November 8, 1993

Here is an improvement to the discussion of the Karoubi operator \( \mathcal{K} \) on \( C(A) \), which arises from the Alexander-Spanier differential.

Notation: \( C(A) = \bigoplus_{n=0}^{\infty} A^{\otimes n+1} \). This can be identified with the graded algebra

\[ T_A(A \otimes A) = A \times C[h], \quad h = \lambda 1. \]

where \( \lambda h = 1 \). \( b' \) is the superderivation of degree \(-1\) such that \( b'(a) = 0, \quad b'(h) = 1 \). Define \( d \) to be the superderivation of degree \(+1\) given by

\[ d(a) = [h, a], \quad d(h) = h^2. \]

Then we have

\[ b'^2 = [b', d] = d^2 = 0. \]

Proof:

\[ [b', d](a) = b' [h, a] = [1, a] - [h, 0] = 0. \]

\[ [b', d](h) = b'(h^2) + d(1) = 1 \cdot h - h \cdot 1 = 0. \]

\[ d^2(a) = d [h, a] = [h^2, a] - [h, [h, a]] = 0. \]

\[ d^2(h) = d (h^2) = (h^2) \cdot h - h (h^2) = 0. \]

Note: \( C(A) \) equipped with \( b' \) is the standard normalized resolution of the \( A \)-bimodule \( A \). \( d \) is the "Alexander-Spanier" differential:

\[ d(a_0, \ldots, a_n) = (1, a_0, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i (\ldots, a_{i-1}, 1, a_i, \ldots) \]

\[ + (-1)^{n+1} (a_0, \ldots, a_n, 1). \]

Why? \( (a_0, \ldots, a_n) = (a_0 h)(a_1 h) \ldots (a_{n-1} h) a_n \)

\[ d(a h) = [h, a] h + a h^2 = h(a h) \]

\[ d \{ (a_0 h) \ldots (a_{n-1} h) a_n \} = h(a_0 h) (a_1 h) \ldots (a_{n-1} h) a_n \]

\[ - (a_0 h) h(a_1 h) \ldots a_n \]

\[ \ldots + (-1)^n (a_0 h) \ldots (a_n h) (h a_n - a_n h) \]
\begin{align*}
&= 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \\
&\quad - a_0 \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \\
&\quad + \cdots \\
&\quad + (-1)^n a_0 \otimes \cdots \otimes a_{n-1} \otimes 1 \otimes a_n \\
&\quad + (-1)^n a_0 \otimes \cdots \otimes a_n \otimes 1.
\end{align*}

Recall that we defined $K$ on $C(A)$ by
\[ K = 1 - [b, s] = \lambda - sc \]
I claim that \[ K = 1 - [b, d] \]

Proof. Since $[b, d] = 0$ we only need to calculate that $[c, d] = 1 - \lambda + sc$. We will use the simplicial structure on $C(A)$:
\[ d_i (a_0, \cdots, a_n) = \begin{cases} 
(a_0, a_1, \cdots, a_{i+1} \cdots) & 0 \leq i < n \\
(a_n, a_0, a_1, \cdots, a_{n-1}) & i = n
\end{cases} \]
\[ s_i (a_0, \cdots, a_n) = (\cdots, a_i, 1, a_{i+1}, \cdots) \quad 0 \leq i \leq n \]

Note $c = (-1)^n d_n$ on $C(A)_n = A \otimes A^{n+1}$ and also $d = s - s_0 + \cdots + (-1)^{n-1} s_n$. Thus
\[ cd = \begin{array}{c}
\text{c}
\end{array} \begin{array}{c}
\text{s}
\end{array} - (-1)^n d_{n+1} + \sum_{i=0}^{n} (-1)^i s_i \\
\begin{array}{c}
\text{c}
\end{array} \begin{array}{c}
\text{s}
\end{array} - \begin{array}{c}
\text{s}
\end{array} \begin{array}{c}
\text{s}
\end{array} - (\sum_{i=0}^{n-1} (-1)^i s_i) (-1)^n d_n
\end{align*}

Thus $cd + dc = -\lambda + 1 + sc$. \qed
We have a canonical algebra homomorphism
\[ \Omega A = T_A(\mathbb{A}^n) \xrightarrow{j} T_A(A \otimes A) = C(A) \]
which is compatible with \( d \). Thus
\[ j(a_0 \otimes a_1 \ldots \otimes a_n) = a_0 \otimes [h_1 a_1] \ldots [h_n a_n] \]
Note that \( b'j = 0 \), so \( bj = cj \).

What is the cross-over term on \( C(A) \) in terms of
the description \( T_A(A \otimes A) \). One has
\[ A^{\otimes n+1} = T_A^n(A \otimes A) = T_A^{n-1}(A \otimes A) \otimes (A \otimes A) \]
so \( c : T_A^n(A \otimes A) \longrightarrow T_A^{n-1}(A \otimes A) \) is given
by
\[ c(j \otimes a) = (-1)^n a j \in T_A^n(A \otimes A) \]

Thus for \( w \in \Omega^n A \) we have
\[ cj(\omega da) = c \{ j(\omega) \cdot (\omega a - a \otimes 1) \} \]
\[ = c \{ j(\omega) \otimes a - j(\omega) a \otimes 1 \} \]
\[ = (-1)^n \{ aj(\omega) - j(\omega) a \} \]
\[ = (-1)^{n-1} \ j(\omega a - a \omega) \]
\[ = j b (\omega da) \]
Thus \( bj = cj = yb \). Summarizing we have
\[ [j, d] = [j, b] = [j, k] = 0 \]
Recall that we have a canonical exact sequence

\[ 0 \to D(A) \to C(A) \xrightarrow{p} \Omega A \to 0 \]

where \( D(A) \) in degree \( n \) is the degenerate subcomplex \( \sum_{i=0}^{n-1} s_i C(A)_{n-i} \). \( p \) is not an algebra homomorphism, but we have \([p, b] = [p, d] = [p, c] = 0\). The formula \([p, d] = 0\) follows from \( d = s - \sum (-1)^i s_i \).

\( j \) gives a splitting of this sequence compatible with \( b, d, c \). It's clear from the formula for \( j \) that its image is the standard complement for \( D(A) \) given by the intersection of the kernels of all face operators by the last. Note that these face operators are the maps \((A \otimes A)^n \to (A \otimes A)^{n-1} \otimes (A \otimes A)\) obtained by collapsing \( A \otimes A \) to \( A \) via multiplication.

The preceding seems not very useful for cyclic homology purposes, since when we pass to the generalized eigenspace for \( k \) and the eigenvalue \( 1 \) we obtain \( P \Omega A \) on which \( B \) is not exact.

