July 3, 1990

Let X be a skew-adjoint operator, say a Dirac operator. In the Friedan-Windey paper there is mentioned the super symmetric time evolution operator

$$e^{\tau X + tX^2} = e^{tX^2} + \tau e^{tX^2} X$$

where t is an ordinary variable and τ is a Grassmann variable. What we have is the representation of the Lie supergroup $\mathbb{R}^{1,1}$ corresponding to the representation of the Lie superalgebra with odd generator acting as X. This Lie superalgebra has basis X, X^2 and these act in the indicated way.

Since the Lie superalgebra is generated by X I have the feeling that the Lie supergroup should be generated by the 1-parameter subgroup corresponding to X in some sense. This 1-param. subgroup consists of elements

$$e^{\tau X} = 1 + \tau X$$

where τ is a Grassmann variable. To make sense of this we consider independent Grassmann variables $\tau_1, \dots, \tau_n$ and take the product

$$\prod_{j=1}^{n} e^{\tau_j X}$$

Now

$$\tau_i X \tau_j X = - \tau_i \tau_j X^2$$

$$\tau_j X \tau_i X = -\tau_j \tau_i X^2 = \tau_i \tau_j X^2$$

so $[\tau_i X, \tau_j X] = -2\tau_i\tau_j X^2$. Note this commutes with any τ_k and X so it's central in algebra of operators generated by X and by the τ_k. Now recall

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}$$

if $[X,Y]$ commutes with X and Y. Thus we have

$$\prod_{j=1}^{n} e^{\tau_j X} = e^{\left(\sum_{j=1}^{n} \tau_j\right)X - \left(\sum_{i<j} \tau_i\tau_j\right)X^2}$$

$$= e^{\left(\sum \tau_j\right)X + \left(\sum_{i,j} a_{ij}\tau_i\tau_j\right)X^2}$$

where $$a_{ij} = \begin{cases} -\frac{1}{2} & i<j \\ 0 & i=j \\ +\frac{1}{2} & i>j \end{cases}$$

What I seem to find then is some sort of way of constructing the operator

$$e^{\tau X + t X^2}$$

as a product of $e^{\tau_j X} = (1+\tau_j X)$ where the τ_j are Grassmann variables. I am aiming for something like a path integral. What you are trying to do is to construct the time evolution operator using fermion increments

It seems interesting to understand the matrix $A = (a_{ij})$ above. Ultimately the bression $\sum a_{ij}\tau_i\tau_j$ is to be replaced by a positive number, or better, a complex number with positive real part.

If we work with the case where $i, j \in \mathbb{Z}$, then a_{ij} depends only on $i-j$ and A is translation invariant. Let T be the forward shift: $T_{ij} = \delta_{i,j+1}$. Then

$$A = \frac{1}{2}(T + T^2 + \cdots) - \frac{1}{2}(T^{-1} + T^{-2} + \cdots)$$

$$= \frac{1}{2}\frac{T}{1-T} - \frac{1}{2}\frac{T^{-1}}{1-T^{-1}}$$

$$= \frac{1}{2}\left(\frac{T}{1-T} - \frac{1}{T-1}\right) = \frac{1}{2}\frac{1+T}{1-T}$$

So A is the skew-adjoint operator with eigenvalue $\frac{1}{2}\frac{1+\zeta}{1-\zeta}$ on the $T=\zeta$ eigenspace.

We can work with $i, j \in \mathbb{Z}/n\mathbb{Z}$ where we consider vectors which are <u>anti periodic</u>, i.e. we have vectors indexed by $\mathbb{Z}/2n\mathbb{Z}$ and shifting by n changes the signs. Here we expect the same expression for A in terms of T, but the only eigenvalues of T are the odd $2n$-th roots of unity.

In the case with $1 \le i, j \le n$ one can see that the kernel of A is small, essentially because $(1-T)A = \frac{1}{2}(1+T)$. Thus a vector in the kernel should have components alternating in sign: $x, -x, x, -x,$ etc. with some adjustments at e ends. Thus the rank of A is large which means that the element $\sum a_{ij}\tau_i\tau_j$ has order roughly $n/2$.

The lesson is that the fermion path really does construct e^{tX^2} to all orders at $t=0$.

Another limiting possibility is to allow the index j to become continuous. In this case $(1-T)A = \frac{1}{2}(1+T)$ shows that A is a Green's operator for something like $\frac{1}{i}\frac{d}{dx}$.

July 5, 1990

Given A such that $\Omega^1 A$ is a projective bimodule, we [redacted] have the lifting property

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & B/I \end{tikzcd}$$

for any algebra B and nilpotent ideal $I \subset B$. Furthermore any two liftings induce the same map $H_0^{DR} A \longrightarrow H_0^{DR} B$.

(Proof of second statement: Goodwillie shows that $B \longrightarrow B/I$ induces an isomorphism on periodic cyclic homology. The assumption that $\Omega^1 A$ is a projective bimodule implies that the canonical map from the even periodic cyclic homology of A to $H_0^{DR}(A)$ is an isomorphism.)

Now I am interested in understanding this in [redacted] more detail. Specifically, suppose F is a functor on algebras and put

$$H_0 F(B) = \operatorname{Ker}\left\{ F(B) \rightrightarrows F(B \oplus \Omega^1 B) \right\}$$

When is it true that the above assertion for H_0^{DR} (the case $F(B) = B/[B,B]$) holds for $H_0 F$?

There are two points to be analyzed - the existence of the lifting and its uniqueness in some sense 'up to homotopy'.

It is natural to look at all liftings and to see what structure the set of these has. When A is a free algebra or free group algebra the set of liftings is naturally an affine space. In these cases one has a set S

of generators for A and a lifting is the same as a lifting of S:

$$\begin{tikzcd} & B \arrow[d] \\ S \arrow[r] \arrow[ur, dashed] & B/I \end{tikzcd}$$

which means that the space of liftings is an affine [illegible] space under I^S. (What structure does a coset $b+I$ have? One has addition by elements of I and multiplication by elements of $1+I$. The latter is not necessarily transitive, e.g. if $b+I=I$ one has a map $1+I \backslash I / 1+I \longrightarrow I/I^2$, so only the addition by I, that is, the affine space structure is relevant.)

Example: $A = \mathbb{C} \oplus \mathbb{C}e$. Here one is given an idempotent in B/I and the space of liftings is the space of all liftings of this idempotent to B.

July 6, 1990

Let A be an algebra such that $\Omega^1 A$ is a projective bimodule. Let B be an algebra, $I \subset B$ an ideal which is nilpotent, let $u_0 : A \longrightarrow B/I$ be a homomorphism. We study the set of liftings of u_0 to a homomorphism $u : A \longrightarrow B$.

Let's begin with the case $I^2 = 0$. Here I becomes an A-bimodule via u_0.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
I \arrow[r, equal] \arrow[d] & I \arrow[d] \\
E \arrow[r] \arrow[d] & B \arrow[d] \\
A \arrow[r] \arrow[d] & B/I \arrow[d] \\
0 & 0
\end{tikzcd}$$

Pull back gives a square zero extension of A, which is classified by an element of $H^2(A, I)$ $= \mathrm{Ext}^2_{A\otimes A^o}(A, I) = \mathrm{Ext}^1_{A\otimes A^o}(\Omega^1 A, I)$. This vanishes because $\Omega^1 A$ is projective. (There's an equivalence between $\Omega^1 A$ projective, $H^2(A,M) = 0$ for all A-bimods M, any square zero extension is a semi direct prod.) Hence liftings exist.

Two liftings differ by a derivation $A \longrightarrow I$ which is unique and can be arbitrary. Thus the set of liftings is an affine space under $\mathrm{Der}(A, I) = \mathrm{Hom}_{A\otimes A^o}(\Omega^1 A, I) = Z^1(A, I)$.

Suppose we want to construct a lifting concretely. We choose a linear lifting $\rho : A \to B$ of u_0 and look at its curvature $\omega : \bar{A}^{\otimes 2} \longrightarrow I$. This is a 2-cocycle:

$$\omega \in Z^2(A, I) \quad \text{or} \quad (b' - \mathrm{ad}(u_0))(\omega) = 0.$$

If this 2-cocycle is a coboundary

$$\omega = (b' - \mathrm{ad}(u_0))(\alpha) \qquad \alpha : \bar{A} \to I$$

then $\rho - \alpha : A \longrightarrow B$ has curvature zero, which means it is a homomorphism. Note:

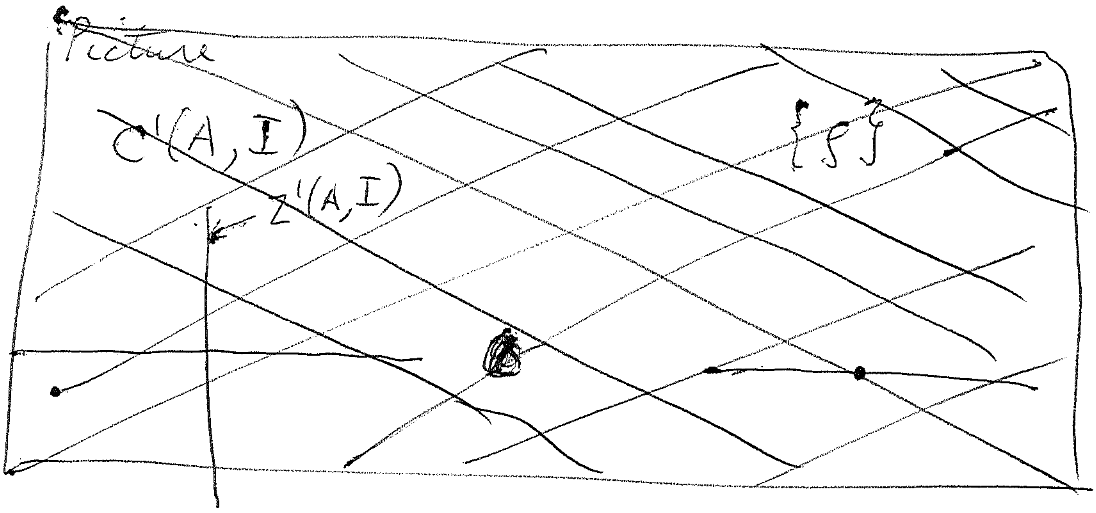
$$0 \to Z^1(A,I) \to C^1(A,I) \longrightarrow Z^2(A,I) \longrightarrow \underset{\substack{\parallel \\ 0}}{H^2(A,I)} \to 0$$

The set of linear liftings ρ is a torsor under $C^1(A,I) = \mathrm{Hom}(\bar{A}, I)$.

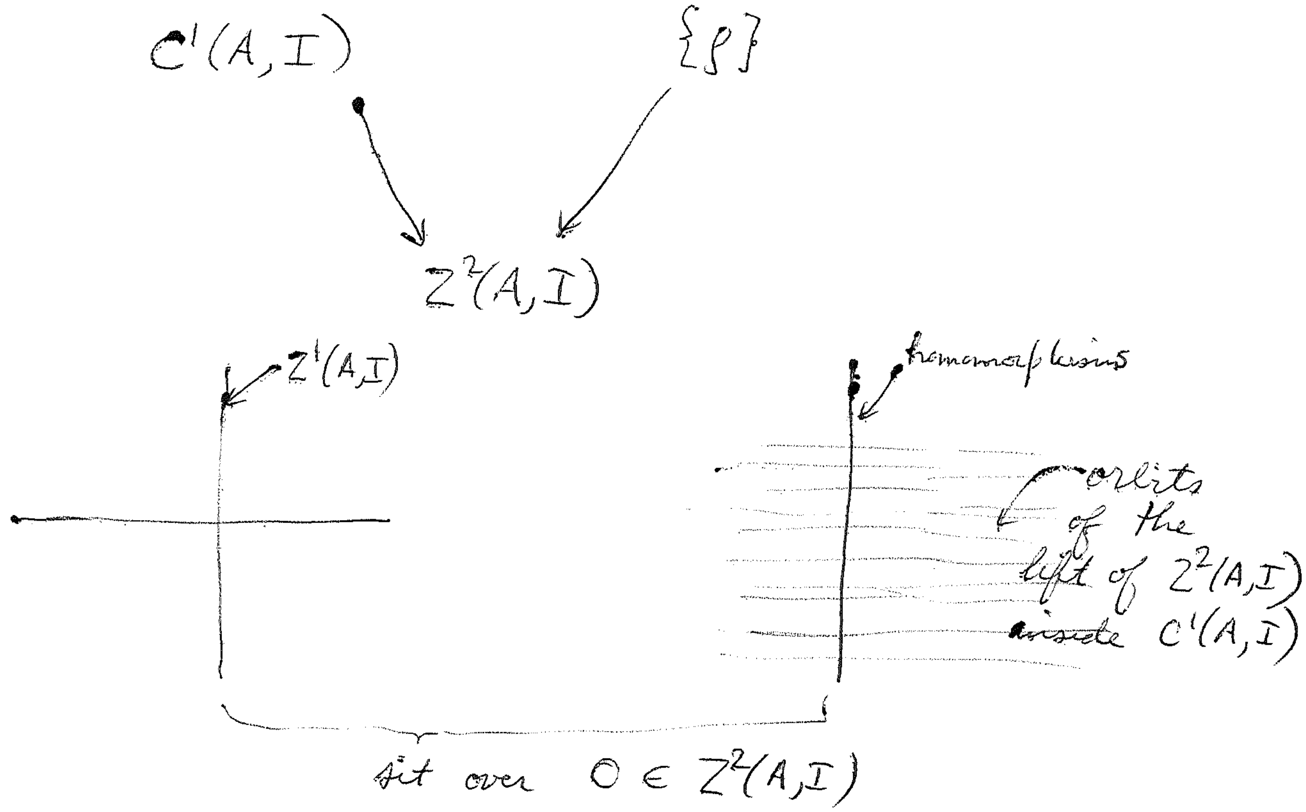
The exact sequence above is obtained by taking bimodule maps from

$$0 \leftarrow \Omega^1 A \longleftarrow A \otimes \bar{A} \otimes A \longleftarrow \Omega^2 A \longleftarrow 0$$

to I. A natural lifting from 2-cocycles to 1-cochains is the same as a splitting of the second exact sequence.



Picture



In other words there is a <u>retraction</u> of the <u>set of linear liftings onto</u> the subset of <u>homomorphisms</u>. This retraction depends on the splitting of the exact sequence

$$0 \longrightarrow \Omega^2 A \longrightarrow A \otimes \bar{A} \otimes A \longrightarrow \Omega^1 A \longrightarrow 0$$

I think this is all one can say about the first order cases.

Let's go on to second order.

$$\begin{tikzcd}
 & B/I^3 \arrow[d] \\
 & B/I^2 \arrow[d] \\
A \arrow[ruu, dashed, "u_2"] \arrow[ru, "u_1"'] \arrow[r, "u_0"'] & B/I
\end{tikzcd}$$

The first thing to note is that we have a principal fibring

$$\begin{array}{ccccc}
Der(A, I^2/I^3) & \longrightarrow & \{u_2\} & \longrightarrow & \{u_1\} \\
 & & & & \wr \\
 & & & & Der(A, I/I^2)
\end{array}$$

which indicates that the <u>space of liftings</u> $\{u_2\}$ <u>is nonsingular</u>.

Another way to see this is to exhibit a retraction of linear liftings onto lifting homomorphisms. Suppose $\rho : A \longrightarrow B/I^3$ is a linear lifting. It induces a homomorphism $\rho^{\#} : RA/IA^3 \longrightarrow B/I^3$. Suppose we fix a lifting homomorphism $\phi : A \longrightarrow RA/IA^3$. Then $\rho^{\#}\phi : A \longrightarrow B/I^3$ is a lifting homomorphism. Thus we have a map

$$\rho \longmapsto \rho^{\#}\phi$$

from linear liftings to lifting homoms. If ρ is already a homom., then $\rho^{\#}$ kills IA so we have

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[dr, equal] & RA/IA^3 \arrow[r, "\rho^{\#}"] \arrow[d, "\pi"] & B/I^3 \\ & A \arrow[ur, "\text{induced hom.}"'] & \end{tikzcd}$$

$$\therefore \quad \rho = \rho^{\#} \circ \rho_{univ} = \rho^{\#} \circ \phi$$

since $\rho_{univ} \equiv \phi \mod IA$. This shows we have a retraction.

The question arises as to whether there is a canonical, or fairly canonical choice for ϕ. The idea is that we have a canonical way to lift in square zero extensions starting from a linear lifting.

The situation is as follows. Suppose we start with

$$\begin{tikzcd}
 & B/I^3 \arrow[d] \\
 & B/I^2 \arrow[d] \\
A \arrow[r, "u_0"'] \arrow[uur, "\rho"] & B/I
\end{tikzcd}$$

Then we have a linear lifting $\rho_1 : A \longrightarrow B/I^2$ which we can modify to a lifting homomorphism $u_1 = \rho_1 - \alpha : A \longrightarrow B/I^2$. Next we need a linear lifting of u_1 and since ρ_1 is covered by ρ, this means we need to lift $\alpha : \bar{A} \longrightarrow I/I^2$ to a map $\bar{A} \longrightarrow I/I^3$.

This process introduces another choice to be made. In the case of RA we have a canonical linear splitting of the IA-adic filtration, so there is a definite way to proceed. However we also have the following possibility.

$$\begin{tikzcd}
 & B/I^8 \arrow[d] \\
 & B/I^4 \arrow[d] \\
 & B/I^2 \arrow[d] \\
A \arrow[r] \arrow[ur, dotted] \arrow[uur, dotted] & B/I
\end{tikzcd}$$

This is analogous to <u>Newton's method</u> vs. the usual implicit fn theorem. It's not clear that one ends up with the same lifting homomorphism.

So the conclusion appears to be that <u>the space of liftings is a nonsingular</u>

variety, a successive fibring by affine spaces.

Let us consider some examples.

In the case $A = \mathbb{C}[e] = \mathbb{C}[F]$ $e^2 = e$, $F^2 = 1$, we have a canonical retraction of linear liftings to homomorphisms. In this case RA is commutative, so there is a unique involution in $\widehat{RA}$ over $F \in A$.

Let's consider the example of a matrix algebra $A = \mathrm{End}(V)$. This is an example of a separable algebra, where separable means that A is projective as an A-bimodule. This is analogous to étale in the same way that $\Omega^1 A$ projective is analogous to smooth. In the commutative theory an etale algebra has unique lifting with respect to nilpotent extensions. This won't be true in the noncommutative case, but one can hope for uniqueness up to inner automorphisms.

Let us take $A = \mathrm{End}(V) = V \otimes V^*$ and show it is projective as an A-bimodule. This is easy because $A \otimes A^\circ$ is a matrix alg. hence any module is projective, but we want to be more specific. Let's construct a bimodule lifting

$$A \otimes A \overset{\longleftarrow}{\underset{\text{mult}}{\longrightarrow}} A$$

Consider

$$V \otimes V^* \otimes V \otimes V^* \longleftarrow V \otimes V^*$$

$$\sum_i v \otimes e_i^* \otimes e_i \otimes \lambda \longleftarrow\!\shortmid\; v \otimes \lambda$$

where e_i is a basis for V and e_i^* the dual basis.

