Review yesterday. Assuming $R$ is a flat algebra over a commutative ring $k$, we can construct a chain complex $P_i$ of $R$-bimodules, which is good flat on the right and which is a resolution of $R$ module bimodules null in the right. One then has for any complex $M$ of modules

$$L_{j_!}(j^*M) = P \otimes_R M$$

In effect, we have $L_{j_!}(j^*M) = F$, where $F$ is a complex of good flat modules, which is a resolution of $M$ module null modules. Then

$$F = R \otimes_R F \xleftarrow{\text{quis}} P \otimes_R F \xrightarrow{\text{quis}} P \otimes_R M$$

When $I \otimes_R I = I$ (equivalently $I \otimes_R R/I = 0$), $P$ is a bimodule resolution of $I$ by bimodules which are good flat on the right. Thus

$$L_{j_!}(j^*M) = I \otimes_R M$$

and we get the diagram

$$
\begin{array}{ccc}
L_{j_!}(j^*M) & \rightarrow & M \\
\uparrow & & \downarrow \\
& \rightarrow & \iota_*Lc^*(M)
\end{array}
$$

Dually, let $M \rightarrow Q$ be an injective resolution. Then with $P$ as above

$$M \xrightarrow{\text{quis}} Q = \text{Hom}_R(R,Q) \xrightarrow{\text{quis}} \text{Hom}_R(P,Q) \xrightarrow{\text{mod null}} \text{Hom}_R(P,Q)$$

whence $\text{Hom}_R(P,Q)$ is a good injective resolution of $M$ module null modules. Thus
Another viewpoint. Suppose we consider the full subcategory of $\text{D}(\text{R}-\text{mod})$ consisting of complexes such that the homology is $I$-torsion. This is a triangulated subcategory: if two objects in a triangle belong to the subcategory, then the third does also by the long exact homology sequence. Let's denote this subcategory by $\text{D}(\text{R}-\text{mod})_{I-\text{tor}}$. Now I think there's a quotient triangulated category $\text{D}(\text{R}-\text{mod})/\text{D}(\text{R}-\text{mod})_{I-\text{tor}}$ defined, which is constructed as a category of fractions.

There are various questions like whether the obvious map

$$1) \quad \text{D}(\text{R}-\text{mod})/\text{D}(\text{R}-\text{mod})_{I-\text{tor}} \xrightarrow{f^*} \text{D}(\text{R}-\text{mod}/I-\text{mod})$$

is an equivalence of triangulated categories. This I feel should be OK. We can also ask whether the obvious map

$$2) \quad \text{D}(\text{R}/I-\text{mod}) \xrightarrow{i^*} \text{D}(\text{R}-\text{mod})_{I-\text{tor}}$$

is an equivalence.
The map $!j^*$ which is exact from $R\text{-mod}$ to $R\text{-mod}/I\text{-null}$. It carries a complex in $R\text{-mod}/I\text{-null}$ into an acyclic complex, which is quasi to $0$. There is a functor in the inverse direction which lifts the complex inductively over the skeleton, putting down to make the differentials have square zero. Check this later.

Look at 2). If we have the triangle

$$
\begin{array}{ccc}
Lj^!(j^*M) & \rightarrow & M \\
\downarrow & & \downarrow \\
M & \rightarrow & L_*L(i^*(M))
\end{array}
$$

then $M \in D(R\text{-mod})_{I\text{-null}}$ (equivalently $j^*M = 0$) implies $M \rightleftarrows L_*L(i^*(M)) = R/I \otimes_R M$ so $M$ is in the image of $L_*$.

Dually given the triangle

$$
\begin{array}{ccc}
L_*Ri^!(M) & \rightarrow & M \\
\downarrow & & \downarrow \\
M & \rightarrow & R_j*(j^*M)
\end{array}
$$

we have $M \in D(R\text{-mod})_{I\text{-null}} \implies j^*M = 0 \implies M \leftarrow L_*Ri^!(M) = R\text{Hom}_R(R/I, M)$.

It seems that we get then

$$R\text{Hom}_R(R/I, M) \rightarrow M \rightarrow R/I \otimes_R M$$

for any complex of $R/I\text{-modules}$.

Let's check some of this directly. Let $M$ be a complex of $R\text{-modules}$ such that $I^H(M) = 0$. Up to quasi-isomorphism we can suppose $M$ is flat. Then we have an exact sequence of complexes
Suppose to begin with that $I$ is right flat. Then we have
\[ H_0(I \otimes_R M) = I \otimes_R H_0(M) = 0 \]
so $I \otimes_R M$ is acyclic and $M$ is quasi-exact.

The complex of $R/I$-modules $M/IM$.

More generally suppose $I \otimes_R I \to I$.

Then we know (under the assumption that $R$ is a flat algebra over a commutative ring) that there is a bimodule resolution $P_\cdot$ of $I$ consisting of good flat right modules. Then
\[ \text{acyclic since each } P_\cdot \otimes_R M \text{ is injective} \]

Thus again we have a quasi-exact $M \to M/IM$.

However notice that under the assumption $I \otimes_R R/I = 0$ we can always construct a resolution $P_\cdot$ of $I$ by good flat right modules, so in any case $I \otimes_R M$ is quasi-exact $P_\cdot \otimes_R M$ which is acyclic and we have $M$ quasi-exact $M/IM$.

Dually we can suppose up to quasi-exact that $M$ is injective. Then we have an exact sequence of complexes:
\[ 0 \to \text{Hom}_R(R/I, M) \to M \to \text{Hom}_R(I, M) \to 0 \]
Assuming $M$ has small homology it follows that $\text{Hom}_R(I, M)$ has small homology.

Suppose $I$ is right flat. Then $\text{Hom}_R(I, M)$ is
a complex of good' injectives. Let's check that this together with the fact that its homology is $I$-null implies $\text{Hom}_R(I, M)$ is acyclic.

Let $E^\bullet$ be a complex of good' injectives, whose homology is $I$-null, and suppose its odd below, say $E^n = 0$ for $n < 0$. Then we have

$$0 \rightarrow H^0(E) \rightarrow E^0 \rightarrow E^1$$

so $H^0(E)$ satisfies both $\text{I}H^0(E) = 0$, $\text{I}H^n(E) = 0$ and thus $H^n(E) = 0$. Then $E^0 \hookrightarrow E^1$ and $E^0$ injective means $E^1 = E^0 \oplus E^1$. It's clear that $E^0$ is acyclic.

In greater generality, assuming only $\text{I}E_R^1 = 0$ we get $P$ a resolution of $I$ by bimodules which are good flat on the right. Then $\text{Hom}_R(I, M) \rightarrow \text{Hom}_R(P, M)$, where $\text{Hom}_R(P, M)$ is good' injective, so again $\text{Hom}_R(P, M)$ is acyclic. Thus $\text{Hom}_R(I, M)$ is acyclic whence $\text{Hom}_R(R/I, M) \rightarrow M$ is a quasi.
June 19, 1994

Some additional comments arising from the past 2 days work:

First let's check that $D(R^{I}_R \text{-mod})$ is a full subcategory of $D(R^{I}_R \text{-mod})$, i.e. if $M_1, M_2$ are exs of $R^{I}_R$-modules then

$$R \text{Hom}_{R^{I}_R}(M_1, M_2) \rightarrow R \text{Hom}_R(M_1, M_2)$$

(I think once you have that $D(R^{I}_R \text{-mod}) \rightarrow D(R^{I}_R \text{-mod})$ is fully faithful, then the essential image is $D(R^{I}_R \text{-mod})$-null, because any object in this last category can be built up a la Posnichov, the point being that fully faithful implies the $k$-invariants always lie in the essential image of $D(R^{I}_R \text{-mod})$).

Let $M_2 \rightarrow Q$ with $Q$ injective $R^{I}_R \text{-mod}$ ex. Then we showed as a consequence of $I \otimes_R I \rightarrow I$ that

$$R \text{Hom}_{R^{I}_R}(R^{I}_I, Q) \rightarrow Q$$

Now $R \text{Hom}_{R^{I}_R}(R^{I}_I, Q)$ is an injective $R^{I}_R \text{-mod}$ ex., so

$$R \text{Hom}_{R^{I}_R}(M_1, M_2) \rightarrow R \text{Hom}_R(M_1, M_2)$$

$$| \text{quis}$$

$$R \text{Hom}_{R^{I}_R}(M_1, R \text{Hom}(R^{I}_I, Q)) \rightarrow R \text{Hom}_R(M_1, Q)$$

adjunction vis.

hence the assertion.
Second idempotent functors and reflections, \( I \overset{\delta}{\otimes} R I \sim I \), etc. Defers this.

Third, note that when \( I \) is right flat, then \( j! \) is exact (because \( j! j^* M = I \otimes_R M \) is exact). Consequently if \( Q \) is an injective module then
\[
\text{Hom}_R(j! U, Q) = \text{Hom}_{R \text{-mod}/I \text{-mod}}(U, j^* Q)
\]
is exact in \( U \) showing that \( j^* \) respects injectives when \( I \) is right flat.

Similarly if \( I \) is left projective then \( j_* \) is exact. In the \( I = I^2 \) situation this is clear from \( j_*(j^* M) = \text{Hom}_R(I \delta, M) \) and \( I \delta = I \).

But it holds in general because the good modules:
\[
M \overset{\sim}{\longrightarrow} \text{Hom}_R(I, M)
\]
evidently form an abelian category with exact forgetful functor to modules. If \( P \overset{\delta}{\to} \) is a projective module, then
\[
\text{Hom}_R(P, j^* U) = \text{Hom}_{R \text{-mod}/I \text{-mod}}(j^* P, U)
\]
is an exact functor of \( U \), showing that \( j^* \) respects projectives when \( I \) is left projective.

It seems that the condition \( I \overset{\delta}{\otimes} R \overset{\sim}{\longrightarrow} I \) depends only on the nonunital ring \( I \). In effect we know this condition is equivalent to the existence of a good flat resolution \( P \) of \( I \). First of all we know that good modules depend only on \( I \). On the other good flat modules are those modules \( M \).
such that \( M = IM \) which satisfy the Cartan–Eilenberg linear equations criterion where the coefficients are in \( I \).

**June 20, 1999**

Prop. \( M \) a complex of \( R \) modules (bold below for lower indexing). Then \( R/I \otimes_R M = 0 \)
\( \iff \) \( M \) quasi to a complex \( P \) of good flat modules (bold below).

**Proof.** (\( \implies \)) \( R/I \otimes_R M = R/I \otimes_R P = R/I \otimes_R P = 0 \).

(\( \impliedby \)) We can suppose \( M \) is a complex of projective modules. Since \( M \) is flat \( R/I \otimes_R M \) is quasi to \( M/IM \), so \( M/IM \) is acyclic. Since \( M \) consists of projective modules, the map \( M \to M/IM \) is null-homotopic. Choosing a null-homotopy and lifting it to a degree 1 operator \( b: M \to M \), we obtain a map \( f = 1 - [d, b]: M \to M \) compatible with a map \( d \) which is homotopic to the identity and whose image is contained in IM. Let

\[
P = \text{lim} \{ \; M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \cdots \}
\]

Then \( P_n \), being a filtered inductive limit of free modules, is flat. Also \( f(M) \subset IM \implies IP = P \). Finally since homology commutes with filtered \( \text{lim} \)'s, we have \( H_x(P) = \text{lim} \{ H_x(M) \xrightarrow{id} H_x(M) \to \cdots \} \) so the obvious map \( M \to P \) is a quasi.
Here's a step toward Morita invariance in general. Let $A$ be a left ideal in a unital algebra $R$. Let $M$ be a good $A$-module: $A \otimes_A M \rightarrow M$. Then $M$ has a unique $R$-module structure extending the $A$-module structure: $r(ab) = (ra)b$, and this $R$-module structure is unital. The composition

$$A \otimes_A M \rightarrow AR \otimes_R M \rightarrow M$$

is an isomorphism, the first map is surjective, hence both maps are isos., showing that $M$ is an $AR$-good $R$-module.

Conversely, let $N$ be an $AR$-good $R$-module: $AR \otimes_R N \rightarrow N$. One has an exact sequence

$$0 \rightarrow K \rightarrow A \otimes_A R \rightarrow AR \rightarrow 0$$

where $KA^2 = 0$: Given $\sum a_i \otimes r_i \in K$, i.e. $\sum a_i r_i = 0$, then $\left(\sum a_i \otimes r_i\right)a' = \sum a_i r_i a \otimes a' = 0$.

Then

$$K \otimes_R N \rightarrow A \otimes_A R \otimes_R N \rightarrow AR \otimes_R N \rightarrow 0$$

$$\downarrow \cong$$

$$A \otimes_A N \rightarrow N$$

and $K \otimes_R N = K \otimes_R AN = K \otimes_R A^2 N = KA^2 \otimes_R N = 0$, showing that $N$ is a good $A$-module.
Again: I ideal in R unital. Let X be a right R-module which is I-good:

\[ X \otimes_R I \rightarrow X. \]

Consider the dense subcategory of \( \text{R-mod} \) consisting of \( M \) such that \( I^n M = 0 \) for some \( n \). I claim that \( X \otimes_R - \) inverts \( I\)-isomorphisms.

To prove this consider an \( I\)-iso \( M_1 \rightarrow M_2 \), so the kernel and cokernel are killed by some power of \( I \). To show \( X \otimes_R M_1 \rightarrow X \otimes_R M_2 \) we can factor the map into a surjection followed by an injection, so it suffices to consider these cases.

If \( M_1 \rightarrow M_2 \) with \( I^n (M_2/M_1) = 0 \), then we have a diagram with exact rows:

\[
\begin{array}{ccccccccc}
I^{(n)} \otimes_R M_1 & \rightarrow & I^{(n)} \otimes_R M_2 & \rightarrow & I^{(n)} \otimes_R (M_2/M_1) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & M_2/M_1 & \rightarrow & 0
\end{array}
\]

One sees easily that there exists a unique dotted arrow such the two triangles including it are commutative. Tensoring with \( X \) yields

\[
X \otimes_R I^{(n)} \otimes_R M_1 \rightarrow X \otimes_R I^{(n)} \otimes_R M_2
\]

\[
\cong \downarrow \quad \quad \quad \quad \quad \downarrow \cong
\]

\[
X \otimes_R M_1 \rightarrow X \otimes_R M_2
\]

where the vertical arrows are isomorphisms, hence \( X \otimes_R M_1 \rightarrow X \otimes_R M_2 \).
On the other hand if \( M_1 \rightarrow M_2 \) is surjective and its kernel \( K \) satisfies \( I^m K = 0 \), then one has

\[
\begin{array}{cccccc}
I^m \otimes K & \rightarrow & I^m \otimes M_1 & \rightarrow & I^m \otimes M_2 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & M_1 & \rightarrow & M_2 & \rightarrow & 0
\end{array}
\]

where the dotted arrow is unique such that the two triangles involving it commute. Again we conclude \( X \otimes_R M_1 \cong X \otimes_R M_2 \).

Here's a more powerful proof. For a flat right \( R \)-module \( P \) which is \( I \)-good (\( PI = P \)), the functor \( P \otimes_R \) is exact and it kills \( M \) such that \( IM = 0 \). Thus it's obvious that \( P \otimes_R \) inverts \( I \)-isomorphisms. But we know that any right \( I \)-good module \( X \) has a presentation \( P_1 \rightarrow P_0 \rightarrow X \rightarrow 0 \), where \( P_0, P_1 \) are right flat \( I \)-good. So if \( M_1 \rightarrow M_2 \) is an \( I \)-isom we have

\[
\begin{align*}
P_1 \otimes_R M_1 & \rightarrow P_0 \otimes_R M_1 \rightarrow X \otimes_R M_1 \rightarrow 0 \\
\downarrow & \cong & \downarrow & \cong & \downarrow \\
P_1 \otimes_R M_2 & \rightarrow P_0 \otimes_R M_2 \rightarrow X \otimes_R M_2 \rightarrow 0
\end{align*}
\]

and it's clear.

This proof is more powerful because it shows that \( X \otimes_R \) inverts a larger class of maps.

Specifically, let us consider three Serre subcategories \( D_i \) for \( i = 0, 1, 2 \) of \( R \)-mod defined as follows.

\( D_0 \) is the category \( I \) above consisting of \( M \in \text{I} \Delta = 0 \)
for some \( n \),

\( S_1 \) is the category of \( \mathcal{T} \)-torsion modules considered previously. Thus \( S_1 \) consists of \( M \) such that \( \text{Hom}_R(M,E) = 0 \) for all injective modules \( E \) such that \( \mathcal{T} E = 0 \). Alternatively \( M \in S_1 \) when the transfinitely defined filtration

\[
F^{x+1}M = \{ m \in M \mid \text{Im} c F^x M \}
\]

\[
F^x M = \bigcup_{\alpha < \alpha} F^\alpha M
\]

exhausts \( M \). Alternatively, for all submodules \( N < M \),

\[
\{ m \mid \text{Im} c N \}
\]

\( S_2 \) is the category consisting of \( M \) such that \( P \otimes_R M = 0 \) for all flat \( \mathcal{T} \)-good right modules \( P \). It suffices to take \( P \) to be a generating flat \( \mathcal{T} \)-good module.

We have the following inclusions

\[
S_0 < S_1 < S_2
\]

Check: Let \( M \in S_1 \), let \( N \) be the largest submodule of \( M \) such that \( P \otimes_R N = 0 \). If \( N < M \) then \( N' = \{ m \in M \mid \text{Im} c N \} \) is \( > N \) and

\[
P \otimes_R N \to P \otimes_R N' \to P \otimes_R (N'/N) \to 0
\]

so we have a contradiction showing \( P \otimes_R M = 0 \), and \( M \in S_2 \).

Notice that \( S_1, S_2 \) are closed under direct sums, hence they are localizing \( \mathcal{T} \) subcategories, i.e. torsion theories.
Let $\mathcal{A} = R$-mod, $J_0 = \ker$-subcategory of $M \in \mathcal{F}$, $I^\infty M = 0$. Let's calculate the maps in $\mathcal{A}/J_0$.

Quite generally one has

$$\text{Hom}_{\mathcal{A}/J_0}(M_1, M_2) = \lim_{\rightarrow} \text{Hom}_{\mathcal{A}}(N_1, M_2/N_2)$$

$$= \lim_{\rightarrow} \text{Hom}_{\mathcal{A}}(M', M_2)$$

$$= \lim_{\rightarrow} \text{Hom}_{\mathcal{A}}(M_1, M'')$$

where the first formula has the advantage that the category over which the limit is taken is small (a directed set in fact).

