/I)$. Not very interesting.

It seems unfruitful to consider $\text{Hom}(M,-)$ for $M$ form. In the noetherian case, situation the thing to consider is $A = \frac{\text{fg mod}(R)}{\text{fg null}(R,I)}$ its and category $\frac{\text{mod}(R)}{\text{I-tors}}$ and its pro category they Deligne.
Suppose \( A \) is a left ideal in \( R \) until
Then we have the Morita context \((\tilde{A} \rightarrow A) \otimes_A \rightarrow R \)
which leads to the equivalence
\[
M \quad \overset{\sim}{\longrightarrow} \quad A \otimes_A M \quad \text{for f rm modules}
\]
\[
N = R \otimes_R N \quad \overset{\sim}{\Longleftarrow} \quad N
\]

the equivalence
\[
M \quad \overset{\sim}{\longrightarrow} \quad \text{Hom}_R^\sim (R, M) \quad \text{for s fd modules}
\]
\[
\text{Hom}_R^\sim (A, N) \quad \overset{\sim}{\Longleftarrow} \quad N
\]

the equivalence
\[
M \quad \overset{\sim}{\longrightarrow} \quad \text{Im} \left\{ A \otimes_A M \twoheadrightarrow \text{Hom}_R^\sim (R, M) \right\}
\]
for mull + c mull f rm modules.

I thought it should be true that \( M \) simple
mmull \( \Rightarrow \) the corresponding \( R \)-module \( N \)
is
naturally isomorphic to \( M \) and hence also
a simple \( R \)-module. This is true when \( A \) is an
ideal (see p 587) but breaks down if \( A \) is only a
left ideal. Let's go over the argument.

Let \( M \in \text{mod}(\tilde{A}) \) \( \Rightarrow M/AM = A \cdot M = 0 \). The

Let \( M \in \text{mod}(\tilde{A}) \) \( \Rightarrow M/AM = A \cdot M = 0 \). The
corresponding mull + c mull-free \( R \)-module \( N \) the ideal \( AR \)
is the image of \( \varphi \) in:

\[
\varphi(a \otimes m)(r) = (ra)m
\]

because \( M/AM = 0 \)

Because \( M/AM = 0 \)

because \( A \cdot M = 0 \)
\( \varphi \) is an \( R \)-module map, the other maps are only \( \tilde{A} \)-module maps in general. When \( A \) is an ideal however \( \tilde{A}(R/A) = 0 \) so that \( \text{Hom}_{\tilde{A}}(R/A, M) = 0 \) and we can identify the image of \( \varphi \) with \( M \) itself.

In fact we can use the Morita context 
\[
\begin{pmatrix} \tilde{A} & A \\ A & R \end{pmatrix}.
\]

Let's shift notation to the ideal \( I \subset R \).

Then we have the Morita context 
\[
\begin{pmatrix} \tilde{I} & I \\ I & R \end{pmatrix}
\]

with \( QP = I^2 \subset \tilde{I} \) and \( PQ = I^2 \subset R \). Thus we have equivalences of categories

\[
\text{fim} (\tilde{I}, I) \cong \text{fim} (R, I)
\]

\[
M \mapsto I \otimes_I M \cong M
\]

\[
N \cong I \otimes_R N \leftrightarrow N
\]

The second functor is restriction of scalars for \( \tilde{I} \to R \).

\[
\text{solid} (\tilde{I}, I) \cong \text{solid} (R, I)
\]

\[
M \mapsto \text{Hom}_{\tilde{I}}(I, M) \cong M
\]

\[
N \cong \text{Hom}_{\tilde{R}}(I, N) \leftrightarrow N
\]

Again the second functor is restriction of scalars.

\[
\begin{pmatrix} \tilde{I} & I \\ I & R \end{pmatrix}
\]

\[
M \mapsto \text{Im} \left( I \otimes_I M \overset{\varphi}{\to} \text{Hom}_{\tilde{I}}(I, M) \right)
\]

\[
\text{Im} \left( I \otimes_R N \to \text{Hom}(I, N) \right) \leftrightarrow N
\]
More on Morita invariance examples

The problem with \( A \triangleleft B, \ ABA = A^2, \ B = \tilde{B} \tilde{A} \tilde{B} \)

is that this is not the appropriate thing to reduce to.

Suppose \( A \) is a subring of \( R \) unital such that \( ARA \subseteq A^2 \), e.g. \( RA \subseteq A \) or \( AR \subseteq A \).
Then one has the Morita context
\[
\begin{pmatrix}
\tilde{A} & AR \\
RA & R
\end{pmatrix}
\]
\[QP = ARA = A^2\]
\[PQ = RA^2R = (RAR)^2\]

whence a Morita equivalence \( A \cong RAR \).

If you want to do this non-unitally, working with \( B = RAR \), then the Morita contexts are
\[
\begin{pmatrix}
A & AB \\
B & B
\end{pmatrix} \subseteq \begin{pmatrix}
\tilde{A} & \tilde{A} \tilde{B} \\
\tilde{B} & B
\end{pmatrix}
\]

Ideals in the smaller are
\[QP = AB^2A = A(RAR)(RAR)A = A^4\]
\[PQ = BA^2B = (RAR)A^2(RAR) = RA^2R = B^4\]

Ideals in the larger are
\[QP = A\tilde{B}A = A^2\]
\[PQ = \tilde{B}A\tilde{B} = (\tilde{B}A\tilde{B})^2\]

But notice that \( \tilde{B}A\tilde{B} = A + BA + AB + B\tilde{A} \tilde{B} \)
\[= A + RA + \tilde{A}R + R\tilde{A}R\tilde{A}R\]
so \( \tilde{B}A\tilde{B} \subseteq B^3 \).
Thus you don't get the situation \( \tilde{B} = \tilde{B}A\tilde{B} \).
Idea. Let $P$ be a projective $R$-module.
Then one gets a Garside torsion theory
where the torsion modules are $M \in \text{Hom}_R(P,M) = 0$.
Choose generators $p_i \in P$, $i \in \Lambda$, and choose a lifting
$s: R^\Lambda \to P$, and choose a lifting
$p \mapsto (f_i(p))$ with $f_i \in \text{Hom}_R(P,R)$.
Then $p = \sum_{\Lambda} f_i(p)_i$ and $f = \sum_{\Lambda} f_i \circ f(p_i)$
so the $f_i$ generate the dual $\text{Hom}_R(P,R)$.

If $M$ is an $R$-module we have

$$
\text{Hom}_R(P,M) \xleftarrow{\sim} \text{Hom}_R(R^\Lambda,M) = M^\Lambda
$$

$p \mapsto \sum f_i(p)_i = f(p)$

so $\text{Hom}_R(P,M) = 0 \iff (\sum f_i(p))M = 0$.

Thus $I = \sum f_i(p) = \sum f(p) = \text{Im} \{ P \otimes_{\mathbb{Z}} \text{Hom}_R(P,R) \to R^2 \}$
also called $\text{Jac}(P)$
is the idempotent ideal corresponding to this torsion theory.
(Its idempotent because $f(p) = \sum f_i(p)_i f(p_i)$
$\Rightarrow I = I^2$.)

You need to look up the Bergman theorem
in Golabi book, which says under some countability hypothesis
that an idempotent ideal is the trace
of a projective module.

Notice that

$p = \sum_{\Lambda} f_i(p)_i \Rightarrow P = IP$ so $P$ is a full
projective module. This means that $I \cap \text{Jac}(R)$
otherwise one has a contradiction of the (generalized)
Kaplansky's theorem. Recall Kaplansky's Theorem.
Says any projective module over a local ring is free, hence it's trace is the whole ring if the module is ≠ 0. So Bergman's thm. must have I ⊆ \text{Jac}(R) as hypothesis.

\[
\text{Hom}_R(M, \text{Hom}_\mathbb{Z}(X, \mathbb{Q})) = \text{Hom}_\mathbb{Z}(X \otimes_R M, \mathbb{Q})
\]

so 

\[X \text{ R-flat, } \mathbb{Q} \mathbb{Z}\text{-injective} \Rightarrow \text{Hom}_\mathbb{Z}(X, \mathbb{Q}) \text{ R-injective} \]

\[
\text{Hom}_R(R/I, \text{Hom}_\mathbb{Z}(X, \mathbb{Q})) = \text{Hom}_\mathbb{Z}(X/I, \mathbb{Q})
\]

so \[X = X/I \Rightarrow \text{Hom}_\mathbb{Z}(X, \mathbb{Q}) \text{ torsion-free} \]

Thus \[X \text{ flat, } \mathbb{Q} \mathbb{Z}\text{-inj} \Rightarrow \text{Hom}_\mathbb{Z}(X, \mathbb{Q}) \text{ solid inj} \]

Suppose \(M\) torsion-free \(R\)-module, let \(0 ≠ m ∈ M\). There exists a sequence \(x = (a_n)\) in \(I\) such that \(F(x) \otimes_R M \to \mathbb{Q}/\mathbb{Z}\) is injective. Thus \(F(x) \otimes_R M \to \mathbb{Q}/\mathbb{Z}\) is injective, hence \(F(x) \otimes_R M \to \mathbb{Q}/\mathbb{Z}\) is injective. Thus we have an \(R\)-module map \(M \to \text{Hom}_R(F(x), \mathbb{Q}/\mathbb{Z})\) s.t. \(m \neq 0\). This means that there are enough solid injectives of the form \(\text{Hom}_R(F(x), \mathbb{Q}/\mathbb{Z})\).

Review the adjoint functors

\[
\text{tor}_1(R, I) \xrightarrow{\text{nat}} \text{mod}(R) \xrightarrow{j^*} \text{mod}(R)/\text{tor}(R, I)
\]

Here \(\text{nat}\) is the inclusion, \(j^*\) the canonical functor to the quotient category. \(\text{tor}_1(M)\) is the torsion submodule.
of $M$, it exists because $\text{tors}(R, I)$ is a Serre subcategory closed under $\oplus$'s. This also implies $M/\text{tors}(R, I)$ is torsion-free.

Let's construct $f^*$, i.e. for each module $N$ we will produce a map $N \to f^* N$ s.t.

$$
\text{Hom}_{M_f}(M, N) \to \text{Hom}_R(M, f^* N) \quad \forall M
$$

Here $M_f = \text{mod}(R)/\text{tors}(R, I)$.

Recall that

1. An injective module $Q$ is solid iff it is torsion-free:

$$
0 \to \text{Hom}_R(R/I, Q) \to \text{Hom}_R(R, Q) \to \text{Hom}_R(I, Q) \to 0
$$

2. The injective hull of a torsion-free module is a torsion-free, hence it can be embedded in a solid injective.

3. A module is solid iff it is the kernel of a map between solid injectives.

Now observe that if $Q$ is a solid injective, then $\text{Hom}_R(-, Q)$ from $\text{mod}(R)$ to $\text{Ab}$ is exact and it kills $\text{tors}(R, I)$, thus it descends to the quotient category. Using

$$
\text{Hom}_{M_f}(M, Q) = \lim_{\text{arrows}} \text{Hom}_R(M', Q)
$$

where the limit is taken over the cat of maps $M' \to M$ which have target $M'$ and which are torsion isomorphisms.
we obtain

\[ \text{Hom}_{M_t} (M, Q) \cong \text{Hom}_R (M, Q) \]

If \( N \) is solid, then choosing a cospresentation

\[ 0 \longrightarrow N \longrightarrow Q^0 \longrightarrow Q^1 \]

with \( Q^i \) solid injective, we see the above implies

\[ \text{Hom}_{M_t} (M, N) \cong \text{Hom}_R (M, N) \]

Now if \( N \) is arbitrary we construct a resolution modulo torsion by solid injectives.

\[ 0 \longrightarrow N/\ell^1 N \longrightarrow Q^0 \longrightarrow N^1 \longrightarrow 0 \]

\[ 0 \longrightarrow N^1/\ell^1 N^1 \longrightarrow Q^1 \longrightarrow N^2 \longrightarrow 0. \]

and define \( j_\ast N \) to be the kernel of \( Q^0 \longrightarrow Q^1 \). Then \( j_\ast N \) is solid.

There is an obvious map \( N \longrightarrow j_\ast N \) whose kernel is \( \ell^1 N \) and whose cokernel is \( \ell^1 N^1 \):

\[ 0 \longrightarrow N/\ell^1 N \longrightarrow Q^0 \longrightarrow N^1 \longrightarrow 0 \]

\[ \downarrow \quad \downarrow \quad \downarrow \]

\[ 0 \longrightarrow j_\ast N \longrightarrow Q^0 \longrightarrow Q^1 \]

Thus \( N \longrightarrow j_\ast N \) is an isom mod torsion where
$J \ast N$ is solid. So we have

$$\text{Hom}_R(M, N) \to \text{Hom}_R(M, J \ast N) \cong \text{Hom}_R(M, J \ast N)$$

(see Jelonek's book on semi-free chapter)

Bergman's thin. says in a ring such that every countable left ideal is projective that any idempotent ideal is the trace of a projective module.
A preradical is a subfunctor $\tau$ of the identity functor on modules. Define

$$
\mathcal{F}_\tau = \{ M | \tau M = M \} \quad \tau\text{-torsion}
$$

$$
\mathcal{F}_\tau^* = \{ M | \tau M = 0 \} \quad \tau\text{-torsion-free}
$$

Then $\mathcal{F}_\tau$ is closed under quotients and $\Theta$'s:

$$
M \to M'' \quad \tau'' M = M'' \quad \bigoplus \tau M_i = \bigoplus \tau M_i \quad \tau (\bigoplus M_i) \quad \text{must be =}
$$

Dually $\mathcal{F}_\tau^*$ is closed under subobjects and $\Theta$'s.

Conversely suppose given a class of modules closed under quotients and $\Theta$'s. Define

$$
\mathcal{T}_{\mathcal{F}} M = \sum_{(N,f)} \text{Im} (f: N \to M) \subset M
$$

Note that $\mathcal{T}_{\mathcal{F}} M$ is a quotient of a direct sum of modules in $\mathcal{F}$ so $\mathcal{T}_{\mathcal{F}} M \in \mathcal{F}$. Clearly $\mathcal{T}_{\mathcal{F}}$ is a subfunctor of the identity, i.e. a preradical. $\mathcal{T}_{\mathcal{F}} M$ is the largest submodule of $M$ belonging to $\mathcal{F}$. It's clear also that

$$
\mathcal{T}_{\mathcal{F}} M = \sum_{N \mathcal{F}} \text{Im} (N \to M) \subset \tau M
$$

$M \in \mathcal{F}_{\mathcal{T}_{\mathcal{F}}} \iff \mathcal{T}_{\mathcal{F}} M = M \iff \mathcal{F} M \

Dually given a class \( \mathcal{F} \) closed under subobjects and quotients put

\[
\overline{\tau}_{\mathcal{F}} M = \bigcap_{(N,f) \in \mathcal{F}} \ker \{ M \xrightarrow{f} N \}
\]

This is a subfunctor of the identity.

Note that \( M/\overline{\tau}_{\mathcal{F}} M \) embeds in a direct product of members of \( \mathcal{F} \), hence \( M/\overline{\tau}_{\mathcal{F}} M \in \mathcal{F} \). \( \overline{\tau}_{\mathcal{F}} M \) is the smallest submodulus of \( M \) such that the quotient belongs to \( \mathcal{F} \).

\[
\overline{\tau}_{\mathcal{F}} M = \bigcap_{(N,f) \in \mathcal{F}} \ker \{ M \xrightarrow{f} N \} \subset \overline{\tau} M
\]

\( N = 0 \)

\[
M \in \mathcal{F} \iff \overline{\tau}_{\mathcal{F}} M = 0 \iff M \in \mathcal{F}
\]

A preradical \( \tau \) is a radical when \( \tau(M/\overline{\tau} M) = 0 \).

