Proof. Let \( A \) be a unital ring, let \( M \) be an \( A \)-module. TFAE:

1) \( M \) is flat

2) For all \( A \)-modules \( N \) of finite presentation one has

\[
(*) \quad \text{Hom}_A(N, A) \otimes_A M \rightarrow \text{Hom}_A(N, M)
\]

Proof of 1) \( \Rightarrow \) 2) \( (*) \) is the map \( f \mapsto f(m) \), and it is a morphism of functors of \( N \), which is an isomorphism for \( N = A \), hence for \( N = A^{\oplus n} \) for any \( n \) (thus for \( N \in P(A) \) and any module \( M \)). When \( M \) is flat the functor on the left is left exact, hence both functors are left exact. Thus \( (*) \) is an isomorphism for any \( N \) of finite presentation.

(Note: \( (*) \) is injective for \( N \) of finite type, since if we choose \( A^n \rightarrow N \), then

\[
\text{Hom}_A(N, A) \otimes_A M \rightarrow \text{Hom}_A(N, M)
\]

\[
\text{Hom}_A(A^n, A) \otimes_A M \rightarrow \text{Hom}_A(A^n, M)
\]

Proof of 2) \( \Rightarrow \) 1). We show first that \( (*) \) implies the category of finite type free \( A \)-modules over \( M \) is a fitting category. We must show given solid arrows in the diagram
there exist \( r \) and the dotted arrows. But if \( N = A^8/A^p a \), then \( m \in \text{Hom}_A(N, M) \), and to express this in the form \( \sum f_i \otimes m_i \in N \otimes_A M \) is the same as factoring: \( m = a'm' \) where \( a a' = 0 \).

We recognize in \((**):\) the linear equations criterion for flatness of Cartan-Eilenberg: Any \( A \)-linear relations \( a_m = 0 \) in \( M \) can be written in terms of linear relations in \( A \): \( m_j = a_j k \), where \( a_j a_j' = 0 \).

**But** \( M = \text{lim} A_n \) and a filtered inductive limit of flat modules is flat. Thus 2) \( \Rightarrow \) 1).

Alternative: To show \( M \) flat \( \Rightarrow \) it suffices to show \( \text{Tor}_1^A(X, M) = 0 \) for any right \( A \)-module of finite presentation \( X \). So given \( 0 \to K \to A^p \to A^8 \to X \to 0 \)

we want to show that \( 0 \to K \otimes_A M \to M^p \to M^8 \to X \otimes_A M \to 0 \) is exact at \( M^p \). Define \( N \) by
\[ A^g \overset{a}{\to} A^p \to N \to 0 \]

so that we have

\[ 0 \to N^* \to A^p \overset{a}{\to} A^g \]

identifying \( N^* \) with \( K \). Now we have

\[ 0 \to \text{Hom}_A(N, M) \to M^p \overset{a}{\to} M^g \]

\[ \overset{\cong}{\to} \text{by hypothesis} \]

\[ N^* \otimes_A M \]

so we conclude the exactness of

\[ 0 \to K \otimes_A M \to M^p \overset{a}{\to} M^g. \]

\[ \text{Corollary: } M \text{ flat and of finite presentation } \implies \]

\[ M \text{ projective}. \]

Because in this case we have

\[ \text{Hom}_A(M, M) \cong M^* \otimes M \]

and writing the identity as \( f \otimes m \) gives a map \( M \overset{f}{\to} A^n \overset{m}{\to} M \) with composition \( id_M \).

Ultimately I want to examine whether there is a good analogue of flat and finite presentation modules in the case of a non-minimal algebra, say \( A \otimes_B eA \).
The observation is that \( A \) is \( H \)-unital iff
\[
A \otimes_A A \rightarrow A \text{ is a quis. } \quad \text{On}
\]
view of
\[
0 \rightarrow A \rightarrow \tilde{A} \rightarrow C \rightarrow 0
\]
we have a triangle
\[
\begin{array}{c}
\text{A} \\
\downarrow \\
\text{\tilde{A}} \otimes_A A \\
\downarrow \\
\text{C} \otimes_A A
\end{array}
\]
so \( A \otimes_A A \rightarrow A \text{ is a quis } \iff \text{ C } \otimes_A A = 0 \)
i.e. \( \text{Tor}_1^A (C, A) = 0 \), and this is equivalent to
the bar construction being acyclic.

For an \( A \)-module \( M \), define \( \tilde{M} \) to be
an \( H \)-unital \( A \)-module when \( A \otimes_A M \rightarrow M \)
as a quis, equivalently \( \text{C } \otimes_A M = 0 \). The condition
\( \text{C } \otimes_A M = 0 \) is nice to work with, e.g. if \( M \) is \( A \)-flat,
then \( M \) is \( H \)-unital iff \( M = AM \).

In particular if \( A \) is \( A \)-flat, then \( A \) is an
\( H \)-unital algebra iff \( A^2 = A \).

Let's now consider Morita invariance of
Hochschild homology in the situation \( A = \text{ae} \otimes_B eA \).
Then
\[
A = \text{ae} \otimes_B eA \overset{\text{quis}}{\leftrightarrow} A \otimes_B eA
\]
provided \( eA \) is a flat \( B \)-module. Then
\[
A \otimes_A \overset{\text{quis}}{\rightarrow} \text{ae} \otimes_B eA \otimes_A = eA \otimes_A A \otimes_B \overset{\text{quis}}{\leftrightarrow} B \otimes_B
\]
where we use that
\[ eA \otimes_A eA \sim eA \otimes_A eA = B \]
as \( eA(eA) \) is a flat left (right) \( A \)-module.

The same assumption: \( eA \) is a flat \( B \)-module implies \( A = H\)-unital:
\[
A \otimes \ A \cong A \otimes (Ae \otimes eA) \cong (A \otimes A) \otimes eA \\
\cong Ae \otimes eA \cong A.
\]
(here we use that \( Ae \) is always \( H\)-unital as \( A \)-module).

The hypothesis \( eA \) flat/B can be weakened to \( Ae \otimes _B eA \cong Ae \otimes eA \), i.e.
\( \text{Tor}_n^B(Ae, eA) = 0 \) for \( n > 0 \).

However one reason for liking the hypothesis that \( eA \) is \( B \) flat is that it implies
\[
A = Ae \otimes _B eA \text{ is } A \text{ flat:}
\]
\[
\begin{array}{cccc}
M & \to & M_A \otimes Ae & \to & (M \otimes Ae)_B \otimes eA \\
& & Me & & M \otimes_A eA
\end{array}
\]

Thus \( A \) is \( A \) flat and such that \( A^2 = A \), so \( A \) is \( H\)-unital.

Continuing with the assumption that \( eA \) is \( B \)-flat, we have that \( M \) \( A \)-flat \( \Rightarrow eM = eA \otimes_A M \) is \( B \)-flat \( \Rightarrow Ae \otimes_B eM = A \otimes_A M \) is \( A \) flat. Thus we have an equivalence between flat \( B \)-modules and flat \( A \)-modules \( M \) such that \( AM = M \).
Let's describe all (minimul) algebras of the form $A = Ae \otimes eA$ where $B = eAe$ is the groundfield $\mathbb{C}$.

First note that if $V$ and $W$ are vector spaces and if $\langle v, w \rangle \mapsto \langle v, w \rangle$, $V \otimes W \to \mathbb{C}$ is a bilinear map, then we obtain an associative product on $V \otimes W$ given by

$$(v_1, w_1)(v_2, w_2) = (v_1 \langle w_1, v_2 \rangle, w_2) = (v_1, \langle w_1, v_2 \rangle w_2)$$

Moreover $V$ (resp. $W$) is naturally a left (resp. right) module over this algebra, which we denote $A$.

Suppose now that $e', e'' \in V$, $e'' \in W$ are elements such that $\langle e'', e' \rangle = 1$. Then $e = (e', e'')$ is an idempotent in $A$. One has

$$(e', e'')(v, w) = (e' \langle e'', v \rangle, w)$$

$$(v, w)(e', e'') = (v, \langle w, e' \rangle e'')$$

so that $eA \cong W$, $Ae \cong V$, $eAe = Ce$.

Put

$$V_1 = \{v \in V \mid \langle e'', v \rangle = 0\}$$

$$W_1 = \{w \in W \mid \langle e', w \rangle = 0\}$$

Then $V = Ce' \oplus V_1$, $W = Ce'' \oplus W_1$. Then we can write $A$ in matrix form

$$A = \begin{pmatrix} C & W_1 \\ V_1 & V_1 \otimes W_1 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Here the product in $A$ is given by the product in $A_1 = V_1 \otimes W_1$, obtained by restricting $\langle , \rangle$ to $W_1 \otimes V_1$, ...
as well as the natural left (resp. right) $A_i$ module structure on $V_i$ (resp. $W_i$).

This discussion makes clear the following

A pair $(A, e)$ consisting of a monounital algebra $A$ and an idempotent $e$ in $A$ such that

\[
Ae \otimes eA \xrightarrow{eAe} A
\]

has the form

\[
A = \begin{pmatrix} B & W_i \\ V_i & A_i \end{pmatrix} \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

where $B = eAe$ can be any unital algebra, $V_i = Ae$ can be any right unital $B$-module, $W_i = eA$ is left unital $B$-module, $A_i = V_i \otimes W_i$ equipped with the product arising from a $B$-bimodule map $W_i \otimes V_i \rightarrow B$ which can be arbitrary.

The product in $A$ is given by the product in $B$ and $A_i$, the $B$-module structures on $V_i, W_i$, and the $A_i$-module structures on $V_i, W_i$.

Let’s go back to the case where $B$ is the groundfield $C$. Then we find that a pair $(A, e)$ such that $eAe = C$ has the form

\[
A = \begin{pmatrix} C & W_i \\ V_i & V_i \otimes W_i \end{pmatrix}
\]
where the product is associated to any pairing \( W_i \otimes V_i \to \mathbb{C} \). The most degenerate case is where the pairing \( \langle , \rangle \) is identically zero. The most non-degenerate case is where \( V_i, W_i \) are finite dimensional and in duality via the pairing (equivalently \( A_i \) is unital). In this case \( A \) is a matrix algebra.

Consider the general situation \( A = A e \otimes e A \), \( B = e A e \). We know that \( A^2 = A \) and that there is an equivalence between \( B \)-modules and good \( A \) modules.

Assume that \( e A \) is a flat \( B \)-module. Then we know that \( A \) is a flat \( A \)-module, hence (since \( A^2 = A \)) that \( A \) is \( H \)-unital: \( A \otimes_A A \simeq A \).

Furthermore we have equivalence of Hochschild (probably also cyclic) homology: \( A \otimes_A \simeq B \otimes_B \).

Now I claim that one also have equivalence on the level of \( K \)-theory, more precisely that the bimodule \( A e \) (which is a representation of \( B \otimes e e A \)) give rise to an isomorphism

\[
K_* (B) \xrightarrow{\sim} K_* (A)
\]

The reason is that \( K \) theory commutes with filtered inductive limits of algebras. Thus \( e A e \) is a flat \( B \)-module, so it is an inductive limit of finitely free \( B \) modules. But recall that the algebra
$A$ has the form

\[ A = \begin{pmatrix} B & W_i \\ V_1 & V_i \otimes W_i \end{pmatrix}, \]

where $V_1$ (resp. $W_i$) is a right (resp. left) $B$-module and the product depends only on a $B$-bimodule map $W_i \otimes V_1 \rightarrow B$. Thus I can suppose, or better reduce to the case where $V_1 = e^* A e$ is finitely presented over $B$ and $W_i = e A e^*$ is finite type free. Then we can apply the Davydov result which tells us that assuming $e A \in \mathcal{P}(B)$, and $A$ an ideal in $R$, we have

\[ K_*(R) = K_*(B) \oplus K_*(R/A) \]

In the present situation we take $R = \tilde{A}$.

I now want to understand concretely why $K_0(B) \rightarrow K_0(A)$ when $A = A e \otimes_B e A$, $B = e A e$ and $e A$ is a flat $B$ module. Recall that

\[ K_0(A) \overset{\text{def}}{=} \ker \left( K_0(\tilde{A}) \rightarrow K_0(e) \right) \cong K_0(\tilde{A})/K_0(e) \]
Consider \( A = A e \otimes e A \), \( B = e A e \). Call an \( A \) module \( M \) of finite type (resp. of finite presentation) when it is so as \( \tilde{A} \) module, i.e. when \( \tilde{A} \tilde{P} \rightarrow M \) (resp. \( \tilde{A} \tilde{P} \rightarrow \tilde{A} \tilde{B} \rightarrow M \rightarrow 0 \)).

Prop. 1) \( \exists \) surjection \( A e \otimes e A \rightarrow M \Leftrightarrow M \) is f.t. and \( A M = M \).

2) \( \exists \) presentation \( A e \otimes e A \rightarrow \tilde{A} \tilde{B} \rightarrow M \rightarrow 0 \) \( \Leftrightarrow M \) is f.p. and \( A \otimes_A M \sim \rightarrow M \).

Proof: The direction \( \Rightarrow \) is easy, so consider \( \Leftarrow \).

Suppose \( \exists \tilde{A} \tilde{P} \rightarrow M \Leftrightarrow \), i.e. \( M \) is generated by elements \( m_i \), \( 1 \leq i \leq p \). Since \( M = A M = A e A M \subset A e M \), there exist \( a_{ij} \in A \), \( m'_j \in M \) such that

\[ m_i = a_{ij} e m'_j \quad \text{(here } 1 \leq j \leq q) \]

Then one has maps

\[ A e \tilde{B} \subset \tilde{A} \tilde{B} \rightarrow M \]

\[ (a_j) \mapsto (a_{11} \otimes \cdots \otimes a_{1q}) (m'_j) = a_{ij} m'_j \]

and the composition \( A e \tilde{B} \rightarrow M \) is surjective since \( (a_{ij} e)'_{1 \leq j \leq q} \rightarrow m_i \). This proves 1) \( \Leftarrow \).