I claim

$$\text{Hom}_{\mathcal{A}/J_0}(M_1, M_2) = \lim_{\rightarrow} \text{Hom}_R(I'(n) \otimes_R M_1, M_2)$$

$$= \lim_{\rightarrow} \text{Hom}_R(M_1, \text{Hom}_R(I'(n), M_2))$$

We just have to check that the objects $\{I'(n) \otimes_R M \rightarrow M_1, n \geq 0\}$ are cofinal in the filtering category of $\text{iso} \cdot M' \rightarrow M$. The dual assertion results by adjointness.

So given an $\text{iso} \cdot M' \rightarrow M$ factor it into surjection followed by injection. On pp 640-641
We've seen there are dotted arrows as follows

\[
\begin{array}{ccc}
I & \rightarrow & \otimes = R M \\
\downarrow & & \\
I^m_R M & \rightarrow & I^{n_2}_R M \\
\downarrow & & \\
M' & \rightarrow & M'' & \rightarrow & M \\
\end{array}
\]

Filling in the top by applying \(I^{(n_2)}_R\) and naturality we win.

Next let's check that the functors \(I\otimes R\) and \(\text{Hom}_R(I, -)\) on \(A\) descend to \(A/\mathfrak{a}_0\) and are inverse. \(\otimes_{\mathfrak{a}_0} : M_1 \rightarrow M_2\) is an \(\mathfrak{a}_0\)-isom. Then

\[
\begin{array}{ccc}
I \otimes_{\mathfrak{a}_0} M_1 & \rightarrow & I \otimes_{\mathfrak{a}_0} M_2 \\
\downarrow_{\text{So-isom}} & & \downarrow_{\text{So-isom}} \\
M_1 & \rightarrow & M_2
\end{array}
\]

shows that \(I \otimes_{\mathfrak{a}_0} M_1 \rightarrow I \otimes_{\mathfrak{a}_0} M_2\) is an \(\mathfrak{a}_0\)-isom. Thus \(I \otimes_{\mathfrak{a}_0}\) descends to \(\mathfrak{a}_0\) and similarly for \(\text{Hom}_R(I, -)\).

Next we have

\[
\begin{array}{ccc}
M & \rightarrow & \text{Hom}_R(I, M) \\
\downarrow_{\text{So-isom}} & & \downarrow_{\text{So-isom}} \\
I \otimes_{\mathfrak{a}_0} M & \rightarrow & I \otimes_{\mathfrak{a}_0} \text{Hom}_R(I, M) \\
\downarrow_{\text{So-isom}} & & \downarrow \\
M & \rightarrow & M
\end{array}
\]

and

\[
\begin{array}{ccc}
\text{Hom}_R(I, I \otimes_{\mathfrak{a}_0} M) & \rightarrow & \text{Hom}_R(I, M) \\
\downarrow_{\text{So-isom}} & & \downarrow \\
M & \rightarrow & M
\end{array}
\]

\[
\begin{array}{ccc}
\chi \otimes m & \rightarrow & \chi \otimes (y_1 \otimes y_m) \\
\downarrow & & \downarrow \chi m \\
\chi m & \rightarrow & (\chi \otimes \chi m)
\end{array}
\]
Thus we have canonical isomorphisms

$$\text{I} \otimes_R \text{Hom}_R(I, M) \longrightarrow M$$

$$M \longrightarrow \text{Hom}_R(I, I \otimes_R M)$$

showing the functors $I \otimes_R -$ and $\text{Hom}_R(I, -)$ are inverse on $Afs_0$. 
For the proof of Morita equivalence we need to replace the ideal $I$ by certain bimodules, something like $I \otimes R I$, which need not be an ideal in $R$.

Consider pairs $(L, \alpha)$ where $L$ is an $R$-bimodule equipped with a bimodule map $\alpha : L \to R$ satisfying $\alpha(l_1) l_2 = l_1 \alpha(l_2)$. We note that the image $\text{im} \alpha$ is an ideal in $R$, and the kernel $\text{Ker} \alpha$ is an $R$-bimodule killed on both sides by the ideal $\text{im} \alpha$:

$\alpha(l_1) \in \text{im} \alpha, \ l_2 \in \text{Ker} \alpha \implies \alpha(l_1) l_2 = l_1 \alpha(l_2) = 0$.

Given two pairs $(L, \alpha)$, $(L', \alpha')$ of this sort, their tensor product is $(L \otimes R', \alpha \otimes \alpha')$, where $\alpha(l \otimes l') = \alpha(l) \alpha(l')$. This map $\alpha : L \otimes R' \to R$ is a well-defined $R$-bimodule map:

$\alpha(l r) \alpha(l') = \alpha(l) \alpha(l') = \alpha(l) \alpha(r l')$

$\alpha(r \otimes l') = \alpha(r l') = r \alpha(l) \alpha(l') = r \alpha(l \otimes l')$

and similarly for right multi. Finally:

$\alpha(l_1 \otimes l'_1) l_2 \otimes l'_2 = \alpha(l_1) \alpha(l'_1) l_2 \otimes l'_2$

$= l_1 \otimes \alpha(l'_1) l_2 \otimes l'_2$

$= l_1 \otimes \alpha(l'_1) \alpha(l_2) l'_2$

$= l_1 \otimes \alpha(l'_1) \alpha(l_2) l'_2$

$= l_1 \otimes l'_1 \alpha(l_2) \alpha(l'_2)$

$= l_1 \otimes l'_1 \alpha(l_2) \alpha(l'_2)$

$= l_1 \otimes l'_1 \alpha(l_2) \alpha(l'_2)$

Given $(L, \alpha)$ we can form an inverse system of bimodules.
\[
\rightarrow \bigotimes_R L \otimes_R L \rightarrow \bigotimes_R L \rightarrow L \rightarrow R
\]

as follows: Note that the condition \( \partial(l_1) l_2 = l_1 \partial(l_2) \) means that the possible face operators
\[
\partial_i : \bigotimes_R^n \rightarrow \bigotimes_R^{n-1} \quad (l_1, \ldots, l_n) \mapsto (l_1, \ldots, \partial_i(l_i), \ldots, l_n)
\]
coincides:
\[
\partial_i(l_1, \ldots, l_n) = (l_1, \ldots, l_i, \partial_i(l_i), l_{i+1}, \ldots, l_n)
= (l_1, \ldots, l_i, l_i \partial(l_{i+1}), \ldots, l_n)
= \partial_i(l_1, \ldots, l_n)
\]

We want to check now that if \((L, \partial)\) is a pair as above such that the ideal \(\partial(L)\) defines the same adic topology as \(I\), i.e.,
\(\partial(L)^n \subset I\), \(I^n \subset \partial(L)\) for some \(n\), then maps in the category \(\text{R-mod}/\{M \mid \exists n, I^n M = 0\}\) are given by
\[
\lim_n \text{Hom}_R(\bigotimes_R^n, M_1, M_2)
= \lim_n \text{Hom}_R(M_1, \text{Hom}_R(\bigotimes_R^n, M_2))
\]

It suffices to check that the maps \(\bigotimes_R M \rightarrow M\) are cofinal in the category of \(I\)-null isomorphisms \(M' \rightarrow M\) with target \(M\). First check that for any module \(M\) the map \(\bigotimes_R M \rightarrow M\) (given by \(\partial\)) is an \(I\)-null isomorphism. (I should have earlier mentioned that we can assume \(\partial L = I\).) We have exact sequences
\[
0 \rightarrow K \rightarrow L \rightarrow I \rightarrow 0
\]
\[
0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0
\]
where \(K\) is killed by \(I\). Then
$K \otimes_R M \rightarrow L \otimes_R M \rightarrow I \otimes_R M \rightarrow 0$

$0 \rightarrow K' \rightarrow I \otimes_R M \rightarrow M \rightarrow M/I \cong M \rightarrow 0$

where $I \cdot K' = 0$ and $I(M/I) = 0$. This shows that $L \otimes_R M \rightarrow M$ is an isomorphism mod $I$-nil.

Next suppose $M' \rightarrow M$ is an $I$-nil isom. and factor it $M' \rightarrow M'' \rightarrow M$; let $K_1$ be the kernel of $M' \rightarrow M''$. For $n$ large $I^n$ kills $K_1$ and $M/M''$ the two dotted arrows exist in

$\begin{align*}
\mathcal{L}_R^n \otimes_R K_1 & \rightarrow \mathcal{L}_R^n \otimes_R M' \rightarrow \mathcal{L}_R^n \otimes_R M'' \rightarrow 0 \\
0 & \rightarrow K_1 \rightarrow M' \rightarrow M'' \rightarrow 0
\end{align*}$

and $\mathcal{L}_R^n \otimes_R M'' \rightarrow \mathcal{L}_R^n \otimes_R M \rightarrow \mathcal{L}_R^n \otimes_R M/M'' \rightarrow 0$

keeping these diagrams commutative. Then

$\begin{align*}
\mathcal{L}_R^n \otimes_R M' & \rightarrow \mathcal{L}_R^n \otimes_R M'' \rightarrow \mathcal{L}_R^n \otimes_R M \\
\downarrow & \downarrow \downarrow L \otimes_R g_2 \\
\mathcal{L}_R^n \otimes_R M' & \rightarrow \mathcal{L}_R^n \otimes_R M'' \rightarrow \mathcal{L}_R^n \otimes_R M
\end{align*}$

$\begin{align*}
M' & \rightarrow M'' \rightarrow M
\end{align*}$

So it works.
June 23, 1997

\[ R = T(V) = \bigoplus_{n \geq 0} V^\otimes n, \quad I = \bigoplus_{n > 0} V^\otimes n = T^>(V) \]

An \( R \)-module is the same as a vector space \( M \) equipped with a linear map \( V \otimes M \rightarrow M \).

\( M \) is I-solid \( \iff \quad V \otimes M \rightarrow M \)

\( M \) is I-cosolid \( \iff \quad M \rightarrow \text{Hom}(V, M) \)

Suppose \( V \) finite-dimensional \( \neq 0 \), let \( x_i \) be a basis for \( V \), \( y_i \) the dual basis for \( V^* \).

A solid \( M \) is the same as a module over

\[ \mathcal{O}_V = T(V \oplus V^*) \bigm/ \langle y \mid x \rangle \]

\[ \sum_i x_i y_i = 1 \]

A cosolid \( M \), i.e. \( M \rightarrow V^* \otimes M \) is the same as a module over

\[ \mathcal{O}_{V^*} = T(V \oplus V^*) \bigm/ \langle y \mid x \rangle \]

\[ \sum_i y_i x_i = 1 \]

Note that

- solid \( \Leftrightarrow \) \( T^>(V) \)-modules = cosolid \( \Leftrightarrow \) \( T^>(V^*) \)-modules

- solid \( \Leftrightarrow \) \( \Leftrightarrow \) cosolid

Recall what we learned about the Cuntz-Krieger algebra \( \mathcal{O}_E \). Here \( A \) is a unital algebra (ring), \( E \) a unital bimodule over \( A \) which is a finitely generated projective generator for \( \text{A-mod} \).

Then

\[ \mathcal{O}_E = T_A(E \oplus E^*) \bigm/ \langle y \mid x \rangle \]

\[ \sum_i x_i y_i = 1 \]

where \( E^* = \text{Hom}_{\text{A-op}}(E, A) \), \( \langle y \mid x \rangle \) is the canonical
map \( E^* \otimes_A E \rightarrow A \) and

\[ \sum_{x_i \otimes y_i} \in E \otimes_A E^* \sim \text{Hom}_{\text{Rep}(E,E)} \]

gives the identity operator on \( E \). The ring \( \mathbb{Z} \mathcal{O}_E \)
is graded \( \mathcal{O}_E = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_E^n \) and

\[ \mathcal{O}_E^{-1} \mathcal{O}_E = \mathcal{O}_E \mathcal{O}_E^{-1} = \mathcal{O}_E \]

so that \( \mathcal{O}_E^{-1} = \mathcal{O}_E \mathcal{O}_E^{-1} \mathcal{O}_E^0 \). Better to say that \( \mathcal{O}_E \) is the
\( \mathbb{Z} \)-graded tensor algebra on the invertible bimodule \( \mathcal{O}_E \) over \( \mathcal{O}_E^0 \).

Go back to \( R = T(V) \), \( I = T^{>0}(V) \otimes \mathcal{M}_2 \)

know in this case that the solid and cosolid
modules form abelian categories which may be identified with \( \mathcal{O}_V \)-mod
and \( \mathcal{O}_V^* \)-mod respectively. Also the forgetful
functors to \( R \)-modules are exact. These forgetful
functors are restriction of scalars associated to
canonical homomorphisms

\[ T(V) \rightarrow \mathcal{O}_V \quad T(V) \rightarrow \mathcal{O}_V^* \]

\[ \text{Hence the inclusion functors} \]

\[ I\text{-solid} \subset R\text{-mod} \quad I\text{-cosolid} \subset R\text{-mod} \]

both have left and right adjoints.
let's make some general observations
that should apply at least in the case
\( R = T(V), \ I = T^0(V). \)

Recall \( M \) is \( I \)–solid when \( M \to \operatorname{Hom}_R(I, M). \)
Given any \( R \)–module \( M \) we have an inductive
system \( M \to \operatorname{Hom}_R(I, M) \to \operatorname{Hom}_R(I^{\otimes 2}, M) \to \cdots \)
and we can take the inductive limit. When \( I \)
is finitely presented as \( R \)–module we know
\( \operatorname{Hom}_R(I, -) \) commutes with filtered \( \operatorname{lim} \)'s and so

\[
\begin{align*}
\operatorname{Hom}_R(I, \operatorname{lim}_n \operatorname{Hom}_R(I^{\otimes n}, M)) \\
= \operatorname{lim}_n \operatorname{Hom}_R(I, \operatorname{Hom}_R(I^{\otimes n}, M)) \\
= \operatorname{lim}_n \operatorname{Hom}_R(I^{\otimes n}, M)
\end{align*}
\]

showing that \( \operatorname{lim}_n \operatorname{Hom}_R(I^{\otimes n}, M) \) is \( I \)–solid.
Moreover the canonical map \( \lim \) from \( M \) to this
limit is an isomorphism modulo \( I \)–torsion modules.
Thus the localization functor when \( I \) is finitely
presented left \( R \)–module is
\( M \to \operatorname{lim}_n \operatorname{Hom}_R(I^{\otimes n}, M) \)

Now suppose \( I \) is finitely generated projective
as \( R \)–module. Then the same is true for
\[
I^{\otimes n} = I \otimes_R I \otimes_R \cdots \otimes_R I
\]
and
\[
\operatorname{Hom}_R(I^{\otimes n}, M) = \operatorname{Hom}_R(I^{\otimes n}, R) \otimes_R M
\]
so
\[
\operatorname{lim}_n \operatorname{Hom}_R(I^{\otimes n}, M) = \left( \operatorname{lim}_n \operatorname{Hom}_R(I^{\otimes n}, R) \right) \otimes_R M
\]
Take now \( R = T(V) \), \( I = T^{>0}(V) \).

Then \[ I^{\otimes n} = R \otimes_V (R \otimes V) \otimes_R \cdots \otimes_R (R \otimes V) = R \otimes V^{\otimes n} \]

as left \( R \)-module, so

\[ \text{Hom}_R(I^{\otimes n}, M) = \text{Hom}_R(V^{\otimes n}, M) = V^{\otimes n} \otimes_R M \]

and the localization functor is

\[ M \mapsto \lim_n \left( M \rightarrow V^{\otimes M} \rightarrow V^{\otimes V^{\otimes M}} \rightarrow \cdots \right) \]

\[ = \lim_n \left( T(V) \rightarrow V^{\otimes T(V)} \rightarrow V^{\otimes 2 \otimes T(V)} \rightarrow \cdots \right) \otimes_R M \]

This should be \( \mathcal{O}_V \).

Next consider the solid case. Given any \( R \)-module \( M \) one has an inverse system

\[ M \leftarrow I_{R,M} \leftarrow I^{\otimes 2}_{R,M} \leftarrow \cdots \]

and one can take the inverse limit. When \( I \)

is finitely generated projective as a right \( R \)-module \( I_{R} \) — commutes with \( \lim_n \)'s, so the functor

\((x)
\[ M \mapsto \lim_n I^{\otimes n}_{R,M} \]

should be right adjoint to the inclusion of solid modules in \( R \)-modules.

It seems like there is some kind of Cuntz–Krieger algebra here in the case of an ideal \( I \subset R \) which is finitely generated projective as right \( R \)-module. Call this algebra \( \mathcal{O}_I \). Its desired property is that there's a homomorphism \( R \rightarrow \mathcal{O}_I \)
such \( \mathcal{O}_I \)-modules are equivalent to \( I \)-solid \( R \)-modules via restriction of scalars.
Assuming I fin. gen. projective as right $R$-module we have

$$I \otimes_R M = \text{Hom}_R (I^*, M)$$

where $I^* = \text{Hom}_{\text{op}} (I, R)$ is the right dual of $I$. Then

$$I \otimes_R I \otimes_R M = \text{Hom}_R (I^*, I \otimes R M)$$

$$= \text{Hom}_R (I^*, \text{Hom}_R (I^*, M))$$

$$= \text{Hom}_R (I^* \otimes_R I^*, M)$$

We know the right adjoint functor to the inclusion $I$-solid $\hookrightarrow R$-mod is

$$M \mapsto \lim_{\longrightarrow} I \otimes_R^n M$$

$$= \lim_{\longrightarrow} \text{Hom}_R \left( I^* \otimes_R^n, M \right)$$

$$= \text{Hom}_R \left( \lim_{\longrightarrow} I^* \otimes_R^n, M \right)$$

Now this should be $\text{Hom}_R (\mathcal{O}_I, M)$ if the inclusion $I$-solid $\hookrightarrow R$-mod is restriction of scalars associated to a homm. $R \rightarrow \mathcal{O}_I$. Thus we should have

$$\mathcal{O}_I = \lim_{\longrightarrow} I^* \otimes_R^n = \lim_{\longrightarrow} \text{Hom}_R (I \otimes_R^n, R)$$

In the case $R = T(V)$, $I = T^* (V)$, then

$$\text{Hom}_{\text{op}} (I \otimes_R^n, R) = \text{Hom}_{\text{op}} (\otimes V^* \otimes^n, R) = R \otimes V^* \otimes^n$$

so that $\mathcal{O}_I = \lim_{\longrightarrow} T(V) \otimes V^* \otimes^n$. This should be the Cuntz $\mathcal{O}_V$, while the algebra encountered with cosolid modules is $\lim_{\longrightarrow} \text{Hom}_R (I \otimes_R^n, R)$. So the
difference between these is whether
we use left or right duals.