A preradical \( \tau \) is idempotent when \( \tau(\overline{\tau} M) = \overline{\tau} M \).

If \( \tau \) is a radical then \( \overline{\tau} \) is closed under extensions:

Given \( 0 \to M' \to M \to M'' \to 0 \) with \( \overline{\tau} M' = M' \) and \( \overline{\tau} M'' = M'' \), then \( M' = \overline{\tau} M' \subset \overline{\tau} M \), so \( M'' = M/M' \) maps onto \( M/\overline{\tau} M \). But \( \overline{\tau} M'' = M'' \) and \( \tau(M/\overline{\tau} M) = 0 \) imply the map \( M'' \to M/\overline{\tau} M \) is zero, hence \( M = \overline{\tau} M \).

Conversely suppose \( \tau \) is a preradical such that \( \overline{\tau} \) is closed under extensions. Given \( M \),
let $M'$ be the inverse image in $M$ of $\tau(M/\tau M)$. We then have
an extension

$$0 \to \tau M \to M' \to \tau(M/\tau M) \to 0$$

If $\tau$ is an idempotent radical, then $\tau M$ and $\tau(M/\tau M) \in \tau\tau$, so $M' \in \tau\tau$, i.e., $\tau M' = M'$. Then $M' \cap \tau M \subseteq \tau M$ implies $M' = \tau M$ so that $\tau(M/\tau M) = 0$.
Thus the converse isn’t so clear.

If $\tau$ is an idempotent radical, then $\tau\tau$ is closed under extensions.

Given $0 \to M' \to M \to M'' \to 0$ with $\tau M'' = 0 = \tau M'$.
Then $\tau M$ goes to zero in $M''$, so $\tau M \cap M' = \tau M$ and $\tau M = \tau \tau M \subseteq \tau M' = 0$.

I guess the good situation is when $\tau$ is an idempotent radical. Then

$F_\tau$ is closed under quotients, extensions, $\oplus$’s

$F_\tau$ subobjects, extensions, $\Pi$’s

and probably the three things $\tau, F_\tau, F_\tau$ are equivalent.

Also $F_\tau$ is closed under subobjects $\Rightarrow$ $F_\tau$ is closed under injective hulls $\Rightarrow$ $\tau$ is left exact.

Now suppose $\tau, \tau'$ are idempotent radicals such that $F_\tau = F_{\tau'}$. This is called TTF, a torsion torsion-free theory. Then $F_\tau$ is a dense subcategory
closed under products, so it's a
Janssen torsion theory: there is a unique
idempotent ideal $I$ in $R$ such that
\[ \mathcal{T}_I = \{ M \mid IM = 0 \} \]
\[ IM = \text{Hom}_R(R/I, M) = I^M \]
Now what is $I'$? $IM$ should be the
kernel of all maps from $M$ to an $I$-null module,
\[ I' M = IM. \]
Thus $\mathcal{T}_I' = \{ M \mid M = IM \}$.
This is not closed under submodules in general,
however it is closed under $\oplus$, quotients, and
extensions:
\[ I \otimes_R M' \rightarrow I \otimes_R M \rightarrow I \otimes_R M'' \rightarrow 0 \]
\[ 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \]
\[ \Rightarrow I \otimes_R M \rightarrow M. \]

What I missed:

equivalence between
$\mathcal{T}$ closed under quotients, $\oplus'$s and idempotent presheaves

$\mathcal{T}$ closed under subobjects, $\oplus'$s and radicals.

Also in the situation $\mathcal{T}_I = \mathcal{T}_I'$, the idempotent
radical $I' M = IM$ commutes with filtered limits,
hence $I'$ is a finite type torsion theory (i.e. if
$R/\alpha$ is torsion: $I(R/\alpha) = R/\alpha$ with $\alpha$ a left ideal
then the same is true for some finitely generated left
ideal $\leq \alpha$. This is obvious: $IR + \alpha = R \iff$ factor: $1-\alpha \in IR$
$\Rightarrow I(R/\alpha) = R/\alpha$.  

* [Note: Redacted or incomplete content]
Question: Suppose $M$ is nullfree: $M = IM$ and $N$ is nullfree: $IN = O$; (nullfree = torsionfree). Is $\text{Hom}_R(M,N) \cong \text{Hom}_{M_t}(M,N)$?

No. Take $(R,I) = (\mathbb{Z}, p\mathbb{Z})$ whence $M_t = \text{mod}(\mathbb{Z}[\frac{1}{p}])$, take $M = \mathbb{Z}[\frac{1}{p}]$, $N = \mathbb{Z}$. Then $\text{Hom}_\mathbb{Z}(M,N) = 0$ but $M$ and $N$ become isomorphic in $M_t$.

So back to the example $R = k[x,y]$, $I = (x,y)$. Recall that we have the Čech resolution

$$0 \to R \to E(R) \to \bigoplus E(R/p) \to E(k) \to 0$$

Then if $F$ is flat and finit, we get the resolution

$$0 \to F \to F \otimes_R E(R) \to \bigoplus F \otimes_R E(R/p) \to 0$$

Since $E(k)$ is $I$-torsion $\iff F \otimes_R E(k) = 0$.

This allows us to replace any complex of flat finit modules by a complex of solid injectives.

Check that flat $\otimes$ injective is injective over a mathean comm. ring. Reason is that a flat module is the filtered inductive limit of free modules, so it suffices to check a filtered inductive limit of injectives is injective. But any ideal $a$ in $k$ is fin. presented so
\[
\operatorname{Hom}(R, \varinjlim \mathbb{Q}_x) = \varinjlim \operatorname{Hom}(R, \mathbb{Q}_x) \\
\operatorname{Hom}(\mathfrak{m}, \varinjlim \mathbb{Q}_x) = \varinjlim \operatorname{Hom}(\mathfrak{m}, \mathbb{Q}_x)
\]

so it's clear.

Next if \( Q \) is injective one has the canonical filtration

\[
0 \subset \Gamma^i(Q) \subset \bigoplus \Gamma^i_p(Q) \subset Q
\]

by injective submodules. When \( Q \) is solid injective \( \Gamma^0(Q) = 0 \), and we get a canonical exact sequence

\[
o \to \bigoplus \Gamma^i_p(Q) \to Q \to E(R) \otimes_R Q \to 0
\]

which splits.

Thus if \( Q \) is a solid injective complex it appears as the \( b \)-fibre of a map

\[
E(R) \otimes_R Q \to \bigoplus \Gamma^i_p(Q)
\]

where

- \( K = E(R) \)
- \( \mathbb{Q} = \text{quotient field of } \mathbb{R} \)
- \( \text{complex of vector spaces over } \mathbb{R} \)
- \( \text{complex of injective torsion modules over the discrete valuation ring } \mathbb{R}_p \)
- \( \text{quotient of the } b \)-fibre of a map

We've seen that any \( \mathbb{R} \)-flat module \( F \) is the kernel of a surjection:

\[
F \otimes_R K \to \bigoplus F \otimes_E E(p)
\]

of solid injectives where the first is a \( \mathbb{K} \)-vector space and the second is a sum of injective hulls \( E(R) \)

of primes of height one.

Conversely, note \( K \) is flat and that
$E(R/p)$ has flat dimension $\leq 1$:

$0 \rightarrow R_p \rightarrow K \rightarrow E(R/p) \rightarrow 0$

Thus it should follow that any firm flat module $F$ has a unique up to canonical isomorphism representation as the kernel of a surjection from a $K$ vector space to a height one injective.

Summarize:

firm flat = kernel of a surjection from a height 0 injective to a height one injective

solid injective = extension of height 0 injective by a height 1 injective

These are canonical descriptions and apply to complexes.
July 21, 1994

For a general torsion theory in $\text{mod}(R)$:

\[
\text{tors} \xrightarrow{L} \text{mod} \xrightarrow{\delta^*} \text{mod/tors}
\]

when do we have a triangle in $D^+(\text{mod})$

\[
L_* R\delta^*(\mathcal{M}) \to \mathcal{M} \to R\delta^*(\mathcal{Q}^*\mathcal{M}) \to ?
\]

Recall that a torsion theory is called stable when the injective hull of any torsion module is torsion. This is equivalent to.

Let $\mathcal{Q}$ be any injective module. The injective hull $E(\mathcal{Q})$ of the torsion submodule $\mathcal{Q}'$ is a summand of $\mathcal{Q}$. Assuming the torsion theory is stable, $E(\mathcal{Q}')$ is torsion, hence contained in $\mathcal{Q}'$. Thus $E(\mathcal{Q}') = \mathcal{Q}'$, showing that $\mathcal{Q}'$ is injective, and that $\mathcal{Q}$ splits into the direct sum of a torsion injective $\mathcal{Q}'$ and a torsion-free injective.

On the other hand, suppose we assume $\mathcal{Q}'$ is injective for every injective $\mathcal{Q}$. If $\mathcal{M}$ is a torsion module, take $\mathcal{Q}$ to be $E(\mathcal{M})$. Then $\mathcal{Q}'$ is an injective submodule of $\mathcal{Q}$ containing $\mathcal{M}$, so $\mathcal{Q}' = \mathcal{Q}$, since $\mathcal{Q}$ is a minimal injective containing $\mathcal{M}$. Thus the injective hull of any torsion module is torsion. Thus we have proved...
A torsion theory is stable iff the torsion submodule of any injective module is injective iff any injective is the direct sum of a torsion injective and a torsion-free injective.

Now assuming that our torsion theory is stable consider $M$ in $D^+(\text{mod})$, and replace it by a quasi-isomorphic injective complex $Q$. Then we have an exact sequence

$$0 \rightarrow \mathcal{I}^! Q \rightarrow Q \rightarrow Q / \mathcal{I}^! Q \rightarrow 0$$

Now $\mathcal{I}^! Q = \mathcal{I}^* R \mathcal{I}^! (M)$ and $Q / \mathcal{I}^! Q$ is a complex of torsion-free injectives which is quasi-isomorphic modulo torsion to $M$, hence $Q / \mathcal{I}^! Q = R \mathcal{I}^* (\mathcal{I}^! M)$. Thus we get the desired $\mathcal{I}$ in $D^+(\text{mod})$.

Consider now the gaussian case $\mathcal{I} = \mathbb{I}^2$

$$\text{mod} (R/I) \xrightarrow{\mathcal{I}^*} \text{mod} (R) \xrightarrow{\mathcal{I}^*} \text{mod} (R) \xrightarrow{\mathcal{I}^*} M$$

$$\mathcal{I}^* (\mathcal{I}^! M) = M / IM \quad \mathcal{I}^* (\mathcal{I}^! M) = \mathbb{I} \otimes_R M$$

Notice the confusion of notation when you use $\mathcal{I}^! M$ for the torsion submodule. Then $\mathcal{I}^!$ has the right adjoint $\mathcal{I}^*$ which is exact, hence $\mathcal{I}^!$ carries injectives to injectives, i.e. it looks like any theory is stable. Instead we probably ought to use $\mathbb{I} M$ for the torsion
submodule \( l_x l^!(M) \). Note that
\[
\text{Hom}_R(M, \mathbb{I}N) = \text{Hom}_R(M/IM, N).
\]
i.e.
\[
\text{Hom}_R(M, l_x l^!(N)) = \text{Hom}_R(l_x l^!(M), N)
\]
so that \( l_x l^! \) is left adjoint to \( l_x l^* \).

Then stability means that \( l_x l^! \) respects injective, and this is equivalent to the left adjoint \( l_x l^*(M) = M/IM \) being exact, i.e. to \( R/I \) being right flat.

---

Question: You know \( \text{finit} \) \( (R,I) \) is abelian when \( I \) is right \( k \)-flat. But \( \text{finit} \) \( (R,I) \) depends only on \( I \), not \( R \).

Can you, given an arbitrary \( (R,I) \), find a Morita equivalence \( \text{finit} \) \( (R,I) \approx \text{finit} \) \( (S,I) \)
where \( I \) is \( S \)-flat?

Are there conditions on \( I \) guaranteeing that \( I \) can be embedded as a right flat ideal in some unital ring \( R \)?
July 23, 1994

There might be a derived category version of the construction $M(R,I) = \text{mod}(R)/\text{null}$, namely, let $D(\text{mod}(R)/\text{null})$ be the full subcat of $D(\text{mod}(R))$ consisting of complexes whose homology groups are null. This is a triangulated subcategory, so I believe there is a quotient triangulated category $D(R,I) = D(\text{mod}(R))/D(\text{mod}(R)/\text{null})$ (modulo set theory problems). One can ask whether this is equivalent to $D(\text{mod}(R)/\text{null})$.

Since Franke has constructed an abelian category whose derived category is "outside the universe" the set theory problems may be real.

One might hope (see p. 709) that

\[ \text{Hom}_D(M,N) = \lim \text{Hom}_D([I_R^k]^n M, N) \]
\[ = \lim \text{Hom}_D(M, R \text{Hom}_R([I_R^k]^n I, N)) \]

(in any case the argument on p. 709 show that the inverse system $[I_R^k]^n M$ is locally essentially zero when M is null.)

It seems that the firm and solid subcategories should be resp.

$R/I \otimes_R M \simeq 0$ equiv. $M \Rightarrow F$ firm flat

$R \text{Hom}_R(R/I, M) \simeq 0$ equiv. $M \Rightarrow Q$ solid injective
Note that earlier arguments using $\text{Hom}_R^2(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ are obsolete. For example, $F$ flat, $\text{IM} = 0 \Rightarrow \text{Ext}_R^*(F, M) = 0$ is proved as follows. Let $P \to F$ be a projective resolution, $M \to Q$ an injective resolution of $R/I$-modules. Then

$$\text{RHom}_R(F, M) \cong \text{Hom}_R(P, M) \cong \text{Hom}_R(P, Q)$$

$$= \text{Hom}_{R/I}(P/IP, Q)$$

is acyclic because $P/IP \cong F/IF = 0$ and $Q$ is injective.

But in fact things are even simpler.

$$\text{RHom}_R(F, M) \cong \text{Hom}_R(P, M)$$

$$= \text{Hom}_{R/I}(P/IP, M)$$

and this is homotopy to zero since $P/IP$ is an acyclic projective complex of $R/I$-modules.
Moore invariant for the firm derived category seems to be a consequence of the fact that it is equivalent to the derived category for the exact category of firm flat modules which we know is Moore invariant. (Understood here is the restriction to complexes bdd below for the lower indexing.) I would like to give a direct proof in the spirit of my work the past few days (specifically the work which is in the notes for the "paper", where the firm derived category is described as consisting of $M$ in $D_+(R)$ such that $\frac{R/I}{R} \otimes_R M = 0$).

So given a Moore context $(R, Q, S)$ with ideals $I = QP$, $J = PQ$ say, I want to prove that $\text{fr}m\ D_+(R, I) \cong \text{fr}m\ D_+(S, J), \ M \mapsto P \otimes_R M$ is an equivalence of categories.

First show this functor is well-defined. Suppose $M$ projective, whence $P \otimes_R M$ quasi $P \otimes_R M$. To show $S/J \otimes_S (P \otimes_R M)$ quasi 0.

To do this we can ignore the left $S$-module structure and choose a projective right $S$-module resolution $E \rightarrow S/J$. Then

$$S/J \otimes_S (P \otimes_R M) = (E \otimes_S P) \otimes_R M$$

and it suffices to show that $E \otimes_S P$ is right $I$-null, because then since $\frac{R/I}{R} \otimes_R M = 0$ we have $\frac{R}{I} \otimes_R M$ kills all complexes of $R^k$-modules which are $I^k$-null. Take a generator $qp$ for $I$. We have
The commutative diagram

\[ \begin{array}{ccc}
E \otimes_S P & \rightarrow & E \otimes_S P \\
\downarrow \quad & & \downarrow \\
E \otimes \mathbb{Q} & \rightarrow & E \otimes \mathbb{Q}
\end{array} \]

Since \( E \) is a resolution of \( S/J \), the inclusion \( E \mathbb{Q} \subset E \) induces the zero map on homology. Thus, right multiplied by \( \mathbb{Q} \) on \( H_x(\otimes_S P) \) is zero, and we win.