Next suppose \( M \) f.p. and \( A \otimes_A M = M \).

Choose a surjection \( A e \otimes e A \rightarrow M \) which is possible by 1) and let \( K \) be the kernel.
\[ 0 \to K \to A\ell \to M \to 0 \]

Then, \( M \) f.p. and \( A\ell \) f.t. \( \Rightarrow K \) f.t.

One has

\[
\begin{align*}
A \otimes A K & \to A \otimes A A\ell \to A \otimes A M \to 0 \\
0 & \to K \to A\ell \to M \to 0
\end{align*}
\]

whence \( A \otimes A K \to K \) so \( AK = K \). Then we know \( \exists A\ell \to K \) proving \( 2) \Leftarrow \).

Let's now consider a flat \( A \)-module \( M \) such that \( AM = M \). We wish to show that it corresponds to a flat \( B \)-module, i.e., that \( eM \) is \( B \)-flat. Recall that if \( N \) is a f.p. \( B \)-module then \( A\ell \otimes B N \) is a f.p. \( A \)-module which is good and conversely. Thus
\[ \text{Hom}_B(N, eM) = \text{Hom}_A(Ae \otimes_B N, M) \]
\[ = \text{Hom}_A(Ae \otimes_B N, \tilde{A}) \otimes_A M \]
\[ = \text{Hom}_A(Ae \otimes_B N, \tilde{A}) \otimes_A Ae \otimes_B eM \]
\[ = \text{Hom}_A(Ae \otimes_B N, \tilde{A}) \otimes_B eM \]
\[ = \text{Hom}_B(N, B) \otimes_B eM \]

which means that \( eM \) is \( B \)-flat. Thus we have proved that good flat \( A \)-modules correspond to flat \( B \)-modules and also (p. 506) that good f.p. \( A \)-modules correspond to f.p. \( B \)-modules.

Putting these together we see good f.t. projective \( A \)-modules correspond to f.t. projective \( B \)-modules.
Suppose \( B \) a monoidal alg, \( V \) a \( B_2 \)-module, \( W \) a \( B \)-module, and \( \langle -,- \rangle : W \otimes V \to B \) a \( B \)-bimodule map.

Let \( A = V \otimes_B W \). Define a product on \( A \) by
\[
(\omega_1, \omega_1)(\omega_2, \omega_2) = (\omega_1, \omega_1 \omega_2, \omega_2)
\]
and left (resp. right) mult. by \( A \) on \( V \) (resp. \( W \)) by
\[
(\omega_1, \omega_1) \cdot \omega = \omega_1 \langle \omega_1, \omega \rangle
\]
\[
\omega \cdot (\omega_1, \omega_1) = \langle \omega, \omega_1 \rangle \omega_1
\]

Then \( A \) becomes a (monoidal) alg and \( V, W \) becomes bimodules: \( A \) \( B \times B \) \( A \) such that \( \langle -,- \rangle \) descends to an \( A \)-bimodule map

\[
p : W \otimes_A V \to B \quad \quad p(\omega, \omega) = \langle \omega, \omega \rangle
\]
satisfying
\[
p(x) y = x p(y)
\]
\( \forall x, y \in W \otimes_A V \).

Check that \( \langle \omega \cdot a, \omega \rangle = \langle \omega, a \cdot \omega \rangle \): Let \( a = (v_i, w_i) \).

Then
\[
\langle \omega, (v_i, w_i) \rangle \cdot \omega = \langle \langle \omega, v_i \rangle, w_i \rangle \cdot \omega = \langle \omega, v_i \rangle \langle \omega, w_i \rangle
\]
\[
\langle \omega, (v_i, w_i) \rangle \cdot \omega = \langle \omega, v_i \langle \omega, w_i \rangle \rangle = \langle \omega, v_i \rangle \langle \omega, w_i \rangle
\]
Next let \( x = (v_i, w_i) , y = (v_j, w_j) \). Then
\[
p(x) y = \langle \omega, y \rangle (v_i, w_i) = (\langle \omega, v_i \rangle, v_j) \langle \omega, w_i \rangle = (v_j (w_i, v_i) \omega_1)
\]
\[
x p(y) = (v_i, w_i) \langle v_j, w_i \rangle = (v_j \langle v_i, w_i \rangle) = (v_j (w_i, v_i) \omega_1)
\]
and these agree in \( W \otimes_A V \).
Assume now that $B$ is unital, that $V$, $W$ are unital $B$ modules, and that $p$ is surjective.

Then $p$ is an isomorphism because if $x = (w_i, \nu_i)$ is such that $p(x) = \langle w_i, \nu_i \rangle = 1$, and if $y \in \ker p$, then $y = p(x)y = x p(y) = 0$.

(Here we have used $1 \langle w, \nu \rangle = \langle 1w, \nu \rangle = (w, \nu)$, i.e. the fact that $W$ is a unital module.)

Furthermore, the elements $\omega_i \in W$, $\sigma_i \in V$ and the $A$-bimodule map $V \otimes W \rightarrow A$

satisfies $
(\nu, \omega_i) \cdot \nu_i = \nu \langle \omega_i, \sigma_i \rangle = \nu 1_B = \nu \omega_i \cdot (\nu, \omega) = \langle \omega_i, \sigma_i \rangle \omega = 1_B \omega = \omega
$

which means that $V \in \tilde{P}(\tilde{A})$, $W \in \tilde{P}(\tilde{A})$

and $V, W$ are dual to each other: $V = \text{Hom}_A(W, A)$, $W = \text{Hom}_A(V, A)$.

Notice that $A$ is good:

$A \otimes A = (V \otimes W) \otimes (V \otimes W) = V \otimes B \otimes W = V \otimes W$

(assuming that $W$ is a unital $B$ module). In fact we have maps

$V \xrightarrow{(\cdot, \omega_i)} A^n \xrightarrow{(\cdot \nu_i)} V$

$W \xrightarrow{((\omega_i, \cdot))} A^n \xrightarrow{(w_i, \cdot)} W$

so that $V$ is a finitely presented module such that $AV = V$, and $W$ is a finitely presented module such that $WA = W$. 

The present situation reduces to the case $A e \otimes_B e A = A$, when $\exists \omega_i \in W$ $\omega_i \in V$ such that $\langle \omega_i, \omega_i \rangle = 1$. In this case $e = (\omega_i, \omega_i) \in A$ is an idempotent: $e^2 = (\omega_i, \omega_i)(\omega_i, \omega_i) = (\langle \omega_i, \omega_i \rangle, \omega_i) = e$.

Note that

$$A \xrightarrow{\cdot \omega_i} V \xrightarrow{(\cdot, \omega_i)} A$$

$$(\omega, \omega) \mapsto \omega \langle \omega, \omega \rangle \mapsto (\omega \langle \omega, \omega \rangle, \omega) = (\omega, \omega)(\omega, \omega) \quad \text{e}$$

showing $V = A e$. Similarly $W = e A e$.

In general given $\langle \omega_i, \omega_i \rangle = 1$ the matrix $e_{ij} = (\omega_i, \omega_j) \in M_n (A)$ is idempotent.

$$e_{ij} e_{jk} = (\omega_i, \omega_j)(\omega_j, \omega_k) = (\omega_i \langle \omega_j, \omega_j \rangle, \omega_k) = (\omega_i, \omega_k) = e_{ik}$$

I think this means that if we replace $B, V_B, W_B, A$ by $B, C^* \otimes V, W \otimes C^*, M_n(A) = (C^* \otimes V) \otimes_B (W \otimes C^*)$, then we effectively reduce to the situation $A = A e \otimes_B e A$. Thus the results about finite presentation and flat modules should carry over to the present situation.
April 18, 1999

Basic construction: Suppose given $B, V_B, BW, (\cdot : W \otimes V \rightarrow B)$ bimod map $/B$.

Then one has an $A = V \otimes_B W$ with an $A$-structure given by

\[(v_1, w_1)(v_2, w_2) = (v_1, \langle w_1, v_2 \rangle, w_2) = (v_1, w_1, v_2, w_2)\]

a left $A$-module structure on $V$ and a right $A$-module structure on $W$, giving by

\[(v_1, w_1) \cdot v = v_1 \langle w_1, v \rangle\]

\[w_1 \cdot (v_1, w_1) = \langle w_1, v_1 \rangle w_1\]

One thereby gets bimodules $AV_B \otimes_B W_A$ and bimodule maps

\[V \otimes_B W \rightarrow A\]

This is the identity

\[W \otimes_A V \rightarrow B\]

\[p((w, v)) = \langle w, v \rangle\]

over $A, B$ respectively. The following squares are commutative

\[\begin{array}{ccc}
V \otimes_B W \otimes_A V & \rightarrow & V \otimes_B B \\
\downarrow & & \downarrow \\
A \otimes_A V & \rightarrow & V
\end{array}\]

\[\begin{array}{ccc}
W \otimes_A B \otimes_W W & \rightarrow & W \otimes_A A \\
\downarrow & & \downarrow \\
B \otimes_B W & \rightarrow & W
\end{array}\]

and $p$ satisfies $p(x) y = x p(y)$

Now make the assumptions

1) $B, V_B, BW$ are good: $B \otimes_B B \rightarrow B, V \otimes_B B \rightarrow V, B \otimes_B W \rightarrow W$, e.g. all unital

2) $W \otimes V \rightarrow B$ (w, v) $\mapsto \langle w, v \rangle$ is surjective
Then we claim that

\[ p : W \otimes_A V \simto B \]

and that \( A, A^V, W_A \) are good.

Proof. In general consider a \( B \)-bimodule map \( M \rightarrow B \) which is surjective and such that \( u(x)y = xu(y) \). From the exact sequence \( 0 \rightarrow K \rightarrow M \rightarrow B \rightarrow 0 \) where \( K = \text{Ker}(u) \) we obtain the exact sequence

\[ \begin{array}{c}
B \otimes_B K \\
\rightarrow \ B \otimes_B M \\
\rightarrow \ B \otimes_B B \\
\rightarrow \ 0
\end{array} \]

Now \( K \) is null \( B \)-bimodule: \( B K = K B = 0 \), so if \( B = B^2 \), then \( B \otimes_B K = B^2 \otimes_B K = B \otimes_B B K = 0 \). Thus \( B \otimes_B M \simto B \otimes_B B \), so if \( M \) and \( B \) are both good \( B \) modules we have \( M \simto B \). This applies to our \( B \) and to \( M = W \otimes_A V \), because \( B \) and \( B \) are assumed good.

Next we have

\[ \begin{array}{c}
V \otimes_B W \otimes_A V \simto V \otimes_B B \\
\downarrow \equiv \quad \downarrow \equiv \\
A \otimes_A V \rightarrow V \\
\text{showing that } A, W_A \text{ are good. This implies } A = V \otimes_B W \text{ is a good } A \text{ module.}
\end{array} \]

Remark: The above argument shows \( B, V, W \) good \( \Rightarrow \) surjective \( \Rightarrow W \otimes_A V \sim B, A^V, A \) good. But one does not get \( B \) good, which would be nice if one wants \( V \otimes_B \) and \( W \otimes_A \) to map into good modules.
Way to think: In a Morita equivalence: 
\( A, B, A^* B, B W_A, V \otimes_B W = A, W \otimes_A V = B \)
you probably want \( A, B, W \) to be good,
and then this implies that \( A, B, V_A, W_A \)
are also good.

Question: When is an algebra Morita equivalent
to a unital algebra?

Suppose \( A, B, A^* B, B W_A, V \otimes_B W = A, W \otimes_A V = B \)
is a (good as above) Morita equivalence, where \( B \) is
a unital. Then \( V_B, B W \) are good \( \Rightarrow \) they are unital
modules.

First we show that \( V \in \mathcal{P}(A) \) and that
\( W = \text{Hom}_A(V, A) \) is the dual \( \mathcal{P}(A^*) \).
Let \( \langle \cdot, \cdot \rangle_A \)
\( \langle \cdot, \cdot \rangle_B \) denote the isomorphisms \( V \otimes_B W \cong A, W \otimes_A V \cong B \).
Let \( w_i, v_i \in W, v_i \in V, 1 \leq i \leq n \) be such that \( \langle w_i, v_i \rangle_B = 1 \).

From
\[
\begin{align*}
V \otimes W \otimes V & \cong V \otimes B \\
\downarrow & \\
A \otimes_A V & \cong V \\
\langle \langle v, w_i \rangle_A, v_i \rangle & \mapsto \langle v, w_i \rangle_A v_i
\end{align*}
\]
and
\[
\begin{align*}
W \otimes V \otimes W & \cong W \otimes A \\
\downarrow & \\
B \otimes_B W & \cong W \\
(1, w) & \mapsto w
\end{align*}
\]

Yielding
\[
V = \langle v, w_i \rangle_A v_i, \quad W = w_i \langle v_i, w \rangle_A
\]
and this implies $V \in P(\tilde{A})$ and that $W$ is its dual.

Note also $v = (v, w_i)_A \implies V = AV$ so that $V$ is a finitely generated projective good $A$-module.

Let's study finitely generated projective good $A$-modules. If $V \in P(\tilde{A})$, then $V$ has the form $\tilde{A}^n e$ where $e^2 = e \in M_n(\tilde{A})$. Then $V = \sum \tilde{A} v_i$, $v_i = (e_{ij})_j$ is the $i$-th row of $e$.

