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Notes on BRS cohomology

A. \( \Omega^*(G) \) where \( G \) is a Lie group

\[ g_f = \text{Lie}(G), \quad X_a \text{ basis for } g_f, \quad [X_a, X_b] = f_{ab}^c X_c \]

The basic object is the Maurer-Cartan form

\[ \Theta = X^a X_a \in \Omega^1(G) \otimes_R g_f \]

Properties:
1) \( \iota_X \Theta = 0 \quad \Rightarrow \quad \iota_{\Theta^a} = \delta^b_a \)

2) \( \text{Ad}(g) R_g^* (\Theta) = 0, \quad \mathcal{L}_X \Theta + [X, \Theta] = 0 \quad \Rightarrow \quad \mathcal{L}_a \Theta^b + f_{ac}^b \Theta^c = 0 \)

3) \( \frac{1}{2} [\Theta, \Theta] = (d \Theta + \Theta^2) = 0 \quad \Rightarrow \quad d \Theta^a + \frac{1}{2} f_{bc}^a \Theta^b \Theta^c = 0 \)

4) \( \mathcal{L}_g \Theta = 0 \quad \Rightarrow \quad \Theta^a \) left invariant

The right translation action \( R_g (g') = g'^g \) makes \( G \) into a principal \( G \)-bundle with base a point. \( \Theta \) is the connection form for the unique connection in this bundle, explaining 1)-3). Left translations are automorphisms of this principal bundle, whence 4).

\[ \begin{align*}
\Omega^*(G) & \xrightarrow{\text{deg}} \wedge^g \Omega^* \\
\Omega^*(G) & \sim \Omega^*(G) \otimes \wedge^g \Omega^*
\end{align*} \]

\( \delta \) on \( \Omega^*(G) \) induces a differential \( \delta \) on \( \wedge^g \Omega^* \) making it a (comm) DG algebra.

6) \( \Omega^*(G)^G = \wedge^g \Omega^* = \wedge^g [\Theta^a] \) with \( \delta \) given by

\[
\delta \Theta^a + \frac{1}{2} f_{bc}^a \Theta^b \Theta^c = 0
\]
B. Lie algebra cohomology

Let $V$ be a vector space and consider the trivial bundle $G \times V \to G$ with $G$ acting by left translation on itself and trivially on $V$.

1) A lift-invariant connection $\nabla = d + \alpha$ on $\tilde{V}$, $\alpha \in \mathfrak{g}^* \otimes \text{End}(V)$ is flat $\iff$ $u: X \mapsto ux$ is a representation of $\mathfrak{g}$ on $V$. (compatible with $[,]$)

One has $\alpha = \Theta^a u_a$ ($u_a = u(X_a)$)

$$d\alpha + \alpha^2 = d\Theta^a u_c + \Theta^a u_c \Theta^b u_b$$

$$= \left(-\frac{1}{2} f^c_{ab} \Theta^a \Theta^b\right) u_c + \frac{1}{2} \Theta^a \Theta^b [u_a, u_b]$$

$$= \frac{1}{2} \Theta^a \Theta^b \left(-f^c_{ab} u_c + [u_a, u_b]\right) \quad \square$$

Fix a rep $u: \mathfrak{g} \to \text{End } V$; one has the flat lift-invariant connection $d + u\Theta$ on $\tilde{V}$, whose complex of lift-invariant $V$-valued forms

$$(\Omega(G) \otimes V)^G = \Lambda \mathfrak{g}^* \otimes V$$

which is a DG module over $\Omega(G)^G = \Lambda \mathfrak{g}^*$. Denote by $\delta$ the differential in the right picture.

$$\delta \Phi = \Theta^a u_a(\iota) \quad \iota \in V$$

The complex $(\Lambda \mathfrak{g}^* \otimes V, \delta)$ is the complex of Lie cochains with values in $V$. 
B1. Lie alg. cochains with values in a representation $U: G \to \text{Aut}(V)$.

View $U$ as a gauge transformation:

$$G \times V \to G \times V \quad (g, v) \mapsto (g, U(g)v)$$

One has an isomorphism

$$(\Omega(G) \otimes V)^{G_L, U} \cong (\Omega(G) \otimes V)^{d + u\Theta}$$

as

$$(U^{-1} \cdot d \cdot U)^\xi = (d + U^{-1} d U)^\xi$$

$u^*$ of MC form on $\text{Aut}(V)$

Thus $(\Lambda^g \otimes V, \delta)$ is isomorphic to the invariants in $(\Omega(G) \otimes V, d)$ for the $G_L, U$ action.


Claim

$$H^*(g, V) = \frac{H^*(\mathfrak{g}) \otimes V}{\mathfrak{g}} \cong H^*(G)$$

Pf. Start with $V = \mathbb{C}$, one has the averaging operator $P = \int g^* \in \mathcal{L}(G)$ which commutes with $d$ and projects onto the left-invariant forms. It induces a projection operator on $H^*(\Omega(G))$, but an argument is needed to see that the induced projection projects onto
the subspace of invariants. Let's proceed directly and show

1) \[ H^*(\Omega(G)^G) \rightarrow H^*(\Omega(G)) \]

Injective: Given \( \omega \) such that \( g^*\omega = \omega \), \( \forall g \) and \( d\omega = 0 \), and suppose \( \omega = dg\eta \), \( \eta \in \Omega(G) \). Then
\[
d(P\eta) = d\left( \int_G g^*\eta \right) = \int_G g^*d\eta = \int_G \omega = 0
\]
where we have used the continuity of \( d \).

Surjective: Given \( \omega \) in \( \Omega(G) \) with \( d\omega = 0 \) and such that its class is invariant: \( \omega - g^*\omega \in \text{Ind } G \), \( \forall g \in G \). Then
\[
d(P\omega) = d\left( \int_G g^*\omega \right) = \int_G g^*d\omega = \int_G (\omega - g^*\omega) \in \text{Ind } G
\]
again by continuity of \( d \). Next
\[
\omega - P\omega = \omega - \int_G g^*\omega = \int_G (\omega - g^*\omega) \in \text{Ind } G
\]
belongs to the closure of \( \text{Ind } G \). Since \( \text{Ind } G \) is closed (elementary proof in Begg's thesis), \( \omega \) is cohomologous to the invariant form \( P\omega \).

Return to our claim. Because \( G \) is connected it acts trivially on \( H^*(\Omega(G)) = H^*(G) \) \((Lx = dx + \chi d \) is trivial on \( H^*(\Omega(G))\)), so 1) above yield \( H^*(\Lambda(g^*), \delta) \sim H^*(G) \)

Next consider general case. We know that \( (\Lambda(g^*), \delta) = (\Omega(G) \otimes V, d + u\theta)^G \)
\[ = (\Omega(G) \otimes V, d)^G, \quad ^G \]
has cohomology equal to
\[ H^*(\mathcal{O}(G) \otimes V, \mathcal{O}_G) \cong \mathcal{O}(G) \otimes V \mathcal{O}_G \] 

by the above averaged operator argument. Thus \( \mathcal{O}(G) \otimes V \mathcal{O}_G = H^*(\mathcal{O}(G) \otimes V) \) since \( G \) acts trivially on \( H^*(\mathcal{O}(G)) \).

Specific map: \( H^*(\mathcal{O}(G) \otimes V) \rightarrow H^*(\mathcal{O}(V)) \) is obvious from the fact that \( H^*(\mathcal{O}(G) \otimes V) \) is a module over \( H^*(\mathcal{O}(G)) \). This given by \( [\omega] \cdot [\xi] \mapsto [\omega \otimes \xi] \).

C. BRS cohomology is Lie algebra cohomology associated to a DG Lie algebra.

Let \( P \) be a \( G \)-manifold, \( G \) acting on \( P \) on the left. One has for each \( X \in \mathfrak{g} \) operators \( L_X : \Omega^* P \rightarrow \Omega^* P \) satisfying:

\[
\begin{align*}
\{ L_X, L_Y \} &= L_{[X,Y]} \\
\{ L_X, \iota_Y \} &= \iota_{[X,Y]} \\
\{ \iota_X, \iota_Y \} &= 0 \\
d, \iota_X &= L_X
\end{align*}
\]

\( L_X \) derivation of degree 0

\( \iota_X \) (anti-) -1

Define \( \tilde{\mathfrak{g}} \) to be the DG Lie algebra with \( \tilde{\mathfrak{g}}^0 = \mathfrak{g}_c \), \( \tilde{\mathfrak{g}}^{-1} = \mathfrak{g}_c \); write \( L_X \in \mathfrak{g}_c \) for the element corresponding to \( X \in \mathfrak{g} \), and similarly \( \iota_X \in \tilde{\mathfrak{g}}^{-1} \). The bracket in \( \tilde{\mathfrak{g}} \) is defined by the above formulas and the differential \( d : \tilde{\mathfrak{g}}^{-1} \rightarrow \tilde{\mathfrak{g}}^0 \) by \( d(\iota_X) = L_X \).

Then \( \Omega^* P \) is a DG module over the DG Lie algebra \( \tilde{\mathfrak{g}} \). In fact one has a DG Lie algebra.
morphism
\[ \hat{\Omega} \rightarrow \text{Der}(\Omega(P)) \]

Analogue of \( \Lambda g^* \) and \( \Theta \in \Lambda g^* \otimes g^* \).
\[ S(\Sigma \hat{g}^*)^* = \Lambda(\hat{g}^*)^* \otimes S(\hat{g}^{-1})^* \]
\[ \Theta = \Lambda g^* \otimes S g^* \]

Let \( x^a \in (\hat{g}^*)^* \) be the basis dual to \( l_a \)
let \( \varphi^a \in (\hat{g}^{-1})^* \)

Then
\[ S(\Sigma \hat{g}^*)^* = \Lambda[x^a] \otimes S[\varphi^a] \]

with \( \deg x^a = 1 \), \( \deg \varphi^a = 2 \).
The analogue of \( \Theta \in \Lambda g^* \otimes g^* \) is
\[ \Theta = x^a l_a + \varphi^a l_a \in S(\Sigma \hat{g}^*)^* \otimes \hat{g}^* \]

and the differential \( \delta \) on \( S(\Sigma \hat{g}^*)^* \) is determined by
\[ \delta \Theta + \Theta^2 = 0 \]

\[ \begin{align*}
0 &= \delta \Theta + \Theta^2 \\
&= \delta x^a l_a + \delta \varphi^a l_a \\
&\quad + (x^b l_b + \varphi^b l_b)(x^c l_c + \varphi^c l_c) \\
&= \delta x^a l_a + \frac{1}{2} x^b x^c (l_b l_c) f_{bc} l_a \\
&\quad + \delta \varphi^a l_a + x^b \varphi^c (l_b l_c - l_c l_b) + \frac{1}{2} \varphi^b \varphi^c (l_b l_c + l_c l_b) \tag{1} \\
&\quad - \delta \varphi^a l_a \\
\end{align*} \]

Next \( d \) on \( \hat{g}^* \) induces a \( d \) on \( S(\Sigma \hat{g}^*)^* \)
such that \( d \Theta = 0 \) for total \( d \) on \( S(\Sigma \hat{g}^*)^* \otimes \hat{g}^* \).

\[ \begin{align*}
\delta x + x^2 &= 0 \\
\delta \varphi + [x \varphi] &= 0 \\
\end{align*} \]
$\Omega = d\Theta = d\chi^a L_a - \chi^a dL_a + dp^a (\gamma a + \Phi b d(\omega))$

$= (d\chi^a + \Psi^a) L_a + (dp^a) \gamma a$

$d\chi^a = -\Psi^a$

$dp^a = 0$

---

$C1.$ Lie cochains on $\gamma$ values in $\Omega(P)$:

\[ \delta(\Sigma \gamma)^* \otimes \Omega(P) = \Lambda[x] \otimes \delta[y] \otimes \Omega(P) \]

This is a DGA over $\delta(\Sigma \gamma)^*$ with $\delta$ given by

\[ \delta(\omega) = \Theta \cdot \omega = \chi^a (L_a \omega) + \Psi^a (\gamma a \omega) \]

Assume now $P$ is a principal $G$-bundle with connection $A = A^a X_a \in \Omega^1(P) \otimes \gamma$, and curvature $F = dA + A^2 = F^a X_a \in \Omega^2(P) \otimes \gamma$. Recall

\[ L_X A + [X, A] = 0 \quad \quad \quad L_X F + [X, F] = 0 \]

\[ L_X A = X \quad \quad \quad L_X F = 0 \]

or

\[ L_a A^b + f^{b}_{ac} A^c = 0 \quad \quad \quad L_a F^b + f^{b}_{ac} F^c = 0 \]

\[ L_a A^b = \delta^b_a \quad \quad \quad L_a F^b = 0 \]

Then

\[ \delta(A^b) = (\chi^a L_a + \Psi^a \gamma a) A^b = -f^{b}_{ac} \chi^a A^c + \Psi^b \]

\[ \delta(F^b) = (\chi^a L_a + \Psi^a \gamma a) F^b = -f^{b}_{ac} \chi^a F^c \]

or

\[ \delta A = -[X, A] + \Psi \]

\[ \delta F = -[X, F] \]
Thm: Let \( P \) be a principal \( G \)-bundle, \( G \) compact connected. Then

1) The \( d \)-cohomology of \( S(\Sigma^g)^* \otimes \Omega(P) \) the BRS algebra is \( H^*(P) \) concentrated in \( (d,\delta) = 0 \), consequently the total or \( d+\delta \) cohomology is \( H^*(P) \).

2) The \( \delta \)-cohomology is \( \Omega(B) \otimes H^g(G) \)
where \( B \) is the base.

Proof. 1) As far as \( d \) is concerned, the BRS alg is the tensor product of \( \Lambda[\mathfrak{g}] \otimes S[q] \), which has trivial cohomology \( C \) in degree \( (0,0) \), and \( \Omega(P) \) which is located in the line \( \delta = 0 \). Thus the \( d \)-cohomology is \( H^*(P) \) located in \( \delta = 0 \), and the spectral sequence collapses giving the same result for the \( d+\delta \) cohomology.