This is compatible with left and right multiplication by elements of A. If we apply multiplication we get

$$\sum_i v \otimes e_i^*(e_i)\lambda = n\, v \otimes \lambda \qquad n = \dim V$$

Thus $v \otimes \lambda \mapsto v \otimes \frac{1}{n}\sum_i e_i^* \otimes e_i \otimes \lambda$ is a canonical section of the multiplication map $A \otimes A \to A$. There are lots of non canonical sections – just take any element of $V^* \otimes V$ which contracts to 1.

~~Why are derivations inner~~

$$\begin{tikzcd} 0 \arrow[r] & V \otimes V^* \arrow[r] \arrow[d, equal] & V \otimes V^* \otimes V \otimes V^* \arrow[r] \arrow[d, equal] & V \otimes V^* \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Omega^1 A \arrow[r] & A \otimes A \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

Why are derivations inner?

Assume A projective as A-bimodule, which means we have a sections of mult: $A \otimes A \longrightarrow A$. This is the same as an element $\sum x_i \otimes y_i \in A \otimes A$ satisfying

$$\sum_i a x_i \otimes y_i = \sum x_i \otimes y_i a \qquad \forall a \in A$$

and

$$\sum_i x_i y_i = 1$$

Example. $A = V \otimes V^*$, take

$$\sum x_i \otimes y_i = \sum e_j \otimes \alpha \otimes e_j^*$$

with $\alpha \in V^* \otimes V$ contracting to 1. Then

$$\sum x_i \otimes y_i a = \sum e_j \otimes \alpha \otimes \underset{\sum_k \langle j|a|k\rangle \underbrace{\langle k|}_{e_k^*}}{\underset{\shortparallel}{e_j^* a}}$$

$$= \sum_j \left(\sum_k e_j \langle j|a|k\rangle \otimes \alpha \otimes e_k^* \right)$$

$$= \sum_k a e_k \otimes \alpha \otimes e_k^* = \sum a x_i \otimes y_i$$

Anyway suppose we have $\sum x_i \otimes y_i$ with these two properties, then we get a splitting of

$$0 \longrightarrow \Omega^1 A \longrightarrow A \otimes A \longrightarrow A \longrightarrow 0$$

as follows: $\underbrace{1 \otimes 1 - \sum x_i \otimes y_i}$ $\longmapsto 0$

specifically

$$= \sum x_i y_i \otimes 1 - x_i \otimes y_i$$

$$= \sum x_i \, dy_i$$

Thus the bimodule map $A \otimes A \longrightarrow \Omega^1 A$ sending $1 \otimes 1$ to $\sum x_i \, dy_i$ should express any derivation as an inner derivation.

Check: Given $D: A \longrightarrow M$, to prove $Da = [a, \sum x_i D y_i]$. We have

$$\sum a x_i \otimes y_i = \sum x_i \otimes y_i a$$

$$\implies \sum a x_i D y_i = \sum x_i D(y_i a) = \sum x_i D y_i \, a + \underbrace{\left(\sum x_i y_i\right)}_{1} Da$$

so it works.

Let us now review the situation. We consider A such that $\Omega^1 A$ is projective and the lifting problem

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] \arrow[ur, dotted] & B/I \end{tikzcd}$$

where I is a nilpotent ideal in B.
We can assume $A = B/I$, i.e. we have an extension of algebras

$$0 \longrightarrow I \longrightarrow B \longrightarrow A \longrightarrow 0$$

with I nilpotent.
We can consider the liftings of A into B. The first point is that this is a nonsingular variety which is an iterated fibration with affine space fibres. Another point it seems is that when A is separable (i.e. A is projective as a bimodule, or equivalently, every derivation of A is inner), then any ~~two~~ liftings are related by by inner automorphisms associated to elements $1+x$ with $x \in I$.

Example. $A = \mathbb{C}[F]$, $F^2 = 1$.
Consider two liftings F, ε of an involution in B/I, which are involutions. One has

$$(F\varepsilon)^{1/2}\varepsilon = F(F\varepsilon)^{1/2} \qquad \text{(see below)}$$

and $(F\varepsilon)^{1/2} \equiv 1 \pmod{I}$, hence $(F\varepsilon)^{1/2} = 1+x$, $x \in I$

conjugates ε and F.

Next consider matrices: $A = M_n\mathbb{C}$. To keep things simple suppose $n=2$ and think of $M_2\mathbb{C}$ as the Clifford algebra C_2 generated by 2 anti-commuting involutions. Suppose we have two liftings γ_1, γ_2 and γ_1', γ_2' which are congruent modulo I. Then we have seen that we can conjugate by an element $1+x$, $x \in I$ and make $\gamma_1 = \gamma_1'$. Next we have

$$(\gamma_2'\gamma_2)^{1/2}\gamma_2 = \gamma_2(\gamma_2\gamma_2')^{1/2}$$

$$= \gamma_2(\gamma_2'\gamma_2)^{-1/2}$$

$$= \gamma_2'(\gamma_2'\gamma_2)(\gamma_2'\gamma_2)^{-1/2} = \gamma_2'(\gamma_2'\gamma_2)^{1/2}$$

but this conjugating element $(\gamma_2'\gamma_2)^{1/2}$ commutes with $\gamma_1 = \gamma_1'$.

Proof above

$$(F\varepsilon)^{1/2}\varepsilon = \varepsilon\,\varepsilon(F\varepsilon)^{1/2}\varepsilon = \varepsilon(\varepsilon F\varepsilon\varepsilon)^{1/2}$$

$$= \varepsilon(F\varepsilon)^{-1/2} = F F\varepsilon(F\varepsilon)^{-1/2} = F(F\varepsilon)^{1/2}$$

Here's a simpler conjugation

$$F\,\frac{1+F\varepsilon}{2} = \frac{F+\varepsilon}{2} = \frac{1+F\varepsilon}{2}\,\varepsilon.$$

The point of $(F\varepsilon)^{1/2} = g$ is that it is the unique g such that $g\varepsilon g^{-1} = F$ and $\varepsilon g \varepsilon = g^{-1}$, whereas $\frac{1+F\varepsilon}{2}$ has only the first property.

Here's a way to handle $M_n(\mathbb{C})$.
Regard it as the Heisenberg algebra of $\mathbb{Z}/n$. The generators are X, Y with relations $X^n = Y^n = 1$, $XYX^{-1} = \zeta Y$ where ζ is a primitive n-th root of 1.
Suppose we have two liftings; the first thing to do is to conjugate the X's.

$$X\left(\frac{1}{n}\sum_{i=0}^{n-1} X^i X_1^{-i}\right) = \left(\frac{1}{n}\sum_{i=0}^{n-1} X^i X_1^{-i}\right) X_1$$

Now when $X = X_1$ we conjugate the Y's in the same way using $\frac{1}{n}\sum_{i=0}^{n-1} Y^i Y_1^{-i}$. Note that this commutes with $X = X_1$, since

$$X\left(Y^i Y_1^{-i}\right) X^{-1} = (\zeta Y)^i (\zeta Y_1)^{-i} = Y^i Y_1^{-i}$$

Let us now return to the general "smooth" case, where $\Omega^1 A$ is projective. Let us consider two liftings and how to link them. Here there appears to be a need for homotopy rather than inner automorphisms, and this seems to be an important distinction

<u>July 7, 1990</u>

A 'smooth', $A \longrightarrow B/I$ given homom. where I nilpotent. Consider lifting homoms.

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[r] \arrow[ur, dashed] & B/I \end{tikzcd}$$

We know these exist and they form a "nonsingular variety" which is connected - it is a retract of the affine space of linear liftings. We now wish to discuss how one can connect two liftings.

First we might as well assume $A = B/I$, in other words, that we are dealing with nilpotent extensions of A. The <u>versal extension of order n</u> is RA/IA^{n+1} which is isomorphic to $\Omega^{ev}/\Omega^{ev,>2n} \simeq \Omega^0 \oplus \Omega^2 \oplus \cdots \oplus \Omega^{2n}$ under $*$ product. The <u>universal extension of order n with two liftings</u> is $QA/JA^{n+1} \simeq \Omega^0 \oplus \Omega^1 \oplus \cdots \oplus \Omega^n$ ($*$ product)

Suppose F is a functor on nilpotent extensions of A, and put $H^0F(A) = \mathrm{Ker}\{F(A) \rightrightarrows F(A \oplus \Omega^1)\}$. We can then ask whether, for any nilpotent extension $B/I = A$ and lifting $A \to B$, is it true that the induced map

$$H^0F(A) \subset F(A) \longrightarrow F(B)$$

is independent of the choice of lifting. Dually suppose F contravariant and put

$$H_0F(A) = \mathrm{Coker}\{F(A\oplus\Omega^1A) \rightrightarrows F(A)\}$$

Is it true that for $B/I = A$, $I^N = 0$, and lifting $A \to B$ the induced map

$$F(B) \longrightarrow F(A) \twoheadrightarrow H_0F(A)$$

is independent of the choice of lifting. If we take $F(X) = \mathrm{Hom}(X, B)$, then

$$F(A) = \mathrm{Hom}(A,B) = \{\text{liftings of } A \text{ into } B\},$$

and $H_0F(A)$ is the quotient of the set of these liftings modulo the equivalence relation generated by the relation where two liftings $A \rightrightarrows B$ are related if they come from a homom. $A\oplus\Omega^1A \to B$

The question then is whether we can join any two liftings $A \rightrightarrows B$ by a chain of liftings each consecutive pair related by a homomorphism $A\oplus\Omega^1A \longrightarrow B$. Let's look at this in the case of the universal 2nd order extension with two liftings $Q/J^3 = A\oplus\Omega^1\oplus\Omega^2$. The two liftings are $a \mapsto a \pm da$. Start with $a \mapsto a+da$ and ask what we can get to.

$$\begin{tikzcd}
A \arrow[r, "a \mapsto a+da"', "\phi"] \arrow[d, shift right, "1"'] \arrow[d, shift left, "1+\delta"] & Q/J^3 \\
A\oplus\Omega^1A \arrow[ur, dashed, leftarrow] &
\end{tikzcd}$$

We are after a derivation $D: A \longrightarrow J/J^3$ such that $(\tilde{D}\,\Omega^1A)^2 = 0$. Now $(\tilde{D}\Omega^1A)^2 = (\phi A) DA\,DA$ so we must have $(DA)^2 = 0$. What can D be? Look modulo J^2: $Q/J^2 = A\oplus\Omega^1A$ and the only derivations $D: A \longrightarrow \Omega^1 A$ around in general are multiples of d. Next the $*$ product on $J/J^3 = \Omega^1 \oplus \Omega^2$ reduces to the ~~obvious~~ usual product $\Omega^1 \otimes \Omega^1 \to \Omega^2$. Thus there seems to be no way to arrange $(DA)^2 = 0$, ~~[illegible]~~ unless we have $DA \subset J^2/J^3$, in which case the lifting to first order is constant.

Thus we learn that it's not possible to join liftings as we have been trying to.

Notice: Consider a 1-parameter family of homoms. $u_t : A \longrightarrow B$ which is linear in t:

$$u_t(a) = u(a) + t\,w(a)$$

Then necessarily w is a derivation with respect to u:

$$\big(u(a_1) + t\,w(a_1)\big)\cdot\big(u(a_2) + t\,w(a_2)\big) = u(a_1 a_2) + t\,w(a_1 a_2)$$

$$\parallel$$

$$u(a_1 a_2) + t\big(u(a_1)\,w(a_2) + w(a_1)\,u(a_2)\big) + t^2\,w(a_1)\,w(a_2)$$

and $w(a_1)w(a_2) = 0$. Thus we see that $A \xrightarrow{1+td} A \oplus \Omega^1 A$ <u>is a universal linear 1-parameter family of homomorphisms.</u>

This shows the necessity of <u>polynomial homotopy</u> in order to join different liftings. The degree is related to the order of nilpotence

July 9, 1990

I would now like to develop periodic cyclic homology from scratch using lifting for smooth algebras relative to nilpotent extensions. Given an algebra A we choose an extension $A = R/I$ with R smooth and consider the homology of $L(R)^\wedge = \varprojlim L(R/I^n)$. We wish to show this is independent of the choice of R up to canonical isomorphism and that it is functorial in A. Supposing this done, let us prove Goodwillie's theorem that if $A = B/J$ with J nilpotent then $HP(B) \xrightarrow{\sim} HP(A)$. Here HP denotes periodic cyclic homology defined by $HP(A) = H(L(R)^\wedge)$ with $L(R)^\wedge$ as above.

Given $A = B/J$, $J^n = 0$ choose an extension $R/I_1 = B$ with R smooth. Then $A = R/I$, where $I_1 \subset I$ and $I/I_1 = J$. Thus $I^n \subset I_1 \subset I$ so the I_1-adic and I-adic completions of $L(R)$ are the same. Thus $HP(A) = HP(B)$ since these are both canonically isomorphic to the homology of $L(R)^\wedge$.

Let us consider the ~~[illegible]~~ sort of thing that occurs in algebraic geometry with nilpotent extensions. The idea I have is the following. I notice that the periodic cyclic homology of A depends on a lot less than A, rather it is the same for A and any nilpotent extension. Let us take the category of algebras and surjections with nilpotent kernel and fix a component in this category. I want to understand this category. The idea is that this category ought to be equivalent to something nonsingular acted on by derivations. The example to imitate is $\mathcal{D}$-modules, where one considers a nonsingular variety with the sheaf of differential operators. Somehow I want to replace A by $\hat{R}$ with derivations acting on it.

Let's adopt the following viewpoint. Given the algebra A we consider ~~smooth the algebras~~ extensions $A = R/I$ with R smooth, and ultimately we want to replace R by its completion. This means we look at topological algebras R which are smooth and which have an ideal of definition I such that $A = R/I$. The next question is what are the morphisms? The idea should be to consider something more general than just homomorphisms over A

$$\begin{tikzcd} R \arrow[rr] \arrow[dr] & & R' \arrow[dl] \\ & A & \end{tikzcd}$$

What could we mean? If A is a nilpotent extension of A_0, then we [redacted] should allow an arbitrary homomorphism over A_0.

$$\begin{tikzcd} R \arrow[r, "u"] \arrow[d] & R' \arrow[d] \\ A \arrow[d] & A \arrow[d] \\ A_0 \arrow[r, equal] & A_0 \end{tikzcd}$$

[redacted] Let's recall that any algebra has a largest nilpotent ideal called the nilradical. [redacted] If we require $Nil(A) = 0$, then $A = A_0$, so we just have homomorphisms over A.

Problem: If A is smooth does it follow that $Nil(A) = 0$? One has to show that if I is an ideal of square 0 in A, then $I = 0$. We know that any ideal in A is left and right flat, and probably left and right projective by a similar argument. Thus $0 \to I \to A$ yields $0 \to I \otimes_A I \to I$ showing that $I \otimes_A I = 0$. This is as far as I got.

So far I have made little progress toward understanding the categorical setup of nilpotent extensions.

Let's return to the problem of showing that if $A = R/I$ with R smooth, then $H_\bullet(L(R)^\wedge)$ is independent of the choice of R. Let's take two of these extensions

$$R/I = A = R'/I'$$

We ought to be able to find a map of inverse systems $R/I^n \longrightarrow R'/I'^n$. This is clear because we can lift successively

$$\begin{tikzcd} & R'/I'^n \arrow[d] \\ R \arrow[ur, dashed, "u_n"] \arrow[r] & A \end{tikzcd}$$

and necessarily $u_n(I) \subset I'/I'^n$ so [redacted] u_n induces a map $R/I^n \longrightarrow R'/I'^n$. The issue is now to show [redacted] the independence of the lifting for the induced map on the homology $H_\bullet(L(R)^\wedge) \longrightarrow H_\bullet(L(R')^\wedge)$.

We've learned something new in the past few days, namely that the liftings u_n can be obtained as follows. First one chooses a linear [redacted] map $\rho : R \to R'$ over A. Then one extends it to $R(R) \longrightarrow R'$; better to say that ρ induces a homomorphism $R(R) \longrightarrow R'$ over A. Then using the fact that R is smooth one has a lifting hom. $R \longrightarrow \varprojlim R(R)/I(R)^n$. Thus we get from ρ

a homomorphism

$$R \longrightarrow \varprojlim R(R)/I(R)^n \longrightarrow \varprojlim R'/I'^n$$

Thus we are retracting a linear lifting $\rho: R \to R'$ to a homomorphism $R^\wedge \longrightarrow R'^\wedge$.

Now a similar process works with two liftings and homotopy.

Let us consider the basic situation we encounter when we try to use "smooth coverings" of A to define periodic cyclic homology. We consider extensions $A = R/I$ with R-smooth and associate to such an extension the complex $L(R)^\wedge$. A morphism $R \to R'$ over A induces a map of complexes $L(R)^\wedge \longrightarrow L(R')^\wedge$. The homotopy class of this map is the same for all maps $R \to R'$. Why? Because we know the space of homomorphisms $R \to R'$ (over A) is a retract of an affine space and we know how to construct a homotopy corresponding to a polynomial path

So the basic situation is shaping up as follows. Let's return to the lifting problem

$$\begin{tikzcd} & B \arrow[d] \\ A \arrow[ur, dashed] \arrow[r] & B/I \end{tikzcd}$$

I nilpotent

A smooth

For each lifting homomorphism we have an induced map of complexes $L(A) \longrightarrow L(B)$. So we have a family of maps of complexes parametrized by our space S of lifting homomorphisms. I guess we have a map from S into $\mathrm{Hom}^0(L(A), L(B))$. A better way to say this is that we have a map of complexes

$$L(A) \longrightarrow L(B) \otimes (\text{fns. on } S)$$

$$\| \text{ (better)}$$

$$\Gamma_{poly}(S, L(B)))$$

It seems that to associate to a tangent vector ξ to S a degree -1 operator ξ which is a contracting homotopy for the derivative map [redacted]

[redacted] $L_\xi : L(A) \longrightarrow L(B)$ associated to ξ

is to define a map

$$L(A) \longrightarrow \Omega^1_{poly}(S, L(B))$$

of degree -1 but not necessarily compatible with differentials.

It would be nice [redacted] if we have a map of complexes

$$L(A) \longrightarrow \Omega_{poly}(S, L(B)).$$

The geometry should be very interesting.

July 10, 1990

Digression to ~~[illegible]~~ recall previous work on tensor products.