Assume $V = AV$, i.e. $v_i = a_{ik} v_k$ with $a_{ik} \in A$, i.e. $(e_{ij})_j = a_{ik} (e_{kj})_j$, or $e_{ij} = a_{ik} e_{jk}$. Then $e \in M_n A$. Conversely, if $e \in M_n A$ is idempotent, then $\tilde{A}^n e \in \tilde{A}^n e e \in \tilde{A}^n e$, so $\tilde{A}^n e = \tilde{A}^n e \in P(\tilde{A})$ and $A \tilde{A}^n e = A \tilde{A}^n e$. Thus we have the first part of Prog.

If $A^2 = A$, then an $A$ module $V$ is a good finitely generated projective module $\iff V = \tilde{A}^n e$ for some $n$ and idempotent $e \in M_n A$. If so then the dual $W = \text{Hom}_A(V, \tilde{A})$ is a finitely generated projective right module.

The last assertion follows from the fact that the dual is given by the transpose matrix:

$$\text{Hom}_A(\tilde{A}^n e, \tilde{A}) = e \tilde{A}^n$$

I think now the following is clear:

Prog. A good algebra $A$ is Morita equivalent to a unital algebra iff there exists a finitely generated projective good $A$ module $V$ such that $W = \text{Hom}_A(V, A)$, then the obvious pairing $V \otimes W \rightarrow A$ is surjective.

I should have noted that when $AV = V$ we
have

\[ 0 \to \text{Hom}_A(V, A) \to \text{Hom}_A(V, \tilde{A}) \to \text{Hom}_A(V, \mathcal{O}) \]

\[ \to 0 \]

Proof: (\(\Rightarrow\)) This we've done.

\(\left(\Leftarrow\right)\). With \(W\) defined this way we have

\(A \otimes_A V, WA\) all good and \(V \otimes W \to A\). So we have a good Morita equivalence with \(B = W \otimes_A V\). But because \(V \in P(\tilde{A})\) and \(W\) is its dual, one knows that \(W \otimes_A V = \text{Hom}_A(V, V)^{op} = \text{Hom}_A(W, W)\). Thus \(B\) is unital.

---

Summarize a few ideas from scratch paper the past few days.

If \(A\) is good, then \(M \to A \otimes_A M\) is right adjoint to the inclusion of good modules in modules. However if only \(A^2 = A\), then the right adjoint should be \(M \to A \otimes_A A \otimes_A M\). \(A \otimes_A M\) need not be good in general, e.g., \(A \otimes_A \tilde{A} = A\). But if \(AM = M\), then \(A \otimes_A M\) is good because

\[ K \otimes_A M \to A \otimes_A A \otimes_A M \to A \otimes_A M \to 0 \]

\(K \otimes_A AM = K \otimes_A AM = 0\)

Thus the functor \(A \otimes_A -\) has to be applied twice to get the adjoint. This is similar to sheaves + presheaves, and it might be worthwhile to explore this example.

You want to work on Wodzicki's result that if \(A < R\) is a left ideal such that \(A^2 = A\), then \(A\) is \(A\)-flat iff \(A\) is \(R\)-flat. In this
Note that if $M$ is a good $A$-module, $A \otimes_A M = M$, then it has a unique $R$-module structure extending the $A$-module structure.
Let $A$ be a nonunital ring such that $A^2 = A$, let Mod$(A)$ be the category of its (left) modules, let Mod$_g(A)$ be the full subcategory of modules which are good: $A \otimes_A M \to M$, let $N$ be the full subcategory of Mod$(A)$ consisting of null modules: $AM = 0$. Let $M \in$ Mod$(A)$.

1) $AM = 0 \iff A \otimes_A M = 0$.

Pf. $\Leftarrow$ Clear since $A \otimes_A M \to AM$.

$\Rightarrow$ Assuming $AM = 0$, to show any $(a, m) \in A \otimes_A M$ is zero. Since $A^2 = A$, one can assume $a = a_1, a_2$. Then $(a_1 a_2 m) = (a_1, a_2 m) = (a_1, 0) = 0$.

2) The kernel and cokernel of $A \otimes_A M \to M$, $µ: (a, m) \mapsto am$ are null modules.

Pf. The cokernel is $NAM$ which is killed by $A$.

If $(a_i, m_i) \in$ the kernel, i.e. $a_i m_i = 0$, then $a(a_i m_i) = (aa_i m_i) = (a, a_i m_i) = 0$.

3) If $AM = M$, then $A \otimes_A M$ is good.

Pf. One has the exact sequence

$$0 \to K \to A \otimes_A M \xrightarrow{µ} M \to 0$$

where $AK = 0$ by 2). This gives the exact sequence

$$A \otimes A \to A \otimes_A A \otimes_A M \xrightarrow{1 \otimes µ} A \otimes_A M \to 0$$

where $A \otimes_A K = 0$ by 1). Now $1 \otimes µ: (a_1, a_2, m) \mapsto (a_1, a_2 m) = (a_1 a_2, m)$ is the same as $µ$ for the module $A \otimes_A M$.

Thus $A \otimes A \otimes_A M \simeq A \otimes_A M$, $(a, (a_i, m)) \mapsto (a a_i, m)$.

4) $A \otimes_A A \otimes_A M$ is good.
Pf. Either because $A(A \otimes_A M) = A \otimes_A M$ and 3), or because $A \otimes_A A$ is good by 3) hence $A \otimes_A (A \otimes_A A \otimes_A M) = A \otimes_A (A \otimes_A A) \otimes_A M = A \otimes_A A \otimes_A M$.

Suppose $M \in \text{Mod}(A)$, $N \in \text{Mod}(A)$.

5) $\text{Hom}_A(N, A \otimes_A M) \sim \text{Hom}_A(N, M)$

Proof. Consider

$$0 \to K \to A \otimes_A M \to M$$

This gives

$$0 \to \text{Hom}_A(N, K) \to \text{Hom}_A(N, A \otimes_A M) \to \text{Hom}_A(N, M)$$

where $\text{Hom}_A(N, K) = \text{Hom}_Z(N/\text{AN}, K) = 0$, since $N$ good $\Rightarrow$ AN = N. It this suffices to show that any module map $N \to M$ lifts to $N \to A \otimes_A M$. But this is clear from $A \otimes_A N \xrightarrow{1 \otimes f} A \otimes_A M$

$$\cong \downarrow \quad \downarrow$$

$$N \xrightarrow{f} M$$

6) $\text{Hom}_A(N, A \otimes_A A \otimes_A M) \sim \text{Hom}_A(N, M)$

Pf. Combine 5) for $M$ and $A \otimes_A M$.

7) One has adjoint functors

$$\text{Modg}(A) \overset{\sim}{\leftarrow} \text{Mod}(A)$$

$A \otimes_A A \otimes_A$

where the upper is left adjoint to the lower.

Immediate from 6).
Our goal next is to show the additive category of good modules is abelian. For this we want to calculate the kernel + cokernel of a map of good modules in the category \( \text{Modg}(A) \).

Let \( f : N_1 \to N_2 \) be a map in \( \text{Modg}(A) \), form the exact sequence in \( \text{Mod}(A) \)

\[
0 \to K \xrightarrow{i} N_1 \xrightarrow{f} N_2 \xrightarrow{p} C \to 0
\]

where \((K,i)\) and \((C,p)\) are the kernel + cokernel of \( f \) in \( \text{Mod}(A) \). Let \( A' = A \otimes_A A \).

8) \( C \) is good and \((C,p)\) is the cokernel of \( f \) in \( \text{Modg}(A) \).

Proof. \( A \otimes_A N_1 \to A \otimes_A N_2 \to A \otimes_A C \to 0 \)

\[
\xrightarrow{i} \quad \xrightarrow{f} \quad \xrightarrow{p} \quad C \to 0
\]

implies \( C \) is good. One has the exact sequence

\[
0 \to \text{Hom}_A(C,M) \to \text{Hom}_A(N_1,M) \to \text{Hom}_A(N_2,M)
\]

for all modules \( M \), in particular, for all good modules, showing \( C \) is the cokernel of \( f \) in \( \text{Modg}(A) \).

9) The kernel of \( f \) in \( \text{Modg}(A) \) is the composite map \( A' \otimes_A K \to K \leftarrow N_1 \).

If. For all good modules \( N \) we have

\[
0 \to \text{Hom}_A(N,K) \to \text{Hom}_A(N,N_1) \to \text{Hom}_A(N,N_2)
\]

\[
\text{Hom}_A(N,A \otimes_A K)
\]

using 6). good
Next let $I$ be the image of $f$ in $\text{Mod}(A)$ so that we have exact sequences

$$0 \rightarrow K \xrightarrow{\phi} N_1 \xrightarrow{\psi} I \rightarrow 0$$
$$0 \rightarrow I \xrightarrow{\chi} N_2 \xrightarrow{\rho} C \rightarrow 0$$

with $f \circ \chi = f$.

From 8) one has $\text{Cok}_g(f) = C$, and from 9) one has $\text{Ker}_g(f) = A' \otimes_A K$, where the $g$ subscript denotes $\text{Ker}, \text{Cok}$ in $\text{Mod}(A)$. Recall that the image and coimage are defined to be

$$\text{Img}(f) = \text{Ker}_g \{ N_2 \rightarrow \text{Cok}_g(f) \}$$
$$\text{Coring}(f) = \text{Cok}_g \{ \text{Ker}(f) \rightarrow N_1 \}$$

and that an additive category (assumes existence of $0$ and finite direct sums = finite direct products) is abelian when $\text{Coring}(f) \simeq \text{Img}(f)$ for all $f$.

We have

$$\text{Img}(f) = \text{Ker}_g \{ N_2 \rightarrow C \} = A' \otimes_A I$$
$$\text{Coring}(f) = \text{Cok}_g \{ A' \otimes_A K \rightarrow N_1 \}$$

But from

$$A' \otimes_A K \rightarrow A' \otimes_A N_1 \rightarrow A' \otimes_A I \rightarrow 0$$
$$0 \rightarrow K \rightarrow N_1 \rightarrow I \rightarrow 0$$

we see $\text{Cok} \{ A' \otimes_A K \rightarrow N_1 \} \simeq A' \otimes_A I$. Thus $\text{Coring}(f) \simeq \text{Img}(f)$ proving

10) $\text{Mod}_g(A)$ is an abelian category.
11) The functor $M \mapsto A'^{\otimes A} M$ from $\text{Mod}(A)$ to $\text{Modg}(A)$ is exact.

Pf. Because this functor is a right adjoint we know it commutes with arbitrary projective limits, so we only have to see that if $M_1 \to M_2$ is a surjection in $\text{Mod}(A)$, then $A'^{\otimes A} M_1 \to A'^{\otimes A} M_2$ is a surjection in $\text{Modg}(A)$. But this is clear because the cokernel in $\text{Modg}(A)$ of $A'^{\otimes A} M_1 \to A'^{\otimes A} M_2$ is its cokernel in $\text{Mod}(A)$.

April 22, 1997:

Let's now discuss limits in $\text{Modg}(A)$.

Let $i : M_i, N_i$ denote functors from a small category $I$ to $\text{Mod}(A), \text{Modg}(A)$ respectively. Write $\varprojlim N_i, \varprojlim N_i$ for the inductive and projective limits in $\text{Modg}(A)$ when these exist.

Let $F : \text{Modg}(A) \to \text{Modg}(A)$ be the inclusion functor and $G(M) = A'^{\otimes A} M$ its right adjoint.

12) $\text{Modg}(A)$ is closed under projective limits and $G$ respects projective limits.

One has

$$\varprojlim N_i = A'^{\otimes A} \varprojlim N_i$$

$$\varprojlim A'^{\otimes A} M_i = A'^{\otimes A} \varprojlim M_i$$

Proof: $\text{Hom}(N, G(\varprojlim M_i)) = \text{Hom}(F(N), \varprojlim M_i)$

$= \varprojlim \text{Hom}(F(N), M_i) = \varprojlim \text{Hom}(N, G(M_i))$

$= \text{Hom}(N, \varprojlim G(M_i))$. This shows the existence of $\varprojlim G(M_i)$, and that
\[ M_i = F(N_i) \text{ gives the existence of } \lim\limits_{i}^g N_i \text{ and that it is } G(\lim\limits_{i}^g F(N_i)). \]

13) \text{ Modg}(A) \text{ is closed under inductive limits and } F \text{ respects inductive limits:}

\[ \lim\limits_{i}^g N_i = \lim\limits_{i} \rightarrow N_i \]

\( Pf. \) Existence of \( \lim\limits_{i}^g N_i \) follows from existence of direct sums, since cokernels exist in any abelian category:

\[ \bigoplus_i N_i \rightarrow \bigoplus_i N_i \rightarrow \lim\limits_{i}^g N_i \rightarrow 0 \]

But \( \bigoplus_i N_i \) is clearly good, so

\[ \bigoplus_i N_i = \bigoplus_i N_i \]

The fact that \( F \) respects inductive limits follows because it is a left adjoint functor.

Remark: The only content to 12) + 13) is the existence of the limits. The rest is obvious properties of adjoint functors. The real surprise is the following

14) \text{ Modg}(A) \text{ respects arbitrary inductive limits:}

\[ A' \otimes_A \lim\limits_{i}^g M_i \xleftarrow{\sim} \lim\limits_{i}^g A' \otimes_A M_i \]

\( Proof. \) Since \( \lim\limits_{i}^g A' \otimes_A M_i = \lim\limits_{i}^g A' \otimes_A M_i \) by 13), this is clear from the fact that \( X \otimes_A - \) is a left adjoint.