Now we know that \( M \rightarrow \mathbb{P}_R^L M \) from \( D^+(R) \) to \( D^+(S) \) carries from \( D^+(R, I) \) into \( D^+(S, J) \). Similarly \( \otimes_S \mathbb{Q} \) gives a functor in the opposite direction. We next want to see these functors are quasi-inverse to each other.

Take \( M \in D^+(R) \) projective, let \( U \rightarrow \mathbb{P}_R^L M \) be a projective resolution, whence \( \mathbb{Q} \otimes_S (\mathbb{P}_R^L M) \approx \mathbb{Q} \otimes_S U \). One has an obvious map

\[ \mathbb{Q} \otimes_S U \rightarrow \mathbb{Q} \otimes_S \mathbb{P}_R^L M \rightarrow M \]

which we want to be a quasi. Again we can ignore the left \( R \)-module structure and choose an \( S \)-split projective resolution \( E \rightarrow \mathbb{Q} \). Then we have

\[ \begin{array}{ccc}
E \otimes_S U & \rightarrow & E \otimes_S \mathbb{P}_R^L M \\
\downarrow \quad & & \downarrow \\
\mathbb{Q} \otimes_S U & \rightarrow & \mathbb{Q} \otimes_S \mathbb{P}_R^L M \rightarrow M
\end{array} \]
We all we need to do is show that 
\[ E \otimes_{S} P \to R \] is an \( I^{ob} \)-null quasi,
\(^{1}\) i.e., the homology of the cone is \( I^{ob} \)-null. This amounts to
\[ H_{n}(E \otimes_{S} P) = \text{Tor}^{S}_{n}(Q, P) \]
being \( I^{ob} \)-null for \( n > 0 \) and
\[ Q \otimes_{S} P \to R \]
being an \( I^{ob} \)-null quasi. The latter we know already: the kernel and cokernel are killed by \( I^{ob} \).

Thus we want to show that the quasi 
\[ E \to Q \] of \( S^{ob} \)-modules goes into \( E \otimes_{S} P \to Q \otimes_{S} P \),
which is a \( I^{ob} \)-null quasi. This amounts to
exactness of the functor
\[ \text{mod}(S^{ob}) \to \text{mod}(R^{ob}) \to \text{mod}(R^{ob})/\text{null}(R^{ob}, I^{ob}) \]
and this follows from the fact that \( \otimes_{S} P \) gives an equivalence 
\[ M(S^{ob}, I^{ob}) \cong M(R^{ob}, I^{ob}) \].

Direct proof. Let \( N \) be an acyclic complex of \( S^{ob} \)-modules. Then we claim the homology of 
\[ N \otimes_{S} P \] is killed by \( I^{ob} \). In effect take a

Generator \( g P \) for \( I^{ob} \). Then
\[ N \otimes_{S} P \to N \otimes_{S} P \]
\[ \otimes_{S} P \]
\[ n \]
\[ \otimes_{S} P \]
\[ \otimes_{S} P \]
\[ \otimes_{S} P \]
\[ \otimes_{S} P \]
\[ \otimes_{S} P \]
\[ \otimes_{S} P \]
\[ \otimes_{S} P \]

commutes and \( N \) has zero homology so it's clear.
Here's a derived category version of Morita equivalence. Suppose \((R, Q)\) is a Morita context. Then we have a triangulated functor
\[
M \mapsto P \otimes_R M \quad K(R) \to K(S)
\]
where \(K\) here means the homotopy category of complexes. Let \(I \subset R, J \subset S\) be ideals such that \(QJP \subset I \subset QP, PIQ \subset J \subset PQ\) as usual.

Then for \(p \in P, a \in I, g \in Q\) one has a comm. diag
\[
\begin{array}{c}
P \otimes_R M \xrightarrow{p \otimes g} P \otimes_R M \\
p \otimes m \quad \rightarrow \quad p \otimes g(p \otimes m)
\end{array}
\]
so that \(I \cdot H^*_X(M) = 0 \Rightarrow PIQ \cdot H^*_X(P \otimes_R M) = 0\).

This means that \(P \otimes_R -\) carries \(I\)-small complexes into \(J\)-small complexes, hence induces a functor on the quotient triangulated categories (assuming these exist). Next check that \(M \mapsto H^*_X(Q_M)\) gives an isomorphism functor on the quotient categories.

I forget in (2) above to point out that if \(I\) is any ideal such that \(I H^*_X(M) = 0\), then \(PIQ \cdot H^*_X(P \otimes_R M) = 0\).

In particular if \(H^*_X(M) = 0\), then \(PQ\) kills \(H^*_X(P \otimes_R M)\).

In general if \(\psi: X \to Y\) is a map of complexes, its \(h\)-fibre is \(F\) such that \(\psi_f = 0\) and \(d_f = (d_x, 0)\). If \(\xi: Y \to X\) is a map of complexes, then
\[
\begin{bmatrix}
(d_x - d_y) & (0 \quad \xi) \\
(\xi - d_y) & (0 \quad 0)
\end{bmatrix} = 
\begin{bmatrix}
(4\xi) & (4\xi - 4d_y) \\
(0) & (0 \quad 4\xi)
\end{bmatrix}
\]
We apply this to

\[ P \otimes_R M \xrightarrow{\varphi} \text{Hom}_R(\mathcal{A}, M) \]

\[ \psi_{\varphi} \downarrow \quad \psi_{\varphi_{\mathcal{P}}} \downarrow \quad \psi_{\varphi_{\mathcal{P}}} \]

\[ P \otimes_R M \xrightarrow{\varphi} \text{Hom}_R(\mathcal{A}, M) \]

Then \( \psi_{\varphi_{\mathcal{P}}} : \varphi \otimes m \mapsto \psi_{\varphi_{\mathcal{P}}}(g, \mapsto (g, \varphi_{\mathcal{P}}) m) = \varphi \otimes (g, \varphi_{\mathcal{P}}) m = \varphi g(\varphi_{\mathcal{P}} m) \)

and \( \varphi \psi_{\varphi_{\mathcal{P}}} : f \mapsto \varphi(f(g)) \leftarrow (g, \mapsto g \varphi f(g)) = \varphi(g, f(g)) = \varphi g \cdot f \)

This shows the cone on \( \varphi \) is killed by \( PQ \).

Another application:

\[ Q \otimes \mathcal{P} \otimes_R M \xrightarrow{\varphi} M \]

\[ Q \otimes \mathcal{P} \otimes_R M \xrightarrow{\varphi} M \]

\[ (g, \varphi_{\mathcal{P}}, \varphi_{\mathcal{Q}} m) = g \otimes \varphi_{\mathcal{P}} (g, \varphi_{\mathcal{Q}}) m = g \otimes \varphi_{\mathcal{P}} g \varphi_{\mathcal{Q}} \varphi_{\mathcal{P}} m = g \varphi_{\mathcal{P}} (g, \varphi_{\mathcal{Q}} m) \]

\( \psi_{\varphi_{\mathcal{P}}} (m) = \varphi(g, \varphi_{\mathcal{P}} m) = (g \varphi) m \)

Thus the cone on \( Q \otimes \mathcal{P} \otimes_R M \xrightarrow{\varphi} M \) is killed by \( QP \).
How to obtain quasi-coherent sheaves over $\mathbb{P}_n$ from a non-unital ring.

Consider $S(V) = \bigoplus_{n \geq 0} S_n(V)$, $\dim(V) < \infty$ over a field $k$.

A graded module $M = \bigoplus_{n \geq 0} M_n$ over $S(V)$ has operators $e_n = \text{projection on } M_n$ and multiplication by elements of $V : v \mapsto v \cdot m$. These satisfy the relations

$$e_n e_m = \begin{cases} e_n & n = m \\ 0 & n \neq m \end{cases}, \quad e_n v = \begin{cases} v e_{n-1} & n \geq 1 \\ 0 & n = 0 \end{cases}$$

Let's choose a basis $V = \sum_{i=1}^d k x_i$, whence we have the basis $x^\lambda$ for $S(V)$. The operators $e_n$ and $x_i$ generate an algebra spanned by $x^\lambda e_n$ for all $n \geq 0$ and multi-indices $\lambda$. The multiplication of these "monomials" is determined by

$$e_n x^\lambda = \begin{cases} x^\lambda e_{n-|\lambda|} & \text{if } |\lambda| \leq n \\ 0 & \text{if } |\lambda| > n \end{cases}$$

So we obtain a twisted tensor product algebra

$$S(V) \otimes (k \oplus \bigoplus_{n \geq 0} k e_n)$$

$$= S(V) \oplus \bigoplus_{n \geq 0} S(V) e_n \approx S(V) \otimes k e_n$$

If we tried to use the monomials $e_n x^\lambda$ instead, i.e. the twisted tensor product in the opposite order, then $e_n S(V) \not\approx S(V)$ since $e_n x^\lambda = 0$ for $|\lambda| > n$. However, we do get a different description

$$S(V) \oplus \bigoplus_{n \geq 0} e_n S_n(V)$$

for this algebra.
Now put \( R = S(V) \otimes (k \otimes \oplus k e_n) \) with this twisted multiplication.

An \( R \)-module \( M \) is an \( S(V) \) module equipped with projectors \( e_n, n \geq 0 \) such that the relations \( e_n v = \begin{cases} v & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases} \) hold. Thus we have a diagram:

\[
\begin{array}{ccc}
\oplus_{n \geq 0} e_n M & \longrightarrow & M \\
\longrightarrow & & \longrightarrow \\
\longrightarrow & & \longrightarrow \\
& & \longrightarrow \\
& & \longrightarrow \\
& & \longrightarrow \\
O & \rightarrow & L^1 T^* e_n M & \rightarrow & M & \rightarrow & L^1 T^* e_n M & \rightarrow & O
\end{array}
\]

of \( S(V) \) modules.

Actually I should probably do things in analogy with sheaf theory. Recall the diagrams of exact sequences when \( R/I \) is \( R \)-flat (e.g. \( \forall a_1, \ldots, a_n \in I \exists \alpha \in I^n : (1 - \alpha_1 a_1) = 0 \)):

\[
\begin{array}{ccc}
O & \rightarrow & L^1 T^* e_n M & \rightarrow & M & \rightarrow & L^1 T^* e_n M & \rightarrow & O \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
& & & & & & & & &
\end{array}
\]

In the module case:

\[
\begin{array}{ccc}
O & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & L^1 T^* \text{Hom}_R (I, M) & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & O \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
& & & & & & & & &
\end{array}
\]

\[
\begin{array}{ccc}
O & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & O \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
& & & & & & & & &
\end{array}
\]

\[
\begin{array}{ccc}
O & \rightarrow & R/I \otimes_R M & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & O \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
& & & & & & & & &
\end{array}
\]

\[
\begin{array}{ccc}
O & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & \text{Hom}_R (I, M) & \rightarrow & O \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
\rightarrow & & \rightarrow & & \rightarrow & & \rightarrow & & \rightarrow \\
& & & & & & & & &
\end{array}
\]
In the case $R = S(V) \otimes (\bigoplus ke_n)$, we check $R/I$ is $R^0$-flat.

We know $I = \bigoplus_{n \geq 0} ke_n \otimes S_{\leq n}(V)$ so $\left\{ \sum e_n \right\}$ is an approximate left (also right) identity.

The corresponding diagram for an $R$-module $M$ is

$$
\begin{array}{c}
\require{AMScd}
\begin{CD}
\oplus e_n M @>>> \oplus e_n M \\
\downarrow @VVV @VVV \\
0 @>>> \Pi^n e_n M @>>> (\Pi/\Theta)(e_n M) \\
\end{CD}
\end{array}
$$

So what seems best is to write the following diagram

$$
\begin{array}{c}
\require{AMScd}
\begin{CD}
M @>>> M/\oplus e_n M \\
\downarrow @V\text{cart.} VV \\
0 @>>> \oplus e_n M @>>> \Pi^n e_n M @>>> (\Pi/\Theta)(e_n M) \\
\end{CD}
\end{array}
$$

linking the triangular diagram on the previous page to description of modules via triples.

So far we have described $W$-graded modules over $S(V)$, and I guess the picture I have so far really amounts to the choice of generators $\{ S(V) \otimes u_n, n \geq 0 \}$, where $u_n$ has degree $n$. 
About Munkholm's talk. He defines an action of the poset $N$ on a category $A$ to be a functor $N \times A \to A$.

Such a functor is equivalent to either

i) a functor $N \to \text{Hom}_{\text{cat}}(A, A)$, i.e. a sequence of functors and maps of functors from $A$ to itself of the form

$$F_0 \to F_1 \to F_2 \to \ldots$$

ii) a functor $A \to \text{Hom}_{\text{cat}}(N, A)$, i.e. a functor sending each $X$ in $A$ to an inductive system

$$F_0(X) \to F_1(X) \to F_2(X) \to \ldots$$

depending functorially in $X$.

It seems that this is not what $N \to A$ means by an action of $N$ on $A$, rather there are some extra "usual axioms" to be satisfied. However, before trying to decipher this, let's discuss $\text{Hom}_{\text{cat}}(N, A)$ a bit. This is the category of sequential inductive systems $\mathcal{X} = (X_0 \to X_1 \to X_2 \to \ldots)$ in $A$.

There is an "Anti-Ree" quotient category $\mathcal{Q}$ with maps

$$\text{Hom}_\mathcal{Q}(\overline{\mathcal{X}}, \overline{\mathcal{Y}}) = \lim_{\longrightarrow} \text{Hom}_N(X, Y(n))$$

where if $\overline{\mathcal{Y}} = (Y_0 \to Y_1 \to Y_2 \to \ldots)$ then $Y(n) = (Y_n \to Y_{n+1} \to Y_{n+2} \to \ldots)$

It should be true that $\mathcal{Q}$ is obtained from $A^{N}$ by inverting the arrows $\mathcal{X} \to \mathcal{X}(1)$ for every $\mathcal{X}$.

In the case where $A$ is abelian, it looks...
looks like we are dividing by the
derre subcategory of $X$ such that
$\exists n.s.t. \ X \to X^{(n)}$ is zero, i.e. $X_k \to X_{k+n}$
is zero for all $k$.

Here seems to be the sort of thing Mauchholz
considers. Suppose $F : A \to A$ is a functor
and $\eta : \text{id} \to F$ is a map of functors such
that the two maps $F \eta, \eta F : F \to F^2 = F \circ F$
coincide, i.e. $\forall X$ in $A$ the two arrows
$F(\eta_X) \to F(F(X))$
$\eta F(X)$

are equivalent to

This seems to be what he calls an
action of $N$ on $A$.

The localization $N^{-1} A$ has the same objects
as $A$, with

$$\text{Hom}_{N^{-1} A}(X, Y) = \lim_{\longrightarrow n} \text{Hom}_A(X, F^n(Y))$$

Thus $N^{-1} A$ should be the category obtained by
inverting the arrows $\eta_X : X \to F(X)$ for all $X$.

Let's check his claim that if $A$ is abelian
and $F$ is left exact, then $N^{-1} A$ is abelian. It
suffices to show that for any short exact
sequence $0 \to X' \to X \to X'' \to 0$ goes into one which is both left and right exact in $N^{-1} A$.

Now $\text{Hom}_A(\_ , F^n(\_))$ transforms this short
exact sequence in $A$ to a left exact sequence
and taking $\lim_{\longrightarrow n}$ yields a left exact sequence. On
the other hand, because $F$ is left exact $\text{Hom}_A(\_ , F^n(\_))$
transforms the given short exact sequence into a left
exact one, so we win.
As a check, we should see that the objects \( Y \) which become zero in \( \mathcal{N} \) form a fere subcategory. The functor \( \lim_{\longrightarrow} \text{Hom}_\mathcal{N} (-, F^n(Y)) \) is zero iff in such that \( Y \to F^n(Y) \) is zero (in which case \( F^k(Y) \to F^{k+n}(Y) \) is zero \( \forall k \)).