2) Since \( q = 8A + [K,A] \), \( q^a = 8A^a + f^a_{bc} X^b A^c \)
we have

\[
\Lambda[\mathfrak{g}] \otimes S[q] \otimes \Omega(P) = \Lambda[\mathfrak{g}] \otimes \Omega(P) \otimes S[8A]
\]

\[
= (\Lambda[\mathfrak{g}] \otimes \Omega_{\text{hor}}(P)) \otimes (\Lambda[A] \otimes S[8A])
\]

Note that if \( \omega \in \Omega_{\text{hor}}(P) \), then

\[
Sw = (X^a L_a + q^a L_a) \omega = X^a L_a \omega \in \Lambda[\mathfrak{g}] \otimes \Omega_{\text{hor}}(P).
\]

\[
(L_x L_y) \omega = \langle L_y [x] - L_x [y] \rangle \omega = -\{x,y\} \omega
\]

so the operators \( L_x \) preserve \( \Omega_{\text{hor}}(P) \). Thus 3) shows the BRS algebra with \( S \) is the tensor product of the Lie cochains on \( g \) acting on \( \Omega_{\text{hor}}(P) \) with a contractible algebra. Thus

\[
H^*(BRS, S) = H^*(g, \Omega_{\text{hor}}(P)) = H^*(g) \otimes \Omega_{\text{hor}}(P) \quad \text{gf}
\]

\[
= H^*(g) \otimes \Omega(B) \quad \text{gf}
\]
Natural question is whether spectral sequence starting with the $d_1$ cohomology $E^1 = H^0(G) \otimes H^*(G)$ and ending with $H^*(P)$ is the same as the Leray spectral sequence.
We have an analogy between the 
\((b, s, 1 - k, B)\) operators on reduced 
cochains and operators \((d, i_x, \lambda_x, p(I_x))\) 
occurring for manifolds with circles 
action. Here \(P = \int_s \exp(i_x)^\ast\) is the 
averaging operator.

Let \(M\) be a manifold with \(S^1\)-action. 
Let's assume the action is free, or at least that 
the isotropy groups are finite, i.e. no fixed points. 
Then \(i_x\) considered as a differential on \(\Omega(M)\) 
is exact: \(\ker i_x = \text{im} (i_x)\). Recall that 
in the cochain setup we have

1) cyclic cochains = \text{im} B = \ker(s) \cap \ker(sb) 
The analogue of this should be

2) basic forms = \text{ker} (i_x) \cap \ker (i_xd) = \text{im} P(I_x) 
The first equality is clear since \(\ker (i_x) = \Omega_{\text{hor}} (M)\) and \(\ker (i_x) \cap \ker (i_xd) = \Omega_{\text{bas}} (M)\) and \(d i_x + i_xd = \lambda_x\). 
The second equality is clear because quite 
generally the averaging operator projects onto the 
invariants, so that applying this principle to 
\(\text{im} i_x = \Omega_{\text{hor}}\) we have \((\text{im} i_x)^{S^1} = p \text{im} i_x = \text{im} P i_x\). 
Next for cochains we have

3) \(\ker B = \text{im} s + \text{im} bs\) 
so the analogue should be

4) \(\ker P i_x = \text{im} i_x + \text{im} d(i_x) = \Omega_{\text{hor}} + d\Omega_{\text{hor}}\)
To check this we have to understand the kernel of $P$ on $\Omega_{\text{hor}}$. In general one should have that \( \text{Ker } P = \text{Im } L_x \), because
\[
1 - P = \int_0^\infty (1 - e^{-tL_x}) dt
= L_x \int_0^\infty \left(1 - \frac{e^{-tL_x}}{L_x} \right) dt
= -\int_0^\infty e^{sL_x} ds.
\]
Thus if $P_x \omega = 0$, then $\langle x, \omega \rangle = L_x(\langle x, \eta \rangle) = \langle x, d\eta \rangle$ for some $\eta$, hence
\[
\omega = (\omega - \langle x, \eta \rangle) + \langle x, \eta \rangle \in \Omega_{\text{hor}} + d\Omega_{\text{hor}}.
\]

Let's check now the

Key Lemma: \( \text{Ker } P_x / \text{Im } P_x \) is acyclic.

Proof. Let $P_x \omega = 0$ and $dw \in \text{Im } P_x$; we have to show that $\omega \in d(\text{Ker } P_x) + (\text{Im } P_x)$. The second condition gives $\langle x, dw \rangle \in \text{Im } (\langle x, P_x \rangle) = 0$ since $\langle x, P_x \rangle = P_x$. Thus $\langle x, dw \rangle = 0$.

Using 4) one has $P_x \omega = 0 \Rightarrow \omega = \eta + d\xi$ with $\eta, \xi$ horizontal. As $\langle x, d\eta \rangle = \langle x, dw \rangle = 0$ we have $\eta \in \text{Ker } P_x \cap \text{Ker } (\langle x, d \rangle = \text{Im } P_x$ using 2). As $\langle x, \xi \rangle = 0$ we have $P_x \xi = 0$ so $\xi \in \text{Ker } P_x$. Thus $\omega = \eta + d\xi \in \text{Im } P_x + d(\text{Ker } P_x)$. \( \square \)

Thus we have a nice analogy of reduced cochains with forms on a manifold with circle
action having no fixed points.

What about the Bianchi forms associated to a vector bundle with connection $(E, \nabla)$ over $M$. Recall that these are
\[
\eta = \text{tr} \ e^{(\nabla + i x)^2} = \text{tr} \ e^{i x + D^2}
\]
and that one has
\[
(d + i x) \eta = \text{tr} \left( [\nabla + i x, e^{(\nabla + i x)^2}] \right) = 0
\]
Notice that $\eta$ is even: $\eta = \sum_{n \geq 0} \eta^{(2n)}$ and that one has
\[
d\eta^{(2n)} = -i x \eta^{(2n+2)}
\]

These odd forms are basic.

An interesting point would be whether the forms $\eta^{(2n)}$ could be non-invariant under the circle action, because this might give some insight into JLO's cocycle. However
\[
(d + i x) \eta = 0 \implies (d + i x)^2 \eta = L_x \eta = 0
\]
Thus cocycles for $d + i x$ are automatically invariant.

\[\text{Observation (Feb 1)} \quad \left( \int_0^1 e^{t L_x} dt \right) L_x = T_{\Pi x}\]

where $\Pi : M \to M/S^1$. 
February 2, 1990

\[ S^1 \rightarrow M \overset{\pi}{\rightarrow} B \quad \text{principal bundle} \]

\[ 0 \rightarrow \ker \pi_* \overset{\rightarrow}{\rightarrow} \Omega(M) \overset{\pi_*}{\rightarrow} \Omega(B) \rightarrow 0 \]

For the analogue of $S^1$ operation, given $\omega \in \Omega(B)$ closed we want $f \in \Omega(M)$ satisfying $\omega \overline{df} = 0$ (analogue of $\omega \overline{df} = 0$) with $\pi_* f = \omega$. Then

\[ f = \Theta \cdot \pi^* \omega \]

Then $\pi_* f = \pi_* (\Theta \cdot \omega) \Rightarrow \text{want } \pi_* (\Theta) = 1$

and

\[ \chi_x (\Theta \pi^* (\omega)) = \chi_x (d \Theta \cdot \pi^* (\omega)) = (\chi_x d \Theta) \cdot \pi^* (\omega) \]

\[ \Rightarrow \text{want } \chi_x d \Theta = 0. \]

Now the easiest way to arrange $\pi_* \Theta = 1$ is to have $\chi_x \Theta = 1$.

In this case $\chi_x \Theta = (d \chi_x + \chi_x d) \Theta = 0$, so $\Theta$ has to be a connection form and our lifting is an invariant form.

\[ 0 \rightarrow \Omega(B) \overset{\pi^*}{\rightarrow} \Omega(P) \overset{\pi_*}{\rightarrow} \Omega(B) \rightarrow 0 \]

This apparently doesn't work with reduced cochains because we can't consider $K$-invariant cochains. However, we can proceed as follows.

Choose $\rho: A \rightarrow k$, $\rho(1) = 1$. Given $z_{n-1}$ a cyclic cocycle, set

\[ g_n (a_0, \ldots, a_n) = \rho(a_0) z_{n-1} (a_1, \ldots, a_n) \]
and set \( f_n = \frac{1}{n+1} \sum_{i=0}^{n} k^i g_n \)

Claim \( k^{n+1} g_n = g_n, \ k f_n = f_n, \ sbf_n = 0 \)

and \( s f_n = \varepsilon_{n-1} \). Thus \( sbf_n \) is a cyclic 
(\( n+1 \))-co-cycle representing \( S[\varepsilon_{n-1}] \).

Proof. Clearly \( g_n \in \tilde{C}_n \) (reduced \( n \)-co-chains) and \( s g_n = \varepsilon_{n-1} \). Thus \( b s g_n = b \varepsilon_{n-1} = 0 \),
so \( k^{n+1} g_n = (1 - bs) g_n = g_n \). Then

\[
(1 - k) f_n = \frac{1}{n+1} (1 - k^{n+1}) g_n = 0
\]

and \( s f_n = \frac{1}{n+1} \sum_{i=0}^{n} k^i s g_n = \varepsilon_{n-1} \).

Finally \( sbf_n = (1 - k - bs) f_n = -b \varepsilon_{n-1} = 0 \).

Interesting question. Suppose \( A = \tilde{A} \) is augmented and that \( f : A \rightarrow C \) is the augmentation. Does this process yield Connes formula for \( S \)?

Transgression: Let \( P \) be a principal \( U \) bundle with base \( B \). We know from considering connections and curvature: \( \text{Chern-Weil theory} \), that there are \( \text{odd degree classes} \ \{ c_S \in H^{2n-1}(P) \} \) defined for \( 2n > \dim B \) which are completely canonical for \( 2n > \dim B + 1 \). (Already when \( 2n-1 = \dim B \) one gets non-uniqueness in the case of \( \text{the} \))
trivial bundle \( B \times G \xrightarrow{p_1} B \)

\( G = U_N \) since gauge transformations will act non-trivially on \( c_{2n-1} \).

**Better:** The odd classes \( c_{2n-1} \) come from the classes \( c_{2n-1} \in H^{2n-1}(G) \) and these for \( 2n-1 \leq \dim B \) will be affected by gauge transformations.

**Question:** How to show these odd classes in \( H^*(P) \) are defined without using differential forms?

**One method is that of Chern-Weil:**
Construct a transgression cochain in a universal situation.

**More concrete method.** We can split off a trivial bundle: Let \( E = P \times U_N \mathbb{C}^r \). Then we have \( E = E_1 \oplus \mathbb{C}^r \) where \( 2 \text{rank}(E_1) \leq \dim B \).

Recall how this is done by obstruction theory.

To reduce the structural group of a principal \( G \) bundle \( P \) to the subgroup \( H \), we need a section

\[ P \times G / H = P / H. \]

The inverse image in \( P \) of the image of this section is then a principal \( H \) bundle \( P_1 \) and \( P = P_1 \times^H G \). Incidentally we get in this case an equivariant map

\[ P \rightarrow H \backslash G \]

which is also equivalent to the section.

So in our case we have

\[ P \rightarrow U_r \backslash U_N \]

cohomology starts in degree \( 2r + 1 \).
So we get odd classes in $H^{2n-1}(P)$ for $2n-1 \geq 2r+1$ where $2r < \dim B$ and hence for $2n-1 > \dim B + 1$, i.e. $2n-1 > \dim B$.

Check minimal degree. Suppose $\dim B = 2n-1$. Then $c_{2n-1} \in H^{2n-1}(P)$ is defined. Also we know that $E = E' \oplus C^k$ with $r = n-1$ so we have $p \to U_{n-1} \setminus U_w$ - first class 2n-1.

If $\dim B = 2n$, then $c_{2n+1} \in H^{2n+1}(P)$ is defined and $E = E' \oplus C^k$ with $r = n$, then get $p \to U_n \setminus U_w$ - first class degree 2n+1.
Lundell (Top. Vol 8) showed that the Bott map $S^2 \times U_n \rightarrow U_{2n}$ can be deformed to a map $S^2 \times U_n \rightarrow U_{n+1}$. He proves that the induced map

$$\pi_{2n}(U_n) \rightarrow \pi_{2n+2}(U_{2n})$$

is injective (in fact $1 \rightarrow \pm(n+1)$) and in some later paper, I think he calculates the homotopy groups of his spectrum and gets $\mathbb{Q}/\mathbb{Z}$'s, maybe.

Here's how one perhaps can view his construction. The basic thing to look at are maps

$$M \times U_n \rightarrow U_m$$

which are families of homomorphisms $U_n \rightarrow U_m$ parametrized by $M$. Put another way, we have a homomorphism

$$U_n \rightarrow U_m^M$$

that is a representation of $U_n$ on the trivial bundle $E^m$ over $M$. Up to conjugacy these can be classified easily, in fact, for $U_n$ replaced by a compact Lie group $G$. We have an equivariant $G$-bundle $E$ that is trivial over $M$, where $G$ acts trivially on $M$, and where $E$ is the trivial bundle $\mathbb{C}^m$. 
We can decompose $E$ with respect to the irreducible reps. of $G$:

$$E = \bigoplus W_{\alpha} \otimes \text{Hom}_G(W_{\alpha}, E)$$

and thus we have a collection of bundles \(\text{Hom}_G(W_{\alpha}, E)\) over $M$ indexed by the irreducible representations such that when added up with the multiplicities $dim W_{\alpha}$ yields the trivial bundle $\tilde{\mathbb{C}}^n$.