Next let's show \( \text{Modg}(A) \) satisfies
Grothendieck’s AB5 axiom. I think this says that if \( N_i \) is a filtering family of subobjects of \( N \) and if \( N' \) is a subobject of \( N_i \), then

\[
N' \cap \bigcup N_i = \bigcup (N' \cap N_i)
\]

I know that AB5 \( \iff \) filtered inductive limits are exact. Observe that (15) is a consequence of filtered \( \operatorname{lim} \)'s exact:

\[
\frac{N'/N' \cap N_i}{N/N_i} \twoheadrightarrow \frac{N/N_i}{N/N_i}
\]

\[
\Rightarrow \lim \frac{N'/N' \cap N_i}{N/N_i} \twoheadrightarrow \lim \frac{N/N_i}{N/N_i}
\]

\[
\Rightarrow \frac{N' \cap \bigcup N_i}{N \cap \bigcup N_i} = \bigcup (N' \cap N_i)
\]

We claim

(16) In \( \text{Mod}(A) \) filtered inductive limits are exact, i.e. one has axiom AB5.

Proof: Let \( N_i' \twoheadrightarrow N_i \) be a filtered system of maps in \( \text{Mod}(A) \), and let \( K_i \) be the kernel of the map \( N_i' \twoheadrightarrow N_i \) in \( \text{Mod}(A) \), so that

\[
0 \rightarrow K_i \twoheadrightarrow N_i' \twoheadrightarrow N_i
\]

is exact in \( \text{Mod}(A) \). Then \( N_i' \twoheadrightarrow N_i \) is a monomorphism in \( \text{Mod}(A) \) iff \( AK_i = 0 \). Assuming this for all \( i \), we have

\[
\lim N_i' \twoheadrightarrow \lim N_i
\]

\[
0 \rightarrow \lim K_i \twoheadrightarrow \lim N_i' \twoheadrightarrow \lim N_i
\]
The point is that $AK_i = 0 \Rightarrow A\begin{array}{c}\longrightarrow\
\end{array}K_i = 0$.

So that $\begin{array}{c}\longrightarrow\\\begin{array}{c}N\\\n\end{array}\end{array} \rightarrow \begin{array}{c}\longrightarrow\\\begin{array}{c}N\\\n\end{array}\end{array}$ is a monom.
in $\text{Mod}(A)$. 

17) \text{$A$ is a generator for } \text{Mod}(A).

Indeed if $N$ is good, then choose a surjection $\begin{array}{c}\oplus A' \rightarrow N\\\rightarrow\\\begin{array}{c}N\\\n\end{array}\end{array}$ in $\text{Mod}(A)$. Since $AN = 0$, one has a surjection of good modules

$\begin{array}{c}\oplus A' = A' \otimes (\oplus A)\\\rightarrow\\\begin{array}{c}A' \otimes A\\\n\end{array}N = N
\end{array}$

which we know is an epimorphism in $\text{Mod}(A)$.

18) \text{$\text{Mod}(A)$ has sufficiently many injectives.}

Follows from Grothendieck $\text{AB5}$ + generators $\Rightarrow$ enough injectives.

Next we would like to give another proof of 18) using existence of injective hulls in $\text{Mod}(A)$. Recall the basic facts about these, fixing the setting to be the category of modules over a (unital) ring.

An injection $M \rightarrow N$ is called \text{essential} when for any submodule $Y \subset N$ one has $Y = 0 \Rightarrow Y \cap M \neq 0$. Composition of essential injections is an essential injection, as well the inductive limit of a filtering family $M \rightarrow N_i$.

$M$ is injective $\iff$ every essential injection $M \rightarrow N$ is an isomorphism. $\iff$ because $N = M \oplus Y$. Choose an embedding $M \rightarrow I$ with $I$ injective; \text{AB5} by Zorn. $\exists Y \subset I$ maximal such that $Y \cap M = 0$ (this uses \text{AB5}).

Then $\exists M \rightarrow I/\sim$ is essential, so $M \sim I/\sim$, i.e. $M \oplus Y = I$

Then $M \rightarrow I/\sim$ is injective.
Given $M \to I$ with $I$ injective, by Zorn $\exists N \subsetneq I$ such that $M \to N$ is essential. If $N \to N_1$ is an essential injection, then $\exists$ a comm. triangle

$$
N \xrightarrow{f} N_1 \xleftarrow{g} I,
$$

since $I$ is injective. The dotted arrow is injective since its kernel intersects $N$ trivially and $N \to N_1$ is essential. Then $N \subset f(N_1)$ is essential and by maximality of $N$ one has $N = f(N_1)$, and so $N \cong N_1$. Thus $N$ is an injective module.

This shows the existence for any $M$ of an injective hull $M \to I$, which can be characterized either as a maximal essential injection, or as a minimal embedding into an injective module. The injective hull is determined up to a canonical isomorphism.

Return now to $\text{Modg}(A)$.


(9) Let $I$ be an injective $A$-module such that $\text{Hom}_A(I, I) = \{y \in I \mid Ay = 0\} = 0$. Then $A' \otimes_A I$
is injective in $\text{Modg}(A)$.

Proof. The functor $M \mapsto \text{Hom}_A(M, I)$ is exact and it kills small modules, so it gives rise to an exact functor on $\text{Modg}(A)$. Thus $\text{Hom}_A(N, I) = \text{Hom}_A(N, A' \otimes_A I)$ is an exact functor of $N \in \text{Modg}(A)$, which means that $A' \otimes_A I$ is injective in $\text{Modg}(A)$.
Let's another proof of the existence of enough injectives in \( \text{Modg}(A) \). Let \( N \) be a good module, let \( \text{ann}_A(N) = \{ n \in N \mid An = 0 \} \), let \( I \) be the injective hull of \( N/\text{ann}_A(N) \):

\[
N/\text{ann}_A(N) \hookrightarrow I
\]

Then \( \text{ann}_A(I) \cap N/\text{ann}_A(N) = 0 \Rightarrow \text{ann}_A(I) = 0 \), so \( A' \otimes_A I \) is injective in \( \text{Modg}(A) \). Moreover, the kernel of \( N \rightarrow A' \otimes_A I \) is contained in \( \text{ann}_A(N) \), so this map is a monomorphism in \( \text{Modg}(A) \).

Actually it seems we can improve (19) to a description of injectives in \( \text{Modg}(A) \).

(20) Let \( J \) be an injective in \( \text{Modg}(A) \). Then \( \text{Hom}_A(A', J) \) is an injective \( A \)-module whose annihilator is zero and one has \( J \cong A' \otimes_A \text{Hom}(A', J) \).

Pf. Let \( \Phi \) be the functor on \( \text{Mod}(A) \) defined by \( \Phi(M) = \text{Hom}_A(A' \otimes_A M, J) \). This is the composite of the exact functor \( M \rightarrow A' \otimes_A M \) from \( \text{Mod}(A) \) to \( \text{Modg}(A) \) and the exact functor \( \text{Hom}(-, J) \) on \( \text{Modg}(A) \). Since

\[
\Phi(M) = \text{Hom}_A(A' \otimes_A M, J) = \text{Hom}_A(M, \text{Hom}(A', J))
\]

it follows that \( \text{Hom}_A(A', J) \) is an injective \( A \)-module. Since \( \Phi(M) = 0 \) when \( M \) is a null module, the \( A \)-annihilator of \( \text{Hom}_A(A', J) \) is zero. Finally restricting to good
modules we have
\[ \mathcal{E}(N) = \text{Hom}_A(N, J) \]
\[ \mathcal{E}(N) = \text{Hom}_A(N, \text{Hom}_A(A', J)) \]
\[ = \text{Hom}_A(N, A' \otimes_A \text{Hom}_A(A', J)) \]
\[ \text{so } J \cong A' \otimes_A \text{Hom}_A(A', J) . \]

What is happening here is that there is another adjoint:

\[
\begin{array}{c}
\text{Modg}(A) \\
\downarrow \text{Hom}_A(A', -) \\
\text{Modl}(A)
\end{array}
\]

such that
\[ A' \otimes_A \text{Hom}_A(A', N) \to N \]

In effect
\[ \text{Hom}_A(N, A' \otimes_A \text{Hom}_A(A', N)) = \text{Hom}_A(N, \text{Hom}_A(A', N)) \]
\[ = \text{Hom}_A(A' \otimes_A N, N) = \text{Hom}_A(N, N) . \]

The point is that instead of the functor
\[ M \mapsto A \otimes_A M \] one can also use \[ M \mapsto \text{Hom}_A(A, M) \] to give a parallel treatment.

1) \[ M \text{ is null } \iff \text{Hom}_A(A, M) = 0 \]

pf: One has the exact sequence
\[ 0 \to \text{ann}_A(M) \to M \to \text{Hom}_A(A, M) \]
\[ \phi(m)(a) = am \]
\[ M \text{ is null } \iff \text{ann}_A(M) = M , \text{ so the implication } \iff \text{ is clear.} \]
Conversely if $M$ is null, and if

$$f \in \text{Hom}_A(A, M), \quad \text{then} \quad f(a_1 a_2) = a_1 f(a_2) = 0$$

so $f = 0$ as $A = A^2$.

2') The kernel and cokernel of $\phi$ are null.

Pf. $\text{ann}_A(M)$ is null obviously. If $f \in \text{Hom}_A(A, M)$, then $af$ is by definition $a' \mapsto f(a'a)$. But

$$f(a'a) = a'f(a) = \phi(f(a))(a').$$

Thus $af = \phi(f(a))$ showing that multiplication by $a$ is zero on the cokernel of $\phi$.

Digress to point out that from

$$0 \to A \to A^+ \to Z \to 0$$

one gets

$$0 \to \text{Tor}_1^A(A, M) \to A \otimes_A M \to A^+ \otimes_A M \to Z \otimes_A M \to 0$$

as well as

$$0 \to \text{Hom}_A(Z, M) \to \text{Hom}_A(A^+, M) \to \text{Hom}_A(A, M) \to \text{Ext}_A^1(Z, M) \to 0$$

$$0 \to \text{ann}_A(M) \to Z \to M \to M/AM \to 0$$

.Call $M$ is good when $\phi$ is an isomorphism.

3') $\text{ann}_A(M) = 0 \Rightarrow \text{Hom}_A(A, M)$ is is good

Pf. $0 \to M \to \text{Hom}_A(A, M) \to \mathbb{C} \to 0$ yields

$$0 \to \text{Hom}_A(A, M) \to \text{Hom}_A(A, \text{Hom}_A(A, M)) \to \text{Hom}_A(A, \mathbb{C})$$

$0$ by $1$.
4') \text{ Hom}_A(A, \text{ Hom}_A(A, M)) = \text{ Hom}_A(A \otimes_A A, M)

is r-good.

Why? Let \( f \in \text{ Hom}_A(A, M) \) be such that \( Af = 0 \).
Then \( (af)(a') = f(a'a) = 0 \) for all \( a, a' \Rightarrow f = 0 \), as \( A^2 = A \).

5') Assume \( N \) is r-good. Then

\[
\text{ Hom}_A(\text{ Hom}_A(A, M), N) \cong \text{ Hom}_A(M, N)
\]

Proof: One has

\[
0 \rightarrow K \rightarrow M \rightarrow \text{ Hom}_A(A, M) \rightarrow C \rightarrow 0
\]

where \( K, C \) are null. This yields

\[
0 \rightarrow \text{ Hom}_A(C, N) \rightarrow \text{ Hom}_A(\text{ Hom}_A(A, M), N) \rightarrow \text{ Hom}_A(M, N)
\]

so it suffices to extend any \( u \in \text{ Hom}_A(M, N) \) to \( \text{ Hom}_A(A, M) \rightarrow N \). But clear from

\[
M \rightarrow \text{ Hom}_A(A, M)
\]

\[
u \downarrow \quad \downarrow u^*
\]

\[
N \rightarrow \text{ Hom}_A(A, N)
\]

6') \text{ Hom}_A(\text{ Hom}_A(A, \text{ Hom}_A(A, M)), N) = \text{ Hom}_A(M, N)

\[
\text{ Hom}_A(A', M)
\]
Existence of enough good flat modules.

Let $M$ be an $A$-module such that $AM = M$. I want to show $M$ is a quotient of a good flat $A$-module.

Starting from a finite subset $m_{i_0}$, $1 \leq i_0 \leq n_0$, we can use $AM = M$ construct successive factorizations

$$m_{i_0} = a^1_{i_0} \cdot m_{i_1} \quad 1 \leq i_1 \leq n_1$$

$$m_{i_1} = a^2_{i_2} \cdot m_{i_2} \quad 1 \leq i_2 \leq n_2$$

This gives the following diagram

$$
\begin{array}{c}
A^{n_0} \xrightarrow{a^1} \tilde{A}^{n_1} \xrightarrow{a^2} \tilde{A}^{n_2} \rightarrow \ldots \\
\downarrow m_{i_0} \hspace{1cm} \downarrow m_{i_1} \hspace{1cm} \downarrow m_{i_2} \hspace{1cm} \\
\hspace{1cm} \hspace{1cm} \hspace{1cm} M
\end{array}
$$

The inductive limit $E$ of the sequence at the top is flat since $E = \lim \tilde{A}^{n_i}$ and it satisfies $AE = E$ since $E = \lim A^{n_i}$. One has a map $E \rightarrow M$ whose image contains the submodule generated by the $m_{i_0}$. Then it's clear that by starting from enough finite subsets to generate $M$ and taking the direct sum of the $E$'s we write $M$ as a quotient of a flat $A$ module $E$ such that $AE = E$. Then $E$ is good since $0 \rightarrow A \rightarrow \tilde{A} \rightarrow Z \rightarrow 0$ yields
$0 \rightarrow \text{Tor}_A^1(\mathbb{Z}, E) \rightarrow A \otimes_A E \rightarrow E \rightarrow E/\text{AE} \rightarrow 0$

I next want to use the existence of enough flat good modules to construct the left-derived functors of the inclusion

$F : \text{Mod}_A(A) \longrightarrow \text{Mod}(A)$

1) Let $E$ be a flat good $A^\text{op}$ module. Then $M \mapsto E \otimes_A M$, $\text{Mod}_A(A) \rightarrow \text{Mod}(\mathbb{Z})$ is exact, and it kills $N$, hence it induces an exact functor $\text{Mod}_A(A) \rightarrow \text{Mod}(\mathbb{Z})$, $N \mapsto E \otimes_A N$.