Consider

\[
\begin{array}{c}
\circlearrowright \to Y' \to Y \to Y'' \to \circlearrowleft \\
\downarrow & \downarrow & \downarrow 0 \\
F' & F & F'' \\
\circlearrowright \to F^n(Y') \to F^n(Y) \to F^n(Y'') \\
\downarrow 0 & \downarrow \\
\circlearrowright \to F^{n+m}(Y') \to F^{n+m}(Y) \\
\end{array}
\]

so it's clear.
Here's what Munkholm means by an $N$-action in a category $A$. Consider $N$ as a monoid object in $A$, with operation given by addition. Thus addition gives a map of posets
\[ N \times N \to N \]
which is associative and commutative. Also the functor $\text{pt} \to N$ sending the unique object to $0$ is an identity for this operation.

On the other hand, for any category $A$, $\text{Hom}_{\text{cat}}(A, A)$ is a monoid object in $\text{cat}$ with the operation given by composition. (To be rigorous one should suppose $A$ small.) A $N$-action on $A$ is a functor
\[ N \to \text{Hom}_{\text{cat}}(A, A) \]
compatible with the product and identity objects. Thus $0 \mapsto \text{id}_A$, and if $1 \mapsto F$, then $n \mapsto F^n$.

Suppose we write $n \times X$ for the action of $n$ on $X$, so $n \times X = F^n(X)$. Let $\eta_X : X \to F(X)$ be
\[ 0 \times X \to 1 \times X \]
the map corresponding to the arrow $0 \to 1$ in $N$. Then we have
\[ 1 \times X = 0 \times (1 \times X) = F(X) \]
\[ 2 \times X = 1 \times (1 \times X) = F(F(X)) \]
\[ 1 \times X = 1 \times (0 \times X) = F(X) \]
showing \( \eta \cdot F = F \cdot \eta \).

To avoid assuming \( A \) small, one writes the action as a \( \lambda \) functor \( N \times A \rightarrow A \) such that one has

i) associativity:

\[
\begin{align*}
N \times N \times A & \xrightarrow{1 \times \mu} N \times A \\
\downarrow \times 1 & \downarrow \mu \\
N \times A & \xrightarrow{\mu} A
\end{align*}
\]

commutes.

ii) identity

\[
\begin{align*}
\xrightarrow{\begin{array}{c}
\downarrow \\
(0, x)
\end{array}} \\
N \times A & \xrightarrow{\mu} A
\end{align*}
\]

Another point is that if \( N \) acts on \( C \) small, and \( C \) is another category, then \( N \) acts on \( \text{Hom}_{\text{cat}}(A, C) \). This is because one has a map \( N \rightarrow \text{Hom}_{\text{cat}}(A, A) \) preserving identity + product, and \( \text{Hom}_{\text{cat}}(A, A) \) acts on \( \text{Hom}_{\text{cat}}(A, C) \).

Consider now some of his examples.

If \( X \) is a metric space, let \( B \) be the poset given by

\( (n, x) \in N \times X \) with the ordering \( (n, x) \leq (n', x') \) \( \Leftrightarrow \)

\( d(x, x') \leq n' - n \).

Then \( y \in B(n, x) \), i.e. \( d(x, y) \leq n \), \( \Rightarrow \)

\( d(x', y) \leq d(x, x') + d(x, y) \leq n' - n + n = n' \), so

\( B(n, x) \subset B(n', x') \). Thus one can roughly think of \( B \) as the poset of balls with integral radii in \( X \).
There's an obvious action of \( N \) on \( B \), namely \( h^*(n, x) = (n^* + n, x) \).

Now consider \( \text{Hom}_{\text{cat}}(B, AB) \). Objects are functors from \( B \) to abelian groups. The \( N \)-action on \( B \) induces one on these functors. Then \( N^+ \text{Hom}_{\text{cat}}(B, AB) \) is by defn. the category of \( \mathbb{Z}B \) modules. Thus we are considering abelian group valued functors on the poset of balls modulo those functors \( (n, x) \mapsto M(n, x) \) such that \( \exists n_0 \text{ s.t. for all } (n, x) \text{ the map } M(n, x) \to M(n_0 + n, x) \) is zero.
Puzzle: Let $A$ be a left ideal in $R$ unital. We have seen that the Morita context $(\tilde{A}, R)$ satisfies

$$ (A, B) \quad M \mapsto A \otimes^R M \quad N \mapsto R \otimes^R N \leftrightarrow N $$

yields an equivalence $m(\tilde{A}, A) \sim m(R, AR)$, $\text{fim}(\tilde{A}, A) \sim \text{fim}(R, AR)$, etc. If we restrict to firm modules however we can use an extension of scalars: $M \mapsto R \otimes^R M$ instead of $A \otimes^R M$. In effect the inclusion $A \hookrightarrow R$ is an isomorphism modulo null $(A^\text{op}, A^\text{op})$, since $(R/A) \cdot A = R^2 + N_A = 0$.

Thus $A \otimes^R M \sim R \otimes^R M$ for $M$ in $\text{fim}(\tilde{A}, A)$.

This is strange because $M \mapsto R \otimes^R M$ doesn't seem to be part of a Morita context, although this is true for a (two-sided) ideal.

Why this arose: I observed in constructing flat firm modules

$$ F(A) = \varinjlim \left( \begin{array}{c} \tilde{A}^\text{no} \\
\tilde{A}^\text{no} \\
\vdots 
\end{array} \begin{array}{c} A^1 \\
A^1 \\
\vdots 
\end{array} \begin{array}{c} A^2 \\
A^2 \\
\vdots 
\end{array} \begin{array}{c} \ldots \\
\ldots \\
\ldots 
\end{array} \right) $$

and

$$ F(A) = \varinjlim \left( \begin{array}{c} R^\text{no} \\
R^\text{no} \\
\vdots 
\end{array} \begin{array}{c} A^1 \\
A^1 \\
\vdots 
\end{array} \begin{array}{c} A^2 \\
A^2 \\
\vdots 
\end{array} \begin{array}{c} \ldots \\
\ldots \\
\ldots 
\end{array} \right) $$

that obviously $R \otimes^R F(A) \sim F(A)$. Since any firm module is a cokernel of a map between direct sums of $F(A)$'s, one has $R \otimes^R M = M$ for $M$ firm.
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It turns out that once we leave the $I = I^2$ situation there are equivalence of module categories (in the generalized sense) which are not Morita equivalences (i.e. obtained from a Morita context).

Consider a Morita context $(R_0, Q, p, s)$, which is unital as usual, such that $PQ = S$. Then we have $(QP)^2 = QSP = QP$, so the ideal $QP$ in $R$ is idempotent. Also we know that $P \in \mathcal{P}(R^0)$, $Q \in \mathcal{P}(R)$ are dual f.g. projective modules, and that $S = P \otimes_R Q = \text{Hom}_{R^0}(P, P) = \text{Hom}(Q, Q)$. This is the picture of a Morita equivalence with a unital ring $S$.

Next consider $R = \mathbb{Z}$, $I = p\mathbb{Z}$, or more generally a commutative ring $R$ and $I = Rf$ where $f$ is a nonzero divisor. Then

$$\text{solid}(R, Rf) \cong \text{mod}(R) / \text{tor}(R, Rf)$$

is equivalent to $\text{mod}(R_f)$, where $R_f = R[f^{-1}]$ is the localization obtained by inverting $f$. The ideal $I$ is not idempotent, so we don't have a Morita equivalence as above.

Suppose we try to write the equivalence $\text{solid}(R, Rf) \cong \text{mod}(R_f)$ using bimodules:

\[ M \longrightarrow \text{Hom}_{R^0}(Q, M) \]

\[ \text{Hom}(P, N) \longrightarrow N \]

Then $P$ must be $R_f$ with obvious left $R_f$-right $R$ bimodule structure. $Q$ as left $R$-module must be such that $R_f \otimes_R Q = R_f$. Hence $R_f$ must also act on the right.
of $Q$ it's fairly clear that after killing any $f$-torsion, we must have $Q = R_f$. But then $QP = R_f R_f = R_f \neq R$.

Another example: Consider first the category of graded modules $M = \bigoplus_{n \in \mathbb{Z}} M_n$ over the Laurent polynomial alg $k[x, x^{-1}]$, where $\deg(x) = 1$. This category is equivalence to $\text{mod}(k)$, the equivalence being given by functors $M \mapsto M_0$ and $N \mapsto k[x, x^{-1}] \otimes_k N$. This situation is an example of a Morita equivalence

$$\begin{pmatrix} R & eR \\ eR & eRe \end{pmatrix} = \begin{pmatrix} R \otimes R & k[x, x^{-1}] e_0 \\ e_0 k[x, x^{-1}] & k \end{pmatrix}$$

$e = e_0$

where $R = k[x, x^{-1}] \otimes \left( \bigoplus_{n \in \mathbb{Z}} ke_n \right)^\sim$ has the $k$-basis $x^p, x^p e_n$ with $p, n \in \mathbb{Z}$ and multiplication given by $e_n x = x e_{n-1}$. Thus $e_0 R$ has basis $x^p e_0, p \in \mathbb{Z}$. $R e_0$ has basis $e_0 x^p, p \in \mathbb{Z}$.

And $R e_0 \otimes e_0 R$ has basis $x^p e_0 \otimes e_0 x^q, p, q \in \mathbb{Z}$ which maps to $x^p e_0 x^q = x^{p+q} e_{-q}$, whence

$$R e_0 \otimes e_0 R \xrightarrow{\sim} R e_0 R = \bigoplus_{n \in \mathbb{Z}} k x^n e_n$$

On the other hand, suppose we try to obtain the same module category (graded $k[x, x^{-1}]$-modules) starting from $\mathbb{Z}$-graded $k[x]$-modules.
Explore the derived category picture. Assume $I = I^2$ and let's start with the solid side, where we have injective resolutions, in order to treat first the simplest situation. Notation

$$M = \text{mod}(R)/\text{mod}(R/I)$$

$$D^+(R) = D^+/\text{mod}(R)$$

$$\text{full subcats of } D^+(R)$$

$$D^+(R)_{\text{null}} = \{ M \in D^+(R) \mid IH^*_R(M) = 0 \}$$

$$D^+(R)_{\text{sol}} = \{ - \mid RH\text{Hom}_R(R/I, M) = 0 \}$$

We have functors

$$D^+(R)_{\text{null}} \longrightarrow D^+(R) \longrightarrow D^+(M)$$

Claim $D^+(R)_{\text{sol}} \longrightarrow D^+(M)$ is an equivalence of Dated cats. Why? $M$ is a Grothendieck cat, so if $\text{Minj}$ is the full subcategory of injectives in $M$ one has an equivalence of Dated cats $C^+(\text{Minj}) \sim D^+(M)$, where $C^+$ is homotopy category of complexes. Next, recall that if $M \longrightarrow E$ is a minimal injective resolution of $M$ and $RH\text{Hom}_R(R/I, M) = 0$, then we know that $E$ is a complex of solid injectives. Thus $D^+(R)_{\text{sol}}$ is equivalent to the full subcat of $D^+(R)$ consisting of solid injective complexes, and as these are injective complexes, the maps are just homotopy classes of maps between them. Thus we have an equiv. of Dated cats.

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\[ C^+(\text{sol} \text{inj}) \sim \rightarrow D^+(\text{sol}) \]

Finally, we have an equivalence of solid injective \( R \)-modules with injectives in \( M \). Thus we have

\[ D^+(\text{sol}) \sim \rightarrow D^+(M) \]
\[ \sim \uparrow \sim \]
\[ C^+(\text{sol} \text{inj}) \sim \rightarrow C^+(M \text{inj}) \]

proving the claim.

As far as I can see the hypothesis that \( I = I^0 \) has not been used except in identifying \( I \)-null modules with \( I \)-torsion modules.

What should happen in general is that we have equivalence

\[ D^+(R)_{\text{sol}} \sim \rightarrow D^+(R)/D^+(R)_{\text{tor}} \sim \rightarrow D^+(M) \]

adjoint functors

\[ D^+(R)_{\text{tor}} \xrightarrow{\iota} D^+(R) \xleftarrow{\rho^*} D^+(M) \]

and a canonical functorial \( \Delta \)

\[ \mathcal{M} \rightarrow M \rightarrow Rf^*(g^*\mathcal{M}) \rightarrow \]

To see this put \( D = D^+(R)/D^+(R)_{\text{tor}} \) and recall that maps in this category are calculated via fractions

\[ \text{Hom}_D(M, N) = \lim_{\rightarrow} \text{Hom}_{D^+(R)}(M', N) \]
\[ = \lim_{\rightarrow} \text{Hom}_{D^+(R)}(M, N') \]
But a basic fact is that
\[ R\text{Hom}_R(T, Q) = 0 \text{ if } T \in D^+(R)_{\text{tor}} \]
\[ \{ Q \in D^+(R)_{\text{sol}} \} \]

In effect we can suppose \( Q \) solid injective

whence \( R\text{Hom}_R(T, Q) = \text{Hom}_R(T, Q) \) and we

can use the increasing Postnikov system of \( T \)

\[
\begin{align*}
T^{-1} & \to Z^0 \to 0 \\
\| & \quad \circ \\
T^{-1} & \to T^0 \to Z^1 \to 0 \\
\| & \quad \circ \\
T^{-1} & \to T^0 \to T' \to Z^2 \to 0
\end{align*}
\]

to reduce to the case where \( T \) is a torsion module

sitting in degree zero, in which case \( \text{Hom}_R(T, Q) = 0 \) (actually

= 0). Using this we have for \( N \in D^+(R)_{\text{sol}} \)

\[
\text{Hom}_{D^+(R)}(M, N) \cong \text{Hom}_{D^+(R)}(M', N)
\]

if \( M' \to M \) has one in \( D^+(R)_{\text{tor}} \). Thus

\[
\text{Hom}_{D^+(R)}(M, N) \to \text{Hom}_{D^+(R)}(M, N) \text{ if } N \in D^+(R)_{\text{sol}}
\]

follows from the first formula for \( \to \). But it

also follows from the second formula, since given

\( N \to N' \) one has \( \text{Hom}_{D^+(R)}(N', N) \to \text{Hom}_{D^+(R)}(N, N) \),

hence the identity \( N \to N \) is cofinal in the filtering

cat of \( N \to N' \).

At this point we know \( D^+(R)_{\text{sol}} \to D^+(R)/D^+(R)_{\text{tor}} \)

is fully faithful. As we also know \( \forall M \exists M \to Q \) torsion quasi, \( Q \) solid injective, it follows

this functor is an equivalence of category.
I think \( E \) should be straightforward. One defines \( TM \) as the fibre of the adjunction arrow \( M \to R\ast_y(y^*M) \). It's then clear that
\[
\text{Hom}_{D^+(R)}(T, TM) \to \text{Hom}_{D^+(R)}(T, M)
\]

\( \text{Tor}^1 \) solid

since \( \text{Hom}_{D^+(R)}(T, R\ast_y(y^*M)) = 0 \).

\text{Everything so far in the torsion context seems to work in general. The real issue is then somehow to understand} \text{the torsion complexes, for instance, how is this related to} \text{D}(\text{tors} (R, I))? \text{In the case I=I^2, these coincide iff the} \text{annihilating condition I\otimes_R I \to I holds. In general one might try to describe} \text{D}(\text{tors} (R, I)) \text{as DG modules over something.}

\text{In the case I=I, then torsion complexes should be something like complexes of R/I module with extra operations. If R=\mathbb{Z} and we are over a field k, then the bar construction is a DG coalgebra whose homology is} \text{Tor}^1(k, k). \text{One can look at DG cocommutative coalgebras over the bar construction.}

\text{Question: In the commutative noetherian case is it true that} \text{D}(\text{tors} (R, I)) = \text{D}(\text{tors} (R, I))?