Note that the fact that the \( d \) homology of \( C(A) \) occurs at the eigenvalue \( 1 \) gives

\[ H(C(A), d) = H(P \Omega A, d) = \mathbb{C}. \]
Note the reference Coquereux-Kastler Pacific J Math 137, in which is constructed a lifting of $\Omega A$ into $\tilde{\Omega} A$ compatible with multiplication but not $d$. I think this can be understood by observing that

$$\Omega'\tilde{A} = e\Omega'\tilde{A}e \oplus e\Omega'\tilde{A}e^+ \oplus e\Omega'\tilde{A}e^+ \oplus e\Omega'\tilde{A}e^+ \oplus \Omega'\tilde{A} \oplus A_0 \oplus \omega A \oplus 0$$

Here $A_0$ denotes the $\tilde{A}$-bimodule on which the left (resp. right) multiplication by $a \in A$ is usual (resp. zero); similarly for $\Omega A$. One has $A_0 = e\Omega'\tilde{A}e^+$, $a \mapsto a\tilde{c} \tilde{d} e$, etc.

Since $\Omega'\tilde{A}$ is a direct summand of $\Omega'\tilde{A}$ as $\tilde{A}$-bimodule the algebra

$$\Omega\tilde{A} = T_{\tilde{A}}(\Omega'\tilde{A})$$

should have $\Omega A = T_{\tilde{A}}(\Omega'\tilde{A}) = T_{\tilde{A}}(\Omega'\tilde{A})$ as subalgebra and quotient algebra, better: $\Omega\tilde{A}$ has $\Omega A$ as retract.

Now consider $\tilde{A}$ non unital and study $\Omega'\tilde{A}$.

Recall $\Omega'\tilde{A} = A \oplus \alpha$

$$\alpha \cdot da_1, da_1 \mapsto a\alpha a_1, \alpha$$

Define $u : \Omega'\tilde{A} \to A_0$ by

$$u(a_0 da_1) = a_0 a_1, \quad u(da_1) = a_1$$

Then $u$ is a bimodule map over $\tilde{A}$. Check:

$$u(a a_0 da_1) = a a_0 a_1 = a_0 u(a da_1),$$

$$u(da_1) = a \alpha a_1 = a_0 u(da_1)$$
\[ u(a_0 d a_1, a) = u(a_0 d(a, a) - a_0 a_1 da) \]
\[ = a_0 a_1 a - a_0 a_1 a \]
\[ = 0 = u(a_0 da_1) a \]
\[ u(d a, a) = u(d(a, a) - a, da) \]
\[ = a_1 a - a_1 a = 0 = u(da_1) a \]

Similarly, there is a bimodule map
\[ v: \Omega^\wedge \to \cot \]

Check:
\[ v(a_0 da_1) = 0 = a_0 v(da_1) \]
\[ v(a da_1) = 0 = a v(da_1) \]
\[ v(a_0 da_1, a) = v(a_0 d(a, a) - a_0 a_1 da) \]
\[ = 0 = v(a_0 da_1) a \]
\[ v(da_1, a) = v(d(a, a) - a_1 da_1) = a_1 a = v(da_1) a \]

We thus have exact sequences of \( \Omega^\wedge \) bimodules:

\[ 0 \to dA \to \Omega^\wedge \xrightarrow{u} A \xrightarrow{v} 0 \]

Note that \( u = v \) when \( A \) has 0 multiplication,
so \( dA = AdA : dA_0 a_1 = -a_0 da_1 \).

However, when \( A \) is unital, these exact
sequences split and are transverse. Why? Let $e$ be the identity of $A$

Then $l_u: a \mapsto ade = aede^+$ is a bimodule lifting $A(\omega) \to \tilde{A}$ for $u$, and similarly $l_v: a \mapsto dea = e^+dea$ is a bimodule lifting $\omega A \to \Omega \tilde{A}$ for $v$. Also $vl_u = 0$ and $u l_v(a) = u(dea) = u(da - e da) = a - ea = 0$. 
November 30, 1993

Remarks about \( K \) in \( C(A) \). First the relation \([c, d] = 1 - K\) \((p.260)\) can be proved using the formula
\[
c(\gamma h a) = (-1)^{1/2} a^\gamma
\]
There's no need to use the \( s_i \).

Next for
\[
0 \rightarrow D(A) \rightarrow C(A) \rightarrow \Omega A \rightarrow 0
\]
one has
\[
p(a_0, h a_1, \ldots, a_n, h a_n) = (a_0 \cdot d, a_1, \ldots, a_{n-1} \cdot d, a_n)\]
\[
= a_0 da_1 \cdots da_n
\]
p arises from the left action of \( C(A) \) on \( \Omega A \)
where \( a \in A \) acts by left multiplication by \( a \) on \( \Omega A \) and \( h \) acts by the operator \( d \) on \( \Omega A \).
Thus this sequence is a sequence of \( C(A) \)-modules
and \( D(A) \) is a left ideal in \( C(A) \). It seems to be the sum of the ideal \( C(A) h^2 C(A) \) and the left ideal \( C(A) h \).

In future: The Coquereaux-Kastler paper suggests looking also at \( \Omega A \).
Let \( M \) be a mixed complex which is homotopy equivalent to zero, i.e. \( \exists h \) of degree 1 such that \([b, h] = 1\), \([B, h] = 0\).

Then \([b, hbh] = 1\), \([B, hbh] = 0\) so we can assume \( h \) special. Then

\[
M = hM \oplus bM \quad \text{where} \quad hM \xrightarrow{h} bM
\]

are inverse, so we have

\[
M = (c \oplus cb) \otimes hM
\]

where \( hM \) is closed under \( b \). This shows a contractible mixed complex has the form \( M = (c \oplus cb) \otimes N \) where \( N \) is a complex with differential \( B \). Splitting \((N, B)\) into a contractible (i.e. \( B\)-acyclic) sub-complex and a minimal sub-complex (i.e. \( B = 0\)), we see \( M \) has Connes's property. Thus \( M \xrightarrow{h} 0 \Rightarrow M \) has Connes's property.

Another way to see this is to note that if a map \( f: M \to M' \) of mixed complexes is \( h \equiv 0 \), then the induced map \( H(M, B) \to H(M', B) \) of complexes with differential \( B \) is also \( h \equiv 0 \). Thus \( M \to H(H(M, B), B) \) is a functor on the homotopy category \( H \) of mixed complexes. Thus a mixed complex \( H \) of mixed complexes also has Connes's property.

Mixed complexes with Connes's property do not form a thick subcategory of \( H \). Example:

\[
\text{condition } Q' \cap \text{Im } B = 0 \quad \text{in the proof of Connes's lemma.}
\]
Note also that the homology $H_n(M/\text{ker}b, b)$ and $H_n(M, b+B)$ are functors in $Ho C^\otimes$, but not $H_n(M/\text{ker}b, B)$. Also $H_n(M/\text{ker}b, b)$ and $H_n(M, b+B)$ are not functors in the derived category $D C^\otimes$.

A better result is that mixed complexes having Connes's property are precisely those which are homotopy equivalent to a $B$-acyclic mixed complex.

Consequence is that a quasi isomorphism between mixed complexes with Connes property is a homotopy equivalence (because this is true for $B$-acyclic mixed complexes).