The point is that if $AM = 0$, then $E \otimes_A M = E \otimes_A AM = 0$, and the rest is obvious.

2) Let $X \in \text{Mod}_A(A^\text{op})$. Then $M \mapsto X \otimes_A M$, $\text{Mod}_A(A) \rightarrow \text{Mod}(\mathbb{Z})$ descends to $\text{Mod}_A(A)/\mathbb{N} \cong X \otimes_A A \cong X$.

Pf. $(\Rightarrow)$ $A \leftarrow \tilde{A}$ becomes an isom. in $\text{Mod}_A(A)/\mathbb{N}$, so $X \otimes_A A \cong X \otimes_A \tilde{A} = X$.

$(\Leftarrow)$ one has $X \otimes_A A \cong X \Rightarrow X \otimes_A A^\prime \cong X$, so $X \otimes_A M = X \otimes_A (A^\prime \otimes_A M)$. Now use the fact that the inverse of $\text{Mod}_A(A) \rightarrow \text{Mod}_A(A)/\mathbb{N}$ is $M \mapsto A^\prime \otimes_A M$.

Recall that for a unital algebra $R$ then an additive functor $F : \text{Mod}_R(R) \rightarrow \text{Mod}(\mathbb{Z})$ has the form $F(M) = X \otimes_R M$ for some unital $R^\text{op}$ module.
iff $F$ commutes with arbitrary lim's. (equivalently $F$ is right exact and commutes with direct sums.) In effect, for any $F$ we have a morphism of functors

$$F(R) \otimes_R M \rightarrow F(M)$$

$$\xi \otimes m \mapsto \xi(m)$$

where $\cdot m : R \rightarrow M$ is $r \mapsto rm$ and $\xi \mapsto \xi(m)_* : F(R) \rightarrow F(M)$ denotes the induced map.

The map 3) is an isomorphism for $M = R$, hence if $F$ commutes with lim's it is an isomorphism for free $M$, and if $F$ also is right exact, it is an isomorphism for any $M$ which is a cokernel of a map between free modules, i.e. any $M$.

At this point, given two rings $A, B$ such that $A^2 = A$ and $B^2 = B$, we can describe all additive functors $\text{Mod}(A)/\mathcal{N}_A \rightarrow \text{Mod}(B)/\mathcal{N}_B$ commuting with lim's as follows. Compose with the equivalence $\text{Mod}(B)/\mathcal{N}_B \cong \text{Mod}_B$, $X \mapsto B \otimes_B X$, and then with the inclusion $\text{Mod}_B \hookrightarrow \text{Mod}(B)$ which we know commutes with lim's (because it's left adjoint to $X \mapsto B \otimes_B X$).

Then we have a functor $\text{Mod}(A) \rightarrow \text{Mod}(A)/\mathcal{N}_A \rightarrow \text{Mod}(B)/\mathcal{N}_B \rightarrow \text{Mod}(B)$ which commutes with lim's. By the above remark this functor has the form

$$M \mapsto X \otimes_A M$$

where $X$ is a bimodule $B \otimes_B \mathcal{N}_B$; $X$ is the image
of $A \in \text{Mod}(A)$. The fact that this functor descends to $\text{Mod}(A)/\mathcal{N}_A$ is equivalent by 2) above to $X \otimes_A A = X$.

The fact that it has values in $\text{Modg}(B)$ means that $B \otimes_B X \to B$.

A simple example:

The trunode $A^! A^!$ yields the inverse $\text{Mod}(A)/\mathcal{N}_A \to \text{Modg}(A)$ of the equivalence going the other way. Notice that this functor is exact, but the trunode is not necessarily $A^!$ flat. (So I don't understand yet exact functors compatible with $\text{Hom}$'s from $\text{Mod}(A)/\mathcal{N}_A$ to $\text{Mod}(B)/\mathcal{N}_B$.)

Let's return to the problem of the left derived functors $L_n F$, where $F: \text{Modg}(A) \to \text{Mod}(A)$ is the inclusion.

Note that the composition

$$
\text{Mod}(A) \xrightarrow{G = A^! \otimes_A} \text{Modg}(A) \xrightarrow{F} \text{Mod}(A)
$$

is given by the trunode $A^! A^!$, so the first guess would be that $L_n F$ is given by $\text{T} \circ \text{Tor}^A_\ast (A^!, -)$. But the difficulty here is that the composite functor situation $FGF$ is not good, $G$ does not take projectives to $F$-acyclic objects, e.g. $G(A) = A^!$ is not $F$-acyclic, as we should see eventually.

However we can construct flat resolutions
for objects in $\text{Modg}(A)$.

Let $M \in \text{Mod}(A)$, put $M_0 = A \otimes_A M$.

Then $A \otimes_A M_0$ is good by p.531,

there is a surjection $P_0 \rightarrow A \otimes_A M_0$ with $P_0$ good and flat. Let $M_1$ be the kernel of $P_0 \rightarrow A \otimes_A M_0$

in $\text{Mod}(A)$, so that we have

$$
A \otimes_A M_1 \rightarrow A \otimes_A P_0 \rightarrow A \otimes_A M_0 \rightarrow 0
$$

$$
\downarrow \quad \downarrow \sim \quad \downarrow \sim
$$

$$
0 \rightarrow M_1 \rightarrow P_0 \rightarrow A \otimes_A M_0 \rightarrow 0
$$

This shows $AM_1 = M_1$, so that we can repeat the process to construct exact sequences

$$
0 \rightarrow M_2 \rightarrow P_1 \rightarrow A \otimes_A M_1 \rightarrow 0
$$

$$
0 \rightarrow M_3 \rightarrow P_2 \rightarrow A \otimes_A M_2 \rightarrow 0
$$

$$
\rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0
$$

where $AM_n = M_n$ and $P_n$ are good and flat. We then have a complex in $\text{Mod}(A)$

consisting of good flat modules, such that

$$
H_0(P_\ast) = A \otimes_A M_0 = A' \otimes_A M
$$

$$
H_n(P_\ast) = \ker \{ A \otimes_A M_n \rightarrow M_n \} \quad n \geq 1
$$

Thus $P_\ast$ is a flat resolution of $A' \otimes_A M$ in $\text{Modg}(A)$; or $P_\ast$ is a flat resolution of $M$ in $\text{Mod}(A)/N_A$. 
Similarly if $x \in \text{Mod}(A^\text{op})$, let $X_0 = XA$ and construct

$$0 \rightarrow X_1 \rightarrow E_0 \rightarrow X_0 \otimes_A A \rightarrow 0$$

$$0 \rightarrow X_2 \rightarrow E_1 \rightarrow X_1 \otimes_A A \rightarrow 0$$

inductively such that $E_n$ is $A^\text{op}$ flat and $X_nA = X_n$ for all $n > 0$. This gives a complex of right flat good modules $E_\cdot$ such that

$$H_0(E_\cdot) = X_0 \otimes_A A = X \otimes A'$$

$$H_n(E_\cdot) = \text{Ker} \{X_n \otimes_A A \rightarrow X_n\} \quad n > 1.$$

Take $X = A$ above and consider the bicomplex $E_p \otimes_A P_\delta$. For $p$ fixed

$$H_{\delta} \left( E_p \otimes_A P_\delta \right) = E_p \otimes_A H_{\delta}(P_\delta)$$

because $E_p$ flat

$$= \begin{cases} E_p \otimes_A A' \otimes_A M = E_p \otimes_A M & \delta = 0 \\ 0 & \delta > 0 \end{cases}$$

because for $\delta > 0$ $H_\delta(P_\delta) \in \mathcal{N}$ and $E_p A = E_p$.

For $\delta$ fixed

$$H_p \left( E_\cdot \otimes_A P_\delta \right) = H_p(E_\cdot) \otimes_A P_\delta$$

$$= \begin{cases} A' \otimes_A P_\delta = P_\delta & p = 0 \\ 0 & p > 0 \end{cases}$$

Thus we get canonical isomorphisms

$$H_n(P_\delta) = H_n(E_\cdot \otimes_A P_\delta) = H_n(E_\cdot \otimes_A M)$$
Where
\[ H_0(P) = H_0(E \otimes_A M) = A' \otimes_A M. \]

At this point we have the left derived functors of the inclusion
\[ \text{Mod}(A)/N_A \hookrightarrow \text{Mod}(A) \to \text{Mod}(A)_A. \]

Namely, \( M \mapsto E \otimes_A M \) is an exact functor from \( \text{Mod}(A)/N_A \) to complexes of abelian groups, so the homology \( H_n(E \otimes_A M) \) is a connected sequence of functors on \( \text{Mod}(A)/N_A \) reducing to the above inclusion for \( n = 0 \).

If \( M \) is good flat, then for \( n > 0 \)
\[ H_n(E \otimes_A M) = H_n(E) \otimes_A M = 0 \]
because \( H_n(E) \) is null and \( M = A \cdot M \). Thus \( H_n(E \otimes_A M) \) is effaceable for \( n > 0 \).

(Notice that \( M \) flat in \( \text{Mod}(A) \) \( \Rightarrow \) \( M \) flat in \( \text{Mod}(A)/N_A \) \( \Rightarrow \) in the sense that the good module \( A' \otimes_A M \) corresponding to \( M \) is flat.)

Here's another way; suppose one \( A \) is given short
an exact sequence in \( \text{Mod}(A) \): \( N_0 \to N_1 \to N_2 \)

This means that \( N_1 \to N_2 \)
is a surjective map of good modules and \( N_0 = A \otimes_A K \)
where \( K \) is the kernel of \( N_1 \to N_2 \) in \( \text{Mod}(A) \).
From

\[ 0 \rightarrow K \rightarrow N_1 \rightarrow N_2 \rightarrow 0 \]

we get

\[ \text{Tor}_1^A(A, N_1) \rightarrow \text{Tor}_1^A(A, N_2) \rightarrow A \otimes_A K \rightarrow N_1 \rightarrow N_2 \rightarrow 0 \]

If \( N_2 \) is flat then

\[ 0 \rightarrow A \otimes_A K \rightarrow N_1 \rightarrow N_2 \rightarrow 0 \]

is exact. This shows that flat good modules are acyclic for \( F \). Moreover, if \( N_1 \) is flat, then we have

\[ 0 \rightarrow \text{Tor}_1^A(A, N_2) \rightarrow A \otimes_A K \rightarrow N_1 \rightarrow N_2 \rightarrow 0 \]

This shows that we have

\[ L_1 F(N) = \text{Tor}_1^A(A, N) \]

Now that we know good flat modules are acyclic for \( F \) and that there are enough of them, we get the existence of \( L_n F \) for all \( n \) and the fact that \( L_n F(N) = H_n(P) \), where \( P \) is the sort of resolution constructed before:

\[ 0 \rightarrow K_1 \rightarrow P_0 \rightarrow N \rightarrow 0 \]

\[ 0 \rightarrow K_2 \rightarrow P_1 \rightarrow A \otimes_A K_1 \rightarrow 0 \]

Thus

\[ L_n F(N) = \text{Ker} \left\{ A \otimes_A K_1 \rightarrow K_2 \right\} \quad \text{for } n \geq 1 \]
April 24, 1994

Book: Torsion Theories by Jonathan Clark.

R unital, \( R\)-mod unital left \( R\)-modules mod-\( R\) unital right \( R\)-modules

A torsion theory \( T \) over \( R \) can be defined
as a discrete subcategory of \( R\)-mod which is
closed under direct sums.

A torsion theory \( T \) is called janssen if
it is closed under arbitrary products.

Prop. Janssen torsion theories correspond bijectively
to ideals \( A \subseteq R \) such that \( A^2 = A \).

Proof. Given \( A \), let \( T_A = \{ M \in R\text{-mod} \mid AM = 0 \} \). This
is a janssen torsion theory: note that if \( AM_j = 0 \), then
\( A \cdot \bigcap M_j = 0 \), because if \( a \in A \), then \( a \cdot \bigcap M_j \subseteq \bigcap a M_j = 0 \).

Given a janssen \( T \), let \( S \) be the set of
cyclic modules \( R/\alpha \), or a left ideal, such that
\( R/\alpha \in T \). Let \( A = \bigcap \alpha \in S \). Then we have

\[
R/A \hookrightarrow \bigcap_{\alpha \in S} R/\alpha \in T
\]

so \( R/A \in T \). Given \( M \in T \) we have \( R/\alpha \cong Rm \leq M \), so \( AM = 0 \). Thus \( A \subseteq \bigcap \) the intersection of all
annihilators of modules in \( T \). On the other hand
this intersection kills \( R/A \) so it is \( A \). Since the
annihilator of an module is an ideal, \( A \) is an
ideal in \( R \). One has \( T \subseteq \{ M \mid AM = 0 \} \), and
the other inclusion \( \{ M \mid AM = 0 \} \subseteq T \) because any such \( M \)
is an \( \bigcap A \) module, thus a quotient of \( \bigoplus R/A \), Thus in \( T \).

Finally \( A^2 = A \), because the exact sequence

\[
0 \rightarrow A/A^2 \rightarrow R/A^2 \rightarrow R/A \rightarrow 0
\]

shows \( R/A^2 \in T \), so \( A(A/R/A^2) = 0 \Rightarrow A \subseteq A^2 \).
In general a torsion theory $T$ corresponds bijectively to a family of left ideals:

$$T \leftrightarrow \{ \alpha \mid R/\alpha \in T \}$$

called a Gabriel filter.

