January 22, 1990

Notes on BRS cohomology

A. $\Omega^{\cdot}(G)$ where G is a Lie group:

$\mathfrak{g} = \mathrm{Lie}(G)$. X_a basis for $\mathfrak{g}$, $[X_a, X_b] = f^c_{ab} X_c$

The basic object is the Maurer-Cartan form $\theta \doteq \theta^a X_a \in \Omega^1(G) \otimes_{\mathbb{R}} \mathfrak{g}$.

$\boxed{\begin{array}{l} \mathcal{L}_a = \mathcal{L}_{X_a} \\ \iota_a = \iota_{X_a} \end{array}}$

Properties: 1) $\iota_X \theta = X \Longrightarrow \iota_a \theta^b = \delta^b_a$

2) $\mathrm{Ad}(g) R_g^*(\theta) = \theta$, $\mathcal{L}_X \theta + [X, \theta] = 0 \Longrightarrow \mathcal{L}_a \theta^b + f^b_{ac} \theta^c = 0$

3) $d\theta + \frac{1}{2}[\theta, \theta] = (d\theta + \theta^2) = 0 \Longrightarrow d\theta^a + \frac{1}{2} f^a_{bc} \theta^b \theta^c = 0$

4) $L_g^* \theta = \theta. \Rightarrow \theta^a$ left invariant

The right translation action $R_g(g') = g'g$ makes G into to principal G-bundle with base a point. θ is the connection form for the unique connection in this bundle, explaining 1)-3). Left translations are automorphisms of this principal bundle, whence 4).

~~B. Lie algebra cohomology~~

5) $\Omega(G)^{G_L} \xleftarrow{\sim} \Lambda \mathfrak{g}^*$ $\qquad \Omega(G) \xleftarrow{\sim} \Omega^0(G) \otimes \Lambda \mathfrak{g}^*$

d on $\Omega(G)$ induces a differential δ on $\Lambda \mathfrak{g}^*$ making it a (comm) DG algebra.

6) $\Omega(G)^{G_L} = \Lambda \mathfrak{g}^* = \Lambda[\theta^a]$ with δ given by

$$\delta \theta^a + \frac{1}{2} f^a_{bc} \theta^b \theta^c = 0$$

B. Lie algebra cohomology

Let V be a vector space and consider the trivial bundle $G \times V \to G$ with G acting by left translation on itself and trivially on V.

1) A left-invariant connection $\nabla = d + \alpha$ on $\tilde{V}$, $\alpha \in \mathfrak{g}^* \otimes \mathrm{End}(V)$ is flat $\Longleftrightarrow$ $u : X \mapsto \iota_X \alpha$ is a representation of $\mathfrak{g}$ on V. (compatible with $[\,,\,]$)

One has $\alpha = \theta^a u_a \quad (u_a = u(X_a))$

$$d\alpha + \alpha^2 = d\theta^c u_c + \theta^a u_a \theta^b u_b$$
$$= \left(-\tfrac{1}{2} f^c_{ab}\, \theta^a \theta^b\right) u_c + \tfrac{1}{2} \theta^a \theta^b [u_a, u_b]$$
$$= \tfrac{1}{2} \theta^a \theta^b \left(-f^c_{ab}\, u_c + [u_a, u_b] \right) \qquad \square$$

Fix a repn $u : \mathfrak{g} \to \mathrm{End}\, V$; one has the flat left-invariant connection $d + u\theta$ on $\tilde{V}$, whence a complex of left-invariant V-valued forms

$$\left(\Omega(G) \otimes V\right)^{G_L} = \Lambda \mathfrak{g}^* \otimes V$$

which is a DG module over $\Omega(G)^{G_L} = \Lambda \mathfrak{g}^*$. Denote by δ the differential in the right picture.

$$\delta \xi = \theta^a u_a(\xi) \qquad \xi \in V$$

The complex $(\Lambda \mathfrak{g}^* \otimes V, \delta)$ is the complex of Lie cochains with values in V.

B1. Lie alg cochains with values in a representation $\mathcal{U}: G \to \mathrm{Aut}(V)$.

View $\mathcal{U}$ as a gauge transformation

$$G \times V \longrightarrow G \times V \qquad (g, \xi) \longmapsto (g, \mathcal{U}(g)\xi)$$

One has an isomorphism

$$\underset{d + u\theta}{\left(\Omega(G) \otimes V\right)^{G_L}} \simeq \underset{d}{\left(\Omega(G) \otimes V\right)^{G_{L,\mathcal{U}}}}$$

as $\left(\mathcal{U}^{-1} \cdot d \cdot \mathcal{U}\right)\xi = \Big(d + \underbrace{\mathcal{U}^{-1} d\mathcal{U}}_{\mathcal{U}^* \text{ of MC form on } \mathrm{Aut}(V)}\Big)\xi$

$$= \left(d + u\theta\right)\xi$$

Thus $\left(\Lambda \mathfrak{g}^* \otimes V, \delta\right)$ is isomorphic to the invariants in $\left(\Omega(G) \otimes V, d\right)$ for the $G_{L,\mathcal{U}}$ action.

B2. G compact connected, V repn. of G.

Claim
$$\boxed{H^*(\mathfrak{g}, V) = \underbrace{H^*(\mathfrak{g})}_{\cong H^*(G)} \otimes V^{\mathfrak{g}}}$$

Pf. Start with $V = \mathbb{C}$. One has the averaging operator $P = \int_G g^*$ on $\Omega(G)$ which commutes with d and projects onto the left-invariant forms. It induces a projection operator on $H^\cdot(\Omega(G))$, but an argument is needed to see that the induced projection projects onto the subspace of invariants.

Let's proceed directly and show

$$1) \qquad H^*(\Omega(G)^G) \xrightarrow{\sim} H^*(\Omega(G))^G$$

Injective: Given ω such that $g^*\omega = \omega$ $\forall g$ and $d\omega = 0$, and suppose $\omega = d\eta$, $\eta \in \Omega(G)$. Then $d(P\eta) = d\left(\int_G g^*\eta\right) = \int_G g^* \underbrace{d\eta}_{\omega} = \int_G \omega = \omega$, where we have used the continuity of d.

Surjective. Given ω in $\Omega(G)$ with $d\omega = 0$ and such that its class in invariant: $\omega - g^*\omega \in \operatorname{Im} d$ $\forall g \in G$. Then

$$d(P\omega) = d\int_G g^*\omega = \int_G d(g^*\omega) = P(d\omega) = 0$$

again by continuity of d. Next

$$\omega - P\omega = \omega - \int_G g^*\omega = \int_G (\omega - g^*\omega) \in \overline{\operatorname{Im} d}$$

belongs to the closure of $\operatorname{Im} d$. Since $\operatorname{Im} d$ is closed (elementary proof in Begg's thesis), ω is cohomologous to the invariant form $P\omega$. □

Return to our claim. Because G is connected it acts trivially on $H^\bullet(\Omega(G)) = H^\bullet(G)$ ($\mathcal{L}_X = d\iota_X + \iota_X d$ is trivial on $H^\bullet(\Omega(G))$), so 1) above yield

$$\begin{array}{c} H^\bullet(\Lambda(\mathfrak{g}^*), \delta) \xrightarrow{\sim} H^\bullet(G) \\ \| \\ H^\bullet(\mathfrak{g}) \end{array}$$

Next consider general case. We know that $(\Lambda(\mathfrak{g}^*) \otimes V, \delta) = (\Omega(G) \otimes V, d + u\theta)^{G_L}$ $= (\Omega(G) \otimes V, d)^{G_L, u}$ has cohomology equal to

$H^\circ(\Omega(G)\otimes V, d)^{G_{L,u}}$ by the above averaging operator argument. This $=$ $\left(H^\circ(G)\otimes V\right)^{G_{L,u}} = H^\circ(G)\otimes V^G$ since G acts trivially on $H^\circ(G)$. ∎

Specific map: $H^\circ(\mathfrak{g})\otimes V^{\mathfrak{g}} \longrightarrow H^\circ(\mathfrak{g}, V)$ is obvious from the fact that $H^\circ(\mathfrak{g}, V)$ is a module over $H^\circ(\mathfrak{g})$. This given by $[\omega]\otimes\xi \longmapsto [\omega\xi]$

C. BRS cohomology is Lie algebra cohomology associated to a DG Lie algebra.

Let P be a G-manifold, G acting on right. One has for each $X\in\mathfrak{g}$ operators L_X, ι_X on $\Omega(P)$. satisfying:

$$[L_X, L_Y] = L_{X,Y} \qquad [L_X, \iota_Y] = \iota_{[X,Y]}$$

$$[\iota_X, \iota_Y] = 0 \qquad [d, \iota_X] = L_X$$

L_X derivation of degree 0

ι_X ~~(anti-)~~ ——— -1

Define $\tilde{\mathfrak{g}}$ to be the DG Lie algebra with $\tilde{\mathfrak{g}}^0 \cong \mathfrak{g}_c$, $\tilde{\mathfrak{g}}^{-1} \cong \mathfrak{g}_c$; write $L_X \in \tilde{\mathfrak{g}}^0$ for the element corresponding to $X\in\mathfrak{g}$, and similarly $\iota_X\in\tilde{\mathfrak{g}}^{-1}$. The bracket in $\tilde{\mathfrak{g}}$ is defined by the above formulas and the differential $d:\tilde{\mathfrak{g}}^{-1}\to\mathfrak{g}^0$ by $d(\iota_X) = L_X$.

Then $\Omega(P)$ is a DG module over the DG Lie algebra $\tilde{\mathfrak{g}}$. In fact one has a DG Lie algebra morphism

$$\tilde{\mathfrak{g}} \longrightarrow \mathrm{Der}(\Omega^\cdot(P))$$

Analogue of $\Lambda \mathfrak{g}^*$ and $\theta \in \Lambda^1\mathfrak{g}^* \otimes \mathfrak{g}$.

$$\mathcal{S}^\cdot(\Sigma^\cdot \tilde{\mathfrak{g}})^* = \Lambda(\tilde{\mathfrak{g}}^0)^* \otimes S(\tilde{\mathfrak{g}}^{-1})^*$$
$$\cong \Lambda \mathfrak{g}^* \otimes S\mathfrak{g}^*$$

Let $\chi^a \in (\tilde{\mathfrak{g}}^0)^*$ be the basis dual to L_a

let $\varphi^a \in (\tilde{\mathfrak{g}}^{-1})^*$ ——————————— ι_a

Then
$$\mathcal{S}(\Sigma \tilde{\mathfrak{g}})^* = \Lambda[\chi^a] \otimes S[\varphi^a]$$
with $\deg \chi^a = 1$, $\deg \varphi^a = 2$.

The analogue of $\theta \in \Lambda^1\mathfrak{g}^* \otimes \mathfrak{g}$ is
$$\theta = \chi^a L_a + \varphi^a \iota_a \in \mathcal{S}(\Sigma\tilde{\mathfrak{g}})^* \otimes \tilde{\mathfrak{g}}$$
and the differential δ on $\mathcal{S}(\Sigma\tilde{\mathfrak{g}})^*$ is determined by
$$\delta\theta + \theta^2 = 0.$$

$$0 = \delta\theta + \theta^2 = \delta\chi^a L_a + \delta\varphi^a \iota_a + (\chi^b L_b + \varphi^b \iota_b)(\chi^c L_c + \varphi^c \iota_c)$$
$$= \delta\chi^a L_a + \tfrac{1}{2}\chi^b\chi^c \underbrace{[L_b, L_c]}_{f^a_{bc} L_a}$$
$$\delta\varphi^a \iota_a + \chi^b\varphi^c \underbrace{(L_b \iota_c - \iota_c L_b)}_{f^a_{bc}\,\iota_a} + \tfrac{1}{2}\varphi^b\varphi^c \underbrace{(\iota_b\iota_c + \iota_c\iota_b)}_{0}$$

$$\boxed{\begin{aligned} \delta\chi^a + \tfrac{1}{2} f^a_{bc}\chi^b\chi^c &= 0 \\ \delta\varphi^a + f^a_{bc}\chi^b\varphi^c &= 0 \end{aligned}} \quad \text{i.e.} \quad \boxed{\begin{aligned} \delta\chi + \chi^2 &= 0 \\ \delta\varphi + [\chi, \varphi] &= 0 \end{aligned}}$$

Next d on $\tilde{\mathfrak{g}}$ induces a d on $\mathcal{S}(\Sigma\tilde{\mathfrak{g}}^*)$ such that $d\theta = 0$ for total d on $\mathcal{S}(\Sigma\tilde{\mathfrak{g}}^*) \otimes \tilde{\mathfrak{g}}$.

$$0 = d\Theta = dX^a L_a - X^a \overbrace{dL_a}^{0} + d\varphi^a \iota_a + \varphi^a \underbrace{d(\iota_a)}_{L_a}$$

$$= (dX^a + \varphi^a) L_a + (d\varphi^a)\,\iota_a$$

$$\boxed{dX^a = -\varphi^a} \qquad \boxed{d\varphi^a = 0}$$

<u>C1.</u> Lie cochains on $\tilde{\mathfrak{g}}$ values in $\Omega(P)$:

$$\mathcal{S}(\Sigma\tilde{\mathfrak{g}})^* \otimes \Omega(P) = \Lambda[X] \otimes S[\varphi] \otimes \Omega(P)$$

This is a bigraded DGA over $\mathcal{S}(\Sigma\tilde{\mathfrak{g}})^*$ with δ given by

$$\delta(\omega) = \Theta \cdot \omega = X^a (L_a \omega) + \varphi^a (\iota_a \omega)$$

Assume now P is a principal G-bundle with connection $A = A^a X_a \in \Omega^1(P) \otimes \mathfrak{g}$, and curvature $F = dA + A^2 = F^a X_a \in \Omega^2(P) \otimes \mathfrak{g}$. Recall

$$\mathcal{L}_X A + [X, A] = 0 \qquad\qquad \mathcal{L}_X F + [X, F] = 0$$

$$\iota_X A = X \qquad\qquad \iota_X F = 0$$

or

$$L_a A^b + f^b_{ac} A^c = 0 \qquad\qquad L_a F^b + f^b_{ac} F^c = 0$$

$$\iota_a A^b = \delta_a^b \qquad\qquad \iota_a F^b = 0$$

Then

$$\delta(A^b) = (X^a L_a + \varphi^a \iota_a) A^b = -f^b_{ac} X^a A^c + \varphi^b$$

$$\delta(F^b) = (X^a L_a + \varphi^a \iota_a) F^b = -f^b_{ac} X^a F^b$$

or

$$\boxed{\begin{aligned} \delta A &= -[X, A] + \varphi \\ \delta F &= -[X, F] \end{aligned}}$$

Thm: Let P be a principal G-bundle, G compact connected. Then

1) The d-cohomology of $S(\Sigma\tilde{\mathfrak{g}})^* \otimes \Omega(P)$ the BRS algebra is $H^\cdot(P)$ concentrated in (δ-degree) $q=0$, consequently the total or $d+\delta$ cohomology is $H^\cdot(P)$.