**Concerning $K$ on $C(A)$.** If you use the Alexander-Spanier differential $d$ instead of $s$, then $K$ commutes with $d$, and there is a canonical special contraction $(1-K)^{-1}d = d(1-K)^{-1}$ on the degenerate complex.
December 2, 1993

Cogneraux-Kastler: A unital algebra with identity element e. Consider the right ideal $e \tilde{\mathfrak{A}} \subseteq \tilde{\mathfrak{A}}$. This is spanned by $e a_0 a_1 \cdots a_n$ for all $n$ and $a_i \in \mathfrak{A}$. Thus $e \tilde{\mathfrak{A}} = C(\mathfrak{A})$ as graded vector spaces. The multiplication on $e \tilde{\mathfrak{A}}$ when transferred to $C(\mathfrak{A})$ is

$$
(a_0, \ldots, a_n) \ast (a_{n+1}, \ldots, a_k) = \sum_{j=0}^{n} (-1)^j \cdot \sum_{i} \left( a_0, \ldots, a_i, a_{n+1}, \ldots, a_k \right)
$$

(1)

$$
= (a_0, \ldots, a_n) \left( a_{n+1}, \ldots, a_k \right)
$$

$$
+ (-1)^n \cdot b'(a_0, \ldots, a_n) \left( e, a_{n+1}, \ldots, a_k \right)
$$

Recall the image of the homomorphism

$$
\psi: \tilde{\mathfrak{A}} \rightarrow C(\mathfrak{A})
$$

$$
a_0 a_1 \cdots a_n \rightarrow a_0 [a_1, a_2] \cdots [a_{n-1}, a_n]
$$

corresponds with the complement $\bigcap \ker d_i$ for $D(\mathfrak{A})$ in $C(\mathfrak{A})$. Thus $b' \psi = 0$. Therefore one has the CK observation that the products on $\psi(\tilde{\mathfrak{A}})$ coming from $\tilde{\mathfrak{A}}$ and from $C(\mathfrak{A})$ coincide.

But actually for this coincidence we only need $b'$ to vanish not the individual faces. Write (1) as

$$
\xi \ast \eta = \xi \eta + (-1)^{|\xi|} b' \xi \cdot s \eta.
$$

So $\xi \ast \eta = \xi \eta$ for all $\eta$ if $b' \xi = 0$. Conversely taking $\eta = e$ we have $\xi \ast e = \xi + (-1)^{|\xi|} b' \xi \cdot (e, e)$ and this implies $b' \xi = 0$; sticking a 1 at the right is injective as $c \lambda^5 = 1$.

So it seems that under the identification
\( e \Omega \alpha = C(\alpha) \) the subspace \( e \Omega \alpha \epsilon \) corresponds to the kernel of \( b' \), and the two products coincide on this subspace.

In future, describe \( e \Omega \alpha \epsilon \) better using \( \Omega' \alpha = \Omega' \alpha \oplus \alpha \oplus \alpha_0 \) and describe the homomorphism. Go back to problem of a lifting for \( \hat{\mathcal{R}} \alpha \rightarrow C \times \hat{\mathcal{R}} \alpha \).
December 17, 1993

Motivation: We know that \( P \Omega A \) is almost B-acyclic; it fails only because of \( C = P \Omega C \).

We have an exact sequence of mixed complexes

\[
0 \rightarrow C \rightarrow P \Omega A \rightarrow \tilde{P} \Omega A \rightarrow 0
\]

where \( \tilde{P} \Omega A \) is B-acyclic (i.e. free). This sequence splits compatibly with \( B \), yielding a map \( \tilde{P} \Omega A \rightarrow C[1] \) of mixed complexes such that \( P \Omega A \) is the \( h \)-fibre. We can lift this map as follows:

\[
\begin{array}{ccc}
\bigotimes & \longrightarrow & B(C)[1] \\
\downarrow & & \downarrow \\
\tilde{P} \Omega A & \longrightarrow & C[1]
\end{array}
\]

Since \( \tilde{P} \Omega A \) is free and the vertical map is a quasi-

Taking the \( h \)-fibre \( F \) of the lift we get

\[
0 \rightarrow \bigotimes B(C) \rightarrow F \rightarrow \tilde{P} \Omega A \rightarrow 0
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & C \longrightarrow P \Omega A \longrightarrow \tilde{P} \Omega A \longrightarrow 0
\end{array}
\]

where the vertical arrows are quasi. Then \( F \) is a minimal free cover of \( P \Omega A \), and perhaps we might locate it inside \( C(A) \) or \( \tilde{A} A \) or \( P \tilde{\Omega} A \).
Now we know that if we split the exact sequence of complexes
\[ 0 \to \mathcal{C}^\bullet[1] \to P\Omega \to \mathcal{C}^\bullet \to 0 \]
then we get an \( S \) operator on \( \mathcal{C}^\bullet \), and
further that \( P\Omega = \Lambda \tilde{\otimes} \mathcal{C}^\bullet \). The adjunction
property for \( \Lambda \tilde{\otimes} \) and \( B(-) \) reduce to lifting
in the diagram of \( S \)-modules
\[
\begin{array}{c}
\mathcal{C}^\bullet \longrightarrow B(C)[1] \\
\mathcal{C}^\bullet \longrightarrow B(C)[1] \downarrow \\
\end{array}
\]
where it's possible because of general properties of
adjoint functors: \( GFG \longleftrightarrow G \).
So there's a canonical way to obtain a
lifting \( \mathcal{C}^\bullet \) once we have an "explicit \( S \)-operator"
on \( \mathcal{C}^\bullet \).

Let's examine this abstractly. Suppose \( M \)
is a free mixed complex. Then \( B h \) of degree \(-1\)
in \( M \) such that \( [B, h] = 1 \). One has
\[
[b, [b, h]] = [b, b] = 0
\]
\[
[B, [b, h]] = -[b, [B, h]] = -[b, 1] = 0
\]
Thus \( [b, h] \) is a map of mixed complexes: \( M \to M[2] \).
In particular, it gives a map of complexes
\( M/\text{BM} \to M/\text{BM}[2] \).

Example. \( M = \Lambda \tilde{\otimes} Q = 1 \otimes Q + B \otimes Q \)
where \( (Q, d, S) \) is an \( S \)-module. Recall
\[ b(\boxplus x + \boxplus y) = 1 \boxtimes dx - \boxplus Sx - \boxplus dy \]
\[ \mathbf{B}(\quad) = \boxplus x. \]

Thus
\[ b = \begin{pmatrix} d & 0 \\ -s & -d \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and if } h = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

then \([\mathbf{B}, h] = 1\), and
\[ [b, h] = \begin{bmatrix} d & 0 \\ -s & -d \end{bmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -s & 0 \\ -s & -d \end{pmatrix} \]

This \( h \) is special, but let's return to a general contraction \( h \) for \( \mathbf{B} \). Exact sequence of \( \mathfrak{g}_5 \):

\[ 0 \longrightarrow \mathfrak{m}/\mathfrak{b} \longrightarrow M \longrightarrow M/\mathfrak{b} \mathfrak{m} \longrightarrow 0 \]

Now \( h \) determines a splitting of this sequence (ignoring \( b \)). Let \( l = hi \)
\[ r = jh \]

Then \( lj = hj = hB \]
\[ r = yh = Bh \]

Also \( lj = jhB = j(1 - Bh) = j \implies jl = 1 \)
as \( j \) is surjective. Similarly,
\[ ri = Bh i = (1 - hB)i = i \implies ri = 1 \]
as \( r \) is injective. Also \( irlj = Bh^2B = h^2B^2 = 0 \)
\( rl = 0 \). Note \( [B, h^2] = [B, h]h - h[B, h] = 1h - h1 = 0 \).