**Gabriel-Popescu theorem:** Let $A$ be a Grothendieck abelian category (AB5 holds), let $I$ be a generator, let $R = \text{End}(I)$. Then $h^I = \text{Hom}(I, -) : A \to R\text{-mod}$ has an exact adjoint, and this yields an equivalence $R\text{-mod}/T \sim A$

where $T$ is a torsion theory on $R$.

In other words Grothendieck categories are exactly those of the form $R\text{-mod}/T$ corresponding to a torsion theory. Now I would like to describe those which correspond to Torsion $T$. These should be the good categories of modules belonging to an unital $A$ such that $A^2 = A$.

Here's an example of an $A$ such that $A^2 = A$ such that there are no finitely generated good modules $\neq 0$. Take $A$ to be germs of continuous functions on $R$ at the origin which vanish there. In other words $A$ is the maximal ideal in a local ring $R=\mathbb{C}[x]$ where $A^2 = A$. Then if $M$ is finitely generated and $AM = M$, Nakayama's lemma $\Rightarrow M = 0$. Recall the proof: finitely generated $\Rightarrow$ maximal submodule in $M$, so we can suppose $M$ simple, whence $M = C = R/A$, which contradicts $AM = M$. 
Let $A \subset I$ be commutative rings, assume $\mathbb{A}^2 = A$, $IA \subset A$ (whence $IA = A$), (thus $A$ is a left ideal in $I$), $AI = I$, (I is the ideal in $I$ generated by $A$), whence $\mathbb{I}^2 = \mathbb{I}$.

Claim the category $A$-mod is of good $A$-modules and the category $I$-mod are canonically equivalent. In fact there is a canonical Morita equivalence between these categories.

If, first note that if $N \in A$-mod, $A \otimes_A N \approx N$ then one has a well-defined $I$-module structure on $N$ defined by $\chi(\alpha n) = (\chi_\alpha) n$.

Also $AN = N \Rightarrow IAN = N$, so $IN = N$.

and consequently $I \otimes_A N$ is a good $I$-module.

If $M \in I$-mod, then $IM = M$, so $\mathbb{A}M = AIM = IM = M$. Consequently $A \otimes_A M$ is a good $A$-module.

Thus we have functors

$$
\begin{align*}
A\text{-mod} & \xrightarrow{F} I\text{-mod} \\
N & \mapsto I \otimes_A N \\
A \otimes_A M & \mapsto I \otimes_A M
\end{align*}
$$

Define natural transformations

$$
\begin{align*}
GF & \rightarrow 1 & A \otimes I \otimes A & \rightarrow N \\
\chi(a, x, n) & \mapsto axn
\end{align*}
$$

$$
\begin{align*}
FG & \rightarrow 1 & I \otimes A \otimes M & \rightarrow M \\
\chi(a, x, n) & \mapsto xan
\end{align*}
$$
We show $GF \cong 1$. One has an exact sequence
\[ 0 \to K \to I \otimes_I N \to N \to 0 \]
of $I$-modules. We know $IK = 0$, hence $AK = 0$.
This yields the right exact sequence
\[ A \otimes K \to A \otimes I \otimes N \to A \otimes N \to 0 \]
\[ (a, x, n) \mapsto (a, xa) \]
But $N$ good $\Rightarrow A \otimes N \cong N$, done.

We show $FG \cong 1$. One has an exact sequence
\[ 0 \to K' \to A \otimes A M \to M \to 0 \]
of $I$-modules. We know $AK' = 0$; now $\text{ann}_I(K')$ is an ideal in $I$ containing $A$; this ideal is $AI = I$, so $IK' = 0$. This yields
\[ I \otimes K' \to I \otimes A \otimes M \to I \otimes M \to 0 \]
\[ (x, a, m) \mapsto (x, am) \]
showing $FG \cong 1$.

Finally check that $FGF \cong F$, $GFG \cong G$ coincide so the are compatible: $FG \cong 1$, $GF \cong 1$ are
\[ I \otimes A \otimes I \otimes N \to I \otimes A \otimes I \otimes M \]
\[ (x, a \cdot x', n) \to (x, ax' n) \]
\[ I \otimes N \to I \otimes M \]
and $(x, a \cdot x', n) = (x, ax' n)$ as $ax' \in AI = I$
\[ A \otimes_A I \otimes A \otimes_A M \implies A \otimes_A M \]

\[
(\alpha, \chi, a', m) \mapsto (a, \chi a'm) \\
(\alpha a', m) \mapsto (\alpha a', m)
\]

and these agree as \( xa' \in IA = A \).

Finally, the Morita equivalence is given by the bimodules:

\[
A\text{-mod} \xrightarrow{\text{I-mod}} \left( \frac{(I \otimes A) \otimes_A}{A \otimes_A I} \right) \text{-mod}
\]

i.e. the bimodules \( I \otimes A, A \otimes_A I \) which are good \( I \)-modules and \( A \)-module resp. The composites are:

\[
(A \otimes_A I) \otimes_A (I \otimes_A I) = A \otimes_A I \otimes_A I \implies A \otimes_A A
\]

\[
(I \otimes_A I) \otimes_A (A \otimes_A I) = I \otimes_A A \otimes A \implies I \otimes_A I
\]

and the isomorphisms are proved as above.
Let $B \subset A$ be rings, assume $B^2 = B$, 
$B = BAB$, $A = ABA$. Then $A = ABA \subset A^3 \subset A^2 \subset A$.

Claim we have a Morita equivalence

$$B\text{-}g\text{mod} \iff A\text{-}g\text{mod}$$

1) 
$$N \rightarrow (A \otimes_A AB) \otimes_B N$$

$$\left( B \otimes_B BA \otimes_A M \right) \leftarrow M$$

Note that, as $A(AB) = AB$, $A \otimes_A AB$ is a good $A$-mod.

Similarly $B \otimes_B BA$ is a good $B$-mod, so the above functors are defined.

We have maps joining the two composites to the identity given by the bimodule maps:

2) 
$$\left( B \otimes_B BA \right) \otimes_A (A \otimes_A AB) \rightarrow B \otimes_B B$$

$$(b, w, a, w) \mapsto (b, aw)$$

3) 
$$\left( A \otimes_A AB \right) \otimes_B (B \otimes_B BA) \rightarrow A \otimes_A A$$

$$(a, w, b, w) \mapsto (a, wbw)$$

Compatibility:

$$(b, w, a, w, b', w') \rightarrow (b, w, a, vbw)$$

$$(B \otimes_B BA) \otimes_A (A \otimes_A AB) \otimes_B (B \otimes_B BA) \rightarrow (B \otimes_B BA) \otimes_A (A \otimes_A A)$$

$$(b, wa, b', w') \mapsto (b, wavb'w)$$

Since $wavb' \in B$
\[(a, v', b, w', a', v') \mapsto (a, v, b, w, a', v')\]

\[
(A \otimes_A AB) \otimes_B (B \otimes_B BA) \otimes_A (A \otimes_A AB) \rightarrow (A \otimes_A AB) \otimes_B (B \otimes_B B)
\]

\[
(a, v, b, w, a', v') \mapsto (a, v, b, w, a', v')
\]

**Remark:** We now show that 2), 3) are isoms.

2): First have \(A \otimes_A AB \rightarrow AB\) surjective and its kernel \(K\) is such that \(AK = 0\).

Then \(BA \otimes_A K = 0\), so

\[
(w, a, v) \mapsto (w, av)
\]

\[
BA \otimes_A A \otimes_A AB \sim BA \otimes_A AB
\]

Consider now the multiplication map

\[
BA \otimes_A AB \rightarrow B
\]

which is surjective as \(BAB = BAB = B\). Its kernel is killed by \(B\): if \((w_i, v_i) \mapsto v_i v_i = 0\), then

\[
b(w_i, v_i) = (bw_i, v_i) = (b, w_i, v_i) = 0\], using \(w_i \in A\).

Thus

\[
B \otimes_B BA \otimes_A AB \sim B \otimes_B B
\]

Combining these two isos, we see 2) is an isom.

3): First \(B \otimes_B BA \rightarrow BA\) is surjective and its kernel \(K\) is killed by \(B\), so using \(AB = (AB)B\) we have \(AB \otimes_B K = 0\), yielding

\[
AB \otimes_B B \otimes_B BA \sim AB \otimes_B BA
\]

Consider the multiplication map

\[
AB \otimes_B BA \rightarrow A
\]
This is surjective as $ABA = AB = A$.

Let $K$ be the kernel. If $(v_i, w_i) \in K$, then $b_1 b_2 (v_i, w_i) = (b_1, b_2 v_i, w_i) = (b_1, b_2 - v_i, w_i) = 0$ in $AB \otimes_B BA$, using the fact that $b_2 v_i \in BAB = B$. Thus $B K = B^2 K = 0$. But $K$ is an $A$-module.

So $A K = AB A K \subset A B K = 0$. We conclude then

$$A \otimes_A AB \otimes_B BA \cong A \otimes_A A$$

and 3) follows by combining the above two isos.

Now suppose $B \subset R$ idempotent, such that $BRB = B$. Let $A = RBR$. Then $B \subset A$,

$BA = BRBR = BR$, $AB = RBRB = RB$,

$B = BB^2 B \subset BAB \subset BRB = B \Rightarrow BAB = B$.

$A = RBR = RB BR = ABBA = ABA$.

Thus one has $B \subset A, B^2 = B, BAB = B, ABA = A$ so we see from the above that $B$-gmod is equivalent to $A$-gmod, where $A$ is the ideal $RBR$ generated by $B$ in $R$. 
Consider functors from $\text{A-mod} \to \text{Ab}$ and ask when they descend to $\text{A-mod}/\text{A-null}$.

1) $X \in \text{mod-A}$. Then $X \otimes_A^- : \text{A-mod} \to \text{Ab}$ descends to $A \iff X \in \text{gmod-A} : X \otimes_A A \to X$.

Pf. Recall $X \in \text{gmod-A} \iff X \otimes_A A^\# \to X$ where $A^\# = A \otimes_A A$. One a commutative triangle

\[
\begin{array}{ccc}
(X \otimes_A A^\#) \otimes_A M & = & X \otimes_A (A^\# \otimes A M) \\
\alpha & \downarrow & \beta \\
X \otimes_A M & & \\
\end{array}
\]

$X$ is good $\iff$ $\alpha$ is an isomorphism (for $\iff$ take $M = \overline{A}$), $X \otimes_A -$ descends to $A \iff \beta$ is an isomorphism for all $M$.

$\implies$ because $A^\# \otimes_A M \to M$ becomes an isom. in $A$,
$\iff$ because $A^\# \otimes_A -$ descends to $A$.

2) Let $N \in \text{A-mod}$. Then $\text{Hom}_A(N, -)$ descends to $A \iff N \in \text{A-gmod}$.

Pf. Commute triangle:

\[
\begin{array}{ccc}
\text{Hom}_A(A^\# \otimes_A N, M) & = & \text{Hom}_A(N, \text{Hom}_A(A^\# M)) \\
\alpha & \downarrow & \beta \\
\text{Hom}_A(N, N) & & \\
\end{array}
\]

$N$ is good $\iff$ $\alpha$ is an isomorphism. $\forall M$.
$\text{Hom}_A(N, -)$ descends to $A \iff \beta$ is an isom,
$\implies$ because $M \to \text{Hom}_A(A^\#, M)$ becomes an isom. in $A$,
$\iff$ because $\text{Hom}_A(A^\#, -)$ descends to $A$.
3) Let $Q \in A$-mod. Then $\text{Hom}_A(-, Q)$ descends to $A$ iff $Q \in A$-gmod: $Q \sim \text{Hom}_A(A, Q)$.

Pf. Comm. triangle

$$
\text{Hom}_A(A \otimes_A M, Q) \cong \text{Hom}_A(M, \text{Hom}_A(A, Q))
$$

$\beta$ is an isom $\forall M \iff A$ is good

$\alpha$ is an isom $\forall M \iff \text{Hom}_A(-, Q)$ descends to $A$,

($\leftarrow$ because $A \otimes_A M \rightarrow M$ becomes an isom in $A$,

$\leftarrow$ because $A \otimes_A -$ descends to $A$)

Write $\text{Hom}_A(M, N)$ for $\text{Hom}_A(M, N)$

where $A = A$-mod/A-null. Similarly write $X \otimes_A M$

for the tensor product functor on

$(\text{mod}-A/\text{null}-A) \times (A$-mod$/A$-null)$

From 1)-3) above one has

$$X \otimes_A M = X \otimes_A M \quad \text{if either} \quad \begin{cases} X \text{ is } A^0\text{-good} & \text{or} \\ M \text{ is } A\text{-good} \end{cases}$$

$$\text{Hom}_A(M, N) = \text{Hom}_A(M, N) \quad \text{if either} \quad \begin{cases} M \text{ is } A\text{-good} & \text{or} \\ N \text{ is } A\text{-good} \end{cases}$$
Now let \( u : A \to B \) be a homomorphism of idempotent rings. One then has

\[
\begin{array}{ccl}
A = A\text{-mod}/A\text{-null} & \overset{u_!}{\leftrightarrow} & B\text{-mod}/B\text{-null} = B
\end{array}
\]

where each functor is left adjoint to the one immediately below. \( u^* \) is the restriction functor; it is induced by restriction of scalars from \( B\text{-mod} \) to \( A\text{-mod} \); this is exact and carries null \( B \)-modules into null \( A \)-modules, so it descends to an exact functor between the quotient categories.