Here's some comments to make the preceding
a bit clearer.

Assuming $I$ finitely generated projective as right
module an $R$-module $M$ is solid: $I \otimes_R M \to M$
iff it is a module over $\varinjlim \text{Hom}_R (I^\otimes n, R)$.

Assuming $I$ fin. gen. proj. as left module an
$R$-module $M$ is cosolid: $M \to \varinjlim \text{Hom}_R (I, M)$ iff it
is a module over $\varprojlim \text{Hom}_R (I^\otimes n, R)$.

Put $I_\wedge^* = \text{Hom}_R (I, R)$, $I_\vee^* = \text{Hom}_R (I, R)$.
Then $M$ solid means one has both $R$ and
$I_\wedge^*$ mapping into $\text{Hom}_R (M, M)$. Also $M$ cosolid
means $M \to I_\vee^* \otimes_R M$ so that
both $R$ and $I_\wedge^*$ map into $\text{Hom}_R (M, M)$.

Q: Is it possible to construct other $R$-algebras
$R \to \mathcal{O}$ by combining the natural transformation
$I \otimes_R M \to M$ and $M \to \text{Hom}_R (I, M)$ to get
something which is inverted exactly in $\mathcal{O}$?

One could hope for enough $\mathcal{O}$ to form the
analogue of an open affine covering of a projective
scheme.
June 25, 1994

Recall

\[ \text{Hom}_{R\text{-mod/\text{-nilp}}} (M_1, M_2) = \lim_{\longrightarrow} \text{Hom}_R (I^\otimes R^n, M_1, M_2) \]

\[ = \lim_{\longrightarrow} \text{Hom}_R (M_1, \text{Hom}_R (I^\otimes R^n, M_2)) \]

Of particular interest is the ring

\[ \mathcal{O} = \text{Hom}_{R\text{-mod/\text{-nilp}}} (R, R) = \text{Hom}_{R\text{-mod/\text{-nilp}}} (I, I) \]

\[ = \lim_{\longrightarrow} \text{Hom}_R (I^\otimes R^n, R) = \lim_{\longrightarrow} \text{Hom}_R (I^\otimes R^n, I) \]

of endomorphisms of the canonical generator \( R \cong I \) for the category \( R\text{-mod/\text{-nilp}} \).

Recall that this ring \( \mathcal{O} \) depends only upon the monomial ring \( I \).

Consider the case \( I = I^2 \). Then

\[ \mathcal{O} = \text{Hom}_R (I \otimes_R I, I) \cong \text{Hom}_R (I \otimes_R I, I \otimes_R I) \]

At this point it would probably be best to adopt the monomial ring viewpoint, favoring in the terminology suggested by Hassewolde.

Let \( I \) be a freering such that \( I^2 = I \), let \( A = I \otimes_R I \) be its canonical solid extension, so that \( I = \frac{1}{I} A/K \), where \( K \) is a null ideal in \( A \). Better: Recall that a freering \( I \) such that \( I^2 = I \) can be canonically written \( I = A/K \) where \( A \) is a solid ring; \( A \otimes_R A \cong A \) and \( K \) is a null ideal in \( A \).
Before continuing with the analysis preceding, it is worthwhile discussing examples of Morita equivalences. The basic result is the following:

Consider a Morita context

\[
\begin{pmatrix}
R & Q \\
P & S
\end{pmatrix}
\]

that is, a unital ring with a 2x2 block matrix decomposition (equivalently a unital ring together with an idempotent element).

Let \( I = QP \subset R \), \( J = PQ \subset S \). These are ideals of \( R \) and \( S \) respectively. Then there are equivalences of categories:

1) \( \begin{align*}
R\text{-mod} \quad & \cong \quad S\text{-mod} \\
I\text{-nilp} \quad & \cong \quad J\text{-nilp}
\end{align*} \)

\[ M \longrightarrow P \otimes_R M \quad \quad \quad q \otimes N \longleftarrow N \]

2) \( (R,I)\text{-solid} \cong (S,J)\text{-solid} \)

3) \( (R,I)\text{-cosolid} \cong (S,J)\text{-cosolid} \)

\[ M \longrightarrow \text{Hom}_R(Q,M) \quad \text{Hom}_S(P,N) \longleftarrow N \]

Examples.

1. Suppose \( S \subset R \) is an inclusion of unital rings, let \( J \) be an ideal in \( S \) which is also a left ideal in \( R \): \( RJ \subseteq J \), and let \( I = JR \) be the ideal in \( R \) generated by \( J \). Then we have a Morita context

\[
\begin{pmatrix}
R & Q \\
P & S
\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} R & J \\
R & S \end{pmatrix} \subseteq \begin{pmatrix} R & R \\
R & R \end{pmatrix}
\]
\[
\begin{pmatrix}
R & J \\
R & S
\end{pmatrix}
\begin{pmatrix}
R & J \\
R & S
\end{pmatrix} =
\begin{pmatrix}
R^2 + JR & RJ + JS \\
R^2 + SR & RJ + S^2
\end{pmatrix} =
\begin{pmatrix}
R & J \\
R & S
\end{pmatrix}
\]

and \( QP = JR = \text{the ideal } I \text{ in } R \)

\( PQ = RJ = \text{the ideal } J \text{ in } S \).

Thus we have the equivalence of categories

\[ M \stackrel{\sim}{\longrightarrow} R \otimes_R M = M \]

restriction of scalars from \( R \) to \( S \)

\[ J \otimes_S N \leftarrow \hspace{1em} N \]

which puts an \( R \)-module structure on any \( J \)-solid \( N: \hspace{1em} J \otimes_S N \rightarrow N \).

Special case: If \( J \) is an ideal in \( S \), then \( I = JR = J \), so we find (at least in the case \( S \subset R \)) that the three categories assoc. to \( (B, J) \) and \( (R, J) \) are the same.

2. Suppose \( S \subset R \) as above, \( J \) an ideal in \( S \) which is a right ideal in \( R \); let \( I = RJ \). Then we have a Monta context

\[
\begin{pmatrix}
R & Q \\
P & S
\end{pmatrix}
\begin{pmatrix}
R & R \\
J & S
\end{pmatrix} \subseteq M_2(R)
\]

Check:

\[
\begin{pmatrix}
R & R \\
J & S
\end{pmatrix}
\begin{pmatrix}
R & R \\
J & S
\end{pmatrix} \subseteq \begin{pmatrix}
R^2 + RJ & R^2 + RS \\
JR + SJ & JR + S^2
\end{pmatrix} = \begin{pmatrix}
R & R \\
J & S
\end{pmatrix}
\]

\( QP = RJ = I \)

\( PQ = RJ = J \)

so this time the equivalence is given by

\[ M \stackrel{\sim}{\longrightarrow} J \otimes_R M \]

\[ R \otimes_S N \leftarrow N \]

case extension from \( S \) to \( R \).
3. Suppose $R/K = S$, the ideal $I$ in $R$ is such that $KI = 0$, and $J$ is the image of $I$ in $S$. Then we have an ideal 
\[
\begin{pmatrix}
0 & K \\
0 & K \\
0 & K
\end{pmatrix} \subseteq \begin{pmatrix}
R & R \\
I & R
\end{pmatrix}
\]

Check:
\[
\begin{pmatrix}
0 & K \\
0 & K
\end{pmatrix} \begin{pmatrix}
R & R \\
I & R
\end{pmatrix} = \begin{pmatrix}
KI & KR \\
KI & KR
\end{pmatrix} = \begin{pmatrix}
0 & K \\
0 & K
\end{pmatrix}
\]
\[
\begin{pmatrix}
R & R \\
I & R
\end{pmatrix} \begin{pmatrix}
0 & K \\
0 & K
\end{pmatrix} = \begin{pmatrix}
0 & RK \\
0 & JK + RK
\end{pmatrix} = \begin{pmatrix}
0 & K
\end{pmatrix}
\]

where we obtain a Morita context:
\[
\begin{pmatrix}
R & Q \\
P & S
\end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix}
R & S \\
I & S
\end{pmatrix} = \begin{pmatrix}
R & R \\
I & R
\end{pmatrix} \bigg/ \begin{pmatrix}
0 & K
\end{pmatrix}
\]

Then $QP = SI = (R/K).I = RI = I$

$PQ = IS = (\text{Image of } I \text{ in } S)S = JS = J$

so the equivalence is given by

\[M \rightarrow I \otimes_R M\] which means for $M$ solid

that $M$ is killed by $K$

hence $M$ is an $S$-module

\[N = S \otimes_S N \leftarrow I N\] restriction of scalars from $S$ to $R$.

Special case: If $I \otimes_K = 0$ so that $I \cong J$,

then we get for a surjection $R \twoheadrightarrow S$ the

independence of the good categories on the embedding as an ideal in a unital algebra.
4. Suppose $R/K = S$, $I$ is an ideal in $R$ such that $IK = 0$, let $J$ be the image of $I$ in $S$. Then we have an ideal:

$$
\begin{pmatrix}
0 & 0 \\
K & K
\end{pmatrix} \subseteq \begin{pmatrix}
R & I \\
R & R
\end{pmatrix}
$$

Check:

$$
\begin{pmatrix}
0 & 0 \\
K & K
\end{pmatrix} \begin{pmatrix}
R & I \\
R & R
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
KR & KI + KR
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
K & K
\end{pmatrix}
$$

$$
\begin{pmatrix}
R & I \\
R & R
\end{pmatrix} \begin{pmatrix}
0 & 0 \\
K & K
\end{pmatrix} = \begin{pmatrix}
IK & IK \\
RK & RK
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
K & K
\end{pmatrix}
$$

hence we obtain a Morita context:

$$
\begin{pmatrix}
R & Q \\
P & S
\end{pmatrix} \equiv \begin{pmatrix}
R & I \\
S & S
\end{pmatrix} = \begin{pmatrix}
R & I \\
R & R
\end{pmatrix}/(0, 0)
$$

Then $QP = IS = I \cdot (R/K) = I$

$PQ = SI = S(\text{Image of } I \text{ in } S) = ST = J$

so the equivalence is given by

$$M \mapsto S \otimes_R M \quad \text{base extn from } R \to S$$

$I \otimes_S N \leftarrow \leftarrow N$
June 26, 1999

Let \( A = A \otimes_A A \) be a solid ring, let \( K \subseteq A \) be an ideal and let \( B = A/K \).

Let's try to relate the right multiplier \( \bullet \) rings \( \text{Hom}_A(A, A) \) and \( \text{Hom}_B(B, B) \).

First we have \( \text{Hom}_B(B, B) = \text{Hom}_A(B, B) \) since \( A \) maps onto \( B \).

Next from the exact sequence

\[
0 \rightarrow K \rightarrow A \rightarrow B \rightarrow 0
\]

one gets

\[
0 \rightarrow \text{Hom}_A(B, B) \rightarrow \text{Hom}_A(A, B) \rightarrow \text{Hom}_A(K, B)
\]

Because \( A \) is solid as \( A \)-module we have

\[
\text{Hom}_A(A, B) \cong \text{Hom}_A(A, A \otimes A B)
\]

solidification of \( B \).

One has also the exact sequence

\[
A \otimes_A K \rightarrow A \otimes_A A \rightarrow A \otimes_A B \rightarrow 0
\]

so that \( A/\text{AK} \cong A \otimes_A B \).

Now take \( K = \text{ann}(A_0) = \{ a | Aa = 0 \} \) largest ideal such that \( AK = 0 \). Then any \( A \)-module map \( f : K \rightarrow B \) has image in \( \{ b \in B | Ab = 0 \} \).

Let \( \pi : A \rightarrow B \) be the canonical surj, suppose \( A \pi(a) = 0 \), i.e. \( \pi(Aa) = 0 \), or \( Aa \subseteq K \). Then \( Aa = A^2 a \subseteq AK = 0 \), so as \( K \) and we conclude \( \text{Hom}_A(K, B) = 0 \).
Thus we have

\[ \text{Hom}_B(B, B) = \text{Hom}_A(B, B) \]

\[ \sim \text{Hom}_A(A, B) \]

\[ \sim \text{Hom}_A(A, A \otimes_A B) \]

\[ \sim \text{Hom}_A(A, A) \]

Something slightly more efficient is to note that since \( AK = 0 \), the surjection \( A \rightarrow B \) is an isomorphism modulo \( A \)-null, hence as \( A \) is solid \( \text{Hom}_A(A, A) \sim \text{Hom}_A(A, B) \).

Here's a check on the above calculation:

Since \( AK = 0 \) we know the equivalence between \( A \)-solid and \( B \)-solid is \( M \rightarrow B \otimes_A M = M/KM \), so that the \( A/KA \) is the \( B \)-solid module corresponding to the \( A \)-solid module \( A \). Then the surjection \( A/KA \rightarrow B \) is an isomorphism modulo \( B \)-null.

But because \( B \cdot B = B \) and \( B = 0 \) we know \( \text{Hom}_{B \text{-mod}}(B, B) = \text{Hom}_B(B, B) \).

In fact this argument can be carried out for an arbitrary \( A \) such that \( A = A^2 \). Namely let \( K = A \) = \( \{a \in A \mid Aa = 0\} \), so that \( AK = 0 \). Then \( A \rightarrow A/K \) is an isomorphism mod \( A \)-null so

\[ \text{Hom}_{A \text{-mod}/A \text{-null}}(A, A) = \text{Hom}_{A \text{-mod}/A \text{-null}}(A/K, A/K) \]

\[ = \text{Hom}_A(A/K, A/K) \] \( \{ \text{since } A \cdot A/K = A/K \) and \( A(A/K) = 0 \)

\[ = \text{Hom}_{A/K}(A/K, A/K) \]
Return to $A = A \otimes_A A$ and let $J$ be any ideal in $A$ such that $AJ = 0$. We want to compare $\text{Hom}_A(A,A)$ with $\text{Hom}_A(A/J,A/J) = \text{Hom}_A(A/J,A/J)$. We have the exact sequence

$$0 \rightarrow \text{Hom}_A(A/J,A/J) \rightarrow \text{Hom}_A(A,A/J) \rightarrow \text{Hom}_A(J,A/J) \rightarrow \text{Hom}_A(A,J).$$

(Since $A$ is solid and $A \rightarrow A/J$ is the mod $A/J$).

Thus

$$\text{Hom}_A(A/J,A/J) \rightarrow \{ \theta \in \text{Hom}_A(A,A) \mid \theta(J) \subset J \}$$

We would now like an example of a solid ring $A$ and $J$ such that $AJ = 0$ where $\text{Hom}_A(A,A)$ does not preserve $J$. Start with any solid ring $B$ such that $B \neq 0$ and put $A = B \oplus B$.

Take $J = \Delta K \subset B \oplus B$, where $\Delta K = \{(a,0) \mid a \in K\}$.

Take $\theta = (1_0)$ on $B \oplus B$.

Further discussion of quotients $A/K = B$ of a ring $A$ (such that $A = A^2$ if necessary).

First let's give the precise formulation of the two Morita equivalence cases:

For $KA = 0$:

$$M \rightarrow A \otimes_A M$$

$$N = B \otimes_B N \leftarrow N$$

$$\begin{pmatrix} A & \tilde{B}_B \\ B_A & B \end{pmatrix} \quad QP = \tilde{B}A = A$$

$$PA = \tilde{B}B = B$$

For $AK = 0$:

$$M \rightarrow \tilde{B} \otimes_A M = M/\text{KM}$$

$$A \otimes_B N \leftarrow N$$

$$\begin{pmatrix} A & \tilde{B} \\ B & B \end{pmatrix} \quad QP = A\tilde{B} = A$$

$$PQ = \tilde{B}A = B$$
Combining these we have the case $\text{Ann}(A) = 0$.

Check that $(0 \ AK) < (A \ KA)$ is an ideal.

\[
\begin{pmatrix}
0 & AK \\
KA & K
\end{pmatrix}
\begin{pmatrix}
\tilde{A} & A \\
A & \tilde{A}
\end{pmatrix}
= \begin{pmatrix}
0 + AK\tilde{A} & AK\tilde{A} \\
KA\tilde{A} + KA & KA\tilde{A} + \tilde{A}
\end{pmatrix}
= \begin{pmatrix}
0 & AK \\
KA & K
\end{pmatrix}
\]

Thus we get a Morita context

\[
\begin{pmatrix}
\tilde{A} & A/KA \\
A/KA & \tilde{A}/K
\end{pmatrix}
= \begin{pmatrix}
\tilde{A} & A/KA \\
A/KA & \tilde{A}/K
\end{pmatrix}
\]

Note the following: Given $A$ let $\text{Ann}(A)$ = \{ $a \in A$ | $aA = 0$ \}, $\text{Ann}_n(A)$ = \{ $a \in A$ | $AA = 0$ \}. Let $\pi : A \to A/\text{Ann}_n(A)$ be the canonical surjection.

Suppose $a \in A$ such that $\pi(a) \in \text{Ann}_n(A/\text{Ann}_n(A))$, that is, $(A/\text{Ann}_n(A))\pi(a) = 0$, equivalently $AA = 0$. Thus

\[
\pi^{-1}(\text{Ann}_n(A/\text{Ann}_n(A))) = \{ a \mid AA = 0 \}
\]

which means that it's the same as the inverse image in $A$ of $\text{Ann}_n(A/\text{Ann}_n(A))$. One has a square of ideals

\[
\text{Ann}_n(A) \cap \text{Ann}_n(A) \subset \text{Ann}_n(A)
\]

\[
\text{Ann}_n(A) \subset \{ a \mid AA = 0 \}
\]

whose quotients are Morita equivalent to $A$. 

Consider next the subring situation $A \subset B$.