Take $M = S^2$. I think the Bott map comes from the following. Let $L = O(-1)$ be the canonical subbundle of $\mathbb{C}^2 \otimes \mathbb{C}^n$ over $S^2 = \mathbb{P}(\mathbb{C}^2) = \mathbb{CP}^1$. For each line $l$ and $g \in U_n$, we consider the unitary action of $U_n$ on $\mathbb{C}^2 \otimes \mathbb{C}^n$ which is $1 \otimes g$ on $l \otimes \mathbb{C}^n$ and the identity on $l^\perp \otimes \mathbb{C}^n$. This gives a family of homomorphisms $U_n \rightarrow U_{2n}$ parametrized by $l \in S^2$. We have the trivial bundle $\tilde{\mathbb{C}}^2 \otimes \mathbb{C}^n = O(-1) \otimes \mathbb{C}^n \oplus O(1) \otimes \mathbb{C}^n$ with $g \in U_n$ acting as $1 \otimes g$ on the first factor and the identity on the second.

What one has done is to take $O(-1) \otimes \tilde{\mathbb{C}}^n$ over $S^2$ with the non-trivial action and added a bundle with trivial action to get a trivial vector bundle. Since the base is $S^2$, we could have added $O(2)$ to $O(-1) \otimes \tilde{\mathbb{C}}^n$ to get a trivial bundle of rank $n+1$.

This is basically what Lurder does I think. However, the families of homomorphisms $M \times U_n \rightarrow U_m$
are not maps \( M \times U_n \rightarrow U_n \)
since \( \mathbb{B} \) at the benefit of \( M \)
once has a nontrivial map. If

the family \( M \times U_n \rightarrow U_n \) is \( \varphi_m(g) \rightarrow \varphi_m(g) \varphi_*(g)^{-1} \)
then \( \varphi_m(g) \varphi_*(g)^{-1} \) is the identity

if either \( m = * \) or \( g = 1 \).

Let's check the Bott map arises in this

way. I recall that Bott's map are families of

minimal geodesics between "antipodal" points. Thus

one has

\[
[0, \pi] \times U_n \longrightarrow \operatorname{Gr}_n(C^{2n})
\]

\[
(\theta, g) \longmapsto F = \cos \theta \varepsilon + \sin \theta \varphi_*(g) \varphi_*(g)^{-1}
\]

which gives the graph \((g, \varepsilon)\) path \( F \) from \( -\varepsilon \leftrightarrow 0 \leftrightarrow \varepsilon \). In effect

\[
\text{graph } T \leftrightarrow F \quad \text{where} \quad F \left( \frac{1}{1+X} \right) = \frac{1}{1+X} \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{1-t} & 1 \end{array} \right) \left( \begin{array}{c} 0 \\ X \end{array} \right)
\]

i.e.

\[
F = (1+X) \varepsilon (1+X)^{-1} = \frac{1+X}{1-X} \varepsilon
\]

and if \( X = t \begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix} \)

\[
\frac{1+X}{1-X} = \frac{(1+X)^2}{1+t^2} = \frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2} \begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}
\]

and

\[
F = \frac{1+X}{1-X} = \frac{1-t^2}{1+t^2} \varepsilon + \frac{2t}{1+t^2} \begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}
\]

Next there is the Bott map

\[
[0, \pi] \times \operatorname{Gr}_n(C^{2n}) \longrightarrow SU_{2n} \quad (g, F) \longmapsto e^{i \theta F}
\]
\[(\Theta, \Phi) \rightarrow e^{i\Theta \Phi} = \cos \Theta + i \sin \Theta \Phi\]

which gives a geodesic from 1 to -1.

(Actually I use \(e^{-i\Theta \Phi}\) in my C.T. paper.)

Note for \(n = 1\), that the first map gives a homeomorphism

\[\bigvee U_1 \rightarrow \mathcal{G}_1(\mathbb{C}^2) = S^2\]

and the second map gives a homeomorphism

\[\bigvee S^2 \rightarrow SU_2 = S^3\]

It's not yet clear why the \[\bigvee\] map I described is related to Bott's map.
Recall that if $P$ is a principal $U_n$-bundle over $B$, then we have defined classes $c_{2n-1} \in H^{2n-1}(P)$ for $2n-1 > \dim B$. Let's check this.

**First method.** Choose a connection $A$ on $P/B$ and use the Chern-Simons deformation $A_t = tA$ to write

$$\text{tr} (F^n_t) = d \left( \int_0^1 \text{tr} \left( A F_t^{n-1} \right) \frac{dt}{(n-1)!} \right) = c_{2n-1}(A)$$

For $2n > \dim B$, $\text{tr} (F^n_t/n!) = 0$ since it comes from the base and so $c_{2n-1}(A)$ is closed. The invariance of its class is derived from this closedness assertion by working over $B \times \mathbb{R}$, so the class is well defined for $2n > \dim (B \times \mathbb{R})$, i.e. $2n-1 > \dim B$.

**Obvious generalization.** Given a vector bundle $E/M$ equipped with a flat partial connection relative to a foliation of $M$, and equipped with a trivialization, there are classes $c_{2n-1} \in H^{2n-1}(M)$ defined for $2n-1 > \text{codim}$ of the foliation.
The second method. Let us try reducing the structural group of $P$ to $U_{n-1}$, i.e., construct a section of $P/U_{n-1}$ over $B$.

The fibre is $U_n/U_{n-1}$, whose cohomology begins in degree $2n-1$, hence one needs a section for $2n-1 > \dim B$ by obstruction theory and it is unique up to homotopy provided $2n-1 > \dim B$. Thus we have in this case a $U_n$-equivariant map

$$
P \to U_{n-1}/U_N$$

unique up to homotopy. Now $H^*(U_{n-1}/U_N) \subset H^*(U_N)$ is the subalgebra generated by the primitive generators $e_{2k-1}$ for $k \geq n$. Pulling back via the above map gives the classes $e_{2n-1}^P \in H^{2n-1}(P)$ for $2n-1 > \dim B$.

Why does this agree with the first method? When we reduce $P$ to $U_{n-1}$, we write the associated vector bundle $E$ as $E_1 \oplus \mathbb{C}^k$ where rank $E_1 = n-1$. Then we can use a connection in $E_1$ together with the 0 connection on $\mathbb{C}^k$.

Since the map $\pi$ above is equivariant we have a comm. square

$$
P \times U_N \quad \xrightarrow{\mu} \quad P \quad \downarrow \quad \downarrow \quad U_{n-1}/U_N \times U_N \quad \xrightarrow{\mu} \quad U_{n-1}/U_N$$

which means that the classes $e_{2n-1}^P \in H^{2n-1}(P)$.
satisfy
\[ \mu^*(e^P_{2n-1}) = e^P_{2n-1} \otimes 1 + 1 \otimes e^U_{2n-1} \in H^{2n}(P \times U_n) \]

This ought to generalize to the case where there's a foliation where we have a flat partial connection on the trival bundle. If we change the trivialization by \( g : M \rightarrow U_n \), then the classes change by \( g^*(e^U_{2n-1}) \).

Let's take a different direction. We considered yesterday families of homomorphisms \( U_n \rightarrow U_m \) parametrized by a manifold \( M \). This is the same as a homomorphism.*

\[ U_n \rightarrow U_M^M = \text{gauge transformations of } C^M \text{ over } M \]

Particularly interesting is the case where \( M = S^2 \) where I think we obtain Bott's periodicity map.

The idea I have is to consider the behavior of left-invariant differential forms with respect to such a homomorphism.* I believe I know something about left-invariant differential forms or something like \( U_M^M \). This is what I learned in studying Atiyah-Singer's paper.
Review the Bott maps

1) \([0, \pi] \times U_n \rightarrow \text{Gr}_n(\mathbb{C}^2 \oplus \mathbb{C}^n)\)
   
   \((\theta, g) \mapsto (\cos \theta) \epsilon + \sin \theta \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\)
   
   \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\)

2) \([0, \pi] \times \text{Gr}(V) \rightarrow \text{U}(V)\)
   
   \((\varphi, F) \mapsto e^{-i\varphi} F = \cos \varphi \otimes -i \sin \varphi F\)

When we compose these we get

3) \([0, \pi] \times [0, \pi] \times U_n \rightarrow \text{U}(\mathbb{C}^2 \oplus \mathbb{C}^n)\)

   \((\varphi, \theta, g) \mapsto \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^{-1} h(\varphi, \theta) \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}\)

where \( h: [0, \pi] \times [0, \pi] \rightarrow SU(2) \) is

\( h(\varphi, \theta) = \cos \varphi - i \sin \varphi \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}\)

Let's look at \( h \) on the boundary of \([0, \pi]^2\):

\[
\begin{pmatrix} e^{i\varphi} 0 \\ 0 e^{-i\varphi} \end{pmatrix}
\]

\[
\begin{pmatrix} 1 0 \\ 0 -1 \end{pmatrix}
\]

\[
\begin{pmatrix} e^{-i\varphi} 0 \\ 0 e^{i\varphi} \end{pmatrix}
\]

\( h(\partial [0, \pi]^2) \subset \text{diagonal maximal torus of } SU_2 \).
We have a map\$
\varnothing \quad \left( 0, \pi \right)^2 / \partial \left( 0, \pi \right)^2 \xrightarrow{h} SU_2 / T = S^2 \$

by evaluating \( h \) on the line \( \mathbb{C}(1) \). Then\$
h(\phi, \theta) (1) = \frac{\cos \phi - i \sin \phi \cos \theta}{-i \sin \phi \sin \theta} \$

\[ = \frac{\cos \theta}{\sin \theta} + i \frac{1}{\sin \theta} \frac{\cos \phi}{\sin \phi} \]

As \( 0 < \theta < \pi \), \( \cos \theta \) goes from \( +\infty \) to \( -\infty \)
and for each \( \theta \in (0, \pi) \), as \( 0 < \phi < \pi \), \( \frac{1}{\sin \phi} \frac{\cos \phi}{\sin \phi} \) goes from \( \infty \) to \( -\infty \), so it is clear that \( \varnothing \) is a homeomorphism.

Let's look at the maps 3) above.
\[ \left( 0, \pi \right)^2 \times U_n \xrightarrow{\pi} U(\mathbb{C} \otimes \mathbb{C}^n) \]

Thus \( \langle \phi, \theta, g \rangle \xrightarrow{} (g, \theta)^{-1} h(\phi, \theta) (g, \theta) \) and for \( (\phi, \theta) \in \mathbb{D}(0, \pi)^2 \), \( h(\phi, \theta) \) commutes with \( (g, \theta) \). So we get a map \( S^2 \times U_n \rightarrow U_{2n} \) which collapses \( \{e\} \times U_n \), but not \( S^2 \times \{e\} \). To get a map \( S^2 \vee U_n \rightarrow U_{2n} \), one considers the commutator
\[ (\phi, \theta, g) \xrightarrow{} (g, \theta)^{-1} h(\phi, \theta) (g, \theta) h(\phi, \theta)^{-1} \]

This is very nice because it can be interpreted as the embeddings \( \mathbb{U}(n) \rightarrow U(\mathbb{C} \otimes \mathbb{C}^n) \), parametr...
by $S^2$ which at the line $L \subset \mathbb{C}^2$ take $g$ in $U(n)$ to $i \otimes g$ on $L \otimes \mathbb{C}^n$ with the identity on $L \otimes \mathbb{C}^n$. So therefore we see the Bott map fits into the form I thought it did when I discussed Ludden's theorem. (p. 214)
Observation: Consider all homomorphisms $U_n \to U_N$ such that $U_n$ with the induced action of $U_n$ is isomorphic to the direct sum of the standard representation of $U_n$ on $\mathbb{C}^n$ and the trivial representation on $\mathbb{C}^{N-n}$. Given $\varphi$, we thus have an isomorphism $\mathbb{C}^n \oplus \mathbb{C}^{N-n} \to \mathbb{C}^N$, which is unique up to $S^1 = U_1$ acting as scalars on $\mathbb{C}^n$ and $U_{N-n}$ acting on $\mathbb{C}^{N-n}$. Put another way, let $V$ be the standard repn. of $U_n$ on $\mathbb{C}^n$. One has a canonical isomorphism associated to $\varphi$:

$$V \otimes \text{Hom}_{U_n}(V, \mathbb{C}^n) \oplus (\mathbb{C}^n)_{U_n} \to \mathbb{C}^N \quad \text{1-diml} \quad \text{N-n diml}$$

Thus $\varphi$ determines an element of

$$U_N/\Delta_n S^1 \times U_{N-n} = (U_N/U_{N-n})/\Delta_n S^1$$

where $\Delta_n S^1 \subset U_n$ is the center. Conversely one sees that any point in this orbit space of the Stiefel manifold $U_N/U_{N-n}$ by the action of scalars determines a homomorphism $U_n \to U_N$. Consider

$$\text{Hom}_{\text{Liegps}}(U_n, U_N) = (U_N/U_{N-n})/\Delta_n S^1$$

Thus for $N-n$ large we have a space for embedding homomorphisms $U_n \to U_N$ which has the homotopy type $BS^1$.  

\[\text{Parameter}\]
Next let us check that a family of homomorphisms: $\mathbf{M} \times G \rightarrow \mathbf{G}'$, parameterized by $\mathbf{M}$ induced a family of maps of classifying spaces: $\mathbf{M} \times \mathbf{B}G \rightarrow \mathbf{B}G'$. The best way to think is that one has an map

$$\mathbf{B}(G'^M) \rightarrow (\mathbf{B}G')^M$$

which is a homotopy equivalence on the component of $(\mathbf{B}G')^M$ corresponding to the trivial $G'$-bundle over $\mathbf{M}$.

Thus one has

$$\mathbf{B}G \rightarrow \mathbf{B}(G'^M) \rightarrow (\mathbf{B}G')^M$$

Alternatively one can take $\mathbf{M} \times \mathbf{P}G$ over $\mathbf{M} \times \mathbf{B}G$ and form a principal $G'$-bundle $(\mathbf{M} \times \mathbf{P}G \times G')_G$ where $G$ acts on $\{m\} \times \mathbf{P}G \times G'$ using the homomorphism $G \rightarrow G'$ at the point $m$.