One has

\[
\begin{align*}
\text{Hom}_A (N, u^*(M)) &= \text{Hom}_A (N, M) & \text{if } N \text{ is } A\text{-good} \\
&= \text{Hom}_A (N, \text{Hom}_B (B, M)) & \text{if } M \text{ is } B\text{-good} \\
&= \text{Hom}_B (B \otimes_A N, M) \\
&= \text{Hom}_B (B \otimes_A N, M)
\end{align*}
\]

Thus \( u^* \) has the left adjoint \( u_! \), given by arbitrary

\[
u_! (N) = B \otimes_A N \text{, when } N \text{ is } A\text{-good}.
\]

Note for \( N \in A\text{-mod} \) that

\[
B(B \otimes_A N) = B \otimes N \text{, hence } B \otimes B \otimes_A N = B^3 \otimes_A N \text{ is the module arising from } N
\]

is \( B\text{-good} \). Thus we have the formula

\[
u_! (N) = B^3 \otimes_A A^3 \otimes_A N
\]

for any \( N \in A\text{-mod} \), where the right side is \( B\text{-good} \).

Next one has

\[
\begin{align*}
\text{Hom}_A (u^*(M), N) &= \text{Hom}_A (M, N) & \text{if } N \text{ is } A\text{-good} \\
&= \text{Hom}_A (B \otimes_B M, N) & \text{if } M \text{ is } B\text{-good}
\end{align*}
\]
\[
\begin{align*}
= \text{Hom}_B(M, \text{Hom}_A(B, N)) \\
= \text{Hom}_B(M, \text{Hom}_A(B, N))
\end{align*}
\]

Thus \( u^* \) has the right adjoint \( u_* \) given by
\[
\begin{align*}
u_*(N) &= \text{Hom}_A(B, N) \quad \text{for } N \text{ A-good}' . \quad \text{Note that for} \\
\wedge \text{Hom}_B(\mathbb{Z}, \text{Hom}_A(B, N)) &= \text{Hom}_A(\mathbb{B} \otimes \mathbb{Z}, N) = 0 , \quad \text{so that} \\
\text{Hom}_B(B, \text{Hom}_A(B, N)) &= \text{Hom}_A(B^g, N) \quad \text{is } B\text{-good} \quad \text{the} \\
\text{module arising from } \text{Hom}_A(B, N) . \quad \text{Thus one has the} \\
\text{formula} \\
u_*(N) &= \text{Hom}_A(B^g, \text{Hom}_A(A^g, N)) \\
&= \text{Hom}_A(A^g \otimes A^g, N)
\end{align*}
\]

for any \( N \in A \), where the right side is \( B\text{-good}' . \)

Think as follows. The usual left adjoint for the restriction of scalars is \( N \rightarrow \mathbb{B} \otimes A N \). To get \( u! \) on the level of good modules, we make it \( B\text{-good} : \mathbb{B} \otimes \mathbb{B} \otimes A N = B^g \otimes A N \) , and then \( B^g \otimes A N \) if we want \( N \in A \). Similarly the usual right adjoint for the restriction of scalars is \( N \rightarrow \text{Hom}_A(B, N) . \) To get \( u_* \) on the level of good' modules we make it \( B^g\text{-good}' : \text{Hom}_B(B, \text{Hom}_A(B, N)) = \text{Hom}_A(B^g, N) \) and then \( \text{Hom}_A(A^g \otimes A, B^g, N) \) if we want \( N \) to
range over \( A \).
Let's consider the 'good' tensor product

\[ X \otimes_A M \overset{\text{def}}{=} X \otimes_A A^g \otimes_A M \]

from \( (\text{mod-}A/\text{null-}A) \times (\text{A-mod}/A\text{-null}) \rightarrow AB \)

Note that if \( X \) is good, then \( X \otimes_A M = X \otimes A M \).

Moreover, if \( X \) is good and flat, then this is an exact functor of \( M \). However, if \( X \) is just flat as an \( A^g \)-module, then \( X \otimes_A M = X \otimes A X \otimes M \) need not be exact in \( M \), e.g., for \( X = \mathcal{A} \) we have \( A X = \mathcal{A} \), unless \( \mathcal{A} \) is a flat \( A^g \)-module.

Recall that there exist enough flat good modules in \( A\text{-gmod} \) and \( \text{gmod-}A \). Thus, the left derived functors of \( \overset{\mathcal{A}}{\otimes} \), denoted \( \text{Tor}_n^A (-, -) \), are defined. Let's recall their construction in analogy with \( \text{Tor}_n^A (-, -) \).

Let \( X \) be a right \( A \)-module. Choose

\[ 0 \rightarrow K_1 \rightarrow E_0 \rightarrow X \quad \text{E}_0 \text{ flat good} \quad \Rightarrow \ K_1 A = K_1 \]

\[ 0 \rightarrow K_2 \rightarrow E_1 \rightarrow K_1 \otimes_A \rightarrow 0 \quad \text{E}_1 \quad \text{K}_2 A = K_2 \]

Then \( E_0 \) is a complex of flat good \( A^g \)-modules which is a resolution of \( X \otimes_A A^g \) in \( \text{gmod-}A \).

Similarly, we can construct a complex \( F_0 \) of good flat \( A \)-modules which is a resolution of \( A^g \otimes_A M \) in the category \( A\text{-gmod} \). Then we have quasi

\[ X \otimes_A F \overset{\mathcal{A}}{=} E \otimes_A F \rightarrow E \otimes_A M \]

In effect \( H_p(E \otimes_A F) = H_p(E) \otimes_A F \)

\[ = \begin{cases} X \otimes_A A^g \otimes A \overset{\mathcal{A}}{=} X \otimes A \overset{\mathcal{A}}{=} & \text{as } F \overset{\mathcal{A}}{=} \text{flat} \\
0 & \end{cases} \]

\[ \overset{\mathcal{A}}{=}_0 & \overset{\mathcal{A}}{=} \text{g} \overset{\mathcal{A}}{=} 0 \overset{\mathcal{A}}{=} 0 \]
as $F_\bullet$ is good and $H_p(E_\bullet)$ is null-$A$ for $p > 0$. Then we define

$$\text{Tor}_n^A(X, M) = H_n(X \otimes_A F_\bullet) = H_n(E \otimes_A F_\bullet) = H_n(E \otimes_A M)$$

This shows the independence of the choice of the resolutions $E_\bullet$, $F_\bullet$. The $\{ \text{Tor}_n^A(X, -) \}$ are the derived functors of $X \otimes_A (-)$. For $X = A$ one has

$$\text{Tor}_n^A(A, M) = A \otimes_A A^\delta \otimes_A M = A^\delta \otimes_A M$$

which is the canonical left exact embedding

$$\text{A-mod}/\text{A-null} \rightarrow \text{A-flat mod} \subset \text{A-mod}$$

$$M \mapsto A^\delta \otimes_A M.$$ 

Call this function $F$, so that we have

$$L_n F(M) = \text{Tor}_n^A(A, M)$$

I next want to relate these $\text{Tor}_n^A(A, M)$. Consider $\text{Tor}_n^A(A, M) = H_n(E \otimes_A M)$ where $E_\bullet$ is a flat resolution of $A^\delta$ in $\text{A-flat mod - A}$. Note that

$$H_n(E_\bullet) = \text{Tor}_n^A(A, A) = L_n F(A)$$

where $L_n F(A^\delta) \rightarrow L_n F(A) \rightarrow L_n F(A)$ since $L_n F$ is defined on the quotient category $\text{A-mod}/\text{A-null}$. Because $E_\bullet$ is a complex of flat $A^{\text{op}}$-modules, it has a spectral sequence

$$E^2_{pq} = \text{Tor}_p^A(H_q(E_\bullet), M) \Rightarrow H_n(E \otimes_A M)$$
yielding the spectral sequence

\[ E^2_{pq} = \text{Tor}^A_p \left( L_q F(A), M \right) \Rightarrow L_n F(M) \]

\[ L_1 F(A) \otimes_A M \]

\[ A^g \otimes_A M \]

\[ \text{Tor}^A_1 (A^g, M) \]

\[ \text{Tor}^A_2 (A^g, M) \]

We get the 5-term sequence

\[ L_2 F(M) \rightarrow \text{Tor}^A_2 (A^g, M) \rightarrow L_1 F(A) \otimes_A M \rightarrow L_1 F(M) \rightarrow \text{Tor}^A_1 (A^g, M) \rightarrow 0 \]

\[ L_1 F(A) \otimes_A M = L_1 F(A) \otimes_A \frac{M}{AM} \]

Thus we get

\[ L_1 F(M) = \text{Tor}^A_1 (A^g, M) \text{ if } M = AM \]

On the other hand suppose we resolve \( M \):

\[ 0 \rightarrow K_1 \rightarrow F_0 \rightarrow A^g \otimes_A M \rightarrow 0 \]

\[ 0 \rightarrow K_2 \rightarrow F_1 \rightarrow A \otimes_A K_1 \rightarrow 0 \]

where \( F_0 \) is flat and good. Then

\[ 0 \rightarrow \text{Tor}_1^A (A, A^g \otimes_A M) \rightarrow A \otimes_A K_1 \rightarrow F_0 \rightarrow A^g \otimes_A M \rightarrow 0 \]

\[ 0 \rightarrow L_1 F(A^g \otimes_A M) \rightarrow F(K_1) \rightarrow F(F_0) \rightarrow F(A^g \otimes_A M) \rightarrow 0 \]

Thus we get

\[ \text{Tor}_1^A (A, A^g \otimes_A M) = L_1 F(M) \]
So we get various expressions for $L_1 F(M)$, namely

$$L_1 F(M) = \text{Tor}_1^A (A^g, A \otimes_A M) = \text{Tor}_1^A (A, A^g \otimes_A M)$$

The only way to be exact is that $M$ be flat or projective.

Claim: $F : A\text{-mod}/A\text{-null} \to A\text{-mod} \subset A\text{-mod}$

$$M \mapsto A^g \otimes_A M \to A^g \otimes_A M$$

is exact iff $A^g$ is a flat $A^{op}$-module.

Why? One has an equivalence between exact functors on $A\text{-mod}/A\text{-null}$ and exact functors on $A\text{-mod}$, killing $A\text{-null}$. Thus $F(M) = A^g \otimes_A M \in A\text{-mod}$ is exact on $A\text{-mod}/A\text{-null}$ if $M \mapsto A^g \otimes_A M$ from $A\text{-mod}$ to $A\text{-mod}$ is exact if $A^g$ is right $A$-flat.

The same holds for $M \mapsto X \otimes_A M$, where $X$ is $A^{op}$-good, from $A\text{-mod}/A\text{-null} \to A\text{-mod}$.

Put another way, a good module $X$ is flat in $\text{mod-}A^{op} \iff$ it is flat in $\text{mod-}A$.

Proof: Let $X \in \text{mod-}A$, $M \in A\text{-mod}$ satisfy $X A = X$, $A M = M$.

Then $X^g \otimes_A M \to X \otimes_A M$.

Pf.: $0 \to K \to A \otimes_A M \to M \to 0$, $AK = 0 \Rightarrow X \otimes_A K = 0$

Thus $X \otimes_A A \otimes_A M \to X \otimes_A M$. Applying this to $A \otimes_A M$

in place of $M$ yields

$$X \otimes_A A \otimes_A A \otimes_A M \to X \otimes_A A \otimes_A M \to X \otimes_A M$$
Morita equivalence - general case.

A Morita equivalence between rings $A, B$ is given by bimodules $P_A, Q_B$ together with pairings $P \otimes_A Q \to B$, $Q \otimes_B P \to A$ satisfying certain conditions of compatibility. These can be expressed by saying one has a ring $R$ with block decomposition

$$ R = \begin{pmatrix} A & Q \\ P & B \end{pmatrix} $$

Now, it turns out that the Morita equivalence between $A, B$ is the composition of Morita equivalences $A \sim R$ and $B \sim R$. We have previously obtained a M.tq. $A \sim R$ assuming that $ARA = A$, $RAR = R$, $A \leq R$ and $A = A^2$. Let's calculate

$$ RA = \begin{pmatrix} A^2 & 0 \\ PA & 0 \end{pmatrix} $$

$$ ARA = \begin{pmatrix} A^3 & 0 \\ 0 & 0 \end{pmatrix} $$

$$ RAR = \begin{pmatrix} A^2 & 0 \\ PA & 0 \end{pmatrix} \begin{pmatrix} A & Q \\ P & B \end{pmatrix} = \begin{pmatrix} A^3 & A^2Q \\ PA^2 & PAQ \end{pmatrix} $$

Thus we want to have

2) $A = A^2$, $P = PA$, $Q = AQ$, $PQ = B$

Similarly for $B \sim R$ we want

3) $B = B^2$, $P = BP$, $Q = QB$, $QP = A$

These 8 conditions reduce to four:

4) $QP = A$, $PQ = B$, $PQP = P$, $QPQ = Q$
Now we should have an equivalence of categories
\[ M \mapsto P_A^B M \]
5) \( A\text{-mod}/A\text{-null} \rightleftharpoons B\text{-mod}/B\text{-null} \)
\[ Q_B^B N \rightleftharpoons N \]

Let's note first that because \( PA = P \), we have
\[ P_A^A \cong P_A^A \]
and similarly \( Q_B^B \cong Q_B^B \).
Thus \( P_A^A M = P_A^A A \otimes_A M \), \( Q_B^B N = Q_B^B B \otimes_B N \).