Thus we have the splitting determined by \( h \).
Also we have

\[ lr = h^i j h = h B h \]

is the special contraction associated to \( h \).

One has \( l' = (lr)i = l \) so we get \( r' = j(lr) = r \)

the same splitting from \( h' = h B h \).

Recall that the splitting \( l, r \) determines \( S \) on \( M/BM \) by

\[ -S = r [b, l] = [b, r] \cdot l \]

(These are equal as \( 0 = [b, rl] = [b, r]l - r [b, l] \) as \( r \) has degree \(-1\).)

\[ [b, h]i = [b, hi] = [b, l] = -i S \]

\[ j [b, h] = [b, jh] = [b, r] = -Sj \]

why?

\[ [b, r] = h^i(n + l_j) = [b, r]l_j \]

\[ j [b, h]l = -Sj \] means \([b, h] \) induces \(-S \) on \( M/BM \). Note that

\[ r [b, h]l = jh (bh + hb) hi = j (h b h^2 + h^2 b h) i \]

vanishes when \( h \) is special.

To summarize we find that when \([b, h] = 1\), then \([b, h] \) induces \(-S \) on \( M/BM \).
December 15, 1993

Small observation about motivating $B$. Starting with $d, b$ on $\Omega$ we get $K$ defined by $1 - K = [b, d]$ and we derive the formulas $K^n = 1 + K^n b d$, $K^{n+1} = -d b$ on $\Omega^n$. Then we get the polynomial relation, the spectral projector $P$ with $PS = \text{Ker} (1 - K)^2$. Then we have on $P\Omega$

\[
K^n = (1 + K^{-1})^n = 1 + n (K^{-1}) = 1 + b d \\
K^{n+1} = (K^n)^{n+1} = 1 + (n+1) (K^{-1}) = 1 - d b
\]

so $-b d = -\frac{1}{n+1} d b$ or $b ((n+1) d) + (n d) b = 0$.

Thus if we define $B = NPd$ we have $bb + B b = 0$ on both $P\Omega$ and $P^* \Omega$, so on $\Omega$. \qed
I propose to analyze standard bimodule resolutions of $A$. The standard normalized resolution is the DG algebra given by the graded algebra

$$A \times \mathbb{C}[\varepsilon] \quad \text{with} \quad 1a = 0, \quad 1\varepsilon = 1, \quad \varepsilon^2 = 0$$

equipped with diff $b'$, where

$$b'(a) = 0, \quad b'(\varepsilon) = 1$$

One has the 1-1 correspondence

$$(a_0, \ldots, a_{n+1}) \leftrightarrow a_0 \varepsilon a_1 \varepsilon \cdots a_n \varepsilon a_{n+1} = a_0 [\varepsilon a_1] \cdots [\varepsilon a_n] \varepsilon a_{n+1}$$

between $A \otimes A \otimes A$ and $(A \times \mathbb{C}[\varepsilon])_{n+1}$. Since

$$b'(a_0 [\varepsilon a_1] \cdots [\varepsilon a_n] \varepsilon a_{n+1}) = (-1)^n a_0 [\varepsilon a_1] \cdots [\varepsilon a_n] a_{n+1}$$

$$= (-1)^n a_0 [\varepsilon a_1] \cdots [\varepsilon a_{n-1}] (\varepsilon a_n a_{n+1} - a_n \varepsilon a_{n+1})$$

we have the correspondence

$$\Omega^n A \otimes A \cong (A \times \mathbb{C}[\varepsilon])_{n+1}$$

such that $b'$ on the left is my formula

$$b'(\omega a \otimes a') = (-1)^{1+1} \omega \otimes a a' - \omega \otimes a a'$$

$$= (-1)^{1+1} \omega \otimes a a' - \omega \otimes a a'$$

On $A \times \mathbb{C}[\varepsilon]$ we have the superderivation of degree +1 and square zero such that

$$[b', \mathrm{ad}(\varepsilon)] = 0$$
Thus $\text{Ker}(b')$ is a graded DG subalgebra of $A \times C[\varepsilon]$ equipped with differ ad(\varepsilon). It can be canonically identified with $\Omega A$. Thus we have subalgebras $C[\varepsilon], \Omega A$ of $A \times C[\varepsilon]$.

Claim one has linear isomorphisms

$$ C[\varepsilon] \otimes \Omega A \cong A \times C[\varepsilon]. $$

$$ \Omega A \otimes C[\varepsilon] \cong A \times C[\varepsilon] $$

given by multiplication in $A \times C[\varepsilon]$. The former follows from the fact that $b'$ and $l_\varepsilon$, left mult. by $\varepsilon$, satisfy the CAR $(b')^2 = (l_\varepsilon)^2 = 0$ $[b', l_\varepsilon] = \varepsilon 1$. The latter is similar with $r_\varepsilon (\alpha) = (-1)^{|\alpha|} \alpha \varepsilon \quad \alpha \in A \times C[\varepsilon]$

That $\Omega A = \text{Ker}(b')$ is a canonical subspace of $A \times C[\varepsilon]$, but we have two choices at least of complements $\varepsilon \Omega A$ and $\Omega A \varepsilon$.

We can also average, namely

$$ [b', \frac{l_\varepsilon + r_\varepsilon}{2}] = 1, \quad \left( \frac{l_\varepsilon + r_\varepsilon}{2} \right)^2 = 0 $$

Check:

$$ l_\varepsilon r_\varepsilon (\alpha) = (-1)^{|\alpha|} l_\varepsilon (\alpha \varepsilon) = (-1)^{|\alpha|} \varepsilon \alpha \varepsilon $$$$ r_\varepsilon l_\varepsilon (\alpha) = r_\varepsilon (\varepsilon \alpha) = (-1)^{|\alpha|+1} \varepsilon \alpha \varepsilon $$$$ b' r_\varepsilon \alpha = (-1)^{|\alpha|} b' (\alpha \varepsilon) = (-1)^{|\alpha|} (b' \alpha) \varepsilon + \alpha $$$$$ r_\varepsilon b' \alpha = (-1)^{|\alpha|+1} (b' \alpha) \varepsilon \quad \therefore [b', r_\varepsilon] = 1 $$

The significance of this is not clear.