2) The δ-cohomology is $\Omega^p(B) \otimes H^q(G)$ where B is the base

Proof. 1) As far as d is concerned the BRS alg is the tensor product of the DGA $\Lambda[\chi] \otimes S[\varphi]$, $d\chi = -\varphi$ which has trivial cohomology $\mathbb{C}$ in degree $(0,0)$, and $\Omega(P)$ which is located in the line $q=0$. Thus the d-cohomology is $H^\cdot(P)$ located on $q=0$, and the spectral sequence collapses giving the same result for the $d+\delta$ cohomology.

2) Since $\varphi = \delta A + [\chi, A]$, $\varphi^a = \delta A^a + f^a_{bc}\chi^b A^c$ we have

$$\Lambda[\chi] \otimes S[\varphi] \otimes \Omega(P) = \Lambda[\chi] \otimes \Omega(P) \otimes S[\delta A]$$

3) $$= \left(\Lambda[\chi] \otimes \Omega_{hor}(P)\right) \otimes \left(\Lambda[A] \otimes S[\delta A]\right)$$

Note that if $\omega \in \Omega_{hor}(P)$, then

$$\delta\omega = (\chi^a L_a + \varphi^a \iota_a)\omega = \chi^a L_a \omega \in \Lambda[\chi] \otimes \Omega_{hor}(P).$$

$(\iota_X L_Y \omega = (L_Y \iota_X - \iota_X L_Y)\omega = -\iota_{[Y,X]}\omega$, so the operators L_X preserve $\Omega_{hor}(P)$.). Thus 3) shows the BRS algebra with δ is the tensor product of the Lie cochains on $\mathfrak{g}$ acting on $\Omega_{hor}(P)$ with a contractible algebra. Thus

$$H^\cdot(BRS, \delta) = H^\cdot(\mathfrak{g}, \Omega_{hor}(P)) = H^\cdot(\mathfrak{g}) \otimes \Omega_{hor}(P)^{\mathfrak{g}}$$
$$= H^\cdot(\mathfrak{g}) \otimes \Omega(B) \qquad \square$$

Natural question is whether spectral sequence ~~starting~~ with the δ cohomology $E^1 = \Omega(B) \otimes H^\cdot(G)$ and ending with $H^\cdot(P)$ is the same as the Leray spectral sequence.

January 25, 1980

We have an analogy between the $(b, s, 1-K, B)$ operators on reduced cochains and operators $(d, i_X, L_X, P\iota_X)$ occurring for manifolds with circles action. Here $P = \int_{S^1} \exp(tX)^*$ is the averaging operator.

Let M be a manifold with S^1-action. Let's assume the action is free, or at least that the isotropy groups are finite, i.e. no fixpts. Then ι_X considered as a differential on $\Omega(M)$ is exact: $\operatorname{Ker} \iota_X = \operatorname{Im} \iota_X$. Recall that in the cochain setup we have

1) cyclic cochains $= \operatorname{Im} B = \operatorname{Ker}(s) \cap \operatorname{Ker}(sb)$

The analogue of this should be

2) basic forms $= \operatorname{Ker}(\iota_X) \cap \operatorname{Ker}(\iota_X d) = \operatorname{Im} P\iota_X$

The first equality is clear since $\operatorname{Ker}(\iota_X) = \Omega_{hor}(M)$ and $\operatorname{Ker}(\iota_X) \cap \operatorname{Ker}(L_X) = \Omega_{bas}(M)$ and $d\iota_X + \iota_X d = L_X$. The second equality is clear because quite generally the averaging operator projects onto the invariants, so that applying this principle to $\operatorname{Im} \iota_X = \Omega_{hor}$ we have $(\operatorname{Im} \iota_X)^{S^1} = P \operatorname{Im} \iota_X = \operatorname{Im} P\iota_X$.

Next for cochains we have

3) $\operatorname{Ker} B = \operatorname{Im} s + \operatorname{Im} bs$

so the analogue should be

4) $\operatorname{Ker} P\iota_X = \operatorname{Im} \iota_X + \operatorname{Im} d\iota_X = \Omega_{hor} + d\Omega_{hor}$

To check this we have to understand the kernel of P on Ω_{hor}. In general one should have that $\operatorname{Ker} P = \operatorname{Im} L_X$, because

$$1-P = \int_0^1 \left(1-e^{tL_X}\right)dt$$

$$= L_X \int_0^1 \underbrace{\left(\frac{1-e^{tL_X}}{L_X}\right)}_{-\int_0^t e^{sL_X}\,ds} dt$$

Thus if $P\iota_X\omega = 0$, then $\iota_X\omega = L_X(\iota_X\eta) = \iota_X d\iota_X\eta$ for some η, hence

$$\omega = (\omega - d\iota_X\eta) + d\iota_X\eta \in \Omega_{hor} + d\Omega_{hor}$$

Let's check now the

Key Lemma: $\operatorname{Ker} P\iota_X / \operatorname{Im} P\iota_X$ is acyclic.

Proof. Let $P\iota_X\omega = 0$ and $d\omega \in \operatorname{Im} P\iota_X$; we have to show that $\omega \in d(\operatorname{Ker} P\iota_X) + (\operatorname{Im} P\iota_X)$. The second condition gives $\iota_X d\omega \in \operatorname{Im}(\iota_X P\iota_X) = 0$ since $\iota_X P = P\iota_X$. Thus $\iota_X d\omega = 0$.

Using 4) one has $P\iota_X\omega = 0 \implies \omega = \eta + d\xi$ with η, ξ horizontal. As $\iota_X d\eta = \iota_X d\omega = 0$ we have $\eta \in \operatorname{Ker}\iota_X \cap \operatorname{Ker}\iota_X d = \operatorname{Im} P\iota_X$ using 2). As $\iota_X\xi = 0$ we have $P\iota_X\xi = 0$ so $\xi \in \operatorname{Ker} P\iota_X$. Thus $\omega = \eta + d\xi \in \operatorname{Im} P\iota_X + d(\operatorname{Ker} P\iota_X)$. $\square$

Thus we have a nice analogy of reduced cochains with forms on a manifold M with circle action having no fixpoints.

What about the Bismut forms associated to a vector bundle with connection (E, ∇) over M. Recall that [redacted] these are

$$\eta = \mathrm{tr}\, e^{(\nabla + \iota_X)^2} = \mathrm{tr}\left(e^{\nabla_X + \nabla^2}\right)$$

and that one has

$$(d + \iota_X)\,\eta = \mathrm{tr}\left(\left[\nabla + \iota_X, e^{(\nabla+\iota_X)^2}\right]\right) = 0$$

Notice that η is even: $\eta = \sum_{n \geq 0} \eta^{(2n)}$ and that one has

$$d\eta^{(2n)} = -\iota_X \eta^{(2n+2)}$$

These odd forms are basic.

An interesting point would be whether the forms $\eta^{(2n)}$ could be non invariant under the circle action, because this might give some insight into JLO's cocycle. However

$$(d + \iota_X)\eta = 0 \implies (d + i_X)^2 \eta = L_X \eta = 0$$

Thus cocycles for $d + \iota_X$ are automatically invariant.

Observation (Feb 1) $\left(\int_0^1 e^{tL_X}\, dt\right)\iota_X = \pi_*$

where $\pi : M \longrightarrow M/S^1$.

February 2, 1990

$S^1 \to M \xrightarrow{\pi} B$ principal bundle

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker}\pi_* \arrow[r] & \Omega(M) \arrow[r, "\pi_*"] & \Omega(B) \arrow[r] & 0 \\ & \Omega(B) \arrow[u, phantom, "\cup"] \arrow[ur, "\pi^*"'] & & & \end{tikzcd}$$

For the analogue of S operation, given $z \in \Omega(B)$ closed we want $f \in \Omega(M)$ satisfying $\iota_X df = 0$ (analogue of $sbf = 0$) with $\pi_* f = z$. Try

$$f = \theta \cdot \pi^*(z)$$

Then $\pi_* f = \pi_*(\theta) \cdot z \implies$ want $\boxed{\pi_*(\theta) = 1}$

and

$$\iota_X d(\theta \pi^*(z)) = \iota_X (d\theta \cdot \pi^*(z)) = (\iota_X d\theta) \cdot \pi^*(z)$$

$\implies$ want $\boxed{\iota_X d\theta = 0.}$. Now the easiest way to arrange $\pi_* \theta = 1$ is to have $\iota_X \theta = 1$. In this case $L_X \theta = (d\iota_X + \iota_X d)\theta = 0$, so θ has to be a connection form and our lifting is an invariant form.

$$\begin{tikzcd} 0 \arrow[r] & \Omega(B) \arrow[r, "\pi^*"] & \Omega(P)_{inv} \arrow[r, "\pi_*"] & \Omega(B) \arrow[r] \arrow[l, bend right, "\theta \cdot \pi^*"'] & 0 \end{tikzcd}$$

This sort of procedure apparently doesn't work with reduced cochains because we can't consider K-invariant cochains. However we can proceed as follows.

Choose $\rho : A \to k$, $\rho(1) = 1$. Given z_{n-1} a reduced cyclic cocycle, set

$$g_n(a_0, \dots, a_n) = \rho(a_0) z_{n-1}(a_1, \dots, a_n)$$

and set $f_n = \frac{1}{n+1} \sum_0^n \kappa^i g_n$

Claim $\kappa^{n+1} g_n = g_n$, $\kappa f_n = f_n$, $sbf_n = 0$ and $sf_n = z_{n-1}$. Thus bf_n is a cyclic $(n+1)$-cocycle representing $S[z_{n-1}]$.

Proof. Clearly $g_n \in \tilde{C}_n$ (reduced n-cochains) and sg_n [redacted] $= z_{n-1}$. Thus bsg_n [redacted] $= bz_{n-1} = 0$, [redacted] so $\kappa^{n+1} g_n$ [redacted] $= (1-bs)g_n = g_n$. Then

$$(1-\kappa)f_n = \frac{1}{n+1}(1-\kappa^{n+1})g_n = 0$$

and

$$sf_n = \frac{1}{n+1}\sum_0^n \underbrace{\kappa^i sg_n}_{\lambda^i z_{n-1}} = z_{n-1}.$$

Finally $sbf_n = (1-\kappa-bs)f_n = -bz_{n-1} = 0$.

Interesting question. Suppose $A = \tilde{\mathcal{Q}}$ is augmented and that $\rho : A \to \mathbb{C}$ is the augmentation. Does this process yield Connes formula for S?

Transgression: Let P be a principal U_N bundle with base B. We know from considering connections and curvature: Chern-Weil theory, that there are odd degree classes $cs_{2n-1} \in H^{2n-1}(P)$ defined for $2n > \dim B$ which are completely canonical for $2n > \dim B + 1$. (Already when $2n-1 = \dim B$ one gets non uniqueness in the case of the trivial bundle $B \times G \xrightarrow{pr_1} B$

$G = U_N$ since gauge transformations will act non trivially on CS_{2n-1}.

Better: The odd classes ~~[illegible]~~ CS_{2n-1} come from the primitive classes $e_{2n-1} \in H^{2n-1}(G)$ and these for $2n-1 \leq \dim B$ will be affected by gauge transformations.

Question: How to show these odd classes in $H^{\cdot}(P)$ are defined without using differential forms?

One method is that of Chern-Simons: Construct a transgression cochain in a universal situation.

More concrete method. We can split off a trivial bundle: Let $E = P \times^{U_N} \mathbb{C}^N$. Then we have $E = E_1 \oplus \mathbb{C}^k$ where $2\,\mathrm{rank}(E_1) \leq \dim B$.

Recall how this is done by obstruction theory. To reduce the structural group of a principal G bundle P to the subgroup H, we need a section of $P \times^G G/H = P/H$. The inverse image in P of the image of this section is then a principal H bundle P_1 and $P = P_1 \times^H G$. Incidently we get in this case an equivariant map

$$P \longrightarrow H\backslash G$$

which is also equivalent to the section.

So in our case we have

$$P \longrightarrow U_r \backslash U_N \qquad r = \mathrm{rank}\, E_1$$

cohomology starts in degree $2r+1$

So we get odd classes in $H^{2n-1}(P)$
for $2n-1 \geq 2r+1$ where $\underbrace{r \text{ is such that}}_{2r \leq \dim B}$
and hence for $2n-1 \geq \dim B + 1$ i.e
$2n-1 > \dim B$.

Check minimal degree. Suppose $\dim B = 2n-1$.
Then $cs_{2n-1} \in H^{2n-1}(P)$ is defined. Also we know that $E = E' \oplus \mathbb{C}^k$ with $r = n-1$
so we have $P \longrightarrow U_{n-1} \backslash U_N$ - first class $2n-1$
If $\dim B = 2n$, then $cs_{2n+1} \in H^{2n+1}(P)$ is defined
and $E = E' \oplus \mathbb{C}^k$ with $r = n$, then get
$P \longrightarrow U_n \backslash U_N$ - first class degree $2n+1$.

February 3, 1990

Lundell (Top. Vol 8) showed that the Bott map $S^2 \wedge U_n \longrightarrow U_{2n}$ can be deformed to a map $S^2 \wedge U_n \longrightarrow U_{n+1}$. He proves that the induced map

$$\begin{array}{ccc} \pi_{2n}(U_n) & \longrightarrow & \pi_{2n+2}(U_{2n}) \\ \| & & \| \\ \mathbb{Z}/n!\mathbb{Z} & & \mathbb{Z}/(n+1)!\mathbb{Z} \end{array}$$

is injective (in fact $1 \to \pm(n+1)$) and in some later paper, I think he calculates the homotopy groups of his spectrum and gets $\mathbb{Q}/\mathbb{Z}$'s maybe.

Here's how one perhaps can view his construction. The basic things to look at are maps

$$M \times U_n \longrightarrow U_m$$

which are families of homomorphisms $U_n \longrightarrow U_m$ parametrized by M. Put another way, we have a homomorphism

$$U_n \longrightarrow U_m^M$$

that is a representation of U_n on the trivial bundle $\underline{\mathbb{C}}^m$ over M. Up to conjugacy these can be classified easily, in fact, for U_n replaced by a compact Lie group G. We have an equivariant G-bundle E ~~which is trivial~~ ~~[illegible]~~ over M, where G acts trivially on M, and where E is the trivial bundle $\underline{\mathbb{C}}^m$.