1. $A$ is a left ideal in $B$ : $BA \subset A$. Then one has a Morita context

\[
\begin{pmatrix}
A & B \\
\tilde{A} & \tilde{B}
\end{pmatrix}
\begin{align*}
M & \to A \otimes A M \\
N & \leftarrow \tilde{B} \otimes 1 N
\end{align*}
\begin{align*}
GP &= \tilde{B}A = A \\
PQ &= \tilde{A}B
\end{align*}
\]

hence a Morita equivalence between $B$ and $\tilde{A}B$ the ideal generated by the left ideal $A$. Another context giving the same Morita equivalence is

\[
\begin{pmatrix}
A & A\tilde{B} \\
\tilde{A} & \tilde{B}
\end{pmatrix}
\begin{align*}
QP &= A\tilde{B}A = A^2 \tilde{A}
\end{align*}
\begin{align*}
PQ &= \tilde{A}B^2 = \tilde{A}B\tilde{A}\tilde{B} = (A\tilde{B})^2 \tilde{A}B
\end{align*}
\]

2. $A$ is a right ideal in $B$ : $AB \subset B$. Then one has the Morita contexts

\[
\begin{pmatrix}
A & A \\
\tilde{B} & B
\end{pmatrix}
\begin{align*}
QP &= A\tilde{B} = A \\
PQ &= B\tilde{A} = the \ ideal \ in \ B \ gen. \ by \ A
\end{align*}
\begin{pmatrix}
A & A \\
\tilde{B}A & B
\end{pmatrix}
\begin{align*}
QP &= \tilde{B}A\tilde{B}A = A^2
\end{align*}
\begin{align*}
PQ &= \tilde{B}A^2 = \tilde{B}A\tilde{B}A = (\tilde{B}A)^2
\end{align*}
\]

giving the same Morita equivalence between $A$ and $\tilde{B}A$.

3. Assume $ABA \subset A$. Then one has the Morita context

\[
\begin{pmatrix}
A & A\tilde{B} \\
\tilde{B}A & B
\end{pmatrix}
\begin{align*}
QP &= A\tilde{B}A
\end{align*}
\begin{align*}
PQ &= \tilde{B}A^2
\end{align*}
\]

Unfortunately $\tilde{B}A^2B$ does not seem to contain any power of $\tilde{B}AB$ in general, so we don't get a Morita equivalence between $A$ and $\tilde{B}A\tilde{B}$, only
However if one makes the stronger hypothesis
\[ ABA \subseteq A^2 \quad (\text{whence } ABA = A^2) \quad \text{then} \]
\[ (\tilde{B}a\tilde{B})^2 = \tilde{B}a\tilde{B}a\tilde{B} = \tilde{B}a^2\tilde{B}, \]

so indeed \( A \) and \( \tilde{B}a\tilde{B} \) are Morita equivalent. The nice Morita context in this case is

\[
\begin{pmatrix}
A & \tilde{B}a \\
\tilde{B}a & B
\end{pmatrix}
\]

\[QP = A\tilde{B}a = A^2 \quad PQ = \tilde{B}a^2\tilde{B} = (\tilde{B}a\tilde{B})^2 \]

Note that the hypothesis \( A\tilde{B}a = A^2 \) is satisfied when \( A \) is either a left or right ideal in \( B \).

Check: Given \( A \subseteq B \) such that \( ABA \subseteq A \), then \( A \) is a left ideal in \( A\tilde{B} \) since \( (A\tilde{B})A \subseteq A \). Thus one has a Morita equivalence between \( A \) and the ideal in \( A\tilde{B} \) generated by \( A \), namely \( A^2\tilde{B} \). Then \( A^2\tilde{B} \) is a right ideal in \( B \), so one has a Morita equiv. of \( A^2\tilde{B} \) with the ideal in \( B \) it generates namely \( \tilde{B}a^2\tilde{B} \). Again one gets a Morita equiv. between \( A \) and \( \tilde{B}a^2\tilde{B} \).
Let $M$ be a chain complex of $R$-bimodules such that $M \otimes_R R/I \cong 0$. Up to quasi-isomorphism, we can suppose $M$ consists of free $R$-bimodules. A free $R$-bimodule is a direct sum $\bigoplus \mathbb{A}_R R$ for some set $\mathbb{A}$. Let's assume that $\mathbb{A}_k R$ is a flat right $R$-module, for example this is true when $R$ is flat over the ground ring $k$. Then $M$ will be flat as right $R$-module and so

$$M \otimes_R R/I \cong M \otimes_R R/I = M/MI$$

is acyclic, equivalently the inclusion $MI \hookrightarrow M$ is a quasi of $R$-bimodules. Since $M$ is free as $R$ bimodule, there is a bimodule map $f : M \rightarrow MI \rightarrow M$ which is homotopic to the identity. Let

$$P = \lim \{ M \xrightarrow{f} M \xrightarrow{f} M \xrightarrow{f} \ldots \}$$

Then $P$ is a flat $R$-bimodule complex such that $P = PI$. Because $P$ is a filtered direct limit of direct sums of copies of $\mathbb{A}_k R$, which is flat as right $R$-module, we know $P$ is flat as right $R$-module. Thus $P$ is a $I$-solid right $R$-module. We also have an evident quasi $M \rightarrow P$.

Note the direction of the arrow, which is unlike the resolution $P \rightarrow I$ constructed when $I \otimes_R R/I = 0$. 
In the sheaf theory situation

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{j^*} & \mathcal{X} \\
\xleftarrow{j_!} & & \xleftarrow{j^*} \\
\mathcal{Y} & \xrightarrow{i^*} & \mathcal{Y} \\
\end{array}
\]

one has that \(i^*\) and \(j_!\) are exact functors and one has the short exact sequence

\[0 \to j_!j^*F \to F \to i_*i^*F \to 0\]

Further a sheaf \(F\) on \(X\) is equivalent to the triple \((i^*F, j^*F, \varphi: i^*F \to j_!j^*F)\). This follows from

\[
\begin{array}{ccc}
l_!F & = & l_!i_*i^*F \\
\downarrow & & \downarrow \\
0 & \to & j_!j^*F \to F \to i_*i^*F \to 0
\end{array}
\]

which shows that the square is cartesian, hence given \((F_Y, F_U, \varphi: F_Y \to j_!j^*F_U)\) this triple corresponds to the sheaf \(F\) defined by the fibre product:

\[
\begin{array}{ccc}
F & \to & l_*F_x \\
\downarrow & & \downarrow \\
l_*\varphi & \to & l_*i_*i^*F
\end{array}
\]

\[
\text{adjunction arrow } 1 \to i_*i^*
\]

Let's now consider the module situation with \(R, I\) with \(I = I^2\).
\[
\begin{align*}
(R/I)_{\text{mod}} & \xrightarrow{\begin{pmatrix} \lambda^* \\ \lambda \end{pmatrix}} R_{\text{mod}} \xrightarrow{\begin{pmatrix} j^* \\ j \end{pmatrix}} M(R, I) \\
\lambda^*(M) = M/I & \quad j_*(j^* M) = I^S \otimes_R M \\
\lambda^*(M) = \text{Hom}_R(R/I, M) & \quad j_*(j^* M) = \text{Hom}_R(I^S, M)
\end{align*}
\]

\(\lambda^*(M) = R/I \otimes_R M\) is exact iff \(R/I\) is flat as a right \(R\)-module, and this we know is equivalent to \(\forall x_1, \ldots, x_n \in I \quad \exists x \in I \ni (1-x)x_i = 0 \quad \forall i\), also equivalent to \(\forall x \in I \exists x \in I \ni (1-x)x_i = 0\).

Assume \(R/I\) is right flat. Then

\[
\begin{array}{c}
0 \to R/I \otimes_R I \to R/I \otimes_R R \\
\downarrow \quad \downarrow \\
I/I^2 \to 0 \to R/I
\end{array}
\]

and the exact sequence \(0 \to I \to R \to R/I \to 0\) shows that \(I\) is also right flat. Thus \(I^S = I \otimes_R I \to I\). It follows then that \(j_*(j^* M) = I \otimes_R M\) is exact in \(M\), hence \(j_*\) is an exact functor.

We also have the exact sequence

\[
\begin{align*}
0 \to I \otimes_R M & \to M \to R/I \otimes_R M \to 0 \\
\downarrow \quad \downarrow \\
\partial \to j_*(j^* M) & \to M \to \lambda^*(M) \to 0
\end{align*}
\]

and the diagram

\[
\begin{align*}
\text{because } I \text{ solid} & \Rightarrow I \otimes_R \text{ is full \& flat} \\
\text{exact as } R/I \text{ flat}
\end{align*}
\]

\[
\begin{align*}
0 \to I \otimes_R M & \to M \to R/I \otimes_R M \to 0 \\
\downarrow & \\
0 \to I \otimes_R \text{Hom}_R(I, M) & \to \text{Hom}_R(I, M) \to R/I \otimes_R \text{Hom}_R(I, M) \to 0
\end{align*}
\]
from which one gets the cartesian square

\[
\begin{array}{ccc}
M & \longrightarrow & R/I \otimes_R M \\
\downarrow & & \downarrow \\
\text{Hom}_R(I;M) & \longrightarrow & R/I \otimes_R \text{Hom}_R(I;M)
\end{array}
\]

This should imply that an $R$-module is equivalent to a triple $(N, Q, \varphi)$ where $N$ is an $R/I$-module, $Q$ is a $I$-coideal $R$-module, and $\varphi: N \longrightarrow R/I \otimes_R Q$ is a map of $R/I$-modules.

Example: $I = \bigoplus k e_\alpha$, $R = k \oplus I$, $Q$ a coideal $I$-module $Q$ has the form $\bigoplus V_\alpha$ where the $V_\alpha$ are vector spaces, and $Q/IQ = \bigoplus V_\alpha$. Thus a triple $(N, Q, \varphi)$ amounts to a vector space $N$, a family of vector spaces $\{V_\alpha\}$, and a map $\varphi: N \rightarrow \bigoplus V_\alpha$. The corresponding $R$-module is given by pull-back

\[
\begin{array}{cccccc}
0 & \longrightarrow & \bigoplus V_\alpha & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus V_\alpha & \longrightarrow & \bigoplus V_\alpha & \longrightarrow & \bigoplus V_\alpha & \longrightarrow & 0
\end{array}
\]

and $M$ amounts to a factorization $\bigoplus V_\alpha \rightarrow M \rightarrow \bigoplus V_\alpha$ of the canonical injection $\bigoplus V_\alpha \hookrightarrow \bigoplus V_\alpha$.

Notice that $k e_\alpha = Re_\alpha$ is a projective $R$-module so $I = \bigoplus k e_\alpha$ is projective, so $R/I$ has projective dimension $1$. $I^* M = \text{Hom}_R(I;M)$ is exact in $M$, hence $I^*$ is exact; this is also clear from the fact that $M(R,I)$ is a product category of $\text{Mod}(k)$ for each $\alpha$, so every object is both injective and projective.

The only local cohomology is in degrees $0, 1$.
Next I want to look at the derived category situation. In the sheaf situation we should have the 3×3 diagram

\[
\begin{array}{cccc}
O & \rightarrow & i_* i^! M & \rightarrow & i_* f^* M & \rightarrow & i_* R i^!(M) & \rightarrow & O \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \rightarrow & \mathbb{TV}_\alpha
\end{array}
\]

where the rows and columns are triangles, and here I should have derived functors \( i_*, i^! \), \( f_* \). It appears from this diagram that \( F \in D(X) \) is equivalent to a triple consisting of \( F_y \in D(Y) \), \( F_u \in D(U) \) and a map \( \varphi : F_y \rightarrow i_* R f_* (F_u) \). In the module case, assuming the \( h \)-unitality: \( I \otimes_R R I = 0 \), we have the diagram

\[
\begin{array}{cccc}
O & \rightarrow & I \otimes_R \mathbb{R} \text{Hom}(R/I, M) & \rightarrow & \mathbb{R} \text{Hom}(RI, M) & \rightarrow & \mathbb{R} \text{Hom}_R(R/I, M) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I \otimes_R M & \rightarrow & M & \rightarrow & R/I \otimes_R M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I \otimes_R \mathbb{R} \text{Hom}_R(I, M) & \rightarrow & \mathbb{R} \text{Hom}_R(I, M) & \rightarrow & R/I \otimes_R \mathbb{R} \text{Hom}_R(I, M)
\end{array}
\]

so that again \( M \) is a suitable homotopy fibre product.
June 29, 1987

Let \( X : \cdots \to X^i \to X^0 \to X^{-1} \to \cdots \) be a complex consisting of injective modules. We want to split off a contractible complex in order to obtain a minimal complex. I recall that a complex \( K \) is contractible iff it has a special contraction \( [a, b] = 1 \), \( b^2 = 0 \), and that a special contraction is equivalent to a choice of complement:

\[
K^n = C^n \oplus Z^n \quad \text{for each } n \in \mathbb{Z},
\]

Fix \( n \) and consider the extension

\[
\begin{array}{c}
Z^n < X^n. \\
\text{Choose } Y^n < X^n \text{ maximal such that } Z^n \cap Y^n = 0.
\end{array}
\]

One knows, because \( X^n \) is injective, that \( Y^n \) is injective and that \( Z^n \to X^n/Y^n \) is an injective hull for \( Z^n \). Notice that \( Y^n \cap Z^n = 0 \) implies \( Y^n < X^n \) and \( X^n/Y^n \) is monic. Thus one has an injection of complexes

\[
\begin{array}{ccccccc}
\to & o & \to & Y^n & \to & Y^n & \to & o & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& X^n & \to & X^{n+1} & \to & X^{n+2} & \to & \\
\end{array}
\]

The top complex is \( \text{Cone}(Y^n \to Y^n) = Y^n[-n-1] \); denote it \( C(Y_n, n) \).

Choosing \( Y_n \) in this way \( Y_n \) we get an injection of

\[
\bigoplus C(Y_n, n) = \bigoplus_n C(Y_n, n) \quad \text{into } X.
\]

But for any complex \( K \) one has

\[
\text{Hom}(K, C(Y_n, n)) = \text{Hom}(K_{n+1}, Y_n)
\]

maps in category of complexes.
so that by choosing a retraction of $X^{n+1}$ onto $dX^n$ for each $n$ we obtain a retraction of $X$ onto $\pi_1 \mathcal{C}(Y_n, \mathbb{Z})$.

In this way we can split off a contractible complex from $X$ and obtain a homotopy equivalent complex $X_{\text{min}}$ having the property that $X_{\text{min}} \to X^n$ is an injective hull of $Z^n$.

Consider now $R \supseteq I$ ideal and let $X$ be a complex bounded below (upper indexing) consisting of injective modules. Then we know

$$RHom_R(R/I, X) \cong \text{Hom}_R(R/I, X) = IX$$

(Here bold below is required, example of complete resolutions used in Tate cohomology).

Suppose that $X$ is minimal as above and that $RHom_R(R/I, X) = 0$, i.e. that $IX$ is acyclic. Look at the lowest degree $n$ such that $IX^n \neq 0$; we can suppose $n=0$. Then $H^0(I_X) = Z^0(I_X) = 0$, i.e. $IX^0 \cap Z^0 = 0$. By minimality, $X^0$ is an injective hull of $Z^0$, so we conclude $IX^0 = 0$ a contradiction. Therefore $X$ is a complex of $I$-cofinal injectives. We've almost proved:

**Prop.** Let $M \in D^+(R\text{-mod})$. Then $RHom_R(R/I, M) = 0$ if and only if $M$ is a complex bounded below of $I$-cofinal injectives.

The direction $\Rightarrow$ is trivial because $RHom_R(R/I, M) = \text{Hom}_R(R/I, X) = 0$. 
Conversely, given $M$ satisfying $R\text{-Hom}(R/I, M) = 0$ we know that $M$ is quasi-isomorphic to a complex built below of injectives which is minimal as above. Then we have seen that $X$ is $I$-coherent.

We have already proved

Prop. Let $M \in D_+(R\text{-mod})$, then $R/I \otimes_R M = 0$ if and only if $M$ is a complex (built below for lower indexing) consisting of $I$-torsion flat modules.
If $M/\text{Ann} M = 0$ and $N = 0$, then

$$\text{Hom}_A(M, N) \xrightarrow{\sim} \text{Hom}_{m(A)}(M, N)$$

Proof. \[ 0 \to K \to A \otimes_A M \to M \to 0 \]

$$0 \to \text{Hom}_A(M, N) \to \text{Hom}_A(A \otimes_A M, N) \to \text{Hom}_A(K, N)$$

and $\text{Hom}_A(K, N) = 0$ because $AK = 0$ and $AN = 0$. Thus

$$\text{Hom}_A(M, N) \xrightarrow{\sim} \text{Hom}_A(A \otimes_A M, N) \xrightarrow{\sim} \text{Hom}_A(A \otimes_A^2 M, N) \xrightarrow{\sim}$$

since $A \otimes_A M \to M \to A \otimes_A M \to A \otimes_A M$ etc. Thus

$$\text{Hom}_A(M, N) \xrightarrow{\text{lim}} \text{Hom}_A(A \otimes_A^n M, N) = \text{Hom}_{m(A)}(M, N)$$

Similarly, for the tensor product defined by

$$M(A_{op}) \times M(A) \xrightarrow{\otimes_A^X} \text{Pro} \text{ Ab}$$

$$X \otimes_A^X M \longmapsto \{X \otimes_A A \otimes_A^X M\}$$

Proof. If $XA = X$ and $AM = M$, then

$$X \otimes_A^X M \xrightarrow{\sim} X \otimes_A M$$

In effect

$$0 \to K \to A \otimes_A M \to M \to 0$$

yields

$$X \otimes_A^X K \to X \otimes_A A \otimes_A^X M \to X \otimes_A M \to 0$$

$$XA \otimes_A K = X \otimes_A AK = 0.$$
Suppose $M$ forms: $I \otimes_R M \rightarrow M$, equivalently $- \otimes_R M$ inverts $I$ so $M$ is finitely projective as right $R$-modules. Now:

$$R \rightarrow \text{Hom}_{R^{\text{op}}}(I, R)$$

has its kernel and cokernel killed by $I$ on the right. (The kernel is $\{r \mid rI = 0\}$.) Thus we have:

$$M \rightarrow \text{Hom}_{R^{\text{op}}}(I, R) \otimes_R M$$

Iterating we have:

$$M \rightarrow \varinjlim \text{Hom}_{R^{\text{op}}}(I^{\otimes n}, R) \otimes_R M$$

the $\sigma_n$ for right modules

$$M(R^{\text{op}}, I^{\otimes n})(R, R)$$

Put another way, the functor $- \otimes_R M$ from $R^{\text{op}}$-mod $\rightarrow$ $\text{Alg}^*$ descends to $M(R^{\text{op}}, I^{\otimes n})$ so its value on $R$, namely $R \otimes_R M = M$, is natural acted on by the endomorphisms of $R$ in the category $M(R^{\text{op}}, I^{\otimes n})$.

