On mixed complexes.
The problem is to find a suitable notation for dealing with the cyclic theory of a mixed complex. What is this?

Consider a mixed complex \((M, b, B)\) where
\[ M = \bigoplus_{n \in \mathbb{Z}} M_n, \quad M_n = 0 \text{ for } n < 0, \quad b \text{ has degree } -1, \quad B \text{ has degree } +1. \]

We think of it as the chain complex \(M\) with differential \(b\), on which one is given an endomorphism \(B\) of degree +1 compatible with the differential and having square 0.
\[ b \text{ is the primary differential.} \]

Define a quasi-isomorphism of mixed complexes to be a map \(M \to M'\) compatible with \(b, B\)
which is a quasi of the underlying complexes:
\[ H(M, b) \cong H(M', b) \]
The cyclic theory attached to \(M\) should consist of all homology constructs constructed from \(M\) which is quasi-isomorphism invariant.

Recall the Husemoller-Kassel idea. A mixed complex is a DG module, \(\otimes\) zero in negative degrees, over the DG algebra \(\mathbb{C}[\varepsilon]\)
where \(\deg(\varepsilon) = 1\) and \(d(\varepsilon) = 0\). So one is looking at the derived category of these DG modules in some sense.
It is natural to consider the bar construction of $C[E]$ which is the coalgebra
\[ C[\eta] \cong \bigoplus_{n \geq 0} C[\eta]^n \]
where $\eta$ has degree 2 and
\[ d(\eta^n) = 0, \quad \Delta(\eta^n) = \sum_{i+j=n} \eta^i \otimes \eta^j. \]

A DG comodule $X$ over $C[\eta]$ can be identified with a complex $X$ having an endomorphism $S$ of degree $-2$. Let's consider right comodules; the comultiplication is a map
\[ X \xrightarrow{\Delta} X \otimes C[\eta] \]
and because $C[\eta]$ is a tensor coalgebra the coproduct is equivalent to a linear map
\[ X \longrightarrow X \otimes C[\eta] \cong \sum^2 X \]
which is just $S$.

Now the point is that the derived category of DG modules over an augmented algebra $A = C \oplus \overline{A}$ (zero in negative degrees) is equivalent to the derived category of right DG comodules over $B(A)$. This equivalence is given by the acyclic complex
\[ B(A) \otimes \overline{A} \]
where $\tau$ is the twisting cochain. In our example, this is
\[ C[\eta] \otimes C[\eta] \]

The differential is
in general, where \( \tau : C \to A \) is a twisting cochain. In our case,

\[
\tau(\eta^n) = \begin{cases} 
\circ & n \neq 1 \\
\varepsilon & n = 1
\end{cases}
\]

So we find

\[
d(\eta^n \otimes 1) = (\otimes m)(\otimes 1)\left(\sum_{m-j} \eta^j \otimes 1\right)
\]

\[
= (\otimes m)(\eta^{n-1} \otimes 3 \otimes 1) = \eta^{n-1} \otimes 3
\]

Given a mixed complex \( M \) the corresponding complex with \( S \) is

\[
C[\eta] \otimes M = \bigoplus_{p>0} \eta^p M
\]

with differential

\[
d(\eta^p m) = \eta^p (\otimes m) + \eta^{p-1} B m
\]

Thus the differential is

\[
d = b \bigcirc + S B
\]

On the other hand, given \( X \) with \( S \), the corresponding mixed complex is

\[
X \otimes C[\eta] \otimes C[\xi] = X \otimes C[\xi]
\]

where \( B \) is multiplication by \( \varepsilon \). What is \( b \)?

We have

\[
X \otimes C[\xi] \xrightarrow{\Delta} X \otimes C[\eta] \otimes C[\xi]
\]

where the differential \( \Delta \) in the latter is \( \delta \otimes 1 + 1 \otimes d \).
We use

\[ X \otimes C[e] \xrightarrow{\Delta \otimes 1} X \otimes C[y] \otimes C[e] \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \text{diffe d} \]

\[ X \otimes C[e] \leftarrow \quad \quad X \otimes C[y] \otimes C[e] \]

\[ \Delta x = x \otimes 1 + Sx \otimes y + \ldots \]

\[ x \otimes 1 \xrightarrow{\Delta \otimes 1} (x \otimes 1 + Sx \otimes y + \ldots) \otimes 1 \]

\[ \xrightarrow{\text{projection}} \quad dx \otimes 1 \otimes 1 + dSx \otimes y \otimes 1 \xrightarrow{\text{proj}} \quad dx \otimes 1 + Sx \otimes e \]

So we get

\[ b(x \otimes 1) = dx \otimes 1 + Sx \otimes e \]

In other words, we get the complex

\[ X_2 \leftarrow X_3 \]

\[ X_2 \xleftarrow{S} X_2 \]

\[ X_0 \xleftarrow{S} X_2 \]

\[ 0 \xleftarrow{} X_1 \]

\[ 0 \xleftarrow{} X_0 \]
Nice cases: If \( B \) is exact on \( M \), then we have a quasi
\[
C[\eta] \otimes M \longrightarrow M/\text{BM}
\]

in pictures:

\[
\begin{array}{cccc}
\uparrow & \uparrow & \uparrow & \uparrow \\
M_2/\text{BM} & M_2 & M_1 & M_0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
M_1/\text{BM} & M_1 & M_0 & M_0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
M_0 & M_0 & M_0 & M_0
\end{array}
\]

To up to homotopy we have a well-defined endomorphism \( S \) on \( M/\text{BM} \) of degree \(-2\).

If \( S \) is surjective on \( X \), then we have a quasi
\[
\text{Ker}(S) \longrightarrow X \otimes C[\eta]
\]

whence a \( B \) map on \( \text{Ker}(S) \) defined up to homotopy.
On \( HC^- \). The problem is to understand \( HC^- \) from the \( \mathbb{Z}/2 \)-graded complex viewpoints. Recall the super notation

\[
(CM)_n = \prod_{p \geq 0} M_{n-2p}
\]

\[
(\hat{C}^{pew}M)_n = \prod_{p \geq 0} M_{n-2p}
\]

\[
(\hat{C}^{-}M)_n = \prod_{p \leq 0} M_{n-2p}
\]

These are all complexes with shift \( b+B \), and we have exact sequences

\[
0 \to \hat{C}^{-}M \to \hat{C}^{pew}(M) \to \Sigma^2 CM \to 0
\]

\[
0 \to \Sigma^2 \hat{C}^{-}M \to \hat{C}^{-}M \to (M, b) \to 0
\]

The second gives a Benson exact sequence

\[
\to H^c_{n+2} M \to H^c_n M \to H^b_n M \to H^c_{n+1} M \to
\]

and the first gives an exact sequence

\[
\to H^c_{n+2} M \to H^p_{n+2} M \to H^c_n M
\]

\[
\to H^c_{n+1} M \to H^p_{n+1} M \to H^c_n M
\]

\[
\to H^c_n M
\]
I feel that \( \{ H_{n}^{c-M} \} \) should be calculable from \( \{ H_{i}(F^n\hat{M}) \} \quad i \in \mathbb{Z}/2 \).

We have an exact sequence

\[
0 \to F^{n+1}\hat{M} \to F^n\hat{M} \to F^n\hat{M}/F^{n+1}\hat{M} \to 0
\]

which gives a 6 term exact sequence

\[
\begin{align*}
&H_{n+1}^{c-M} \\
&\xrightarrow{b} H_{n+1}^{c-M} \xrightarrow{\partial} H_{n+1}^{d-M} \\
&H_{n+1}^{d-M} \xrightarrow{\partial} H_{n+1}^{c-M} \xrightarrow{b} H_{n+1}^{c-M} \\
&H_{n+1}^{c-M} \xrightarrow{\partial} H_{n+1}^{d-M} \xrightarrow{b} H_{n+1}^{c-M}
\end{align*}
\]

One can therefore hope that

\[
\begin{align*}
H_{n+1+2\mathbb{Z}}^{c-M} &= H_{n+1}^{c-M} \\
H_{n+1+2\mathbb{Z}}^{d-M} &= H_{n+1}^{d-M}
\end{align*}
\]

(Observe - to get \( H_{n}^{c-M} \) you take \( H_{n+2\mathbb{Z}}^{c-M} \).)

To keep things straight, think of \( H_{n}^{c-M} \) as calculated from the \( p \leq 0 \) part of the bicomplex. Thus there is an canonical map

\[
H_{n}^{c-M} \to H_{n}
\]

Proof of \( \ast \).