We can decompose E with respect to the irreducible repns. of G:

$$E = \bigoplus_{\alpha} W_\alpha \otimes \mathrm{Hom}_G(W_\alpha, E)$$

and thus we have a collection of bundles $\mathrm{Hom}_G(W_\alpha, E)$ over M indexed by the irreducible representations such that when added up with the multiplicities $\dim W_\alpha$ yields the trivial bundle $\widetilde{\mathbb{C}^m}$.

Take $M = S^2$. I think the Bott map comes from the following. Let $L = \mathcal{O}(-1)$ be the canonical subbundle of $\tilde{\mathbb{C}}^2$ over $S^2 = \mathbb{P}(\mathbb{C}^2) = \mathbb{C}P^1$. For each line ℓ and $g \in U_n$ we consider the unitary autom of $\mathbb{C}^2 \otimes \mathbb{C}^n$ which is $1 \otimes g$ on $\ell \otimes \mathbb{C}^n$ and the identity on $\ell^\perp \otimes \mathbb{C}^n$. This gives a family of homos. $U_n \longrightarrow U_{2n}$ parametrized by $\ell \in S^2$. We have the trivial bundle $\tilde{\mathbb{C}}^2 \otimes \tilde{\mathbb{C}}^n = \mathcal{O}(-1) \otimes \tilde{\mathbb{C}}^n \oplus \mathcal{O}(1) \otimes \tilde{\mathbb{C}}^n$ with $g \in U_n$ acting as $1 \otimes g$ on the first factor and the identity on the second.

What one has done is to take $\mathcal{O}(-1) \otimes \tilde{\mathbb{C}}^n$ over S^2 with the nontrivial action and added a bundle with trivial action to get a trivial vector bundle. Since the base is S^2 we could have added $\mathcal{O}(n)$ to $\mathcal{O}(-1) \otimes \tilde{\mathbb{C}}^n$ to get a trivial bundle of rank $n+1$.

This is basically what Lundell does, I think. However the families of homom. $M \times U_n \longrightarrow U_m$

are not maps $M \wedge U_n \longrightarrow U_m$ since at the basept of M one has a nontrivial map. If the family $M \times U_n \longrightarrow U_m$ is $(m,g) \longmapsto \varphi_m(g)$, then $(m,g) \longmapsto \varphi_m(g)\,\varphi_*(g)^{-1}$ is the identity if either $m = *$ or $g = 1$.

Let's check the Bott map arises in this way. I recall that Bott's map are families of minimal geodesics between "antipodal" points. Thus one has

$$[0,\pi] \times U_n \longrightarrow Gr_n(\mathbb{C}^{2n})$$

$$(\theta\ ,\ g) \longmapsto F = \cos\theta\,\varepsilon + \sin\theta\begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

which gives the graph (tg) path from $\varepsilon \leftrightarrow \mathbb{C}^n \oplus 0$ to $-\varepsilon \leftrightarrow 0 \oplus \mathbb{C}^n$. In effect

$$\text{graph } T \leftrightarrow F \qquad \text{when} \qquad F\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -T^* \\ T & 1 \end{pmatrix}}_{1+X}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{i.e. } F = (1+X)\,\varepsilon\,(1+X)^{-1} = \frac{1+X}{1-X}\,\varepsilon$$

and if $X = t\begin{pmatrix} 0 & -g^* \\ g & 0 \end{pmatrix}$ $\quad \dfrac{1+X}{1-X} = \dfrac{(1+X)^2}{1+t^2} = \dfrac{1-t^2}{1+t^2} + \dfrac{2t}{1+t^2}\begin{pmatrix} 0 & -g^* \\ g & 0 \end{pmatrix}$

and

$$F = \frac{1+X}{1-X} = \frac{1-t^2}{1+t^2}\,\varepsilon + \frac{2t}{1+t^2}\begin{pmatrix} 0 & g^* \\ g & 0 \end{pmatrix}$$

Next there is the Bott map

$$[0,\pi] \times Gr_n(\mathbb{C}^{2n}) \longrightarrow SU_{2n} \qquad (\theta, F) \longmapsto e^{i\theta F}$$

$$(\theta, F) \longmapsto e^{i\theta F} = \cos\theta + i\sin\theta F$$

which gives a geodesic from 1 to −1. (actually I use $e^{-i\theta F}$ in my C.T. paper.)

~~[illegible]~~

Note for $n=1$, that the first map gives a homeomorphism

$$\Sigma U_1 \xrightarrow{\sim} Gr_1(\mathbb{C}^2) = S^2$$

and the second map gives a homeomorphism

$$\Sigma S^2 \xrightarrow{\sim} SU_2 = S^3$$

It's not yet clear why the ~~[illegible]~~ map I described is related to Bott's map.

<u>February 4, 1990</u>

Recall that if P is a principal U_N-bundle over B, then we have defined classes $CS_{2n-1} \in H^{2n-1}(P)$ for $2n-1 > \dim B$. Let's check this.

<u>First method.</u> Choose a connection A on P/B and use the Chern-Simons deformation $A_t = tA$ to write

$$\operatorname{tr}\left(\frac{F^n}{n!}\right) = d \underbrace{\int_0^1 \operatorname{tr}\left(A\frac{F_t^{\,n-1}}{(n-1)!}\right) dt}_{CS_{2n-1}(A)}$$

For $2n > \dim B$, $\operatorname{tr}(F^n/n!) = 0$ since it comes from the base and so $CS_{2n-1}(A)$ is closed. The invariance of its class is derived from this closedness assertion by working over $B \times \mathbb{R}$, so the class is well defined for $2n > \dim(B \times \mathbb{R})$ i.e. $2n-1 > \dim B$.

Obvious generalization: Given a vector bundle E/M equipped with a flat partial connection relative to a foliation of M and equipped with a trivialization, there are classes $c_{2n-1} \in H^{2n-1}(M)$ defined for $2n-1 > \operatorname{codim}$ of the foliation.

<u>2nd method</u>. Let us try reducing the structural group of P to U_{n-1}, i.e. construct a section of P/U_{n-1} over B. The fibre is U_N/U_{n-1} whose cohomology begins in degree $2n-1$, hence one has a section for $2n-1 \geq \dim B$ by obstruction theory and it is unique up to homotopy provided $2n-1 > \dim B$. Thus we have in this case a U_N-equivariant map

$$* \qquad P \longrightarrow U_{n-1}\backslash U_N$$

unique up to homotopy. Now $H^*(U_{n-1}\backslash U_N) \subset H^*(U_N)$ is the subalgebra generated by the primitive generators e_{2k-1} for $k \geq n$. Pulling back via the above map gives the classes $e_{2n-1} \in H^{2n-1}(P)$ for $2n-1 > \dim$.

Why does this agree with the first method? When we reduce P to U_{n-1} we write the associated vector bundle E as $E_1 \oplus \mathbb{C}^k$ where rank $E_1 = n-1$. Then we can use a connection in E_1 together with the 0 connection on $\mathbb{C}^k$.

??

Since the map $*$ above is equivariant we have a comm. square

$$\begin{tikzcd} P \times U_N \arrow[r, "\mu"] \arrow[d] & P \arrow[d] \\ U_{n-1}\backslash U_N \times U_N \arrow[r] & U_{n-1}\backslash U_N \end{tikzcd}$$

which means that the classes $e^P_{2n-1} \in H^{2n-1}(P)$

satisfy

$$\mu^*\left(e^P_{2n-1}\right) = e^P_{2n-1}\otimes 1 + 1\otimes e^{U_N}_{2n-1} \in H^{2n-1}(P\times U_N)$$

This ought to [redacted] generalize to the case where there's a foliation, where we have a flat partial connection on the trivial bundle. If we change the trivialization by $g: M \longrightarrow U_N$, then the classes change by $g^*(e^{U_N}_{2n-1})$.

Let's take a different direction. We considered yesterday families of homomorphisms $U_n \longrightarrow U_m$ parametrized by a manifold M. This is the same as a homomorphism

$$* \qquad U_n \longrightarrow U_m^M = \text{gauge transformations of } \tilde{\mathbb{C}}^m \text{ over } M$$

Particularly interesting is the case where $M = S^2$ where I think we obtain [redacted] Bott's periodicity map.

The idea I have is to consider [redacted] the [redacted] behavior of left-invariant differential forms with respect to such a homomorphism $*$. I believe I know something about left-invariant differential forms on something like U_m^M. This is what I learned in studying Atiyah-Singer's paper.

[redacted]

February 6, 1990

Review the Bott maps

1) $[0,\pi] \times U_n \longrightarrow Gr_n(\mathbb{C}^2 \otimes \mathbb{C}^n)$

$$(\theta, g) \longmapsto (\cos\theta)\varepsilon + \sin\theta \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

$$= \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

2) $[0,\pi] \times Gr(V) \longrightarrow U(V)$

$$(\varphi, F) \longmapsto e^{-i\varphi F} = \cos\varphi - i\sin\varphi F$$

When we compose those we get

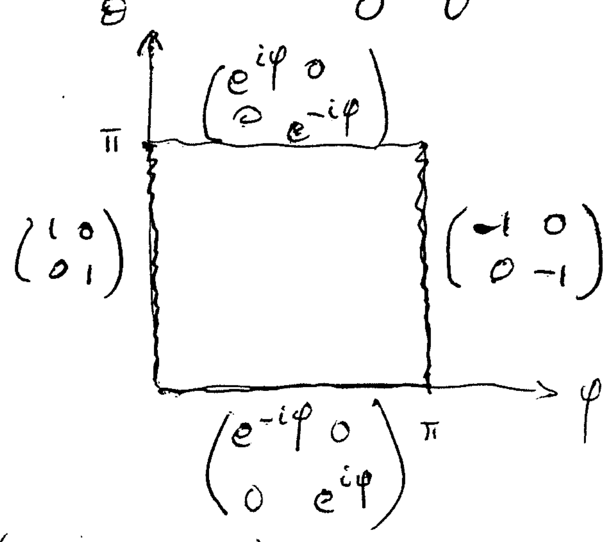
3) $[0,\pi] \times [0,\pi] \times U_n \longrightarrow U(\mathbb{C}^2 \otimes \mathbb{C}^n)$

$$(\varphi, \theta, g) \longmapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}^{-1} h(\varphi,\theta) \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

where $h: [0,\pi] \times [0,\pi] \longrightarrow SU(\mathbb{C}^2)$ is

$$h(\varphi,\theta) = \cos\varphi - i\sin\varphi \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Let's look at h on the boundary of $[0,\pi]^2$:



$\therefore\ h(\partial\{[0,\pi]^2\}) \subset T$ diagonal maximal torus of SU_2.

We have a map

$$(*) \quad [0,\pi]^2/\partial\{[0,\pi]^2\} \xrightarrow{\ h\ } SU_2/T = S^2$$

by evaluating h on the line $\mathbb{C}\begin{pmatrix}1\\0\end{pmatrix}$. Then

$$h(\varphi,\theta)(1) = \frac{\cos\varphi - i\sin\varphi\cos\theta}{-i\sin\varphi\sin\theta}$$

$$= \frac{\cos\theta}{\sin\theta} + i\,\frac{1}{\sin\theta}\,\frac{\cos\varphi}{\sin\varphi}$$

As $0<\theta<\pi$, $\cos\theta$ goes from $+\infty$ to $-\infty$ and for each $\theta\in(0,\pi)$, as $0<\varphi<\pi$, $\frac{1}{\sin\theta}\frac{\cos\varphi}{\sin\varphi}$ goes from $+\infty$ to $-\infty$, so it is clear that $(*)$ is a homeomorphism.

Let's look at the map 3) above.

$$\begin{array}{ccc}
[0,\pi]^2\times U_n & \longrightarrow & U(\mathbb{C}^2\otimes\mathbb{C}^n)\\
\downarrow & \nearrow & \\
[0,\pi]^2/\partial\{[0,\pi]^2\}\times U_n & &
\end{array}$$

since $(\varphi,\theta,g)\longmapsto \begin{pmatrix}g&0\\0&1\end{pmatrix}^{-1} h(\varphi,\theta)\begin{pmatrix}g&0\\0&1\end{pmatrix}$ and for $(\varphi,\theta)\in\partial\{[0,\pi]^2\}$, $h(\varphi,\theta)$ commutes with $\begin{pmatrix}g&0\\0&1\end{pmatrix}$. So we get a map $S^2\times U_n\longrightarrow U_{2n}$ which collapses $\{*\}\times U_n$, but not $S^2\times\{1\}$. To get a map $S^2\wedge U_n\longrightarrow U_{2n}$ one considers the commutato[r]

$$(\varphi,\theta,g)\longmapsto \begin{pmatrix}g&0\\0&1\end{pmatrix}^{-1}\underbrace{h(\varphi,\theta)\begin{pmatrix}g&0\\0&1\end{pmatrix}h(\varphi,\theta)^{-1}}$$

This is very nice because it can be interpreted as the family of embeddings $U(\mathbb{C}^n)\longrightarrow U(\mathbb{C}^2\otimes\mathbb{C}^n)$, parametriz by S^2 which at the line $L \subset \mathbb{C}^2$ take g in $U(\mathbb{C}^n)$ to $1 \otimes g$ on $L \otimes \mathbb{C}^n$ with [redacted] the identity on $L^\perp \otimes \mathbb{C}^n$. So therefore we see the Bott map fits into the form I thought it did when I discussed Lundell's theorem. (p 214)

February 10, 1980

Observation: Consider all homomorphism $U_n \xrightarrow{\varphi} U_N$ such that $\mathbb{C}^N$ with the induced action of U_n is isomorphic to the direct sum of the standard representation of U_n on $\mathbb{C}^n$ and the trivial representation on $\mathbb{C}^{N-n}$. Given φ we thus have an isomorphism $\mathbb{C}^n \oplus \mathbb{C}^{N-n} \xrightarrow{\sim} \mathbb{C}^N$ which is unique up to $S^1 = U_1$ acting as scalars on $\mathbb{C}^n$ and U_{N-n} acting on $\mathbb{C}^{N-n}$. Put another way, let V = standard repn. of U_n on $\mathbb{C}^n$. One has a canonical isomorphism associated to φ

$$V \otimes \underbrace{\mathrm{Hom}_{U_n}(V, \mathbb{C}^N)}_{1\text{-diml}} \oplus \underbrace{(\mathbb{C}^N)^{U_n}}_{N-n \text{ diml}} \xrightarrow{\sim} \mathbb{C}^N$$

Thus φ determines an element of

$$U_N / \Delta_n S^1 \times U_{N-n} = (U_N/U_{N-n})/\Delta_N S^1$$

where $\Delta_n S^1 \subset U_n$ is the center. Conversely one sees that any point in the orbit space of the Stieffel manifold U_N/U_{N-n} by the action of scalars determines a homomorphism $U_n \to U_N$. Conclude.

$$\mathrm{Hom}^{\text{special embedding type}}_{\text{Lie gps}}(U_n, U_N) = (U_N/U_{N-n})/\Delta_N S^1)$$

Thus for $N-n$ large we have a parameter space for "embedding" homomorphisms $U_n \to U_N$ which has the homotopy type BS^1.