Next consider applying $M \mapsto M \otimes_A M$ to the resolution $A \times C[\varepsilon]$. Begin with the
standard isomorphism

\((\Omega^n A \otimes A)_n \equiv \Omega^n A\)

\(b(a_0 da_1 \ldots da_n \otimes a_{n+1}) \mapsto a_{n+1} a_0 da_1 \ldots da_n\)

This gives the isomorphism

\(((A \times C[\varepsilon])_{n+1})_n \equiv \Omega^n A\)

\(b(a_0 [\varepsilon a_1] \ldots [\varepsilon a_n] \varepsilon a_{n+1}) \mapsto a_{n+1} a_0 da_1 \ldots da_n\)

Let's compute the composition

\[\Omega A \xrightarrow{\sim} \text{Ker}(b') \subset A \times C[\varepsilon] \xrightarrow{b} \Omega A [1] \oplus A\]

\(a_0 da_1 \ldots da_n \xrightarrow{b} a_0 [\varepsilon a_1] \ldots [\varepsilon a_n] = a_0 [\varepsilon a_1] \ldots [\varepsilon a_{n-1}] \varepsilon a_n - a_0 [\varepsilon a_1] \ldots [\varepsilon a_{n-1}] a_n \varepsilon\)

\(\xrightarrow{b} a_0 a_0 da_1 \ldots da_{n-1} - a_0 da_1 \ldots da_{n-1} a_n = (-1)^n b(a_0 da_1 \ldots da_n)\)

What's clear is that

\[\Omega A \xrightarrow{\sim} (\Omega A) \varepsilon \subset A \times C[\varepsilon] \xrightarrow{b} \Omega A [1] \oplus A\]

\(a_0 da_1 \ldots da_n \xrightarrow{b} a_0 [\varepsilon a_1] \ldots [\varepsilon a_n] = a_0 [\varepsilon a_1] \ldots [\varepsilon a_{n-1}] \varepsilon a_n + a_0 \varepsilon [\varepsilon a_1] \ldots [\varepsilon a_{n-1}][\varepsilon a_n] + (-1)^{n+1} b(d a_0 \ldots d a_n) + (-1)^n a_0 da_1 \ldots da_n\)

is the identity essentially. On the other hand

\[\Omega A \xrightarrow{\sim \varepsilon} \varepsilon (\Omega A) \subset A \times C[\varepsilon] \xrightarrow{\varepsilon b} \Omega A [1] \oplus A\]

\(a_0 da_1 \ldots da_n \xrightarrow{\varepsilon b} \varepsilon a_0 [\varepsilon a_1] \ldots [\varepsilon a_n] = \frac{(\varepsilon 1)^n}{\varepsilon [\varepsilon a_1][\varepsilon a_2] \ldots [\varepsilon a_{n-1}][\varepsilon a_n]} \varepsilon a_0 \varepsilon [\varepsilon a_1] \ldots [\varepsilon a_{n-1}][\varepsilon a_n] + (-1)^{n+1} b(d a_0 \ldots d a_n) + (-1)^n a_0 da_1 \ldots da_n\)
This map is \( 1 - bd = K + db \)

up to sign.

December 24, 1993

Using \( A \times C[\varepsilon] = \Omega A \oplus \Omega A \otimes \varepsilon \), let’s calculate \((A \times C[\varepsilon]) \otimes A\) and \((A \times C[\varepsilon]) / [-, -]\). As usual suppress \( \otimes \) signs. We have seen

\[
(A \times C[\varepsilon])_n \otimes A = \begin{cases} A & n = 0 \\ (\Omega^{n-1} A) \varepsilon & n > 0 \end{cases}
\]

but let’s check this directly. One has

\[
[a, \omega + \eta \varepsilon] = [a, \omega] + [a, \eta] \varepsilon - \eta da
\]

so

\[
[A, A \times C[\varepsilon]] = \left[ A, \Omega A \right] + \left\{ -\eta da + [a, \eta] \varepsilon \mid \eta \in \Omega A \right\}
\]

This should be complementary to \( \Omega \varepsilon \) in degree \( 26 \).

Suppose

\[
[a, \omega] \otimes -\eta da + [a, \eta] \varepsilon = \xi \varepsilon.
\]

Actually things looks nicer if we put \( a \) on the right:

\[
[\omega_1 + \eta \varepsilon, a_i] = [a_i, a_i] + \eta da_i + [\eta, a_i] \varepsilon
\]

Suppose this = \( \xi \varepsilon \) (use summation convention)

Then \( \xi = [\xi, a_i] \) and \( [\omega_1, a_i] + \eta da_i = 0 \).

Apply \( b \) to the latter and use \( b [\omega_1, a_i] = 0 \) as \( b^2 = 0 \).

Get \( [\eta, a_i] = 0 \) so \( \xi = 0 \). Thus

\[
[A \times C[\varepsilon], A] \cap (\Omega A) \varepsilon = 0
\]

On the other hand the sum of these subspaces
contains \((\Omega A)\varepsilon\) and all
\([\omega, a] + \gamma da\) with \(a \in A, \omega, \gamma \in \Omega A\).

Taking \(\omega = 0\) get \(\Omega^{>0}A\), so
\[[A \times C[\varepsilon], A] + (\Omega A)\varepsilon = ([A, A] + \Omega^{>0}A) \oplus (\Omega A)\varepsilon\]

Next \([\varepsilon, \omega + \eta \varepsilon] = d\omega + d\eta \varepsilon\). We want to calculate \((A \times C[\varepsilon]) \otimes A\) modulo the image of \(d\Omega A + (d\Omega A)\varepsilon\). Under the isomorphism \((A \times C[\varepsilon]) \otimes A \cong \Omega A \langle 1 \rangle \oplus A\eta\)

\((d\Omega A)\varepsilon\) goes into \(d\Sigma A\). Let's calculate what happens to \(d\omega da_0 \cdots da_n = d\omega_0 \cdots da_{n-1} (\varepsilon a_n, -a_n \varepsilon)^r\).

The image in \((A \times C[\varepsilon]) \otimes A\) is
\(b([\omega_0, \omega_0 da_0 \cdots da_{n-1}, -da_0 \cdots da_{n-1}, a_n] \varepsilon)\)

which corresponds to the following elt of \(\Omega A\)

\([- [da_0 \cdots da_{n-1}, a_n] = (-1)^{n+1} b d (a_0 da_0 \cdots da_n)\]

Thus
\[A \times C[\varepsilon]/[-A] + [-\varepsilon] = (d\Omega A/d\Omega A + bd\Omega A) \langle 1 \rangle \oplus A\eta\]

But \(\Omega A/d\Omega A + bd\Omega A = \Omega A/d\Omega A + (1-\varepsilon)\Omega A\) is reduced.

The cyclic complex except in degree zero where it is \(A\) instead of \(\Omega A\).
Let us next consider trinodele resolutions for \( \tilde{A} \) and \( \hat{A} \). There are four
\[
\begin{array}{cc}
\hat{A} \times C[h] & A \times C[h] \\
\downarrow & \downarrow \\
\tilde{A} \times C[\varepsilon] & A \times C[\varepsilon]
\end{array}
\]

The point is that \( \tilde{A} \times C[\varepsilon] \) and \( A \times C[h] \) are very close.