We have \( H_{n}^{c-M} = \) \( 0 \)

\[
Z_{n}(\hat{c}-M)/B_{n}(\hat{c}-M)
\]

where

\[
Z_{n}(\hat{c}-M) = \{ (x_{n}, x_{n+2}, \ldots) \mid bx_{n} = Bx_{n} + bx_{n+2} = \ldots = 0 \}
\]

\[
B_{n}(\hat{c}-M) = \{ (by_{n+1}, by_{n+1} + by_{n+3}, \ldots) \}
\]
\[(\mathbb{F}^m)_{n+1,\mathbb{Z}} = \left\{ (x_n, x_{n+2}, \ldots) \mid x_n \in bM_{n+1} \right\} \]

\[(\mathbb{F}^n)_{n+1,\mathbb{Z}} = \left\{ (y_{n+1}, y_{n+3}, \ldots) \right\} \]

\[Z_{n+1,\mathbb{Z}}(\mathbb{F}^n) = \left\{ (y_{n+1}, y_{n+3}, \ldots) \mid b y_{n+1} = b y_{n+1} + b y_{n+3} = \ldots = 0 \right\} = Z_{n+1}(\mathbb{E}^{-M}) \]

\[B_{n+1,\mathbb{Z}}(\mathbb{F}^n) = \left\{ (B x_n + b x_{n+2}, \ldots) \mid x_n \in bM_{n+1} \right\} \]

\[= \left\{ \left( \frac{B b y_{n+1} + b x_{n+2}}{b(\mathbb{E}^{-B y_{n+1}} + x_{n+2})} \right)^2 \right\} \]

\[= B_{n+1}(\mathbb{E}^{-M}). \quad \text{Thus we have} \quad H_{n+1,\mathbb{Z}}(\mathbb{F}^n) = H_{n+1}^{c^{-M}}. \]

Next,

\[Z_{n+1,\mathbb{Z}}(\mathbb{F}^n) = \left\{ (x_n, x_{n+2}, \ldots) \mid x_n \in bM_{n+1}, b y_n + b x_{n+2} = b y_{n+2} + b x_{n+4} = \ldots = 0 \right\} \]

\[= \ker \left\{ Z_n(\mathbb{E}^{-M}) \longrightarrow H_n^{b M} \right\} \]

\[\left( x_n, x_{n+2}, \ldots \right) \rightarrow [x_n] \quad b y_n = b x_n + b x_{n+2} = \ldots = 0 \]

\[B_{n+1,\mathbb{Z}}(\mathbb{F}^n) = \left\{ (b y_{n+1}, b y_{n+1} + b y_{n+3}, \ldots) \right\} = B_n(\mathbb{E}^{-M}) \]

Thus,

\[H_{n+1,\mathbb{Z}}(\mathbb{F}^n) = \ker \left\{ H_n^{c^{-M}} \longrightarrow H_n^{b M} \right\} \]
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Differential operators.

Let's recall the program for universal enveloping algebras and algebras of differential operators.

Generalized enveloping algebras can be described the following ways:

1) An algebra $A$ with increasing filtration $F^n A$ such that $0 = F^{-1} A < \cdots < U F^n A = A$, $F^p A : F^q A < F^{p+q} A$ and such that $\text{gr} A$ is the symmetric algebra $\text{gr} A$.

2) An extension of Lie algebras

\[ 0 \to C \to \tilde{g} \to g \to 0 \tag{*} \]

3) An affine space equipped with a Poisson manifold structure such that the Poisson bracket of (affine) linear functions is a linear function. (Technically this description requires $g = F^1 A / F^0 A$ to be finite dimensional.)

The affine space in 3) is the splitting of $(*)$, as vector spaces. Then $\tilde{g} = F_1 A$ appears as the linear (degree 1) might be better terminology) functions on the affine space. The differential forms with polynomial coefficients on this affine space:

\[ S^* (\tilde{g}) \left( \frac{S^* (\tilde{g})}{s^1 \tilde{g}} \right) \otimes \Lambda g \]

form a mixed complex with $d$ as usual and $b$ from the Poisson structure. The basic result to understand properly is the fact that
the cyclic theory of $A$ is given by this mixed complex.

One presumably starts with the basic $A$-bimodule resolution of $A$:

$$
\rightarrow A \otimes \Lambda^2 \mathfrak{g} \otimes A \rightarrow A \otimes \mathfrak{g} \otimes A \rightarrow A \otimes A \rightarrow A \rightarrow 0
$$

which is a Koszul type resolution, except there are some subtleties such as the bracket in $\mathfrak{g}$ and the fact that $\mathfrak{g}$ acts by left + right multiplication on $A$. Applying $\otimes A$ gives the complex

$$
\rightarrow A \otimes \Lambda^2 \mathfrak{g} \rightarrow A \otimes \mathfrak{g} \rightarrow A
$$

computing the Lie algebra homology $H_*(\mathfrak{g}, A)$. We have to identify $\mathfrak{g}$ (I think) with $(\mathfrak{g} \otimes \Lambda \mathfrak{g}, b)$, which means we bring in a PBW type linear isomorphism $\mathfrak{g} \otimes \mathfrak{g} \cong A$.

After we understand the Hochschild homology, we still have to worry about the cyclic theory, i.e. how to link $B$ to the $d$ in the above de Rham complex.

Let's consider differential operators. $D = \text{differential operators on } M$, $\mathcal{O} = \text{functions}$, $\mathcal{F} = \Gamma(M, TM) = \text{vector fields}$. Consider the increasing filtration $F_i \mathcal{D} = \text{operators of order } \leq i$, exact sequence of Lie algebras

$$
0 \rightarrow \mathcal{O} \rightarrow F_1 \mathcal{D} \rightarrow \mathcal{F} \rightarrow 0
$$

which splits canonically since $\mathcal{F} \subset F_1 \mathcal{D}$ as the
subspace of operators which kill $1 \in O$. However, these Lie algebras have $O$ bimodule structures to be taken into account, so it seems not a good idea to think of $F_{O[D]}$ as $O \otimes T$. Anyway the goal (I think) is to show that the cyclic theory of $O$ is given by the mixed complex of polynomial coefficient differential forms on $T^*M$, where the $b$ operator comes from the symplectic structure on $T^*M$.

Presumably one starts with calculating the Hochschild homology. It seems that we have a bimodule resolution

$$O \otimes T \otimes O \longrightarrow O \otimes O \otimes O \longrightarrow O \otimes O \longrightarrow O \longrightarrow 0$$

of Koszul type. Given $v \in T$ we assign to $v$ the operator $x \otimes y \mapsto x_0 \otimes y - x \otimes 0 y$ on $O \otimes O$. Check well-defined $x [f, v] + x_0 f$

$$xf \otimes y \mapsto xf \otimes y - xf \otimes vy$$

$$x \otimes fy \mapsto x_0 \otimes fy - x \otimes 0 fy$$

$$[vf]y + fvy$$

$$xf \otimes y \mapsto x[f, v] \otimes y + xvf \otimes y - xf \otimes vy$$

$$-x \otimes fy \mapsto -x_0 \otimes fy + x \otimes [v, f] y - x \otimes fy$$

since $[f, v] \in O$.

A better proof might be to consider the map
\[
\Phi : \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{D}
\]
\[
x \otimes \tilde{u} \otimes y \mapsto x \tilde{u} \otimes y - x \otimes \tilde{y}
\]

It's clear that \( \Phi(x, \tilde{u}, y) = 0 \) if \( \tilde{u} \in \mathcal{O} \), so
\[
\Phi(x, \tilde{u}, y) \text{ is defined } \forall \tilde{u} \in \mathcal{F}_1 \mathcal{D} / \mathcal{O} = \mathcal{T}.
\]
Now
\[
\Phi(xf, \tilde{v}, y) = xf \tilde{v} \otimes y - xf \otimes \tilde{y}
\]
\[
= \Phi(x, f\tilde{v}, y)
\]
and
\[
\Phi(x, \tilde{u}, fy) = x \tilde{u} \otimes fy - x \otimes \tilde{f}y
\]
\[
= \Phi(x, \tilde{u}, f\tilde{y})
\]
But \( f\tilde{v} = \tilde{u}f \in \mathcal{F}_1 \mathcal{D} / \mathcal{O} = \mathcal{T} \). Thus \( \Phi \) induces a well-defined map
\[
\mathcal{D} \otimes \mathcal{T} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{D}
\]

Let's assume we really do have our resolution
\[
\rightarrow \mathcal{D} \otimes \Lambda^2 \mathcal{T} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{T} \otimes \mathcal{D} \longrightarrow \mathcal{D} \otimes \mathcal{D} \longrightarrow 0
\]
Then we get a spectral sequence
\[
E_1^{pq} = \text{Tor}_q \mathcal{D} \otimes \mathcal{D} (\mathcal{D}, \mathcal{D} \otimes \Lambda^p \mathcal{T} \otimes \mathcal{D}) \Rightarrow \mathcal{H}_{p+q} \mathcal{D}.
\]
\[
\ll
\]
\[
\text{Tor}_q \mathcal{D} \otimes \mathcal{D} (\mathcal{D}, \Lambda^p \mathcal{T})
\]
\[
\text{Tor}_q \mathcal{D} \otimes \mathcal{D} (\mathcal{D}, \mathcal{O}) \otimes \Lambda^p \mathcal{T}
\]
Thus we seem to get
\[ E^1_{pq} = H_2(C, \Theta) \otimes \Lambda^p T \]

Now it seems that
\[ H_2(C, \Theta) = \begin{cases} \ast & q \neq n \\ \Lambda^n T^* = \Omega^n & q = n \end{cases} \]

Example. \( \Theta = \mathbb{C}[x, p] \), \( [x, p] = 1 \), \( \Theta = \mathbb{C}[x] \).