Next let us check that a family of homomorphisms: $M \times G \longrightarrow G'$ parametrized by M induced a family of maps of classifying spaces: $M \times BG \longrightarrow BG'$. The best way to think is that one has an map

$$B(G'^M) \longrightarrow (BG')^M$$

which is a homotopy equivalence on the component of $(BG')^M$ corresponding to the trivial G'-bundle over M. Thus one has

$$BG \longrightarrow B(G'^M) \longrightarrow (BG')^M$$

alternatively one can take $M \times PG$ over $M \times BG$ and form a principal G'-bundle $(M \times PG \times G')_G$ where G acts on $\{m\} \times PG \times G'$ using the homomorphism $G \to G'$ at the point m.

So in the situation of

$$* \qquad (U_N/U_{N-n} \times \Delta S^1) \times U_n \longrightarrow U_N$$

it seems that we get a map

$$(U_N/U_{N-n} \times \Delta S^1) \times BU_n \longrightarrow BU_N$$

which must be compatible with the map

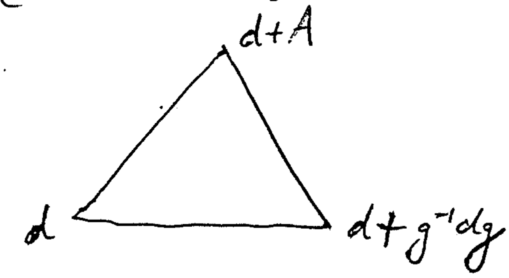
$$BS^1 \times BU_n \longrightarrow BU_n$$

given by tensoring a line bundle + a vector bundle.

Sometime it might be interesting to work out the effect of $*$ on cohomology. Can let $n, N \to \infty$.

Let's recall the idea that the Chern-Simons forms on a principal U_n-bundle should be viewed as special cases of Chern-Simons forms which can be associated to a vector bundle equipped with a flat partial connection [redacted] relative to a foliation and a trivialization. [redacted] Extending the flat partial connection to a connection which can be written $d+A$ relative to the trivialization, the Chern-Simons forms are obtained by using the linear path $d+tA$.

Next suppose we change the trivialization by a gauge transformation g. The new CS forms result from the linear path joining $g^{-1}\cdot d\cdot g = d+g^{-1}dg$ to $d+A$. Actually any path in the space of connections should give the same cohomology classes, so if we consider



we find that the CS ~~forms~~ classes under the gauge transformation get added by the primitive ~~forms~~ classes on U_n pulled back via the gauge transformation.

Another question is to relate the CS forms defined in $H^{odd}(P)$ by ~~the~~ connections with the forms defined via obstruction theory. The latter are obtained as follows. ~~[illegible]~~ One can reduce the structural group of P from U_N to U_r when $\dim B \leq 2r+2$, because one can construct a section of P/U_r over B and the fibre begins in degree $2r+1$. Having done this, one has [redacted] a U_N-equivariant map

$$P = P' \times^{U_r} U_N \longrightarrow U_r \backslash U_N$$

and so the odd generators of $H^*(U_r \backslash U_N)$ give odd cohomology classes for P.

So we ought to describe the odd generators of $H^*(U_r \backslash U_N) = H^*(U_N/U_r)$. The space U_N/U_r occurs as the fibre

$$U_N/U_r \longrightarrow BU_r \longrightarrow BU_N$$

and so over U_N/U_r there is a [redacted] universal pair consisting of a rank r vector bundle E and an [redacted] isomorphism $E \oplus \widetilde{\mathbb{C}}^{N-r} \cong \widetilde{\mathbb{C}}^N$. To construct odd forms on U_N/U_r we [redacted] look at the trivial connection d on $\widetilde{\mathbb{C}}^N$ and the connection which is the direct sum of the Grassmannian connection on E and the trivial connection on $\widetilde{\mathbb{C}}^{N-r}$. Then one uses the invariant polynomials ϕ giving rise to the Chern classes, i.e. [redacted] $tr(\Lambda^n \alpha) = \phi(\alpha)$, $\alpha \in gl_N$.

[redacted] This construction leaves much to be desired. The foliation method seems to produce CS classes for [redacted] all invariant polys. of degree > codim of the foliation. There's probably a Gelfand–Fuks type algebra [redacted] which gives the cohomology of the principal U_N-bundle over BU_N restricted to the $2n$-skeleton. Take Weil algebra and kill the appropriate [redacted] power of the ideal generated by the components of the curvature.

<u>Variation maps</u> The general idea is to consider the evaluation maps

$$ev_m : G^M \longrightarrow G$$

for each $m \in M$. These induce maps

$$\begin{array}{ccc} ev_m^* : \Omega^\bullet(G) & \longrightarrow & \Omega^\bullet(G^M) \\ \cup & & \cup \\ \Lambda^\bullet \mathfrak{g}^* & \longrightarrow & \Lambda^\bullet(\mathfrak{g}^M)^* \end{array}$$

more precisely $C^\bullet(\mathfrak{g}^M; \mathbb{C})$

So given a form ω on G one has a smooth family of forms $ev_m^*(\omega)$ on G^M, whence we have a map

$$\begin{array}{ccc} \Omega^\bullet(G) & \longrightarrow & \Omega^{\bullet,0}(G^M \times M) \\ \cup & & \cup \\ \Lambda^\bullet \mathfrak{g}^* & \longrightarrow & C^\bullet(\mathfrak{g}^M, \Omega^0(M)) \end{array}$$

(*) [redacted]

compatible with d on the left and $d' = d_{G^M}$ on the right.

(Perhaps it is useful to note that we

have a DGA map induced by $ev: G^M \times M \to G$

$$ev^*: \Omega^\bullet(G) \longrightarrow \Omega^\bullet(G^M \times M)$$

and that the map

$$\Omega^\bullet(G) \longrightarrow \Omega^{\bullet,0}(G^M \times M)$$

results by passage to the quotient.)

Note that $\mathfrak{g}^M = \mathfrak{gl}_n(\Omega^0(M))$, $\mathfrak{g} = \mathfrak{gl}_n$
hence the [redacted] map $\circledast$ on Lie cochains is [redacted] a DGA map

$$\circledast \qquad \Lambda^\bullet \mathfrak{gl}_n^* \longrightarrow C^\bullet(\mathfrak{gl}_n(A), A)$$

where $A = C^\infty(M)$. I claim such a map exists for any algebra.

Proof. Let $\theta_{ij} \in C^1(\mathfrak{gl}_n(A), A) = \mathrm{Hom}(\mathfrak{gl}_n(A), A)$ be the map such that $\iota_X \theta_{ij} = X_{ij}$ if $X = (X_{ij}) \in \mathfrak{gl}_n$.
Then

$$\iota_Y \iota_X \Big(\delta \theta_{ik} + \sum_j \theta_{ij}\theta_{jk}\Big) = -\underbrace{\theta_{ik}([X,Y])}_{[X,Y]_{ik}} + \sum_j X_{ij}Y_{jk} - Y_{ij}X_{jk} = 0$$

Thus $\delta\theta_{ik} + \sum_j \theta_{ij}\theta_{jk} = 0$ which means that the obvious homomorphism $\circledast$ sending $\theta_{ij} \in \mathfrak{gl}_n^*$ to the $\theta_{ij} \in C^1(\mathfrak{gl}_n A, A)$ is compatible with differentials.

February 12, 1980

Notes on Feigin-Tsigan on Lie algebra cohomology and Riemann-Roch.

They define $W^*(\mathfrak{g})$, the Weil algebra, as the cochains on the DGA $\mathfrak{g}[\varepsilon] = \mathfrak{g} \oplus \varepsilon\mathfrak{g}$ where degree $\varepsilon = -1$ and $d(\varepsilon) = 1$. This exhibits the Weil algebra as a bigraded differential algebra and suggests that there is a larger context into which the Weil algebra fits. In effect so far our understanding of $W(\mathfrak{g})$ comes from connections on principal bundles and when we form equivariant forms we just tensor $W(\mathfrak{g})$ and $\Omega(M)$. But apparently there is also the possibility of a twisted tensor product. ? Also the bigrading is ad hoc.

Dual version $W_*(\mathfrak{g}) = \Lambda\mathfrak{g} \otimes S\mathfrak{g}$ of Lie chains. This is better for $\mathfrak{g}$ infinite-dimensional such as $\mathfrak{gl}(A)$. In the case of $\mathfrak{gl}(A)$ one obtains the Lie chains on $\mathfrak{gl}$ applied to the DGA $A \oplus \varepsilon A$, which is a special case of $R \oplus \varepsilon I$ I studied in the case of extensions. In particular we know that the δ homology of $W_*(\mathfrak{g})$ is the cyclic homology of the semi-direct product $A \oplus \varepsilon A$ (super conventions with degree $\varepsilon = 1$ for the lower grading). By Goodwillie this cyclic homology is

$$HC_{\cdot}(A) \oplus \varepsilon H_{\cdot}(A,A) \oplus \underbrace{H_{\cdot}\left(\left(\varepsilon A \overset{\mathbb{L}}{\otimes}_A\right)^2_\lambda\right)}_{\varepsilon^2 H_{\cdot}(A,A)} \oplus \underbrace{\cdots\cdots\cdots}_{\varepsilon^3 H_{\cdot}(A,A)} \oplus \cdots$$

On the cohomology side this says that the δ cohomology

$$H^\bullet(gl(A),\ S(gl(A))^*)$$

is (up to duality problems) the free commutative algebra generated by $HC(A)$ and by various copies of $H_\cdot(A,A)$. In principal this means one knows $(S(gl(A))^*)^{gl(A)}$ which is the Weil candidate for the cohomology of $B\mathcal{G}$. (?)

Let's discuss what FT do. Let $\mathfrak{g} = gl(A)$ and let $\mathfrak{h}$ be a reductive subalgebra in $\mathfrak{g}$, better Lie subalgebra reductive in $\mathfrak{g}$ so that $\mathfrak{g}$ is a semisimple $\mathfrak{h}$ module. Then consider $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{g} + \varepsilon\mathfrak{g}$ and form relative cochains. This is the same as $\mathfrak{h}$-basic elements of $W(\mathfrak{g})$:

$$W(\mathfrak{g},\mathfrak{h}) = \left(\Lambda(\mathfrak{g}/\mathfrak{h})^* \otimes S(\mathfrak{g})^*\right)^{\mathfrak{h}}$$

$$= \Big(\underbrace{\Lambda(\mathfrak{g}/\mathfrak{h}) \otimes S(\mathfrak{g}/\mathfrak{h})}_{\text{contractible}} \otimes S(\mathfrak{h})\Big)^{*,\mathfrak{h}}$$

Thus $W(\mathfrak{g},\mathfrak{h})$ gives $H^*(BH)$ and we have a spectral sequence starting with the δ cohomology

$$H^\bullet(\mathfrak{g},\mathfrak{h}; S(\mathfrak{g})^*) \Longrightarrow H^\bullet(BH)$$

FT's goal is to use $HH(A) = H(A,A)$ to produce a "characters" for ~~the~~ the representation

$$h \longrightarrow \mathfrak{g} = \mathfrak{gl}(A)$$

and the character lies in $H^{\cdot}(BH)$.

They assume that the Hochschild cohomology is concentrated in degree $2n$. By means of a suitable decreasing filtration on $W_*(\mathfrak{g}, h)$, they show that if j is minimal with $H_j(\mathfrak{g}, h; (S^{>0}\mathfrak{g})^*)$ $\neq 0$, then there are maps

$$H_{j+2q}(W_*(\mathfrak{g}, h)) \longrightarrow H_j(\mathfrak{g}, h; S^q \mathfrak{g})$$

Dually we get maps

$$H^j(\mathfrak{g}, h; (S^q\mathfrak{g})^*) \longrightarrow H^{j+2q}(BH)$$

We have a spectral sequence

$$H^p(\mathfrak{g}, h; (S^k\mathfrak{g})^*) \otimes H^q(h^*) \Longrightarrow H^{p+q}(\mathfrak{g}, (S^k\mathfrak{g})^*)$$

so maybe this means that the classes in ↑ associated to Hochschild cohomology give rise to relative classes in $H^{\cdot}(\mathfrak{g}, h, (S\mathfrak{g})^*)$, and hence to classes in $H^{\cdot}(BH)$.

One further idea is that a Lie homomorphism $h \longrightarrow \mathfrak{gl}_n A$ extends to an algebra hom.

$$U(h) \longrightarrow M_n A$$

hence induces

$$H(U(h), U(h)) \longrightarrow H(M_n A, M_n A) = H(A, A)$$

$$\|$$

$$H_{\bullet}(h, U(h)) = H_{\bullet}(h, S(h))$$

Dually if $H^{\cdot}(A,A)$ is concentrated in degree $2n$ we get a map

$$H^{2n}(A,A) \longrightarrow H^{2n}(\mathfrak{h}, S(\mathfrak{h})^*) = H^{2n}(\mathfrak{h}) \otimes (S(\mathfrak{h})^*)^{\mathfrak{h}}$$

Apparently this is related to the above, although I don't see how. When $A = \mathbb{C}$ and $n=0$ we obtain in $S(\mathfrak{h}^*)^{\mathfrak{h}} = H^{\cdot}(BH)$ the usual character of the induced bundle over BH.

February 15, 1980

Program: To understand the character of Feigin + Tsygan and related ideas.

Let $\mathfrak{g} = gl(A)$ where A is a unital algebra such that $HH^{\cdot}(A) \doteq \mathbb{C}[2n]$. If $h \subset \mathfrak{g}$ is a "reductive subalgebra" (means red. h and $\mathfrak{g}$ semi-simple as h-module), they define a "character" in $S(h^*)^h$. (It suffices to have a representation $h \longrightarrow \mathfrak{g}$ whose image is a reductive subalgebra).

Let's take A unital and $h = gl(\mathbb{C})$ and try to understand what's happening.