Given two algebras A, B consider their free product $A*B$. This admits a filtration indexed by words in two letters without repetitions

$$\begin{tikzcd}
 & A \arrow[r, hook] \arrow[dr, hook] & AB \arrow[r, hook] \arrow[dr, hook] & ABA \arrow[r, hook] \arrow[dr, hook] & {} \\
\mathbb{C} \arrow[ur, hook] \arrow[dr, hook] & & & & \\
 & B \arrow[r, hook] \arrow[uur, hook] & BA \arrow[r, hook] \arrow[uur, hook] & BAB \arrow[r, hook] \arrow[uur, hook] & {}
\end{tikzcd}$$

and the associated graded is the direct sum of

$$\begin{array}{cccc}
 & \bar{A} & \bar{A}\otimes\bar{B} & \bar{A}\otimes\bar{B}\otimes\bar{A} \\
\mathbb{C} & & & \\
 & \bar{B} & \bar{B}\otimes\bar{A} & \bar{B}\otimes\bar{A}\otimes\bar{B}
\end{array}$$

To prove this one wants to make $A*B$ act in some nice way on the direct sum and then ~~[illegible]~~acting on 1 gives an additive isom. This requires splitting $0\to\mathbb{C}\to A\to\bar{A}\to 0$ and similarly for B. However the following can be done without making such a choice

Use the cochains $a\otimes b \longmapsto [a,b]$, $\bar{A}\otimes\bar{B}\longrightarrow A*B$ to set up a canonical map

$$\bigoplus_{n\geq 0} A\otimes B\otimes(\bar{A}\otimes\bar{B})^{n} \longrightarrow A*B$$

$$(a_0, b_0, a_1, \dots, b_n) \longmapsto a_0b_0[a_1,b_1]\cdots[a_n,b_n]$$

One shows this is an isomorphism by making $A*B$ act on the direct sum on the right. Rules: $a(a_0b_0) = (aa_0)b_0$

$$b\,a_0b_0 = b[a_0,b_0] + bb_0a_0$$

$$= b[a_0,b_0] - [a_0, bb_0] + a_0bb_0$$

$$\boxed{b\,a_0b_0 = a_0(bb_0) - [a_0, bb_0] + b[a_0,b_0]}$$

Thus the rules for the action are

$$a(a_0,b_0,a_1,\ldots,b_n) = (aa_0, b_0, a_1,\ldots, b_n)$$

$$\begin{aligned} b(a_0,b_0,a_1,\ldots,b_n) = {} & (a_0, bb_0, a_1,\ldots,b_n) \\ & - (1,1,a_0,bb_0,a_1,\ldots,b_n) \\ & + (1,b,a_0,b_0,a_1,\ldots,b_n) \end{aligned}$$

One can check this gives an action of A and of b on the direct sum, etc.

<u>Brief idea:</u> If one chooses splittings of $0 \to C \to A \to \bar{A} \to 0$ and ditto for B, presumably one can ~~defi~~ construct an isom of $A*B$ with $C \oplus \bar{A} \oplus \bar{B} \oplus \cdots$. If we do this for RA it might have some bearing on the Chern-Simons deformation.

Choose $\pi : A \to k$ with $\pi 1 = 1$.

Get $T(\bar{A}) = \bigoplus_{n\geq 0} \bar{A}^{\otimes n} \longrightarrow RA$, $(a_1,\ldots,a_n) \longmapsto (\rho a_1 - \pi a_1)\cdots(\rho a_n - \pi a_n)$

Action of ρa given by the rule

$$\rho a(\rho a_1 - \pi a_1) = (\rho a - \pi a)(\rho a_1 - \pi a_1) + \pi a(\rho a_1 - \pi a_1)$$

or

$$\rho a(a_1,\ldots,a_n) = (a, a_1,\ldots,a_n) + \pi a(a_1,\ldots,a_n)$$

What to do about $A\otimes B$:
Let's think of A, B as smooth so that their cyclic theory ~~arises~~ given by the mixed complex

$$\Omega^1 A_\natural \rightleftarrows A \qquad \Omega^1 B_\natural \rightleftarrows B$$

The tensor product of these mixed complexes

$$\Omega^1 A_\natural \otimes \Omega^1 B_\natural \rightleftarrows \Omega^1 A_\natural \otimes B \oplus A \otimes \Omega^1 B_\natural \rightleftarrows A\otimes B$$

should then give the cyclic theory of $A\otimes B$. On the other hand $A*B$ is also ~~smooth~~ as

$$\Omega^1(A*B) = (A*B)\otimes_A \Omega^1 A \otimes_A (A*B)$$
$$\oplus\ (A*B)\otimes_B \Omega^1 B \otimes_B (A*B)$$

so if $I = \operatorname{Ker}\{A*B \longrightarrow A\otimes B\}$, then we obtain the periodic cyclic homology of $A\otimes B$ by I-adically completing the mixed complex $\Omega^1 R_\natural \rightleftarrows R$, where $R = A*B$

The other thing we know is this. Put $S = A\otimes B$, $R = A*B$, $S = R/I$ where I is generated by $[A,B]$. We have an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow S\otimes_R \Omega^1 R \otimes_R S \longrightarrow \Omega^1 S \longrightarrow 0$$

where $S\otimes_R \Omega^1 R \otimes_R S = \Omega^1 A\otimes(B\otimes B)\oplus(A\otimes A)\otimes \Omega^1 B$ and $\Omega^1 S = \Omega^1(A\otimes B)$,

and $I/I^2 \simeq \Omega^1 A \otimes \Omega^1 B$.

It appears therefore that ~~Hochschild~~ the two resolutions for the $S\otimes S^\circ$ bimodule S

$$0 \to I/I^2 \to S\otimes_R \Omega^1 R \otimes_R S \to S\otimes S \to S \to 0$$

$$(0\to\Omega^1A\to A\otimes A \to A\to 0)\otimes(0\to\Omega^1 B\to B\otimes B\to B\to 0)$$

are the same.

The issue seems to be the following. We have $L(R) = (\Omega^1 R_\natural \rightleftarrows R)$ with its I-adic filtration which we want to relate to $(\Omega^1 A_\natural \rightleftarrows A)\otimes(\Omega^1 B_\natural \rightleftarrows B)$. This is going to be possible because the Hochschild homology stops at degree 2. But it first seems necessary to link the I adic filtration on $L(R)$ to Hochschild homology, which is an old problem.

Example. $A = k[\mathbb{Z}]$, $B = k[\mathbb{Z}]$. Here there is interesting cyclic homology in degree 2.

Additional problems.

Does any algebra have the same periodic cyclic homology as some smooth algebra? (Perhaps need Hoch homology $=0$ for degrees $\gg 0$).

"<u>Smooth coverings</u>" Here one might use partitions of unity, or some version of Morita equivalence, besides writing ~~the~~ the algebra as a quotient of a smooth alg.

"<u>Partitions of unity</u>" and the Tate setup where $R = I + J$.

July 12, 1990

Let A be an algebra. Using the tensor product operation $X \otimes_A Y$ on bimodules we can define a coalgebra in the category of A-bimodules to be a bimodules C equipped with bimodule maps

$$C \xrightarrow{\varepsilon} A \qquad \Delta : C \longrightarrow C \otimes_A C$$

satisfying the usual conditions:

$$C \xrightarrow{\Delta} C \otimes_A C \underset{1 \otimes \varepsilon}{\overset{\varepsilon \otimes 1}{\rightrightarrows}} C$$ both compositions are 1

$$C \xrightarrow{\Delta} C \otimes_A C \underset{1 \otimes \Delta}{\overset{\Delta \otimes 1}{\rightrightarrows}} C \otimes_A C \otimes_A C$$ both compositions coincide

An algebra R in the category of A-binods is a bimodule with maps

$$A \longrightarrow R \qquad R \otimes_A R \longrightarrow R$$

satisfying the usual conditions. This means the image of $1 \in A$ in R is the identity of R, and that $A \longrightarrow R$ is an algebra homom. Thus R is just an algebra under A.

Given such a C and R we consider bimodule maps $C \longrightarrow R$, and define the "convolution" product $\varphi * \psi$:

$$C \xrightarrow{\Delta} C \otimes_A C \xrightarrow{\varphi \otimes \psi} R \otimes_A R \longrightarrow R.$$

This makes $\operatorname{Hom}_{A\text{-binod}}(C, R)$ into an algebra

Example: Let $C = A \otimes A$ with $A \otimes A \xrightarrow{\mu} A$ the multiplication and

$$A \otimes A \xrightarrow{\Delta} (A \otimes A) \otimes_A (A \otimes A)$$
$$(a_1, a_2) \longmapsto (a_1, 1) \otimes_A (1, a_2)$$

Then $\mathrm{Hom}_{A\text{-bim}}(A \otimes A, R) = R$ as algebras. (Notice that a bimodule map $\varphi : A \otimes A \longrightarrow R$ is the same as the element $\varphi(1 \otimes 1) \in R$, and that $\Delta(1,1) = (1,1) \otimes_A (1,1)$ so that

$$(\varphi * \psi)(1,1) = \varphi(1,1)\,\psi(1,1).)$$

Example. Tensor coalgebra on a bimodule. Let M be an A-bimodule and C an A-bimodule coalgebra. Suppose we have a bimodule morphism $u : C \longrightarrow M$. Then we obtain maps

$$u^{\otimes n} : C \longrightarrow M \otimes_A \overset{n}{\cdots} \otimes_A M$$

for $n \geq 0$ given by the composition

$$C \xrightarrow{\Delta^{(n)}} C \otimes_A \overset{n}{\cdots} \otimes_A C \xrightarrow{u \otimes_A \overset{n}{\cdots} \otimes_A u} M \otimes_A \cdots \otimes_A M.$$

Put $M^{(n)} = M \otimes_A \overset{n}{\cdots} \otimes_A M$. We have canon. isomorphisms

$$M^{(i+j)} = M^{(i)} \otimes_A M^{(j)}$$

which allow us to define a coproduct

$$\bigoplus_{n \geq 0} M^{(n)} \xrightarrow{\Delta} \Big(\bigoplus_{i \geq 0} M^{(i)}\Big) \otimes_A \Big(\bigoplus_{j \geq 0} M^{(j)}\Big)$$

or $T_A(M) = \bigoplus_{n\geq 0} M^{(n)}$. We should check that this defines an $\underset{\text{A-bimodule}}{}$ coalgebra.

Δ is given by $(pr_i \otimes_A pr_j)\Delta = pr_{i+j}$.

Thus

$$(pr_i \otimes_A pr_j \otimes_A pr_k)(\Delta \otimes_A 1)\Delta = (pr_{i+j} \otimes_A pr_k)\Delta = pr_{i+j+k}$$

$$(\quad " \quad)(1 \otimes_A \Delta)\Delta = (pr_i \otimes_A pr_{j+k})\Delta = pr_{i+j+k}$$

Thus $T_A(M)$ is simultaneously an A-bimodule coalgebra and algebra. We should have

$$\operatorname{Hom}_{\substack{A\,\text{bim}\\ \text{coalgs}}}(C, T_A(M)) = \operatorname{Hom}_{A\,\text{bim}}(C, M)$$

assuming some condition like the primitive filtration on C is exhaustive.

Question: What sort of compatibility should be expected between the coalgebra and algebra structures on $T_A(M)$?

Another example: Let us consider a coalgebra C and a C-bicomodule, all of these being in the category of A-bimodules. Then it should be possible to form a different tensor coalgebra

$$C \oplus E \oplus (E \otimes^C E) \oplus (E \otimes^C E \otimes^C E) \oplus \cdots$$

To be specific let us consider $C = A \otimes A$ as above and $E = A \otimes \bar{A} \otimes A$ with left and right comultiplication defined by

$$E \xrightarrow{\Delta_L} C \otimes_A E$$

$$A\otimes\bar{A}\otimes A \longrightarrow (A\otimes A)\otimes_A (A\otimes\bar{A}\otimes A)$$

$$(x, a, y) \longmapsto (x, 1)\otimes_A (1, a, y)$$

$$E \xrightarrow{\Delta_R} E\otimes_A C$$

$$A\otimes\bar{A}\otimes A \longrightarrow (A\otimes\bar{A}\otimes A)\otimes_A (A\otimes A)$$

$$(x, a, y) \longmapsto (x, a, 1)\otimes_A (1, y)$$

Let's compute $E\otimes^C E = \mathrm{Ker}\left\{ E\otimes_A E \underset{1\otimes_A\Delta_L}{\overset{\Delta_R\otimes_A 1}{\rightrightarrows}} E\otimes_A C\otimes_A E \right\}$

$$E\otimes_A E \qquad\qquad E\otimes_A C\otimes_A E$$

$$(A\otimes\bar{A}\otimes A)\otimes_A (A\otimes\bar{A}\otimes A) \underset{1\otimes\Delta_L}{\overset{\Delta_R\otimes 1}{\rightrightarrows}} (A\otimes\bar{A}\otimes A)\otimes_A (A\otimes A)\otimes_A (A\otimes\bar{A}\otimes A)$$

$$(x, a, y)\otimes_A (x', a', y') \xmapsto{\Delta_R\otimes 1} (x, a, 1)\otimes_A (1, y)\otimes_A (x', a', y')$$

$$= (x, a, 1)\otimes_A (1, 1)\otimes_A (yx', a', y')$$

$$'' \xmapsto{1\otimes\Delta_L} (x, a, y)\otimes_A (x', 1)\otimes_A (1, a', y')$$

$$= (x, a, yx')\otimes_A (1, 1)\otimes_A (1, a', y')$$

Now let us identify

$$E\otimes_A E = A\otimes\bar{A}\otimes A\otimes\bar{A}\otimes A$$

$$(x, a, y)\otimes_A (x', a', y') \longleftrightarrow (x, a, yx', a', y')$$

$$E\otimes_A C\otimes_A E = A\otimes\bar{A}\otimes A\otimes A\otimes\bar{A}\otimes A$$

$$(x, a, y)\otimes_A (1, 1)\otimes_A (x', a', y') \longleftarrow (x, a, y)\otimes(x', a, y')$$

Then we have the maps

$$A \otimes \bar{A} \otimes A \otimes \bar{A} \otimes A \overset{\Delta_R \otimes 1}{\underset{1 \otimes \Delta_L}{\rightrightarrows}} A \otimes \bar{A} \otimes A \otimes A \otimes \bar{A} \otimes A$$

$$(x, a, yx', a', y') \longmapsto (x, a, 1, yx', a', y')$$
$$\longmapsto (x, a, yx', 1, a', y')$$

~~...~~ Thus we have the two maps

$$A \rightrightarrows A \otimes A \qquad z \longmapsto 1 \otimes z, \quad z \longmapsto z \otimes 1$$

tensored on the left with $A \otimes \bar{A}$ and on the right with $\bar{A} \otimes A$. It follows that

$$E \otimes^C E = A \otimes \bar{A} \otimes \bar{A} \otimes A$$

$$(x, a_1, 1) \otimes_A (1, a_2, y) \longleftarrow (x, a_1, a_2, y)$$

July 13, 1990

Basic principle is that projective resolutions exist, that there is a map from one to another which is unique up to homotopy. Apply this principle in the case of a projective resolution P of the $A \otimes A^{\circ}$-module A. We have a canonical map $\varepsilon : P \longrightarrow A$, and we can form other projective resolutions

$$P \otimes_A P \otimes_A \cdots \otimes_A P \xrightarrow{\varepsilon \otimes \cdots \otimes \varepsilon} A.$$

There have to be maps unique up to homotopy between them. So one has a map $P \longrightarrow P \otimes_A P$ and a homotopy joining $\varepsilon \otimes 1$ and $1 \otimes \varepsilon : P \otimes_A P \longrightarrow P$.

These resolutions form some sort of contractible 'category' or space on which one has an associative product.

Next one forms the 'fixpoints' $(P \otimes_A)^n$, $n \geq 0$. These complexes are all homotopy equivalent, but they can be assembled somehow to produce cyclic homology. We don't really understand this mechanism from the derived category viewpoint.

Let us take B to be the standard normalized bar resolution: $B_n = A \otimes \bar{A}^{\otimes n} \otimes A$, differential b'. According to yesterdays calculations this ought to be a tensor coalgebra in the category of A-bimodules.

Recall that $B_0 = A \otimes A$ is such a coalgebra and notice that $V \otimes A$, $A \otimes W$ are right and left $A \otimes A$-comodules:

$$V \otimes A \longrightarrow (V \otimes A) \otimes_A (A \otimes A)$$
$$(v, a) \longmapsto (v, 1) \otimes (1, a) \qquad \text{etc.}$$

(V, W are vector spaces)

<u>Lemma:</u> $(V \otimes A) \otimes^{(A \otimes A)} (A \otimes W) = V \otimes W$

<u>Proof.</u> On the left is the kernel of the pair of maps

$$(V \otimes A) \otimes_A (A \otimes W) \rightrightarrows (V \otimes A) \otimes_A (A \otimes A) \otimes_A (A \otimes W)$$

[redacted] [redacted]

$$(v, a_1) \otimes (a_2, w) \longmapsto (v, 1) \otimes_A (1, a_1) \otimes_A (a_2, w)$$

[redacted]

$$\longmapsto (v, a_1) \otimes_A (a_2, 1) \otimes_A (1, w)$$

Let's identify the spaces on the right + left in the obvious way. Then we get

$$V \otimes A \otimes W \rightrightarrows V \otimes A \otimes A \otimes W$$
$$(v, a, w) \longmapsto (v, 1, a, w)$$
$$\longmapsto (v, a, 1, w)$$

and the kernel is easily seen to be $V \otimes W$.

Now take $E =$ [redacted] $A \otimes V \otimes A$ and consider it as a bicomodule over $C = A \otimes A$. Form the tensor coalgebra

$$T = C \oplus E \oplus (E \otimes^C E) \oplus \cdots$$

By the lemma $E \otimes^C E = A \otimes V^{\otimes 2} \otimes A$, hence $T_n = A \otimes V^{\otimes n} \otimes A$ for all n. It's clear that the coproduct is

$$\Delta(a, v_1, \ldots, v_n, a') = \sum_{i=0}^{n} (a, v_1, \ldots, v_i, 1) \otimes_A (1, v_{i+1}, \ldots, v_n, a')$$

[redacted] Next take $V = \bar{A}$. We have a bimodule map $E \to C$:

$$A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A$$

which should [redacted] extend as a degree -1 (anti-) coderivation on T of square zero. Assume this works and that it gives b'. Thus $\mathcal{B}$ has this DG-coalgebra-in-[redacted] the-category-of-A-bimodules interpretation.

Look at the cocommutator subspace

$$\mathrm{Ker}\left\{ \mathcal{B}\otimes_A \xrightarrow{\Delta - \sigma\Delta} (\mathcal{B}\otimes_A)^2 \right\}$$

Take degree 0

$$\begin{tikzcd}
(a_1, a_2) \in (A \otimes A)\otimes_A \arrow[r, "\Delta"] \arrow[d, mapsto] \arrow[d, equal, shift left=8] & (A\otimes A)\otimes_A (A \otimes A)\otimes_A \arrow[d, equal] & (a_1, a_2)\otimes_A (a_3, a_4)\otimes_A \arrow[d, mapsto] \\
a_2 a_1 \qquad A & A \otimes A & (a_4 a_1, a_2 a_3)
\end{tikzcd}$$

$$\begin{tikzcd}
(a, 1) \arrow[r, "\Delta"] & (a, 1)\otimes_A (1, 1)\otimes_A \arrow[r, "\sigma"] & (1,1)\otimes_A (a,1)\otimes_A \arrow[d, equal] \\
a \arrow[u, mapsto] & (a \otimes 1) \arrow[u, mapsto] & (1, a)\otimes_A (1,1)\otimes_A \\
 & & (1, a) \arrow[u, mapsto]
\end{tikzcd}$$

Thus we can identify $\Delta - \sigma\Delta$ in degree 0 with

$$A \longrightarrow A \otimes A, \quad a \longmapsto (a, 1) - (1, a)$$

so the cocommutator space in degree 0 is $\mathbb{C}$.