Recall that the canonical map

$$\text{Hom}_R(I, R) \otimes_R M \rightarrow \text{Hom}_R(I, M)$$

is an isomorphism when $I$ is finitely generated projective as left $R$-module, or when $I$ is finitely presented as left $R$-module and $M$ is flat. Notice that this gives a different dual of $I$. 

Notation:

left dual $I^*_L = \text{Hom}_R(I, R)$

right dual $I^*_R = \text{Hom}_{R^{\text{op}}}(I, R)$
Consider the diagram
\[ \begin{array}{ccc}
R \otimes_R M & \longrightarrow & I^*_R \otimes_R M \\
\| & & \downarrow \\
M & \longrightarrow & \text{Hom}_R(I, M)
\end{array} \]

Then we have

Prop: Assume \( R \to I^*_R = \text{Hom}_R(I, R) \) has kernel + cokernel killed by \( I^n \) on the right for some \( n \).
Assume \( I \) is f.g. proj \( R \)-module, or a fin. pres \( R \)-module and that \( M \) is flat.
Then \( M \) firm \( \Rightarrow \) \( M \) cofirm.

Thus in the comm. noetherian case we have

\( \text{firm} + \text{flat} \Rightarrow \text{cofirm} \).

Consider next the square
\[ \begin{array}{ccc}
\text{Hom}_R(I^*_R, M) & \longrightarrow & \text{Hom}_R(R, M) \\
\| & & \downarrow \\
I \otimes_R M & \longrightarrow & M
\end{array} \]

Then we have

Prop: Assume \( R \to I^*_R = \text{Hom}_R(I, R) \) has kernel + cokernel killed by \( I^n \) on the left for some \( n \).
Assume \( I \) is f.g. gen proj as right \( R \)-module
Then \( M \) cofirm \( \Rightarrow \) \( M \) firm

Example: \( R = T(V), \; I = T^{(0)}(V), \) assuming \( \dim V > 2 \)
Here the hypothesis that \( R \to I^*_R \) be a right mod nilp. isom.
and that \( R \to I^*_R \) be a left mod nilp. isom. fail.
Why? \( R \rightarrow \text{Hom}_R(I,I) \subset \text{Hom}_R(I,R) \) is found to be

\[
R \rightarrow \text{Hom}_R(I,I) \subset \text{Hom}_R(I,R) \\
\| \quad \| \\
R \rightarrow V^* \otimes V \otimes R \subset V^* \otimes R \\
1 \rightarrow v_i^* \otimes v_i \otimes 1
\]

so \( \text{Hom}_R(I,I)/I \) is a free right \( R \)-module \( \neq 0 \), hence not killed by any \( I^n \).
I ideal $\mathfrak{I} \subset \mathbb{R}$, recall there is a torsion theory $\mathfrak{T}_I$ on $\mathbb{R}$-modules whose torsion-free modules are the $M$ such that $IM = 0$, and whose torsion modules are the $M$ such that $\text{Hom}(M, \mathbb{E})$ for every torsion-free injective. Torsion theories of the form $\mathfrak{T}_I$ for some ideal are called regular torsion theories in Golan's book (Ch. 30). One has the following description of the torsion modules.

**Proof:** $M$ is $\mathfrak{T}_I$-torsion $\iff \forall m \in M, \forall$ sequence $a_1, a_2, \ldots \in \mathfrak{I}$, $\exists n \in \mathbb{N}$ s.t. $a_n a_{n-1} \ldots a_1 m = 0$.

**Proof:** ($\Rightarrow$) $M$ is torsion $\iff \forall N < M$ one has $I(M/N) \neq 0$. One can then define a filtration by transfinite induction: $F^0(M) = 0$

$$F^{\alpha+1}M / F^{\alpha}M = I(M/F^{\alpha}M)$$

$$F^{\alpha}M = \bigcup_{\beta < \alpha} F^{\beta}M$$ a limit ordinal

Suppose given $m \in M$ and $(a_n) \in I$, let $a_n$ be the least s.t. $a_n a_{n-1} \ldots a_1 m \in F^{a_n}M$.

If $a_n a_{n-1} \ldots a_1 m \neq 0$ then $a_n > 0$ and $a_n$ is not a limit ordinal, so $a_{n+1} \ldots a_1 m \in F^{a_n+1}M$, whence $a_{n+1} < a_n$.

But one can't have an infinite decreasing sequence $x_0 > x_1 > \ldots$ of ordinals, so $a_n \ldots a_1 m = 0$ for some $n$.

$\iff$ If $M$ is not torsion, $\exists N < M$ such that
Let \( m \in M - N \). We will construct \((a_n)\) in 
\[ a_n = a, m \notin N. \]
By replacing \( M \) by \( M/N \) we can suppose \( IM = 0 \). Thus, \( Im \neq 0 \), so 
\[ \exists a \text{, } q, m \neq 0. \]
Then \( Ia + a, m \neq 0 \). Then \( Ia + a, m \neq 0 \). Then \( Ia + a, m \neq 0 \).

That's the proof I saw yesterday. Here's a more efficient one.

Let \((a_n)\) be a sequence in \( I \). The condition 
\[ \exists a \text{ s.t. } a_n \neq 0 \]
means that \( M \) is in the kernel of the cokernel map:
\[
M \rightarrow \lim_{n \to \infty} (M \xrightarrow{a_1} M \xrightarrow{a_2} \cdots) \\
\downarrow \\
(\lim_{n \to \infty} R \xrightarrow{a_1} R \xrightarrow{a_2} \cdots) \otimes_R M
\]

Denote this \( F(a_n) \)

Note that \( F(a_n) \) is a free flat right \( R \)-module.

Consider the category of all \( M \) such that 
\[ F(a_n) \otimes_R M = 0 \]
for all sequences \((a_n)\) in \( I \).
Because \( F(a_n) \) is flat this category is closed under submodules. It is a hereditary subcategory closed under \( \oplus \)'s. It clearly contains I-module modules, hence all I-torsion modules. This gives the implication \((\Rightarrow)\) in the above proposition.
IF AE

2) \( X \otimes_R M = 0 \) for all right modules \( X \) s.t. \( X = X_1 \).
3) \( F(\ell_\infty) \otimes_R M = 0 \) for all sequences \( \ell_\infty \) in \( I \).

It remains to check 1) \( \Rightarrow \) 2). Because \( X \otimes_R - \) respects \( \lim \)'s, there is a largest submodule \( N \subset M \) such that \( X \otimes_R N = 0 \). If \( N \subset M \), then, as \( M \) is assumed torsion-free, \( \exists \) \( N' \subset M \) and \( I(N'/N) = 0 \). Then we have

\[
X \otimes_R N \to X \otimes_R N' \to X \otimes_R (N'/N) \to 0
\]

\( \Rightarrow \) \( N = M \) and \( X \otimes_R M = 0 \).

Here's an attempt to show that firm modules form an abelian category at least in the case where \( R \) is noetherian commutative.

Recall that in general, an \( R \)-functor \( R\text{-mod} \to \text{Ab} \) respecting \( \ell_\infty \)'s (i.e., \( \otimes \)'s and right exact) has the form \( X \otimes_R - \), where the right module \( X \) is \( F(R) \).

Next, \( X \otimes_R - \) descends to \( R\text{-mod}/I\text{-tors} \) \( \leftrightarrow \) \( X \) is firm. Thus firm right modules are equivalent to functors \( R\text{-mod}/I\text{-tors} \to \text{Ab} \) respecting \( \lim \)'s. (I think here one needs that the canonical functor \( R\text{-mod} \to R\text{-mod}/I\text{-tors} \) respects \( \lim \)'s because it has a right adjoint.)

Now suppose \( R \) left noetherian. Then the finitely generated \( R \)-modules form a noetherian abelian...
category whose associated ind-object category (also the associated locally noetherian category) is $R$-mod. This should also be true for the quotient categories:

$R$-mod/I-tors is the locally noetherian cat. assoc. to the noetherian category $fg$R-mod/$fg$I-tors. I think then that right antiequors (resp. left's) functors $R$-mod/I-tors $\rightarrow$ Ab are equivalent to functors $fg$R-mod/$fg$I-tors $\rightarrow$ Ab which are right exact.

If all this holds, then we have an equivalence between finite modules $X$ and right exact functors $fg$R-mod/$fg$I-tors $\rightarrow$ Ab. In fact this should be clear directly.

So now the idea was that right exact functors $\mathcal{A} \rightarrow$ Ab, where $\mathcal{A}$ is a small abelian category, should form an abelian category. I think Gabriel proves the result for left exact functors.

I think there's a problem with right exact functors because the left-derived functor $L_0F \rightarrow F$ is constructed using inverse limits, as opposed to the right-derived functor $F \rightarrow R^nF$, which is constructed using direct limits.
July 7, 1994

Given an ideal \( I \subset R \) we have a canonical functor

\[
I^\op \text{firm} \longrightarrow \lim \text{ cont Fun} (R\text{-mod}/I\text{-tors}, \text{Ab})
\]

\[
X \longmapsto (M \mapsto X \otimes_R M)
\]

Why? The functor \( X \otimes_R - : R\text{-mod} \longrightarrow \text{Ab} \), where

\( X \) is an \( R^\op \)-module, is \( \lim \) continuous as it has

the right adjoint \( N \mapsto \text{Hom}_R(X, N) \). On the other hand we have adjoint functors

\[
\begin{array}{ccc}
I\text{-tors} & \overset{L^*}{\longrightarrow} & R\text{-mod} & \overset{\gamma^*}{\longrightarrow} & R\text{-mod}/I\text{-tors} \\
\mid & \mid & \mid & \mid & \mid \\
' & \mid & \text{localization} & \mid & I\text{-cofirm} \\
/ & \mid & \text{inclusion} & \mid & \\
& & & & \\
\end{array}
\]

from the theory of torsion theories. In particular

the canonical map \( \gamma^* \) to the quotient category

is \( \lim \) continuous as it has a right adjoint.

When \( X \) is \( I^\op \)-firm we know that

\( X \otimes_R - \) inverts isoms modulo \( I\text{-tors} \). Thus

this functor descends to a functor \( R\text{-mod}/I\text{-tors} \longrightarrow \text{Ab} \).

To see it is \( \lim \) cont suppose given \( \epsilon \)

functor \( C \longrightarrow R\text{-mod}/I\text{-tors} \). Then using \( \gamma^* \) it can be

lifted to a functor \( i \mapsto M_i : C \longrightarrow R\text{-mod} \) and

we have \( \gamma^*(\lim_i M_i) = \lim_i \gamma^* M_i \). Then

\[
X \otimes_R \gamma^*(\lim_i M_i) = X \otimes_R \gamma^*(\lim_i M_i) = X \otimes_R \lim_i M_i = \lim_i X \otimes_R M_i .
\]
This seems okay.
Next we want to show that 1) is an equivalence of categories. To start with \( \Phi : R\text{-mod}/I\text{-tors} \to \text{Ab} \), then \( \Phi j^* : R\text{-mod} \to \text{Ab} \) is \( \text{lim cnt} \), continuous.
Now there is a canonical map

\[
\Phi j^*(M) \otimes_R M \to \Phi j^*(M)
\]
(for an arbitrary functor this is true) which is an isomorphism iff the functor is \( \text{lim cnt} \). Thus we have

\[
\Phi j^*(M) = X \otimes_R M
\]
with \( X = \Phi j^*(R) \).
Since \( \Phi \) descends it疑问红mod I-tors isoms, hence \( X \) is \( I^0\text{-fim} \).

Thus we have an equivalence of categories

\[
I^0\text{-fim} \xrightarrow{\sim} \text{lim cnt Fun}(R\text{-mod}/I\text{-tors}, \text{Ab})
\]

**Proof.** TFAE

1) \( I \) is (left) \( T \)-nilpotent (\( \forall \) sequence \( (a_n) \) in \( I \))

\[

\exists n > a_n a_{n-1} \cdots a_1 = 0
\]

2) \( I\text{-cofim} = 0 \)

3) \( I^0\text{-fim} = 0 \)

**Proof.** Hence \( I\text{-cofim} \cong R\text{-mod}/I\text{-tors} \), \( I\text{-cofim} = 0 \) means every \( R \)-module is in \( I\text{-tors} \), equivalently \( R \) is \( \text{I-tors} \).

1) \( \Rightarrow 2 \). If \( I\text{-cofim} \neq 0 \), then there exists a torsion-free module \( N \), i.e. \( \exists N = 0 \). Pick \( n \in N, n \neq 0 \). Then \( I_n \neq 0 \) so can pick \( a, e.I \) s.t. \( a, n \neq 0 \), then \( I_a n \neq 0 \).
can pick \( a_2 \in I \) s.t. \( a_2 \cap n \neq 0 \), etc.

showing \( I \) is not \( T \)-nilpotent.

3) \( \Rightarrow \) 1) If \( I \) is not \( T \)-nilpotent,

there is a sequence \( a_n \) in \( I \) such that

\[ \forall n \quad a_n \ldots a_1 \neq 0. \]

This means that

\[ F = \lim_{\rightarrow} \left\{ R \xrightarrow{a_1} R \xrightarrow{a_2} \cdots \right\} \neq 0. \]

We know that \( F \) is a flat and finitely-generated right module, so \( I^k \text{-} \text{f.g.} \neq 0 \).

2) \( \Rightarrow \) 3) If \( I \)-cof.g. = 0, then \( R \)-mod/I-tors = 0 so \( I^k \text{-} \text{f.g.} \), which is the cat.

right exact functors from \( R \)-mod/I-tors to \( Ab \) is zero.

As a check, suppose there exists a nonzero \( I \)-f.g. right module \( F \). Then consider the class of modules \( M \) such that \( F \otimes_R M = 0 \). This is clearly a Serre subcat of \( R \)-mod closed under \( \oplus \)'s. It contains any \( I \)-nil. modules, hence all \( I \)-torsion modules.

But it doesn't contain \( R \) so we see \( I \)-tors < \( R \)-mod.

Thus \( I^k \text{-} \text{f.g.} \neq 0 \Rightarrow I \)-cof.g. = 0, so 2) \( \Rightarrow \) 3).

Here's seems to be the standard example of an \( I \) which is \( (\text{left}) \)-\( T \)-nilpotent but not \( \text{right} \)-\( T \)-nilpotent. Consider infinite strictly upper triangular matrices with finite support and entries say in \( k \). Given a sequence \( a_1, \ldots \in I \), note that because \( a_1 \) has finite support \( a_1 \) is contained in a left ideal which is finite dimensional, namely matrices supported in columns \( i < j \leq m \) for some \( m \).
This left ideal is killed by $I^{m-1}$ so it's clear that $q_m \cdots q_1 = 0$.

On the other hand, denoting by $e_i$, $i \leq j$, the basis matrix with 1 in the $(i, j)$-th position we have

$$e_1 e_2 e_3 \cdots e_{n-1} e_n = e_{1n} \neq 0$$

for all $n$ so $I$ is not right $T$-nilpotent.

Then for such a ring $I$ we have

$I^{ob}$-firm $= \emptyset$, $I$-cofirm $= \emptyset$

$I$-firm $\neq 0$, $I^{ob}$-cofirm $\neq 0$.

Let's calculate these categories. Take $R$ to be the path algebra of the quiver

$$
\begin{array}{cccc}
1 & \rightarrow & 2 & \rightarrow \\
\quad & \uparrow & \quad & \\
3 & \rightarrow & 4 & \rightarrow \\
\end{array}
$$

and $I$ the ideal of paths of length $\geq 1$. This $R$ is non-unital. It is a tensor algebra

$$R = S \oplus B \oplus B \otimes S \oplus \cdots$$

where the summands can be visualized as the matrices supported in the various diagonals starting the main diagonal (which gives $S = \bigoplus_{i=1}^{\infty} \mathbb{K} e_i$).

An $I$-firm $R$-module should be the same as a firm $S$ module $M$ equipped with an isomorphism

$$(*) \quad B \otimes S M \cong M$$

of $S$ modules. So $M$ we know is $\bigoplus_{i=1}^{\infty} M_i$ where
$M_i = e_{ii} M$. Then (x) gives

$$e_{i+1} \otimes M_{i+1} \sim M_i$$

so a firm module in this case is a representation of the quiver

$$M_1 \sim M_2 \sim M_3 \sim$$

such that the arrows are isomorphisms. I should be more careful:

$$e_{12} \otimes M_2 \sim M_1$$

$$e_{23} \otimes M_3 \sim M_2$$

Consider now an $I^{ob}$ firm $R^{ob}$ module, which should be the same as an $S^{ob}$-module $M$ together with an isomorphism of right $S$-mods

$$M \otimes_S B \sim M$$

This time we have

$$\bigoplus_{i=1}^{\infty} M_i \otimes e_{ii} \sim \bigoplus_{i=0}^{\infty} M_{i+1}$$

which means

$$0 \sim M_1$$

$$M_1 \otimes e_{12} \sim M_2$$

$$M_2 \otimes e_{23} \sim M_3$$

hence the only $I^{ob}$-firm module is zero.

Note this is consistent with $I$ $T$-nilpotent

$\iff I^{ob}$-firm $= 0$.  

686
Notice the following consequence of $I^{op}$-fibration $\cong \text{lin} \text{cnt, } \text{Fun} (I^{op}\text{-fibrm}, \text{Ab})$.

Namely if $I^{op}\text{-fibrm} \cong \mathcal{O} \text{-mod}$ for some ring $\mathcal{O}$, then $I^{op}\text{-fibrm} \cong \mathcal{O}^{op} \text{-mod}$.

Let's consider the case where $I$ is a finitely generated proj $R$-module. Then

$$\text{Hom}_R (I, M) \cong \text{Hom}_R (I, R) \otimes_R M$$

so an $I^{op}\text{-fibrm}$ module is an $R$-module such that $M \cong I^{*} \otimes_R M$. This means that besides the operators $T_r : m \mapsto rm$ for $r \in R$ we also have operators $T^*_\varphi : M \rightarrow I^{*} \otimes_R M \rightarrow M$ for $\varphi \in I^{*} = \text{Hom}_R (I, R)$.

An $I^{op}\text{-fibrm}$ module is an $R^{op}$-module $X$ such that $X \otimes R I \rightarrow X$. This means in addition to the operators $T_r : X \rightarrow X_R$ we have operators $T^*_\varphi : X \rightarrow X \otimes_R I \rightarrow X$.

An interesting point here is supposedly these types of modules depend only on $I$ and not $R$. How do you see this? It is obviously meaningless to expect $I$ to be finitely generated projective over $\mathcal{O}$.
July 8, 1974

About pure exact sequences + pure injective (= algebraically compact) modules. References:
Precht - Model theory + modules.