Consider $A + A\varepsilon$; this is the DGA with A in degree 0, ε in degree 1 for the lower indexing and with $d(\varepsilon) = 1$. We have

$$gl(A + A\varepsilon) = \mathfrak{g}[\varepsilon] = \mathfrak{g} + \mathfrak{g}\varepsilon$$

and $W^*(\mathfrak{g})$ is the bigraded diff algebra of Lie cochains on $\mathfrak{g}[\varepsilon]$, whereas $W_*(\mathfrak{g})$ is the bigraded differential coalgebra of chains. Feigen + Tsygan use the relative complexes $W^*(\mathfrak{g}, h)$, etc. which have the form

$$W_*^{\otimes}(\mathfrak{g}, h) = \left(\Lambda\, \mathfrak{g}/h \otimes S(\mathfrak{g})\right)_h \qquad W^* = (W_*)^*$$

In the case $h = gl(\mathbb{C}) \subset \mathfrak{g} = gl(A)$, then invariant theory says

$$W_*(\mathfrak{g}, h) = S_{super}\left(\Sigma \overline{CC}(A[\varepsilon])\right)$$

which computes the homology from the reduced cyclic homology of $A[\varepsilon]$. Better it tells us that $W_*(g,h)$ is the free comm. coalgebra generated by the reduced cyclic complex of $A[\varepsilon]$, + this is compatible with the double complex structure.

The δ cohomology of $\sum \bar{C}C(A[\varepsilon])$ is

$$\begin{array}{|llll}
\vdots & \vdots & \vdots & \vdots \\
\overline{HC}_2 & HH_2 & HH_1 & HH_0 \\
\overline{HC}_1 & HH_1 & HH_0 & \\
\overline{HC}_0 & HH_0 & & \\
\hline
\end{array}$$

and the d cohomology is

$$\begin{array}{|lllll}
 & & & & \cdots \\
 & & & \mathbb{C} & \\
 & & \mathbb{C} & & \\
 & \mathbb{C} & & & \\
\mathbb{C} & & & & \\
\hline
\end{array}$$

This is known because the δ cohomology depends only on the alg. structure of $A + A\varepsilon$ which is the semi direct product of A and the bimodule $A\varepsilon$

Consider what happens if $HH_\bullet(A) = \mathbb{C}[2n]$. We have from the long exact sequence

$$\longrightarrow HC_i(\mathbb{C}) \longrightarrow HC_i(A) \longrightarrow \overline{HC}_i(A) \longrightarrow HC_{i-1}(\mathbb{C})$$

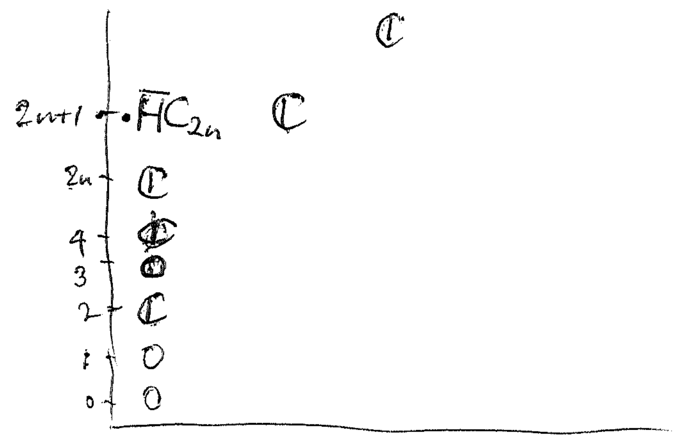
and the fact that $HC_i(A) = \begin{cases} \mathbb{C} & i = 2n' \quad n' \geq n \\ 0 & \text{otherwise} \end{cases}$

that
$$\bar{HC}_{2i}(A) = 0 \qquad i<n$$
$$\bar{HC}_{2i-1}(A) = \mathbb{C} \qquad 1 \le i < n$$

and

$$0 \to \bar{HC}_{2n+1}(A) \to \underset{\mathbb{C}}{\underset{\parallel}{HC_{2n}(\mathbb{C})}} \to \underset{\mathbb{C}}{\underset{\parallel}{HC_{2n}(A)}} \to \bar{HC}_{2n}(A) \to 0$$

In the spectral sequence starting with the δ cohomology we have



We know the abutment, so we conclude $\underset{\bar{HC}_{2n}(A)=0.}{\underset{\parallel}{\bar{HC}_{2n+1}(A)}}$

<u>Lemma:</u> If $HH_\bullet(A) = \mathbb{C}[2n]$, then $HC_j(k) \xrightarrow{\sim} HC_j(A)$ for $j \ge 2n$ and so

$$\bar{HC}_j(A) = \begin{cases} \mathbb{C} & j = 2i-1 \qquad i = 1, \dots, 2n \\ 0 & \text{otherwise} \end{cases}$$

Once we know $\bar{HC}_{2n}(A) = \bar{HC}_{2n+1}(A) = 0$, what happens in higher degrees follows via S which is an isomorphism.

February 16, 1990

Let h be a finite diml reductive Lie algebra and let $h \to gl_N(A)$ be a Lie alg homomorphism such that A^N is a "reductive repn." of h, that is, a sum finite dimensional irreducible representations of h. Then we can decompose

$$A^N = \bigoplus_\alpha W_\alpha \otimes \mathrm{Hom}_h(W_\alpha, A^N)$$

where the W_α are the inequivalent fd irred repns of h. Thus we are led to consider reductive repns of h on finite projective (right) A-modules.

The FT character for such a repn should be additive and natural, whence to [redacted] compute this character it should suffice to consider the case of $h = gl_k$ acting in the obvious way on $\mathbb{C}^k \otimes E$, where E is a finite projective A-module. In this case we have a homomorphism of algebras

$$M_k \longrightarrow \mathrm{End}_{A^{op}}(\mathbb{C}^k \otimes E) = M_k \otimes \mathrm{End}_{A^{op}}(E).$$

Thus [redacted] the FT character should be computable from what it does in the case of the homomorphism

$$gl_k \longrightarrow gl_{km}(A)$$

where $\mathbb{C} \longrightarrow$ [redacted] $M_m(A)$ is a nonunital algebra homom.

[redacted]

February 20, 1990

Lecture on Lie algebra cohomology.

G Lie group. It acts on itself by both left and right translations and these actions commute. One has

left invariant vector fields = infinitesimal right translations.

$Lie(G)$ is the space of left-invariant vector fields under bracket. Put $\mathfrak{g} = Lie(G) \otimes_{\mathbb{R}} \mathbb{C}$. One has

$$\Lambda \mathfrak{g}^* \xrightarrow{\sim} \Omega(G)^{G(\text{left})}$$

Thus $\Lambda \mathfrak{g}^*$ has a differential δ corresponding to d on $\Omega(G)$. If ω is a left-invariant 1-form one has

$$\iota_Y \iota_X d\omega = X\omega(Y) - Y\omega(X) - \omega([X,Y])$$

$$= -\omega([X,Y])$$

Thus if X_a is a basis for $\mathfrak{g}$ and θ^a is the dual basis for $\mathfrak{g}^*$, and $[X_a, X_b] = f^c_{ab} X_c$ one has $d\theta^a + \frac{1}{2} f^a_{bc} \theta^b \theta^c = 0$.

Maurer-Cartan form $\quad \theta = \theta^a X_a \in \Omega^1(G)^{G(\text{left})} \otimes \mathfrak{g}$

Properties: left-invariant

$$\iota_X \theta = X$$

$$Ad(g) R_g^* \theta = \theta \qquad L_X \theta + [X, \theta] = 0$$

$$d\theta + \tfrac{1}{2}[\theta, \theta] = 0$$

It is the unique connection form in G considered as a principal bundle over $\{pt\}$.

Put $C^\bullet(\mathfrak{g}) = \Lambda \mathfrak{g}^*$ with δ

$$H^\bullet(\mathfrak{g}) = H^\bullet(C(\mathfrak{g}))$$

One has canon. map $H^\bullet(\mathfrak{g}) \longrightarrow H^\bullet_{DR}(G)$.

Next want Lie alg. cohom. with coefficients in a $\mathfrak{g}$-module.

Consider the trivial bundle $\tilde{V} = G \times V / G$ with G acting $g(g_1, v) = (gg_1, v)$. This is an equivariant v. bundle over G w.r.t. the left-translation action, and any such equivariant bundle E is canonically isomorphic to $\tilde{V}$ with $V =$ ~~the~~ fibre of E over $1 \in G$. One has

$$\Omega^\bullet(G, \tilde{V}) = \Omega(G) \otimes V$$

$$\Omega^\bullet(G, \tilde{V})^G = \Omega(G)^{G(\text{left})} \otimes V = \Lambda \mathfrak{g}^* \otimes V$$

An invariant connection on $\tilde{V}$ is of the form $d + A$ with $A \in \Omega^1(G)^{G(\text{left})} \otimes \mathrm{End}(V)$. A is the same as a linear map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ via $\rho(X) = \iota_X A$, $A = \rho\theta = \theta^a \rho(X_a)$. One has

$$(d+A)^2 = dA + A^2 = \rho\, d\theta + \rho\theta\rho\theta$$

$$= \rho\left(-\tfrac{1}{2}[\theta,\theta]\right) + \tfrac{1}{2}[\rho\theta, \rho\theta]$$

The connection is flat $\Longleftrightarrow$ ρ is a Lie homom.

Thus if V is a $\mathfrak{g}$-module with $\mathfrak{g}$-action given by ρ we have a diffl δ on

$$\Omega^\bullet(G, \tilde{V})^G = \Lambda \mathfrak{g}^* \otimes V$$

induced by $d + \rho\theta$ on $\tilde{V}$. Put

$$C^\bullet(\mathfrak{g}; V) = \Lambda \mathfrak{g}^* \otimes V \quad \text{with } \delta$$

$$H^\bullet(\mathfrak{g}; V) = H^\bullet(C(\mathfrak{g}; V)).$$

[redacted] Formulas

$$C^\cdot(\mathfrak{g}) = \Lambda[\theta^a] \quad \text{with} \quad \delta\theta^a + \tfrac{1}{2} f^a_{bc}\theta^b\theta^c = 0$$

$$C^\cdot(\mathfrak{g}; V) = \Lambda[\theta^a] \otimes V$$ [redacted]

[redacted] is the DG module over $C^\cdot(\mathfrak{g})$ with

$$\delta v = \rho(\theta)v = \theta^a \rho(X_a) v$$

<u>Chevalley–Eilenberg Thm.</u> Assume G compact connected, and let V be a f.d. representation of G. Then

$$H^\cdot(\mathfrak{g}; V) \cong H^\cdot_{DR}(G) \otimes V^G$$

In particular $H^\cdot(\mathfrak{g}) \xrightarrow{\sim} H^\cdot_{DR}(G)$.

Proof. We are given a Lie grp hom. $\phi: G \to \mathrm{Aut}(V)$ and the associated inf. repn $\rho: \mathfrak{g} \to \mathrm{End}\, V$ is $\rho\theta = \phi^{-1}d\phi$. Consider ϕ as gauge transformation, i.e. automorphism of $\tilde{V}$:

$$\begin{array}{ccc} G\times V & \xrightarrow{\sim} & G\times V \\ (g,v) & \longleftrightarrow & (g, gv) \end{array}$$

As $\phi^{-1}\cdot d\cdot\phi = d + \phi^{-1}(d\phi) = d + \rho\theta$, one has an isomorphism

$$\phi : \Omega(G)\otimes V \xrightarrow{\sim} \Omega(G)\otimes V$$

Connection: $\quad d+\rho\theta \longleftrightarrow d$

G-action: left transl. on G trivial on V $\longleftrightarrow$ left transl. on G ϕ action on V

Thus we have

$$H^\cdot(\mathfrak{g}, V) = H^\cdot\{(\Omega(G)\otimes V, d)^{G(\text{left},\phi)}\}$$

We have a canonical map

$$H^\cdot\left\{ (\Omega(G)\otimes V, d)^{G(\text{left},\phi)} \right\} \longrightarrow H^\cdot(\Omega(G)\otimes V)^{G(\text{left},\phi)}$$

which we show is an isomorphism

injective: Given ω with $d\omega = 0$ and $g^*\omega = \omega$, suppose $\omega = d\eta$. To prove ω is d of an invariant form. But

$$d\left(\int_G g^*\eta\right) = \int_G d(g^*\eta) = \int_G g^*\omega = \int_G \omega = \omega$$

since d continuous, so this is clear.

surjective: Given $\omega \in \Omega(G)\otimes V$ with $d\omega = 0$ suppose the class of ω is G-invariant: $\omega - g^*\omega \in \operatorname{Im} d$ for all g. To prove ω cohomologous to an invariant form. But

$$\omega - \int_G g^*\omega = \int_G (\omega - g^*\omega) \in \overline{\operatorname{Im} d}$$

and one knows that $\operatorname{Im} d$ is closed. (Hodge theory, Beggs' characterization of elements of $\operatorname{Im} d$ as forms integrating to zero over all closed cycles.)

So far we have used G compact. Now as it is connected, we have

$$H^\cdot(\Omega(G)\otimes V)^{G(\text{left},\phi)} = \left(H^\cdot(G)\otimes V\right)^{G(\text{left},\phi)}$$

$$= H^\cdot(G)\otimes V^G$$

since G acts trivially on $H^\cdot(G)$ in this case.

February 21, 1990

Let $G \to P \xrightarrow{\pi} B$ be a principal bundle. Let's recall how the Leray spectral sequence for this fibring arises. ~~[illegible]~~ One has an exact sequence

$$0 \longrightarrow S \longrightarrow T \longrightarrow Q \longrightarrow 0$$

where $T = T_P$, $S = T_\pi$, $Q = \pi^* T_B$. Dually one gets

$$0 \longrightarrow Q^* \longrightarrow T^* \longrightarrow S^* \longrightarrow 0$$

The ideal in $\Gamma(P, \wedge T^*)$ generated by $\Gamma(P, Q^*)$ is stable under d (this is integrability for the tangent subbundle to a foliation). Call this ideal J; one has the J-adic filtration

$$(*) \quad \Omega(P) \supset J \supset J^2 \supset \cdots$$

where $J^p / J^{p+1} = \Gamma(P, \wedge^p Q^* \otimes \wedge S^*)$. Recall that $\wedge^p Q^*$ is flat along the leaves; in this case $\wedge^p Q^* = \pi^*(\wedge^p T_B)$. Moreover d on $\Omega(P)$ induces on J^p/J^{p+1} the "Dolbeault" differential which amounts to the family of DR complexes on the fibres with coefficients in the flat bundle $\wedge^p Q^*$. Thus one has

$$E_1^{pq} = H^{p+q}(J^p/J^{p+1}) = \Omega^p(B) \otimes H^q(G)$$

and it should be possible to identify $E_1^{pq} \xrightarrow{d_1} E_1^{p+1,q}$ with the effect of d_B.