We are assuming that is unital so that
\[
\tilde{A} \simrightarrow C \times A \ ; \text{let} \ 1 \text{ denote the identity of} \ A.\]

Recall the GNS algebra and the result
\[
\Gamma(A \to RA \times C) = A \times \tilde{C}
\]

If we apply this when \( A', C \) are respectively \( C[\varepsilon], \tilde{A} \) we get
\[
\Gamma(C[\varepsilon] \to C[h] \times A) = C[\varepsilon] \times \tilde{A}
\]

Here \( \Gamma(C[\varepsilon]) = C[h] \) with \( h = \rho(\varepsilon) \). This means that standard normalized resolution D6 algebra for \( \tilde{A} \) is the GNS algebra for dilating \( h \) in the standard unnormalized resolution of \( A \) to a special contraction. In terms of D6 modules:

A D6 module over \( \tilde{A} \times C[\varepsilon] \) is a complex of \( \tilde{A} \) modules equipped with special contraction independent of the \( \tilde{A} \) module structure, \( E \) then splits \( eE \oplus e^*E \), where \( eE \) is a complex of \( A \) modules and \( A \) acts on \( e^*E \) by zero. Then \( \varepsilon \) is a dilation of \( C[\varepsilon] \) on \( eE \).
Recall that in the GNS algebra \( \Gamma = \Gamma(A' \to B) = A \oplus A \otimes B \otimes A \)
the element \( e = 1 \otimes (\otimes) \) is idempotent and
\( e \Gamma e = B \).
Thus from \( * \) we get
\[
e(\tilde{A} \times C[e]) e = A \times C[h]
\]
and this is easy to understand because in degree \( n \) we have
\[
e(\tilde{A} \times C[e])_n e = e(\tilde{A} \otimes A \otimes A)_n e = A \otimes A^{n+1}
\]
Thus we have homomorphisms
\[
\tilde{A} \times C[e] \overset{\text{nonunit}}{\subseteq} e(\tilde{A} \times C[e]) e \overset{\sim}{\longrightarrow} A \times C[h] \overset{\sim}{\longrightarrow} A \times C[e]
\]
Let's now explore Kadison's viewpoint. The idea is to work relative to the separable subalgebra
\( S = \tilde{C} \subset \tilde{A} \).
\[
\tilde{A} \otimes \tilde{A} \overset{\pi}{\longrightarrow} \tilde{A} \otimes \tilde{C} \overset{\pi}{\longrightarrow} \tilde{A} \times C[h]/\langle [e,h] \rangle
\]
\( \tilde{A} \times C[h] \) 
standard resolution
relative still resolution
Check. \( \tilde{A} \otimes \tilde{C} \tilde{A} \) is the quotient of the bimodule \( \tilde{A} \otimes \tilde{A} \) given by the relations \( s(1 \otimes 1) = (1 \otimes 1)s \) for \( s \in \tilde{C} \), i.e.
\( e h = h e \).
We get a lifting for this surjection \( \pi \) by sending \( h \in \tilde{A} \times C[h]/\langle [e, h] \rangle \) to the element \( ehe + e^4 he \) \( \in \tilde{A} \times C[h] \) which commutes with \( e \) and satisfies \( b'(ehe + e^4 he) = e^2 + (e^4)^2 = 1 \).

When we try to do something similar for the normalized resolutions \( \mathcal{C}[\varepsilon] \times (A \times C[\varepsilon]) \),

\[
\tilde{A} \times C[\varepsilon] \xrightarrow{\pi} \tilde{A} \times C[\varepsilon]/\langle [e, \varepsilon] \rangle
\]

it doesn’t work because \( (e\varepsilon e + e^4 \varepsilon e)^2 \neq 0 \).

In fact,

\[
(ehe)^2 = ehehe = eh[e, h] + he \quad (ehe)^2 = eh[e, h] + he
\]

\[
= eh[e, h] + eh^2 e
\]

\[
e h(1-e)[e, h] = e[h, 1-e][e, h] = e[h, e]^2
\]

\[
(ehe)^2 = eh^2 e + e[h, e]^2
\]

\[
(e\varepsilon e)^2 = e[e, e]^2
\]

Yet \( \pi \) should be a homotopy equivalence of \( \tilde{A} \)-bimodule resolutions of \( \tilde{A} \).

Critical case to examine: \( \tilde{A} = C \), whence

\[
(*) \quad \mathcal{C}[\varepsilon] \times \mathcal{C}[\varepsilon] \longrightarrow \mathcal{C}[\varepsilon] \otimes \mathcal{C}[\varepsilon]
\]
The problem is to construct a section of the surjection of normalized resolutions
\[ \tilde{A} \times C[\varepsilon] \longrightarrow \tilde{A} \times C[\varepsilon]/([\varepsilon, \varepsilon]) \]
We know how to do this on the level of the unnormalized resolutions, so we are led to examine liftings from the normalized to the unnormalized resolutions. We look at this for a general algebra $A$, not just $\tilde{A}$.
We want a bimodule section of
\[ A \times C[h] \longrightarrow A \times C[\varepsilon] \]
The simplicial normalization theorem gives us two canonical sections, namely
\[
\begin{align*}
& a_0, \varepsilon a_1, \ldots, \varepsilon a_n, a_{n+1} \\
& a_0, \varepsilon[a_1, \varepsilon], \ldots, \varepsilon[a_n, \varepsilon], a_{n+1} 
\end{align*}
\]
\[
\begin{align*}
& a_0 [h, a_1] - [h, a_n] h a_{n+1} \\
& a_0 [h, [a_1, h]] - [a_n, h] a_{n+1}
\end{align*}
\]
\[
\begin{align*}
& \text{killed by } d_0, \ldots, d_{n-1} \\
& \text{killed by } d_0, \ldots, d_n
\end{align*}
\]
The Alexander-Hamilton differential $d$ on $A \times C[h]$ (the super derivation of degree +1 defined by $d(a) = [h, a]$), $d(h) = h^2$) gives rise to a DG algebra lifting $\Delta A \longrightarrow A \times C[h]$
\[
\begin{align*}
& a_0 d_{a_1} - d_{a_n} \longrightarrow a_0 [h, a_1] - [h, a_n]
\end{align*}
\]
which makes $A \times C[\hbar]$ into a bimodule over $\Omega A$. I have to check that the two liftings given by the sum of maps coincide with this lifting on $\Omega A$.

To do this, and even more, define a left action of $A \times C[\hbar]$ on $A \times C[\hbar]$ by

$$a \cdot x = ax$$

$$\epsilon \cdot x = \hbar b'(x \hbar) = \hbar x - \hbar^2 b'(x)$$

Check that $\epsilon \cdot (\epsilon \cdot x) = 0$.

For $\beta \in \text{Ker}(b') \subset A \times C[\hbar]$, we have

$$\epsilon \cdot (a \cdot \beta) - a \cdot (\epsilon \cdot \beta) = \hbar a \beta - ah \beta = [\hbar, a] \beta$$

Thus

$$\left(a_0 \epsilon [a_1 \epsilon] \ldots [a_n \epsilon] a_{n+1}\right) \beta = a_0 \hbar [a_1, \hbar] \ldots [a_n, \hbar] a_{n+1} \beta$$

which means that acting on $1$ yields the second lifting to $\bigcap_i \text{Ker} a_i$.