We now want to see that the canonical map
\[(P_A^A A) \otimes_A (Q_B^B B) \otimes_B N \rightarrow (B \otimes_B B) \otimes_B N\]
\((p, a, q, b, n) \mapsto (pa, b, n)\)
is an isomorphism of functors of \( N \). It's suffices to show that
6) \( P_A^A A \otimes_A Q \otimes_B B \rightarrow B \otimes_B B \)
\((p, a, q, b) \mapsto (pa, b)\)
is an isomorphism. Now \( PA = P, Aq = Q \Rightarrow P_A^A A \otimes_A Q \rightarrow P_A^A Q \).
Consider
7) \( P_A^A Q \xrightarrow{\pi} B \)
\((p, q) \mapsto pq\)

This is a map of \( B\)-bimodules which is surjective as \( PQ = B \) by hypothesis. Also
\[ p'q' (p \otimes q) = (p'q'p, q) = (p', q'p, q) \]
\[ (p, q)p'q' = (p, qp'q') = (pqp', q') \]
as \( QP = A \) and the tensor product \( P_A^A Q \) is over \( A \).
This implies that the kernel of \( \pi \) is null as both
left + right $B$-module. Thus if

$$(pi, q_i) \in \text{Ker}(\pi),$$ i.e. $p_iq_i = 0$, then

$$(p_i q_i) p' q' = (p_i q_i p') q' = 0$$

whence $(p_i q_i) B = 0$ as $P A = B$.

Since $\pi$ is surjective with kernel killed by $B$ we conclude

$$P \otimes_A Q \otimes_B B \rightarrow B \otimes_B B$$

Thus 6) which is the composite

$$P \otimes_A A \otimes_A Q \otimes_B B \rightarrow P \otimes_A Q \otimes_B B \rightarrow B \otimes_B B$$

is an isomorphism.

It's clear now that 5) is an equivalence.

If instead of the quotient categories we use the good module categories we have the equivalence

$$\begin{align*}
A \text{-gmod} & \overset{\sim}{\longrightarrow} B \text{-gmod} \\
A \otimes_A Q \otimes_B B & \otimes_A B \otimes_A P \otimes_A A \\
B \otimes_B P \otimes_A A & \otimes_A B \otimes_B B
\end{align*}$$

Note that $B P = P \Rightarrow B \otimes_B P$ is a good left $B$ module

$A Q = Q \Rightarrow A \otimes_A Q \rightarrow A \otimes_A A \rightarrow A$.
Concept of a generator \( U \) in an abelian category \( A \): By definition \( U \) is a generator when the functor \( h^U = \text{Hom}(U, -) \) from \( A \) to \( \text{Ab} \) is faithful, i.e. \( \forall X, Y \)

\[
\text{Hom}(X, Y) \rightarrow \text{Hom}(h^U(X), h^U(Y))
\]
equivalently \( \exists U \rightarrow X \) such that \( U \rightarrow X \rightarrow Y \) is \( \neq 0 \). Observe this last condition depends only on the image of \( X \rightarrow Y \), so that \( U \) is a generator \( \iff \forall X \rightarrow Y \neq 0 \exists P \rightarrow X \) such that \( P \rightarrow X \rightarrow Y \) is \( \neq 0 \), equivalently \( \forall X', \exists P \rightarrow X \) with image not contained in \( X' \).

Thus \( U \) is a generator \( \iff \forall X \) the smallest subobject of \( X \) containing the images of all \( U \rightarrow X \) is \( X \). If \( A \) has direct sums this means \( X \) is a quotient of \( \bigoplus \limits_{i \in I} U \) for some set \( I \).

Note that any generator for \( A\text{-mod} \) (i.e. \( U \)) such that \( A \) is a summand of \( \bigoplus \limits_{i \in I} U \) for some \( I \) is a generator for \( A = A\text{-mod}/A\text{-null} \). However \( A \) is not a generator for \( A\text{-mod} \), but it is a generator for \( A \), since given \( M \) such that \( AM = M \), one has \( A \otimes_A M \rightarrow M \), so a set of generators for \( M \) gives rise to a surjection \( \oplus A \rightarrow M \), \( (a_i) \rightarrow a_i \cdot m_i \).
An \( A \)-module \( U \) is a generator for \( A \) iff \( A \) is a quotient of \( \bigoplus_{I} A^g \otimes_A U \) for some set \( I \).

**Proof:** As \( A \) generates \( A \), \( U \) generates \( A \) if there is an epimorphism \( \bigoplus_{I} U \rightarrow A \) in \( A \)-mod, equivalently there is an epimorphism \( A^g \otimes_A (\bigoplus_{I} U) = \bigoplus_{I} A^g \otimes_A U \rightarrow A^g \) in \( A \)-mod. Since \( A^g \rightarrow A \) we get \( (\Rightarrow) \). Conversely, if we have \( \bigoplus_{I} A^g \otimes_A U \rightarrow A \), then tensoring with \( A \) yields \( \bigoplus_{I} A^g \otimes_A U = \bigoplus_{I} A \otimes (A^g \otimes_A U) \rightarrow A \otimes A = A^g \) so \( \bigoplus_{I} U \rightarrow A \) in \( A \).

Consider a Morita equivalence

\[
\begin{align*}
\begin{pmatrix}
A & Q \\
P & B
\end{pmatrix}
\end{align*}
\]

The condition \( QP = A \Rightarrow \) \( I \) surjection \( \bigoplus_{I} Q \rightarrow A \) in \( A \)-mod for some \( I \). Thus \( Q \) is a generator for \( A \)-mod/\( A \)-null, and similarly \( P \) is a generator for \( \text{mod-} A/\text{null-} A \).

\[
\begin{align*}
A &= A^2 = QP, & Q &= AQ = QB, \\
P &= PA = BP, & B &= B^2 = PQ
\end{align*}
\]

Another way to write this: \( R_+ = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \), \( R_- = \begin{pmatrix} 0 & Q \\ P & 0 \end{pmatrix} \)

Then we have a superalgebra such that

\[
(R_-)^2 = R_+ \quad (R_-)^3 = R_-
\]
Proof of the equivalence of categories

2) \((A\text{-gmod}) \xrightarrow{B \otimes_B P \otimes_A} (B\text{-gmod})\) \\
\[A \otimes_A Q \otimes_B\]

Note: \(BP = P \Rightarrow B \otimes_B P\) is a good \(B\)-module \(\Rightarrow B \otimes_B P \otimes_A M\) is a good \(B\)-module \(\forall M\). Thus these functors are defined.

Next have canonical surjection of \(B\)-bimodules

3) \(P \otimes_A Q \twoheadrightarrow B\)

whose kernel is killed by \(B\) on either side:

If \(\sum (p_i q_i) \mapsto \sum p_i q_i = 0\), then

\[p \otimes \sum (p_i q_i) = \sum (p \otimes p_i q_i) = (p \otimes \sum p_i) q_i = 0\]

Thus get

4) \(B \otimes_B P \otimes_A Q \twoheadrightarrow B \otimes_B B\)

and hence

5) \((B \otimes_B P) \otimes_A (A \otimes_A Q) \twoheadrightarrow B \otimes_B P \otimes_A Q \twoheadrightarrow B \otimes_B B\)

where the first isom results from \(X \otimes_A A \otimes_A M \twoheadrightarrow X \otimes_A M\)

if \(XA = X, AM = M\).

Similarly have canonical surjection of \(A\)-bimodules

6) \(Q \otimes_B P \twoheadrightarrow A\)

whose kernel is killed by \(A\) on either side. Thus

7) \(A \otimes_A Q \otimes_B P \twoheadrightarrow A \otimes_A A\)

8) \((A \otimes_A Q) \otimes_B (B \otimes_B P) \twoheadrightarrow A \otimes_A Q \otimes_B P \twoheadrightarrow A \otimes_A A\).
The iso. 5), 8) prove the equivalence of categories 2).

From 3) we also get
\((P \otimes A) \otimes_B P \simeq B \otimes_B P\)
\(Q \otimes_B (P \otimes A) \simeq Q \otimes B\)

and from 6) we get
\((Q \otimes_B P) \otimes_A Q \simeq A \otimes_A Q\)
\(P \otimes_A (Q \otimes_B P) \simeq P \otimes A\)

Thus we have comm. squares

\[
\begin{array}{ccc}
P \otimes_A Q \otimes_B P & \simeq & P \otimes_A A \\
\downarrow \cong & & \downarrow \cong \\
B \otimes_B P & \longrightarrow & P
\end{array}
\]
\[
\begin{array}{ccc}
Q \otimes_B (P \otimes_A Q) & \simeq & Q \otimes B \\
\downarrow \cong & & \downarrow \cong \\
A \otimes_A Q & \longrightarrow & Q
\end{array}
\]

In a similar way we can start with

10) \(P \otimes_A A \longrightarrow P\)

which is a surjection of \(B, A\) bimodules whose kernel is null on both sides: if \((p_i, q_i) \in P \otimes_A A\) is such that \(p_i = 0\), and \(p_q \in B\), then
\[p_q (p_i, q_i) = (p_q p_i, q_i) = (p_q, p_q q_i) = 0.\]

Then 10) yields

11) \(B \otimes_B P \otimes_A A \longrightarrow B \otimes_B P\)

showing that \(B \otimes_B P\) is right \(A\)-good.
Similarly

12) \( B \otimes_B P \rightarrow P \)

is a surjection of \( B_2, A_1 \) bimodules whose kernel is null in both sides, whence

\[
\begin{array}{ccc}
\otimes & B \otimes_B P \otimes_A A & \rightarrow & P \otimes_A A \\
\end{array}
\]

showing \( P \otimes_A A \) is left \( B \)-good.

Thus we learn that \( P \otimes_A A \cong B \otimes_B P \)
is the good form of the bimodule \( P \), while
\( A \otimes_A Q \cong Q \otimes_B B \) is the good form of the bimodule \( Q \), so that we can write the equivalences as follows.

\[
\begin{array}{ccc}
\otimes & \text{A-gmod} & \rightarrow & \text{B-mod/B-null} \\
\otimes & \text{A \otimes Q \otimes_B} & \\
\otimes & \text{A-mod/A-null} & \rightarrow & \text{B-gmod} \\
\otimes & \text{B \otimes_B P \otimes_A} & \\
\otimes & \text{Q \otimes_B} & \\
\end{array}
\]

Notice that this implies that the \( B \)-bimodule
\( P \otimes_A A \otimes_A Q = P \otimes_A Q \) gives a well-defined functor on \( B \)-mod/B-null, and this functor is the identity. This suggests that \( P \otimes_A Q \) is \( B \)-good. Check:

\[
P \otimes_A Q \otimes_B B = P \otimes_A A \otimes_A Q = P \otimes_A Q
\]

But then since \( P \otimes_A Q \rightarrow B \) has null kernel we must have \( P \otimes_A Q = B \otimes_B B \).
\[
\begin{align*}
B \otimes B & \leftarrow B \otimes B \\
\text{because} & \\
& B, \text{Ker}(P \Rightarrow A) = 0 \\
& \cap \text{Ker}(A \Rightarrow P) : A = 0
\end{align*}
\]

So we find for any Morita equivalence data \((A, Q, P, B)\) that

\[
\begin{align*}
P \otimes A & = B \otimes B \\
Q \otimes B & = A \otimes A
\end{align*}
\]

\[
\begin{align*}
P \otimes A \otimes A & = B \otimes B \\
Q \otimes B \otimes A & = A \otimes A
\end{align*}
\]

Let us say that Morita equivalence data \((A, Q, P, B)\) is good when instead of just the relations

\[
\begin{align*}
A & = A^2 = QP \\
Q & = AQ = QB \\
P & = PA = BP \\
B & = B^2 = PQ
\end{align*}
\]

we have the stronger relations

\[
\begin{align*}
A & = A \otimes A \otimes A = Q \otimes B \\
Q & = A \otimes A \otimes Q = Q \otimes B \\
P & = P \otimes A = B \otimes B \\
B & = B \otimes B = P \otimes A
\end{align*}
\]
Then it's pretty clear that given any Morita equiv. data \((A, Q, \rho, B)\), then it can be replaced by good Morita equiv. data:

\[
\begin{pmatrix}
A \otimes_A A & A \otimes_A Q = Q \otimes_B B \\
\rho \otimes_A A &= B \otimes_B P \quad B \otimes_B B
\end{pmatrix}
\]

In terms of the superalgebra \(R\) the Morita equivalence is good when

\[
R^- \otimes R^+ \sim R^+ \quad R^- \otimes R^+ \otimes R^- \sim R^-
\]
Wodzicki result: Let $A$ be a left ideal in a ring $R$ and let $M$ be an $R$-module such that $AM = M$. Then $M$ is flat over $A$ if and only if $M$ is flat over $R$.

Proof. Easy direction ($\Rightarrow$): If $X \in \text{mod-} R$ one has $X \otimes_A M \cong X \otimes_R M$ because $x \otimes_A a \cdot m = x \otimes_A A a \cdot m = x \otimes_A \text{ram}$.

Assuming $M$ is $A$-flat the functor $X \mapsto X \otimes_A M = X \otimes_R M$ from $\text{mod-} R$ to $\text{Ab}$ is exact, hence $M$ is $R$-flat.

($\Leftarrow$): Use the Cartan-Eilenberg linear equation criterion for flatness: Given solid arrows

\[
\begin{array}{ccc}
\tilde{A}^p \xrightarrow{a} & A^b & \xrightarrow{a'} \longrightarrow A^r \\
\downarrow m & & \downarrow m' \\
M & = & M 
\end{array}
\]

The dotted arrows exist. Here $a, a'$ are matrices over $\tilde{A}$ and $m, m'$ are column vectors with entries in $M$. We can look at the linear equations $am = 0$ over $R$ and use the fact that $M$ is $R$-flat to obtain the solid arrows.

\[
\begin{array}{ccc}
\tilde{R}^p \xrightarrow{a} & \tilde{R}^b & \xrightarrow{r} \tilde{R}^c \\
\downarrow m & & \downarrow m' \\
M & = & M = M 
\end{array}
\]

Using $AM = M$ the dotted arrows exist, where $\alpha$ has
entries in $A$.

Note here we use the fact that $A$ is a left ideal to conclude that $R^S \ni x \mapsto xx \in R^S$ has its image in $A^S$.

Then putting $a' = ra$ we get the desired completion of $1$.