\text{Take M in} \text{D}(\text{tors} (R, I)). \text{Up to quasi we can suppose M is a minimal injective complex, i.e.} \text{M}^n = \text{injective hull of} \ Z^n \text{for all} n. \text{Let's proceed by induction on} n \text{to show that} \text{M}^n \text{is torsion for all} n, \text{this being obvious for} n < 0. \text{Assuming}
$M^{n-1}$ is torsion we see from the exact sequence
\[
\frac{M^{n-1}}{\text{tors}} \longrightarrow Z^n \longrightarrow H^*(M) \longrightarrow 0
\]
that $Z^n$ is torsion. But for a stable torsion theory the injective hull of a torsion module is torsion, so $M^n$ is torsion. Thus $M$ is a complex of torsion modules.

To finish note that for $\text{tors} = \text{tors}(R, I)$, the injective objects are injective $R$-modules. So we have an equivalence
\[
C^+(\text{tors inj}) \cong D^+(\text{tors})
\]
Now $C^+(\text{tors inj})$ is a full subcat of $C^+(\text{inj}) \hookrightarrow D^+(R)$, so one has a full embedding $C^+(\text{tors inj}) \hookrightarrow D^+(R)_{\text{tors}}$. On the other hand we have seen that any $M$ in $D^+(R)_{\text{tors}}$ is quasi to a torsion injective complex. Thus one has
\[
D^+(R)_{\text{tors}} \longrightarrow D^+(\text{tors})
\]
when the $I$-torsion theory on $\text{mod}(R)$ is stable, e.g. $R$ comm. and Noetherian.
August 9, 1994

Let's summarize some of yesterday's work about the solid picture and then proceed to the firm picture.

Let's begin with \( \text{mod} = \text{mod}(R) \), the full subcats

\[
\begin{align*}
\text{tors} & = \text{tors}(R, I) \\
\text{sol} & = \text{sol}(R, I)
\end{align*}
\]

of \( I \)-torsion and \( I \)-solid \( R \)-modules, and \( M_\ell = \text{mod}/\text{tors} \).

Key points:

i) \( N \) solid \( \Rightarrow \) \( \text{Hom}_R (\_, N) \) coarsens tors isos.

ii) \( \forall M \exists \) torsion iso \( M \to M^\# \) with \( M^\# \) solid.

These imply \( \text{Hom}_R (M, N) \cong \text{Hom}_R (M^\#, N) \) for all solid \( N \), hence the inclusion \( \text{sol} \hookrightarrow \text{mod} \) has the left adjoint \( M \mapsto M^\# \).

Since \( \text{Hom}_{M_\ell} (M, N) = \lim_{M \to M} \text{Hom}_R (M', N) \)

i) implies \( \text{Hom}_R (M, N) \cong \text{Hom}_{M_\ell} (M, N) \) if \( N \) solid,

in particular \( \text{sol} \hookrightarrow M_\ell \) is fully faithful.

ii) implies this functor is essentially surjective,

whence one has an equivalence \( \text{sol} \cong M_\ell \).

Next consider \( D^+ R \), the full subcats

\[
\begin{align*}
D^+ R_{\text{tors}} & = \{ M \mid H_* (M) \text{ torsin} \} \\
D^+ R_{\text{sol}} & = \{ M \mid R \text{Hom}_R (R/I, M) = 0 \}
\end{align*}
\]

Key points

i) \( M \in D^+ R_{\text{tors}}, N \in D^+ R_{\text{sol}} \Rightarrow R \text{Hom}_R (M, N) = 0 \).

ii) \( \forall M \in D^+ R \exists \text{tors-quot} \to M \to M^\# \text{ st } M^\# \in D^+ R_{\text{sol}} \).

In fact we know that \( M^\# = R g_* (g^* M) \), specifically...
\[ M^\# = j^* (j_!) \], where \( j^* M \to I \) is an injective resolution of \( \mathbb{Z} \) complexes over \( M_t \). Now i), ii) imply

\[ \text{Hom}_{D^+ R}(M, N) \cong \text{Hom}_{D^+ R}(M^\#, N) \]

for \( N \in D^+ R_{s_s} \), hence \( M \mapsto M^\# = Rj_*(j^* M) \) is left adjoint to the inclusion \( D^+ R_{s_s} \subset D^+ R \). It's clear we also have an equivalence

\[ D^+ R_{s_s} \xrightarrow{\sim} D^+ R / D^+ R_{t_0} \]

by the same sort of formal arguments using i), ii).

There's an extra point here, namely the equivalence

\[ D^+ R / D^+ R_{t_0} \to D^+ M_t \]

This is perhaps true quite generally, maybe restricting to bounded complexes, namely

\[ D(A/s) = DA / DA_s \]

complexes with homology in the Serre subcategory \( S \).

One can see \( \ast \) holds because \( D^+ R_{s_s} \) and \( D^+ M_t \) each have equivalent injective complex subcategories, and these subcats are equivalent.
Now I want to assume \( I = I^2 \), whence \( \text{tor} = \text{null} \), and I want to consider the finer picture.

Consider \( D^+_R \) and the full \( \Delta_1 \)-sets

\[
D^+_R \text{null} = \{ M \mid I^2H_R(M) = 0 \}, \quad D^+_R \text{fin} = \{ M \mid R/I^2M \isom 0 \}.
\]

The key points are:

i) If \( M \in D^+_R \text{fin} \), \( N \in D^+_R \text{null} \), then \( R \text{Hom}_R(M, N) \isom 0 \).

ii) For any \( M \in D^+_R \), \( \exists \text{ null quasi-} M^\# \to M \) with \( M^\# \in D^+_R \text{fin} \).

Proof of i):

Can suppose \( M \) projective. Consider the Postnikov system of \( N \):

\[
\begin{array}{ccc}
0 & \longrightarrow & N_1/B_1 \longrightarrow N_0 \longrightarrow N_1 \\
& \downarrow & \downarrow \downarrow \varepsilon & \downarrow \\
& \longrightarrow & 0 \longrightarrow N_0/B_0 \longrightarrow N_1 & \\
\end{array}
\]

This is an inverse system of quotients \( N^{(p)} \) of \( N \) such that \( N = \varprojlim N^{(p)} \) and \( \text{ker}(N^{(p)} \to N^{(p+1)}) = H^p(N) \). Since a surjective sequential inverse system of acyclic complexes is acyclic (Milnor exact sequence), it suffices to show \( \text{Hom}_R(M, N) \isom 0 \), when \( N \) is a null module in degree zero.

By hypothesis, \( R/I \otimes_R M = M/I\text{im}R \) is acyclic. As \( M/I\text{im}R \) is a pro-\( \Delta_1 \)-complex of \( \Delta_1 \)-modules, it is homotopic to zero. Thus \( \text{Hom}_R(M, N) = \text{Hom}_R(M/I\text{im}R, N) \) is acyclic.

(Note that this argument does not work in the general torsion context since it is possible to have nonzero maps from a flat module to a torsion module.)
ii) follows from the existence of flat resolutions in \( M \) which after lifting via \( f^! \) become firm flat complexes.

Formally it should follow from i) + ii) that the inclusion \( D^+_f \text{firm} \subset D^+_f \) has a right adjoint: \( M \to M^\# = Lf^!(f^*M) \). Moreover we should have an equivalence of categories:

\[
D^+_f \text{firm} \cong D^+_f / D^+_f \text{null}
\]

I want now to check the extra point:

\[
D^+_f \text{firm} \cong D^+_f M
\]

Because of the existence of enough flat objects in \( M \), we should be able to construct \( Lf^! : D^+_f M \to D^+_f \text{firm} \). This seems to be a 'resolution' theorem, going from all complexes in \( M \) to complexes in the exact category \( \text{Mflat} \). The idea then is

\[
\begin{array}{ccc}
D^+_f \text{firm} & \cong & D^+_f M \\
\uparrow \alpha & & \uparrow \sim \\
D^+_f (\text{firm flat}) & = & D^+_f (\text{Mflat})
\end{array}
\]
August 10, 1994

I want to check claims about flat resolutions. Consider $M = M(R,I)$, $M_{\text{flat}}$ the full subcategory of flat modules. I want to check that one has an equivalence of derived categories

$$D_+(M_{\text{flat}}) \xrightarrow{\sim} D_+(M)$$

\[ \text{def} \quad \text{def} \]

$$C_+(M_{\text{flat}})/\text{acyc.} \quad C_+(M)/\text{acyc.}$$

First check fully faithful. We have

$$\text{Hom}_{D_+(M_{\text{flat}})}(F,G) = \lim_{\mathbf{F} \to F} \text{Hom}_{C_+(M_{\text{flat}})}(F',G)$$

means quasi

$$\text{Hom}_{D_+(M)}(F,G) = \lim_{M' \to F} \text{Hom}_{C_+(M)}(M',G)$$

These agree because $C_+(M_{\text{flat}}) \to C_+(M)$ is fully faithful (these are homotopy categories), provided for any $M'$ quasi $F' \to M'$ with $F'$ flat, since then the limit over $M' \to F$ can be taken over the cofinal category of $F' \to F$. The same condition (for $M' \to F$ with $F$ flat) implies the functor $D_+(M_{\text{flat}}) \to D_+(M)$ is essentially surjective.

Let's prove this condition holds by constructing flat Cartan-Eilenberg resolutions. Given $M \in D_+(M)$ consider its Postnikov system. To simplify suppose $M_n = 0$ for $n < 0$. 

Choose a flat resolution $F(H_0) \to H_0$.

Then consider $M_0 \times_{H_0} F(H_0)$ and note that

$$M_0 \times_{H_0} F(H_0) \xrightarrow{pr_2} F(H_0)$$

$$\downarrow \text{(surj)}$$

$$\text{surj} \quad H_0$$

implies $pr_1$ is a (surj) quasi and $pr_2$ is surjective.

Choose $F(M_0)$ to be a flat complex with a surj. quasi $F(M_0) \to M_0 \times_{H_0} F(H_0)$, and let $F(B_0) = \text{Kernel of } F(M_0) \to F(H_0)$. Then we have

$$0 \to F(B_0) \to F(M_0) \to F(H_0) \to 0$$

$$\downarrow \quad \downarrow \text{(surj)} \quad \downarrow \text{(surj)}$$

$$0 \to B_0 \to M_0 \to H_0 \to 0$$

and because $F(M_0)$ maps onto the fibre product we conclude $F(B_0) \to B_0$ is a (surjective) quasi. (The surjective in parentheses is obvious when $B_0$ is a single module, and even in the case of a complex is probably not essential.) The important point is that $F(B_0)$ being the kernel of a surjection of flat complexes is flat.

Now repeat this construction: Choose $F(H_0) \to M_0 \times_{H_0} F(B_0)$.

$$0 \to F(Z_1) \to F(M_1) \to F(B_0) \to 0$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$0 \to Z_1 \to M_1 \to B_0 \to 0$$

And so on.
Now continue the process following the Postnikov system of $M$:

\[ 0 \rightarrow M_1/B_1 \rightarrow M_0 \rightarrow B_0 \rightarrow 0 \]
\[ 0 \rightarrow M_1/Z_1 \rightarrow M_0 \rightarrow B_0 \rightarrow 0 \]

Thus we construct $H_0$.

\[ 0 \rightarrow F(H_1) \rightarrow F(M_1/B_1) \rightarrow F(B_0) \rightarrow 0 \]
\[ 0 \rightarrow H_1 \rightarrow M_1/B_1 \rightarrow B_0 \rightarrow 0 \]

by choosing $F(M_1/B_1)$ flat mapping to $(M_1/B_1) \times B_0$.

Then

\[ 0 \rightarrow F(b_1) \rightarrow F(Z_1) \rightarrow F(H_1) \rightarrow 0 \]
\[ 0 \rightarrow B_1 \rightarrow Z_1 \rightarrow H_1 \rightarrow 0 \]

and then

\[ 0 \rightarrow F(H_2) \rightarrow F(M_2/B_2) \rightarrow F(B_1) \rightarrow 0 \]
\[ 0 \rightarrow H_2 \rightarrow M_2/B_2 \rightarrow B_1 \rightarrow 0 \]

But we haven't got $F(M_1)$?

So construct:

\[ F(M_1) \rightarrow F(M_1/B_1) \rightarrow F(B_0) \rightarrow 0 \]
\[ \downarrow \text{quis} \downarrow \text{quis} \downarrow \text{quis} \]
\[ M_1 \rightarrow M_1/B_1 \rightarrow B_0 \rightarrow 0 \]
Then define
\[ F(B_1) = \text{Ker} \{ F(M) \to F(M/B_1) \} \]
\[ F(Z_1) = \text{Ker} \{ F(M) \to F(B_0) \} \]
\[ F(H_1) = \text{Ker} \{ F(M/B_1) \to F(B_0) \} \]

whence we have the exact sequence
\[ 0 \to F(B_1) \to F(Z_1) \to F(H_1) \to 0 \]

This seems to proceed to yield the required C.E. resolution.

However, I don't think it's essential for the derived category purposes to have C.E. resolutions. One can construct a quasi-F \to M with F flat stupidly by first expressing M as the quotient of a k-flat complex:
\[ \cdots \to P_2 \otimes F_2 \to P_1 \otimes F_1 \to P_0 \otimes F_0 \to 0 \]
\[ \to N_2 \to N_1 \to N_0 \to 0 \]

then doing the same for the kernel, etc.
Beilinson theory about sheaves on $\mathbb{P}^n$. Apparently Beilinson's short paper (\textit{c.}1978) is about a derived category correspondence between certain modules and certain sheaves. This is some sort of tilting business. A Polish mathematician has some improvements & complements, and his papers are reviewed by Jeremy Rickard. Here is what I have pieced together from the reviews.

Let $R = S(V)$, $I = VS(V)$, $k = R/I$. We have the Koszul resolution

$$0 \to S(V) \otimes k^d \to \cdots \to S(V) \otimes V \to S(V) \to k \to 0$$

for computing $\text{Tor}_*(R, k)$ and $\text{Ext}_R^*(k, -)$.

I'm going to concentrate on the Ext. This is because $\text{Ext}_R^*(k, M)$ is a graded module over $\text{Ext}_R^*(k, k) = N(V^*)$. Otherwise I would be working with $\text{Tor}_R^*(k, M)$ which is a graded comodule over the coalgebra $\text{Tor}_R^*(k, k) = N(V)$. One has a canonical isomorphism

$$\text{Tor}_R^*(k, M) = \text{Ext}_R^{d-*}(k, M) \otimes k^d$$

Given an $R$-module $M$ we get a d.g. $N(V^*)$ module $N(V^*) \otimes M$ with differential $\sum \delta_i \otimes \delta_i$, which computes $\text{Ext}_R^*(k, M)$. Obviously this extends to a functor from complexes of $R$-modules, i.e. d.g. $R$-modules, to d.g. $N(V^*)$ modules. It should be clear that it descends to a functor between bold derived categories.

Conversely a d.g. $N(V^*)$ module $N$ gives rise
to a d.g. $R$-module $S(V) \otimes N$ with differential $1 \otimes d + \sum v_i \otimes \sigma_i^*$, and this also should descend to a functor between bdd derived categories.

When we compose these functors

$$M \mapsto \bigotimes \Lambda(V^*) \otimes M \mapsto S(V) \otimes \Lambda(V^*) \otimes M$$

one can't expect this to give the identity on $D^b(R)$, because $R \text{Hom}_R(k, -)$ sees nothing away from the origin. In fact $R \text{Hom}_R(k, -)$ vanishes exactly on the firm = solid subcategory of $D^b(R)$.