Thus we get the Leray spectral sequence from the decreasing filtration $(*)$. In the principal bundle situation we are studying $S = \tilde{\mathfrak{g}}$, so

$$J^p / J^{p+1} = \Gamma(P, \pi^* \wedge^p T_B) \otimes \wedge^{\cdot} \mathfrak{g}^*$$

$$\simeq \Omega^p(P)_{hor} \otimes \Lambda^{\cdot} \mathfrak{g}^*$$

and d can be identified with the Lie algebra cohomology differential δ associated to the $\mathfrak{g}$ action on $\Omega(P)_{hor}$ by the operators L_X. So

$$E_1^{pq} = H^q(\mathfrak{g}, \Omega^p(P)_{hor}) = \Omega^p(P)_{bas} \otimes H^q(G).$$

This holds even without assuming G compact connected. In effect $\Omega(P)_{hor}$ is locally $\Omega(B) \otimes \Omega^0(G)$, and $\Omega^0(G) \otimes \Lambda \mathfrak{g}^*$ with δ is $\Omega^{\cdot}(G)$.

Now we want to show that the spectral sequence arising from the bigraded diffl alg

$$\Omega(P) \otimes C(\mathfrak{g}[\varepsilon]) = \Omega(P) \otimes \Lambda \mathfrak{g}_X^* \otimes S \mathfrak{g}_Y^*$$

starting with the δ cohomology coincides with the Leray spectral sequence. In the former spectral sequence one uses the decreasing filtration with F^p = all columns of degree $\geq p$.

February 22, 1990

We wish to define a map of DG algebras

$$\circledast \qquad \Omega(P) \longrightarrow \Omega(P) \otimes \Lambda \mathfrak{g}_X^* \otimes S \mathfrak{g}_\varphi^*$$

where the latter is given the total differential $d+\delta$.

Consider first the case where $\Omega(P)$ is replaced by $W(\mathfrak{g})$. Then the homomorphism we want is equivalent to a connection in $W(\mathfrak{g}) \otimes \Lambda \mathfrak{g}_X^* \otimes S \mathfrak{g}_\varphi^*$. Recall the Russian formula

$$(d+\delta)(A+X) + (A+X)^2 = dA + A^2 = F.$$

Thus [redacted] the desired homomorphism is given by

$$A \longmapsto A + X$$

$$F \longmapsto F$$

[redacted] This suggests the formula for $\circledast$:

$$\omega \longmapsto \omega + X^a \iota_a \omega + \frac{1}{2} X^a X^b \iota_b \iota_a \omega + \cdots$$

$$= e^{X^a \iota_a} \omega$$

This should be a homomorphism because $X^a \iota_a$ is a derivation. We can check this as follows. Recall that we have an isom.

$$\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \xrightarrow{\sim} \Omega(P)$$

Now $e^{X^a \iota_a}$ is the identity on $\Omega(P)_{hor}$ and we have $e^{X^a \iota_a} A = A + X$. Thus $e^{X^a \iota_a}$ combines the inclusion of $\Omega(P)_{hor}$ with the [redacted] homomorphism $\Lambda \mathfrak{g}_A^* \longrightarrow \Lambda \mathfrak{g}_A^* \otimes \Lambda \mathfrak{g}_X^*$ sending A to $A+X$.

Next we want to check that $e^{\chi^a\iota_a}$ is compatible with differentials. This is clear on elements in $\Lambda\mathfrak{g}^*_A$, since they come from $W(\mathfrak{g})$, and we've checked the case of $W(\mathfrak{g})$ using the Russian formula.

So we consider $\omega \in \Omega(P)_{hor}$. We have

$$(d+\delta)\, e^{\chi^a\iota_a}\omega = (d+\delta)\omega = d\omega + \chi^a L_a\omega - \cancel{\varphi^a\iota_a\omega}$$

$$e^{\chi^a\iota_a}\, d\omega = d\omega + \chi^a\iota_a d\omega + \tfrac{1}{2}\chi^a\chi^b\iota_b\iota_a d\omega + \cdots$$

$$= d\omega + \chi^a L_a\omega$$

because $\iota_a d\omega = L_a\omega - \cancel{d\iota_a\omega}$ and $\iota_b\iota_a d\omega =$
$\iota_b L_a\omega = \pm(L_a\iota_b - \iota_{[X_a,X_b]})\omega = 0$.

Thus we have a DGA morphisms

$$e^{\chi^a\iota_a} : \Omega(P) \longrightarrow \Omega(P)\otimes\Lambda\mathfrak{g}^*_\chi\otimes S\mathfrak{g}^*_\varphi$$

Next we want to check the filtrations. Recall that the ideal $J \subset \Omega(P)$ is generated by $\Omega(P)^{>0}_{hor}$. The ideal to consider in $\Omega(P)\otimes\Lambda\mathfrak{g}^*_\chi\otimes S\mathfrak{g}^*_\varphi$ is generated by $\Omega^{>0}(P)$ and by φ; its the ideal of elements of d-degree >0. Thus $e^{\chi^a\iota_a}$ is compatible with filtrations.

Let's compute the map on the associated graded algebras. We have

$$\mathrm{gr}^J\,\Omega(P) = \Omega(P)_{hor}\otimes\Lambda\mathfrak{g}^*_A$$

with d given by $dA + A^2 = 0$

$$d\omega = A^a L_a\omega$$

In effect if ω is horizontal, then $d\omega$ needn't be; its horizontal part is

$$d\omega - A^a\iota_a d\omega + \tfrac{1}{2}A^aA^b\iota_b\iota_a d\omega - \cdots$$

$= d\omega - A^a L_a \omega$

Thus if $\omega \in \Omega^p(P)_{hor}$ one has

$$d\omega - A^a L_a \omega \in \Omega^{p+1}(P)_{hor}$$

showing that $d\omega = A^a L_a \omega$ in $gr^J \Omega(P)$.

The map on gr's is

$$\begin{array}{ccc} \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* & \longrightarrow & \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \otimes \Lambda \mathfrak{g}_\chi^* \otimes \Lambda \mathfrak{g}_\varphi^* \\ \cup\!\!\!\!\!\!\;\; & & \cup\!\!\!\!\!\!\;\; \\ \omega & \longmapsto & \omega \\ A & \longmapsto & \chi \end{array}$$

χ = leading part of $A + X$

Check compatibility of d on the left with δ on the right:

$$d\omega = A^a L_a \omega \longmapsto \chi^a L_a \omega = \delta \omega$$

$$dA = -A^2 \longmapsto -\chi^2 = \delta \chi$$

Conclude that the induced map on gr's is the inclusion

$$\left(\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^*\right) \longrightarrow \left(\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_\chi^*\right) \otimes \left(\Lambda \mathfrak{g}_A^* \otimes \Lambda \mathfrak{g}_\varphi^*\right)$$

$$\cong \left(\text{———}\right) \otimes \left(\Lambda \mathfrak{g}_A^* \otimes S \mathfrak{g}_{\delta A}^*\right)$$

and so induces an isomorphism on δ-cohomology.

Question. If P/B is a principal G-bundle, is it possible to use the Weil algebra to construct cohomology classes in B? I want to do this when G is not compact, for example when $G = C^\infty(M, U_N)$

The idea here is that the standard Chern-Weil construction yields cohomology classes from $S(\mathfrak{g}^*)^G$, and there is Bott's spectral sequence

$$H^{\bullet}_{diff}(G, Sg^*) \Longrightarrow H^{\bullet}(BG)$$

It might be the case that one has a canonical map

$$H^{\bullet}(g, Sg^*) \longrightarrow H^{\bullet}_{diff}(G, Sg^*)$$

although perhaps this is unreasonable (the obvious map maybe goes the other way as the formal group at $e \subset G$.)

First map: suppose P/B has a connection. Then we have a map $W(g) \longrightarrow \Omega(P)$ whence a map of bigraded diffl algebras

$$W(g) \otimes \Lambda g^*_{\chi} \otimes S g^*_{\varphi} \longrightarrow \Omega(P) \otimes \Lambda g^*_{\chi} \otimes S g^*_{\varphi}$$

This gives a map on δ cohomology which is

$$E_1^{pq} = \begin{cases} H^q(g, S^{\frac{p}{2}} g^*) & p \text{ even} \geq 0 \\ 0 & p \text{ odd} \end{cases} \longrightarrow E_1^{pq} = \Omega^p(B) \otimes H^q(G)$$

Then we should have

$$E_2^{pq} = \begin{cases} H^q(g, S^{p/2} g^*) \\ 0 \end{cases} \longrightarrow E_2^{pq} = H^p(B, H^q(G))$$

Second map: Consider $W(g) \longrightarrow \Omega(P)$ as a map of filtered algebras, where we filter $W(g)$ by powers of the ideal J generated by the F^a. The map on gr^J is

$$\Lambda g^*_A \otimes S g^*_F \longrightarrow \Lambda g^*_A \otimes \Omega(P)_{hor}$$

$dA + A^2 = 0$ $\qquad\qquad$ $dA + A^2 = 0$

$dF + [A,F] = 0$ $\qquad\qquad$ $d\omega = A^a L_a \omega$

These are consistent because

$$[A,F] = A^a[X_a,F] = -A^a L_a F$$

The map on E_1 is then what? The degrees are funny, and it appears that one wants F^a to have degree 2. Basically one seems to get the same map on E_1-terms.

[redacted]

The real issue appears to be whether we can construct interesting cohomology classes in B. I would like to be able to transgress Lie cohomology classes to BG. [redacted] The feeling is that there ought to be giant transgression cochains.

February 24, 1990

Let's consider a vector bundle E over $Y \times M$ such that $E_y \simeq \mathbb{C}^N \times M/M$ for all $y \in Y$. Let $\mathcal{G} = \mathrm{Aut}(\mathbb{C}^N \times M/M)$ $= U_N(C^\infty(M))$, let P/Y be the principal $\mathcal{G}$-bundle with $P_y = \mathrm{Isom}_M(\mathbb{C}^N \times M, E_y)$, $\pi : P \to Y$ the canonical map. Then $\pi^* E$ is canonically trivial.

Let's consider a connection D in E. When pulled up to $P \times M$ it has the form

$$\delta + d + \eta + A$$

where $\eta \in \Omega^{1,0}(P \times M) \otimes M_N = \Omega^1(P, \underbrace{\Omega^0(M) \otimes M_N}_{\tilde{\mathfrak{g}}})$

is a connection form in P/Y and where

$$A \in \Omega^{0,1}(P \times M) \otimes M_N = \Omega^0(P, \Omega^1(M) \otimes M_N)$$

can be identified with a $\mathcal{G}$-equivariant map from P to the space of connections on $\mathbb{C}^N \times M/M$. This equivariance condition should be

$$g^* A = g^{-1} \cdot (d + A) \cdot g - d$$

$$g^* A = g^{-1} dg + g^{-1} A g$$

(Why: $\mathcal{G}$ acts ~~to~~ the right on P and to the left on $\mathcal{A} = \Omega^1(M) \otimes M_N$; invariance of $A \in \Omega^0(P, \Omega^1(M) \otimes M_N)$ means $g * (g^* A) = A$ or $g^* A = g^{-1} * A$.) Thus

$$L_X A = dX + [A, X] \qquad X \in \tilde{\mathfrak{g}}$$

or $\quad \iota_X \delta A = d \iota_X \eta + [A, \iota_X \eta] = -\iota_X (d\eta + [A, \eta])$

i.e. $i_X\{\delta A + d\eta + [A,\eta]\} = 0$

We therefore arrive at the following

$$\delta + d + \eta + A \qquad \eta \in \Omega^{1,0}(\mathcal{P}\times M)\otimes M_N$$
$$A \in \Omega^{0,1}(\mathcal{P}\times M)\otimes M_N$$

$$\iota_X \eta = X \qquad X \in \Omega^0(M)\otimes M_N = \tilde{\mathfrak{g}}$$

$$L_X\eta + [X,\eta] = 0 \iff \iota_X(\delta\eta + \eta^2) = 0$$

$$L_X A = dX + [A,X] \iff \iota_X(\delta A + d\eta + [A,\eta]) = 0$$

The curvature is

$$(\delta + d + \eta + A)^2 = (\delta\eta + \eta^2) + (\delta A + d\eta + [\eta, A]) + (dA + A^2)$$

Thus we have [redacted] an invariant connection [redacted] on an equivariant bundle over $\mathcal{P}\times M$ which is not only flat along the $\mathcal{G}$-orbits (this means $\iota_X\iota_X(\delta\eta + \eta^2) = 0$) but satisfies something stronger.

February 25, 1990

Consider $M = S^1$, $\mathcal{G} = U_N^{S^1}$, $B\mathcal{G} = (BU_N)^{S^1}$, $\tilde{g} = gl_N(C^\infty(S^1))$. In this case you ought to be able to describe easily the odd generators in $H^\cdot(B\mathcal{G})$. The problem is that these do not occur in $H^0(\tilde{g}, S\tilde{g}^*)$.

Here's perhaps a derivation of Bott's spectral sequence

$$E_2 = H^\cdot_{dif}(G, Sg^*) \Longrightarrow H^\cdot(BG).$$

It proceeds as for the van Est spectral sequence. Recall this. Form double complex

$$C^p_{diff}(G, \Omega^q(G))$$

Because $\Omega^q(G) = \Omega^0(G) \otimes \Lambda g^*$ is an induced module, it is acyclic for diff cohomology, so

$$H^p_h\{C^\cdot(G, \Omega^\cdot(G))\} = \begin{cases} 0 & p > 0 \\ \Omega(G)^G = \Lambda^\cdot g^* & p = 0 \end{cases}$$

and so we have a quis

$$\Lambda g^* = \Omega^\cdot(G)^G \subset C^\cdot_{diff}(G, \Omega^\cdot(G))$$

The other spectral sequence has $E_2^{pq} = H^p_{diff}(G, H^q(G))$

~~Now let $P \to B$ be a principal G bundle with P highly connected, and consider the double complex~~

$$\cancel{C^\cdot_{diff}(G, \Omega^\cdot(P)_{hor}\, [redacted])} \qquad \cancel{\Omega^\cdot(P)_{hor} \text{ not a complex}}$$

~~Again $\Omega^\cdot(P)_{hor}$ is an induced G-module, so one has a quis~~

Let $P \longrightarrow B$ be a principal G-bundle. I want to use the same sort of argument for

$$C^{\bullet}_{diff}(G, \Omega^{\bullet}(P)_{hor})$$

The problem is that $\Omega^{\bullet}(P)_{hor}$ is not a complex; it is not closed under d.

Principle: $\Omega^{\bullet}(B)$ is the fixpt subalgebra for the action of $G, \mathfrak{g}\varepsilon$ on $\Omega^{\bullet}(P)$. It should be possible to form differentiable cochains on $(G, \mathfrak{g}\varepsilon)$ with coefficients $\Omega^{\bullet}(P)$. Moreover the total complex of these cochains should be quis to $\Omega^{\bullet}(B)$.