Proof. For an exact sequence \( 0 \to M' \to M \to M'' \to 0 \)

1) For any fin pres. \( R \)-module \( X \), the functor \( X \otimes_R - \) applied to this sequence is exact.

2) For any fin pres \( R \)-module \( N \), the functor \( \text{Hom}_R(N, -) \) applied to the sequence is exact.

3) The sequences of \( R \)-modules

\[
0 \to \text{Hom}_Z(M'', \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}_Z(M', \mathbb{Q}/\mathbb{Z}) \to 0
\]

is split exact.

4) The given exact sequence is a filtered inductive limit of split exact sequences.

Proof. First discuss the equivalence of 1) and 2). Consider

\[
\begin{array}{ccccccccc}
0 & \to & \text{Hom}_R(N, M') & \to & \text{Hom}_R(N, M) & \to & \text{Hom}_R(N, M'') & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M' \otimes R & \to & M \otimes R & \to & M'' \otimes R & \to & 0 \\
& & \downarrow r_1 & & \downarrow r_0 & & \downarrow r_2 & & \\
0 & \to & M' \otimes R & \to & M \otimes R & \to & M'' \otimes R & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X \otimes M' & \to & X \otimes M & \to & X \otimes M'' & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & & & & & & & & 0
\end{array}
\]
Here starting with \( R^b \xrightarrow{f} R^p \xrightarrow{g} M \xrightarrow{h} 0 \), we define \( X \) by \( R^p \xrightarrow{r^*} R^b \xrightarrow{g} X \xrightarrow{} 0 \).

Assuming 1) we have \( \text{Hom}_R(N, M) \xrightarrow{} \text{Hom}_R(N, M') \). By the Serre-Filmberg lemma \( \text{Hom}_R(N, M) \xrightarrow{} \text{Hom}_R(N, M') \).

Thus 1) \( \Rightarrow \) 2), and the other direction is similar.

Check that the two cases of 1) are equivalent. The point is that any module is a filtered inductive limit of finitely presented modules. Choose a presentation

\[
R^{(\Lambda)} \xrightarrow{q} R^{(\Lambda')} \xrightarrow{} M \xrightarrow{} 0
\]

Then consider the poset of finite subsets pair \((S', S)\) such that \( S' \subset \Lambda', S \subset \Lambda \) and \( q(R^S) \subset R^{S'} \). This poset is directed and \( M = \lim_{(S', S)} \text{Coker}(R^{S'} \xrightarrow{S} R^{S})\).

This result I think identifies the ind-category of finitely presented modules with the category of \( R \)-modules. Actually, the important point is that for \( N \) finitely presented \( \text{Hom}_R(N, -) \) commutes with filtered lim's.

Next write \( M'' \) as a direct limit of finitely presented modules \( N_i \) and considers the pull-back sequence

\[
0 \xrightarrow{} M' \xrightarrow{} M \xrightarrow{} M'' \xrightarrow{} 0
\]

Assuming 2) the upper sequence splits, so we see 2) \( \Rightarrow \) 1). The converse is obvious.
Finally we have
\[ \text{Hom}_{\mathcal{R}^\text{op}}(X, \text{Hom}_\mathbb{Z}(-, \mathbb{Q}/\mathbb{Z})) = \text{Hom}_\mathbb{Z}(X_\mathcal{R}^\text{op}, \mathbb{Q}/\mathbb{Z}) \]

for any $\mathcal{R}^\text{op}$ module $X$. Assuming 1) the right side applied to our given exact sequence is exact for any $X$, hence so is the left side and this implies 3). Conversely 3) implies the right side applied to the given sequence is exact, and then because $\mathbb{Q}/\mathbb{Z}$ is a faithful injective, 1) follows.

The conditions of the prop above define the notion of pure exact sequence. Then this leads to the notion of pure projective and pure injective modules.

Notice that any f.p. module is pure-proj by 2), hence any summand of a direct sum of f.p. modules is pure-proj. Given any module $M$, because f.p. pure proj. modules form a small category (essentially) we can manufacture an epimorphism

\[ 0 \to K \to \bigoplus N_i \to M \to 0 \]

where the $N_i$ are f.p., such that every map from a f.p. module to $M$ lifts. Thus the above exact sequence is pure exact, and $M$ is a pure quotient of a pure projective module. If $M$ is pure projective this sequence splits, so pure projective is equivalent to summand of a direct sum of fin. pres. modules.

It's clear one can construct pure exact pure projective resolutions unique up to
homotopy.

Next let's examine pure-injectives.

From

\[ \text{Hom}_R(M, \text{Hom}_Z(X, E)) = \text{Hom}_Z(X \otimes_R M, E) \]

it is clear that \( R^{\text{op}} \)-module \( X \) and injective \( \mathbb{Z} \) module \( E \) then \( \text{Hom}_Z(X, E) \) is a pure-injective.

(Note that \( X \text{ flat} \Rightarrow \text{Hom}_Z(X, E) \) is injective.)

Again we can manufacture an injection

\[ 0 \rightarrow M \rightarrow \text{Tor}_i^\mathbb{Z}(X, \mathbb{Q}/\mathbb{Z}) \rightarrow C \rightarrow 0 \]

such that for any f.p. right module \( X \), the induced map

\[ X \otimes_R M \rightarrow X \otimes_R \text{Tor}_i^\mathbb{Z}(X, \mathbb{Q}/\mathbb{Z}) \]

is injective. Thus the exact sequence above is pure exact, so any module \( M \) is a pure submodule of a pure injective. Then we can construct a pure injective resolution unique up to homotopy and do "pure" homological algebra.

The next stage is to consider the functor categories of covariant and contravariant functors from \( \text{fp mod}(R) \) to \( \text{Ab} \). We have the two embeddings

\[ \Phi : \text{mod}(R) \rightarrow \text{Fun}(\text{fp mod}(R)^{\text{op}}, \text{Ab}) \]

\[ M \mapsto h_M = \text{Hom}_R(-, M) \]

\[ \Psi : \text{mod}(R^{\text{op}}) \rightarrow \text{Fun}(\text{fp mod}(R), \text{Ab}) \]

\[ X \mapsto X \otimes_R - \]
where the images are the left exact and right exact functors, resp. I recall that there are canonical maps for any $F$:

contra.

\[ F(M) \quad \longrightarrow \quad \text{Hom}_{\mathcal{R}}(M, F(R)) \]

cov.

\[ F(R) \otimes_{\mathcal{R}} M \quad \longrightarrow \quad F(M) \]

which are isomorphisms for $M$ finitely presented iff $F$ is left exact (resp. right exact).

However in this "pure" game one improves this characterization as follows.

$\Phi$ induces equivalence

\[ \text{fg mod}(\mathcal{R}) \quad \sim \quad \text{fg proj functors} : \text{fp mod}(\mathcal{R}) \overset{\Phi}{\longrightarrow} \text{Ab} \]

\[ \text{pure proj}(\mathcal{R}) \quad \sim \quad \text{proj functors} \]

\[ \text{mod}(\mathcal{R}) \quad \sim \quad \text{flat functors} \]

A fin. gen. functor is a quotient of a representable one, a fin. pres. functor thus has a presentation $h^i_M \rightarrow h^i_M \rightarrow E \rightarrow 0$, a flat functor $F$ is such that any map $E \rightarrow F$ with $E$ fin. pres. factors through a representable functor. A flat functor is a filtered inductive limit of representable functors, so it's clear that flat functors come from modules.

$\Phi$ induces equivalences

\[ \text{mod}(\mathcal{R}) \quad \sim \quad \text{fp-injective functors} : \text{fp mod}(\mathcal{R}) \rightarrow \text{Ab} \]

\[ \text{pure-inj}(\mathcal{R}) \quad \sim \quad \text{injective functors} \]
A finite-presentation injective functor is such that any map \( U \to Q \) where \( U \) is a f.g. subfunctor of a representable functor \( h^E = \text{Hom}(E, -) \), \( E \) finitely presented, can be extended to a map \( h^E \to Q \).

Question: Assume I neither left T-nilpotent nor right T-nilpotent. Does there exist a nonzero module \( M \) such that \( M/IM = \_M = 0 \)?

For example if we take \( I \) to be matrices \((a_{ij})\) with \((a_{ij}) \in \mathbb{Z} \times \mathbb{Z}\) of finite support and upper triangular: \( a_{ij} = 0 \) for \( i > j \), then finitely \( I \)-modules should be representations of the quiver

\[
\rightsquigarrow M \rightsquigarrow M_0 \rightsquigarrow M_1 \rightsquigarrow \]

This needs checking at some point.

Let's study

\[
\mathbb{F}: \text{mod}(R) \longrightarrow \text{Fun}(\text{fpmod}(R^\text{op}), \text{Ab})
\]

\[
M \longmapsto (X \longmapsto X \otimes_R M)
\]

We wish to understand injective functors. Given an \( X \in \text{fpmod}(R^{\text{op}}) \) and injective abelian group \( E \), we have an exact contravariant functor \( \text{Fun} \longrightarrow \text{Ab} \)

\[
F \longmapsto \text{Hom}_R(F(X), E)
\]

This is linear continuous, so it should be representable:

\[
\text{Hom}_Z(F(X), E) = \text{Hom}_{\text{Fun}_R}(F, G)
\]
for some $G \in \text{Fun}$. Taking $F = h^y = \text{Hom}_R(Y, -)$ we then have

$$G(Y) = \text{Hom}_{\text{Fun}}(h^y, G)$$

$$= \text{Hom}_Z(h^y(X), E)$$

$$= \text{Hom}_Z(\text{Hom}_R(Y, X), E)$$

Now $Y$ is finitely presented, so we have exact sequences

$$R^p \rightarrow R^q \rightarrow Y \rightarrow 0$$

$$X^p \leftarrow X^q \leftarrow \text{Hom}_R(Y, X) \leftarrow 0$$

(E inj)

$$\text{Hom}_Z(X, E)^p \rightarrow \text{Hom}_Z(X, E)^q \rightarrow \text{Hom}_Z(\text{Hom}_R(Y, X), E) \rightarrow 0$$

Thus

$$G(-) = - \otimes_R \text{Hom}_Z(X, E).$$

Now we know (p 691) that for any right $R$-mod $X$ and injective $\mathbb{Z}$-module $E$ that $\text{Hom}_Z(X, E)$ is a pure injective module. Moreover any pure injective is a summand of a product of $\text{Hom}_Z(X_i, \mathbb{Q}/\mathbb{Z})$ for $X_i$ fin pres. Thus we have shown that $- \otimes_R \mathbb{Q}$ is an injective functor for $\mathbb{Q}$ pure injective.
Yesterday I wrote a proof that an injective functor in $\text{Fun}(\text{fpmod}(R^{op}), AB)$ is of the form $X \mapsto X \otimes_R Q$ with $Q$ would purely injective and conversely. I like to have a direct proof that $\text{fpmod}(R^{op})$ with $Q$ pure injective is injective in the functor category, but the proof is indirect in the sense that $Q$ pure injective $\iff$ $Q$ summand of $\bigoplus_i \text{Hom}_Z(X_i, Q/2)$ for some family $\{X_i\}$ in $\text{fpmod}(R^{op})$, and so one reduces to the case $Q = \text{Hom}_Z(X, Q/2)$. In this case one has a funny double dual argument.

The proof in Jensen-Lenzing is different and proceeds by characterizing functors of the form $-\otimes_R M$ as fp-injective in the functor category. Clearly injective $\implies$ fp injective, so an injective functor has the form $-\otimes_R Q$. Now $0 \to M' \to M \to M'' \to 0$ pure exact $\iff 0 \to (-\otimes_R M') \to (-\otimes_R M) \to (-\otimes_R M'') \to 0$ exact $\iff 0 \to \text{Hom}_R(M'', Q) \to \text{Hom}_R(M, Q) \to \text{Hom}_R(M', Q) \to 0$ exact (since the embedding $M \to -\otimes_R M$ is fully faithful; $M$ can be recovered from the value of this functor on $R$). Thus $-\otimes_R Q$ injective $\implies Q$ is pure injective.

I still need a proof that $Q$ pure-inj $\implies -\otimes_R Q$ is an injective functor. But first let's examine fp-injective functors...
A functor $G : \text{Fun} \rightarrow \text{Fun}(\text{fp-mod}(\mathbb{K}^\text{op}), \mathbb{K})$ is fp-injective when $\text{Ext}^1(F, G)$ for any fin pres functor $F$. For $F$ to be finitely presented means it is a cokernel

$$h' \rightarrow h^x \rightarrow F \rightarrow 0$$

of representable functors. To calculate $\text{Ext}^1_{\text{Fun}}(F, G)$ choose an epimorphism $h^x \rightarrow F$ and we find $G$ is fp-injective iff any diagram

$$\begin{array}{ccc}
U & \rightarrow & h^x \\
\downarrow & & \downarrow \\
G & \rightarrow & F
\end{array}$$

can be completed where $U$ is a fin gen functor (quotient of a representable one).

I claim fp-inj $\Rightarrow$ right exact. In effect given $X_1 \rightarrow X_0 \rightarrow X \rightarrow 0$ exact in $\text{fp-mod}(\mathbb{K}^\text{op})$ we wish to show $G(X_1) \rightarrow G(X_0) \rightarrow G(X) \rightarrow 0$ is exact. But

$$0 \rightarrow h^x \rightarrow h^x \rightarrow h^{x_1}$$

is exact. So

$$\begin{array}{ccc}
h^x & \rightarrow & h^x_0 \\
\downarrow & & \downarrow \\
G & \rightarrow & F
\end{array}$$

and

$$\begin{array}{ccc}
h^x_0 \rightarrow h^x & \rightarrow & h^{x_1} \\
\downarrow & & \downarrow \\
G & \rightarrow & F
\end{array}$$

$\Rightarrow G(X_0) \rightarrow G(X) \rightarrow \text{ker}(G(X) \rightarrow G(X_0))$. 

Finally $F$ right exact $\Rightarrow$ canon map $\bigotimes_R F(R) \rightarrow F(X)$ is an isomorphism for $X \in \text{fp} \text{mod}(R^{op})$.

The missing argument that $Q$ pure-injective $\Rightarrow - \otimes_R Q$ is an injective functor goes as follows. Use the fact that the functor category $\text{Fun}$ has enough injectives to embed $- \otimes_R Q$ into an injective functor $E$. Then injective $\Rightarrow$ fp injective so we know $E = - \otimes_R M$ for $M = E(R)$. Then $- \otimes_R Q \rightarrow - \otimes_R M$ means that $Q$ is a pure submodule of $M$, so because $Q$ is assumed pure injective, we know $Q$ is a summand of $M$, hence $- \otimes_R Q$ is a summand of $- \otimes_R M = E$, so $- \otimes_R Q$ is injective.

The argument I gave yesterday in effect constructs enough injectives of the form $\text{Hom}_Z(X, Q/Z)$ and shows explicitly by the double dual argument:

$$X \otimes_R \text{Hom}_Z^2(X, Q/Z) \rightarrow \text{Hom}_Z^2(\text{Hom}_{R^{op}}(Y, X), Q/Z)$$

(because isom. for $Y=R$ and both sides right exact), also

$$\text{Hom}_\text{Fun}(F, \text{Hom}_Z^2(X, Q/Z)) = \text{Hom}_Z^2(F(X), Q/Z)$$

that the functor $- \otimes_R \text{Hom}_Z(X, Q/Z)$ is injective in $\text{Fun}$. 
I have left out the implication that $- \otimes_R M$ is always fp-injective.

Suppose $U \subset h^X$ is a finitely generated subfunctor of a representable functor $h^X$. Then we have an epimorphism $h^X \to U$. Then if $X = \text{Coker} (X_0 \to X_1)$ we have

$$(\ast) \quad 0 \to h^X_1 \to h^X_0 \to h^X$$

whence $U = h^X_0 / h^X_1$. Then given $U \to M$ is equivalent to an element of the kernel of $X_0 \otimes_R M \to X_1 \otimes_R M$, which by exactness of

$$X_1 \otimes_R M \to X_0 \otimes_R M \to X \otimes_R M \to 0$$

comes from an elt. of $X_1 \otimes_R M$, i.e., a map $h^X_1 \to - \otimes_R M$ extending $\varphi$.

The construction of $(\ast)$ shows that any finitely generated functor in $\text{Fun}(\text{fpmod}(R^0), \text{Ab})$ has a projective resolution of length $\leq 2$.

I want now to examine again the case $R = k[x, y], \ I = (x, y)$, keeping in mind also the graded module situation. Recall that $R\text{-mod}/I\text{-tors} \cong \text{quasi-coherent sheaves on the affine plane with origin removed}$. In the graded situation we get the category of quasi-coherent sheaves on $\mathbb{P}^1$.

Recall we have

$$I\text{-firing } \cong \text{lim cont. } \text{Fun}(R\text{-mod}/I\text{-tors}, \text{Ab})$$

On the other hand, because $R$ is noetherian comm.

it should be true that

\[ \text{lim cont.} \]
functor \[ R\text{-mod}/I\text{-tors} \to \text{Ab} \] should be equivalent to a right exact functor
\[ fg\text{-mod}(R)/fgI\text{-tors} \to \text{Ab} \]

I want to look at the other functor in this situation:
\[ R\text{-mod}(R)/I\text{-tors} \to \text{Fun}(I\text{-fim}, \text{Ab}) \]

I recall that there is a canonical isomorphism:
\[ \text{Tor}^R_n(k, M) = \text{Ext}^2_R(k, M) \]

so that \( \text{flat} \Rightarrow I\text{-cofim} \)
\( \text{inj} \Rightarrow I\text{-fim} \).
Here's an improvement concerning flat
form resolution.

Let $M$ be an $R$-module. Then
$\text{Tor}_j^R(R/I,M) = 0$ for $0 \leq j \leq n$ iff $E$ a
resolution

$\cdots \to E_n \to E_{n-1} \to \cdots \to E_0 \to M \to 0$

where $E_j$ is flat $0 \leq j \leq n$.

Proof: $(\leq)$ One has because $E_0, \ldots, E_n$ are flat
\[ \text{Tor}_j^R(R/I,M) = H_j(E/IE) \]
and $E_j/IE_j = 0$ in degrees $\leq n$ because they are
flat.