So in the situation of

$$\mathbf{X} \times (\mathbf{U}_n/\mathbf{U}_{n-1} \times \Delta S^1) \times \mathbf{U}_n \rightarrow \mathbf{U}_n$$

it seems that we get a map

$$(\mathbf{U}_n/\mathbf{U}_{n-1} \times \Delta S^1) \times \mathbf{B}U_n \rightarrow \mathbf{B}U_n$$

which must be compatible with the map

$$BS^1 \times \mathbf{B}U_n \rightarrow \mathbf{B}U_n$$

given by tensoring a line bundle + a vector bundle. Sometime it might be interesting to work out the effect of $\times$ on cohomology. Can let $\mathbf{n}, \mathbf{N} \rightarrow \infty$.
Let's recall the idea that the Chern-Simons forms on a principal $U$-bundle should be viewed as special cases of Chern-Simons forms which can be associated to a vector bundle with a flat partial connection relative to a foliation and a trivialization.

Extending the flat partial connection to a connection which can be written $d + A$ relative to the trivialization, the Chern-Simons forms are obtained by using the linear path $d + t A$.

Next suppose we change the trivialization by a gauge transformation $g$. The new CS forms result from the linear path joining $g^{-1} d g = d + g^{-1} d g$ to $d + A$. Actually any path in the space of connections should give the same cohomology classes, so if we consider

we find that the CS classes under the gauge transformation get added by the primitive on $H^3$ pulled back via the gauge transformation.

Another question is to relate the CS forms defined in $H^3(P)$ by connections with the forms defined via obstruction theory. The latter are obtained as follows. One
can reduce the structural group of $P$ from $U_N$ to $U_r$ when $\dim B \leq 2r+2$

because one can construct a section of $\bar{P}/U_r$ over $B$ and the fibre begins in degree $2r+1$. Having done this, one has a $U_N$-equivariant map

$$\bar{P} = \bar{P} \times U_r U_N \rightarrow U_r \backslash U_N$$

and so the odd generators of $H^\ast(U_r \backslash U_N)$ give odd cohomology classes for $\bar{P}$.

So we ought to describe the odd generators of $H^\ast(U_r \backslash U_N) = H^\ast(U_N/U_r)$. The space $U_N/U_r$ occurs as the fibre

$$U_N/U_r \rightarrow BU_r \rightarrow BU_N$$

and so over $U_N/U_r$ there is a universal pair consisting of a rank $r$ vector bundle $E$ and an $\mathbb{L}$ isomorphism $E \oplus \mathbb{C}^{r-2} \cong \mathbb{C}^N$. To construct odd forms on $U_N/U_r$, we look at the trivial connection on $\mathbb{C}^N$ and the connection which is the direct sum of the Grassmannian connection on $E$ and the trivial connection on $\mathbb{C}^{N-r}$. Then one uses the invariant polynomials giving rise to the Chern classes, i.e.

$$\text{tr} (\wedge^a) = \phi(a) \cdot \deg \mathbb{L}_N$$

This construction leaves much to be desired. The foliation method seems to produce CS classes for all invariant polynomials of degree $\geq \text{codim}$
of the foliation. There's probably a Helfand–Fuchs type algebra which gives the cohomology of the principal $\mathbb{U}_N$-bundle over $\mathbb{U}_N$ restricted to the $2\pi$-skeleton. Take Weil algebra and kill the appropriate power of the ideal generated by the components of the curvature.

Variation maps. The general idea is to consider the evaluation maps

$$\omega_m : G^M \to G$$

for each $m \in M$. These induce maps

$$\omega_m^* : \Omega^k(G) \to \Omega^k(G^M)$$

and

$$\Lambda^* g^* \to \Lambda^* (g^M)^*$$

more precisely,

$$C^*(g^M, C)$$

so given a form on $G$ one has a smooth family of forms $\omega_m^* (\omega)$ on $G^M$, whence we have a map

$$\Omega^k(G) \to \Omega^k \cdot (G^M)^x M$$

and

$$\Lambda^* g^* \to \Lambda^k (g^M, \Omega^0(M))$$

compatible with $d$ on the left and $d' = d + n$ on the right.

(Perhaps it is useful to note that we
have a map induced by $\omega: G^M \times M \to G$

$\omega^*: \Omega^*(G) \to \Omega^*(G^M \times M)$

and that the map

$\Omega^*(G) \to \Omega^*(G^M \times M)$

results by passage to the quotient.

Note that $g^M = gl_n(\Omega^M)$, $g^M = gl_n$

hence the map $\otimes$ on Lie cochains is a DGA map

$\Lambda^* gl_n^* \to C^*(gl_n(A), A)$

where $A = \mathcal{C}(M)$. I claim such a map exists for any algebra.

Proof: Let $\Theta_{ij} \in C^1(gl_n(A), A) = \text{Hom}(gl_n(A), A)$ be the map such that

$x \Theta_{ij} = X_{ij}$ if $X = (x_{ij}) \in gl_n$.

Then

$\chi (\delta \Theta_{ik} + \sum_j \Theta_{ij} \Theta_{jk}) = -\Theta_{ik}(x_{ij}) + \sum_j x_{ij}y_{jk} - y_{ij}x_{jk} = 0$

$[x, y]_{ik}$

Thus $
\delta \Theta_{ik} + \sum_j \Theta_{ij} \Theta_{jk} = 0$ which means

that the obvious homomorphism $\otimes$ sending $\Theta_{ij} \in gl_n^*$

to $\Theta_{ij} \in C^1(gl_n A, A)$ is compatible with differentials.
Notes on Feigin-Tsigan on Lie algebra cohomology and Riemann-Roch.

They define $W^*(g)$, the Weil algebra, as the cochains on the DGA $g[[\varepsilon]] = g \oplus \varepsilon g$ where degree $\varepsilon = -1$ and $d(\varepsilon) = 1$. This exhibits the Weil algebra as a bigraded differential algebra and suggests that there is a larger context into which the Weil algebra fits. In effect so far our understanding of $W(g)$ comes from connections on principal bundles and when we form equivariant forms we just tensor $W(g)$ and $\Omega(M)$. But apparently there is also the possibility of a twisted tensor product.?

Also the bigrading is ad hoc.

Dual version $W_*(g) = \Lambda g \otimes \delta g$ of Lie chains. This is better for $g$ infinite-dimensional such as $o\ell(A)$. In the case of $o\ell(A)$ one obtains the Lie chains on $\delta g$ applied to the DGA $A \oplus A \delta$, which is a special case of $R \oplus \varepsilon I$ studied in the case of extensions. In particular we know that the $\delta$ homology of $W_*(g)$ is the cyclic homology of the semi-direct product $A \oplus \delta A$ (super conventions with degree $\varepsilon = 1$ for the lower grading). By Goodwillie this cyclic homology is

$$H_\varepsilon(A) \oplus \mathbb{H}(A,A) \oplus H_*((\delta A \otimes A)^2) \oplus \ldots \oplus \mathbb{H}_\delta(A,A) \oplus \mathbb{H}_\varepsilon(A,A) \oplus \mathbb{H}_3(A,A).$$
On the cohomology side this says that the S cohomology
\[ H^*(\text{gl}(A), S^0\text{gl}(A))^* \]

is (up to duality problems) the free commutative algebra generated by \( H(C(A)) \) and by various copies of \( H(A, A) \). In principal this means one knows \( (S^0\text{gl}(A))^* ) \otimes \mathcal{H}(V) \) which is the Weil candidate for the cohomology of \( B\mathcal{G} \) (?)

Let’s discuss what FT do. Let \( g = \text{gl}(A) \) and let \( h \) be a reductive subalgebra in \( g \), better Lie subalgebra reductive in \( g \) so that \( g \) is a semisimple \( h \)-module. Then consider \( h \subset g \subset g + \mathfrak{g} \mathfrak{g} \) and form relative cochains. This is the same as \( h \)-basic elements of \( W(g) \):

\[ W(g, h) = (\bigwedge (g/h) \otimes S(g))^* \otimes^h \]

\[ = \left( \frac{\bigwedge (g/h) \otimes S(g/h) \otimes S(h)}{\text{contractible}} \right)^* \otimes^h \]

Thus \( W(g, h) \) gives \( H^*(Bh) \) and we have a spectral sequence starting with the \( S \) cohomology
\[ H^*(g, h; S(g)^*) \Rightarrow H^*(Bh) \]

FT’s goal is to use \( HH(A) = H(A, A) \) to produce a “character” for \( \mathcal{G} \) the representation
\[ h \rightarrow \text{deg}_f(A) \]

and the character lies in \( H^*(BH) \).

They assume that the Hochschild cohomology is concentrated in degree \( 2n \). By means of a suitable decreasing filtration on \( W_*(\text{deg}_f, h) \), they show that if \( f \) is minimal with \( H_j(\text{deg}_f, h^*; S^{\geq 0}g) \neq 0 \), then there are maps

\[ H_{j+2g}(W_*(\text{deg}_f, h)) \rightarrow H_j(\text{deg}_f, h^*; S^{\geq 0}g) \]

Dually we get maps

\[ H^j(\text{deg}_f, h^*; S^{\geq 0}g) \rightarrow H^{j+2g}(BH) \]

We have a spectral sequence

\[ \text{H}^j(\text{deg}_f, h^*; S^{\geq 0}g) \otimes H^k(BH) \rightarrow \text{H}^{j+k}(\text{deg}_f, h^*; S^{\geq 0}g) \]

so maybe this means that the classes in \( \text{H}^* \) associated to Hochschild cohomology give rise to relative classes in \( H^*(\text{deg}_f, h^*; S^{\geq 0}g) \), and hence to classes in \( H^*(BH) \).

One further idea is that a homomorphism \( h \rightarrow \text{deg}_f A \) extends to an algebra hom.

\[ U(h) \rightarrow M_n A \]

hence induces

\[ H(U(h), U(h)) \rightarrow \text{BH}(M_n A, M_n A) = H(A, A) \]

\[ H_*(\frac{h}{h}, U(h)) = H_*(h, S(h)) \]
Dually of $H^*(A,A)$ is concentrated in degree 2n we get a map

$$H^{2n}(A,A) \rightarrow H^{2n}(h, S(h)^*) = H^{2n}(h) \otimes (S(h)^*)^h$$

Apparently this is related to the above, although I don't see how. When $A=C$ and $n=0$ we obtain in $S(\mathbf{g})^h = H^*(BH)$ the usual character of the induced bundle over $BH$. 
Program: To understand the character of Feigin + Tsygan and related ideas.

Let \( g_f = \text{gl}(A) \) where \( A \) is a unital algebra such that \( \text{HH}^1(A) \cong \mathbb{C}[2n] \). If \( h \subset g_f \) is a "reductive subalgebra" (meaning \( g_f \) is semi-simple as \( h \)-module), they define a "character" in \( S(h^*)^h \). (It suffices to have a representation \( h \to g_f \) whose image is a reductive subalgebra).

Let's take \( A \) unital and \( h = \text{gl}(\mathbb{C}) \) and try to understand what's happening.

Consider \( A + A\varepsilon \); this is the DGA with \( A \) in degree 0, \( \varepsilon \) in degree 1 for the lower indexing and with \( d(\varepsilon) = 1 \). We have

\[
\text{gl}(A + A\varepsilon) = g_{[\varepsilon]} = g_f + \varepsilon \varepsilon
\]

and \( W_*(g_f) \) is the bigraded algebra of a certain Lie algebra whose cochains on \( g_{[\varepsilon]} \), whereas \( W_*(g_f) \) is the bigraded differential algebra of chains. Feigin + Tsygan use the relative complexes \( W_*(g_f, h) \), etc., which have the form

\[
W_*(g_f, h) = (\wedge g_f/h \otimes S(g_f))_h, \quad W^* = (W_*)^*
\]

In the case \( h = \text{gl}(\mathbb{C}) \subset g_f = \text{gl}(A) \), then invariant theory says

\[
W_*(g_f, h) = S_{\text{sup}} \left( \sum \mathbb{C}(A[\varepsilon]) \right)
\]

which computes the homology from the reduced cyclic homology of \( A[\varepsilon] \). Better it tells us...
that $W_k(g, h)$ is the free comm.
corlgebra generated by the reduced
cyclic complex of $A[\varepsilon]$, + this is
compatible with the double complex
structure.

The $S$ cohomology of $\sum \tilde{C}(A[\varepsilon])$ is

\[
\begin{array}{c}
H_2 \\
H_1 \\
H_0
\end{array}
\]

and the $d$ cohomology is

\[
\begin{array}{c}
\mathbb{C} \\
\mathbb{C} \\
\mathbb{C}
\end{array}
\]

This is known because the $S$ cohomology depends
only on the adj. structure of $A + A\varepsilon$, which is the
semi direct product of $A$ and the bimodule $A\varepsilon$

Consider what happens if $HH_i(A) = \mathbb{C}[2n]$. We have from the long exact sequence

\[ H^i(C) \rightarrow H^i(C) \rightarrow H^i(C) \rightarrow H^i(C) \rightarrow H^i(C) \]

and the fact that $H^i(C) = \{ \mathbb{C} \quad i = 2n^2 \quad n' > n \}$

otherwise
That $\overline{H}_{2i}(A) = 0$ \quad i < n
$\overline{H}_{2i-1}(A) = 0$ \quad 1 \leq i < n

and

$0 \to \overline{H}_{2n+1}(A) \to \overline{H}_{2n}(A) \to 0$

In the spectral sequence starting with the $\delta$ cohomology we have

\[
\begin{array}{c|c}
2n+1 & \overline{H}_{2n} \\
\downarrow & \downarrow \\
2n & \overline{H}_{2n-1} \\
\downarrow & \downarrow \\
2n-1 & \overline{H}_{2n-2} \\
\downarrow & \downarrow \\
\vdots & \vdots \\
2 & \overline{H}_1 \\
\downarrow & \downarrow \\
1 & \overline{H}_0 \\
\downarrow & \downarrow \\
0 & \overline{H}_i \\
\end{array}
\]

We know the abutment, so we conclude $\overline{H}_{2n}(A) = 0$.