Look in degree 1.

$$\mathcal{B}_1 \otimes_A \xrightarrow{\Delta} (\mathcal{B}_0 \otimes_A \mathcal{B}_1 \oplus \mathcal{B}_1 \otimes_A \mathcal{B}_0) \otimes_A$$

$$(a_0, a_1, 1) \otimes_A \xmapsto{\Delta} (a_0, 1) \otimes_A (1, a_1, 1) \otimes_A + (a_0, a_1, 1) \otimes_A (1, 1) \otimes_A$$

$$\xmapsto{\sigma\Delta} (1, 1) \otimes_A (a_0, a_1, 1) \otimes_A + \cancel{}$$

$$(1, a_1, 1) \otimes_A (a_0, 1) \otimes_A$$

Identify $\mathcal{B}_1 \otimes_A = (A \otimes \bar{A} \otimes A) \otimes_A \xrightarrow{\sim} A \otimes \bar{A}$

$$\mathcal{B}_1 \otimes_A \mathcal{B}_0 \otimes_A = (A \otimes \bar{A} \otimes A) \otimes_A (A \otimes A) \otimes_A \xrightarrow{\sim} A \otimes \bar{A} \otimes A$$

(use σ for $\mathcal{B}_0 \otimes_A \mathcal{B}_1 \otimes_A$)

and we get

$$A \otimes \bar{A} \underset{\sigma\Delta}{\overset{\Delta}{\rightrightarrows}} A \otimes \bar{A} \otimes A \oplus A \otimes \bar{A} \otimes A$$

$$(a_0, a_1) \longmapsto (1, a_1, a_0) + (a_0, a_1, 1)$$

$$\longmapsto (a_0, a_1, 1) + (1, a_1, a_0)$$

Thus $\operatorname{Ker}(\Delta - \sigma\Delta)$ in degree 1 is $\bar{A}$.

Look in degree n.

$$\mathcal{B}_n \otimes_A \xmapsto{\Delta} \mathcal{B}_0 \otimes_A \mathcal{B}_n \otimes_A + \cdots + \mathcal{B}_n \otimes_A \mathcal{B}_0 \otimes_A$$

$$(a_0, \ldots, a_n, 1) \longmapsto (a_0, 1) \otimes_A (1, a_1, \ldots, a_n, 1) \otimes_A + \cdots + (a_0, \ldots, a_n, 1) \otimes_A (1, 1) \otimes_A$$

Project into $\mathcal{B}_0 \otimes_A \mathcal{B}_n \otimes_A$ and we get

$$\mathcal{B}_n \otimes_A \xrightarrow{p_{0n}(\Delta - \sigma\Delta)} \mathcal{B}_0 \otimes_A \mathcal{B}_n \otimes_A$$

$$\begin{array}{ccccc} \| & & \| & & \\ A \otimes \bar{A}^{\otimes n} & \longrightarrow & (A \otimes A) \otimes_A (A \otimes \bar{A}^{\otimes n} \otimes A) \otimes_A & \xrightarrow{\sim} & A \otimes \bar{A}^{\otimes n} \otimes A \end{array}$$

$$(a_0, \ldots, a_n) \longmapsto (1, a_1, \ldots, a_n, a_0) - (a_0, a_1, \ldots, a_n, 1)$$

The condition ~~means that~~ $p_{0n}(\Delta - \sigma\Delta)\xi = 0$ means therefore that $\xi \in 1 \otimes \bar{A}^{\otimes n} \subset A \otimes \bar{A}^{\otimes n}$

So take $a_0 = 1$ and look at the other components: $\rho_{i,n-i}(\Delta\xi) =$

[redacted] $(1, a_1, \dots, a_i, 1) \otimes_A (1, a_{i+1}, \dots, a_n, 1) \otimes_A \in B_i \otimes_A B_{n-i} \otimes_A$

[redacted] $\rho_{i,n-i}(\sigma\Delta\xi) =$ [redacted]

$$(-1)^{i(n-i)} (1, a_{n-i+1}, \dots, a_n, 1) \otimes_A (1, a_1, \dots, a_{n-i}) \otimes_A$$

It seems that all we have are the two [redacted] maps

$$\begin{array}{ccccc} \bar{A}^{\otimes n} & = & \bar{A}^{\otimes i} \otimes \bar{A}^{\otimes n-i} & \longrightarrow & 1 \otimes \bar{A}^{\otimes i} \otimes 1 \otimes \bar{A}^{\otimes (n-i)} \\ \shortparallel & & & \nearrow & \\ \bar{A}^{\otimes n-i} \otimes \bar{A}^{\otimes i} & \underset{(-1)^{i(n-i)}}{\simeq} & \bar{A}^{\otimes i} \otimes \bar{A}^{\otimes n-i} & & \end{array}$$

<u>Conclusion</u>: The cocommutator subcomplex of $\mathcal{B}$ is $\bigoplus_{n \geq 0} \bar{A}^{\otimes n, \lambda}$ with differential induced by b'. In other words it is the reduced cyclic complex with a degree shift and [redacted] $\mathbb{C}$ in degree zero.

Consequence: Notice that we obtain the reduced cyclic complex without $T(\bar{A})$ being a DG coalgebra.

If R is an algebra [redacted] and $\theta : A \to R$ is a homomorphism, then we know that

$$\mathrm{Hom}_{\mathbb{C}}(T(\bar{A}), R) \simeq \prod_{p \geq 0} C^p_{norm}(A, R)$$

is a DG algebra with the standard Hochschild differential $\delta = ad(\theta) - b'$.

Evidently we have written

$$\mathrm{Hom}_{\mathcal{C}}(T(\bar{A}), R) = \mathrm{Hom}_{A\text{-bimod}}(\mathcal{B}, R)$$

and argued that the cup product of cochains is equivalent to a coalgebra structure on $\mathcal{B} = A \otimes T(\bar{A}) \otimes A$. (The signs are of course confused.)

So far we have discussed the maps

$$\mathcal{B} \xrightarrow{\Delta} \mathcal{B} \otimes_A \mathcal{B} \qquad\qquad \mathcal{B} \xrightarrow{\varepsilon} A$$

where $\mathcal{B}$ is the standard normalized resolution of the A-bimodule A. We have the other resolutions $\mathcal{B}^{(n)} = \mathcal{B} \otimes_A \overset{n}{\cdots} \otimes_A \mathcal{B}$ and maps between them obtained by applying Δ to a copy of $\mathcal{B}$ or ε to a copy of $\mathcal{B}$. Thus we have n operators from $\mathcal{B}^{(n)}$ to $\mathcal{B}^{(n+1)}$ and n operators from $\mathcal{B}^{(n)}$ to $\mathcal{B}^{(n-1)}$ where here $n \geq 1$ an $\mathcal{B}^{(0)} = A$. I guess the structure of these operators is semi-simplicial

$$\cdots\ \mathcal{B}\otimes_A\mathcal{B}\otimes_A\mathcal{B} \ \substack{\longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\ \mathcal{B}\otimes_A\mathcal{B} \ \substack{\xleftarrow{\ \Delta\ } \\ \xrightarrow{\varepsilon\otimes 1} \\ \xrightarrow[1\otimes\varepsilon]{}}\ \mathcal{B}$$

This reminds one of

$$\cdots\cdots\ X\times X\times X \ \substack{\longleftarrow \\ \longleftarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\ X\times X \ \substack{\longleftarrow \\ \longrightarrow \\ \longrightarrow}\ X$$

but somehow the ~~simplicial~~ semi-simplicial framework is too confusing to be useful at present.

Next let's consider the other element of structure, namely the homotopies between different maps. In particular there has to be a homotopy joining the two arrows

$$\begin{tikzcd} B \otimes_A B \arrow[r, shift left=2, "\varepsilon\otimes 1"] \arrow[r, shift right=2, "1\otimes\varepsilon"'] \arrow[r, phantom, "\Downarrow h"] & B \end{tikzcd}$$

and it is natural to expect $h.\Delta = 0$. (Actually such a homotopy ought to exist on general grounds since $(\varepsilon\otimes 1)\Delta = 1 = \Delta(\varepsilon\otimes 1)$.)

Canonical candidate for h - see p498-505 January 1988; this is work on trying to understand the double complex $CC(A \leftarrow A)$ as made of columns which are h.eq. to Hochschild complex. See also p451+456, August 1989; this is work on dilating Z to F and is relevant to partitions of 1.

Consider the DGA of chain alg. type (differentials of degree -1) generated by A in degree 0 and an element η of degree 1 such that $d\eta = 1$. This DGA is the tensor algebra

$$T_A(A\otimes A) = \bigoplus_{n\geq 0} A^{\otimes n+1}$$

with the differential b'. The point is that $A\otimes A \xrightarrow{\sim} A\eta A$ and hence when we write $(a_0, \dots, a_n)$ we think of the commas as being η.

Next look at η^2. One has $d(\eta^2) = d\eta\,\eta - \eta\, d\eta = 1\cdot\eta - \eta\cdot 1 = 0$, hence the ideal generated by η^2 is a differential ideal and we can form the ~~quotient~~ DG algebra. It's clear the quotient is the normalized resolution:

$$\xrightarrow{b'} A\otimes\bar{A}\otimes A \xrightarrow{b'} A\otimes A \xrightarrow{b'} A \longrightarrow 0$$

(In the quotient the product $a_0\eta a_1\eta\cdots\eta a_n$ vanishes if any of $a_1,\dots,a_{n-1}$ are $=1$.)

Let us put $R = T_A(A\otimes A)/(\text{ideal gen. by } \eta^2)$. We then have an exact sequence

$$0 \longrightarrow A \longrightarrow R \longrightarrow \Sigma B \longrightarrow 0$$

and the product on R gives a map of complexes

$$R\otimes_A R \longrightarrow R$$

carrying $A = A\otimes_A A$ to A by the identity. This gives an induced map of complexes

$$R\otimes_A R/A\otimes_A A \longrightarrow R/A$$

and we have

$$0 \longrightarrow R\otimes_A A + A\otimes_A R \to R\otimes_A R \to (R/A)\otimes_A(R/A) \to 0$$

hence an exact sequence of complexes

$$\begin{array}{c} 0 \longrightarrow R/A \oplus R/A \longrightarrow R\otimes_A R/A \longrightarrow (R/A)\otimes_A(R/A) \to 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad R/A \end{array}$$

If we use the unique lifting of ΣB into R, then we get a map

$$\Sigma B \otimes_A \Sigma B \longrightarrow \Sigma B$$

which is not compatible with with differential. Remove Σ's and we should get a map

$$h : B \otimes_A B \longrightarrow B$$

of degree $+1$ which is a homotopy between $\varepsilon \otimes 1$ and $1 \otimes \varepsilon$. Formula:

$$h\{(a_0, a_1, a_p, a_{p+1}) \otimes_A (a'_0, a'_1, \ldots, a'_q, a'_{q+1})\}$$

$$= (-1)^p (a_0, a_1, \ldots, a_p, a_{p+1}a'_0, a'_1, \ldots, a'_q, a'_{q+1})$$

the sign being due to taking the second Σ in $\Sigma B \otimes_A \Sigma B \longrightarrow \Sigma B$ out to the left.

Here's a way to check the sign. We believe $h : \varepsilon \otimes 1 \Longrightarrow 1 \otimes \varepsilon$ on $(B\otimes_A)^2 \to B\otimes_A$ is a homotopy. Hence $h\sigma : 1 \otimes \varepsilon \Longrightarrow \varepsilon \otimes 1$ is a homotopy and $h + h\sigma : \varepsilon \otimes 1 \Longrightarrow \varepsilon \otimes 1$, so $h + h\sigma$ should be a map of complexes $\Sigma(B\otimes_A B \otimes_A) \longrightarrow B\otimes_A$. Let's compose this map with

$$\Delta : B\otimes_A \longrightarrow B\otimes_A B \otimes_A$$

$$\Delta(a_0, a_1, \ldots, a_n, 1)\otimes_A$$
$$= \sum_{i=0}^{n} (a_0, \ldots, a_i, 1)\otimes_A (1, a_{i+1}, \ldots, a_n, 1)\otimes_A$$

when h is applied one gets 0 because there's a 1 where it gets killed. But

$$\sigma\Delta(a_0, a_1, \ldots, a_n, 1)\otimes_A$$
$$= \sum_{i=0}^{n} (-1)^{(n-i)i} (1, a_{i+1}, \ldots, a_n, 1)\otimes_A (a_0, \ldots, a_i, 1)\otimes_A$$

$$h\sigma\Delta(a_0, \ldots, a_n, 1)\otimes_A$$
$$= \sum_{i=0}^{n} \underbrace{(-1)^{(n-i)i} (-1)^{n-i}}_{(-1)^{(n-i)(i+1)} = (-1)^{n(i+1)}} (1, a_{i+1}, \ldots, a_n, a_0, \ldots, a_i, 1)\otimes_A$$

Thus $\quad (h + h\sigma)\Delta = B$ on $B\otimes_A$.

We see therefore that it ought to be possible to define B as $A \overset{!}{\otimes}_A A$ by choosing a projective bimodule resolution $\varepsilon : P \longrightarrow A$ and using a map ~~[illegible]~~ of resolutions $P \longrightarrow P\otimes_A P$ and a homotopy

$$P\otimes_A P \; \underset{1\otimes\varepsilon}{\overset{\varepsilon\otimes 1}{\underset{\Downarrow h}{\longrightarrow}}} \; P.$$

<u>Question</u>: Can one obtain the cyclic complex with all its structure — this effectively means the b, B bicomplex —

by using derived category methods on the category of A-bimodules with its tensor product?

This is related to Neeman's question of "totalizing" ~~maps~~ a sequence of maps in a triangulated category

$$\cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

with successive maps composing to zero.

Important point ~~from~~ August 1989 - the reduced cyclic complex can also be obtained apparently from the commutator quotient space of $T_A(A\eta A)/(\eta^2)$. So you have the reduced cyclic complex as both a cocommutator subspace and commutator subspaces, which is a step toward a proof of the Connes exact sequence.

July 14, 1990

Tensor products (sequel to 475-78).

Let A, B algebras, [redacted] $S = A\otimes B$, $R = A * B$. $I \subset R$ the ideal $\ni$ $S = R/I$.

The problem is to identify the homology of

$$(*) \qquad (\Omega^1 A_\natural \rightleftarrows A) \otimes (\Omega^1 B_\natural \rightleftarrows B)$$

with the periodic cyclic homology of $A\otimes B$ assuming A, B smooth.

Let's try to identify the $\mathbb{Z}/2$ graded complex $(*)$ with

[redacted]

$$(**) \qquad R/I^2+[R,I] \rightleftarrows (\Omega^1 R/I\Omega^1 R)_\natural$$

(Actually in $(*)$ it would have been better to put $A \rightleftarrows \Omega^1 A_\natural$ etc. since one usually writes the even space [redacted] first when working with $\mathbb{Z}/2$-graded complexes.)

Recall

$$\Omega^1 R = R\otimes_A \Omega^1 A \otimes_A R \;\oplus\; R\otimes_B \Omega^1 B \otimes_B R$$

so [redacted] any element of $\Omega^1 R$ is a sum of elements of the form $r_1 da r_2$ and $r_1 db r_2$. One has

$$\begin{aligned}\Omega^1 R / F_I^1 \Omega^1 R &= S\otimes_R \Omega^1 R \otimes_R S \\ &= (A\otimes B)\otimes_A \Omega^1 A \otimes_A (A\otimes B) \oplus \cdots \\ &= \Omega^1 A \otimes (B\otimes B) \oplus (A\otimes A)\otimes \Omega^1 B\end{aligned}$$

$$\left(\Omega^1 R / I\Omega^1 R\right)_{\natural} = (\Omega^1 A)_{\natural} \otimes B \oplus A \otimes (\Omega^1 B)_{\natural}$$

This gives us a canonical isomorphism between the odd spaces in (*) and (**). In order to compare the even spaces consider the extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R & \longrightarrow & S & \longrightarrow & 0 \\ & & & & \shortparallel & & \shortparallel & & \\ & & & & A * B & \longrightarrow & A \otimes B & & \end{array}$$

and recall the lifting $\rho : A \otimes B \longrightarrow A * B$ $\rho(a \otimes b) = ab$ which has curvature

$$\omega(a_1 \otimes b_1, a_2 \otimes b_2) = a_1 a_2 b_1 b_2 - a_1 b_1 a_2 b_2$$
$$= a_1 [a_2, b_1] b_2$$

We have seen that R admits the direct sum decomposition

$$R = \bigoplus_{n \geq 0} (A \otimes B) \otimes (\bar{A} \otimes \bar{B})^{\otimes n}$$

given by the cochains $\rho\alpha^n$, where $\alpha(a, b) = [a, b]$. Truncating we have an exact sequence and splitting

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & R/I^2 & \longrightarrow & S & \longrightarrow & 0 \\ & & \shortparallel & & \shortparallel & & \shortparallel & & \\ 0 & \longrightarrow & \Omega^1 A \otimes \Omega^1 B & \longrightarrow & R/I^2 & \underset{\rho}{\longrightarrow} & A \otimes B & \longrightarrow & 0 \end{array}$$

$$a_1 da_2 \otimes db_1 b_2 \longmapsto a_1 [a_2, b_1] b_2$$

The role of $\Omega^1 A$, $\Omega^1 B$ should be clarified a bit. The point is that we have a square zero extension with lifting, ~~hence~~ which is equivalent to a 2-cocycle on $A \otimes B$ and the 2 cocycle is the external cup product of the 1-cocycles $d: A \to \Omega^1 A$, $d: B \to \Omega^1 B$.

At this point we have isomorphisms of the even and odd spaces appearing in (*) and (**), so we want to calculate the differentials.

Let us consider

$$b: (\Omega^1 R / I\Omega^1 R)_\natural \longrightarrow R/I^2 + [R,I]$$

$$\begin{array}{ccccc}
\| & & & & \| \\
(\Omega^1 A)_\natural \otimes B & + & A \otimes (\Omega^1 B)_\natural & & A\otimes B \oplus \Omega^1 A_\natural \otimes \Omega^1 B_\natural \\
\cup\!\!\!| & & \cup\!\!\!| & & \\
a_1 da_2 \otimes b & & a \otimes db_1 b_2 & & \\
\downarrow b & & \downarrow b & & \\
[ba_1, a_2] & , & [b_2 a, b_1] & \in & R/I^2 + [R,I] \\
\downarrow & & \downarrow & & \downarrow \\
[a_1,a_2]\otimes b & & a\otimes[b_2,b_1] & & R/I = A \otimes B
\end{array}$$

This is just what we ~~expect~~ have for the map $\Omega^1 A_\natural \otimes B \oplus A \otimes \Omega^1 B_\natural \longrightarrow A \otimes B$ from (*).

To find the components in $\Omega^1 A_\natural \otimes \Omega^1 B_\natural$ we must use the lifting ρ.

$$[a_1,a_2]\otimes b \stackrel{\rho}{\longmapsto} [a_1,a_2]b$$

so we remove this from $[ba_1,a_2]$ obtaining

$$[ba_1,a_2] - [a_1,a_2]b \quad \in \quad I/I^2$$

We would like to obtain $-a_1 da_2 \otimes db \in \Omega^1 A_\natural \otimes \Omega^1 B_\natural$

the minus sign being due to d and $-a_1 da_2$ being odd.