On the other hand, let us define a right action of $A \times C[\hbar]$ on $A \times C[\hbar]$ by

$$a \cdot \alpha = ax$$

$$\alpha \cdot \epsilon = (-1)^{ax} b'((x \hbar) \hbar) = \alpha \hbar + (-1)^{ax} b'(x) \hbar^2$$

Check: $(\alpha \cdot \epsilon) \cdot \epsilon = (-1)^{\alpha \hbar + 1} b'((x \hbar) \hbar) = 0$.

Again if $b'(\beta) = 0$ then

$$(\beta \cdot \epsilon) \cdot a - (\beta \cdot a) \cdot \epsilon = \beta \hbar a - \beta a \hbar = \beta [\hbar, a]$$
so that
\[ \beta : (a_0 [e, a_1 \ldots [e, a_n] e_{n+1}]) = \beta a_0 \cdot [h, a_1 \ldots [h, a_n] h_{n+1} \]
which means that acting on 1 gives the first lifting to \( \mathfrak{H} \) Kerd \( i \). last

Now let's return to
\[
\tilde{A} \times \mathbb{C}[h] \xrightarrow{\mu} \tilde{A} \times \mathbb{C}[h] / \langle [h, e] \rangle
\]
\[
\downarrow
\]
\[
\tilde{A} \times \mathbb{C}[e] \xrightarrow{\lambda} \tilde{A} \times \mathbb{C}[e] / \langle [e, e] \rangle
\]

\( \mu \) is the DG alg homom. such that \( \mu(a) = a \)
\( \mu(h) = h^4 = e^4h^{-1}e + e^{-4}h e^{-1} \). Note \( [e, h^4] = 0 \) and \( h^4 \rightarrow h \) under the surjection \( \rightarrow \). \( \mu \) commutes with \( b' \) since \( b'(h^4) = e^2 + (e^4)^2 = 1 \). Recall we have defined \( \lambda : e \) on \( \tilde{A} \times \mathbb{C}[h] / \langle [h, e] \rangle \) by
\[
\lambda \cdot e = (-1)^{|h|} b'(h) h
\]

Thus \( \mu(u \cdot e) = (-1)^{|h|} b'(u(h)) h^4 \), so we get a right action of \( \tilde{A} \times \mathbb{C}[e] / \langle [e, e] \rangle \) on \( \tilde{A} \times \mathbb{C}[h] \) by defining
\[
\lambda \cdot e = (-1)^{|h|} b'(h) h^4
\]

Finally we define an action of \( \tilde{A} \times \mathbb{C}[e] / \langle [e, e] \rangle \)
on \( \tilde{A} \times \mathbb{C}[e] \) by
\[
\lambda \cdot e = (-1)^{|h|} b'(e^4) \cdot e^4
\]
\( e^4 = eee + e^4 e^4 \)
Check well-defined

\[(\alpha e)_0 e = (-1)^{|x_1|} b' \cdot (\alpha e \cdot \frac{e_1}{e_1})^2 = (-1)^{|x_1|} b' \cdot (\alpha e \cdot \frac{e_1}{e_1})^2 e_1 e \]

\[= (\alpha e_0) e \]

\[(\alpha e_0) (e_0) = (-1)^{|x_1| + 1} b' \cdot (\alpha e_0 \cdot \frac{e_1}{e_1})^2 e_1^2 \]

\[= (-1)^{|x_1| + 1} b' \cdot (\alpha e_0 \cdot \frac{e_1}{e_1})^2 e_1^2 = 0 \]

both killed by \(b'\).

Thus we have the section

\[\tilde{A} \times C[\varepsilon] \rightarrow \tilde{A} \times C[\varepsilon] \rightarrow \tilde{A} \times C[\varepsilon]/(\varepsilon, e I)\]

\[a_0 [\varepsilon_{q_1}, \ldots, [\varepsilon_{q_n}, \ldots, \varepsilon_{q_{n+1}]} \]

This is a bimodule map compatible with \(b'\).
Recall that yesterday we described the two canonical bimodule liftings (given by the simp. norm. thm) for the map of std resolutions

\[ A \times \mathbb{C}[h] \longrightarrow A \times \mathbb{C}[\varepsilon] \]

using a left & right action of \( A \times \mathbb{C}[\varepsilon] \) on \( A \times \mathbb{C}[h] \). The right action is given by

\[ \alpha \cdot a = \alpha a \]

\[ \alpha \cdot \varepsilon = (-1)^{|\alpha|} h'(\alpha h) h = \alpha h + (-1)^{|\alpha|} b'(\alpha) h^2 \]

and yields the lifting

\[ l(a_0 \varepsilon a_1 \varepsilon \cdots a_n \varepsilon a_{n+1}) = a_0 [h, a_1] \cdots [h, a_n] h a_{n+1} \]

We used this to get a lifting for the Kashiwara maps for normalized resolutions \( \otimes \)

Here \( e^h = eee + e^h ee^h \), and the lifting for \( \otimes \) is \( l \).
\[
\begin{align*}
ul (a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1}) \\
= u (a_0 [h, a_1] \cdots [h, a_n] h a_{n+1}) \\
= a_0 [\varepsilon h, a_1] \cdots [\varepsilon h, a_n] \varepsilon h a_{n+1}
\end{align*}
\]

Now restrict to sub-bimodules on which \( \varepsilon \) is the identity.

Here

\[
\begin{align*}
A \times \mathbb{C}[h] \xrightarrow{\varepsilon} A \times \mathbb{C}[\varepsilon] \\
u a_0 h \cdots h a_{n+1} = a_0 \varepsilon \cdots \varepsilon a_{n+1}
\end{align*}
\]

is an isomorphism.

\[\text{We get the lifting}\]

\[
\begin{align*}
ul (a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon a_{n+1}) \\
= u (a_0 [h, a_1] \cdots [h, a_n] h a_{n+1}) \\
= a_0 [\varepsilon \varepsilon \varepsilon, a_1] \cdots [\varepsilon \varepsilon \varepsilon, a_n] \varepsilon \varepsilon \varepsilon a_{n+1} \\
= a_0 [\varepsilon a_1] \varepsilon [\varepsilon a_2] \varepsilon \cdots \varepsilon [\varepsilon a_n] \varepsilon \varepsilon a_{n+1}
\end{align*}
\]

Next consider what happens on commutator quotient spaces.

\[
\begin{align*}
(\tilde{\mathcal{A}} \times \mathbb{C}[\varepsilon]) \otimes_{\mathcal{A}} \Omega \tilde{\mathcal{A}} \cong (\tilde{\mathcal{A}} \times \mathbb{C}[\varepsilon]/(\varepsilon, \varepsilon)) \otimes_{\mathcal{A}} \Omega \tilde{\mathcal{A}}
\end{align*}
\]