**Lemma:** If $\overline{H}_i(A) = \mathbb{C}[2n]$, then $\overline{H}_{2j}(k) \Rightarrow \overline{H}_{2j}(A)$ for $j > 2n$ and so

$\overline{H}_{2i}(A) = \begin{cases} 
\mathbb{C} & j = 2i-1 \\
0 & \text{otherwise}
\end{cases}$

Once we know $\overline{H}_{2n}(A) = \overline{H}_{2n+1}(A) = 0$, what happens in higher degrees follows by a $S$ which is an isomorphism.
Let $\mathfrak{h}$ be a finite-dimensional reductive Lie algebra and let $\mathfrak{h} \to \mathfrak{gl}(A)$ be a Lie algebra homomorphism, such that $A^N$ is a "reductive repn." of $\mathfrak{h}$, that is, a semisimple finite-dimensional irreducible representation of $\mathfrak{h}$. Then we can decompose

$$A^N = \bigoplus_x W_x \otimes \text{Hom}_\mathfrak{h}(W_x, A^N)$$

where $W_x$ are the inequivalent f.d. irreducible reps of $\mathfrak{h}$. Thus we are led to consider reductive reps (right) of $\mathfrak{h}$ on finite projective $A$-modules.

The FT character for such a repn should be additive and natural, whence to compute this character it should suffice to consider the case of $\mathfrak{h} = \mathfrak{gl}_k$ acting in the obvious way on $C^k \otimes E$, where $E$ is a finite projective $A$-module. In this case we have a homomorphism of algebras

$$M_k \to \text{End}_{A^P}(C^k \otimes E) = M_k \otimes \text{End}_{A^P}(E).$$

Thus the FT character should be computable from what it does in the case of the homomorphism

$$\mathfrak{gl}_k \to \mathfrak{gl}_{km}(A)$$

where $C \to \mathbb{M}_m(A)$ is a nonunital homomorphism.
Lecture on Lie algebra cohomology.

G Lie groups. It acts on itself by both left and right translations and these actions commute. One has

left invariant vector fields = infinitesimal right translations.

\( \text{Lie}(G) \) is the space of left invariant vector fields under bracket. Put \( g = \text{Lie}(G) \otimes \mathbb{R} \). One has

\[ \Lambda g^* \rightarrow \Omega(G) \otimes \mathbb{R}(\text{left}) \]

Thus \( \Lambda g^* \) has a differential \( \delta \) corresponding to \( d \) on \( \Omega(G) \). If \( \omega \) is a left invariant 1-form one has:

\[
\omega(x) \ dx = \frac{1}{2} \left[ \omega(\partial_x), \partial_x \right] = \omega([X,Y]) = -\omega([X,Y])
\]

Thus if \( X_a \) is a basis for \( g \) and \( \Theta^a \) is the dual basis for \( g^* \), and \( [X_a, X_b] = f_{ab}^c X_c \) one has

\[ d\Theta^a + \frac{1}{2} f_{bc}^a \Theta^b \Theta^c = 0 \]

Manin - Cartan form

\[ \Theta = \Theta^a X_a \in \Omega^1(G) \otimes g \]

Properties: left invariant

\[ \mathfrak{L}_X \Theta = X \Theta \]

\[ \text{Ad}(g) \mathfrak{R}_g^* \Theta = \Theta \]

\[ d\Theta + \frac{1}{2} [\Theta, \Theta] = 0 \]

It is the unique connection form in \( G \) considered as a principal bundle over \( E_{pt} G \).
Let $C^i(g) = \Lambda^i g^* \otimes V$ with $S$

$H^i(g) = H^i(C(g))$

One has a natural map $H^i(g) \rightarrow H^i_{DR}(G)$.

Next want Lie alg. coh. with coefficients in as $g^*$-module.

Consider the trivial bundle $\tilde{V} = G \times V / G$

with $G$ acting $g(g_1, v) = (g g_1, v)$. This is an equivariant $G$-bundle over $G$ w.r.t. the left-translation action, and any equivariant bundle $E$ is canonically isomorphic to $\tilde{V}$ with $V$ the fibre of $E$ over $1 \in G$. One has

$\Omega^i(G, \tilde{V}) = \Omega^i(G) \otimes V$

$\Omega^i(G, \tilde{V})^{g} = \Omega^i(g \cdot \text{left}) \otimes V = \Lambda^i g^* \otimes V$

An invariant connection on $\tilde{V}$ is of the form $d + A$ with $A \in \Omega^1(G) \otimes \text{End}(V)$. $A$ is the same as a linear map $\rho : g \rightarrow \text{End}(V)$ via

$\rho(x) = \chi A$

$A = \rho \Theta = \Theta \rho(x)$. One has

$(d + A)^2 = dA + A^2 = \rho d\Theta + \rho \Theta \rho \Theta$

$= \rho (-\frac{1}{2}[\Theta, \Theta] + \frac{1}{2}[\rho \Theta, \rho \Theta])$

The connection is flat $\iff$ $\rho$ is a Lie homomorph.

Thus if $V$ is a $G$-module with $g$-action given by $\rho$, we have a diff $S$ on

$\Omega^i(G, V)^G = \Lambda^i g^* \otimes V$

induced by $d + \rho \Theta$ on $\tilde{V}$. Put

$C^i(g; V) = \Lambda^i g^* \otimes V$ with $S$

$H^i(g, V) = H^i(C(g; V))$. 
Formulas

\[ C^\bullet(g) = \Lambda[\mathfrak{g}^\bullet] \quad \text{with} \quad \delta \mathfrak{g}^\bullet + \frac{1}{2} f_{bc} \mathfrak{g}^b \mathfrak{g}^c = 0 \]

\[ C^\bullet(g; V) = \Lambda[\mathfrak{g}^\bullet] \otimes V \]

in the DG module over \( C^\bullet(g) \) with

\[ \delta v = f(\mathfrak{g}) v = \mathfrak{g} f(\mathfrak{g}_v) v \]

Chevalley–Eilenberg Thm. Assume \( G \)
compact connected, and let \( V \) be a f.d.
representation of \( G \). Then

\[ H^\bullet(g; V) \cong H^\bullet_{DR}(G) \otimes V \]

In particular

\[ H^\bullet(g) \cong H^\bullet_{DR}(G) \]

Proof. We are given a Lie \( g \)-map \( \phi : g \to \text{End}(V) \)
and the associated repn \( V : g \to \text{End}(V) \) is

\[ \phi^\ast \phi^{-1} \phi = \phi^{-1} d\phi \]

Consider \( \phi \) as gauge transformation,

\[ \mathfrak{g} \to \text{aut} \text{orphism of } V \]

As \( d \phi \cdot \phi = d + \phi^{-1} d\phi = d + f(\mathfrak{g}) \) one has

an isomorphism

\[ \phi : \Omega(g) \otimes V \to \Omega(g) \otimes V \]

Connection:

\[ d + f(\mathfrak{g}) \leftrightarrow d \]

G-action:

\[ \text{left trans. on } G \leftrightarrow \text{left trans. on } G \]

trivial on \( V \)

Thus we have

\[ H^\bullet(g; V) = H^\bullet\left\{ (\Omega(g) \otimes V, d) \right\} \]

We have a canonical map
which we show is an isomorphism

injective: Given \( \omega \) with \( d\omega = 0 \) and \( g^*\omega = \omega \), suppose \( \omega = d\eta \). To prove \( \omega \) is \( d \) of an invariant form. But

\[
\int \sum_{g \in G} d(g^*\eta) = \int \sum_{g \in G} d(g^*\omega) = \int g^*\omega = \int \omega = 0
\]

since \( d \) continuous, so this is clear.

surjective: Given \( \omega \in \Omega(G) \otimes V \) with \( d\omega = 0 \) suppose the class of \( \omega \) is \( G \)-invariant: \( \omega - g^*\omega \in \text{Ind} \) for all \( g \). To prove \( \omega \) cohomologous to an invariant form. But

\[
\omega - \int g^*\omega = \int (\omega - g^*\omega) \in \text{Ind}
\]

and one knows that \( \text{Ind} \) is closed. (Hodge theory, Bega, characterization of elements of \( \text{Ind} \) as forms integrating to zero over all closed cycles.)

so far we have used \( G \) compact. Now as it is connected, we have

\[
H^* (\Omega(G) \otimes V) \cong (H^*(G) \otimes V) \otimes V^G
\]

since \( G \) acts trivially on \( H^*(G) \) in this case.
Let \( G \to P \xrightarrow{\pi} B \) be a principal bundle. Let's recall how the Leray spectral sequence for this fibering arises.

One has an exact sequence

\[
0 \to S \to T \to Q \to 0
\]

where \( T = T_p \), \( S = T_\pi \), \( Q = \pi^* T_B \). Ideally one gets

\[
0 \to Q^* \to T^* \to S^* \to 0
\]

The ideal in \( \Gamma(P, \Lambda T^*) \) generated by \( \Gamma(P, Q^*) \) is stable under \( d \) (this is integrability for the tangent subbundle to a foliation). Call this ideal \( J \); one has the \( T \)-adic filtration

\[
\mathcal{O}(P) \supset J \supset J^2 \supset \cdots
\]

where \( J^p / J^{p+1} = \Gamma(P, \Lambda^p Q^* \otimes \Lambda S^*) \). Recall that \( \Lambda^p Q^* \) is flat along the leaves; in this case \( \Lambda^p Q^* = \pi^* \Lambda^p T_B \). Moreover, \( d \) on \( \mathcal{O}(P) \) induces \( d \) on \( J^p / J^{p+1} \) the "Dolbeault" differential which amounts to the complex associated to the fibres with coefficients in the flat bundle \( \Lambda^p Q^* \). Thus one has

\[
E_p^{p,0} = H^{p+1}(J^p / J^{p+1}) = \Lambda^p(B) \otimes H^0(G)
\]

and it should be possible to identify \( E_p^{p,0} \) with the effect of \( d_B \).

Thus we get the Leray spectral sequence from the decreasing filtration \( \mathcal{O} \). In the principal bundle situation one is studying \( S = \mathcal{O} \), so

\[
J^p / J^{p+1} = \Gamma(P, \pi^* \Lambda^p T_B) \otimes \Lambda^p \mathcal{O}^*
\]
\[ \Omega^P(P \vert \text{hor}) \otimes \Lambda^* \mathfrak{g}^* \]

and \(d\) can be identified with the Lie algebra cohomology differential \(S\) associated to the \(g\) action on \(\Omega^P(P \vert \text{hor})\) by the operators \(L_x\). So

\[ E^P_\lambda = H^\lambda(g, \Omega^P(P \vert \text{hor})) = \Omega^P(P \vert \text{bas}) \otimes \mathfrak{h}^* \]

This holds even without assuming \(G\) compact connected. In effect \(\Omega^P(P \vert \text{hor})\) is locally \(\Omega(B) \otimes \Omega^0(G)\), and \(\Omega^0(G) \otimes \Lambda^* \mathfrak{g}^*\) with \(S\) is \(\Omega^0(G)\).

Now we want to show that the spectral sequence arising from the \(\Delta\)-bigraded diffealg

\[ \Omega^P(P) \otimes C(g[\leq 1]) = \Omega^P(P) \otimes \Lambda^* \mathfrak{g}^* \otimes \mathfrak{h}^* \]

starting with the \(S\) cohomology coincides with the Leray spectral sequence. In the former spectral sequence, one uses the decreasing filtration with \(F^p = \text{all columns of degree} \geq p\).
February 22, 1990

We wish to define a map of DG algebras

\[ \Omega(P) \rightarrow \Omega(P) \otimes \Lambda \log^* \otimes S \log^* \]

where the latter is given the total differential \( d + \delta \).

Consider first the case where \( \Omega(P) \) is replaced by \( W(\log) \). Then the homomorphism we want is equivalent to a connection in \( W(\log) \otimes \Lambda \log^* \otimes S \log^* \). Recall the Russian formula

\[ (d + \delta)(A + X) + (A + X)^2 = dA + A^2 = F. \]

Then the desired homomorphism is given by

\[ A \mapsto A + X \]
\[ F \mapsto F \]

This suggests the formula for \( \otimes \):

\[ \omega \mapsto \omega + x^a \partial_a \omega + \frac{1}{2} x^a x^b \partial_a \omega + \ldots = e^{x^a \partial_a} \omega \]

This should be a homomorphism because \( x^a \partial_a \) is a derivation. We can check this as follows. Recall that we have an \( \alpha \)-map.

\[ \Omega(P)_{av} \otimes \Lambda \log^*_A \rightarrow \Omega(P) \]

Now \( e^{x^a \partial_a} \) is the identity on \( \Omega(P)_{av} \) and we have \( e^{x^a \partial_a} A = A + X \). Thus \( e^{x^a \partial_a} \) combines the inclusion of \( \Omega(P)_{av} \) with the homomorphism \( \Lambda \log^*_A \rightarrow \Lambda \log^*_A \otimes \Lambda \log^*_A \).
Next we want to check that $e^{x^a} \xi_a$ is compatible with differentials. This is clear on elements in $\Lambda \mathfrak{g}^*$, since they come from $\Omega(\mathfrak{g})$, and we've checked the case of $\mathfrak{g}(\mathfrak{g})$ using the Russian formula.

So we consider $\omega \in \Omega(\mathfrak{g})_{h\omega}$. We have

\[(d+\delta) e^{x^a} \omega = (d+\delta) \omega = dw + x^a_La \omega - \varphi \xi_a \omega \]

\[e^{x^a} dw = dw + x^a_La dw + \frac{1}{2} x^a x^b b^a b^b (\omega) + \ldots \]

because $L_a dw = L_a \omega - d \xi_a \omega$ and $b^a L_b \omega = \frac{1}{2} \xi_a \big[ L_a, b^b \big] \omega = 0$.