Now

$$\begin{array}{ccc} -a_1 da_2 \otimes db & \longleftrightarrow & -a_1[a_2, b] \\ \cap & & \cap \\ \Omega^1 A \otimes \Omega^1 B & \cong & I/I^2 \end{array}$$

and

$$\underbrace{[ba_1, a_2]}_{\shortparallel} - [a_1, a_2]b$$

$$\underbrace{[b, a_2]a_1} + b[a_1, a_2]$$

$$\equiv a_1 \underbrace{[b, a_2]}_{\in I} = -a_1[a_2, b] \quad \text{mod } [R, I]$$

Also $\quad b[a_1, a_2] - [a_1, a_2]b = [b, [a_1, a_2]]$

$$= [[b, a_1], a_2] + [a_1, [b, a_2]] \in [R, I]$$

so indeed it works for $a_1 da_2 \otimes b$.

Next the image of $a \otimes db_1 b_2 \in A \otimes (\Omega^1 B)_\natural$ in $I/I^2 + [R, I]$ is

$$[b_2 a, b_1] - a[b_2, b_1]$$

and we would like to get $da \otimes db_1 b_2 \in \Omega^1 A_\natural \otimes \Omega^1 B_\natural$ which corresponds to $[a, b_1]b_2$.

$$[b_2 a, b_1] - a[b_2, b_1] = b_2[a, b_1] + [[b_2, b_1], a]$$

$$\equiv [a, b_1]b_2 + \cancel{[[b_2, a], b_1] + [b_2, [b_1, a]]} \quad \text{mod } [R, I]$$

So we win.

Next we consider the map

$$R/I^2+[R,I] \xrightarrow{d} (\Omega^1 R/I\Omega^1 R)_\natural$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$A\otimes B \oplus \Omega^1 A_\natural \otimes \Omega^1 B_\natural \qquad \Omega^1 A_\natural \otimes B + A\otimes\Omega^1 B_\natural$$

$a\otimes b$ lifts to $ab \in R/I^2 \overset{d}{\longmapsto} da\otimes b + a\otimes db$

$a_1 da_2 \otimes db_1 b_2 \in$ becomes $a_1[a_2,b_1]b_2 \in I/I^2$

$$d\{a_1[a_2,b_1]b_2\} = \cancel{da_1\underbrace{[\ ,\ ]}_{\in I}b_2} + a_1\, d[a_2,b_1]\,b_2 + \cancel{a_1[\ ]db_2}$$

$$= a_1[da_2,b_1]b_2 + a_1[a_2,db_1]b_2$$

$$= a_1 da_2\, b_1 b_2 - a_1 b_1\, da_2\, b_2$$
$$+ a_1 a_2\, db_1 b_2 - a_1\, db_1\, a_2 b_2 \qquad \in (\Omega^1 R/I\Omega^1 R)_\natural$$

At this point we must go back to the isomorphism

$$\Omega^1 R/F^1\Omega^1 R = \Omega^1 A\otimes(B\otimes B) \oplus (A\otimes A)\otimes\Omega^1 B$$

$$d\{a_1[a_2,b_1]b_2\} = a_1 da_2\otimes(1\otimes b_1 b_2) + (a_1 a_2\otimes 1)\otimes db_1 b_2$$
$$- a_1 da_2\otimes(b_1\otimes b_2) - (a_1\otimes a_2)\otimes db_1 b_2$$

and so when we pass to commutator quotient space we obtain

$$a_1 da_2 \otimes \underbrace{[b_1,b_2]}_{-\text{image of } db_1 b_2} + [a_1,a_2]\otimes db_1 b_2$$

$$\in (\Omega^1 A)_\natural\otimes B\oplus A\otimes(\Omega^1 B)_\natural$$

This is exactly what we want, the minus sign is expected since $a_1 da_2$ and $b: \Omega^1 B_\natural \to B$ are odd.

Thus we have proved

Proposition: Let $R = A * B$, $R/I = A \otimes B$. Then the $\mathbb{Z}/2$ graded complexes

$$\left(A \rightleftarrows \Omega^1 A_\natural\right) \otimes \left(B \rightleftarrows \Omega^1 B_\natural\right)$$

and $$\left(R/I^2 + [R,I] \rightleftarrows (\Omega^1 R / I\Omega^1 R)_\natural\right)$$

are canonically isomorphic.

This result was proved before: see p50-51 August 1989.

July 16, 1990

Review of standard facts about square zero extensions of algebras.

Let us consider an algebra extension

$$(*) \qquad 0 \longrightarrow M \longrightarrow E \longrightarrow A \longrightarrow 0$$

where $M^2 = 0$. Claim there is a short exact sequence of A-bimodules

$$(**) \qquad 0 \longrightarrow M \longrightarrow \Omega^1 E / M\Omega^1 E + \Omega^1 E M \longrightarrow \Omega^1 A \longrightarrow 0$$

$$\Omega^1 E / M\Omega^1 E + \Omega^1 E M = A \otimes_E \Omega^1 E \otimes_E A$$

The right exactness is obvious from considering derivations with values in an arbitrary bimodule. Injectivity at the left in $(**)$ is equivalent to being able to find an embedding $M \hookrightarrow N$ of A-bimodules and a derivation $D: E \to N$ which restricts to this embedding ~~Such a pair $(M \hookrightarrow N, D: E \to N)$ [illegible] equivalent to~~ Such a derivation D is the same thing as a splitting of the induced extension

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d] & E \arrow[r, "\pi"] \arrow[d, "\iota"] & A \arrow[r] \arrow[d, equal] \arrow[dl, dashed, "s"'] & 0 \\
0 \arrow[r] & N \arrow[r] & E \amalg^{M} N \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

In effect given the splitting s, there are two alg. homomorphisms ι and $s\pi$ whose difference is a derivation $D: E \to N$, etc.

Thus the injectivity on the left in (**) means we can kill square zero extension classes by suitable bimodule embeddings $M \hookrightarrow N$, or equivalently then there is such an embedding which extends to a derivation $D: E \longrightarrow N$.

To see that the latter is always true choose a linear lifting

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & E \arrow[r, "\pi"'] & A \arrow[r] \arrow[l, bend right, dashed, "\rho"'] & 0 \end{tikzcd}$$

and define $D : E \longrightarrow \mathrm{Hom}_{\mathbb{C}}(A, M)$

by $\quad (Dx)(a) = [x, \rho](a) = x\rho(a) - \rho(xa).$

Then $\quad D(xy) = [xy, \rho] = x[y,\rho] + [x,\rho]y.$

$$= x(Dy) + (Dx)y$$

provided $\mathrm{Hom}_{\mathbb{C}}(A, M)$ is considered an A-bimodule with left mult given by left mult. of A on M and with right mult. given by left mult. of A on itself. Check:

$$(x(Dy) + (Dx)y)(a) = x\cdot(Dy)(a) + (Dx)(ya)$$

$$= x(y\rho(a) + \rho(ya)) + x\rho(ya) - \rho(xya)$$

$$= (xy)\rho(a) - \rho(xya) = D(xy)(a)$$

If $x \in M$, then $(Dx)(a) = x\rho(a) = x \cdot a$

Thus D restricted to M is the map

$$M \longrightarrow \mathrm{Hom}_{\mathbb{C}}(A, M) \qquad m \longmapsto (a \longmapsto ma)$$

and this is a bimodule homomorphism for the bimodule structure on $\mathrm{Hom}_{\mathbb{C}}(A, M)$.

Goal: Suppose we are given an extension of algebras

$$0 \longrightarrow I \longrightarrow R \longrightarrow A \longrightarrow 0$$

with R smooth. We want to derive the Connes exact sequence for the cyclic homology of A using this extension. We have the A-bimodule resolution of A given by

$$0 \to I/I^2 \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \longrightarrow A \otimes A \longrightarrow A \to 0$$

Tensoring, better applying $I/I^2 \otimes_A$ successively, gives exact sequences

$$(*) \quad 0 \longrightarrow T^n_A(I/I^2) \longrightarrow T^{n-1}_A(I/I^2) \otimes_R \Omega^1 R \otimes_R A \longrightarrow T^{n-1}_A(I/I^2) \otimes A \longrightarrow T^{n-1}_A(I/I^2) \longrightarrow 0$$

We know that I/I^2 is left and right A-projective and the same is true of $T^n_A(I/I^2)$ which incidently is $\cong I^n/I^{n+1}$. Thus the A-bimodules

$$T^{n-1}_A(I/I^2) \otimes_R \Omega^1 R \otimes_R A \quad , \quad T^{n-1}_A(I/I^2) \otimes A$$

are also projective. Thus by concatenating the sequences $(*)$ we obtain a projective A-bimodule resolution of A which can be used to compute the Hochschild homology.

Notice that if $R = RA$, then $I/I^2 = \Omega^2 A$ and the resolution we have is made of the spaces $\Omega^n A \otimes A$.

The point is as follows. We have a projective A-bimodule resolution:

$$\longrightarrow I^n/I^{n+1} \otimes_R \Omega^1 R \otimes_R A \longrightarrow I^n/I^{n+1} \otimes A \longrightarrow$$

which upon applying $\otimes_A$ gives

$$\begin{tikzcd}
\to (I^n\Omega^1/I^{n+1}\Omega^1)_\natural \arrow[r, "b"] & I^n/I^{n+1} \arrow[r] \arrow[d, equal] & (I^{n-1}\Omega^1/I^n\Omega^1)_\natural \arrow[r] \arrow[d, equal] & I^{n-1}/I^n \\
& I/I^2 \otimes_A \overset{n}{\cdots} \otimes_A I/I^2 \arrow[r] & I/I^2 \otimes_A \overset{n-1}{\cdots} \otimes_A I/I^2 \otimes_R \Omega^1 R \otimes_R &
\end{tikzcd}$$

bracket map

here you apply d to the last factor I/I^2.

On the other hand in the short exact sequence of complexes

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & (I^n\Omega^1/I^{n+1}\Omega^1)_\natural \arrow[r, "b"] \arrow[d] & I^n/I^{n+1} \arrow[r, "d"] \arrow[d] & (I^{n-1}\Omega^1/I^n\Omega^1)_\natural \arrow[r] \arrow[d] & {} \\
\arrow[r] & (\Omega^1/I^{n+1}\Omega^1)_\natural \arrow[r, "b"] \arrow[d] & R/I^{n+1} \arrow[r] \arrow[d] & (\Omega^1/I^n\Omega^1)_\natural \arrow[r] \arrow[d] & {} \\
\arrow[r] & (\Omega^1/I^n\Omega^1)_\natural \arrow[r, "b"] \arrow[d] & R/I^n \arrow[r] \arrow[d] & (\Omega^1/I^{n-1}\Omega^1)_\natural \arrow[r] \arrow[d] & {} \\
& 0 & 0 & 0 &
\end{tikzcd}$$

the map d appearing is d applied to all factors of I in I^n in a suitable sense.

So the issue is this change or symmetrization in d.

Lets study this more closely. We have exact sequences

$$\begin{tikzcd} \left(I^n\Omega^1/I^{n+1}\Omega^1\right)_{\natural} \arrow[r, "b"] & I^n/I^{n+1} \arrow[r, "\pi"] & \left(I/I^2\otimes_A\right)^n \arrow[r] \arrow[dll, bend left, "N"'] & 0 \\ 0 \arrow[r] & HH_{2n} \arrow[r] & \left(I/I^2\otimes_A\right)^n \arrow[r, "\text{last factor } d"'] & \mathrm{Ker}\left\{\left(I^{n-1}\Omega^1/I^n\Omega^1\right)_{\natural} \xrightarrow{b} I^{n-1}/I^n\right\} \end{tikzcd}$$

call this ∂

and the d which is ∂ applied to all factors in $\partial N\pi$. So we have two ~~extra~~ copies of

$$\mathrm{Ker}\, N = \mathrm{Coker}\, N \simeq (1-\sigma)\left(I/I^2\otimes_A\right)^n$$

in the homology of the first row in the diagram at the bottom of the preceding page.

I seem to recall that the image of $0 \to HH_{2n} \to \left(I/I^2\otimes_A\right)^n$ is contained in the subspace $\left(I/I^2\otimes_A\right)^{n,\sigma}$. This is a consequence of the action of the cyclic group on $(A\otimes_A)^n$ being trivial mod homotopy. Thus we have an ~~[illegible]~~ embedding

$$\begin{array}{ccc} (1-\sigma)\left(I/I^2\otimes_A\right)^n & \hookrightarrow & \left(I^{n-1}\Omega^1/I^n\Omega^1\right)_{\natural} \\ & & \cap \\ & & \left(\Omega^1/I^n\Omega^1\right)_{\natural} \end{array}$$

so we can form the quotient space which we will write $(\Omega^1/I^n\Omega^1)_{\natural,\sigma}$. Then we have a complex

$$\longrightarrow (\Omega^1/I^{n+1}\Omega^1)_{\natural,\sigma} \longrightarrow R/I^{n+1} \longrightarrow (\Omega^1/I^n\Omega^1)_{\natural,\sigma} \longrightarrow R/I^n \longrightarrow$$

which should complete $HC_n(A)$.

Let's examine this on the subcomplex side, where we consider

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & (I^{n+1}\Omega^1)_\natural \arrow[r] \arrow[d] & I^{n+1} \arrow[r] \arrow[d] & (I^n\Omega^1)_\natural \arrow[r] \arrow[d] & {} \\
\arrow[r] & (I^n\Omega^1)_\natural \arrow[r] \arrow[d] & I^n \arrow[r] \arrow[d] & (I^{n-1}\Omega^1)_\natural \arrow[r] \arrow[d] & {} \\
\arrow[r] & (I^n\Omega^1/I^{n+1}\Omega^1)_\natural \arrow[r] \arrow[d] & I^n/I^{n+1} \arrow[r] \arrow[d] & (I^{n-1}\Omega^1/I^n\Omega^1)_\natural \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Here we know at least when R is free that the the middle complex gives the even reduced cyclic homology at the points $(I^n\Omega^1)_\natural$, but that at I^n it gives $\overline{HC}_{2n-1}A \oplus (1-\sigma)(I/I^2\otimes_A)^n$:

Recall
$$I^n/[R,I^n] = (I\otimes_A)^n = (I\otimes_A)^{n,\sigma} \oplus (1-\sigma)(I\otimes_A)^n$$
$$(I\otimes_A)^{n,\sigma} = I^n/[I,I^{n-1}] \qquad (1-\sigma)(I\otimes_A)^n = (1-\sigma)(I/I^2\otimes_A)^n$$

There is therefore an obvious thing to do, namely decrease I^n by the amount $(1-\sigma)(I/I^2\otimes_A)^n$.

Specifically notice we have inclusions

$$\begin{tikzcd}
& I^n \arrow[dl, dash] \arrow[dr, dash] & & \\
{[I, I^{n-1}]} \arrow[dr, dash] \arrow[rr, phantom, "\text{bicart}"] & & I^{n,\sigma} \arrow[dl, dash] \arrow[dr, dash] & \\
& {[R, I^n]} & & I^{n+1}
\end{tikzcd}$$

where $I^{n,\sigma}$ is defined so that $I^{n,\sigma}/[R,I^n] = (I\otimes_R)^{n,\sigma}$; note [redacted]

$$[I,I^{n-1}]/[R,I^n] = (1-\sigma)(I\otimes_R)^n$$

Suppose we pick a complement V for $[R,I^n]$ in $[I,I^{n-1}]$, whence we have maps of complexes

$$\begin{tikzcd}
{} \arrow[r] & (I^n\Omega^1 R)_\natural \arrow[r] \arrow[d, equal] & I^{n,\sigma} \arrow[r] \arrow[d] & (I^{n-1}\Omega^1)_\natural \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & (I^n\Omega^1)_\natural \arrow[r] \arrow[d, equal] & I^n \arrow[r, "d"] \arrow[d] & (I^{n-1}\Omega^1)_\natural \arrow[r] \arrow[d, equal] & {} \\
{} \arrow[r] & (I^n\Omega^1)_\natural \arrow[r] & I^n/V \arrow[r, "d"] & (I^{n-1}\Omega^1)_\natural \arrow[r] & {}
\end{tikzcd}$$

V goes to zero under d, since [redacted] d kills $(1-\sigma)(I\otimes_R)^n$. The composition is the identity so we have split off the extra $(1-\sigma)(I\otimes_R)^n$ in the middle [redacted] complex.

At this point it is clear that things ought to work, but we really need to work out efficient proofs.

July 17, 1990

Idea: When $A = R/I$, R smooth, we have a different way to compute the Hochschild homology. We have

$$gr_I(R) = \bigoplus_{n\geq 0} I^n/I^{n+1} = T_A(I/I^2)$$

and an exact sequence

$$(*) \qquad 0 \longrightarrow I/I^2 \longrightarrow A\otimes_R \Omega^1 R \otimes_R A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

whence exact sequences of A-bimodules

$$0 \to I^n/I^{n+1} \longrightarrow I^{n-1}/I^n \otimes_R \Omega^1 R \otimes_R A \longrightarrow I^{n-1}/I^n \otimes A \to I^{n-1}/I^n \to 0$$

which can be concatenated to give a projective A-bimodule resolutions of A. Compare this with the exact sequences

$$(**) \qquad 0 \longrightarrow \Omega^1 A \longrightarrow A\otimes A \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow \Omega^n A \longrightarrow \Omega^{n-1}A\otimes A \longrightarrow \Omega^{n-1}A \longrightarrow 0$$

and concatenating to obtain the standard normalized bar construction. We have seen that when it comes to relating the concatenated resolutions associated to $(*)$, $(**)$ to traces on R or RA, there are similar technical problems with cyclic symmetry, which I feel are ultimately related to the fact that the natural cyclic action on $(A \overset{!}{\otimes}_A)^n$ is trivial mod homotopy.

These two examples indicate that there is something more basic happening on the level of A-bimodules than what we have understood so far. A fundamental mystery remains, namely how cyclic homology is generated by Hochschild homology with all its structure ~~in~~ the derived category level.

To be more specific let K be the A-bimodule complex $(*)$

$$\longrightarrow 0 \longrightarrow I/I^2 \longrightarrow A \otimes_R \Omega^1 R \otimes_R A \longrightarrow \overset{\circ}{A \otimes A} \longrightarrow 0 \longrightarrow \cdots$$

Then concatenation - which is a form of Yoneda product - produces a bimodule resolution of A, call it P. We should be able to produce maps analogous to ones considered for the normalized bar resolution:

$$P \xrightarrow{\Delta} P \otimes_A P \underset{1 \otimes \varepsilon}{\overset{\varepsilon \otimes 1}{\rightrightarrows}} P$$

At this point there are the following things we can do. 1) We can review the cup product structure on Hochschild cohomology + and the commutativity of $H^\cdot(A,A)$. This appears to be analogous to Kasparov cup product theory. It turns out I think that more is needed than the above arrows Δ, h. ?

2) We can look for an appropriate B operator for our new Hochschild complex based on R, I. Perhaps we can find an analogue of the invariant complex.