Start with \( a_0 da_1 \cdots da_n \in \Omega^n \mathcal{A} \), go to

\[\begin{align*}
b (a_0 [\varepsilon, a_1] \cdots [\varepsilon, a_n] \varepsilon) \in (\tilde{\mathcal{A}} \times \mathbb{C}[\varepsilon]/(\varepsilon, \varepsilon)) \otimes_{\mathcal{A}} \Omega \tilde{\mathcal{A}}
\end{align*}\]
Apply $u \circ l$ to get

$$b \left( \{a_0 \leq_{e \leq} \ldots \leq_{e \leq} e \} \in (\tilde{A} \times C[e]) \otimes \tilde{A} \right)$$

is

$$a_0 da_1 e \ldots e da_n e \in \Omega \tilde{A}$$

Put another way, we have

$$b \left( a_0 \leq_{k \leq} \ldots \leq_{k \leq} k \right)$$

$$(A \times C[k]) \otimes A = C(A)$$

$$\downarrow$$

$$e(\tilde{A} \times C[e]) \otimes A \rightarrow (\tilde{A} \times C[e]) \otimes A = \Omega \tilde{A}$$

I guess I know that

$$e(\Omega \tilde{A}) \otimes A$$

are algebra homomorphisms, so that it should be possible to transport the Alexander Spanier differential to $e(\Omega \tilde{A}) e$. No, $u$ isn't, but

$$e(\Omega \tilde{A}) e \sim b \circ C(A)$$

should be an isomorphism.
Key diagram of $A$-bimodule resolutions

$$\begin{array}{c}
A \times \mathbb{C}[\hbar] \\
p \\
\downarrow \\
A \times \mathbb{C}[\varepsilon] \end{array}$$

$$u(h) = eee$$

$$e(A \times \mathbb{C}[\varepsilon]) e \twoheadrightarrow A \times \mathbb{C}[\varepsilon]$$

$u$ and the two surjections are DG algebra maps.

$$u(a_0 \cdot h a_1 \cdots a_n \cdot h a_{n+1}) = a_0 [\varepsilon, q_1] \cdots [\varepsilon, q_n] e a_{n+1}$$

Beware, in using this formula, not to write $1$ for the identity of $A$. Thus

$$u(h^2) = u(e e h e h e) = e e e e e e = e [\varepsilon, e] e e$$

$$= [\varepsilon, e] (1 - e) e e = [\varepsilon, e] (1 - e) [\varepsilon, e]$$

$$= e [\varepsilon, e]^2$$

$$l (a_0 [\varepsilon, q_1] \cdots [\varepsilon, q_n] e a_{n+1}) = a_0 [h, q_1] \cdots [h, q_n] h a_{n+1}$$

$$u l (a_0 [\varepsilon, q_1] \cdots [\varepsilon, q_n] e a_{n+1}) = a_0 [e e e, q_1] \cdots [e e e, q_n] e e e a_{n+1}$$

$$= a_0 [\varepsilon, q_1] e \cdots [\varepsilon, q_n] e e a_{n+1}$$

$l$ is not compatible with multiplication, but arises from the right action of $A \times \mathbb{C}[\varepsilon]$ on $A \times \mathbb{C}[\hbar]$ defined by

$$\alpha \cdot a = \alpha a$$

$$\alpha \cdot \varepsilon = (-1)^{[\alpha]} b'(\alpha h) h = \alpha h + (-1)^{[\alpha]} b'(\alpha) h^2$$
When we apply $\otimes_A$ and make standard identification we get the diagram

\[
\begin{array}{c}
\xymatrix{
\tilde{A} \times C[h] \ar[rd]_u \ar[dd]_p \ar@/^1pc/[rr]^l & \\
\tilde{A} \ar[r]^e & \Omega \tilde{A} \\
\end{array}
\]

\[u(a_0h\cdots h a_n) = a_0 da_1 \cdots da_n \in e \Omega \tilde{A}\]

\[p(\quad) = \quad \in \Omega A\]

\[l(a_0 da_1\cdots da_n) = a_0 [h, a_1] \cdots [h, a_n]\]

\[ul(a_0 da_1\cdots da_n) = a_0 da_1 e \cdots da_n e\]

\[u\] is not a homomorphism, but an isomorphism of algebras $\tilde{A} \times C[h] \cong e(\Omega \tilde{A})e$. In particular $ul$ is a multiplicative lifting of $\Omega A$ into $e \Omega \tilde{A} \subset \Omega \tilde{A}$; this is a result of Coquereaux-Kastler.

Further work: Analyze

\[a_0 da_1 \cdots da_n = a_0 da_1 (e^t) \cdots da_n (e^t)\]

using $e^t da e^{-t} = 0$.

\[da_1 e^t da_2 = a_1 da_2 e^t\]

This is related to $\Omega' \tilde{A} = e \Omega' \tilde{A} e \oplus e^t \Omega' \tilde{A} e \oplus \Omega' \tilde{A} e^{-t}$ is $\Omega' \tilde{A}$.

\[\Omega' \tilde{A} = T_A(\Omega' \tilde{A}),\]

and $\Omega \tilde{A} = T_A(\Omega' \tilde{A})$. 
There's also the GNS result

\[ \Gamma(C[\varepsilon] \to A \ast C[h]) = \tilde{A} \ast C[\varepsilon] \]

and its interpretation as using complexes of \( A \) modules equipped with contraction independent of the \( A \) module structure.
December 30, 1993

$M$ a $B$-acyclic mixed complex, $h$ an operator of degree $-1$ such that $[B, h] = 1$. Then $[b, h]$ has degree $-1$ and it commutes with both $b$ and $B$, hence we obtain an $S$-module structure on $(M/BM, b)$ with $S = -[b, h]$.

Define

$u : \Lambda \otimes (M/BM) \rightarrow M$

by

$u(1 \otimes \bar{m}) = hBm$

$u(B \otimes \bar{m}) = Bm$

This is well-defined, compatible with $B$ as $BhB = B(1-Bh) = B$, compatible with $d = 1 \otimes b - S \otimes B$ on the left and $b$ on the right:

$bu(1 \otimes \bar{m}) = bhBm = [b, h]Bm - hhBm$

$= hB(bm) - B([-b, h])m$

$= u(1 \otimes bm - B \otimes Sm) = ud(1 \otimes \bar{m})$

$bu(B \otimes \bar{m}) = BBm = Bbm = u(-B \otimes bm)$

$= ud(B \otimes \bar{m})$.

and bijective, the inverse being

$v(m) = 1 \otimes \bar{m} + B \otimes \bar{hm}$

Check $uv(m) = u(1 \otimes \bar{m} + B \otimes \bar{hm}) = hBm + Bhm = m$

$vu(1 \otimes \bar{m}) = v(hBm) = 1 \otimes \bar{BM} + B \otimes \bar{h^2Bm}$

$= 1 \otimes \bar{m} - B \otimes \bar{hm} + B \otimes \bar{Bhm} = 1 \otimes \bar{m}$

$vu(B \otimes \bar{m}) = v(Bm) = 1 \otimes \bar{Bm} + B \otimes \bar{hBm} = B \otimes \bar{m} - B \otimes \bar{hm} = B \otimes \bar{hm}$.