Then \( \Theta = \bigoplus \mathbb{C}[x] p^n \)
\( [\Theta, \Theta] = \bigoplus \mathbb{C}[x] [x, p^n] = \bigoplus \mathbb{C}[x] p^n = \Theta \).

And in degree 1 you get a kernel \( \Theta \).

Assume \( \bigcirc \) holds. Then our spectral lives on the row \( q = n \) and
\[ E^1_{pn} = \Lambda^n T^* \otimes \Theta \Lambda^p T = \Lambda^n T^* = \Omega^{n-p} \]

Let's guess that \( d : E^1_{pn} \longrightarrow E^1_{p-1, n} \) is the de Rham \( d : \Omega^{n-p} \longrightarrow \Omega^{n-p+1} \). If so then we have
\[ HH_k(\Theta) = \bigoplus E^2_{k-n, n} = H_{DR}^{2n-k}(M) \]

The answer for the Hochschild homology is supposed to be the b-homology of \( \Omega^*_\Theta(T^*M) \). Because \( T^*M \) is symplectic one has a symplectic \( \ast \) operator interchanging \( b, d \). Thus the \( b \) homology in degree \( k \) is isomorphic to
the d homology in the complementary degree $2n-k$. 
Suppose $A$ smooth commutative.

We are concerned with constructing an $A$-bimodule resolution of $A$ which is minimal. Such a resolution is probably not unique but becomes unique if we complete along the diagonal. Thus we look at

$$(A \otimes A) = \lim_{\leftarrow} \left( A \otimes A / J_n / \Delta \right)$$

Now this coincides with the infinite jets $J_\infty$. Recall that we have a canonical exact sequence

1) $0 \to A \to J_\infty \to \Omega_A \otimes J_\infty \to \Omega_A^2 \otimes J_\infty \to \ldots$

(Spencer sequence). We can understand this Spencer sequence as follows. Recall one has an exact sequence

$$0 \to A \to A \otimes A \to \Omega_A \otimes A \to \Omega_A^2 \otimes A \to$$

where the exactness is clear since $0 \to A \to \Omega_A \to \ldots$ is exact; also we have $b'$ such that $[b', d \omega] = 1$. Let's use the maps

$$\Omega_A \otimes A \to \Omega_A \otimes (A \otimes A) \to \Omega_A^2 \otimes J_\infty$$

Then 1) should be a quotient of 2). Note that $\Omega_A J_\infty$ should be $\lim \Omega_A J_n$ in the above.
1) \( J_\infty \longrightarrow \mathbb{L}_A^1 \otimes_{A} J_\infty \longrightarrow \mathbb{L}_A^2 \otimes_{A} J_\infty \)

should be the DR complex associated to the local system given by \( T_\infty \) with a certain left connection. Let's work this out in coordinates using the fact that \( J_\infty : \mathcal{A} \longrightarrow J_\infty \) gives the flat sections. This is the map:

\[
\begin{align*}
\mathcal{A} & \longrightarrow \mathcal{A} \otimes \mathcal{A} \\
\alpha & \longmapsto 1 \otimes \alpha
\end{align*}
\]

Local coordinates, \( x = (x^i) \) so that roughly, \( \mathcal{A} = \mathbb{C}[x] \) and \( J_\infty = \mathbb{C}[x][[\delta x]] \) and

\[
\begin{align*}
\mathcal{A} \otimes \mathcal{A} & \longrightarrow J_\infty \\
x \otimes 1 & \longmapsto x \\
1 \otimes x & \longmapsto x + \delta x
\end{align*}
\]

Thus, \( J_\infty (x) = x + \delta x \), so

\[
J_\infty (f(x)) = f(x + \delta x) = \sum \frac{1}{n!} f^{(n)}(x) \delta x^n
\]

(n multi-index in general).

Now \( J_\infty \longrightarrow \mathbb{L}_A^1 \otimes_{A} J_\infty \)

should be of the form \( d = dx^i \partial_i \) and it should kill \( J_\infty (\mathcal{A}) \). Thus

\[
\partial_i f(x, \delta x) = 0 \quad \text{if} \quad f(x, \delta x) = f(x + \delta x)
\]
This gives
\[ \partial_v f(x, \delta x) = (\partial_x^i - \partial_{\delta x}^i) f(x, \delta x) \]

and so
\[ d = dx^i (\partial_x^i - \partial_{\delta x}^i) : T_x \rightarrow \Omega^1_{A} \otimes_A T_x \]

Next consider the problem of finding a \( \delta \) operator on \( \Omega_{A} \otimes_A A \). We note that \( \Omega_{A} \otimes_A A = \Lambda (\Omega_{A} \otimes_A A) \) is an exterior algebra, hence we get an interior product operator \( i(\xi) \) on \( \Omega_{A} \otimes_A A \) associated to any \( A \otimes A \)-module map
\[ \Omega^1_{A} \otimes_A A \xrightarrow{\xi} A \otimes A \]

Now \( \xi \) is equivalent to a left \( A \)-module map
\[ \Omega^1_{A} \xrightarrow{\xi_0} A \otimes A \]

In order to obtain an \( A \otimes A \) resolution of \( A \) from the Koszul complex, we want the image of \( \xi \) to be \( \Omega^1_A = \ker \{ A \otimes A \rightarrow A \} \). In fact we really want \( \xi \) to come from a lifting:
\[ \Omega^1_A \xrightarrow{\xi_0} \Omega^1_A \xrightarrow{\xi} A \]
so that we get the correct identification of \( T^1 \) with \( \Omega A \otimes A \).

Let's analyze exactly what \( \xi \) is. First an \( A \otimes A \)-module map

\[ T_A \otimes A \to A \otimes A \]

by duality corresponds to a section of

\[ T \otimes A = \text{pr}_1^* T \]

where \( T \) is the tangent bundle on \( M \times \text{Var}(A) \).

Thus we have a vector field on \( M \times M \) which is horizontal, or if we want a field over \( M \times M \) which gives at each point \((x, y)\) a tangent vector to \( M \) at \( x \). The properties of this field are that it vanishes on the diagonal, and that for \( y \) approaching \( x \) it agrees up to first order with \( y \mapsto y - x \). What this means is that the map \((x, y) \mapsto g(x, y) \in T_x M \) should vanish on the diagonal and the 1-jet along the diagonal, which is a map from \( T M \) to \( TM \) should be the identity.

The typical picture for this map is

\[ g(x, y) = \exp_{T_x}^{-1}(y) \]

where \( \exp_x : T_x M \to M \) is the exponential map at \( x \) associated to a connection on \( M \).
Here's another viewpoint.

Let's look at the left $A$-module map,

$$i_o : \Omega_A^1 \longrightarrow A \otimes A$$

This associates to a cotangent vector $\xi \in T^*_xM$ a functor $g(x, \xi)(y)$ on $M$, which should vanish when $y = x$ and whose 1-jet at $y = x$ is the linear function on $T_x$ given by $\xi$. This is therefore a way of assigning to a cotangent vector $\xi$ at $x$ an actual function near $x$ extending $\xi$ considered as an infinitesimal function.

But we have some new ideas about how to do these tubular neighborhood things. In particular we want to make use of the multiplicative group.