Thus we have a DGA morphism

\[e^{x^a} : \Omega(\mathfrak{g}) \rightarrow \Omega(\mathfrak{g}) \otimes \Lambda \mathfrak{g}^* \otimes \mathfrak{g}^* \]

Next we want to check the filtrations. Recall that the ideal $\mathfrak{J} \subset \Omega(\mathfrak{g})$ is generated by $\Omega(\mathfrak{g})_{h\omega}$. The ideal to consider in $\Omega(\mathfrak{g}) \otimes \Lambda \mathfrak{g}^* \otimes \mathfrak{g}^*$ is generated by $\Omega^0(\mathfrak{g})$ and by $\varphi$; it's the ideal of elements of $d$-degree $> 0$. Thus $e^{x^a}$ is compatible with filtrations.

Let's compute the map on the associated graded algebras. We have

\[\varphi^* \Omega(\mathfrak{g}) = \Omega(\mathfrak{g})_{h\omega} \otimes \Lambda \mathfrak{g}^* \]

with $d$ given by $dA + A^2 = 0$

\[d \omega = A^a_La \omega \]

In effect, if $\omega$ is horizontal, then $d \omega$ needn't be; its horizontal part is

\[d \omega - A^a_La d \omega + \frac{1}{2} A^a A^b b^a b^b \xi_a d \omega \ldots \]
\[
= d\omega - A^a L_a \omega
\]

Thus if \( \omega \in \Omega^0(P)_{\text{hor}} \) one has

\[ d\omega - A^a L_a \omega \in \Omega^{+1}(P)_{\text{hor}} \]

showing that \( d\omega = A^a L_a \omega \) in \( \Omega^1_{\text{Hor}}(P) \).

The map on gk's is

\[ \Omega(P)_{\text{hor}} \otimes \Lambda g_A^* \rightarrow \Omega(P)_{\text{hor}} \otimes \Lambda g_A^* \otimes \Lambda g_x^* \otimes \Lambda g_y^* \]

\( \omega \) \( \mapsto \omega \)

\( A \) \( \mapsto X \) = leading part of \( A + X \)

Check compatibility of \( d \) on the left with \( d \) on the right:

\[ d\omega = A^a L_a \omega \quad \mapsto \quad X^a L_a \omega = \delta \omega \]

\[ dA = -A^2 \quad \mapsto \quad -X^2 = \delta X \]

Conclude that the induced map on gk's is the inclusion

\[ (\Omega(P)_{\text{hor}} \otimes \Lambda g_A^*) \rightarrow (\Omega(P)_{\text{hor}} \otimes \Lambda g_A^*) \otimes (\Lambda g_A^* \otimes \Lambda g_x^*) \]

\[ \cong (\quad) \otimes (\Lambda g_A^* \otimes S g_A^*) \]

and so induce an isomorphism on \( H \)-cohomology.

---

Question. If \( P/B \) is a principal \( G \)-bundle, is it possible to use the Weil algebra to construct cohomology classes in \( B \)? I want to do this when \( G \) is not compact, for example when \( G = C^\infty(M, \mathfrak{g}) \).

The idea here is that the standard Chern-Weil construction yields cohomology classes from \( S(\mathfrak{g}^*)^G \) and...
there is Bott's spectral sequence
\[ H^\text{diff} (G, Sg^*) \Rightarrow H^* (BG) \]
It might be the case that one has a canonical map
\[ H^* (G, Sg^*) \Rightarrow H^\text{diff} (G, Sg^*) \]
although perhaps this is unreasonable (the obvious map might go the other way as the formal group at \( e \) in \( G \)).

First map: Suppose \( P/B \) has a connection. Then we have a map \( W(g) \rightarrow \Omega (P) \) whence a map of bigraded differential algebras
\[ W(g) \otimes \Lambda g^* \otimes Sg^* \rightarrow \Omega (P) \otimes \Lambda g^* \otimes Sg^* \]
This gives a map on \( S \) cohomology which is
\[ E^0_2 = \left\{ \begin{array}{ll}
H^0 (g, Sg^*) & \text{even} \\
0 & \text{odd}
\end{array} \right. \rightarrow E^0_2 = \Omega^0 (B) \otimes H^0 (G) \]
Then we should have
\[ E^2_2 = \left\{ \begin{array}{ll}
H^0 (g, S^{P^2}g^*) & \text{even} \\
0 & \text{odd}
\end{array} \right. \rightarrow E^2_2 = H^0 (B, H^0 (G)) \]

Second map: Consider \( W(g) \rightarrow \Omega (P) \) as a map of filtered algebras, where we filter \( W(g) \) by powers of the ideal \( \mathcal{J} \) generated by the \( F^a \). The map on \( g^* \) is
\[ \Lambda g^*_A \otimes Sg^*_F \rightarrow \Lambda g^*_A \otimes \Omega (P)_{\text{hor}} \]
\[ dA + A^2 = 0 \]
\[ df + [A, F] = 0 \]
\[ d\omega = A^a L_\omega \omega \]
These are consistent because
\[ [A, F] = A^a \left[ x_a, F \right] = -A^a \partial_a F \]

The map \( E_j \) is then what? The degrees are funny, and it appears that one wants \( F^a \) to have degree 2. Basically, one seems to get the same map on \( E^1 \)-terms.

The real issue appears to be whether we can construct interesting cohomology classes in \( B \). I would like to be able to transgress Lie cohomology classes to \( BG \). The feeling is that there ought to be giant transgression cochains.
Let's consider a vector bundle $E$ over $Y \times M$ such that $E_y \simeq C^N \times M / M$ for all $y \in Y$. Let $\mathcal{G} = \text{Aut}(C^N \times M / M) = \mathcal{U}_N(\mathcal{C}^\infty(M))$, let $P / Y$ be the principal $\mathcal{G}$-bundle with $\mathcal{P}_y = \text{Isom}(C^N \times M, E_y)$, $\pi : P \to Y$ the canonical map. Then $\pi^* E$ is canonically trivial.

Let's consider a connection $D$ in $E$. When pulled up to $P \times M$, it has the form $\delta + d + \eta + A$

where

$$\eta \in \Omega^{1, 0}(P \times M) \otimes M_N = \Omega^1(P, \Omega^0(M) \otimes M_N)$$

is a connection form in $P / Y$ and where

$$A \in \Omega^{0, 1}(P \times M) \otimes M_N = \Omega^0(P, \Omega^1(M) \otimes M_N)$$

can be identified with a $\mathcal{G}$-equivariant map from $P$ to the space of connections on $C^N \times M / M$. This equivariance condition should be

$$g^* A = g^{-1} \cdot (d + A) \cdot g - d$$

$$g^* A = g^{-1} dg + g^{-1} A g$$

(Why: $\mathcal{G}$ acts to the right on $P$ and to the left on $A = \Omega^1(M) \otimes M_N$; invariance of $A \in \Omega^0(P, \Omega^1(M) \otimes M_N)$ means $g^* (g^* A) = A \circ g^* A = g^{-1} \cdot A$.) Thus

$$L_X A = dX + [A, X] \quad X \in \mathfrak{g}$$

$$\delta \quad L_X \delta A = d_x \eta + [A, x \eta] = -x \cdot (d \eta + [A, \eta])$$
\[ \left( \delta + d + \eta + A \right)^2 = \left( \delta \eta + \eta^2 \right) + \left( dA + d\eta + [\eta, A] \right) + \left( dA + A^2 \right) \]

Thus, we have a connection on an equivariant bundle over $P \times M$ which is not only flat along the $M_1$-orbits (this means $\chi_x \left( \delta \eta + \eta^2 \right) = 0$) but satisfies something stronger.
Consider $M = S^1$, $\mathfrak{g} = \mathfrak{u}(\mathbb{C}^1)$, $B\mathfrak{g} = \langle B(\mathfrak{g}) \mathfrak{g}^* \rangle$, $\tilde{\mathfrak{g}} = \mathfrak{gl}(C^*(S^1))$. In this case you ought to be able to describe easily the odd generators in $H^*(B\mathfrak{g})$. The problem is that these do not occur in $H^0(\Omega\tilde{\mathfrak{g}}, S\tilde{\mathfrak{g}}^*)$.

Here's a derivation of Bott's spectral sequence

$E_2 = H^i(G, S\tilde{\mathfrak{g}}^*) \Rightarrow H^i(BG)$.

It proceeds as for the van Est spectral sequence. Recall this. Form double complex

$C^i(B, \Omega^j(B))$.

Because $\Omega^j(B) = \Omega^j(G) \otimes \Lambda \tilde{\mathfrak{g}}^*$ is an induced module, it is acyclic for diff cohomology, so

$H^j_h\left[ C^\cdot(B, \Omega^i(B)) \right] = \begin{cases} 0 & p > 0 \\ \Omega^i(B) = \Lambda \tilde{\mathfrak{g}}^* & p = 0 \end{cases}$

and so we have a quasi

$\Lambda \tilde{\mathfrak{g}}^* = \Omega^i(G)^G \subset C^\cdot(B, \Omega^i(B))$.

The other spectral sequence has $E_{p, q}^2 = H^p_\text{diff}(B, H^q(G))$.

Now let $P \rightarrow B$ be a principal $G$-bundle with $P$ highly connected, and compose the double complex $C^\cdot(B, \Omega^i(B))$ with $\Lambda \tilde{\mathfrak{g}}^*$ to obtain a complex $C^\cdot(B, \Omega^i(B)) \otimes (\Lambda \tilde{\mathfrak{g}}^*)_{\otimes p}$, which has a quasi

$\Lambda \tilde{\mathfrak{g}}^* = \Omega^i(G)^G \subset C^\cdot(B, \Omega^i(B)) \otimes (\Lambda \tilde{\mathfrak{g}}^*)_{\otimes p}$.
Let $P \to B$ be a principal $G$-bundle. I want to use the same sort of argument for $C^\infty \left( G, \Omega^*(P)_{\text{hor}} \right)$.

The problem is that $\Omega^*(P)_{\text{hor}}$ is not a complex; it is not closed under $d$.

**Principle:** $\Omega^*(B)$ is the first subalgebra for the action of $G$, $g_e$ on $\Omega^*(P)$. It would be possible to form differentiable cochains on $(G, g_e)$ with coefficients $\Omega^*(P)$. Moreover the total complex of these cochains should be quasi to $\Omega^*(B)$.

But actually it might not be important to discuss cochains on $(G, g_e)$. Rather one can look just as taking fixed points. One first wants to take fixed points under $g_e$ which is normal in $(G, g_e)$. This gives $\Omega^*(P)_{\text{hor}}$. Then one takes $G$-invariants obtaining $\Omega^*(P)_{\text{bas}} = \Omega^*(B)$. The problem is that $g_e$ is not closed under $d$, and $\Omega^*(P)_{\text{hor}}$ is not closed under $d$.

Nevertheless something like this ought to work—perhaps some version of homological perturbation theory. (Milgram said given a group extension one can splice resolutions and this leads to perturbation theory.)

Let's examine $\Omega^*(P)_{\text{hor}}$. Recall that for
\[ \omega \in \Omega(P)_{\text{hor}} \] one has that
\[ d\omega \in \Omega(P)_{\text{hor}} + \Omega(P)_{\text{hor}} \otimes g^*, \]

More precisely
\[ d\omega - A^a L_a \omega = d\omega - A^a L_a \omega \in \Omega(P)_{\text{hor}} \]

Thus on \( \Omega(P)_{\text{hor}} \) we have a degree one derivation \( d - A^a L_a \).

One has
\[
(d - A^a L_a)^2 = (d - A^a L_a)(d - A^b L_b)
\]
\[ = d^2 - A^a L_a d - d A^b L_b + A^a L_a A^b L_b
\]
\[ = -A^a L_a d + A^b d L_b - (\partial d L_b) L_a + \frac{1}{2} A^a A^b [L_a, L_b] + A^a f_{bc} A^b L_c + A^a A^b L_a L_b \]
\[ = (dA^a + \frac{1}{2} f_{bc} A^b A^c) L_a = -F^a L_a \]

Thus on \( \Omega(P)_{\text{hor}} \) we have the following structure: \( P \), Algebra, derivation of degree 1, action of \( g \), \( F \in \Omega^2(P)_{\text{hor}} \otimes g^* \).

Properties: \( \nabla, F \) are \( g \)-invariant

Check:
\[ [L_a, \nabla] = [L_a, d - A^b L_b] = \partial f_{ab} A^c L_c + A^b f_{ac} L_c = 0 \]

\( g \)-invariance of \( F = dA + A^2 \) is obvious from

\( g \)-invariance of \( A \).

\[ \nabla^2 = -F L_a \]

Check:
\[ \nabla(F^a) = (d - A^b L_b) F^a \]
\[ = -(L_a F)^a + A^b f_{bc} F^c \]
\[ = -f_{bc} A^b F^c + f_{bc} A^b A^c F^c = 0 \]
Conversely suppose given a graded commutative algebra \( \Omega_h \) with \( \nabla \) and \( g \)-action and \( F \in \Omega^2 \otimes g \) having the above properties. Set

\[ \Omega = \Omega_h \otimes \Lambda \mathfrak{g}^* \]

and define the obvious \( g \)-action on \( \Omega \) and define \( d \) by

\[
\begin{align*}
&dw = \nabla w + A^q L_q w \quad \omega \in \Omega_h \\
&dA^a + \frac{1}{2} f^a_{bc} A^b A^c = F^a
\end{align*}
\]

Check that \( d^2 = 0 \).

\[
\begin{align*}
d(dw) &= d(\nabla w + A^b L_b w) \\
&= (\nabla + A^b L_b) \nabla w + dA^b L_b w - A^b (\nabla + A^q L_q) L_b w \\
&= \nabla^2 w + dA^c L_c w + \frac{1}{2} A^b A^c [L_b, L_c] w \\
&= \nabla^2 w + F^c L_c w = 0
\end{align*}
\]

\[
\begin{align*}
d(F^a) &= (\nabla + A^b L_b) F^a = A^b (\nabla - f^a_{bc} F^c) = -f^a_{bc} A^b F^c \\
\therefore \quad dF &= -[A, F] \quad \text{and so} \quad d^2 A = d(F - A^2) = 0.
\end{align*}
\]
The problem is still to find a derivation of the Bott spectral sequence using 
\( \text{C}^{\text{diff}}(G, \Omega(P)_{\text{hor}}) \).