One expects that for $M$ perfect the total homology of $\Lambda(V^*) \otimes M$ should be finite diml. This is clear because $M$ perfect $\Rightarrow$ $M$ quasi-f.g. a finitely generated free $R$-module complex (recall f.g. projectives over $S(V)$ are stably-free (Serre) and even free (Serre conjecture)). Thus this assertion reduces to the case $M = R$, i.e., clear.

Since $\Lambda(V^*)$ is finite-dimensional a d.g. $\Lambda(V^*)$ module with f.d. homology should be quasi-f.g. d.g. $\Lambda(V^*)$ module which is f.g. Clearly a f.g. d.g. $\Lambda(V^*)$ module $N$ gives rise to a perfect $R$-module complex $S(V) \otimes N$.

So far I haven't paid any attention to the grading on $S(V)$. So now restrict to graded modules over $S(V)$, which means that d.g. $S(V)$-modules are bigraded in some way, also d.g. $\Lambda(V^*)$-modules.

Then one can expect the functors between
perfect complexes of graded $S(V)$-modules and f.d. (bi)graded $\Lambda(V^*)$-modules to give an equivalence of categories.

Finally under this equivalence, perfect complexes whose homology is finite-dimensional correspond to free dg f.d. $\Lambda(V^*)$-modules. Thus one has an equivalence between perfect complexes over $P(V^*)$ and the derived cat of bigraded f.d. $\Lambda(V^*)$-modules modulo free such modules. This seems to be Beilinson's theorem, although his correspondence is perhaps slightly different (maybe related to Mumford's regular sheaves).

What's needed here is to check the claims, say in the case $d=1$, just to be certain of the convergence. The problem of interest for me concerns where the firm = solid derived category might come in. The firm = solid derived category is equivalent to the derived category of quasi-coherent sheaves on $P(V^*)$, so the question is how to bring in the perfect complexes. You need some link - how to recognize when a solid complex corresponds to a perfect complex on $P(V^*)$.

No: Bernstein-Gelfand + Gelfand prove their result: derived category of graded $S(V)$ modules, and stable category (kill projectives) of suitable bigraded f.d. $\Lambda(V^*)$ modules, and derived cat of coherent sheaves on $P(V^*)$ are equivalent.

Beilinson gets a true tiling example: I find bigraded algebra $A$ such that both $f_{*}$, derived cat of $A$-modules $\Rightarrow$ derived cat of coh sheaves on $P_{n}$.
Problems:

1) In the comm. noetherian case, where $M_n$ is the category of quasi-coherent sheaves on $\mathcal{U} = \text{Sp}(R) - \text{Sp}(R/I)$, one has the notion of perfect complex in $D^b(M_n)$. In fact Grothendieck in SGA 6 has described perfect complexes intrinsically as objects in the derived category which are of finite presentation in a suitable sense. You want to develop this idea, e.g. find out whether there is an intrinsic notion of perfect complex in general, whether it depends only on $I$ and whether it is Morita invariant.

2) Suppose $R$ quasi-free, does it follow for any ideal $I$ that the excision result holds, namely that any complex with torsion homology is quasi-isomorphic to a complex of torsion modules. Maybe null instead of torsion.
Belkin's Thm. First do $P^1$. Let $T = O \otimes O(-1)$.

Then $T$ generates the derived category of coherent sheaves on $P^1$. In effect

\[ 0 \rightarrow \Lambda^2 V \otimes O(-2) \rightarrow V \otimes O(-1) \rightarrow O \rightarrow 0 \]
\[ 0 \rightarrow \Lambda^3 V \otimes O(-3) \rightarrow V \otimes O(-2) \rightarrow O(-1) \rightarrow 0 \]

shows $O(-1), O(-2), \ldots$ lie in the $A$-category generated by $O, O(-1)$. Similarly

\[ 0 \rightarrow \Lambda^2 V \otimes O(-1) \rightarrow V \otimes O \rightarrow O(1) \rightarrow 0 \]

shows $O(1), O(2), \ldots$ lie in this $A$-category.

Next $\text{Ext}^n(T, T)$.

\[ \text{Hom}(T, T) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & O(1) \\ 0 & 0 & 0 \end{pmatrix} \]

\[ \Gamma\left( \text{Hom}(T, T) \right) = \begin{pmatrix} k & 0 \\ 0 & k \\ 0 & 0 \end{pmatrix} \text{ call this } A \]

\[ H^1\left( \text{Hom}(T, T) \right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ as } H^1(O(-1)) = 0 \]

Thus $\text{Ext}^n(T, T) \cong H^n\left( \text{Hom}(T, T) \right) = \begin{cases} A & n = 0 \\ 0 & n \neq 0 \end{cases}$

So by tilting theory we should have an equivalence between $D_{\text{coh}}^b(P^1)$ and $D^b_A(A)$.

In general for $P(V^*)$ the same thing works. Let $T = O \oplus O(-1) \oplus \ldots \oplus O(-d)$, $d = \dim V$

and recall $H^0(P(V^*), O(n)) = 0$ unless $\frac{d}{d-1} \leq n < 0$.
We have the basic exact sequence

\[ 0 \rightarrow \Lambda^d V \otimes \mathcal{O}(-d) \rightarrow \cdots \rightarrow V \otimes \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0 \]

which represents a canonical generator of

\[ H^{d-1}(\mathbb{P}^V, \mathcal{O}(-d)) \otimes \Lambda^d V \]

and the rest comes from Serre duality.

We have

\[ \text{Hom}(T, T) = \begin{pmatrix} 0 \\ \mathcal{O}(-d+1) \end{pmatrix} \otimes \begin{pmatrix} \mathcal{O}(-d+1) \\ \mathcal{O}(-d+2) \end{pmatrix} \]

so that

\[ \text{Ext}^g(T, T) = \begin{cases} 0 & \text{if } g \neq 0 \\ \begin{pmatrix} k^g V \oplus \cdots \oplus V \oplus \mathbb{A}^g \end{pmatrix} & \end{cases} \]

It might be better instead of \( T \) to take the thing that occurs with Severi-Brauer varieties. This means using \( \Lambda^d V \otimes \mathcal{O}(-j) \) for \( j = 0, \ldots, d-1 \).

Thus for \( d = 2 \),

\[ A = \begin{pmatrix} k & V \otimes V^* \\ 0 & k \end{pmatrix} \]

for \( d = 3 \):

\[ A = \begin{pmatrix} k & V \otimes V^* & S^2 V \otimes V^* \\ 0 & k & V \otimes V^* \\ \mathbb{A}^3 & 0 & k \end{pmatrix} \]

Q: Is \( A \) of finite global dimension, i.e., homological dimension?
Basic definition: Given a class $S$ of objects in a abelian category one defines the right
and left $\perp$ categories by

$$S^\perp = \{ M \mid \text{Hom}(T, M) = \text{Ext}^1(T, M) = 0 \quad \forall T \in S \}$$

$$\perp S = \{ M \mid \text{Hom}(M, T) = \text{Ext}^0(M, T) = 0 \quad \forall T \in S \}$$

Consider $G = \text{mod}(R)$, $S = \text{mod}(R/\mathfrak{I})$. Then

$$S^\perp = \text{solid}(R, \mathfrak{I})$$

$$\perp S = \text{frin}(R, \mathfrak{I})$$

Check: Given a module $M$, choose $0 \to K \to P \to M \to 0$
with $P$ projective. Then

$$0 \to \text{Hom}_R(M, T) \to \text{Hom}_R(P, T) \to \text{Hom}_R(K, T) \to \text{Ext}^1_R(M, T) \to 0$$

\[\text{Hom}_R(P/\mathfrak{I}P, T) \xrightarrow{\text{Hom}} \text{Hom}_R(K/\mathfrak{I}K, T)\]

$0 \to M \to \perp S \iff K/\mathfrak{I}K \to P/\mathfrak{I}P$, which by

$0 \to \text{Tor}_1^R(K/\mathfrak{I}K, M) \to K/\mathfrak{I}K \to P/\mathfrak{I}P \to M/\mathfrak{I}M \to 0$

is equivalent to $M \in \text{frin}(R, \mathfrak{I})$.

Similarly if $0 \to M \to Q \to C \to 0$ with $Q$ injective, one has
\[ 0 \to \text{Hom}_R(T, M) \to \text{Hom}_R(T, Q) \to \text{Hom}_R(T, C) \to \text{Ext}_R^1(T, M) \to 0 \]

\[ \text{Hom}_R(T, I) \xrightarrow{\sim} \text{Hom}_R(T, I) \quad \text{which is equivalent to} \quad \text{Ext}_R^1(R | I, M) = 0 \quad \text{for} \quad d = 0, 1. \]

i.e. to \( M \in \text{sol}(R, I) \).

Consider a homomorphism \( R \to U \).

Suppose restriction of scalars \( \text{mod}(U) \to \text{mod}(R) \)

is fully faithful: for all \( U \)-modules \( M, N \)

we have

\[ \text{Hom}_U(M, N) \to \text{Hom}_R(M, N) \]

\[ \text{Hom}_U(U \otimes_R M, N) \]

Since this holds \( \forall N \), we must have \( U \otimes_R M \cong M \)

in particular \( U \otimes_R U \cong U. \)

Conversely

\[ U \otimes_R U \cong U \Rightarrow U \otimes_R M \cong M \]

\[ \Rightarrow \text{mod}(U) \to \text{mod}(R) \text{ fully faithful.} \]

Suppose that \( R \to U \) is an epimorphism in the category of rings, i.e. \( \text{Hom}_\text{rings}(U, S) \to \text{Hom}_\text{rings}(R, S) \) is injective for all \( S \). Recall that the \( U \)-bimodule \( \Omega_R^1 U = \text{Ker} \{ U \otimes_R U \to U \} \)

is universal for derivations \( D: U \to B_p \), where \( B_p \) is a \( U \)-bimodule.
If $\Omega^1_R U \neq 0$, then we have a nontrivial derivation $d$ of $U$ in $R$

$$u \mapsto \Omega^1_R u \in \Omega^1_R U$$

hence two homomorphisms $1, 1 + d : U \to U \otimes_R \Omega^1_R U$

which agree on $R$. Thus $R \to U$ an epimorphism of rings $\Rightarrow U \otimes_R U \cong U$.

Conversely if $f, g : U \to S$ are two nontrivial homomorphisms agreeing on $R$, then $f - g : U \to S$

is a derivation from $U$ to $S$ considered as a $U$-bimodule via $f$ on one side and $g$ on the other. This implies $\Omega^1_R U \neq 0$, hence $U \otimes_R U \cong U$

is not an isomorphism. Thus we have

Prop. TFAE for a homomorphism $R \to U$.

1) Restriction of scalars $\text{mod}(U) \to \text{mod}(R)$ is fully faithful.

2) $U \otimes_R U \cong U$

3) $\Omega^1_R U = 0$

4) $R \to U$ is an epimorphism in the category of rings.

In Geigle-Lenzing there is the notion of homological epimorphisms $R \to U$ of rings involving $U \otimes_R U \cong U \Rightarrow D(U) \to D(R)$ fully faithful.

You've examined the case $U = R/I$. 

A abelian category, \( T \), \( U \) full subcategories such that

1) \( \text{Ext}^j(T, U) = 0 \) \( j = 0, 1 \) \( T \in T \) \( U \in U \)

2) \( \forall M \in A \) \( \exists \tilde{M} \in U \) and \( \tilde{\varepsilon}_M : M \to \tilde{M} \)

such that \( \text{Ker}(\tilde{\varepsilon}_M) \), \( \text{Coker}(\tilde{\varepsilon}_M) \) are in \( T \).

Then \( 0 \to M' \to M \to M'' \to 0 \) we get

\[
\begin{align*}
&0 \to \text{Hom}(M'', U) \to \text{Hom}(M', U) \to \text{Hom}(M, U) \to \text{Ext}^1(M'', U) \\
&\text{Thus } M' \text{ in } T \implies \text{Hom}(M'', U) \to \text{Hom}(M, U)
\end{align*}
\]

\( M'' \text{ in } T \implies \text{Hom}(M, U) \to \text{Hom}(M', U) \)

so \( \text{Hom}(-, U) \) inverts any map with \( \text{Ker} \) \( \text{Coker} \) in \( T \).

In particular \( \tilde{\varepsilon}_M \) so

\[
\begin{align*}
&\exists \tilde{\varepsilon}_M^* : \text{Hom}(\tilde{M}, U) \to \text{Hom}(M, U) \quad \forall U \in U
\end{align*}
\]

This implies \( (\tilde{M}, \tilde{\varepsilon}_M) \) unique up to canonical isom., also that \( M \to \tilde{M} \) is left adjoint to \( U \in A \).

Next TFAE for \( M \) in \( A \)

\[
\begin{align*}
&1) \downarrow \\
&(2) \text{Hom}(M, U) = 0 \quad \forall U \in U \\
&(3) \tilde{M} = 0
\end{align*}
\]

Finally \( \tilde{M} = 0 \) \( \implies M \to 0 \) has kernel in \( T \) so \( M \in T \).

Returning to (\#) one then sees that \( M \) in \( T \)
\( \iff M' \text{ and } M'' \text{ in } T \). (note \( M \in T \implies \text{Hom}(M, U) = 0 \)
\( \forall U \implies \text{Hom}(M'', U) = 0 \) \( \forall U \implies M'' \in T \implies \text{Ext}(M'', U) = 0 \),
so then \( \text{Hom}(M', U) = 0 \) \( \forall U \implies M' \in T \).)

Thus \( T \) is a dense subcategory. As before we have \( \text{Hom}_{\text{rel}}(M, U) = \varinjlim \text{Hom}(M, U) = \text{Hom}(M, U) \)
showing that $\mathcal{U} \rightarrow A/\mathcal{T}$ is fully faithful, then essentially surjective by 2).

Also defining $\mathcal{I}M$ by

$$0 \rightarrow \mathcal{I}M \rightarrow M \xrightarrow{\eta} \tilde{M}$$

we have $\text{Hom}(T, \mathcal{I}M) \cong \text{Hom}(T, M) \forall T$

so $T$ is right adjoint to the inclusion $\mathcal{T} \hookrightarrow A$.

Next suppose $A$ Grothendieck (generator + AB5). Then $M \in \mathcal{T} \iff \text{Hom}(M, U) = 0$ implies $\mathcal{T}$ is closed under direct sums. Then we know that $A/\mathcal{T}$ is Grothendieck, hence also $\mathcal{U}$ and that injectives in $\mathcal{U}$ are the same as $\mathcal{T}$-free injectives in $A$. 
Thick subcategory $\mathcal{V}$ of a triangulated category $\mathcal{D}$ according to Verdier (SGA 4 1/2) is a full subcategory closed under translation and cones, such that for any map $X \to Y$ in $\mathcal{D}$, if the cone is in $\mathcal{V}$ and the map factors through an object of $\mathcal{V}$ then both $X, Y$ are in $\mathcal{V}$.

Check that if $V, W \in \mathcal{D}$ satisfy
1) $\text{Hom}(V, W) = 0 \quad \forall V \in \mathcal{V}, \; W \in \mathcal{W}$
2) $\forall X \in \mathcal{D} \exists \Delta \quad V \to X \to W \to$
then both $V$ and $W$ are thick.

First note that these conditions imply for $V \to X \to W \to$ as in 2) that

\[
\text{Hom}(V', V) \xrightarrow{\sim} \text{Hom}(V', X) \quad \forall V' \in \mathcal{V}
\]

\[
\text{Hom}(W', W) \xrightarrow{\sim} \text{Hom}(X, W') \quad \forall W' \in \mathcal{W}
\]

which means that $V \subset \mathcal{D}$ has right adjoint $X \to V$ and $W \subset \mathcal{D}$ has left adjoint $X \to W$.