$(\Rightarrow)$ Let $\longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a projective
resolution of $M$. Then $\text{Tor}_j^R(R/I,M) = H_j(P/IP) = 0$
for $j \leq n$. Let $\pi : P \longrightarrow P/IP$ be the canary surjection,
let $P(\leq n)$ be the $n$-skeleton of $P$. Then because
$P(\leq n)$ consists of proj modules and $H_j(P/IP) = 0$
for $j \leq n$, the restriction $P(\leq n) \longrightarrow P/IP$ of $\pi$
is null homotopic. Choose a null homotopy and
lift it to an operator of degree one on $P$
s.t. $h(P_j) = 0$, $j > n$, and let $f = 1 - [d, h]$. Then
$\pi(f) = \pi - [d, \pi(h)] = 0$ on $P_j$ for $j \leq n$, so
$f(P_j) < IP_j$ for $j \leq n$. Let
\[ E = \lim_{\longrightarrow} (P \xrightarrow{+} P \xrightarrow{+} \cdots) \]
$E$ is flat, form in degrees $\leq n$, and $\alpha$-resolution of
\[ M \quad \text{since} \quad H_\ast(E) = \lim_\rightarrow (H_\ast(p) \to H_\ast(p)) \to . \]

In the case of a noetherian, commutative ring \( R \), we can try to show that
\[
\text{mod}(R) / \text{I-tors} \to \text{Fun}(\text{I-fun}, \text{Ab})
\]
is fully faithful as follows. Let \( I = \sum_{i=1}^{n} R q_i \).

Then we have the open affine covering
\[ \text{Sp}(R) = \bigcup \text{Sp}(R q_j) \] which leads to

the Čech formula
\[
0 \to j_* j^* M \to \prod_j \Gamma(\text{Sp}(q_i), j^* M) \to \prod_j \Gamma(\text{Sp}(R q_j), j^* M)
\]
or
\[
0 \to j_* j^* M \to \prod_j R_{q_j} \otimes_R M \to \prod_j R_{q_j} \otimes_R M
\]

Thus if \( \overline{F} : - \otimes_R M \to - \otimes_R N \) is a map of functors, where say \( M \) and \( N \) are coherent, then
\[
0 \to M \to X_0 \otimes_R M \to X_1 \otimes_R M
\]
\[
0 \to N \to X_0 \otimes_R N \to X_1 \otimes_R N
\]
yields a map \( f : M \to N \). We now want to show that \( \overline{F} = 1 \otimes f \). Subtracting, we can assume \( f = 0 \), i.e. \( \overline{F} : - \otimes_R M \to - \otimes_R N \) is a map which vanishes for \( - = R q_j \).

At this point we need to know something more about flat modules.
Suppose I finitely generated as right $R$-module: $I = \sum a_i R$. Consider an $R$-module $M$ such that $IM = M$. To fix the ideas suppose $R$ is a $k$-algebra, $k$ a field, let $V$ be a vector space with basis $v_1, \ldots, v_n$ and let $v_1^*, \ldots, v_n^*$ be the dual basis for $V^*$.

The idea is to construct a flat form $R$-module mapping into $M$, but always using the fact that $M = IM = \sum q_i M$, so that any $m \in M$ can be written $m = \sum q_i m_i$ for some choice of $m_i$, $1 \leq i \leq n$.

Let $\phi$ be the map of $R$-modules

$$R \rightarrow R \otimes V \otimes V^* \rightarrow R \otimes V^*$$

$$r \mapsto \sum_i r \otimes v_i \otimes v_i^* \mapsto \sum_i r a_i \otimes v_i^*$$

We construct module maps

$$R \xrightarrow{\phi} R \otimes V^* \xrightarrow{\rho \otimes 1} R \otimes V^* \otimes V^*$$

$$\downarrow \phi \otimes \text{id}_V \downarrow \phi \otimes \text{id}_{V^*}$$

$$M = M = M = M$$

Choose $m \in M$ and let $u_0(r) = rm$. Choose $m_i \in M$ such that $m = \sum q_i m_i$.

Choose $m_i \in M$ such that $m_i = \sum_j q_j m_{ij}$. Then

$u_0(\phi(r)) = u_1(\sum r a_i \otimes v_i^*) = \sum r a_i m_i = rm = u_0(r)$. 

Choose $m_i \in M$ such that $m_i = \sum j q_j m_{ij}$. 

$u_1(\phi(r)) = u_1(\sum r a_i \otimes v_i^*) = \sum r a_i m_i = \sum q_j m_{ij} = u_0(r)$.
and put \( u_2(v_j \otimes u^*_j \otimes u^*_i) = \sum_{m_j} m_{ij} \).

Then \[
u_2 \circ (\psi \otimes 1 \otimes u^*_i) = u_2(\sum_{m_j} m_{ij} u^*_j \otimes u^*_i) = \sum_{m_j} m_{ij} = m_i = u_1(1 \otimes u^*_i),
\]

Thus \( u_2 \circ (\psi \otimes 1) = u_1 \). It's clear the construction continues with choosing \( m_{ij} = \sum_{m_k} m_{ik} \), etc.

This construction shows that
\[
\lim_{\rightarrow} \left( R \xrightarrow{u} R \otimes V^* \xrightarrow{\psi} R \otimes V^* \otimes V^* \xrightarrow{\ldots} \right)
\]
is a generator for the category of firm flat modules. In the case \( R \subseteq T(V) \), \( a_i \leftrightarrow v_i \), the above limit is the Aunty algebra \( O_V \) with generators \( T_v, T^*_v \) for \( v \in V \), \( \lambda \in V^* \) subject to the relations \( T^*_v T_v = \langle \lambda, v \rangle \), \( \sum_{i} T^*_v T^*_v = 1 \).

The reason for this is the fact that a firm module is equivalent to a vector space equipped with an isomorphism \( V \otimes M \cong M \).

It seems that the above inductive limit is just the base extension of \( O_V \):
\[
R \otimes T(V)
\]
and moreover it should be describable also as \( R \otimes T(V^*)/\sum a_i u^*_i = 1 \).

Given \( M \) such that \( M \cong \text{im} \)

as above, so that \( V \otimes M \rightarrow M \) is
surjective, if we choose a lifting then we obtain operators $T_i \in \text{V}^*$

\[
\begin{array}{c}
M \xrightarrow{\text{lifting}} V \otimes M \xrightarrow{\Delta_1} M \\
\end{array}
\]

such that $\sum a_i T_{v_i}^* = 1$. (Note the lifting is $m \mapsto v_i \otimes T_{v_i}^* m$.)

Thus $M$ becomes a module over $R \otimes_{\text{V}} R^*$. In general $M$ is a quotient of a direct sum of copies of the $R$-module $R \otimes_{\text{V}} R^*$. The choice of the lifting gives a systematic choice of solutions for $m = \sum a_i m_i$, $m_i = \sum a_i^* v_i$, etc.

Consider now the case $R = S(\text{V})$, the polynomial ring. I would like to understand the firm modules in this case, in particular whether they form an abelian category. It would be nice if there were a smaller generator than $R \otimes_{\text{V}} R^* = R \otimes_{\text{V}} T(\text{V}) / \sum v_i v_i^* = 1$.

Notice that there is commutative version of this, namely $S(\text{V}) \otimes S(\text{V}^*) / \sum v_i v_i^* = 1$.

There are various questions to ask. For example, one might restrict to graded modules and consider $\mathbb{Z}$-graded $R = S(\text{V})$, algebras $S = \bigoplus S_n$ such that $V S_n = S_0$. In this case it seems that $S_1 = S_0 V$ is an invertible $S_0$ module with inverse $S_{-1}$. So we are looking at rings over $S(\text{V})$ which invert the ideal $S(\text{V}) V$.

Examples: $S(\text{V})_0 = S(\text{V})[A^*] / (A^* V = 1)$

or $S(\text{V}) \otimes S(\text{V}^*) / \sum v_i v_i^* = 1$. 
Notice that $S(V) \otimes S(V^*) / \sum v_i v_i^* = 1$ has the associated variety consisting of pairs $\lambda \in V^*$, $\nu \in V$ such that $\nu(\lambda) = 1$.

This is an affine variety which fibres with affine space fibres over both $V^* - 0$ and $V - 0$.

There's something reminiscent of Morita equivalence here.
Pre-additive category = additive category without the existence of $0$ and $\oplus$.

A small pre-additive category $A$ is the same as a monoidal ring $A$ with a matrix decomposition

$$A = \bigoplus_{\alpha, \beta} A_{\alpha \beta}$$

such that $A_{\alpha \beta} A_{\gamma \delta} = \begin{cases} 0 & \beta \neq \gamma \\ A_{\alpha \delta} & \beta = \gamma \end{cases}$

such that $\exists e_\alpha \in A_{\alpha \alpha}$ such that

$e_\alpha f = f$ for $f \in A_{\alpha \beta}$

$f e_\alpha = f$ for $f \in A_{\beta \alpha}$

Thus if $\operatorname{Ob} A$ is finite, $A$ is the same as a monoidal ring with a matrix decomposition.

In particular, a M"{o}bius context

$$\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$$

is the same as a pre-additive category with two objects. Some other examples:

$$\begin{pmatrix} R & Q & R & R \\ P & S & P & P \\ R & Q & R & R \\ R & Q & R & R \end{pmatrix} \begin{pmatrix} R_0 & R_1 & R_2 \\ \vdots & R_{-1} & R_0 & R_1 \\ R_2 & R_1 & R_0 \end{pmatrix}$$

"Hankel" matrix ring

where $R = \bigoplus_{n \in \mathbb{Z}} R_n$ is $\mathbb{Z}$-graded.
Suppose that $A = \bigoplus_{x \in X} A_{A_{\lambda}}$ corresponds to a preadditive category. One has

$$A_{e_\lambda} = \bigoplus_{x \in X} A_{A_{\lambda}}$$

so as left $A$-module one has

$$A = \bigoplus_{\beta} A_{e_\beta}$$

which means that $A$ is a projective $\tilde{A}$-module.

Recall the picture

$$\text{null} \leftrightarrow \text{mod}(\tilde{A}) \leftrightarrow M(A)$$

In this situation, $\tilde{A}/A$ is a flat $A_{e_\lambda}$ module so $f^*(M) = \tilde{A}/A \otimes_{A_{e_\lambda}} M = M/M_{\lambda}M$ is exact. Because $A$ is a projective $A_{e_\lambda}$ module, $f^* (\bullet) = \text{Hom}_{A_{e_\lambda}} (A_{e_\lambda}, M)$ is exact, so $f^*$ is exact.

We should have an identification

$$M(A) = \text{Additive functor } (A, A\text{-mod})$$

with

$$f^!(F) = \bigoplus_{\alpha} F(A_{\alpha}), \quad f_*(F) = \Gamma \Gamma F(\alpha).$$

Also

$$0 \to \mathcal{L}_{e_{\lambda}} (M) \to M \to f_* f^* M \to \mathcal{L}_{e_{\lambda}} R^1 \mathcal{L}^1 (M) \to 0$$

and

$$R^g f_* (f^* M) \to \mathcal{L}_{e_{\lambda}} R^g M = 0 \text{ for } g \geq 1,$$

because $f^*$ is exact.
How to calculate $L_j!(j^*M)$.

First remark is that the construction of a firm flat resolution module mod-$M$ makes sense for a complex held below, say $M$, is a chain complex of $R$-modules, and pick an surjection $F_0' \to I^e_{R, M_0}$. Then

$\cdots \to F'_2 \to F'_1 \to F'_0 \to 0 \to$

$\cdots \to I^e_{R, M_1} \to I^e_{R, M_0} \to 0 \to$

gives a surjection of complexes $F_0' \to I^e_{R, M}$ such that $F$ is firm flat. Now proceed as before

$0 \to K_1 \to F_0 \to I^e_{R, M_0} \to 0$

$0 \to K_2 \to F_1 \to I^e_{R, K_1} \to 0$

and we obtain a double complex $F_\cdot$ of firm flat modules together with an augmentation $F_0 \to M$, which horizontally resolves $M$.

Now if $E \to R$ is a right $R$-module resolution module mod-$M$-module, with $E$ firm flat, then one has quasi

$E \otimes_R M \leftarrow E \otimes_R F \to F$

and any of these complexes represents $L_j!(j^*M)$.

Remark that $F$ is a complex of $R$-modules, but $E \otimes_R M$ is only a complex of abelian groups.
There's a problem it seems in constructing $E$ as a bimodule complex.

Here's another way to obtain $L_j f^* M$:

$$L_j (f^* M) = \lim \left\{ \cdots \rightarrow I \otimes I^j M \rightarrow I^{j-1} M \rightarrow M \right\}$$

where the inverse system is essentially constant in the sense that the maps become more and more connected. To prove this formula let $F \rightarrow M$ be as above and let $C$ be the cone on this map. Then $F$ is from flat modules, so

$$I \otimes_R F \rightarrow IF = F$$

is a quasi, so the inverse system in the case of $F$ is constant up to quasi. Next the triangle

$$I \otimes_R C \rightarrow R \otimes_R C \rightarrow R/I \otimes_R C$$

shows that the homology group of $I \otimes_R C$ are null, since $C$ is (and obviously $R/I \otimes_R C$) have this property. But if the lowest homology group of $C$ is in degree $n$, then $H_j (I \otimes_R C) = 0$ for $j < n$ and $H_n (I \otimes_R C) = I \otimes_R H_n (C) = 0$ since $I = I^2$ and $IH_n (C) = 0$. Thus $I \otimes_R C$ is at least 1 more connected than $C$. Thus $\{ I \otimes_R C \}$ is essentially zero, and

$$F = \{ I \otimes_R F \} \sim \{ I \otimes_R M \}$$

establishing the formula.
We continue with the derived category situation. In $D_+(\text{mod}(R))$ we have a canonical distinguished triangle

$$I \otimes_R M \rightarrow R \otimes_R M \rightarrow R/I \otimes_R M \rightarrow$$

1) $\sim$ $\sim$

$$M \rightarrow L_x L^!(M)$$

Thus, to produce a canonical distinguished triangle

2) $L_{j_!} j^!(M) \rightarrow M \rightarrow L_x L^!(M)$

we must produce a canonical isomorphism.

3) $L_{j_!} j^!(M) \cong I \otimes_R M$

In particular, we must have

4) $0 \cong I \otimes_R R/I$

Assuming 4), we now construct 2) + 3).

$L_{j_!} j^! M$ can be calculated using any complex in $C(M)$ which is quasi $j^! M$ and which consists of flat objects.

$L_{j_!} j^! M \cong F$ where $F$ is any complex (below) of flat free modules equipped with a quasi $F \rightarrow M$ modulo null modules. Construction of such an $F$ using the fact that any complex $N$ such that $IN = N$ is a quotient $F \rightarrow N$ of a flat free complex:

$$0 \rightarrow K_1 \rightarrow F_0 \rightarrow I \otimes_R M \rightarrow 0$$

$$0 \rightarrow K_2 \rightarrow F_1 \rightarrow I \otimes_R M \rightarrow 0$$
This yields a double complex \( F \) of flat \( R \)-modules together with a "horizontal" augmentation \( F \rightarrow M \) which is a quasi mod null.

Suppose now that \( M \) is a complex of projective modules. The condition \( I \otimes_R I = 0 \) means that \( I \) has a resolution \( E \) by flat right \( R \)-modules. Consider the diagram

\[
\begin{array}{ccc}
E \otimes_R F & \xrightarrow{\alpha} & I \otimes_R F = F \\
\downarrow \phi & & \downarrow \delta \\
E \otimes_R M & \xrightarrow{\varepsilon} & I \otimes_R M
\end{array}
\]

The map \( \alpha \) is a quasi because \( E \rightarrow I \) is a quasi and \( F \) is flat; similarly \( \delta \) is a quasi because we are assuming \( M \) projective.

The map \( \beta \) is a quasi because \( E \) is flat and \( H_x(F) \rightarrow H_x(M) \) is an \( \lim \) mod null modules.

Thus \( \delta \) is a quasi. But \( \delta \) is a map of \( R \)-module complexes (unlike \( \beta, \alpha, \delta \)). Thus we have

\[
L_\varepsilon^* M \rightarrow F \rightarrow I \otimes_R M = I \otimes_R M
\]

Next consider \( R_{j\ast}(g^* M) \). In \( D^+(\text{mod}(R)) \) we have a canonical dist \( \Delta \)

\[
\begin{array}{cccc}
R \text{Hom}_R(R/I, M) & \xrightarrow{\sim} & R \text{Hom}_R(R_jM) & \xrightarrow{\sim} & R \text{Hom}_R(I_j M) \\
\downarrow \alpha & & \downarrow \alpha & & \downarrow \alpha \\
L_j R^j \ast(M) & \rightarrow & M
\end{array}
\]
so that to have a canonical distinguished \( \Delta \)

\[
\varphi: R \Gamma ! (M) \to M \to Rf_* f^*(M) \to
\]

we need a canonical isom.

\[
R \text{Hom}_R (I, M) \cong Rf_* f^*(M)
\]

Up to isom. in \( R \text{D}^+ (\text{mod}(R)) \) we can suppose \( M \) injective.

Recall that \( Rf_* f^* M \cong Q \) where \( Q \) is a complex of cofiber injective modules equipped with a quasi \( M \to Q \) module null modules. Consider the diagram of complexes (of abelian groups)

\[
\begin{array}{ccc}
\text{Hom}_R (I, M) & \xrightarrow{\alpha} & \text{Hom}_R (F, M) \\
\downarrow{\beta} & & \downarrow{\gamma} \\
Q = \text{Hom}_R (I, Q) & \xrightarrow{\delta} & \text{Hom}_R (F, Q)
\end{array}
\]

where \( F \) is a finite flat left module resolution of \( I \). Because \( M, Q \) are injective and \( F \to I \) is a quasi, the maps \( \alpha, \beta \) are quasi. To see \( F \) is a quasi it's equivalent to show that \( \text{Hom}_R (F, c) \) is quasi \( 0 \) where \( C \) is the cone on \( M \to Q \). Thus \( C \) is a complex of injectives whose homology is null.

So we reach the problem of showing \( R \text{Hom}_R (F, N) = 0 \) where \( F \) is a flat finite complex and \( N \) is complex with null homology. But it's clear that I shouldn't have introduced \( F \).
Let's start again with the whole derived category business.

We wish to construct a canonical functional distinguished triangle in $D^b(\text{mod}(R))$

1) \[ L^1 j^! f^*(M) \rightarrow M \rightarrow L^1 L^* f^*(M) \rightarrow \]

Since one has the distinguished \( \Delta \)

\[ I \otimes_R M \rightarrow R \otimes_R M \rightarrow R/I \otimes_R M \rightarrow \]

\[ M \rightarrow L^1 i^* f^*(M) \]

our task amounts to constructing a cannon isomorphism.