There should be a total differential defined on this space even though \( \Omega(P)_{\text{hor}} \) is not a complex.

To get some insight we can consider the Lie analogue
\[ C^0(g, \Omega(P)_{\text{hor}}) = \Omega(P)_{\text{hor}} \otimes \Lambda g^* \, . \]

We know that if a connection on \( P \) is given, then we have an isomorphism 
\[ \Omega(P)_{\text{hor}} \otimes \Lambda g^* \xrightarrow{\sim} \Omega(P) \]
and hence we have a differential \( d \) on \( C^0(g, \Omega(P)_{\text{hor}}) \). We know this total differential splits 
\[ d = (d - A^a L_a) + A^a L_a \]
into horizontal and vertical derivation, but it's not a bigraded differential algebra since 
\[ \nabla^2 = (d - A^a L_a)^2 = -F^a L_a \]
on \( \Omega(P)_{\text{hor}} \).

Something else we ought to be able to do is to form also 
\( \text{C}^{\text{diff}}(G, \Omega(P) \otimes S(g^*)) \)
and get the same result.
Recall that $\Omega$ is a $gl(\mathfrak{sl})$-DG alg. (commutative) with a connection $A \in \Omega^1 \otimes \mathfrak{g}^*$, then $\Omega_{hor}$ is a graded $gl$-algebra, equipped with a degree one derivation $\nabla$ and $F \in \Omega^2_{hor}$ such that $\nabla F$ are $gl$-invariant and $\nabla^2 = -F^a L_a$. Conversely given a graded $gl$-algebra $\Omega_h$ with $\nabla F$ having these properties, we obtain a $gl(\mathfrak{sl})$ DGA

$$\Omega = \Omega_h \otimes \Lambda \mathfrak{g}_A^* \quad d = \nabla + A^a L_a$$

Here are some examples.

Take $\Omega = W(\mathfrak{g}_F) = \Lambda \mathfrak{g}_A^* \otimes S\mathfrak{g}_F^*$. Then $\Omega_{hor} = S(\mathfrak{g}_F^*)$ with $\nabla = 0$ and $F = F^a a \in S(\mathfrak{g}_F^*) \otimes \mathfrak{g}_A^*$. Let's check the condition $\nabla^2 = -F^a L_a$. One has

$$-F^a L_a (F^b) = F^a R^b_{ac} F^c = \frac{1}{\text{sym in } a,c} \frac{1}{\text{antis in } a,c}$$

Take $\Omega = \Omega(\mathfrak{g}) \otimes \Omega(\mathfrak{g}_F)$, where $\mathfrak{g}$ is a $gl$-manifold. Then

$\Omega_{hor} \cong \Omega(\mathfrak{g}) \otimes S(\mathfrak{g}_F^*)$

where the isomorphism is induced by sending $A^a \mapsto 0$ and the inverse isomorphism by

$$\omega \in \Omega(\mathfrak{g}) \mapsto \prod_a (1 - A^a c_a) \omega = (e^{-A^a c_a}) \omega$$

$$= \omega - A^a c_a \omega + \frac{1}{2} A^a A^b c_b c_a \omega + \frac{1}{3} A^a A^b A^c c_b c_c c_a \omega + \ldots$$
Let's calculate $\nabla F$ in
$(\Omega(P) \otimes W(g))_{\text{hor}}$. Now the connection
form $\Omega$ belongs to $W(g)$ and so does the curvature, so the
curvature $F$ is the canonical elements

$$F^a X_a \in S^1(g^*) \otimes g.$$ 

on $(\Omega(P) \otimes W(g))_{\text{hor}}$ is induced by
d$-A^a i_a$. We know $\nabla = 0$ on $W(g)_{\text{hor}}$
and so let us now calculate $\nabla$ on
the element of $(\Omega(P) \otimes W(g))_{\text{hor}}$ corresponding
to $\omega \in \Omega(P) < \Omega(P) \otimes S(g^*)$. This elt is

$$e^{-A^a i_a}, \omega = \omega - A^a i_a + \frac{1}{2} A^a A^b i_b i_a \omega.$$ 

We apply $d - A^a i_a$, then apply the inverse
isomorphism which sends $A^a \mapsto 0$ and $dA = F - A^2$
to $F$. We obtain

$$(d - A^a i_a) \left( \omega - A^a i_a + \ldots \right) \downarrow$$

$$d\omega - \otimes F^a i_a \omega$$

Thus

$$\nabla(\omega) = (d - F^a i_a) \omega \quad \omega \in \Omega(P)$$

$$\nabla(F) = 0 \quad \text{on } S(g^*).$$

Check:

$$\nabla^2(\omega) = \nabla \left( d\omega - F^b i_b \omega \right)$$

$$= (d - F^a i_a)(d\omega) - F^b \left( d - F^a i_a \right) i_b \omega$$

$$= -F^a i_a d\omega - F^b d i_b \omega + F^b F^a i_a i_b \omega.$$
\[ \Delta^2(\omega) = -F^a L_a \omega \text{ for } \omega \in \Omega(P) \text{ and } \Delta^2 = -F^a L_a = 0 \text{ on } S(\sigma_j^*) \].

Remark: The inverse of the isomorphism

\[ \Omega(P)^{\text{hor}} \otimes \Lambda \sigma_j^* \cong \rightarrow \Omega(P) \]

is obtained as follows:

\[ \Omega(P) \xrightarrow{e^{\phi \omega}} \Omega(P) \otimes \Lambda \sigma_j^* \xrightarrow{\pi \otimes (\Pi \rightarrow A)} \Omega(P)^{\text{hor}} \otimes \Lambda \sigma_j^* \]

\[ \Omega(P)^{\text{hor}} \otimes \Lambda \sigma_j^* \longrightarrow \Omega(P)^{\text{hor}} \otimes \Lambda \sigma_j^* \otimes \Lambda \omega^* \longrightarrow \Omega(P)^{\text{hor}} \otimes \Lambda \omega^* \]

\[ \omega \quad \longrightarrow \quad \omega \quad \longrightarrow \quad \omega \]

A \quad \longrightarrow \quad A + \phi \quad \longrightarrow \quad A \]

Recall that \( \Pi : \Omega(P) \rightarrow \Omega(P)^{\text{hor}} \) is

\[ \Pi = \sum_{j} \eta_j A_j^* = \sum_{j} (1 - A_j^* \eta_j) \]

\[ \xi_1 \ldots \xi_n A^n \ldots A^1 \]
Bott's spectral sequence. The idea is roughly that \((G, \mathfrak{g}_e)\) modules of the form \(\Omega(M)\) with \(M\) a free \(G\)-manifold should be acyclic for the "differentiable cohomology." Thus we have the functor of taking \(G\)-invariants for the \((G, \mathfrak{g}_e)\) action. This is left exact and we take its derived functors using \(\Omega(M)\) as acyclic objects. Let's ignore foundations and see what happens.

Consider a principal \(G\)-bundle \(P\). Then we can filter

\[
\Omega(P) \supset J_2 \supset J_3 \supset \cdots
\]

where \(J_i\) is the ideal generated by \(\Omega^i(P)_{\text{hor}}\). The quotients are

\[
J_i/J_{i+1} = \Omega_{\text{hor}}^i \otimes \Lambda \mathfrak{g}_e^*.
\]

and these should be acyclic for the "differentiable \((G, \mathfrak{g}_e)\) cohomology," because the \(\Lambda \mathfrak{g}_e^*\) is free for \(\mathfrak{g}_e\), and \(\Omega_{\text{hor}}^i\) being a module over \(\Omega^i(P)\) is acyclic for differentiable \(G\) cohomology. We have

\[
(\Omega_{\text{hor}}^i \otimes \Lambda \mathfrak{g}_e^*)_{\text{bas}} = \Omega_i^p(P)_{\text{bas}} = \Omega_i^p(B)
\]

and so we get

\[
E_1^{pq} = \begin{cases} \Omega^p(B) & q = 0 \\ 0 & q \neq 0 \end{cases} \implies H^i(B)
\]

which is OK.
Next let's consider
$$\Omega(P) \otimes W(g) \subset I \supset I^2 \supset \ldots$$
where $I$ is the ideal generated by $\Lambda g^*_F$. One has
$$I^p/I^{p+1} = \Omega^p(P) \otimes (\Lambda A^* \otimes S^p g^*_F)$$
where $d$ in the latter factor is
$$dA + A^2 = 0 \quad df + [A, F] = 0$$
These quotients should be acyclic for the $(\mathcal{O}, g^*_F)$ cohomology. Taking basics gives
$$\left(\frac{I^p}{I^{p+1}}\right)_{\text{hor}} = \left(\Omega^p(P) \otimes S^p g^*_F\right)^G$$
Actually we should do this carefully. Let's start with $\Omega(P) \otimes W(g)$ again and take horizontal elements; this gives
$$\left(\Omega(P) \otimes W(g)\right)_{\text{hor}} = \Omega(P) \otimes S g^*_F$$
with $D = d - F^* L_a$ where $D(F) = 0$.
If we are interested in the associated graded with respect to the $T$-adic filtration, we have $D = d$ since $F^* L_a$ raises $T$-order. Thus
$$D = d$$
$$\left(\frac{I^p}{I^{p+1}}\right)_{\text{hor}} = \Omega^p(P) \otimes S^p g^*_F$$
Now if $P = \text{universal bundle } B G$, then $\Omega^p(P)$ is an acyclic resolution of $\mathcal{O}$ for the
differentiable cohomology with $G$. Thus

$$H^0\left( \frac{T^* T^{*+1}}{T^*} \right)_{bas} = H^0 \text{diff} (G, S\log^*)$$

and we therefore get both spectral sequence

$$H^0 \text{diff} (G, S\log^*) \to H^*(BG)$$

Problem: What is the relation between

$$C(g[t], \Omega(P)) = \Omega(P) \otimes \log^* \otimes S\log^*$$

and

$$\Omega(P) \otimes W(g)$$?

Recall we have a map

$$\Omega(P) \to \Omega(P) \otimes \log^* \otimes S\log^*$$

$$\omega \mapsto \omega + \sum x^a \omega + \sum x^a x^b \omega + \cdots$$

This is an algebra homomorphism such that

$$\omega \in \Omega(P)_{hor} \mapsto \omega$$

$$A \mapsto A + X$$

and it is compatible with differentials since for $\omega \in \Omega(P)_{hor}$ one has

$$d\omega = A^a L_a \omega + \partial (d\omega - A^a L_a \omega)$$

$$\mapsto d\omega + x^a L_a \omega$$
and
\[ A \xrightarrow{i} (x + A)^2 \xrightarrow{d + 8 \gamma} (d + 8 \gamma)x + x^2 = dx = + \gamma \]

Actually it should be possible to give a proof without choosing \( A \).
So we also have a DGA map
\[ W(g) \rightarrow \Lambda g^0 \otimes S_{g^0} \]

sending the universal connection to \( X \) and the universal curvature to
\[ (d + 8 \gamma)x + x^2 = dx = + \gamma \]

Then we can put these together to obtain
\[ \Omega(P) \otimes W(g) \rightarrow \Omega(P) \otimes \Lambda g^0 \otimes S_{g^0} \]

This is a DGA isomorphism:
\[ \Omega(P)_{h_A} \otimes \Lambda g^0_A \otimes \Lambda g^0_X \otimes S_{g^0} \rightarrow \Omega(P)_{h_A} \otimes \Lambda g^0_A \otimes \Lambda g^0_X \otimes S_{g^0} \]

The curious point concerns the \( L_a, l_a \) operations. These are defined diagonally on the left by
\[ l_a \omega = 0 \quad l_a A = X_a \quad l_a \theta = X_a \quad l_a \varphi = 0 \]

and so we must define on the right
\[ l_a \omega = 0 \quad l_a A = 0 \quad l_a X = X_a \quad l_a \varphi = 0. \]
Recall that $\Lambda_{\mathfrak{g} \mathfrak{x}} \otimes S_{\mathfrak{g} \mathfrak{x}}$ is the analogue for $\mathfrak{g}[E]$ of $\mathcal{C}(\mathfrak{g}) = \Lambda(\mathfrak{g}^*)$, on which $\mathfrak{g}[E]$ operates. Thus for $X, X \in \mathfrak{g}[E]$ we have operators $L_X, L_X$ and $c_X, c_X$ on $\mathcal{C}(\mathfrak{g}[E])$. Let's compute them. Recall

$$0 = X^b X^c + (-\varphi^b) X^c$$

so

$$c_X 0 = X \implies c_X (\varphi^b) = 0$$

$$c_X (\delta^b_a) = \delta^b_a$$

Then

$$L_X = \delta c_X + c_X \delta$$

implies

$$L_X (\varphi^b) = \delta c_X (\varphi^b) + c_X (\delta \varphi^b)$$

$$= c_X (-\frac{1}{2} f^b_{cd} X^c X^d)$$

$$= -\frac{1}{2} f^b_{cd} X^c X^d + \frac{1}{2} f^b_{ea} X^e = -f^b_{ac} X^c$$

$$L_X (\varphi^b) = c_X (\varphi^b) = c_X (-\frac{1}{2} f^b_{cd} X^c \varphi^d)$$

$$= -f^b_{ad} \varphi^d$$

This tells us that if we identify

$$W(\mathfrak{g}) = \mathcal{C}(\mathfrak{g}[E])$$

$$A \leftrightarrow X$$

$$F \leftrightarrow \varphi$$

then the natural of $L_X, c_X$ on $W(\mathfrak{g})$ correspond to the operators $L_X, c_X$ on $\mathcal{C}(\mathfrak{g}[E])$ associated to the subalgebra $\mathfrak{g} \subset \mathfrak{g}[E]$. 
From $\xi_x = X^\varepsilon$ we conclude

$$L_{X^\varepsilon} = -i_{X^\varepsilon} \delta + s_{X^\varepsilon} \text{ yields.}$$