Then $\text{Hom}(V', X) = 0 \quad \forall V' \Rightarrow X \cong W \in \mathcal{W}$

$\text{Hom}(X, W') = 0 \quad \forall W' \Rightarrow V \cong X \in \mathcal{V}$

Now suppose given $f : X \to Y$ in $\mathcal{D}$. If the cone on $f$ is in $\mathcal{V}$ we have

\[
\text{Hom}(Y, W') \xrightarrow{\sim} \text{Hom}(X, W') \quad \forall W'
\]

and if $f$ factors $X \xrightarrow{\sim} V' \to Y$ then this isom. factors through $\text{Hom}(V, W') = 0$. i.e. $X, Y \in \mathcal{V}$.
Similarly $W$ is thick.

The significance of the factorization condition is not clear to me. It must somehow be used in constructing the desired structure on $D/V$. All you need to define this category is the class of maps to be inverted, and these are the ones whose cones are in $V$. Such $V$-icos are closed under composition by the octahedral axiom.

Let's go back to a Grothendieck, $\mathcal{T}$ a dense subcategory, $\mathcal{A}/\mathcal{T}$ the quotient Grothendieck category. We have adjoint functors

$$
\mathcal{T} \xleftarrow{\mathcal{T}^*} \mathcal{A} \xrightarrow{\mathcal{A}/\mathcal{T}} \mathcal{A}/\mathcal{T}
$$

and an equivalence $\mathcal{T}^* \sim \mathcal{A}/\mathcal{T}$.

Consider next the derived category situation:

There are functors

$$
\text{D}^+(\mathcal{A}) \xleftarrow{\mathcal{R}^*} \text{D}^+(\mathcal{A}) \xrightarrow{\mathcal{J}^*} \text{D}^+(\mathcal{A}/\mathcal{T})
$$

because $\mathcal{A}$, $\mathcal{A}/\mathcal{T}$ are Grothendieck, they have sufficiently many injectives, hence $\mathcal{R}_i^*$, $\mathcal{J}_*^*$ are calculated by injective resolutions.

We know that $\text{D}^+(\mathcal{A}) \sim \text{K}^+(\text{Inj}(\mathcal{A}))$ and $\text{D}^+(\mathcal{A}/\mathcal{T}) \sim \text{K}^+(\text{Inj}(\mathcal{A}/\mathcal{T}))$. Moreover $\mathcal{J}^*$ induces an equivalence $\text{Inj}(\mathcal{A}/\mathcal{T}) \sim [\text{D}^+(\mathcal{A})]$.
I want to check that fits into the $V, W \in D$ discussion. Here $D = D^+(A), V = D^+(A)/T$, and $W$ is the full subcategory consisting of complexes $W$ satisfying $\mathbb{H}^{\bullet}(T, W) = 0$ for all $T \in T$, $T$ being considered as a complex supported in degree zero.

First check that $\mathbb{H}^{\bullet}(V, W) = 0$ for $V \in V, W \in W$. This follows from the Postnikov filtration of $V$, which increases, adding one homology group at a time.

Next given $X \in D$ we want to construct a triangle $\triangledown: V \rightarrow X \rightarrow W \rightarrow$. Here $W$ will be $Rj^*(j^*X)$. More precisely, we choose an injective resolution of $j^*X$, which is a complex in $\text{Inj}(A/\mathcal{T}) = \{ \text{Inj} \} \cup \{ X \text{ injective in } \mathcal{C} \} \cup \{ \text{Hom}(T, X) = 0 \forall T \in \mathcal{T} \}$.

Thus we have a complex $Q$ of $T$-free injectives in $\mathcal{C}$ and a quasi $j^*X \rightarrow j^*Q$, equivalently a map $X \rightarrow j^*(j^*Q) = Q$ whose cone has homology $j^*Q$ in $\mathcal{T}$. Thus the desired triangle $\triangledown: V \rightarrow X \rightarrow W \rightarrow$ is given by $W = Q$ and $V = \text{Cone}(X \rightarrow W)$.

The rest now should fall in place: $\text{Ri}^!, \text{Rj}^*$ are right adjoints of $i^*, j^*$. Equivalence of $\mathcal{W} = \{ W \in D | \mathbb{H}^{\bullet}(T, W) = 0 \forall T \in \mathcal{T} \}$ with $D^+(A/\mathcal{T})$. 

---
Let's try to make things clearer in the derived category situation. Suppose I replace $D^+(A)$ with the equivalent category $K^+(\text{Inj}(A))$. Similarly replace $D^+(A/T)$ by $K^+(T\text{-free Inj}(A))$.

Notation:

$D = K^+(\text{Inj}(A))$, $W = K^+(T\text{-free Inj}(A))$

$V = K^+(\text{Inj}(A))_F$ complexes of injectives with homology in $T$.

Note $V$, $W$ are full subcats. of $D$, $S$.

Then the conditions $\text{Hom}_D(V, W) = 0$, $\forall M \in A$

$V \rightarrow M \rightarrow W$ are satisfied, so one knows that there are adjoints

$V \rightleftarrows D \rightleftarrows W$

such that both

$V \rightarrow D \rightarrow W$

$W \rightarrow D \rightarrow V$

are 'exact', equivalently: $W \rightarrow D/V$ and $V \rightarrow D/W$ are equivalences of $A$-stalk categories.

So the issue remaining here is the relation of $D^+(A)_F$ with $D^+(T)$. Restricting to injective complexes, I want to know whether a complex of injectives $V$ with homology in $T$ is quasi-isomorphic to a complex in $D$. 
Let's consider $A = \text{mod}(R)$, $\mathcal{T} = \text{tors}(R/I)$. We want conditions sufficient that $D^+(R)_{\text{tors}} \leftarrow D^+(\mathcal{T})$. In other words, given an injective (held to the left) complex of $R$-mods with torsion homology, we would like it to be quasi-isomorphic to a complex of torsion modules.

1st condition is stability (i.e. tors is a stable torsion theory). This means $E$ injective $R$-module $\Rightarrow \tau E$ is injective $R$-module.

This condition implies

2nd condition: $E$ injective $R$-module $\Rightarrow E/\tau E$ is an injective $R$-module.

Suppose the 2nd condition holds. Then if $E$ is an injective complex with torsion homology we have an exact sequence of complexes

$$0 \rightarrow \tau E \rightarrow E \rightarrow E/\tau E \rightarrow 0$$

where $E/\tau E$ is torsion-free & injective. But we know that $R\text{Hom}_R(E, E/\tau E) \approx 0$ because $E$ has torsion homology and $E/\tau E$ is solid injective. Thus $E \rightarrow E/\tau E$ is homotopic to zero, whence $\exists E \rightarrow \tau E$ such that $E \rightarrow \tau E \rightarrow E$ is homotopic to the identity.

It's simpler to look at homology where we have short exact sequences

$$0 \rightarrow H^+(E/\tau E) \rightarrow H^+(\tau E) \rightarrow H^+(E) \rightarrow 0$$

Because $E/\tau E$ solid injective its first non-zero homology
This means $E/\mathcal{I}E$ must be acyclic, and so $\mathcal{I}E \to E$ is a quasi-isomorphism.

Observe that the 2nd condition holds if $\text{proj. dim.}(R) \leq 1$, since then any quotient of an injective module is injective.

In the case $I = I^2$, we know the first condition (stability) is equivalent to $R/I$ being a flat $R^{op}$-module.

Similarly as $E/\mathcal{I}E = \text{Hom}_R(\mathcal{I}, E)$, and

$$\text{Hom}_R(M, \text{Hom}_R(\mathcal{I}, E)) = \text{Hom}_R(\mathcal{I} \otimes_R M, E)$$

we see that $E/\mathcal{I}E$ injective for all $E \iff \mathcal{I} \otimes_R -$ exact $\iff I$ is $R^{op}$-flat.

In the case $I = I^2$, we have a necessary and sufficient condition for $D^+(\text{tors}) \to D^+(R)_\text{tors}$, namely $I \otimes_R R/I \cong \mathcal{O}$, and we know this condition depends only on the grading $I$.

In general we have the criterion that $\text{proj. dim.}(R) \leq 1$, also stability holds when $R$ is commutative noetherian.

It's possible that Joachim's approximate $h$-unitality is relevant here. One can pose the question of whether $X \in D^b(R)$ will be quasi a complex in $D^b(R/I^n)$ for some $n$. Idea of using the bar construction to obtain the derived category of modules in some adic sense (e.g. pro-nilpotent completion of $\mathcal{O}$).
Return to the question of whether
\[ M(R, I) \rightarrow \text{add}(\text{res}^{\mu}(R^b, I^b), \mathbb{R}) \]
\[ M \rightarrow - \otimes_R M \]
is fully faithful, hopefully the essential image consists of the right continuous functors.

The idea I have is to show that I can recover \( \widetilde{M} \) (= \( \text{res} \mu K^* \otimes M \)) from the family of abelian groups \( F \otimes_R M \) with \( F \) a firm flat right module. In fact we would like to use \( F \) of the form \( F(\alpha), \alpha \) a sequence in \( I \).

Some evidence that this might be possible.

First we know that the tensor elements \( m \in M \) are the \( T \)-nilpotent elements; a sequence \( a, \ldots, a_m = 0 \). This means that \( \ell M \) is the intersection of the kernels of the canonical maps \( M \rightarrow F(\alpha) \otimes_R M \), where \( \alpha \) runs over all sequences in \( I \). (A related result is that there are enough solid injectives of the form \( \text{Hom}_R(F(\alpha), \mathbb{R}) \).

Why? If \( m \not\in \ell M \), then \( \exists \alpha \) such that \( m \) does not go to zero under \( M \rightarrow F(\alpha) \otimes_R M \). Then there is a character \( \chi \) such that \( m \) is not killed by \( M \rightarrow F(\alpha) \otimes_R M \rightarrow \mathbb{R} / \mathbb{R} \), By
\[ \text{Hom}_R(M, \text{Hom}_R(F(\alpha), \mathbb{R})) = \text{Hom}_R(F(\alpha) \otimes_R M, \mathbb{R}) \]
we get a map \( M \rightarrow \text{Hom}_R(F(\alpha), \mathbb{R}) \), such that the composition with \( \text{Hom}_R(F(\alpha), \mathbb{R}) \rightarrow \text{Hom}_R(R, \mathbb{R}) \) does
not kill \( M \).

Further evidence comes from the commutative \( R \) finitely generated \( I \) case.

If \( I = \sum_{i=1}^{n} Rf_i \), then we know that

\[
\tilde{M} = \Gamma \left( \text{Spec}(R) - \text{Spec}(R[I]), \text{sheaf assoc.} \right)
\]

\[
= \text{Ker} \left\{ \prod_{i} M_{f_i} \to \prod_{ij} M_{f_if_j} \right\}
\]

where \( M_{f_i} = F(f_i, f_i \ldots) \otimes_R M \).

Let's look at the case \( I = I^2 \). Here we know the final result holds: namely equivalence:

\[
M(R, I) \to \text{rt cont add} \left( \text{firm}(R^{ob}, I^{ob}), \text{Ab} \right)
\]

\[
\text{firm}(R, I) \to \text{rt cont add} \left( M(R^{ob}, I^{ob}), \text{Ab} \right)
\]

because is an equivalence in general.

But in any case we can try to see if our approach works, namely to express

\[
\tilde{M} = \text{Hom}_R \left( I^{(2)}, M \right)
\]

somehow in terms of \( F(\alpha) \otimes_R M \).

Example. \( I = ReR \) where \( e^2 = e \). Then we know \( ReR = Re \otimes_S eR \) where \( S = eRe \)

\[
\tilde{M} = \text{Hom}_R \left( Re \otimes_S eR, M \right) = \text{Hom}_S \left( eR, \text{Hom}_R \left( Re, M \right) \right)
\]
\[
\tilde{M} = \text{Hom}_S(eR, eM)
\]

If we choose a presentation
\[
S^{(1)} \to S^{(1_0)} \to eR \to 0
\]
then we get
\[
0 \to \tilde{M} \to (eM)^{\Lambda_0} \to (eM)^{\Lambda_1}
\]
where \( eM = F(e, e, \ldots) \otimes_R M \).

If we choose a presentation of \( eR \) by free \( S \)-\( R \) bimodules \( S \otimes_S R \), then \( \tilde{M} \) is a kernel of a map between products of the \( R \)-module
\[
\text{Hom}_S(S \otimes_S R, eM) = \text{Hom}_S(R, eM).
\]

Another example: Suppose \( I \) is a

form flat right module. Then we have
\[
\tilde{M} = \text{Hom}_R(I, I \otimes_R M)
\]
esince \( \text{Hom}_R(I, -) = \text{Hom}_R(I^{(2)}, -) \) inverts the null-kernel \( I \otimes_R M \to M \). Choosing a presentation
\[
R^{(1)} \to R^{(1_0)} \to I \to 0
\]
of left modules, we get
\[
0 \to \tilde{M} \to (I \otimes_R M)^{\Lambda_0} \to (I \otimes_R M)^{\Lambda_1}
\]
where \( I \otimes_R M \) has the form \( F \otimes_R M \) at least.

Q: Is it possible to reduce to this case by Morita equivalence?
Consider the derived category $\mathcal{A}$ in more generality. Start with a Grothendieck category $\mathcal{A}$ and localizing some subcategory $\mathcal{T}$:

$$
\mathcal{T} \xleftarrow{i_*} \mathcal{A} \xrightarrow{j^*} \mathcal{A}/\mathcal{T}
$$

$$i^! i_* = 1 \quad j^* j_* = 1$$

For any $M \in \mathcal{A}$ we have an exact sequence

$$0 \rightarrow i_* i^! M \rightarrow M \rightarrow j^* j_* M$$

where the last map is surjective when $M$ is an injective object. NOT CLEAR!

Let's check this last point. It suffices to take any map $N \rightarrow j^* j_* M$, then show $\exists$

$$
\begin{array}{ccc}
N & \longrightarrow & M \\
\downarrow & & \downarrow \\
N & \longrightarrow & j^* j_* M
\end{array}
$$

for some $N \rightarrow N$ surjective. The arrow $f$ is equivalent to a map in

$$\text{Hom}_{\mathcal{A}/\mathcal{T}}(j^* N, j^* M) = \lim_{N' \rightarrow N \text{ Fin}} \text{Hom}_{\mathcal{A}}(N', M)$$

so $f$ can be represented by a correspondence

$$
\begin{array}{ccc}
N' & \rightarrow & \mathcal{M} \\
\downarrow & & \\
N & \leftarrow & M
\end{array}
$$
Let us factor $N' \to N$ into an injection followed by a surjection:

$$N' \hookrightarrow N_1 \twoheadrightarrow N$$

say via Grothendieck's graph:

$$N' \subset N \times U \to N$$

Then we have

$$N' \longrightarrow M$$

$$\downarrow \quad \exists \text{ if } M \text{ is injective}$$

$$N_1 \twoheadrightarrow N$$

I need to arrange that $N \leftarrow N_1 \to M$ also represents $f$, and it's clear then that I want $N_1 \to N$ to have kernel in $T$. So I have to be able to factor a $T$-isomorphism into an injection followed by surjection where these are $T$-iso.

This doesn't seem to work, but perhaps it is not needed.
Start with $\mathcal{A}$ a Grothendieck category $\mathcal{T}$ a Serre subcategory closed under $\Theta$'s, whence we have adjoint functors

$$
\mathcal{T} \xleftarrow{l^{\ast}} \mathcal{A} \xrightarrow{j^{\ast}} \mathcal{A}/\mathcal{T}
$$

such that $l^{\ast}, j^{\ast}$ are exact

$$
l^{\ast}l^{\ast} = 1 \quad j^{\ast}j^{\ast} = 1
$$

$$
o \to l^{\ast}l^{!}M \to M \to j^{\ast}j^{\ast}M
$$

exact. VM

Now consider

$$
D^{+}(_{\mathcal{T}}) \xleftarrow{l^{\ast}} D^{+}(\mathcal{A}) \xrightarrow{j^{\ast}} D^{+}(\mathcal{A}/\mathcal{T})
$$

Check these are adjoint functors as indicated.