Let's recall the tubular nbhd. business. Suppose $Y$ is a submanifold of $M$, $I \subset \mathcal{O}_M$ the ideal of functions vanishing in $Y$. One has exact sequences

$$0 \longrightarrow I \longrightarrow \Gamma_M \stackrel{\delta}{\longrightarrow} \Gamma_Y \longrightarrow 0$$

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_A^1 \otimes B \longrightarrow \Omega_B^1 \longrightarrow 0$$

One chooses a splitting of the bottom sequence. This gives an $A$-module map $\Omega_A^1 \longrightarrow I/I^2$.
which, since $\Omega^1_A$ is projective, can be lifted to a $A$-module map $\Omega^1_A \to I$. This gives a derivation $A \xrightarrow{D} I \subset A$ which we know leads to an isomorphism

$$\hat{A} = \bigoplus_{n \geq 0} I^n/I^{n+1} = \bigoplus_{n \geq 0} \text{Sym}^n_B (I/I^2)$$

The derivation $D$ is a vector field on $M$ which vanishes on $Y$ and where 1-jet along $Y$ viewed as an endomorphism of $TM|_Y$ is the projection onto the normal bundle (considered as a subbundle of $TM|_Y$.)

Let's now look at the case of the diagonal

$$0 \to I \to A \otimes A \to A \to 0$$

$$0 \to I/I^2 \to \Omega^1_A \oplus \Omega^1_A^+ \to \Omega^1_A \to 0$$

There are two obvious splittings of the second sequence. Picture.

Let's identify the normal bundle of $\Delta M \subset M \times M$ with $pr_1^*(TM)$. Now we want a vector field on $M \times M$ which vanishes on $\Delta M$ and whose 1-jet is the projection of $TM|_{\Delta M}$ onto the horizontal space.
Then we want to extend the this 1-jet to a vector field on \( M \times M \). Along the diagonal this vector field is horizontal, so the obvious thing to do is to extend it keeping it horizontal. Then for each \( y \in M \) we have a vector field \( X_{\{y\}} \) on \( M \) which vanishes at \( y \) and whose 1-jet at \( y \) is the identity on \( T_y M \).

If we use this sort of extension it is clear that the corresponding lifting of \( A \) into \( A \otimes A \) which is thus killed by the vector field is \( a \mapsto 1 \otimes a \).

It now appears that we have constructed \( b' = i(X) \) on \( \Omega_A \otimes A \otimes A \) as well as \( d \otimes 1 \). Then \( b' \) and \( d \otimes 1 \) should commute with multiplication by elts of \( P^cT^*A = 1 \otimes A \). Roughly speaking everything happens on \( M \times \{y\} \) for each \( y \in M \).

The same should be true for \( [b', d \otimes 1] \). This should be the Lie derivative on forms \( \Omega_A \) associated to the vector field \( X_{\{y\}} \) for each leaf \( M \times \{y\} \).

Notice that if we could arrange \( X_{\{y\}} \) to vanish only at \( y \) and this vector field has flow defined for all \( t \), then \( M \) had to be diffeomorphic to \( T_y M \).
Consider the blowup of $(0,0) \in \mathbb{A} \times M$. This is described in alg. geom. by $\text{Proj}$ of the graded algebra

$$\bigoplus_{n>0} (\mathcal{O}_{\mathbb{A}[h]} + m[h])^n$$

which appears

$\deg 0$:

$$0 \oplus \mathcal{O}h \oplus \mathcal{O}h^2 \oplus \cdots$$

$\deg 1$:

$$m \oplus \mathcal{O}h \oplus \mathcal{O}h^2 \oplus \cdots$$

$$m^2 \oplus mh \oplus \mathcal{O}h^2 \oplus \cdots$$

Take the $\mathbb{A}$ located in degree 1 and localize; more precisely take the direct limit under multiplication by $h$. This gives the algebra

$$R = \bigoplus m^2 h^{-2} \oplus mh^{-1} \oplus \mathcal{O} \oplus \mathcal{O}h \oplus \cdots$$

Note the $R$ is graded; this is related to $\boxtimes$ being bigraded.

Let us find the variety of $R$. If $X : R \rightarrow C$ is a homomorphism, the $X(h)$ is non-zero or zero. For $X(h) \neq 0$, we have a point of $R_h = \mathcal{O}[\mathbb{A}[h], h^{-1}]$, i.e., we have $C^x \times \text{Var}(O)$. If $X(h) = 0$, then $X$ is a point of

$$R/R_h = \cdots \oplus (m/m^2) h^{-1} \oplus \mathcal{O}/m \cong \text{Sym}(m/m^2)_{m/m}$$

i.e., a point of the tangent space to $M$ at $0$.  

Thus $R$ describes the part of the blowup $\widetilde{X} \times M$ complementary to $\widetilde{M}$.

Another way to look at $R$ is that it is obtained by adjoining to $O$ element $h, x^i$ such that $v_i h = x^i$ is a system of parameters generating $M$. Thus the ratios $\frac{x^i}{h}$ are well-defined as $h \to 0$.

The next point is to understand to what extent $R$ gives $O$ as a deformation of $\text{Sym}(m/m^2)$. We have

\[ R = m^2 h^{-2} \oplus mh^{-1} \oplus O \oplus Oh \oplus Oh^2 \]

\[ RH = \cdots \oplus m^2 h^{-1} \oplus m \oplus Oh \oplus \cdots \]

\[ RH^2 = m^2 \oplus RH \oplus \cdots \]

Thus $q_2 R = \bigoplus_{n > 0} RH^n/RH^{n+1}$ is

\[
(m^2/m^2) h^{-2} \oplus \left(\frac{m^2}{m^2}\right) h^{-1} \oplus C
\]

\[ \oplus \left(\frac{m^2}{m^2}\right) h^{-1} \oplus m^2/m^2 \oplus C h
\]

\[ \oplus m^2/m^2 \oplus \left(\frac{m^2}{m^2}\right) h \oplus C h^2
\]

which is $\left(\bigoplus m^{n/m+n+1}\right) \otimes C[h]$. Obviously we should now look into applying the tubular model, ideas. This means looking for a derivation $D: R \to hR$ such that $D$ induces the identity on $hR/h^2 R$. It suffices that $D$ extends $h D_h$ on $C[h]$. Then $D$ induces a derivation on $R^*_h = \mathcal{O} \otimes C[h,h]$.  

which is determined by a derivation \( O \rightarrow O \otimes \mathbb{C}[h, h^{-1}] \)

Geometrically we want something like a radial vector field

\[ x \frac{\partial}{\partial x} + h \frac{\partial}{\partial h} \]

i.e., \( x \frac{\partial}{\partial x} + h \frac{\partial}{\partial h} \). When we change to \( h \rightarrow h_0 \) we get the vector field \( h \frac{\partial}{\partial h} \):

\[
\begin{align*}
  f(h, x) & \leftrightarrow f(h, 0) \\
  \downarrow & \downarrow \\
  h \frac{\partial}{\partial h} + x \frac{\partial}{\partial x} & \leftrightarrow h \frac{\partial}{\partial h} \\
  h_0 \frac{\partial}{\partial h}(h, x) + x \frac{\partial}{\partial x} f(h, x) & \rightarrow h f_1(h, 0) + h_0 f_2(h, 0)
\end{align*}
\]

Thus the derivation \( D \) of \( R \) I am after should be of the form \( h \frac{\partial}{\partial h} + D' \) where \( D' : O \rightarrow O \) is independent of \( h \). Recall

\[ R = \cdots \oplus m h^{-1} \oplus O \oplus Oh \oplus \cdots \]

We need \( D'(m) \subset m \) in order that \( D \) be defined on \( R \). We also want \( D \) to be zero on \( R/hR \) which means \( D' O \subset m \) and \( D' = 1 \) on \( m/m^2 \) so that \( D \) is \( 0 \) on \( (m/m^2) h^{-1} \).

Thus we reach the kind of
derivation $D : \mathfrak{g} \to \mathfrak{m}$ associated to the tubular nbhd thm. for $0 \in M$.

Let's now state the conclusions. We have been studying the tubular nbhd theorem for an embedding of nonsingular varieties of manifolds. Suppose we have $B = A/I$ with $A, B$ smooth. The formal tubular nbhd theorem uses a derivation $D : A \to \mathcal{I}A$ such that $D = 1$ on $I/I^2$. Next consider the "deformation to the normal bundle" which is the algebra

$$R = \bigoplus_{n>0} I^n h^{-n} \oplus \bigoplus_{n>0} A h^n$$

over $\mathbb{C}[h]$. Given $D$ as above on $A$, let

$$\tilde{D} = h\partial_h + D$$
on $R$.

Claim then that $\tilde{D}$ gives the tubular nbhd thm. for $R/hR = \bigoplus_{n>0} (I^n/I^{n+1}) h^{-n}$.