But actually it might not be important to discuss cochains on $(G, \mathfrak{g}\varepsilon)$. Rather one can look just as takeing fixpoints. One first wants to take fixpts under $\mathfrak{g}\varepsilon$ which is normal in $(G, \mathfrak{g}\varepsilon)$. This gives $\Omega^{\bullet}(P)_{hor}$. Then one takes G-invariants obtaining $\Omega^{\bullet}(P)_{bas} = \Omega^{\bullet}(B)$. The problem is that $\mathfrak{g}\varepsilon$ is not closed under d, and $\Omega^{\bullet}(P)_{hor}$ is not closed under d.

Nevertheless something like this ought to work - perhaps some version of homological perturbation theory. (Milgram said given a group extension one can splice resolutions and this leads to perturbation theory.)

Let's examine $\Omega(P)_{hor}$. Recall that for

$\omega \in \Omega(P)_{hor}$ one has that
$d\omega \in d\Omega(P)_{hor} + \Omega(P)_{hor} \otimes \mathfrak{g}^*$.

More presisely

$$d\omega - A^a \iota_a d\omega = d\omega - A^a L_a \omega \in \Omega(P)_{hor}$$

Thus on $\Omega(P)_{hor}$ we have a degree one derivation $d - A^a \iota_a$. One has

$$(d - A^a L_a)^2 = (d - A^a L_a)(d - A^b L_b)$$

$$= \cancel{d^2} - A^a L_a d - d A^b L_b + A^a L_a A^b L_b$$

$$= \cancel{-A^a L_a d + A^b d L_b} - (dA^b) L_b$$

$L_X A = -[X, A]$

$$= A^a f^b_{ac} A^c L_b + \underbrace{A^a A^b L_a L_b}_{\frac{1}{2} A^a A^b [L_a, L_b]}$$

$$= -\left(dA^a + \tfrac{1}{2} f^a_{bc} A^b A^c\right) L_a = -F^a L_a$$

Thus on $\Omega(P)_{hor}$ we have the following structure: Gr. comm. Algebra, derivation ∇ of degree 1, action of $\mathfrak{g}$, $F \in \Omega^2(P)_{hor} \otimes \mathfrak{g}$.

Properties: $\boxed{\nabla, F \text{ are } \mathfrak{g}\text{-invariant}}$ Check:

$$[L_a, \nabla] = [L_a, d - A^b L_b] = + f^c_{ab} A^b L_c - A^b f^c_{ab} L_c = 0$$

$\mathfrak{g}$ invariance of $F = dA + A^2$ is obvious from $\mathfrak{g}$ invariance of A.

$$\boxed{\nabla^2 = -F^a L_a} \qquad \boxed{\nabla(F^a) = 0}$$

Check: $\nabla(F^a) = (d - A^b L_b) F^a$

$$= -([A, F])^a + A^b f^a_{bc} F^c$$

$$= -f^a_{bc} A^b F^c + f^a_{bc} A^b F^c = 0$$

Conversely suppose given a graded commutative algebra Ω_h with ∇ and $\mathfrak{g}$-action and $F \in \Omega_h^2 \otimes \mathfrak{g}$ having the above properties, set

$$\Omega = \Omega_h \otimes \Lambda \mathfrak{g}_A^*$$

and define the obvious $\mathfrak{g}$ action on Ω and define d by

$$d\omega = \nabla\omega + A^a L_a \omega \qquad \omega \in \Omega_h$$

$$dA^a + \tfrac{1}{2} f^a_{bc} A^b A^c = F^a$$

Check that $d^2 = 0$.

$$
\begin{aligned}
d(d\omega) &= d\left(\nabla\omega + A^b L_b \omega\right) \\
&= \left(\nabla + \cancel{A^a} L_a\right)\nabla\omega + dA^b L_b \omega - A^b\left(\cancel{\nabla} + A^a L_a\right) L_b \omega \\
&= \nabla^2\omega + dA^c L_c \omega + \tfrac{1}{2} A^a A^b [L_a, L_b]\omega \\
&= \nabla^2 \omega + F^c L_c \omega = 0
\end{aligned}
$$

$$d(F^a) = \left(\nabla + A^b L_b\right) F^a = A^b\left(-f^a_{bc} F^c\right) = -f^a_{bc} A^b F^c$$

$\therefore\ dF = -[A, F]$ and so $d^2 A = d(F - A^2) = 0$.

February 26, 1990

The problem is still to find a derivation of the Bott spectral sequence using

$$C_{diff}(G, \Omega(P)_{hor})$$

There should be a total differential defined on this space even though $\Omega(P)_{hor}$ is not a complex.

To get some insight we can consider the Lie analogue

$$C(\mathfrak{g}, \Omega(P)_{hor}) = \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}^* .$$

We know that if a connection in P is given, then we have an isomorphism

$$\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}^*_A \xrightarrow{\sim} \Omega(P)$$

and hence we have a total differential d on $C^\cdot(\mathfrak{g}, \Omega(P)_{hor})$. We know this total differential splits

$$d = (\underbrace{d - A^a L_a}_{\nabla}) + A^a L_a$$

into horizontal and vertical derivation, but it's not a bigraded diff'l algebra since

$$\nabla^2 = (d - A^a L_a)^2 = -F^a L_a$$

on $\Omega(P)_{hor}$.

Something else we ought to be able to do is to form also

$$C_{diff}(G, \Omega(P) \otimes S(\mathfrak{g}^*))$$

and get the same result.

February 27, 1990

Recall that if $\Omega^\cdot$ is a $\mathfrak{g}[\varepsilon]$-DG alg. (commutative) with a connection $A \in \Omega^1 \otimes \mathfrak{g}$, then Ω_{hor} is a graded $\mathfrak{g}$-algebra, equipped with a degree one derivation ∇ and $F \in \Omega^2_{hor} \otimes \mathfrak{g}$ such that ∇, F are $\mathfrak{g}$-invariant and $\nabla^2 = -F^a L_a$. Conversely given a graded $\mathfrak{g}$-algebra Ω_h with ∇, F having these properties, we obtain a $\mathfrak{g}[\varepsilon]$-DGA

$$\Omega = \Omega_h \otimes \Lambda \mathfrak{g}_A^* \qquad d = \nabla + A^a L_a$$

Here are some examples.

Take $\Omega = W(\mathfrak{g}) = \Lambda \mathfrak{g}_A^* \otimes S\mathfrak{g}_F^*$. Then $\Omega_{hor} = S(\mathfrak{g}_F^*)$ with $\nabla = 0$ and $F = F^a X_a \in S^1(\mathfrak{g}_F^*) \otimes \mathfrak{g}^*$. Let's check the condition $\nabla^2 = -F^a L_a$. One has

$$-F^a L_a(F^b) = F^a f^b_{ac} F^c = \underbrace{f^b_{ac}}_{\text{antis in } a,c} \underbrace{F^a F^c}_{\text{sym in } a,c} = 0$$

Take $\Omega = \Omega(P) \otimes W(\mathfrak{g})$, where P is a $\mathfrak{g}$-manifold. Then

$$\Omega_{hor} \xrightarrow{\sim} \Omega(P) \otimes S(\mathfrak{g}^*)$$

where the isomorphism is induced by sending $A^a \mapsto 0$ and the inverse isomorphism by

$$\omega \in \Omega(P) \longmapsto \prod_a (1 - A^a \iota_a)\, \omega = (:e^{-A^a \iota_a}:)\, \omega$$

$$= \omega - A^a \iota_a \omega + \frac{1}{2!} A^a A^b \iota_b \iota_a \omega - \frac{1}{3!} A^a A^b A^c \iota_c \iota_b \iota_a \omega + \cdots$$

Let's calculate ∇, F in $(\Omega(P)\otimes W(\mathfrak{g}))_{hor}$. Now the connection form A belongs to $W(\mathfrak{g})$ and so does the curvature, so the curvature F is the canonical elements $F^a X_a \in S^1(\mathfrak{g}^*)\otimes\mathfrak{g}$. ∇ on $(\Omega(P)\otimes W(\mathfrak{g}))_{hor}$ is induced by $d - A^a\iota_a$. We know $\nabla = 0$ on $W(\mathfrak{g})_{hor}$ and so let us now calculate ∇ on the element of $(\Omega(P)\otimes W(\mathfrak{g}))_{hor}$ corresponding to $\omega \in \Omega(P) \subset \Omega(P)\otimes S(\mathfrak{g}^*)$. This elt is

$$:e^{-A^a\iota_a}:\omega = \omega - A^a\iota_a + \tfrac{1}{2}A^aA^b\iota_b\iota_a\omega - \cdots$$

We apply $d - A^a\iota_a$, then apply the inverse isomorphism which sends $A^a \longmapsto 0$ and $dA = F - A^2$ to F. We obtain

$$(d - A^a\iota_a)(\omega - A^a\iota_a + \cdots)$$
$$\downarrow$$
$$d\omega - F^a\iota_a\omega$$

Thus
$$\nabla(\omega) = (d - F^a\iota_a)\omega \qquad \omega\in\Omega(P)$$
$$\nabla(F) = 0 \qquad \text{on } S(\mathfrak{g}^*).$$

Check:
$$\begin{aligned}\nabla^2(\omega) &= \nabla(d\omega - F^b\iota_b\omega)\\ &= (d - F^a\iota_a)(d\omega) - F^b(d - F^a\iota_a)\iota_b\omega\\ &= -F^a\iota_a d\omega - F^b d\iota_b\omega + \cancel{F^bF^a\iota_a\iota_b\omega}\end{aligned}$$

So $\nabla^2(\omega) = -F^a L_a \omega$ for $\omega \in \Omega(P)$ and $\nabla^2 = -F^a L_a = 0$ on $S(\mathfrak{g}^*)$.

Remark: The inverse of the isom.

$$\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \xrightarrow{\sim} \Omega(P)$$

is obtained as follows

$$\begin{array}{ccccc}
\Omega(P) & \xrightarrow{e^{\varphi^a L_a}} & \Omega(P) \otimes \Lambda \mathfrak{g}_\varphi^* & \xrightarrow{\pi \otimes (\varphi \mapsto A)} & \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \\
\uparrow S & & \uparrow S & & \| \\
\Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* & \longrightarrow & \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \otimes \Lambda \mathfrak{g}_\varphi^* & \longrightarrow & \Omega(P)_{hor} \otimes \Lambda \mathfrak{g}_A^* \\
\omega & \longmapsto & \omega & \longmapsto & \omega \\
A & \longmapsto & A+\varphi & \longmapsto & A
\end{array}$$

$$\begin{pmatrix} A \longmapsto 0 \\ \varphi \longmapsto A \end{pmatrix}$$

Recall that $\pi : \Omega(P) \to \Omega(P)_{hor}$ is

$$\pi = \prod_j \iota_j A^j = \prod_j (1 - A^j \iota_j)$$

$$\| $$

$$\iota_1 \cdots \iota_n A^n \cdots A^1$$

no summation convention

Bott's spectral sequence. The idea is roughly that $(G, \mathfrak{g}^{\varepsilon})$ modules of the form $\Omega(M)$ with M a free G-manifold should be acyclic for the "differentiable cohomology". Thus we have the functor of taking basics or invariants for the $(G, \mathfrak{g}^{\varepsilon})$ action. This is left exact and we take its derived functors using $\Omega(M)$ as acyclic objects. Let's ignore foundations and see what happens.

Consider a principal G-bundle P. Then we can filter

$$\Omega(P) \supset J \supset J^2 \supset \cdots$$

where J is the ideal generated by $\Omega^{>0}(P)_{hor}$. The quotients are

$$J^p / J^{p+1} = \Omega^p_{hor} \otimes \Lambda \mathfrak{g}^*$$

and these should be acyclic for the "differentiable $(G, \mathfrak{g}^{\varepsilon}_{\wedge})$ cohomology," because the $\Lambda \mathfrak{g}^*$ is cofree for $\mathfrak{g}^{\varepsilon}$, and Ω^p_{hor} being a module over $\Omega^0(P)$ is acyclic for differentiable G cohomology.

We have

$$\left(\Omega^p_{hor} \otimes \Lambda \mathfrak{g}^*\right)_{bas} = \Omega^p(P)_{bas} = \Omega^p(B)$$

and so we get

$$E_1^{pq} = \begin{cases} \Omega^p(B) & q = 0 \\ 0 & q \neq 0 \end{cases} \Longrightarrow H^n(B)$$

which is OK.

Next let's consider

$$\Omega(P) \otimes W(\mathfrak{g}) \supset I \supset I^2 \supset \cdots\cdots$$

where I is the ideal generated by $S^1(\mathfrak{g}_F^*)$. One has

$$I^p/I^{p+1} = \Omega^{\bullet}(P) \otimes \left(\Lambda \mathfrak{g}_A^* \otimes S^p \mathfrak{g}_F^*\right)$$

where d in the latter factor is

$$dA + A^2 = 0 \qquad dF + [A, F] = 0$$

These quotients should be acyclic for the $(G, \mathfrak{g}_E)$ cohomology. Taking basics gives

$$\left(I^p/I^{p+1}\right)_{bas} = \left(\Omega(P) \otimes S^p \mathfrak{g}_F^*\right)^G$$

Actually we should do this carefully. Let's start with $\Omega(P) \otimes W(\mathfrak{g})$ again and take horizontal elements: This gives

$$\left(\Omega(P) \otimes W(\mathfrak{g})\right)_{hor} = \Omega(P) \otimes S(\mathfrak{g}^*)$$

with $\nabla = d - F^a L_a$ where $\nabla(F^a) = 0$

If we are interested in the associated graded with respect to the I-adic filtration, we have $\nabla = d$ since $F^a l_a$ raises I-order. Thus $\nabla = d$ on

$$\left(I^p/I^{p+1}\right)_{hor} = \Omega^{\bullet}(P) \otimes S^p \mathfrak{g}^*$$

Now if $P =$ universal bundle BG, then $\Omega^{\bullet}(P)$ is an acyclic resolution of $\mathbb{C}$ for the

differentiable cohomology w.r.t. G.
Thus

$$H^q\left((I^p/I^{p+1})_{bas}\right) = H^q_{diff}(G, S^p\mathfrak{g}^*)$$

and we therefore get Bott's spectral sequence

$$H^q_{diff}(G, S^p\mathfrak{g}^*) \Longrightarrow H^\bullet(BG)$$

Problem: What is the relation between

$$C(\mathfrak{g}[\varepsilon], \Omega(P)) = \Omega(P) \otimes \Lambda \mathfrak{g}^*_\chi \otimes S\mathfrak{g}^*_\varphi$$

and $\quad \Omega(P) \otimes W(\mathfrak{g})$?