Let $X, Y, Z$ be left bounded complexes of injective objects in $\mathcal{A}, \mathcal{T}, \mathcal{A}/\mathcal{T}$ resp.

$$
R\text{Hom}_{\mathcal{A}}(X, Rj^{\ast}(Z)) = R\text{Hom}_{\mathcal{A}}(X, j^{\ast}Z)
$$

$$
= \text{Hom}_{\mathcal{A}}(X, j^{\ast}Z)
$$

$$
= \text{Hom}_{\mathcal{A}/\mathcal{T}}(j^{\ast}X, Z)
$$

$$
= R\text{Hom}_{\mathcal{A}/\mathcal{T}}(j^{\ast}X, Z)
$$

$Z$ inj

$J^{\ast}Z$ inj

since $J^{\ast}$ has exact left adjoint

$$
R\text{Hom}_{\mathcal{A}^{\mathcal{T}}}(Y, Ri^{\ast}(X)) = R\text{Hom}_{\mathcal{A}}(Y, i^{\ast}X)
$$

$X$ inj

$i^{\ast}X$ inj as

$l^{\ast}$ has exact left adj.
\[ \text{Hom}_a(\xi Y, X) = R\text{Hom}_a(\xi Y, X) \]

(The above works obviously for a pair \((f^*, f_*^*)\) with \(f^*\) exact; the fact that one or the other is injective is irrelevant.)

Now we know \(f^* Rf_* = 1\), since \(f^* Rf_* (Z) = f^* f_* Z = Z\). Thus

\[ R\text{Hom}_a(Rf_* (Z_1), Rf_* (Z_2)) = R\text{Hom}_a(f^* Rf_* (Z_1), Z_2) = R\text{Hom}_a(Z_1, Z_2) \]

showing \(Rf_*\) is fully faithful.

But \((Rc^!_*)\xi_*\) need not be the identity since \(\xi_*\) need not preserve injectives.

Assume \((Rc^!_*)\xi_* = 1\).

Then

\[ R\text{Hom}_a(\xi^* Y_1, \xi^* Y_2) = R\text{Hom}_a(Y_1, Rc^!(\xi^* Y_2)) = R\text{Hom}_a(Y_1, Y_2) \]

so that \(\xi_*\) is fully faithful.

I should have noted earlier before making the above assumption the following

\[ f^* c_* = 0 \quad \text{and} \quad Rc^! Rf_* = 0 \]

\[ Rc^! Rf_* (Z) \xrightarrow{2 \text{ inj}} Ri^! f_* Z = c^! j_* Z = 0. \]
Then we have the orthogonality
\[ R\text{Hom}_\mathcal{A}(L_x Y, Rf_*(Z)) = R\text{Hom}_{\mathcal{A}/\mathcal{I}}(\mathcal{F}^* L_x Y, Z) = 0 \]

Finally we want the canonical diagram
\[ L_x R\xi!(X) \to X \to Rf_*(f^*X) \to \]

Let's define \( U \) to be the cofibre of the adjunction map \( L_x R\xi!(X) \to X \).

so that we have the triangle
\[ L_x R\xi!(X) \to X \to U \to \]

Apply \( R\xi! \) to get the triangle
\[ R\xi! L_x R\xi!(X) \to R\xi!(X) \to R\xi!(U) \to \]

\[ \beta: R\xi! \to R\xi!(X) \]

and recall that \( \beta \) is an isomorphism by assumption.

Thus we find \( R\xi!(U) = 0 \).

Up to quasi we can assume \( U \) is a minimal injective complex (left-bounded). Then \( R\xi!(U) = 0 \) means \( \xi^! U \) is acyclic. Assuming \( \xi^! U^{n'} = 0 \) for \( n' < n \) we have
\[ 0 \to \xi^! U^n \to \xi^! U^{n+1} \to \]
\[ 0 \to U^n \to U^{n+1} \]
so \( Z^n \cap \xi^! U^n = 0 \). But for \( U \) to be minimal
means that $2^n \subset U^n$ is an essential extension, so we find $i^! U^n = 0$.

Thus $i^! U = 0$ which means that $U$ is a complex of $T$-free injectives.

Thus $U = j_* (j^* U)$ where $j^* U$ is a complex of injectives in $A(T)$.

$\therefore U = Rj_* (j^* U)$.

Finally apply $j^*$ to the triangle $i_* R_i^! (X) \to X \to U$ given $j^* X \to j^* U$, so we are done.
Recall the misunderstanding of a thick subcategory of a triangulated category. Start with motivation for the actual definition. If $F : \mathcal{X} \to \mathcal{X}'$ is an exact functor between triangulated categories, then the kernel $\mathcal{K}$ of $F$ is thick: For any map $X \to Y$ having one in the kernel $\mathcal{K}$ and factoring through an object in $\mathcal{K}$ has both $X, Y$ in $\mathcal{K}$.

But my idea was to invert maps in $\mathcal{K}$ whose cone lies in a given full subcategory $S$. I think that maps in the localization of $\mathcal{X}$ with $S$-isomorphisms can be calculated by left or right fractions, provided $S$ is closed under cones. (Assume $S$ contains 0.)

Now suppose given $S$ closed under cones but not thick. Look at $\mathcal{K}[ (1\text{-isos})^{-1} ]$. Since $S$ is not thick, there is a map $X \to Z$ whose cone is in $S$, which factors $X \to Y \to Z$ where $Y$ is in $S$. Think of these maps as inclusions and look at octahedron.
Then we have in $X[(f\text{-isos})^{-1}]$

\[
X \rightarrow Z \rightarrow Z/y \rightarrow \Sigma (y/x) \rightarrow \Sigma^2 X
\]

Let's check what we need to compute the maps in the localization by fractions. Any map in $X[(f\text{-isos})^{-1}]$ is a product of maps $f_i$ and inverts $s_i^{-1}$. To be able to represent these by $f s^{-1}$ (right fractions) we need to be able to convert $s^{-1} f$ to this form.

\[
\begin{array}{ccc}
W & \rightarrow & Z \\
\downarrow f' & & \downarrow s \\
X & \rightarrow & Y
\end{array}
\]

This can always be done by one of the TR axioms I think. In any case for the examples I know one can assume up to isomorphism that $Z$ is the h-fibre of $Y \rightarrow S$ where $S \in S$, then define $W$ to be $f^* Z = X \times_Y Z$, and $W$ h-fibre of $X \rightarrow S$, etc.

Similarly we can represent maps in the localization category in the form $f s^{-1}$.
Next to have a calculus of right fractions we need \( sf = 0 \Rightarrow \exists s' \) such that \( fs' = 0 \). Thus given

\[
X \xrightarrow{s} Y \xrightarrow{s'} Z
\]

we know \( f \) factors: \( X \rightarrow h\text{-}fibre(Y \rightarrow Z) \rightarrow Y \)

this \( h\text{-}fibre \) being an object of \( S \).

Then take \( W = h\text{-}fibre(X \rightarrow S) \), whence \( W \xrightarrow{s'} X \) is an \( S\)-iso such that \( W \xrightarrow{s'} X \rightarrow S \) is zero, hence \( fs' = 0 \).

I've left out the requirement that \( s'^{-1} s^{-1} = (s_2 s_1)^{-1} \) with \( s_2 s_1 \) an \( S\)-iso. Thus I need \( S\)-isos to be closed under composition, which means \( S \) closed under cones.

The problem I'm working on is to show when \( I^2 \subset R \) one has an equivalence of \( \Delta \)-left categories.

\[
\text{firm} D(R, I) \sim D^{-}(M(R, I))
\]

Here \( \text{firm} D(R, I) \) is the full subcat of \( D^{-}(R) \) consisting of complexes \( M \) such that \( L_\Lambda^\cdot(M) = R/I_\Lambda M \otimes_R \) is \( \sim 0 \).

We know such on \( M \) is quasi a firm flat complex (right bld). Moreover we have an equivalence

\[
\text{flat firm}(R, I) \sim \text{flat} M(R, I)
\]
given by \((j!, j^*)\). Also we know \( M(R, I) \) has sufficiently many flat objects, so that for any
right-bdd complex in \( M(R, I) \) \( F \to N \) with \( F \) a complex in \( \text{flat} M(R, I) \).

This missing ingredient is two equivalences:

\[
\text{D}(\text{flat } M) \cong \text{D}(M)
\]

\[
\text{D}(\text{flat } \text{firm}) \cong \text{firm D}
\]

where the derived categories are suitably defined, something like the quotient category by the acyclic complex subcategories.
Problem: I have two candidates for a firm derived category as follows. First is the full subcategory of $D^-(R)$ consisting of complexes $U$ satisfying $\left( R/I \otimes_R U \right) \otimes_R U = 0$, up to equivalence. This is the same as the full subcategory of $D^-(R)$ consisting of firm flat complexes.

On the other hand, one can consider the homotopy category of firm flat complexes $K^- (\text{firm flat})$, and the thick subcategory of acyclic complexes, and form the quotient $K^- (\text{firm flat}) / \text{acyc}$.

Let's denote the former by $\text{firmD}(R, I)$, so that we have $\text{firmD} \subset D^-(R)$. Denote the latter by $D^-(\text{firm flat})$. Note that the derived category makes sense for an exact category in my sense (full subcategory of an abelian category closed under extensions).

There is an obvious functor

* $D^-(\text{firm flat}) \rightarrow \text{firmD}$

and the question is whether it's an equivalence of categories. I know it's essentially surjective. In effect, given $U$ such that $R/I \otimes_R U = 0$, choose a quasi $P \rightarrow U$ with $P$ projective, then $P/IP$ is acyclic, and one can deform the identity operator on $P$ to $f: P \rightarrow IP \subset P$, then form $P[f, f]$ to
The question is then whether $x$ is fully faithful. Now maps $F_1 \rightarrow F_2$ in firm $D$ (i.e., in $D^-(R)$), where $F_1$, $F_2$ are firm flat, are represented by

$$X \xrightarrow{\delta} F_1 \quad X \xrightarrow{\delta} F_2$$

It follows that $X \in$ firm $D$. Our problem is to replace $X$ by a firm flat $F$. We have given $X \Leftarrow P \xrightarrow{\delta} F$ as above, can suppose $X = P$.

Then we reach the situation

$$P \xleftarrow{\delta} \xrightarrow{\delta} F \xrightarrow{\delta} F_1 \quad F_2$$

and it's not clear what to do,
Recall for a Morita context \((R, Q, P, S)\) and ideals \(I \subset R, J \subset S\) the condition
\[\ast\ (\Pi Q \supset J^k, \ QJ^p \supset I^k \text{ for some } k \geq 0.\]

I claim this condition means that the topology \(\tau_s\) defined by the ideals \(\Pi^n Q, n > 0\) is the same as the \(J\)-adic topology, as well as the topology on \(R\) defined by \(QJ^n P, n > 0\) is the same as the \(I\)-adic topology.

Let's check the condition \(\ast\) implies this.

We have
\[
\Pi^n Q \supset (\Pi Q)^n \quad (\text{QP} \subset R \text{ and IB = I})
\]

Also \(\ast\Rightarrow\ QP \supset I^k\), so

\[
(\Pi Q)^n = P(\Pi Q)^n \supset P(I^{1+k})^n \supset Q = \Pi^{k+n-k} Q
\]

Thus \(\{\Pi^n Q\}\) gives the \(\Pi Q\)-adic topology on \(S\).

Next we have \(\Pi Q \supset J^k\) and
\[
\Pi^{kn} Q \subset P(QJ^p)^k Q = PQ(JPQ)^k Q \subset J^n.
\]

Simpler is \((\Pi Q)^k \subset \Pi^k Q \subset PQJPQ \subset J\).

So we see that \(\Pi Q\)-adic and \(J\)-adic topologies on \(S\) coincide. Thus \(\{\Pi^n Q\}\) gives the \(J\)-adic topology on \(S\). Similarly \(\{QJ^np\}\) gives the \(I\)-adic topology on \(R\).

Conversely these imply \(\Pi Q \supset J^k\) and
\[
QJP \supset J^k \text{ for some } k.
\]
Assume again $QJP \supset I^k$ and $PIQ \subset J^k$ for some $k \geq 0$.

Let $P_1 = PI, \quad Q_1 = Q$. Then

$P_1Q_1 = PIQ$, which has the same adic top as $J$;

$J^k \subset PIQ$

$(PIQ)^k \subset PI^kQ \subset PQJPQ \subset J$

and $Q_1P_1 = QPI$, which has the same adic topology as $I$;

$I^{k+1} \subset QJPI \subset QPI$

$QPI \subset I$.

Thus we can suppose $QP = I$, $PQ = J$ by adjusting the ideals without changing the modules.
Consider again a Noetherian context \((\overline{R, S})\) and ideals \(I \subset R, J \subset S\). Write \(I \sim I'\) when the ideals \(I, I'\) give the same adic topology, i.e. \(I \supset I^k, I' \supset I'^k\) for some \(k\).

Consider the conditions

1) \(Q \sim I\) and \(PQ \sim J\).
2) \(QJP \supset I^k\) and \(PIQ \supset J^k\) for some \(k\).
3) \(QJP \sim I\) and \(PIQ \sim J\).
4) \(QJP \sim I\) and \((JP)Q \sim J\).

Then one has the implications

Why? Assume 1) i.e. \(Q \sim I^k\), \(PQ \sim J^k\) for some \(k\).

Then \(QJP \supset (Q(PA))^kP = (QP)^{k+1} \supset I^{k(k+1)}\) and similarly \(PIQ \supset J^{k(k+1)}\), proving 2).

3) \(\Rightarrow\) 2) is obvious since \(QJP \sim I \Rightarrow QJP \supset I^k\) same.

2) \(\Rightarrow\) 3) Assume 2). Then \(PIQ \supset J^k\) is given and \((PIQ)^k \subset PI \subset PQJPQ \subset J\).

so \(PIQ \sim J\). Similarly (really by symmetry) \(QJP \sim I\), so 3) holds.

2) \(\Rightarrow\) 4). Assume 2). Then \(J \supset JPQ \supset JPIQ \supset J^{1+k}\) \(\Rightarrow J \sim JPQ\).

\(QJP \supset I^k\) is given and \((QJP)^k \subset QJ^kP \subset QPIQP \subset I\) \(\Rightarrow I \sim QJP\)

(whence 4)
Significance of the above

2) \( \Leftrightarrow 3 \) says that my hypothesis for Mörta equivalence means \( QIP \sim I \) and \( PIQ \sim J \).

3) \( \Rightarrow 4 \) says that if I replace the given Mörta context with \( (R, Q, S) \), where \( P_i \)
is \( JP \) (also \( PI \) works), then I have the situation 1).

When 1) holds I can assume \( QP = I \) and \( PQ = J \), which was my operating assumption for a long time.

Why \( (\begin{array}{cc} R & I \\ R & I \end{array}) \) is a Mörta context.

Consider \( M_2(R) = (\begin{array}{cc} \overline{R} & \overline{R} \\ \overline{R} & \overline{R} \end{array}) \) and use \( e \) for the identity of \( R \). Then \( (e \ 0) \) is an idempotent in \( M_2(R) \) and so

\[
(\begin{array}{cc} e & 0 \\ 0 & 1 \end{array}) (\begin{array}{cc} \overline{R} & \overline{R} \\ \overline{R} & \overline{R} \end{array}) (\begin{array}{cc} e & 0 \\ 0 & 1 \end{array}) = (\begin{array}{cc} e\overline{R} & e\overline{R} \\ e\overline{R} & e\overline{R} \end{array}) = (\begin{array}{cc} R & R \\ R & R \end{array})
\]
is a ring with the identity \( (e \ 0) \). Clearly

\( (\begin{array}{cc} R & I \\ R & I \end{array}) \) is a subring of \( (\begin{array}{cc} R & R \\ R & R \end{array}) \).