The next stage is to understand the case of Lie algebras. I need to see how $b$ appears
Poisson groups: (conversation with Weinstein and Lu).

A Poisson Lie group turns out to be a Lie group $G$ with a 1-cocycle $G \to \Lambda^2 \mathfrak{g}$ where $G$ acts on $\Lambda^2 \mathfrak{g}$ via the adjoint action.

(The definition is group object in the category of Poisson manifolds.) A Poisson-Lie algebra is a Lie algebra $\mathfrak{g}$ with a 1-cocycle $\mathfrak{g} \to \Lambda^2 \mathfrak{g}$. It turns out that there is a natural duality operator on Poisson-Lie algebras $\mathfrak{g} \to \mathfrak{g}^*$. It would seem that the adjoint of the 1-cocycle for $\mathfrak{g}$ gives the Lie bracket on $\mathfrak{g}^*$ and that the adjoint of the bracket on $\mathfrak{g}$ is the 1-cocycle for $\mathfrak{g}^*$.

Example: If $\mathfrak{g}$ is a Lie algebra with 0 Poisson structure, then the dual is $\mathfrak{g}^*$ with zero brackets and the familiar Poisson manifold structure.

Manin construction. A Poisson Lie algebra $\mathfrak{g}$ determines a symplectic Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are both Lie subalgebras and isotropic in some way. Conversely if we have a symplectic Lie algebra which is a direct sum of isotropic subalgebras, then the two subalgebras are dual Poisson Lie algebras. Example: Complex-Lie groups and Gram-Schmidt decomposition (or Hausdorff decomposition) $G = K \cdot A N$. Then $K$ and $AN$ are dual Poisson Lie groups.

Duality for Poisson-Lie groups so linked to Lie algebras that it is meaningful only for 1-connected groups.
Mainstein mentions symplectic groupoids.

This is a groupoid \( \Gamma \xrightarrow{\pi} \mathcal{O} \) such that both maps are Poisson, I think. A key point is that Lagrangian bisections (simultaneously sections of \( \alpha \) and \( \beta \)) form a group, infinite-dimensional, whose Lie algebra is 1-forms under Poisson bracket.

Example of such a \( \Gamma \) is \( G \times \mathfrak{g}^* \) with \( G \) a Lie group of 0 Poisson structure, where \( G \) acts on \( \mathfrak{g}^* \) via coadjoint repn. i.e. \( \mathfrak{g}^* = \mathcal{O} \).

More generally one could take \( G \times \mathfrak{g}^* \) with \( G \) acting on \( \mathfrak{g}^* \). (Also \( \mathfrak{g}^* \times G \) and there is some sort of compatibility.)

Quantizing \( \Gamma \). The composition is a relation which is a Lagrangian submanifold of \( \mathcal{F} \times \mathcal{F} \times \Gamma \), so if \( V \) is the vector space obtained by quantizing \( \Gamma \) one should associate \( V^* \otimes V^* \otimes V \) to \( \mathcal{F} \times \mathcal{F} \times \Gamma \) and an element of \( \Gamma \) assoc. to the Lagrangian submanifold, whence we get an algebraic structure on \( V \).
On cyclic objects.

Motivation: Let $P$ be a projective $A$-bimodule resolution of $A$. Then

$F : [n] \rightarrow [P \otimes_A ]^{n+1}$

is a contravariant functor from the "cyclic category without degeneracies" $\overline{\text{Cyc}}$ to complexes. $\overline{\text{Cyc}}$ is the category of finite non-empty cyclically ordered sets and injective maps preserving the cyclic order. The above functor takes maps to quasi-isomorphisms, so one expects a spectral sequence

$E_2^{pq} = H_p(\text{Cyc}, H_q(A)) \Rightarrow H(\lim \rightarrow F)$

Also $B\overline{\text{Cyc}}$ should be $BS^1$ up to homotopy equivalence. Thus $\lim \rightarrow F$ should give the cyclic homology of $A$.

Example: A separable, $P = A$, then we have the constant functor $[n] \mapsto A$.

I would like to understand how to prove that $B\overline{\text{Cyc}} \sim BS^1$. The idea should be to construct a "circle bundle" over $B\overline{\text{Cyc}}$ and show the total space is contractible.

Here's an attempt: Let $\Delta$ be the simplicial category without degeneracies, i.e., finite non-empty totally ordered sets, and inclusions preserving the order. We have a functor $f : \Delta \rightarrow \overline{\text{Cyc}}$ which sends
a totally ordered set into the corresponding cyclic ordered set, where the successor of the last element is the first element. Then we get a fibred category over $\mathbf{Cyc}$ with fibres $\tilde{f}/[n]$, $n > 0$.

What is $\tilde{f}/[n]$? It is the poset whose elements are non-empty subsets $\sigma \subset [n] = \{0, \ldots, n\}$ equipped with total order compatible with the cyclic order on $\sigma$ induced by the cyclic order on $[n]$; the ordering is given by inclusion compatible with the total ordering.

1. $n = 0$. Then $\tilde{f}/[0]$ is a point.

2. $n = 1$. Then $\tilde{f}/[1]$ has four elements: the subsets $\{0\}, \{1\} \subset \{0, 1\}$, and the subset $\{0, 1\}$ with the two total orderings. The poset is this

$$
\begin{array}{c}
\{0 < 1\} \\
\{0\} \\
\{1\} \\
\{0 > 1\}
\end{array}
$$

So the homotopy type is $S^1$.

For $n > 1$, $\tilde{f}/[n] \simeq S^1$ as follows. Embed $[n]$ inside $S^1$ as cyclic subset. For each subset $\sigma \subset [n]$ with total order, associate the interval of (or arc) of $S^1$ starting with the first element of $\sigma$ and ending with the last. If $|\sigma| = 1$, this arc reduces to a point. Then we have a functor from $\tilde{f}/[n]$ to contractible subspaces of $S^1$. 
Given \( z \in S' \), consider the subposet of \( T \) such that \( z \in x_0 \). This poset has a least element which reduces to a point if \( z \in \{n\} \), and otherwise to a pair of consecutive points.

It is clear this gives a homotopy equivalence of \( f/\{n\} \) with \( S' \) for \( n \geq 1 \). It's also clear that any map \( \{n\} \to \{n'\} \) induces a hom
\[
f/\{n\} \longrightarrow f/\{n'\}, \text{ again for } n \geq 1.
\]
So we have to deal with the problem that \( f/\{0\} = pt \).
November 1, 1991

Some ideas from Goodwillie. Fibre square:

$$\begin{align*}
\text{HC}^- & \longrightarrow \text{HP} \\
\downarrow & \quad \quad \downarrow \\
\text{HH} & \longrightarrow \text{HC}^-
\end{align*}$$

HC for a cyclic module is $\text{L}_{\text{Klein}}$

HC$^-$ should be $\text{R}^\leftarrow_{\text{Klein}}$

These correspond to the homotopy orbit space and homotopy fixpoint space for a circle action. There ought to be a way to see that the Dennis trace $K \rightarrow \text{HH}$ factors through the homotopy fixpoint space, and thus becomes a map $K \rightarrow \text{HC}^-$. HP should be some kind of Tate homology with respect to the circle action. Apparently for $G$ spectra, $G$ compact, there is a notion of Tate (maybe Tate-Farrell) cohomology.

There is a cyclic bar construction for a category in which simplices are loops. When one realizes it one gets the space whose elements are subdivisions of $S^1$ with objects corresponding to arrows and arrows corresponding to vertices. This seems strange. Apparently the self maps on the circle act on the realization of cyclic set
November 15, 1991

Sir, Frieed's top QFT. This is a baby model of Witten's Chern-Simons QFT where the compact comm. Lie group $G$ is replaced by a finite group. One supposes given a class in $H^3(G, \mathbb{R}/\mathbb{Z}) = H^3(BG, \mathbb{Z})$.

If $X$ is a closed 3-manifold with principal $G$-bundle, one gets a number in $\mathbb{Q}/\mathbb{Z}$ by pairing with the fundamental class; this gives the action $\bar{S}(P)$ and one sums $\exp i \bar{S}(P)$ over the possible $P$.

The object of the QFT is to do something for closed 2-manifolds $Y$, more precisely to attach a line to $Y$ and an element of the line when $Y$ is given as the boundary of a 3-manifold. Then later one tries to reduce to codimension 2.
After Tsygan.