Let's review things we learned about $H^\circ(A,M)$ using the normalized bar resolution $\mathcal{B}$. An element of $H^p(A,M)$ is a homotopy class of maps

$$\mathcal{B} \longrightarrow \Sigma^p M$$

of bimodule complexes. The cup product

$$H^p(A,M) \otimes H^q(A,N) \longrightarrow H^{p+q}(A, M \otimes_A N)$$

comes from $\Delta : \mathcal{B} \longrightarrow \mathcal{B} \otimes_A \mathcal{B}$ as follows

$$\mathcal{B} \xrightarrow{\Delta} \mathcal{B} \otimes_A \mathcal{B} \xrightarrow{f \otimes_A g} \Sigma^p M \otimes_A \Sigma^q N = \Sigma^{p+q}(M \otimes_A N)$$

More generally if X, Y are complexes of bimodules (suitably bounded), then an element of $R^p\,\mathrm{Hom}(X,Y)$ can be identified with a homotopy class of maps

$$\mathcal{B} \otimes_A X \longrightarrow \Sigma^p Y$$

Suppose we also have an element of $R^q\,\mathrm{Hom}(Y,Z)$ represented by

$$\mathcal{B} \otimes_A Y \longrightarrow \Sigma^q Z.$$

Then we can compose these maps as follows

$$\mathcal{B} \otimes_A X \xrightarrow{\Delta \otimes 1} \mathcal{B} \otimes_A \mathcal{B} \otimes_A X \longrightarrow \mathcal{B} \otimes_A \Sigma^p Y = \Sigma^p(\mathcal{B} \otimes_A Y) \longrightarrow$$

$$\longrightarrow \Sigma^p(\Sigma^q Z) = \Sigma^{p+q} Z$$

Now actually I am confusing things. One has the derived category of complexes of A-bimodules with its composition (or Yoneda product) structure. On the other hand one has the tensor product operation

$$X \overset{!}{\dot{\otimes}}_A Y$$

on this derived category. Since $X \overset{!}{\dot{\otimes}}_A A = A \overset{!}{\dot{\otimes}}_A X = X$ one has possibly two actions of $H^{\bullet}(A,A)$ on X. If we use $A \overset{!}{\dot{\otimes}}_A X = A \overset{!}{\dot{\otimes}}_A X \overset{!}{\dot{\otimes}}_A A = X \overset{!}{\dot{\otimes}}_A A$, we see these actions coincide.

Next consider the two actions on $H^{\bullet}(A,X) = R^{\bullet}\mathrm{Hom}(A,X)$ of $H^{\bullet}(A,A) = R^{\bullet}\mathrm{Hom}(A,A)$ we get from composition and tensor product. These coincide

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & A \arrow[r, "g"] \arrow[d, equal] & X \arrow[d, equal] \\
A \wedge A \arrow[r, "f \wedge 1"] \arrow[rr, bend right, "(-1)^{|f||g|} f \wedge g"'] & A \wedge A \arrow[r, "1 \wedge g"] & A \wedge X
\end{tikzcd} \qquad \wedge = \overset{!}{\dot{\otimes}}_A$$

Similarly $R^{\bullet}\mathrm{Hom}(X,Y)$ is naturally a module over the commutative algebra $R^{\bullet}\mathrm{Hom}(A,A)$; (left + right coincide).

Suppose we now try to be specific and concrete about this general nonsense. (Mainly I am interested in the properties needed for $\mathcal{B}$).

A first property of $\mathcal{B}$ is that one has the canonical map $\varepsilon : \mathcal{B} \to A$ which is a homotopy equivalence as either left or right A-modules. Thus for any X one has that $\mathcal{B} \otimes_A X \xrightarrow{\varepsilon \otimes 1} X$ and $X \otimes_A \mathcal{B} \xrightarrow{1 \otimes \varepsilon} X$ are quasi-isomorphisms.

Let us consider a resolution

$$\to \underbrace{0 \to X_n \to X_{n-1} \to \cdots \to X_0}_{X} \to A \to 0$$

where X is a complex of bimodules such that $X_0, \dots, X_{n-1}$ are projective. Set $M = X_n$. We then have maps of complexes

$$\begin{tikzcd}
X \arrow[r, "\varepsilon"] \arrow[d] & A \\
M[n]
\end{tikzcd}$$

where ε is a quis. Perhaps we should also assume that ε is a homotopy equivalence of complexes of either left or right A-modules.

We now iterate:

$$\begin{tikzcd}
X^{(2)} \arrow[r, dashed] \arrow[dd, dashed] & X^{(1)} \arrow[r, dashed] \arrow[d, dashed] & X \arrow[r, "\varepsilon"] \arrow[d] & A \\
 & M[n] \otimes_A X \arrow[r, "1\otimes\varepsilon", "\text{quis}"'] \arrow[d] & M[n] & \\
{} \arrow[r] & M[n] \otimes_A M[n] & &
\end{tikzcd}$$

If we take the successive fibre products $X^{(i)}$ we obtain a concatenated bimodule resolution. We could also use tensor product.

$$\to X \otimes_A X \otimes_A X \xrightarrow{1\otimes 1\otimes\varepsilon} X \otimes_A X \xrightarrow{1\otimes\varepsilon} X \xrightarrow{\varepsilon} A$$

It seems that the tensor product $X \otimes_A \overset{n}{\cdots} \otimes_A X$ maps onto the concatenated resolution. Notice that we could have constructed other concatenated resolutions:

$$\begin{tikzcd}
0 \arrow[r, dashed] \arrow[dd, dashed] & 0 \arrow[r, dashed] \arrow[d, dashed] & X \arrow[r, "\varepsilon"] \arrow[d] & A \\
& X \otimes_A M[n] \arrow[r, "\varepsilon \otimes 1"] \arrow[d] & M[n] & \\
M[n] \otimes_A X \otimes_A M[n] \arrow[r, "1 \otimes \varepsilon \otimes 1"] \arrow[d] & M \otimes_A M[2n] & & \\
{} & & &
\end{tikzcd}$$

and these should appear as different quotients of the tensor products $X \otimes_A \cdots \otimes_A X$. So the question arises as to what to do with all of this.

Let's return to the resolution

$$\begin{tikzcd}
& & & \Omega^1 A \arrow[dr, hook] & & & \\
0 \arrow[r] & \underbrace{I/I^2}_{N} \arrow[r] & \underbrace{A \otimes_R \Omega^1 R \otimes_R A}_{P} \arrow[rr] \arrow[ur] & & A \otimes A \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Denote: N $\quad$ P

Put $N^{(n)} = N \otimes_A \overset{n}{\cdots} \otimes_A N$. We have the resolution of the A-bimodule A

$$\begin{tikzcd}
{} \arrow[r] & N^{(n)} \otimes A \arrow[rr] \arrow[dr] & & N^{(n-1)} \otimes_A P \arrow[rr] \arrow[dr] & & N^{(n-1)} \otimes A \arrow[r] & {} \\
& & N^{(n)} \arrow[ur, hook] & & N^{(n-1)} \otimes_A \Omega^1 A \arrow[ur, hook] & &
\end{tikzcd}$$

which is projective when P is, in particular if R is smooth. Apply $? \otimes_A$ to get a complex computing the Hochschild homology.

$$\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[d] & & & & & \\
 & & & HH_{2n} \arrow[dr] & & 0 & & & \\
 {} \arrow[dr] & & & & (N\otimes_A)^n \arrow[ur] \arrow[dr] & & & & {} \\
 {} \arrow[r] & N^{(n)}\otimes_A P\otimes_A \arrow[rr] \arrow[dr] & & N^{(n)} \arrow[rr] \arrow[ur] & & N^{(n-1)}\otimes_A P\otimes_A \arrow[rr] \arrow[dr] & & N^{(n-1)} \arrow[r] \arrow[ur] & {} \\
 & & N^{(n)}\otimes_A \Omega A\otimes_A \arrow[ur] \arrow[dr] & & & & N^{(n-1)}\otimes_A \Omega A\otimes_A \arrow[ur] \arrow[dr] & & \\
 & HH_{2n+1} \arrow[ur] & & 0 & & HH_{2n-1} \arrow[ur] & & 0 & \\
 0 \arrow[ur] & & & & 0 \arrow[ur] & & & &
\end{tikzcd}$$

The key lemma I need is that the image of $HH_{2n} \hookrightarrow (N\otimes_A)^n$ is contained in $(N\otimes_A)^{n,\sigma}$. Assuming this true, we see that

$$(1-\sigma)(N\otimes_A)^n \text{ injects in } N^{(n-1)}\otimes_A P\otimes_A$$

so if we define $\underbrace{N^{(n)\sigma} \subset N^{(n)}}$ to be the inverse image of $(N\otimes_A)^{n,\sigma} \subset (N\otimes_A)^n$, ~~then~~ we get ~~a~~ sequence

$$\longrightarrow N^{(n)}\otimes_A P\otimes_A \big/ \operatorname{Im}(1-\sigma)(N\otimes_A)^{n+1} \longrightarrow N^{(n)\sigma} \longrightarrow N^{(n-1)}\otimes_A P\otimes_A \big/ \operatorname{Im}(1-\sigma)(N\otimes_A)^n \longrightarrow$$

which also gives the Hochschild homology

July 18, 1990

Let $A = R/I$ with R smooth.
Recall we have a cocartesian square

$$\begin{tikzcd}[column sep=small] & I^n \arrow[dl, dash] \arrow[dr, dash] & \\ {[I, I^{n-1}]} \arrow[dr, dash] & & I^{n,\sigma} \arrow[dl, dash] \\ & {[R, I^n]} & \end{tikzcd}$$

where

$$I^{n\sigma}/[R,I^n] = (I \otimes_R)^{n,\sigma}$$

$$[I, I^{n-1}]/[R,I^n] = (1-\sigma)(I\otimes_R)^n = (1-\sigma)(I/I^2 \otimes_A)^n$$

and that $I^{n,\sigma} \supset I^{n+1}$.

Recall also that we have the complex computing the Hochschild homology

$$\longrightarrow N^{(n)} \otimes_R \Omega^1 \otimes_R \xrightarrow{\ b\ } N^{(n)} \xrightarrow{\ \partial\ } N^{(n-1)} \otimes_R \Omega^1 \otimes_R \longrightarrow \cdots$$

where $N = I/I^2$, $N^{(n)} = T_A(N)^n$, and ∂ is given by $d: N \hookrightarrow A \otimes_R \Omega^1 \otimes_R A$ on the last factor of N in $N^{(n)}$. We have seen, thanks to the lemma that $HH_{2n}(A) \subset (N\otimes_A)^{n,\sigma}$, that ∂ induces an injection

$$(1-\sigma)(N\otimes_A)^n \hookrightarrow N^{(n-1)} \otimes_R \Omega^1 \otimes_R .$$

Here's what seems true. We have two complexes computing the cyclic homology of A:

$$\begin{tikzcd} \arrow[r] & R/I^n \arrow[r] & \left(\Omega^1/I^{n-1}\Omega^1\right)_{\natural} \big/ \partial(1-\sigma)(N\otimes_A)^n \arrow[r] & R/I^{n-1,\sigma} \arrow[r] & {} \\ \arrow[r] & R/I^{n,\sigma} \arrow[r] \arrow[u] & \left(\Omega^1/I^{n-1}\Omega\right)_{\natural} \arrow[r] \arrow[u] & R/I^{n-1} \arrow[r] \arrow[u, two heads] & {} \end{tikzcd}$$

It's clear that the surjection is a quis.

Next we have an exact sequence of complexes

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & I^{n,\sigma}/I^{n+1} & \longrightarrow & N^{(n-1)}\otimes_R \Omega^1 \otimes_R / \partial(1-\sigma)(N\otimes_A)^n & \longrightarrow & I^{n-1,\sigma}/I^n & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & R/I^{n+1} & \longrightarrow & (\Omega^1/I^n\Omega^1)_\natural / \partial(1-\sigma)(N\otimes_A)^n & \longrightarrow & R/I^n & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & R/I^{n,\sigma} & \longrightarrow & (\Omega^1/I^{n-1}\Omega^1)_\natural & \longrightarrow & R/I^{n-1,\sigma} & \longrightarrow \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

From yesterday's work the top row ought to give the Hochschild homology, hence we obtain the Connes exact sequence.

Notice that the sequence

$$\xrightarrow{b} I^{n,\sigma}/I^{n+1,\sigma} \xrightarrow{d} (I^{n-1}\Omega^1/I^n\Omega^1)_\natural \xrightarrow{b} I^{n-1,\sigma}/I^{n,\sigma} \longrightarrow$$

should give the Hochschild homology.

Here's a proof of the lemma used above. Choose $\rho : A \to R$ as usual and consider the induced map $RA \to R$, $IA \to I$, etc. We have a commutative triangle

$$\begin{array}{ccc}
 & & (IA/IA^2 \otimes_A)^n \\
 & \nearrow & \downarrow \\
HH_{2n}(A) & & \\
 & \searrow & \\
 & & (I/I^2 \otimes_A)^n
\end{array}$$

so in order to see the image of $HH_{2n}(A)$ lands in $\mathbb{C}$ invariants one is reduced to the case of RA. In this case $IA/IA^2 = \Omega^2 A$ and

$$\left(IA/IA^2 \otimes_A\right)^n = \Omega^{2n}A\otimes_A = \Omega^{2n}A/b\Omega^{2n+1}A$$

so we are looking at

$$0 \longrightarrow HH_{2n}(A) \longrightarrow \Omega^{2n}A/b\Omega^{2n+1}A \xrightarrow{\quad b \quad} \Omega^{2n-1}A$$
$$\Omega^{2n}A/b\Omega^{2n+1}A = \left(\Omega^1 A\otimes_A\right)^{2n}$$

From $1-K = bd+db$ we know that $K=1$ on the image of $HH_{2n}(A)$.

At this point, although there are various details remaining, I basically understand Hochschild homology and the Connes exact sequence in the case of a presentation $A = R/I$ with R smooth. The situation is not much different in the general case as in the case RA.

In fact it might be useful to work out in detail what we learned in the general case for this example. In particular lets show the complex

$$\longrightarrow I^n/I^{n+1} \xrightarrow{\ \partial\ } I^{n-1}/I^n \otimes_R \Omega^1 R\otimes_R \xrightarrow{\ b\ } I^{n-1}/I^n \longrightarrow$$

in the case $R=RA$, $I=IA$ can be identified with the b-complex ΩA with b.

Consider

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r, "d"] & A \otimes_R \Omega^1 R \otimes_R A \arrow[r] \arrow[d, equal] & \Omega^1 A \arrow[r] & 0 \\
& \Omega^2 A \arrow[u, "\wr"] & A \otimes \bar{A} \otimes A & &
\end{tikzcd}$$

$$\begin{array}{ccccc}
\rho(a_0)\omega(a_1,a_2) & & a_0\, d\rho(a_1)\, a_2 & \longmapsto & a_0 da_1 a_2 \\
\updownarrow & & \updownarrow & & \\
a_0 da_1 da_2 & & (a_0, a_1, a_2) & &
\end{array}$$

Now $\rho(a_0)\omega(a_1,a_2) \overset{d}{\longmapsto} \rho(a_0)\left(d\rho(a_1a_2) - d\rho(a_1)\rho(a_2) - \rho(a_1)d\rho(a_2)\right)$

$$\updownarrow$$

$$(a_0, a_1a_2, 1) - (a_0, a_1, a_2) - (a_0a_1, a_2, 1)$$

Thus $\Omega^2 A \longrightarrow A \otimes \bar{A} \otimes A$ sends $da_1 da_2$ to

$$-(a_1a_2, 1) + (1, a_1a_2, 1) - (1, a_1, a_2) = -b'(1, a_1, a_2, 1)$$

and if we identify $A \otimes \bar{A} \otimes A \simeq \Omega^1 A \otimes A$ we get $\Omega^2 A \longrightarrow \Omega^1 A \otimes A$ sending $da_1 da_2$ to

$$-a_1 da_2 \otimes 1 + d(a_1a_2) \otimes 1 - da_1 \otimes a_2 = da_1(a_2 \otimes 1 - 1 \otimes a_2)$$

This is $-b'$ and yields $-b$ on the commutator quotient space.

Next consider

$$\begin{tikzcd}
I^n/I^{n+1} \arrow[r, "\partial"] \arrow[d, "\wr\wr\ \rho\omega^n"'] & I^{n-1}/I^n \otimes_R \overset{R \otimes d\rho(A) \otimes R}{\Omega^1 R} \otimes_R \arrow[d, "\wr\wr\ \rho\omega^{n-1}d\rho"] \\
\Omega^{2n} A & \Omega^{2n-1} A
\end{tikzcd}$$

$$\underbrace{\partial\left(\rho(a_0)\omega^n(a_1,\dots,a_{2n})\right)}_{\substack{\updownarrow \\ a_0 da_1 \cdots da_{2n}}} = \rho(a_0)\omega^{n-1}(a_1,\dots,a_{2n-2}) \left\{ \begin{array}{l} \partial\rho(a_{2n-1}a_{2n}) \\ -\rho(a_{2n-1})\partial\rho(a_{2n}) \\ -\partial\rho(a_{2n-1})\rho(a_{2n}) \end{array} \right\}$$

The right side corresponds to the element of $\Omega^{2n-1}A$

$$a_0\, da_1 \cdots da_{2n-2} \left\{ \begin{array}{l} d(a_{2n-1}a_{2n}) \\ -\, a_{2n-1}\, da_{2n} \end{array} \right\}$$

$$-\, a_{2n} a_0\, da_1 \cdots da_{2n-1}$$

$$= [a_0 da_1 \cdots da_{2n-1}, a_{2n}] = -b(a_0 da_1 \cdots da_{2n})$$

The other map

$$\begin{array}{ccc} I^{n-1}/I^n \otimes_R \Omega^1 R \otimes_R & \xrightarrow{b} & I^{n-1}/I^n \\ s\uparrow & & s\uparrow \\ \Omega^{2n-1}A & & \Omega^{2n-2}A \end{array}$$

$$\begin{array}{ccc} \rho(a_0)\,\omega^{n-1}(a_1,\dots,a_{2n-2})\,d\rho(a_{2n-1}) & \longmapsto & [\rho(a_0)\omega^{n-1}(a_1,\dots,a_{2n-2}), \rho(a_{2n-1})] \\ \updownarrow & & \updownarrow \\ a_0\, da_1 \cdots da_{2n-1} & & [a_0\, da_1 \cdots da_{2n-2}, a_{2n-1}] \\ & & \| \\ & & b(a_0 da_1 \cdots da_{2n-1}) \end{array}$$

coincides with b.

Therefore we have identified the ∂, b complex giving $HH(A)$ with the actual Hochschild complex (Ω, b) in the case $R = RA$. (Incidently the slight difference in signs fits or checks with the formulas

$$(\tau b)_{2n-1} = b\tau_{2n-2} - (1+\lambda)s\tau_{2n}$$

$$(Td)_{2n} = -nP(\tilde{K}^2)bT_{2n-1} + BT_{2n+1} \;)$$

Now we have seen how to reduce the ∂, b complex in general to

$$\longrightarrow I^n_\sigma/I^{n+1} \longrightarrow I^{n-1}/I^n \otimes_R \Omega^1 R \otimes_R \big/ \partial(1-\sigma)(\dots) \longrightarrow \cdots$$

How does this compare to the $\tilde{K}^2$-invariant complex? It seems larger - apparently one has taken $\Omega^{2n}A / b\Omega^{2n+1}A$ and taken the part averaging to zero, killed this and its image in $b(\Omega^{2n}A)$ ($=$ Ker b on $\Omega^{2n-1}A$ except for the $\tilde{K}^2 = 1$ eigenspace). But one has not removed the part of $b\Omega^{2n+1}A$ $=$ Ker b on $\Omega^{2n}A$ averaging to zero under $\tilde{K}^2$.

It appears that one has gone exactly halfway towards the $\tilde{K}^2$-invariant subcomplex.

July 19, 1990: (Erica is 12!)