Then

$$L_{X^\varepsilon} (X^b) = -i_{X^\varepsilon} \delta X^b = -i_{X^\varepsilon} (-\frac{1}{2} f^b_{\ cd} X^c X^d) = 0$$

$$L_{X^\varepsilon} \phi^b = -i_{X^\varepsilon} \delta \phi^b = \frac{1}{2} f^b_{\ cd} X^c \phi^d$$

$$= -f^b_{\ ca} X^c = f^b_{\ ac} X^c$$

**Summary:** The action of $\gamma, \delta \gamma, \gamma \in \mathfrak{g} + \mathfrak{g}^*$ on $C(\mathfrak{g} \oplus \mathfrak{g}^*)$ is given by

$$L_X X = X, \quad L_X \phi = 0$$

$$L_X X = -[X, X], \quad L_X \phi = -[X, \phi]$$

$$L_{\xi_x} X = 0, \quad L_{\xi_x} (-\phi) = X$$

$$L_{\xi_x} X = 0, \quad L_{\xi_x} (\phi) = +[X, \phi]$$

Check

$$[d, L_{\xi_x}] X = d L_{\xi_x} X - L_{\xi_x} dX = -L_{\xi_x} \phi = X = L_X X$$

$$[d, L_{\xi_x}] \phi = d L_{\xi_x} \phi - L_{\xi_x} d\phi = -d \phi = 0 = L_X \phi$$

Thus

$$[d, L_{\xi_x}] = L_X$$

$$[d, L_{\xi_x}] X = (d L_{\xi_x} + L_{\xi_x} d) X = L_{\xi_x} \phi = +[X, X]$$

$$[d, L_{\xi_x}] \phi = d L_{\xi_x} \phi + L_{\xi_x} d\phi = d[X, \phi] = [X, \phi]$$
Thus \[ [d, L_{x_{\epsilon}}] = -L_{x} \]

Further check:
\[
[d, L_{x_{\epsilon}}] = [d, [\delta, L_{x_{\epsilon}}]] = [-[\delta, [d, L_{x_{\epsilon}}]]] = \quad [\delta, L_{x} = -L_{x}
\]

---

Summary + discussion: We originally started trying to understand BRS cohomology for a $G$-manifold $P$. This is a bigraded differential algebra which we have now connected up with $\Omega(P) \otimes W(g)$, the sort of thing encountered in equivariant cohomology. From this gadget we can obtain the Leray spectral sequence for $G \rightarrow P \rightarrow B$, as well as Bott's spectral sequence.
Review van Est and Bott spectral sequences. The basic idea is that one can calculate differentiable cohomology using $\Omega^*(P)$, where $P$ is a universal $G$-bundle.

In the case of van Est, consider

$$\left( \Omega^p(P) \otimes \Omega^*(G) \right)^G$$

This is a double complex. One has

$$H^P_h = \begin{cases} 0 & p > 0 \\ \lambda^g & p = 0 \end{cases}$$

$$H^b_v = \left( \Omega^*(P) \otimes H^8(G) \right)^G$$

Here one is using exactness of $(M \otimes ?)^G$, where $M$ is $\Omega^p(P)$ or $\Omega^8(G)$.

In the case of Bott's spectral sequence consider

$$\left( \Omega^p(P) \otimes W(g) \right)_{bas} = \left( \Omega^p(P) \otimes S^8(g^*) \right)^G$$

This is a complex. Ignoring differentials it is

$$\oplus \left( \Omega^p(P) \otimes S^8(g^*) \right)^G,$$

and it's a filtered complex with $g^{-P} = \left( \Omega^p(P) \otimes S^8(g^*) \right)^G$. The differential is induced by $d - F^a = d$ on $g^P$. Finally, one must show that

$$\Omega(B) = \Omega(P)_{bas} \to \left( \Omega(P) \otimes W(g) \right)_{bas}$$

is a quasi. This should be easy if there is a connection in $P$. 
Review: Given $\Omega(P)$ introduce the $J$-adic filtration, where $J$ is the ideal generated by $\Omega(\nu_{hor})$. The $J$ cohomology is the cohomology of $gr^J\Omega(P)$, which is $H^*(g; \Omega^*(P)_{hor}) = \Omega^*(B, H^*(g))$.

If we start with $\Omega(P) \otimes W(g)$ and use the isomorphism $\Omega(P) \otimes W(g) \cong \Omega(P) \otimes S(g^*) \otimes \Lambda g^*$

$$
\begin{align*}
\Omega(P) & \otimes \Lambda g^* \otimes \Lambda g_{hor}^* \\
\Omega(P)_{hor} & \otimes \Lambda g^* \otimes \Lambda g_{hor}^* \otimes S(g^*)
\end{align*}
$$

$$
\begin{align*}
\omega & \mapsto \omega \\
A & \mapsto A + X \\
\Theta & \mapsto X \\
\varphi & \mapsto \varphi = dX
\end{align*}
$$

Then $J$ is exactly the ideal of elements of degree $> 0$.

The other thing we can do is to take basic elements first. Applied to $\Omega(P) \supset J \supset J^2 \supset \ldots$, this gives $\Omega(B) \supset \Omega^2(B) \supset \Omega^3(B) \supset \ldots$, the "skeletal" filtration. Applied to $\Omega(P) \otimes W(g)$, it gives the "skeletal" filtration of $(\Omega(P) \otimes S(g^*))$. To get Bott's spectral sequence we use a different ideal $\varnothing$ in $\Omega(P) \otimes W(g)$, namely the ideal generated by $S^1(g^*)$. 

Let us now consider
\[ Y = C^\infty(M, UN) = \text{Aut}(C^\infty_M) \]
\[ g = \text{Lie}(Y) = C^\infty(M) \otimes M \]

Recall that a principal \( Y \)-bundle \( P \to B \) is equivalent to a hermitian vector bundle \( E \) over \( B \times M \) such that \( E_y \cong (C^\infty)^M \) for each \( y \in B \). A connection in \( E \) is equivalent to a connection in \( P \) together with an \( Y \)-equivariant map from \( P \) to the space \( A \) of connections on \((C^\infty)^M\).

We obtain characteristic classes in \( H^*(B) \) associated to \( P \) by integrating characteristic classes of \( E \) over homology classes in \( M \). If we fix a connection on \( E \) and a cycle (closed current) on \( M \), then we obtain closed forms on \( B \). The idea is to describe this construction abstractly, that is, in the spirit of the Weil algebra.

The bundle \( E \) pulled up to \( P \times M \) is canonically trivial, and the connection on \( E \) up on \( P \times M \) is
\[ \delta + \frac{d}{dM} + X + A \]
where \( X \in \Omega^1(P \times M) \otimes M \) is the connection form in \( P \) and
\[ A \in \Omega^0(P \times M) \otimes M \]
is the equivariant map.

We want to play the Weil algebra game...
more generally. Observe that $X$

$X \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$

induces $\mathfrak{g}^* \rightarrow \Omega^1(P)$ which extends to a DGA morphism

$W(\mathfrak{g}) \rightarrow \Omega(P)$

Thus we seek to endow $W(\mathfrak{g})$ so as to incorporate $A$. This means we adjoin to $W(\mathfrak{g})$ something like $A^*$ in degree 0. Thus we probably want all polynomial functions on the affine space $A$, and all polynomial coefficient differential forms. At this rate we end up with just $W(\mathfrak{g}) \otimes \Omega(A)$

and the complex of equivariant differential forms.

Note the curvature is

$(\delta^2 + X + A)^2 =$

$(\delta X + x^2) + (\delta A + dX + [X, A]) + (dA + A^2)$

and that these three components are generally nonzero.

---

The next project is to do a cyclic or large $N$ version.

In the case of the Weil algebra we take the dual of the bar construction on $\{R \rightarrow R\}$ considered as a DGA.

Dually we have the tensor coalgebra generated by $R$ in degree 1 and $R$ in degree 2.

By analogy we expect to add $\Omega^1_R$ in degree 0 and $\Omega^1_R$ in degree 1.

Let's consider the DG algebra traditionally
encountered in anomalies (e.g. Bonora Connes-Leamasino), namely

$\Omega^0(A) \otimes \Lambda \log x$

Lie cochains on the infinitesimal gauge transformations with values in functionals on connections. The differential is given by

$\delta A = -D_A(X) = -dX - [A, X]$

$\delta X + X^2 = 0$

Let us take $g = \Omega^0(M) \otimes \mathcal{M}_N$, $A = \Omega^1(M) \otimes \mathcal{M}_N$

and consider

$(\Omega^0(M) \otimes \mathcal{M}_N)^* \otimes \Lambda \log x$

which should be a Hopf algebra. (Here we consider poly functionals on $A$). Proceeding in known fashion we consider

$\{(\Omega^0(M) \otimes \mathcal{M}_N)^* \otimes \Lambda (\Omega^1(M) \otimes \mathcal{M}_N)^* \otimes \mathcal{M}_N\}^{\otimes N}$

By invariant theory this should contain a tensor algebra on $\Omega^1(M)^*$ in degree 0 and $\Omega^0(M)^*$ in degree 1, and there is a differential $\delta$.

This free noncommutative DG algebra is the "noncommutative" version of $\Omega^0(A) \otimes \Lambda \log x$. To calculate $\delta$ we look at the canonical twisting cochain

$\left(\Lambda (\Omega^0 \otimes \mathcal{M}_N)^* \otimes \Omega^1 \otimes \mathcal{M}_N \right)^{\otimes N}$

$U$ [

$\left(\Omega^0 \otimes \Omega^1 \right)^* \otimes (\Omega^0 \otimes \Omega^1)^*$

$\Theta = X + A$
Then the twisting cochain condition gives
\[(\delta + d)(\chi + A) \cup (\chi + A)^2 = 0\]
\[\left(\delta\chi + \chi^2\right) + (\delta\chi + dA + [\chi, A]) + (dA + A^2)\]
values in \(\Omega^0\)
values in \(\Omega^1\)
values in \(\Omega^2 = 0\)
\[\delta\chi + \chi^2 = 0 = \delta\chi + dA + [\chi, A]\]

Therefore if our procedure is \textbf{OKAY} we should have that
\[(S(A)^* \otimes \Lambda^\ast_{\mathfrak{g}})^{gl_n}\]
is a Hopf algebra whose primitive part is the cyclic cochain complex in the DG algebra

\[
\begin{array}{ccc}
\Omega^0 & \delta \rightarrow & \Omega^1 \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}
\]

The cyclic chain complex looks as follows.

You first write the DGA

\[
\begin{array}{ccc}
\Omega^1 & \leftarrow & \Omega^0 \\
-1 & & 0
\end{array}
\]

Then you take the cyclic complex which has rows the \(\otimes^n\) powers.
In the study of anomalies one encounters $\Omega^0(\mathfrak{g}) \otimes \Lambda g^*$, the complex of Lie cochains for the infinitesimal gauge transformations acting on functions on the space of connections. Actually the physicists want $\Omega^0(\mathfrak{g})$ replaced by $\Gamma_{\text{loc}}$ (the "local" functionals, integrals of polynomials in $A$ and its derivatives). Anomalies are elements of $H^1(\mathfrak{g}, \Gamma_{\text{loc}})$.

For example,

$$W(A) = \log \det \delta_A$$

defined by regularization is nonlocal yet

$$\delta W(A) = \text{"Tr"} \left( \delta_A^{-1} \delta A \right)$$

is local (here $\delta A = [\mathcal{E}, \delta A]$ and the above expression is a local expression related to the index theorem). Thus $\delta W(A)$ can represent a nontrivial element of $H^1(\mathfrak{g}, \Gamma_{\text{loc}})$.

One has

$$H^0(\mathfrak{g}, \Omega^0(\mathfrak{g})) = \Omega^0(\mathfrak{g} / \mathfrak{h}) \otimes H^0(\mathfrak{g})$$

if we restrict to gauge transformations = 1 at a basepoint of $M$. This relates anomalies to $H^1(\mathfrak{g})$:

$$H^1(\mathfrak{g}, \Gamma_{\text{loc}}) \rightarrow H^1(\mathfrak{g}, \Omega^0(\mathfrak{g})) = \Omega^0(\mathfrak{g} / \mathfrak{h}) \otimes H^1(\mathfrak{g})$$

which in turn is related to $H^2(\mathfrak{g} / \mathfrak{h})$ and determinant line bundles.
Point: Lie algebra cohomology for the action on $\Omega^0(\mathfrak{a})$ is directly related to $\tilde{H}^\ast(\mathfrak{a})$:

$$H^\ast(\mathfrak{g}, \Omega^0(\mathfrak{a})) = \Omega^0(\mathfrak{a}/\mathfrak{g}) \otimes \tilde{H}^\ast(\mathfrak{g}).$$

Yesterday I thought that in the case $\mathfrak{g} = U_N(\mathbb{C}(M))$, $N$ large, I could relate $\Omega^0(\mathfrak{a}) \otimes \Lambda^\ast \mathfrak{g}$ to the cyclic cochains for the DGA $\Omega^0(M)$

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \cdots$$

(Actually I looked at $\Omega^0(\mathfrak{a})$ replaced by the polynomial functions).

Let's consider the reduced complex which corresponds to the condition $g = 1$ at the basepoint. Then $d: \overline{\Omega}^0 \to \Omega^1$ is injective and so the $d$-homology in the cyclic chains

$$\begin{align*}
\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2
\end{align*}$$

is concentrated on the diagonal. It looks therefore as if one obtains only something in degree 0 like $\Omega^0(\mathfrak{a}/\mathfrak{g})$ and nothing in positive degrees.

Thus it is necessary to be cautious in this case.