Let $g$ act on $A_0$, let $T$ be an invariant trace on $A_0$. He defines a map of complexes
\[ C^*(g) \rightarrow C^*_{per}(A_0) \]
which is given roughly by
\[ D_1 \cdot \ldots \cdot D_n \rightarrow \sum (-1)^T(a_0 D_1 a_1 \cdots D_n a_n) \]
If this map is denoted $c \mapsto \tau_c$, then we have
\[ b\tau_c = 0 \quad B\tau_c = \tau_{2c} \]

What’s going on here is perhaps that we form the DB algebra $C^*(g, A_0)$ and use $\tau$ as a trace
\[ C^*(g, A_0) \rightarrow C^*(g) \]
Return to the derivation of \( RA \) given by
\[ Da = \phi a \]
where \(-S\phi = d\omega \). Then \( D \) is of degree 2 relative to the grading of \( RA = \Omega^* A \) by degree, so it is possible to describe the lifting homomorphism \( l : A \to RA \) such that \( DL = 0 \). Observe that
\[
D(da, da_2) = D(a_1 q_2 - a_1 o q_2)
= \phi(a_1 q_2) - \phi a_1 o q_2 - a_1 o \phi q_2
= \phi(a_1 q_2) - \phi a_1 q_2 - a_1 \phi q_2 + d\phi a_1 da_2 + da_1 d\phi q_2
\]
In fact recall that \( D \) extends to \( QA = (\Omega^* A, 0) \) via
\[ D(da) = \frac{1}{2} da + d\phi a. \]
Thus
\[
D(a_0 da_1 \cdots da_n)
= (Da_0) \cdot da_1 \cdots da_n + \sum_{j=0}^n (a_0 da_j - da_j (\frac{1}{2} da_j + d\phi a_j)) da_{j+1} \cdots da_n
= (\phi a_0) da_1 \cdots da_n + \sum_{j=0}^n a_0 da_j - da_j (\frac{1}{2} da_j + d\phi a_j) da_{j+1} \cdots da_n
= \frac{1}{2} (a_0 da_1 \cdots da_n) + \left\{ \phi a_0 da_1 \cdots da_n + \sum_{j=0}^n a_0 da_j da_{j-1} d\phi a_j da_{j+1} \cdots da_n \right\}
\]
Call this \( L(a_0 da_1 \cdots da_n) \)
in analogy with Lie derivative.

Then \[ D = \frac{N}{2} + L \]
as operators on \( \Omega^* A \).
Now
\[ l_a = a + l_a + \ldots \]
with \( l_n a \in L^n a \), so
\[ 0 = D(l_a) = \sum_{n \geq 0} n l_n a + \sum_{n \geq 0} l_n a \]
which gives the recursion relation
\[ n l_n a + l_{n-1} a = 0 \]
or
\[ l_n a = \frac{(-1)^n}{n!} l_n a \]
Thus
\[ l_a = e^{-l} a \]

Notice that \( L^n : \hat{A} \rightarrow \Omega^n \hat{A} \) is an interesting cochain constructed from \( \phi \), which means that your statement that the only linear maps constructed from \( \phi \) in the universal lifting section is wrong.

The question is whether there might be a smaller derivation than \( D = \phi e \). We would like to preserve the fact that \( D \) extends to \( \hat{A} \).

This means that we want a derivation \( D : A \rightarrow \hat{A} \) relative to \( \Theta \), i.e.

1) \[ D(a, a_2 + d(a_1 a_2)) = D(a_1 + da_1) \circ (a_2 + da_2) + (a_1 + da_1) \circ D(a_2 + da_2) \]
2) \[ D(a_2) = D(a_2) + a_1 \circ D(a_2) + D(a_1) da_2 + da_1 D(a_2) \]
3) \[ D(a_2) = D(a_1) a_2 + a_1 \circ D(a_1) + D(a_1) da_2 + a_1 da_2 + da_1 D(a_2) \]
d(Dq_1 a_1 a_2) = Dq_1 a_1 d_2 + Dq_1 d_2 a_2 + da_1 Dq_1 a_2 + a_1 dDq_1 a_2 + (dDq_1 a_1 d_2 - da_1 (dd_1) a_2)

4) Subtracting 4) from 3) we have

\[ [D_D d](a_1 a_2) = [D_D d](a_1) a_2 + d(Dq_1 a_1) d_2 \]
\[ + a_1 [D_D d] a_2 - da_1 (dd_1) a_2 \]

Thus we see that \[ [D_D d] a_1 a_2 = D(\overline{d} a_1) - d\overline{D} a_1 \]
is a derivation \( \overline{A} \rightarrow \overline{\text{odd}} \overline{A} \).

Let’s write

\[ D(a + da) = f(a) = \sum_n f_n(a) \]

where \( f_n : \overline{A} \rightarrow \overline{\text{odd}} \overline{A} \). Then the derivation condition is

\[ f(a_1 a_2) = (a_1 + da_1) f(a_2) + f(a_1) (a_2 + da_2) \]
\[ = a_1 f(a_2) + f(a_1) a_2 - da_1 df(a_2) - (-1)^{|f|} df(a_1) da_2 \]
\[ + da_1 f(a_2) + f(a_1) da_2 \]

i.e.

5) \[ \delta f = - duf - fud + dudf + (-1)^{|f|} dfud \]

Recall in general that

6) \[ (d_d d_d) f = - duf - (-1)^{|f|} fud \]

From 5) we get

\[ d(\delta f) = dudf - dfud \]

Note that \( Da = f_+(a) \quad D(da) = f_-(a) \)
so that
\[ \delta d[a] = f_-(a) - df_+(a). \]

Let's check that \([\delta d] \] is a derivation
\[ \delta f_+ = -dudf_+ - f_+ ud + dudf_- + df_- ud \]
\[ \delta(df_+) = d(\delta f_+) + (\delta d - df) f_+ \]
\[ = dudf_+ - df_+ ud + df_- f_+ - f_+ ud \]
\[ \therefore \delta(f_+ - df_+) = 0 \]

Recursive construction of \( f_+ \):
Start with \( f_{2n-1} \), \( \delta f_{2n-1} = 0 \). Then
\[ \delta (dudf_{2n-1} - f_{2n-1} ud) = 0 \]
and since \( A \) is quasi-free, we can find \( f_{2n} \)
\[ \delta f_{2n} = -dudf_{2n-1} - f_{2n-1} ud \]
Then you can check that one can continue with
\[ f_{2n+1} = df_{2n}, \quad f_{2n+2} = f_{2n+3} = \ldots = 0. \]

Ambiguity in the choice of \( f_{2n} \) is a derivation \( \omega \) values in \( \Omega^{2n-1} A \); ambiguity in choice of \( f_{2n} \) is a derivation \( \omega \) values in \( \Omega^{2n} A \).

Note that if we take \( D \) of the form
\[ Da = f_+ a, \quad D(da) = \frac{1}{2} da + df_+(a), \]
then we can remove the derivation with \( a \rightarrow \phi a \), \( da \rightarrow \frac{1}{2} da + df \phi a \)
where \( \phi a = f_-(a) \). Then we have \( f_-(a) = df_+(a) \),
and the relations are
\[ \delta f_+ = -d \delta f_+ + d f_+ d + d \delta f_+ + d f_+ d = 0 \]
\[ \delta f_- = -d \delta f_- + f_+ d + d \delta f_- - d f_+ d \]
and the latter is satisfied automatically since
\[ \delta f_- = \delta df_+ = (ds - ds) f_+ = -d \delta f_+ - f_+ d \]

Thus derivatives of the form
\[ D a = f_+(a) \]
\[ D(da) = \frac{1}{2} d a + d f_+(a) \]
with \( f_+ \) of order 2 are the same as a family of derivatives \( A \longrightarrow \Omega^{2n} A \) for \( n \geq 2 \).

Consider constructing a lifting homomorphism inductively:

\[ \begin{array}{c}
\Lambda \\
\downarrow \phi \\
a \phi_a
\end{array} \]

This gives a 2-cocycle
\[ a_1 a_2 - \phi(a_1 a_2) = (a_1 - \phi a_1) \circ (a_2 - \phi a_2) \]

\[ = a_1 a_2 - \phi^2(a_1 a_2) = a_1 a_2 + a_1 da_2 + a_2 da_1 + \phi a_1 a_2 - \phi a_1 da_2 \\
+ a_1 \phi a_2 - da_1 d \phi a_2 + a_2 \phi a_1 + \phi a_1 \phi a_2 \\
= \phi a_1 \phi a_2 - d \phi a_1 a_2 - da_1 d \phi a_2 \\
\]

We thus have a bimodule map \( \Omega^2 A \longrightarrow \Omega^4 A \) which can be composed with
\[ \phi : A \to \Omega^2 A \] to obtain a correction yielding a lifting homomorphism \( A \to RA/IA^3 \).