Let us consider the $\mathbb{Z}/2$ graded complex

$$R/I^{m+1}+[R,I^m] \underset{d}{\overset{b}{\rightleftarrows}} (\Omega^1R/I^m\Omega^1R)_\natural$$

and compute the ~~c~~homology for $R=RA$, $I=IA$. A linear functional τ on the left is equivalent to a family of normalized cochains $\tau_0,\tau_2,\dots,\tau_{2m}$ such that $b\tau_{2m}=0$. A linear functional T on the right is equivalent to a family of normalized cochains $T_1,T_3,\dots,T_{2m-1}$. Formulas

$$(\tau b)_{2n+1} = b\tau_{2n} - (1+\lambda)s\,\tau_{2n+2} \qquad \tau_{2m+2}=0$$

$$(Td)_{2n} = -nP(\tilde{K}^2)b\,T_{2n-1} + BT_{2n+1}$$

Split the cochains into those invariant under $\tilde{K}^2$ and those averaging to zero under $\tilde{K}^2$. Consider the latter cochains where one has

$$(\tau b)_{2n+1} = b\tau_{2n} - (1+K)s\,\tau_{2n+2}$$

$$(Td)_{2n} = 0$$

Since $(b-(1+K)s)^2 = -(1-\tilde{K}^2)$ is $-(1-\tilde{K}^2)$ which is inv. on the cochains killed by $P(\tilde{K}^2)$, we know that $b-(1+K)s$ is an isomorphism from even to odd forms in $\operatorname{Ker} P(\tilde{K}^2)$. Given $T=(T_1,T_3,\dots)$ in $\operatorname{Ker} P(\tilde{K}^2)$ we have a unique $\tau=(\tau_0,\tau_2,\dots)$ this in kernel such that $T_{2n+1} = b\tau_{2n}-(1+\lambda)s\,\tau_{2n+2}$. Suppose

$$T_{2m+1} = T_{2m+3} = \cdots = 0. \quad \text{Then}$$

$$b\tau_{2m} = (1+\lambda)s\tau_{2m+2}$$

so $b\tau_{2m}$ is killed by b and $(1+\lambda)s$ hence by $1-K^2$. Thus $b\tau_{2m} = 0$ since $1-K^2$ is invertible on $\mathrm{Ker}(P(\tilde{K}^2))$.

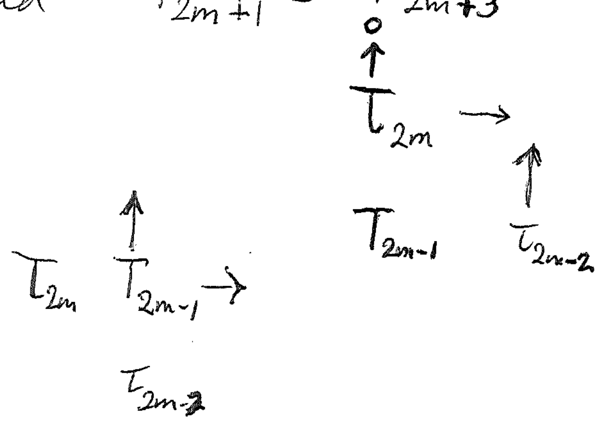
Similarly we have $b\tau_{2m+2} = (1+\lambda)s\tau_{2m+4}$ is killed by both b and $(1+\lambda)s$ and hence it is zero. Thus τ_{2m+2} is killed by b and $(1+\lambda)s$ so it is also zero. etc.

Thus we see that the $\mathrm{Ker}\, P(\tilde{K}^2)$ part of the cohomology of the $\mathbb{Z}/2$ graded complex is 0. As for the $\mathrm{Im}\, P(\tilde{K}^2)$ part the formulas become

$$(\tau b)_{2n+1} = b\tau_{2n} - \frac{1}{n+1} B\tau_{2n+2}$$

$$(Td)_{2n} = -nbT_{2n-1} + BT_{2n+1}$$

so we have something isomorphic to the usual b-B $\mathbb{Z}/2$-graded complex except it comes with these support conditions: $b\tau_{2m}=0$, $\tau_{2m+2} = \tau_{2m+4} = \cdots = 0$ and $T_{2m+1} = T_{2m+3} = \cdots = 0$.



The even cohomology should be $HC^{2m}(A)$; the odd cohomology should be $HC^{2m-1}(A)/B\,HH^{2m}(A)$

Suppose the $HH_n(A) = 0$ $n \geq 3$. Then from the Connes exact sequence

$$0 \to HC_4 \xrightarrow{\sim} HC_2 \longrightarrow 0 \longrightarrow HC_3 \xrightarrow{S} HC_1 \xrightarrow{B} HH_2 \to$$

$$\to HC_2 \xrightarrow{S} HC_0 \xrightarrow{B} HH_1 \xrightarrow{I} HC_1 \to 0$$

and so the periodic cyclic homology is

$$HC^{per}_{even} = HC_2 \qquad H^{per}_{odd} = \mathrm{Ker}\{HC_1 \xrightarrow{B} HH_2\} = HC_3$$

But this is the same as the homology of

$$R/I^2 + [R,I] \rightleftarrows (\Omega^1 R / I\Omega^1 R)_4 \qquad R = RA$$

This checks with what we expect to be true for $R = A * B$, $R/I = A \otimes B$, A, B smooth.

General discussion: The persistent problem is the Hochschild complex with all its structure. The structure is what's needed to construct the cyclic complex.

Actually what I want are not explicit complexes but suitable derived category objects. Thus I should be able to start with a simple way of calculating HH and construct a corresponding simple variant for HC. There are three kinds of cohomology and arrows

Hochschild $\leftarrow$ cyclic $\longrightarrow$ periodic

In homology the arrows run the other way.

The business about extensions seems not too well suited to cyclic theory, but very suited to periodic theory.

Idea: Suppose we start with an algebra A having a simple projective A-bimodule resolution. Then we would like to use this to give a complex computing the Hochschild homology and to find a way to improve it to a double complex or filtered complex giving the cyclic homology, and eventually the periodic cyclic homology. There should be a map from K-classes into the periodic homology.

Notice that a mixed complex which is bounded gives a simple uncompleted $\mathbb{Z}/2$-graded complex for the periodic homology.

Idea: Let's consider an algebra A such that the bimodule $\Omega^{2n}A$ is projective. Then taking $R = RA$, $I = IA$, I know that I get a very nice model for the periodic cyclic homology of A given by the $\mathbb{Z}/2$-graded complex

$$R/I^{n+1}+[R,I^{n}] \rightleftarrows (\Omega^1R/I^{n}\Omega^1R)_\natural$$

In fact I should perhaps forget $\Omega^{2n}A$ projective and just look at these $\mathbb{Z}/2$-graded complexes. We know that there are projection operators on any of these complexes which project onto invariant chains and the subcomplexes of invariant chains has the same homology. Moreover when this is done the differentials become $b - B$ up to some rescaling.

Now suppose A commutative whence we have a map from $(\Omega A, b, B)$ to $(\Omega_A, 0, d)$. It appears therefore that we have a canonical map from the $\mathbb{Z}/2$ graded complex $L(R)^\wedge$ to the $\mathbb{Z}/2$-graded complex given by Ω_A^{even}, Ω_A^{odd} and d. If this is true, then associated to idempotents and invertibles in A should be canonical even and odd forms. This is not very interesting because one really needs matrices over A to get interesting idempotents.

But the assumption that A be comm. is not essential. One has a map from $(\Omega A, b, B)$ to $(\Omega A/[\ ,\ ], 0, d)$ which will serve as well.

The moral seems to be that Chern character forms appear naturally by lifting idempotents and invertibles in A to $RA^\wedge$. Somehow there's some Fedosov insight around here which I am missing.

1

July 21, 1990

An interesting point (2.16) in my extension paper is the fact that there are, when I is either left or right flat, natural cyclic group $\mathbb{Z}/n$ actions on $H_\cdot(R, I^n)$ and $H_\cdot(R, R/I^n)$ compatible with the long exact sequence

$$H_i(R,R) \longrightarrow H_i(R, R/I^n) \longrightarrow H_{i-1}(R, I^n) \longrightarrow H_{i-1}(R,R)$$

and the trivial action on $H_\cdot(R,R)$. Thus we have

$$(1-\sigma)H_i(R, R/I^n) = (1-\sigma)H_{i-1}(R, I^n)$$

$$(1-\sigma)H_1(R, R/I^n) = (1-\sigma)\left(I\otimes_R\right)^n = (1-\sigma)\left(I/I^2\otimes_A\right)^n$$

On the other have we have

$$0 \to HC_{2n}(A) \to HC_0(R/I^{n+1}) \to H_1(R, R/I^n)_\sigma \to HC_{2n-1}(A) \to 0$$

$$0 \to HC_2(R/I^n) \to HC_0(R/I^{2n}) \to H_1(R, R/I^n) \to HC_1(R/I^n) \to 0$$

Thus we expect to have

$$HC_1(R/I^n) = HC_{2n-1}(A) \oplus (1-\sigma)\left(I/I^2\otimes_A\right)^n$$

Another point of view is that the $\mathbb{Z}/2$ graded complex $R^\wedge \rightleftarrows \Omega^1 R_\natural{}^\wedge$ has two natural families of quotient complexes, namely

$$R/I^{n+1}+[R,I^n] \rightleftarrows \left(\Omega^1 R/I^n\Omega^1 R\right)_\natural$$

and

$$R/I^n \rightleftarrows \Omega^1(R/I^n)_\natural$$

It is interesting to work out the homology of these $\mathbb{Z}/2$-graded complexes.

The first one we've done and found the homology to be

$$\begin{aligned} \text{even} &: \quad HC_{2n}(A) \\ \text{odd} &: \quad \operatorname{Ker}\{HC_{2n-1}(A) \xrightarrow{B} HH_n(A)\} \end{aligned}$$

Let's calculate the homology of the second. We have ~~have~~

$$\begin{aligned} \text{odd} &= H_{odd}\left(X(R/I^n)\right) = HC_1(R/I^n) \\ &= HC_{2n-1}(A) \oplus \left((1-\sigma)(I/I^2 \otimes_A)^n\right) \\ \text{even} &= H_{ev}\left(X(R/I^n)\right) = H_0^{DR}(R/I^n) \end{aligned}$$

Now

$$\begin{aligned} \Omega^1(R/I^n) &= \Omega^1 R / I^n\Omega^1 R + \Omega^1 R I^n + dI^n \\ \Omega^1(R/I^n)_\natural &= \Omega^1 R / I^n \Omega^1 R + \Omega^1 R I^n + dI^n + [R,\Omega^1 R] \\ &= \Omega^1 R / I^n \Omega^1 R + [R,\Omega^1 R] + dI^n \end{aligned}$$

Consider the maps of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & HC_{2n}(A) \arrow[r] \arrow[d, "\alpha"] & HC_0(R/I^{n+1}) \arrow[r, "d"] \arrow[d] & \Omega^1/I^n\Omega^1 + [R,\Omega^1] \arrow[d] \\ 0 \arrow[r] & H_0^{DR}(R/I^n) \arrow[r] \arrow[d, hook] & HC_0(R/I^n) \arrow[r] \arrow[d, equal] & \Omega^1/I^n\Omega^1 + [R,\Omega^1] + dI^n \arrow[d] \\ 0 \arrow[r] & HC_{2n-2}(A) \arrow[r] & HC_0(R/I^n) \arrow[r] & \Omega^1/I^{n-1}\Omega^1 + [R,\Omega^1] \end{tikzcd}$$

A short diagram chase shows that α is surjective; this uses that ~~[illegible]~~ elements of dI^n can be lifted back. Thus we conclude

$$H_0^{DR}(R/I^n) = \operatorname{Im}\left(HC_{2n}(A) \xrightarrow{S} HC_{2n-2}(A)\right)$$

$$= \operatorname{Ker}\{B: HC_{2n-2}(A) \longrightarrow \cancel{H}H_{2n-1}(A)\}.$$

July 27, 1990

$\hat{Q} = QA$ can be considered either as an algebra or a superalgebra and there are associated $\mathbb{Z}/2$ graded complexes $X(Q)$ and $X^s(Q)$. These can be split into even and odd parts. We know

$$X^s(Q)^- = X(Q)^-$$

because odd traces = odd supertraces. We have calculated the homology of $X^s(Q)^{\pm}$, so let's look at $X(Q)^+$.

We look dually at linear functions. Even linear functions τ on Q are described by even cochains: $\tau_{2n} = \tau(\rho q^{2n})$. Even traces T on $\Omega^1 Q$ are described by traces on $\Omega^{10} = Q \otimes_A \Omega^1 A \otimes_A Q$, or by A-bimodule traces on $Q \otimes_A \Omega^1 A$, or by families $T_1, T_2, T_4, \ldots$ which are arbitrary such that $bT_1 = 0$. $T_{2n} = T(\theta q^{2n-1} d\theta)$ and we have

$$bT_{2n} = 2T(\theta q^{2n} d\theta)$$

$$T(\theta q^{2n} d\theta) = \tfrac{1}{2} bT_{2n}$$

$$T(q^{2n}\,d\theta) = \tfrac{1}{2} sb\, T_{2n} = T(q^{2n}\,dq)$$

$$T(\rho q^{2n-1}\,d\theta) = (1-\tfrac{1}{2} sb)\, T_{2n} = T(\rho q^{2n-1}\,dq)$$

We claim the differentials are:

$$\boxed{\begin{aligned}(\tau b)_{2n} &= (2-bs)\,\tau_{2n}\\ (Td)_{2n} &= n\,P(-\tilde{\kappa})\, bs\, T_{2n}\end{aligned}}$$

Check first

$$(\tau b)_{2n} = (\tau b)(\theta q^{2n-1} d\theta) = \tau(\theta q^{2n-1}\theta) - \lambda\,\tau(\theta^2 q^{2n-1})$$

$$= \tau(\rho q^{2n} + q^{2n}\rho) - \lambda\,\tau(\rho q^{2n} + q\rho q^{2n-1})$$

~~Recall~~

$$\cancel{b\tau_{2n} - (1+\lambda)s\tau_{2n+2} = \tau(\rho q^{2n}\rho) + \lambda\tau(\rho^2 q^{2n})}$$

~~so~~ $$\cancel{sb\tau_{2n} = \tau(q^{2n}\rho) + \underbrace{s\lambda\tau(\rho^2 q^{2n})}_{-\lambda\tau(\rho q^{2n})}}$$

$$bs\,\tau_{2n} = b\,\tau(q^{2n})$$

$$= \tau\big((\rho q + \cancel{q\rho})q^{2n-1} - q(\cancel{\rho q^{2n-1}} + q^{2n-1}\rho)\big)$$

$$+ \lambda\,\tau\big((\rho q + q\rho)q^{2n-1}\big)$$

$$= \tau(\rho q^{2n}) - \tau(q^{2n}\rho) + \lambda\,\tau(\rho q^{2n} + q\rho q^{2n-1})$$

$$\therefore\ (\tau b)_{2n} + bs\,\tau_{2n} = 2\tau(\rho q^{2n}) = 2\tau_{2n}$$

$$(\tau b)_{2n} = (2-bs)\,\tau_{2n}$$

As a check note that $\tau b = 0$ implies $\tau_{2n} = \frac{1}{2} bs\tau_{2n}$ whence $b\tau_{2n} = 0$ and $\kappa\tau_{2n} = (1-bs-sb)\tau_{2n}$ $= (\tau_{2n} - 2\tau_{2n}) = -\tau_{2n}$. This is the usual description for even traces on Q.

Next

$$(Td)_{2n} = Td(\rho q^{2n}) = \sum_{j=0}^{2n-1} T(\rho q^{2n-1-j} dq\, q^j) + T(d\rho\, q^{2n})$$

$$\kappa^j T(\rho q^{2n-1} dq) = \cancel{\lambda^j} (-1)^j T(q^j \rho q^{2n-1-j} dq)$$

$$= (-1)^j \underbrace{1}_{T\text{-sign}} \underbrace{1}_{\substack{\lambda\text{-sign}\\ \text{odd no}}} T(\rho q^{2n-1-j} dq\, q^j)$$

$$\therefore (Td)_{2n} = \sum_{j=0}^{2n-1} (-\kappa)^j \underbrace{T(\rho q^{2n-1} dq)}_{(1-\frac{1}{2}sb)T_{2n}} + \lambda^{2n} \underbrace{T(q^{2n} d\rho)}_{\frac{1}{2}sbT_{2n}}$$

$$1 - \tfrac{1}{2}sb = 1 - \tfrac{1}{2}(1-\kappa-bs) = \frac{1+\kappa}{2} + \frac{1}{2}bs$$

$$\sum_{j=0}^{2n-1} (-\kappa)^j \left(\frac{1+\kappa}{2} + \frac{1}{2}bs\right) T_{2n} = \left(\frac{1-\kappa^{2n}}{2} + \frac{1}{2}b\sum_{j=0}^{2n-1}(-\lambda)^j s\right) T_{2n}$$

$$= \left(\frac{1-\kappa^{2n}}{2} + \frac{1}{2}b\,2nP(-\tilde{\kappa})s\right) T_{2n}$$

$$\therefore (Td)_{2n} = nP(-\tilde{\kappa})bs\,T_{2n} + \underbrace{\frac{1-\kappa^{2n}}{2}T_{2n} + \frac{1}{2}\kappa^{2n}sbT_{2n}}_{\frac{1}{2}T_{2n} + \frac{1}{2}\kappa^{2n}(-1+\underset{-\kappa-bs}{\underset{\shortparallel}{sb}})T_{2n}}$$

$$= \frac{1}{2}T_{2n} - \frac{1}{2}\underbrace{\kappa^{2n+1}}_{1-bs}T_{2n} - \frac{1}{2}b\cancel{\kappa^{2n}}sT_{2n} = 0$$

$$\therefore\ (Td)_{2n} = nP(-\vec{K})\, bs\, T_{2n}.$$

One can check the operators $(2-bs)$ and $nP(-\vec{K})\,bs$ kill each other.

To calculate the cohomology recall

$$\begin{tikzcd}
0 \arrow[r] & sC^{2n+1} \arrow[r, hook] \arrow[d, "1"] & C^{2n} \arrow[r, "s"] \arrow[d, "1-\frac{1}{2}bs"] & sC^{2n} \arrow[r] \arrow[d, "\frac{1+K}{2} = \frac{1+\lambda}{2}"] & 0 \\
0 \arrow[r] & sC^{2n+1} \arrow[r] & C^{2n} \arrow[r, "s"] & sC^{2n} \arrow[r] & 0
\end{tikzcd}$$

so the operator $2-bs$ is invertible on all eigenspaces of $\vec{K}$ corresponding to eigenvalues $\neq -1$. Where $\tilde{K}=-1$ also $K=-1$ so

$$\frac{1}{2}bs + \frac{1}{2}sb = \frac{1-K}{2} = 1.$$

so $\frac{1}{2}bs$, $\frac{1}{2}sb$ are complementary idempotents and we have

$$(Td)_{2n} = 2n\ \frac{1}{2}bs\, T_{2n}$$
$$(\tau b)_{2n} = sb\tau_{2n}$$

and so <u>there is no homology</u>. This ignores whatever happens at $n=0$.