2) \[ L^1 j^! f^*(M) \rightarrow I \otimes_R M \]

Now 2) implies that if \( M \) is a complex (both below) with null homology, then \( I \otimes_R M = 0 \). In particular

2) \[ I \otimes_R R/I = 0. \]

Conversely assume \( I \otimes_R R/I = 0 \), i.e., \( \text{Tor}^R_{\ast}(I, R/I) = 0 \). I claim then that \( \text{Tor}^R_{\ast}(I, N) = 0 \) for all null modules \( N \). In effect assume \( \text{Tor}^R_{\ast}(I, N) = 0 \) for all \( i < n \) and null module \( N \). Choose an exact sequence \( 0 \rightarrow N_i \rightarrow (R/I)^a \rightarrow N \rightarrow 0 \). Then \( \text{Tor}^R_{\ast}(I, N) \rightarrow \text{Tor}^R_{n-1}(I, N_1) = 0 \).

It follows then from the spectral sequence \( E^2_{pq} = \text{Tor}^R_{p}(I, H^q(M)) \Rightarrow H_n(I \otimes_R M) \)

that \( I \otimes_R M = 0 \) for any complex \( M \) with null homology.
Finally $L_f j^* M$ can be calculated using a flat resolution of $j^* M$ in $M$.
This amounts to a form flat complex $F$ together with a quasi $F \to M$ modulo small modules. The cone $C$ for this map then has null homology so $I \otimes_C R \to 0$, whence
$I \otimes_F R \to I \otimes_M R$ is a quasi. (I forgot to mention that $L_f j^* M = F$ above). Then we have

$$L_f j^* M = F \xRightarrow{\text{quis}} I \otimes_R F \xrightarrow{\text{quis}} I \otimes_M R$$

yielding 2).

Next we wish to obtain a canonical distinguished $\Delta$ in $D^+(\text{mod}(R))$

$$3) \quad \otimes R\text{-}
\text{Hom}(M) \to M \to R_f j^* j^* M \to$$

and since one has the distinguished $\Delta$

$$R\text{Hom}_R (R I, M) \to R\text{Hom}_R (R j, M) \to R\text{Hom}_R (I, M) \to$$

$$\otimes \text{RHom}(M) \to M$$

we need a canonical isom

$$4) \quad R_f j^* j^* M \cong R\text{Hom}_R (I, M)$$

Note that 4) $\Rightarrow R\text{Hom}_R (I, M)$ when $j^* M = 0$, i.e. $M$ has null homology. In particular $\text{Ext}^1_R(I, N) = 0$ for all null modules $N$.

Let $P \to I$ be a projective resolution. Take $N = \text{Hom}_R(P, I, \mathbb{Q}/\mathbb{Z})$, where left mult. of $R$ on $N$ comes from the right mult in $R/I$. Then
\[
\text{Ext}^*_R(I, N) = H^*_R \text{Hom}_R(P, \text{Hom}_\mathbb{Z}(R/I, Q/Z)) \\
= H^*_R \text{Hom}_\mathbb{Z}(R/I \otimes_R P, Q/Z) \\
= \text{Hom}_\mathbb{Z}(H^*_R(R/I \otimes_R P), Q/Z) \\
\cong \text{Tor}_R^*(R/I, I)
\]

Thus 4) \Rightarrow

5) \text{Tor}_R^*(R/I, I) = 0 \quad \text{i.e.} \quad R/I \otimes_R I \cong 0.

Conversely, assume 5). Then we claim \text{Ext}^*_R(I, N) = 0 for all null modules \(N). By the preceding Ext calculation we know this holds for \(N\) of the form \(\text{Hom}_\mathbb{Z}(R/I, Q)\) with \(Q\) any injective \(\mathbb{Z}\) module and any \(\mathbb{Z}\) null module in such a module. So again we can argue that if \text{Ext}^*_R(I, N) for all \(\mathfrak{m}\) null and \(\mathfrak{n}\) null modules \(N\), then upon embedding \(N\) into some \(\text{Hom}_\mathbb{Z}(R/I, Q)\) and letting \(N_1\) be the cokernel, we have \(0 = \text{Ext}^*_R(I, N_1) \cong \text{Ext}^*_R(I, N)\), proving the claim.

From the spectral sequence
\[
E_2^{p, q} = \text{Ext}^p_R(I, H^q(M)) \Rightarrow H^{p+q}(R \text{Hom}_R(I, M))
\]
we conclude that \(R \text{Hom}_R(I, M) \cong 0\) if \(M\) has null homology.

Finally, \(R \otimes_R j^*M\) can be calculated using an injective resolution of \(j^*M\) in \(M\), which amounts to an complex \(Q\) of coform injectives together with a quasi \(M \to Q\) module null modules. One has
\[
R \otimes_R j^*M = Q.
\]
Then
\[ Rf_j \star j^* M = Q \longrightarrow R\text{Hom}_R(I, Q) \quad \leftarrow \quad R\text{Hom}_R(I, M) \]

The first map is a quasi because \( Q \) is cofibrant injecting, the second because the arc \( \alpha \) on \( M \rightarrow Q \) has null homology, hence \( R\text{Hom}_R(I, C) = 0 \). Thus we get the desired isom.

\[ Rf_j \star j^* M \cong R\text{Hom}_R(I, M) \]

Observe we can replace \( I \) by a complex \( U \) in the above arguments. Two cases:

1) If \( U \) is a complex of \( R^{ab} \) modules (bold below) then \( U \otimes_R R/I = 0 \Leftrightarrow U \otimes_R \) kills complexes with null homology (complexes in \( D^+(\text{mod}(R)) \)).

2) If \( U \) is a complex of \( R \) modules (bold below) then \( R/I \otimes_R U = 0 \Leftrightarrow R\text{Hom}_R(U, \_ \_) \) kills complexes with null homology (complexes in \( D^+(\text{mod}(R)) \)).

Proof of 1). Enough to prove \( \Rightarrow \). Given \( M \) bold below with null homology, the Postnikov system of \( M \) reduces to the case where \( M \) is a null module \( N \). Writing \( N \) as a quotient of \( (R/I)^{(n)} \) with kernel \( N_1 \), then repeating to obtain \( N_2 \), etc.

we have
\[ U \otimes_R N \longrightarrow U \otimes_R N_1[1] \longrightarrow U \otimes_R N_2[2] \longrightarrow \]

But these are getting more and more connected, so all homology groups of \( U \otimes_R N \) are zero.
Proof of 2) Let $P$ be a proj. resolution of $U$, let $Q$ be any injective $\mathbb{Z}$-module. Then
\[
\text{RHom}_R (U, \text{Hom}_\mathbb{Z} (R/I, Q))
\]
\[
\cong \text{Hom}_R (P, \text{Hom}_\mathbb{Z} (R/I, Q))
\]
\[
\cong \text{Hom}_\mathbb{Z} (R/I \otimes_R P, Q)
\]
\[
\cong \text{Hom}_\mathbb{Z} (R/I \otimes_R U, Q)
\]

If $\text{RHom}_R (U, -)$ kills complexes with null homology, then this shows $R/I \otimes_R U = 0$, so we obtain the direction $\subseteq$.

Conversely assume $R/I \otimes_R U = 0$. To prove $\text{RHom}_R (U, M) = 0$ for any complex in $D^+ (\text{mod}(R))$ with null homology, we can reduce to the case where $N$ is a null module via the Postnikov system of $N$. The above calculation gives $\text{RHom}_R (U, N) = 0$ for $N$ of the form $\text{Hom}_\mathbb{Z} (R/I, Q)$, with $Q$ any injective $\mathbb{Z}$-module. We can embed $N$ in some $\text{Hom}_\mathbb{Z} (R/I, Q)$; then if $N'$ is the cokernel, embed $N'$ similarly to obtain $N''$; etc. Then
\[
\text{RHom}_R (U, N) \cong \text{RHom}_R (U, N') [-1] \cong \text{RHom}_R (U, N'') [-2] \cong \cdots
\]
so all the homology groups of $\text{RHom}_R (U, N)$ are zero. This proves $\supseteq$.

Notice that the above proof holds without assuming $I = I^2$, provided we use the fact that any null module $I^n N = 0$ is an extension of modules killed by $I$. 


Morita invariance examples.

Suppose \( A \triangleleft B \), \( B = \tilde{B}A\tilde{B} \) the ideal generated by \( A \). Factor
\[
A \subset A\tilde{B} \subset \tilde{B}A\tilde{B} = B
\]
The second inclusion is such that \( A\tilde{B} \) is a right ideal in \( B \) such that \( B \) is the ideal \( \tilde{B}A\tilde{B} \) generated by \( A\tilde{B} \).

Assume now that \( A \) is a left ideal in \( A\tilde{B} \): \( A\tilde{B}A \subset A \) and that \( A\tilde{B} \) is the ideal in \( A\tilde{B} \) generated by \( A \): \( A\tilde{B} = A\tilde{B}A\tilde{B} = \tilde{A}\tilde{B} \subset A + A^2\tilde{B} \).

Then \( A\tilde{B}A = A^2 + A^2\tilde{B}A \). So \( A\tilde{B}A = A^2 \).

Thus we get the conditions \( \tilde{B}A\tilde{B} = B \), \( A\tilde{B}A = A^2 \).

Recall the cases:

- \( A \) a left ideal in \( B \)
- \( A \) right ideal in \( B \)

Combine in the above situation:

\[
\begin{pmatrix}
A & A\tilde{B} \\
A & A\tilde{B} \\
\tilde{B}A & B
\end{pmatrix}
\]

\[
\begin{pmatrix}
A & \tilde{B}A \\
\tilde{B}A & \tilde{B}A
\end{pmatrix}
\]

Giving the composite Morita equivalence
\[
\begin{pmatrix}
A & \tilde{B}A \\
\tilde{B}A & B
\end{pmatrix}
\]

\[
QP = \tilde{B}A\tilde{B}A = A\tilde{B}A = A^2 \subset A
\]

\[
PQ = \tilde{B}A^2\tilde{B} = \tilde{B}A\tilde{B}A\tilde{B} = (\tilde{B}A\tilde{B})^2
\]

\[
= B^2 \subset B
\]
Suppose \( A/K = B \) where \( AK_A = 0 \).

Recall the two cases,

\[
K'A = 0 \quad M \rightarrow A \otimes_A M \quad \begin{pmatrix} A & B \\ \tilde{B} & \tilde{B} \end{pmatrix} = \begin{pmatrix} A & A \\ \tilde{B} & \tilde{B} \end{pmatrix} / \begin{pmatrix} 0 & K' \\ 0 & K' \end{pmatrix}
\]

\[
AK'' = 0 \quad X \rightarrow X \otimes_A \tilde{A} \quad \begin{pmatrix} A & A \\ \tilde{A} & \tilde{A} \end{pmatrix} = \begin{pmatrix} A & A \\ \tilde{A} & \tilde{A} \end{pmatrix} / \begin{pmatrix} 0 & 0 \\ K'' & K'' \end{pmatrix}
\]

Factor \( A \rightarrow A/K' \rightarrow A/K \quad \text{want} \quad AK \subset K' \)

\[
\begin{pmatrix} A & A/K' \\ A & A/K' \\ A/K \end{pmatrix} \quad \tilde{A}/K' \otimes_{A/K'} A/K' = A/K'
\]

\[
\begin{pmatrix} A/K'' & A/K \\ A/K' & A/K \\ A/K_A & A/K \\ A/K & A/K \end{pmatrix} \quad \tilde{A}/K \otimes_{A/K} A = A/K_A
\]

yielding the composite Morita equivalence

\[
\begin{pmatrix} A & A/K' \\ A/K'' & A/K \end{pmatrix} = \begin{pmatrix} A & A \\ A & A \end{pmatrix} / \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix}
\]

Here I have replaced \( KA \) by \( K'' \). The conditions for to be a Morita context are

\[
\begin{pmatrix} C & K' \\ K'' & K \end{pmatrix} \begin{pmatrix} A & A \\ A & A \end{pmatrix} = \begin{pmatrix} K'A & K'A \\ K''A + KA & K''A + KA \end{pmatrix} \subset \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix}
\]

\( K'A = 0 \quad KA \subset K'' \quad \text{assuming } K', K'' \subset K \quad \text{they are ideals in } A \)

\( \text{Also} \quad \begin{pmatrix} A & A \\ A & A \end{pmatrix} / \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix} = \begin{pmatrix} AK'' & AK'+AK \\ AK'' & AK'+AK \end{pmatrix} \subset \begin{pmatrix} 0 & K' \\ K'' & K \end{pmatrix} \)

\( AK'' = 0 \quad AK \subset K' \)

Starting from \( AK_A = 0 \) we can take \( K'' = KA \quad K' = AK \)
This is a smallest possibility, which leads to a largest $P, Q$. We also have

$$K' = \{ k \in K \mid kA = 0 \} \quad K'' = \{ k \in K \mid Ak = 0 \}$$

which leads to the smallest $P, Q$.

---

Given a Morita context $(R, Q, p, s)$ and ideals $I \subset R \subset J \subset S$. Assume

(*) \[ QJP < I \subset QP \quad \text{PIQ} \subset J \subset PQ \]

Then \[ I^3 \subset QPIQP \subset QJP \subset I \quad J^3 \subset PQJPQ \subset PIQ \subset J \]

shows that $I \sim QJP$ and $J \sim PIQ$.

Moreover given $I \subset QP$ if we set $J = PIQ$, then (*) holds: $PIQ = J \subset PQ$.

QJP = Q(PIQ)P \subset RIR = I \subset QP

---

Let us now go over the details of Morita equivalence when $(R, Q, p, s)$, $QJP \subset I \subset QP$, $PIQ \subset J \subset PQ$.

We first need to see

$$M(R, I) \longrightarrow M(S, J)$$

$$M \longmapsto P \otimes_R M$$

is well-defined. It suffices to show the functor $P \otimes_R - : \text{mod}(R) \rightarrow \text{mod}(S)$ carries $I$-null isomorphisms into $J$-null isomorphisms.

Suppose $M \stackrel{\epsilon}{\rightarrow} N$ is a map of $R$-modules
where kernel and cokernel are killed by I. Better to do the kernel and cokernel separately.

I claim that \( \text{Im} \circ \text{ker}(\varepsilon) = 0 \Rightarrow \text{PIQ} \cdot \text{ker}(\imath \circ \varepsilon) = 0 \), where \( \imath \circ \varepsilon : P \odot M \to P \odot N \). Take \( p \in \text{PIQ} \) and \( p \odot m \in P \odot N \). Then

\[
\text{pa}(p \odot m) = p \odot (q \odot m) = p \odot a(q \odot m).
\]

But \( a \circ \text{ker}(\varepsilon) = 0 \) means \( a(q \odot m) = \varepsilon(m) \) for some \( m \).

Then \( \text{pa}(p \odot m) = p \odot \varepsilon(m) = \imath(\varepsilon)(p \odot m) \).

Next I show \( \text{Im} \circ \text{ker}(\varepsilon) = 0 \Rightarrow \text{PIQ} \cdot \text{ker}(\imath \circ \varepsilon) = 0 \).

Take \( p \in \text{PIQ} \) and \( \sum p_i \otimes m_i \in \text{ker}(\imath \circ \varepsilon) \), i.e. \( \sum p_i \otimes \varepsilon(m_i) = 0 \). We have a well-defined map \( P \otimes R N \to N \) \( p \otimes m \mapsto (q \odot m) \).

Thus \( 0 = \sum (q \odot p_i) \varepsilon(m_i) = \varepsilon \sum (q \odot p_i) m_i \).

Then

\[
\text{pa} \sum p_i \otimes m_i = \sum p \odot (q \odot p_i) \otimes m_i = \imath \circ \varepsilon \sum (q \odot p_i) m_i = 0
\]

using the fact that \( a \circ \text{ker}(\varepsilon) = 0 \). Thus \( \text{PIQ} \cdot \text{ker}(\imath \circ \varepsilon) = 0 \).

Alternative approach: First I show for any \( R \)-module \( M \) that the canonical map

\[
P \otimes R M \to \text{Hom}_R (Q, M) \hspace{1cm} \varphi : p \otimes m \mapsto q \mapsto (q \odot m)
\]

has its cokernel and kernel killed by \( \text{PIQ} \), hence by \( I \).

Let \( \sum p_i \otimes m_i \in \text{ker}(\varphi) \) i.e. \( \sum (q \odot p_i) m_i = 0 \) \( \forall q \)

Then \( \text{pa} \sum p_i \otimes m_i = \sum p \odot (q \odot p_i) m_i = 0 \) showing \( \text{PIQ} \cdot \text{ker}(\varphi) = 0 \).
Let $f \in \text{Hom}_R(Q,M)$. Then
\[
(pg)f(q') = f(g'(pg)) = f(g'p\theta) = g'pf(q) = \varphi(p\circ f(q))(q')
\]
Thus $(pq)f = \varphi(p\circ f(q))$ showing $PQ \circ \text{Coker}(\varphi) = 0$.

Now let $\varepsilon : M \rightarrow N$ be a map of $R$-modules, let $K = \ker(\varepsilon)$, $C = \text{Coker}(\varepsilon)$, and consider the diagram
\[
P \otimes_R M \xrightarrow{i \otimes \varepsilon} P \otimes_R N \xrightarrow{\varphi} P \otimes_R C \rightarrow 0
\]
\[
0 \rightarrow \text{Hom}_R(Q,K) \rightarrow \text{Hom}_R(Q,M) \xrightarrow{\varepsilon_*} \text{Hom}_R(Q,N)
\]
Take $p \in \text{PIQ}$, $p \otimes \varepsilon \\ p \otimes C$. Assuming $I \cdot C = 0$, then
\[
p\otimes f(p \otimes C) = p \otimes f(g'p)c = 0
\]
showing $P \otimes C$ is killed by $P \otimes \text{PIQ}$. 

Let $f \in \text{Hom}_R(Q,K)$. Assuming $I \cdot K = 0$ we have
\[
((pq)f)(q') = f(g'pq) = f(g'p)f(q) = 0
\]
so $\text{PIQ}$ kills $\text{Hom}_R(Q,K)$.

Now $\text{PIQ} \supset J^3$, and we have shown that $\varphi$ maps are $J$-null isomorphisms. Thus we can conclude that $\varepsilon$ is $J$-null epic (resp. monic) $\Rightarrow 1 \otimes \varepsilon$ and $\varepsilon_* \otimes 1$ are $J$-null epic (resp. monic).