However this is different from the canonical lifting homomorphism
\[ a \to a - \phi a + \frac{1}{2} \partial^2 a. \]

Recall
\[
\begin{align*}
\overline{A} & \xrightarrow{L=\phi} \Omega^2 A \\
& \xrightarrow{L} \Omega^4 A \\
& \xrightarrow{\cdot} \Omega^5 A \\
\end{align*}
\]

This \( L \) on \( \Omega^2 A \) does not seem to be a bimodule map.
\[ D(da_1, da_2) = \frac{1}{2}(\phi(a_1) - \phi(a_2) - \phi(a_1) - \phi(a_2) + d\phi(a_1) da_2 + da_1 d\phi(a_2) \]

\[ = da_1 da_2 + d\phi(a_1) da_2 + da_1 d\phi(a_2). \]

\[ \therefore \text{Observe this is consistent with} \]

\[ D(da_1, da_2) = \frac{1}{2}(da_1 + d\phi(a_1)) da_2 + da_1 \left( \frac{1}{2} da_2 + d\phi(a_2) \right) \]

Thus

\[ D(a_0 da_1 \ldots da_n) = \frac{1}{2}(a_0 da_1 \ldots da_n) + L(a_0 da_1 \ldots da_n) \]

where

\[ L(a_0 da_1 \ldots da_n) = \phi(a_0 da_1 \ldots da_n) \]

\[ + \sum_{j=1}^{n} a_0 da_1 \ldots da_j d\phi(a_j da_{j+1} \ldots da_n) \]

In doing this calculation you should note that

\[ D(da) = \frac{1}{2} da + d\phi(a) \] is closed hence

\[ D(a_0 da_1 \ldots da_n) = D(a_0 da_1 \ldots da_n) \]

\[ = Da_0 \circ da_1 \ldots da_n + \sum_{j=1}^{n} a_0 da_1 \ldots da_j D\phi(a_j da_{j+1} \ldots da_n) \]

\[ = Da_0 da_1 \ldots da_n + \sum_{j=1}^{n} a_0 da_1 \ldots da_j \left( \frac{1}{2} da_j + d\phi(a_j) \right) \]

etc.

Next put \( H = \frac{1}{2} \omega \) in \( \Omega^\omega A \). Then since \( \omega \) is of degree 2 we have \( [H, L] = L : [H, L] \omega = \frac{1}{2} |L| \omega |L\omega - \frac{1}{2} |L| \omega = L \omega. \)
Thus we have
\[ e^{-L}He^L = e^{-L}(e^LHe^L + [H,e^L]) \]

\[ \int_0^1 e^{(1-t)L}[H,L]e^{tL}dt = Le^L \]

so
\[ e^{-L}He^L = H + L = D \]

Thus \( e^{-L} \) carries \( \text{Ker}(H-\frac{1}{2}n) \) to \( \text{Ker}(D-\frac{1}{2}n) \).

Now be careful and recall that we have an algebra isomorphism
\[ \hat{\Omega}A \cong \hat{Q}A \]
given by sending \( \omega \in \Omega^nA \) to the unique eigenvector of \( D \) with eigenvalue \( \frac{1}{2}n \) whose leading term in \( \omega \). Recall that \( \hat{D} \)

\[ \text{Ker}(D-\frac{1}{2}n) \subset \Omega^nA \]

Now \( e^{-L}w \) belongs to \( \Omega^nA \) and

\[ D(e^{-L}w) = e^{-L}Hw = \frac{n}{2} e^{-L}w \]

so it's all clear.
November 17, 1991

Question. Consider the algebra

$$M_\infty C = \lim \limits_{\longrightarrow} M_n C$$

of matrices with finite support, and adjoin an identity. This is not separable, but is it quasi-free? What are modules over $$(M_\infty C)^\ast$$?

Dec 9, 1991

Notes from various sheets of paper:

G groups. Consider universal extension:

$$G = F(G)/N(G)$$, where $$F(G)$$ = free group on $$G \setminus \{1\}$$.

Do there exist universal G-modules for cocycles? What is $$FG/(NG, NG)$$ the universal abelian extension?

Problem: In the setup of Kadison's theorem: $$A \otimes A$$ projective A bimodule, to construct an explicit homotopy equivalence of the resolutions $$(\partial A \otimes A, b')$$ and $$(\partial A \otimes A, b')$$. (It should be easy but I got tied up in knots.)
December 13, 1991

Madsen's talk about the cyclotomic trace. The end result is (for p odd)

\[ TC(Z_p, p) \hat{\rightarrow} (\text{Im} J \times \text{BIm} J \times SU) \hat{\rightarrow} \]

and

\[ K(Z_p) \hat{\rightarrow} \text{cyclotomic trace} \rightarrow TC(Z_p, p) \hat{\rightarrow} \]

is split surjective. Now one knows

\[ K_{et}(Z_p) \hat{\rightarrow} (\text{Im} J \times \text{BIm} J \times SU) \hat{\rightarrow} \]

so one has

\[ K(Z_p) \hat{\rightarrow} \rightarrow TC(Z_p, p) \hat{\rightarrow} \]

although the commutativity is not clear at the moment. Thus the "topological cyclic homology at p": \[ TC(Z_p, p) \] of \( Z_p \) is supposed to calculate the K-theory of \( Z_p \).

There is something called topological Hochschild homology due to Bökstedt. This is defined for any ring spectrum \( R \) and denoted \( TH(R) \). One forms a simplicial spectrum

\[ \ldots \Rightarrow R \wedge R \Rightarrow R \Rightarrow R \]

analogous to what one does for an algebra, then takes the colimit (or realization). Perhaps the good way to say it is that one has a
cyclic spectrum

\[ [n] \mapsto R \wedge [n+1] R \]

Suppose we take \( R = \mathbb{Z} \), more precisely, \( R = \mathbb{E} \) Eilenberg–MacLane spectrum. Note that \( R \wedge R \) is \( K(\mathbb{Z}) \wedge K(\mathbb{Z}) \), which is not \( K(\mathbb{Z}) \), because \( \pi(K(\mathbb{Z}) \wedge K(\mathbb{Z})) = H(K(\mathbb{Z}), \mathbb{Z}) \).

So even in this case \( T(\mathbb{Z}) \), the top Hoch homology is nontrivial. I believe Bökstedt has computed \( \pi(T(\mathbb{Z}_p)) \) (mod \( p ? ? \)) and found the answer to be a polynomial ring on generator of degree \( 2p \) tensored with an exterior algebra of degree \( 2p-1 \) (?).

The next thing is the topological cyclic homology, which is probably a negative cyclic theory, i.e. taking homotopy fixed points for the circle actions. There is a subdivision process on \( T(\mathbb{R}) \) which allows one to concentrate on the subgroup \( \mathbb{G} \) of the circle, and by taking suitable fixed points, to define \( T(\mathbb{R}, \mathbb{G}) \).
December 19, 1991

Gunnar Carlsson's talk:

There is an assembly map

$$B\Gamma + K\mathbb{A} \rightarrow K\mathbb{A}[\Gamma]$$

One would like to prove it is a split injection of spectra, and ultimately an equivalence in good cases. The strategy is to realize $K\mathbb{A}[\Gamma]$ as the first spectrum for a spectrum $M$ with $\Gamma$-action. Then we have a canonical map

$$K\mathbb{A}[\Gamma] = \begin{array}{c} M^\Gamma \rightarrow M^{h\Gamma} = \text{Hom}_\text{Maps}(E\Gamma, M)^\Gamma \\ \text{homotopy} \\ \text{fixes spectrum} \end{array}$$

For some reason, because homotopy fixests can be analyzed via spectral sequences, one might be able to define a map from $M^{h\Gamma}$ to $B\Gamma + K\mathbb{A}$. In the case $\Gamma$ finite of order $N$ we have an embedding

$$A[\Gamma] \subset M_n\mathbb{A}$$
given by the left regular representation, and

$$A[\Gamma] = (M_n\mathbb{A})^\Gamma$$

where $\Gamma$ acts on matrices via right multiplication. This gives what we want

$$K\mathbb{A}[\Gamma] = K(M_n\mathbb{A})^\Gamma$$

Now $K(M_n\mathbb{A}) \sim K\mathbb{A}$ by Morita equivalence, so
it seems that

\[ K(M \otimes A)^{h\Gamma} \cong KA^{h\Gamma} = \text{Map}(B\Gamma, KA) \]

since \( \Gamma \) acts trivially on \( KA \). The question is why should there be a map

\[ \text{Map}(B\Gamma, KA) \rightarrow B\Gamma \wedge KA? \]