August 9, 1990

$$E = EA = Q \tilde{\otimes} k[F] = A * k[F]$$

Recall a Fredholm module gives a homorphism $\theta : A \longrightarrow L = \mathcal{L}(H)$ and an involution $F \in L$, whence one has a homomorphism $\phi : E \longrightarrow L$. The cohomologically interesting things ~~[illegible]~~ appear to be supertraces on E relative to the grading ε defined by ~~[illegible]~~ $a^\varepsilon = a$, $F^\varepsilon = -F$. One can show that odd + even supertraces on E correspond to even + odd supertraces on Q. Recall, however, that E is not the superalg. tensor product $E \hat{\otimes} C_1$, so this statement about supertraces has to be checked by hand:

~~[illegible]~~

Even: 1-1 Correspondence between even τ on E and arbitrary τ' on Q given by

$$\tau(x + yF) = \tau'(x)$$

One has

$$\tau\big((x_1 + y_1 F)(x_2 + y_2 F)\big) = \tau'(x_1 x_2) + \tau'(y_1 y_2^\gamma)$$

$$\tau\big((x_2 - y_2 F)(x_1 + y_1 F)\big) = \tau'(x_2 x_1) - \tau'(y_2 y_1^\gamma)$$

Thus τ is an ε-trace $\iff$ $\tau'(x_1 x_2) = \tau'(x_2 x_1)$ i.e. τ' is a trace and $\tau'(y_1 y_2^\gamma) = -\tau'(y_2 y_1^\gamma) = -\tau'(y_1^\gamma y_2)$ i.e. τ' is odd.

<u>Odd</u>: 1-1 Correspondence between odd τ on E and arbitrary τ' on Q given by

$$\tau(x+yF) = \tau'(y)$$

One has

$\tau((x_1+y_1$ [redacted] $F)(x_2+y_2F)) = \tau'(x_1y_2) + \tau'(y_1x_2^\gamma)$

$$\tau((x_2+y_2F)(x_1+y_1F)) = \tau'(x_2y_1) + \tau'(y_2x_1^\gamma)$$

so τ is a trace iff τ' is a γ-trace. A γ-trace is necessarily even, so τ' is an even [redacted] supertrace.

In the Fred situation one takes the operator trace tr and makes it odd with respect to ε.

$$\tau(x+yF) = tr(yF)$$

Thus we get the even supertrace $tr(Fy)$ on Q. In the graded case, where ε on E is realized by an operator [redacted] in L, one has the even supertrace $tr(\varepsilon\,?)$ on L which pulls back to

$$\tau(x+yF) = tr\,\varepsilon(x+yF) = tr(\varepsilon x).$$

Thus we get the odd trace $tr(\varepsilon x)$ on Q.

The next step is to discuss homotopy. We have

$$\Omega^1 E = \underbrace{E \otimes_A \Omega^1 A \otimes_A E}_{EdAE} \oplus \underbrace{E \otimes_{k[F]} \Omega^1 k[F] \otimes_{k[F]} E}_{EdFE}$$

[redacted] This decomposition corresponds to the fact that a variation of $\phi: E \to L$ is the sum of the variations due to varying $\theta : A \to L$ and $F \in L$ separately.

Let us consider the situation for odd traces on E $\Leftrightarrow$ even supertraces on Q. Thus we ~~look~~ at linear functions on

$$(*) \qquad \begin{array}{c} E^{-} \underset{d}{\overset{b}{\rightleftarrows}} (\Omega^1 E \otimes_E)^{-} \\ \| \\ QF \end{array}$$

We have $(EdFE)\otimes_E = E \otimes_{k[F]} \Omega^1 k[F] \otimes_{k[F]}$

$$= EdF \otimes_{k[F]} = EdF / [F, EdF]$$

$$= QdF + QFdF / [F, QdF + QFdF]$$

Better $\quad EdF = (Q^{-}dF + Q^{-}FdF) \quad$ comm. w. F

$\qquad\qquad (+\, Q^{+}dF + Q^{+}FdF) \quad$ anticomm w. F

so $\quad EdF/[F, EdF] \cong \underset{\text{odd rel. } \varepsilon}{Q^{-}dF} + \underset{\text{even rel } \varepsilon}{Q^{-}FdF}$

and $\quad (EdFE\otimes_E)^{-} = Q^{-}dF.$ Thus

$$b(EdFE\otimes_E)^{-} = Q^{-}F$$

Better $\quad b : (EdFE \otimes_E)^{-} \xrightarrow{\sim} Q^{-}F.$

It looks as if we obtain a quotient complex of $(*)$ above obtained by dividing by the acyclic complex

$$Q^{-}F \underset{0}{\overset{\sim}{\rightleftarrows}} (EdFE\otimes_E)^{-}$$

which is

$$(**) \quad Q^+F \rightleftarrows (EdAE\otimes_E)^-$$

Now $\quad EdAE\otimes_E = E\otimes_A \Omega^1 A\otimes_A$

$$= (Q+FQ)\otimes_A \Omega^1 A\otimes_A$$

$$\cong \underbrace{Q\otimes_A \Omega^1 A\otimes_A}_{\text{even rel. to } \varepsilon} \oplus \underbrace{{}_\gamma Q\otimes_A \Omega^1 A\otimes_A}_{\text{odd rel } \varepsilon}$$

$$\therefore \quad (EdAE\otimes_E)^- \simeq {}_\gamma Q\otimes_A \Omega^1 A\otimes_A$$

which we have seen is the universal space for even supertraces on $\Omega^1 Q$. Therefore it is rather likely that $(**)$ is the complex $X^s(Q)^+$.

Before taking up the calculation, let's look at the case of even supertraces on E.

$$\begin{array}{l} E^+ \rightleftarrows ({}_\varepsilon\Omega^1 E\otimes_E)^+ \\ \parallel \\ Q \end{array}$$

$${}_\varepsilon EdAE\otimes_E = E\otimes_A \Omega^1 A\otimes_A = (Q+FQ)\otimes_A \Omega^1 A\otimes_A$$

$= Q\Omega^1 A\otimes_A \oplus$ [redacted] $FQ\Omega^1 A\otimes_A \cong {}_\gamma Q\otimes_A \Omega^1 A\otimes_A$

$$\therefore ({}_\varepsilon EdAE\otimes_E)^+ \simeq Q\otimes_A \Omega^1 A\otimes_A$$

${}_\varepsilon EdFE\otimes_E = {}_\varepsilon E\otimes_{k[F]}$ [redacted] $\Omega^1 k[F]\otimes_{k[F]}$

$$= EdF\otimes_{k[F]} = EdF/[F, EdF]_s$$

$$EdF = Q^-dF + Q^-FdF \qquad \text{comm. with } F$$
$$Q^+dF + Q^+FdF \qquad \text{anticomm. w. } F$$

$$\therefore\ EdF/[F, EdF]_s = Q^+dF + Q^+FdF$$

$$({}_\varepsilon EdFE\otimes_E)^+ = Q^+FdF$$

$$b({}_\varepsilon EdF\cancel{E}\otimes_E)^+ = Q^+ \qquad (b = b_s)$$

In fact $b : ({}_\varepsilon EdFE\otimes_E)^+ \xrightarrow{\sim} Q^+$, so forming the quotient complex gives

$$Q^- \rightleftarrows Q\otimes_A \Omega^1 A \otimes_A$$

which ought to be [redacted] isomorphic to $X(Q)^-$. In this case (even supertraces on E) we have a map

$$X(Q) \longrightarrow X^s(E)^+$$

because supertraces restrict to traces on the even subalgebra.

August 10, 1990

Consider $E = A * k[F]$ as a superalgebra with involution $\varepsilon : a \mapsto a$, $\varepsilon : F \mapsto -F$. We want to study supertraces and their homotopy on E because of the relation with Fredholm modules, which we now recall.

Given $A \xrightarrow{\theta} L$, $F \in L$ one gets an induced homom. $\phi : E \longrightarrow L$. A trace Tr on L pulls back to a trace on E which we can make odd:

$$\tau(x + yF) = Tr(yF) \qquad x, y \in Q$$

In the graded case this is identically zero, and we consider the pull back of the even supertrace $Tr\,\varepsilon$ on L, which gives an even supertrace on E:

$$\tau(x + yF) = Tr\{\varepsilon(x + yF)\} = Tr(\varepsilon x)$$

Suppose we have a first order variation: $\dot{\theta} : A \longrightarrow L$ derivation rel θ, $\dot{F} \in L$ s.t. $F\dot{F} + \dot{F}F = 0$. This induces a derivation $\dot{\phi} : E \longrightarrow L$ rel ϕ, hence a bimodule morphism $\Omega^1 E \longrightarrow L$, so we can pull back Tr + make it odd, or pull back $Tr\,\varepsilon$ in the graded case. In this way we obtain an odd (or even) supertrace on $\Omega^1 E$.

Goal is to calculate $E^{\pm} \underset{d}{\overset{b}{\rightleftarrows}} (\Omega^1 E_{\natural})^{\pm}$.

A key idea is to use a "gauge transformation" to make the variation $\dot{F}$ of F zero. (Here we use separability of $k[F]$, i.e., the fact that any derivation with values in a bimodule is inner: Given $D: k[F] \to M$ put $X = \frac{1}{2}FD(F)$. Then

$$D(F) = [F, \tfrac{1}{2}FDF] = [F, X]$$

showing $D = -ad(X)$.)

In concrete terms this means we replace $d: E \to \Omega^1 E$ with

$$\Delta = d + ad(X) : E \to \Omega^1 E$$
$$X = \tfrac{1}{2}FdF$$

which is a derivation satisfying $\Delta(F) = 0$.

Consider the induced bimodule map $\tilde{\Delta}: \Omega^1 E \to \Omega^1 E$. It kills the subbimodule $EdFE$, and modulo this subbimodule one has $\Delta \equiv d$, hence $\tilde{\Delta} \equiv \tilde{d} =$ identity of $\Omega^1 E$. Thus we have a splitting of the exact sequence

$$0 \to EdFE \to \Omega^1 E \underset{\tilde{\Delta}}{\rightleftarrows} \Omega^1 E / EdFE \to 0$$

giving

$$\Omega^1 E = \tilde{\Delta}(\Omega^1 E) \oplus EdFE$$

Thus

$$\Omega^1 E_{\natural s} = \left(\tilde{\Delta}(\Omega^1 E)\right)_{\natural s} \oplus (EdFE)_{\natural s}$$

Now in the present case, X is even, so that

$\operatorname{ad}(X)$ is the superbracket with X. This means that $d = \Delta$ from E to $\Omega^1 E_{\natural s}$, so the image of d should be contained in $(\tilde{\Delta}(\Omega^1 E))_{\natural s}$.

Because $E = A * k[F]$ we know that

$$\Omega^1 E = \left(E \otimes_A \Omega^1 A \otimes_A E\right) \oplus \underbrace{\left(E \otimes_{k[F]} \Omega^1 k[F] \otimes_{k[F]} E\right)}_{E\,dF\,E}$$

hence

$$\Omega^1 E / E\,dF\,E \cong E \otimes_A \Omega^1 A \otimes_A E .$$

General discussion (link with Kadison's Thesis). Suppose S is a separable subalgebra of an algebra E. Separable means that the sequence of S-bimodules

$$(*) \qquad 0 \longrightarrow \Omega^1 S \longrightarrow S \otimes S \longrightarrow S \longrightarrow 0$$

splits (i.e. S is projective as an S-bimodule). So we get exact sequences

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & E \arrow[u] \arrow[r, equal] & E \arrow[u] & \\
0 \arrow[r] & E \otimes_S \Omega^1 S \otimes_S E \arrow[r] & E \otimes E \arrow[u] \arrow[r] & E \otimes_S E \arrow[u] \arrow[r] \arrow[l, dashed, bend right] & 0 \\
 & & \Omega^1 E \arrow[u] \arrow[r] & \Omega^1(E/S) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

which show that we have a split exact sequence

$$\begin{tikzcd}
0 \arrow[r] & E \otimes_S \Omega^1 S \otimes_S E \arrow[r] & \Omega^1 E \arrow[r] & \Omega^1(E/S) \arrow[r] \arrow[l, dashed, bend right] & 0
\end{tikzcd}$$

where the splitting essentially is due to the splitting of $(*)$ for S.

This gives us then a splitting

$$\Omega^1 E_\natural = \left(\Omega^1(E/S)_\natural\right) \oplus \left(E \otimes_S \Omega^1 S \otimes_S\right)$$

where the image of d should land in the first factor. What we would hope for is an isomorphism

$$b : E \otimes_S \Omega^1 S \otimes_S \xrightarrow{\sim} [E,S] \subset E,$$

which is more or less clear by tensoring ⊛ with E over $S \otimes S^{op}$. This means we have the direct sum situation

$$\begin{tikzcd}
E/[E,S] \arrow[r, shift right, "d"'] & \Omega^1(E/S)_\natural \arrow[l, shift right, "b"'] \\
\oplus & \oplus \\
{[E,S]} \arrow[r, shift right, "0"'] & E \otimes_S \Omega^1 S \otimes_S \arrow[l, shift right, "\cong"'] \\
\| & \| \\
E \arrow[r, shift right] & \Omega^1 E_\natural \arrow[l, shift right]
\end{tikzcd}$$

One should not have to assume S is a subalgebra of E.

The picture is pretty clear now for $E = A * k[F]$, but things are more complicated because of supertraces. First let's compute $(E\,dF\,E)_{\natural s} = E\,dF/[F, E\,dF]_s$.

$$\left(E\,dF/[F,E\,dF]_s\right)^- = Q\,dF/[F, Q\,dF]_- = Q^-\,dF$$

$$\left(E\,dF/[F,E\,dF]_s\right)^+ = QF\,dF/[F, Q\,dF]_+ = Q^+\,F\,dF$$

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Note that

$$b_s : Q^- dF \xrightarrow{\sim} Q^- F \subset E^-$$

$$b_s : Q^+ F dF \xrightarrow{\sim} Q^+ \subset E^+$$

leaving over $E^-/Q^-F = Q^+F$, $E^+/Q^+ = Q^-$.

The next point to analyze is the other piece which comes from

$$\tilde{\Delta}(\Omega^1 E) \cong \Omega^1(E/k[F]) \cong E \otimes_A \Omega^1 A \otimes_A E$$

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$$E = A * k[F] = Q \tilde{\otimes} k[F]$$

is a superalgebra with $a^\varepsilon = a \qquad F^\varepsilon = -F$.

$$\left|\begin{matrix}\text{odd} \\ \text{even}\end{matrix}\right. \text{ supertraces on } E = \left|\begin{matrix}\text{even} \\ \text{odd}\end{matrix}\right. \text{ supertraces on } Q$$

via $\quad \tau(x+yF) = \tau'(y)$

resp. $\quad \tau(x+yF) = \tau'(x)$

To calculate $E \rightleftarrows \Omega E_{\natural S}$. Put $S = k[F]$. We have exact sequence of E-bimodules

$$\begin{tikzcd}
0 \arrow[r] & E \otimes_S \Omega^1 S \otimes_S E \arrow[r] \arrow[d, phantom, "\parallel"] & \Omega^1 E \arrow[r] & \Omega^1(E/S) \arrow[r] \arrow[d, phantom, "\wr"] & 0 \\
& E\,dF\,E & & E \otimes_A \Omega^1 A \otimes_A E \arrow[ul, dashed] &
\end{tikzcd}$$

relative differentials

which splits as $E = A * S$. This splitting, which is given by the map $\Omega^1(E/S) \longrightarrow \Omega^1(E)$ associated to the derivation $F \mapsto 0$, $a \mapsto da$ is different from the splitting given by $\tilde{\Delta}$, associated to $\Delta(F) = 0$, $\Delta(a) = da + [X, a]$. The difference of these two liftings is the map $\Omega^1(E/S) \longrightarrow E\,dF\,E$ associated to the derivation with $F \mapsto 0$, $a \mapsto [X, a]$.

We consider the relative complex

$$E/[F,E]_s \rightleftarrows \Omega^1(E/S)_{\natural S}$$

Note $\quad [F,E]_s = [F, Q+QF]_s = FQ^- \oplus Q^+$

$$E/[F,E]_s = Q^- \oplus FQ^+$$

Also

$$\Omega^1(E/S) \xleftarrow{\sim} E\otimes_A \Omega^1 A \otimes_A E$$

$$\Omega^1(E/S)_{\natural s} \longleftarrow \left(E\otimes_A \Omega^1 A\otimes_A\right)^+ \oplus \left(E\otimes_A \Omega^1 A\otimes_A\right)^-$$

$$\begin{array}{ccc} \| & & F\cdot \, \| \\ Q\otimes_A \Omega^1 A\otimes_A & & {}_\gamma Q\otimes_A \Omega^1 A\otimes_A \end{array}$$

According to this isomorphism an odd trace T on $\Omega^1(E/S)$ is equivalent to a trace on $FQ\,\Omega^1 A \leftarrow {}_\gamma Q\otimes_A \Omega^1 A$ over A, hence it is equivalent to an arbitrary family of odd cochains via

$$T_{2n+1} = T(F\theta^\gamma q^{2n}\, d\theta) = T(\theta q^{2n} F d\theta)$$

Claim we have an isomorphism

$$(\circledast)\qquad \begin{tikzcd} Q^+ \arrow[r, shift left] \arrow[d, "\cong"] & (\Omega^1 Q_{\natural s})^+ \arrow[l, shift left] \arrow[d, "\cong"] \\ (E/[F,E]_s)^- \arrow[r, shift right] & (\Omega^1(E/S)_{\natural s})^- \arrow[l, shift right] \end{tikzcd}$$

where the vertical maps are given by $\alpha \mapsto F\alpha$ essentially. First we have the commutative square

$$\begin{tikzcd} Q \arrow[r, "d"] \arrow[d, "F\cdot"'] & \Omega^1 Q \arrow[d, "F\cdot"] \\ E \arrow[r, "d"] & \Omega^1(E/S) \end{tikzcd}$$

since $dF = 0$ in $\Omega^1(E/S)$. Next observe we have an induced map

$$\Omega^1 Q/[Q, {}_\gamma\Omega^1 Q] \xrightarrow{F\cdot} \Omega^1 E/[E, \Omega^1 E]$$

$$x^\gamma\omega - \omega x \longmapsto F(x^\gamma\omega - \omega x) = x(F\omega) - (F\omega)x$$

This gives us the d-part of ⊛. As for b we have

$$b_s(x\,dy) = xy - y^{\gamma}x \qquad x, y \text{ same parity}$$

$$\downarrow F\cdot$$

$$b_s(Fx\,dy) = Fxy - yFx$$

so it works. The fact that the vertical maps are isomorphisms in ⊛ is clear from our calculations.

In the other case we immediately get a commutative diagram

$$\begin{tikzcd} Q \arrow[r, shift left] \arrow[d] & \Omega^1 Q_\natural \arrow[l, shift left] \arrow[d] \\ E \arrow[r, shift left] & \Omega^1 E_{\natural s} \arrow[l, shift left] \end{tikzcd}$$

since Q is the even subalgebra of E. This induces

$$\begin{tikzcd} Q^- \arrow[r, shift left] \arrow[d] & (\Omega^1 Q_\natural)^- \arrow[l, shift left] \arrow[d] \\ (E/[F,E]_s)^+ \arrow[r, shift left] & (\Omega^1(E/S)_{\natural s})^+ \arrow[l, shift left] \end{tikzcd}$$

using inclusion ~~of~~ for the ~~top~~ row + projection for the bottom. [redacted] The vertical maps should be isoms.