The Tate idea is to use \( N: H_0(G, M) \rightarrow H^0(G, M) \), so the map seems to go in the opposite direction.
Madsen's Constructions

Cyclotomic trace maps

\[
\text{Trc}: \quad K(R) \longrightarrow TC(R, p) \\
\text{Trc}: \quad A(X) \longrightarrow TC(X, p)
\]

Big Thm about cyclotomic trace is

\[
\text{Trc}: \tilde{A}(X)_p \cong \tilde{TC}(X, p)_p
\]

Basic properties

\[
\tilde{TC}(R, p)_p \cong \tilde{TC}(R \otimes \mathbb{Z}_p, p)_p \\
\tilde{TC}(M, R, p)_p \cong \tilde{TC}(R, p)_p
\]

Topological Hochschild homology:

\[
\tilde{TC}(R) = \left| B_{\text{spec}}^g (R, R) \right|
\]

\[
\begin{align*}
\text{sub} & \quad \sim \\
\text{by division} & \quad \left| B_{\text{spec}}^g (R^p, R^p_n) \right|
\end{align*}
\]

\[
\text{cyclic shift}
\]

This gives action of \( \mathbb{F} p_n = \mathbb{Z}/p^n \).

**FACTS:**

1. \( \tilde{TC}(R) \) \( C_{p^n} \)-equivariant spectrum

2. \( \Phi: \tilde{TC}(R)^{C_{p^n}} \longrightarrow \tilde{TC}(R)^{C_{p^n}} \) (some sort of transfer?)

3. \( D: \tilde{TC}(R)^{C_{p^n}} \longrightarrow \tilde{TC}(R)^{C_{p^n}} \) (inclusion of fixed subspaces)

4. \( D \Phi = D \Phi \), \( h\tilde{\text{fix}}(\Phi) = \tilde{TC}(R) \wedge EC_{p^n} \)

\[
\tilde{TC}(R, p) = \left( \text{holim}_{D} \tilde{TC}(R)^{C_{p^n}} \right)^{h\Phi} \quad \text{(This is a total holim over } \Phi, D \text{ maps)}
\]
Example \( R = QS^0 \)

\[
T(R)^{C_p^\infty} = Q(BG_{p^\infty}^+ \times \cdots \times Q(BG_{p^+}) \times Q(S^0)
\]

\[
\Phi(x_n, \ldots, x_0) = (x_{n+1}, \ldots, x_0)
\]

\[
D(x_n, \ldots, x_0) = (\text{trf}(x_n), \ldots, \text{trf}(x_1) + x_0)
\]

\[
\text{trf} : Q(BG_{p^k}) \to Q(BG_{p^{k+1}})
\]

transfer roll \( BG_{p^k} \to BG_{p^{k+1}} \).

\( \Phi \) is related to the geometric operation \( f : S^V \to S^V \) such that

\[
(f : S^V \to S^V) \mapsto (f^{C_p} : S^{V_{C_p}} \to S^{V_{C_p}})
\]

Note \( \lim_{D} \inf T(QS^0)^{C_p^\infty} = (QS^0)^{S^1} = \prod_{-\infty}^{\infty} Q(Z_{C_p}) \)

by a version of Segals calculation.

\[
(QS^0)^{C_p^G} = \prod_{H \leq G \leq H \in G} Q(BN_{C_p} H)
\]

if \( G \) finite.
\[ T(R) \]

\[ T(R)^{G_p^n} \]

\[ \xrightarrow{\text{wed}} T(C(R))^{G_{n-1}} \]

\[ T(R)^{G_p^n} \xrightarrow{\Phi} T(C(R))^{G_{n-1}} \]

\[ \downarrow \downarrow \]

\[ \Omega \]

\[ \ell \]

\[ T_k(R) \xrightarrow{\Phi} \text{Map}(S^{n_p^1} \land \ldots \land S^{n_k^1}, (RS^{n_p^1})^{G_p^n} \land \ldots \land (RS^{n_k^1})^{G_k^n}). \]

\[ f : S^{V_{G_p^n}} \rightarrow S^{V_{G_p^n}} \]

\[ \phi(t) = f_{G_p^n} : S^{V_{G_p^n}} \rightarrow S^{V_{G_p^n}} \]

\[ \text{holim} T(R)^{G_p^n} \cong \overline{\Omega} \]

\[ \text{holim} T(C(R))^{G_{n-1}} = (\Omega)^{G_{n-1}} \]

\[ \frac{\text{holim} T(C(R))^{G_{n-1}}}{\overline{\Omega}} = T(\Omega \times G_p^n) \]

\[ (\Omega)^{G_G} = T(\Omega \times G_H) \]

\[ \text{(finite)} \]
December 16, 1991

End homology theory of Pedersen-Winkel.

\( M = \) category of metric spaces and eventual Lipschitz maps. \( f: M \to M' \) is essential Lipschitz where \( E_k, l \) depending on \( f \) such that

\[ d(f(x), f(y)) \leq k \cdot d(x, y) + l \quad \forall x, y \in M \]

such an \( f \) need not be continuous.

\( X = \) category of compact subsets of \( S^\infty \)

contained in \( S^m \) for some \( m \) and Lipschitz maps. There is a functor \( X \to M \), \( x \mapsto O(x) \)

where \( O(x) \) is the cone on \( \mathcal{X} \).

Key point: If \( \mathcal{F} \mathcal{C} \) is a permutable category, one can define a K-spectrum \( K(M, \mathcal{F} \mathcal{C}) \) by using

chains \( \bigoplus_{x \in M} A_x \) which are locally finite and a suitable kind of bounded equivalence.
(Carlsson described this in the case of finitely generated free modules as follows. Take infinitely generated free modules with given basis \( s \), a map \( s \to M \) which is locally-finite. Use operators that are like infinite matrices with finitely many entries in each row and columns; maybe it's more like having finite band around the diagonal.

Basic result is that \( K(\mathbb{R}^n, \mathcal{F} \mathcal{C}), n \geq 0, \)

which are connected spectra (so view them as infinite loop spaces) is a non-connected delooping of \( K \mathcal{C} \).
as in Wagner-Karoubi delooping.

The way to say this maybe is that the Pedersen-Heichel bounded homology theory is a far reaching generalization of the Wagner-Karoubi result.

In certain cases one has (or would like)

\[ \text{locally finite } (M, K_{02}) \sim K(M, 02) \]

One needs $M$ to be locally contractible in some sense.

Important picture related to surgery (or $L$-theory)

\[ R \times M \]

\[ O(\text{pt} + M) \]
December 22, 1991

We have embeddings

\[ \text{[non-unital]} \rightarrow \text{[unital]} \rightarrow \text{[non-unital]} \]

\[ A \rightarrow \tilde{A} = \text{C} \otimes \text{A} \]

\[ A \rightarrow A \]

Given a functor \( F \) on unital algebras, let

\[ F'(A) = \text{Ker} \{ F(\tilde{A}) \rightarrow F(A) \} = \text{Coker} \{ F(A) \rightarrow F(\tilde{A}) \} \]

Given a functor \( G \) on non-unital algebras we just restrict it to unital algebras.

We've seen that there is a canonical map:

\[ F'(A) \rightarrow F(A) \]

\[ \downarrow \]

\[ F(\tilde{A}) \rightarrow F(A) \otimes F(\text{C}) \]

\[ \downarrow \]

\[ F(A) \rightarrow F(\text{C}) \]

so that \( F'(A) \rightarrow F(A) \iff F(A \otimes \text{C}) \rightarrow F(A) \otimes F(\text{C}) \).

I have adopted the view that cyclic theory, say \( HC \), is defined for unital algebras, and that because of the direct product theorem it extends to non-unital algebras. But suppose you took the other view that cyclic theory is defined for non-unital algebras primarily, say via Connes-Tsygan complex, or maybe by cyclic object theory. Then the issue is to see that

\[ 0 \rightarrow HC(A) \rightarrow HC(\tilde{A}) \rightarrow HC(\text{C}) \rightarrow 0 \]

is exact. This is a type of excision result.
Actually if HC is defined via the CT bicomplex, then one might be able to establish $\otimes$ by the method used in [10]. (Removing the $b'$-columns, then normalizing).

Perhaps what is worthwhile in this situation to observe is that it might be possible to understand degeneracies for cyclic objects (better: get some insight about degeneracies) using the cyclic objects $A^\otimes n$, $n \geq 0$ for $A$ non-unital.