Recall we have a map

$$\Omega(P) \longrightarrow \Omega(P) \otimes \Lambda \mathfrak{g}^*_\chi \otimes S\mathfrak{g}^*_\varphi$$

$$\omega \longmapsto e^{\chi^a \iota_a}\omega = \omega + \chi^a \iota_a \omega + \tfrac{1}{2}\chi^a\chi^b \iota_b \iota_a \omega + \dots$$

This is an algebra homomorphism such that

$$\omega \in \Omega(P)_{hor} \longmapsto \omega$$

$$A \longmapsto A + \chi$$

and it is compatible with differentials since for $\omega \in \Omega(P)_{hor}$ one has

$$\begin{array}{ccc} \omega & \longmapsto & \omega \\ \downarrow d & & \downarrow d+\delta \\ & & d\omega + \chi^a L_a \omega \\ d\omega = A^a L_a \omega + (d\omega - A^a L_a \omega) & \longmapsto & d\omega - A^a L_a\omega + (\chi^a + A^a) L_a \omega \\ & & = d\omega + \chi^a L_a \omega \end{array}$$

and

$$\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, "d"] & (\chi+A) \arrow[r, bend left, "d+\delta"] & (d+\delta)(\chi+A)^2 \arrow[d, equal] \\
F-A^2 \arrow[rr, mapsto] & & F-(A+\chi)^2
\end{tikzcd}$$

Russian formula

Actually it should be possible to give a proof without choosing A.

So we also have a DGA map

$$W(\mathfrak{g}) \longrightarrow \Lambda \mathfrak{g}^*_\chi \otimes S\mathfrak{g}^*_\varphi$$

sending the universal connection to χ and the universal curvature to

$$(d+\delta)\chi + \chi^2 = d\chi = +\varphi$$

Then we can put these together to obtain

$$\Omega(P) \otimes W(\mathfrak{g}) \longrightarrow \Omega(P) \otimes \Lambda \mathfrak{g}^*_\chi \otimes S\mathfrak{g}^*_\varphi$$

This is a DGA isomorphism:

$$\begin{array}{ccc}
\Omega(P)_{hor} \otimes \Lambda\mathfrak{g}^*_A \otimes \Lambda\mathfrak{g}^*_\theta \otimes S\mathfrak{g}^*_\Omega & \longrightarrow & \Omega(P)_{hor} \otimes \Lambda\mathfrak{g}^*_A \otimes \Lambda\mathfrak{g}^*_\chi \otimes S\mathfrak{g}^*_\varphi \\
\omega & \longmapsto & \omega \\
A & \longmapsto & A+\chi \\
\theta & \longmapsto & \chi \\
\Omega & \longmapsto & +\varphi
\end{array}$$

The curious point concerns the L_a, ι_a operations. These are defined diagonally on the left by

$$\iota_a \omega = 0 \qquad \iota_a A = X_a \qquad \iota_a \theta = X_a \qquad \iota_a \Omega = 0$$

and so we must define on the right

$$\iota_a \omega = 0 \qquad \iota_a A = 0 \qquad \iota_a \chi = X_a \qquad \iota_a \varphi = 0.$$

Recall that $C(\mathfrak{g}[\varepsilon]) = \Lambda \mathfrak{g}_x^* \otimes S \mathfrak{g}_\varphi^*$ is the analogue for $\mathfrak{g}[\varepsilon]$ of $C(\mathfrak{g}) = \Lambda(\mathfrak{g}^*)$, on which $\mathfrak{g}[\varepsilon]$ operates.

Thus to $X, X\varepsilon \in \mathfrak{g}[\varepsilon]$ we have operators $L_X, L_{X\varepsilon}$ and ι_X, $\iota_{X\varepsilon}$ on $C(\mathfrak{g}[\varepsilon])$. Let's compute them. Recall

$$\theta = \chi^b X_b + (-\varphi^b) X_b \varepsilon$$

so $\iota_X \theta = X \implies$

$$\iota_{X_a}(\chi^b) = \delta_a^b$$
$$\iota_{X_a}(\varphi^b) = 0$$

Then $L_X = \delta \iota_X + \iota_X \delta$ implies

$$L_{X_a}(\chi^b) = \delta(\underbrace{\iota_{X_a}\chi^b}_{\text{scalar}}) + \iota_{X_a}(\delta\chi^b)$$

$$= \iota_{X_a}\left(-\tfrac{1}{2} f^b_{cd} \chi^c \chi^d\right)$$

$$= -\tfrac{1}{2} f^b_{ad} \chi^d + \tfrac{1}{2} f^b_{ca} \chi^c = -f^b_{ac} \chi^c$$

$$L_{X_a}(+\varphi^b) = \iota_{X_a} \delta \varphi^b = \iota_{X_a}(-f^b_{cd} \chi^c \varphi^d)$$

$$= -f^b_{ad} \varphi^d$$

This tells us that if we identify

$$W(\mathfrak{g}) = C(\mathfrak{g}[\varepsilon])$$
$$A \longleftrightarrow \chi$$
$$F \longleftrightarrow +\varphi$$

then the natural of L_X, ι_X on $W(\mathfrak{g})$ correspond to the operators L_X, ι_X on $C(\mathfrak{g}[\varepsilon])$ associated to the subalgebra $\mathfrak{g} \subset \mathfrak{g}\varepsilon$

From $\iota_{X\varepsilon}\theta = X\varepsilon$ we conclude

$$\iota_{X_a\varepsilon}(x^b)=0 \qquad \iota_{X_a\varepsilon}(-\varphi^b)=\delta_a^b$$

Then $L_{X\varepsilon} = -\iota_{X\varepsilon}\delta + \delta\iota_{X\varepsilon}$ yields. — because $\iota_{X\varepsilon}$ even

$$L_{X_a\varepsilon}(x^b) = -\iota_{X_a\varepsilon}\delta x^b = -\iota_{X_a\varepsilon}\left(-\tfrac{1}{2}f^b_{cd}x^cx^d\right) = 0$$

$$L_{X_a\varepsilon}\varphi^b = -\iota_{X_a\varepsilon}\delta\varphi^b = \overset{\text{even}}{\iota_{X_a\varepsilon}}\left(+f^b_{cd}x^c\varphi^d\right)$$

$$= -f^b_{ca}x^c = f^b_{ac}x^c$$

Summary: The action of ι_Y, L_Y, $Y\in\mathfrak{g}+\mathfrak{g}\varepsilon$ on $C(\mathfrak{g}[\varepsilon])$ is given by

$$\iota_X x = X \qquad \iota_X\varphi = 0$$

$$L_X x = -[X,x] \qquad L_X\varphi = -[X,\varphi]$$

$$\iota_{X\varepsilon}x = 0 \qquad \iota_{X\varepsilon}(-\varphi) = X$$

$$L_{X\varepsilon}x = 0 \qquad L_{X\varepsilon}(\varphi) = +[X,x]$$

Check

$$[d,\iota_{X\varepsilon}]x = d\iota_{X\varepsilon}x - \iota_{X\varepsilon}dx = -\iota_{X\varepsilon}\varphi = X = \iota_X x$$

$$[d,\iota_{X\varepsilon}]\varphi = d\iota_{X\varepsilon}\varphi - \iota_{X\varepsilon}\cancel{d\varphi} = -dX = 0 = \iota_X\varphi$$

Thus $\boxed{[d,\iota_{X\varepsilon}] = \iota_X}$

$$[d,L_{X\varepsilon}]x = (d\cancel{L_{X\varepsilon}} + L_{X\varepsilon}d)x = L_{X\varepsilon}\varphi = +[X,x]$$

$$[d,L_{X\varepsilon}]\varphi = dL_{X\varepsilon}\varphi + L_{X\varepsilon}\cancel{d\varphi} = d[X,x] = [X,\varphi]$$

Thus $$\boxed{[d, L_{X\varepsilon}] = -L_X}$$

Further check:

$$[d, L_{X\varepsilon}] = [d, [\delta, \iota_{X\varepsilon}]] = -[\delta, [d, \iota_{X\varepsilon}]]$$
$$= -[\delta, \iota_X] = -L_X$$

Summary + discussion: We originally started trying to understand BRS cohomology for a G-manifold P. This is a bigraded differential algebra which we have now connected up with $\Omega(P) \otimes W(\mathfrak{g})$, the sort of thing encountered in equivariant cohomology. From this gadget we can obtain the Leray spectral sequence for $G \to P \to B$, as well as Bott's spectral sequence

February 28, 1990

Review van Est and Bott spectral sequences. The basic idea is that one can calculate differentiable cohomology using $\Omega^\cdot(P)$, where P is a universal G-bundle.

In the case of van Est consider

$$\left(\Omega^p(P) \otimes \Omega^q(G)\right)^G$$

This is a double complex. One has

$$H_h^{p\cdot} = \begin{cases} 0 & p > 0 \\ \Lambda^\cdot \mathfrak{g}^* & p = 0 \end{cases}$$

$$H_v^{\cdot q} = \left(\Omega^\cdot(P) \otimes H^q(G)\right)^G$$

Here one is using exactness of $(M \otimes ?)^G$ where M is $\Omega^p(P)$ or $\Omega^q(G)$.

In the case of Bott's spectral sequence consider

$$\left(\Omega(P) \otimes W(\mathfrak{g})\right)_{bas} = \left(\Omega(P) \otimes S(\mathfrak{g}^*)\right)^G$$

This is a complex. Ignoring differentials it is $\bigoplus_p \left(\Omega(P) \otimes S^p(\mathfrak{g}^*)\right)^G$, and it's a filtered complex with $gr_p = \left(\Omega^\cdot(P) \otimes S^p(\mathfrak{g}^*)\right)^G$. The differential is induced by $d - F^a \iota_a = d$ on gr_p. Finally one must show that

$$\Omega(B) = \Omega(P)_{bas} \longrightarrow \left(\Omega(P) \otimes W(\mathfrak{g})\right)_{bas}$$

is a quis. This should be easy if there is a connection in P.

<u>Review</u>: Given $\Omega(P)$ introduce the **J-adic** filtration, where J is the ideal generated by $\Omega(P)^{>0}_{hor}$. The δ cohomology is the cohomology of $gr^J \Omega(P)$, which **is** $H^q(\mathfrak{g}, \Omega^p(P)_{hor}) = \Omega^p(B, H^q(G))$.

If we start with $\Omega(P)\otimes W(\mathfrak{g})$ and use the isomorphism

$$\Omega(P)\otimes W(\mathfrak{g}) \xrightarrow{\ \sim\ } \overbrace{\Omega(P)\otimes S(\mathfrak{g}^*_\varphi)}^{hor} \otimes \Lambda \mathfrak{g}^*_\chi$$

$$\begin{array}{ll} \parallel & \\ \Omega(P)_{hor}\otimes \Lambda\mathfrak{g}^*_A \otimes \Lambda\mathfrak{g}^*_\theta \otimes S\mathfrak{g}^*_\Omega & \qquad \Omega(P)_{hor}\otimes \Lambda\mathfrak{g}^*_A \otimes \Lambda\mathfrak{g}^*_\chi \otimes S(\mathfrak{g}^*_\varphi) \end{array}$$

$$\begin{aligned} \omega &\longmapsto \omega \\ A &\longmapsto A+\chi \\ \theta &\longmapsto \chi \\ \Omega &\longmapsto \varphi = d\chi \end{aligned}$$

then J is exactly the ideal of elements of d-degree >0.

The other thing we can do is to take basic elements first. Applied to $\Omega(P)\supset J\supset J^2\supset\cdots$ this gives $\Omega(B)\supset\Omega^{\geq 1}(B)\supset\Omega^{\geq 2}(B)\supset\cdots$, the "skeletal" filtration. Applied to $\Omega(P)\otimes W(\mathfrak{g})$ it gives the "skeletal" filtration of $(\Omega(P)\otimes S(\mathfrak{g}^*))^G$. To get Bott's spectral sequence we use a different ideal in $\Omega(P)\otimes W(\mathfrak{g})$, namely the ideal generated by $S^1(\mathfrak{g}^*)$.

Let us now consider

$$\mathcal{G} = C^\infty(M, U_N) = \operatorname{Aut}\{(\mathbb{C}^N)_M\}$$

$$\mathfrak{g} = \operatorname{Lie}(\mathcal{G})_c = C^\infty(M) \otimes M_N$$

Recall that a principal $\mathcal{G}$-bundle $P \to B$ is equivalent to a hermitian vector bundle E over $B \times M$ such that $E_y \simeq (\mathbb{C}^N)_M$ for each $y \in B$. A connection in E is equivalent to a connection in P together with an $\mathcal{G}$-equivariant map from P to the space $\mathcal{A}$ of connections on $(\mathbb{C}^N)_M$.

We obtain characteristic classes in $H^\cdot(B)$ associated to P by integrating characteristic classes of E over homology classes in M. If we fix a connection on E and a cycle (closed current) on M, then we obtain closed forms on B. The idea is to describe this construction abstractly, that is, in the spirit of the Weil algebra.

The bundle E pulled up to $P \times M$ is canonically trivial and the connection on E up on $P \times M$ is

$$\overbrace{d_P}^{\delta} + \overbrace{d_M}^{d} + X + A$$

where

$$X \in \Omega^{1,0}(P \times M) \otimes M_N = \Omega^1(P, \underbrace{\Omega^0(M) \otimes M_N}_{\mathfrak{g}})$$

is the connection form in P and

$$A \in \Omega^{0,1}(P \times M) \otimes M_N = \Omega^0(P, \underbrace{\Omega^1(M) \otimes M_N}_{\mathcal{A}})$$

is the equivariant map.

We want to play the Weil algebra game

more generally. Observe that χ

$$\chi \in \Omega^1(P, \mathfrak{g}) = \Omega^1(P) \otimes \mathfrak{g}$$

induces $\mathfrak{g}^* \longrightarrow \Omega^1(P)$ which extends to a DGA morphism

$$W(\mathfrak{g}) \longrightarrow \Omega(P)$$

Thus we seek to endarge $W(\mathfrak{g})$ so as to incorporate A. This means we adjoin to $W(\mathfrak{g})$ ~~something~~ like $\mathcal{A}^*$ in degree 0. Thus we [redacted] probably want at least all polynomial functions on the affine space $\mathcal{A}$, and all polynomial coefficient differential forms. At this rate we end up with just

$$W(\mathfrak{g}) \otimes \Omega(\mathcal{A})$$

and the complex of equivariant differential forms.

Note the curvature is

$$(\delta + d + \chi + A)^2 =$$

$$(\delta\chi + \chi^2) + (\delta A + d\chi + [\chi, A]) + (dA + A^2)$$

and that these three components are generally nonzero.

[redacted] The next project is to do a cyclic or large N version. [redacted] In the case of the Weil algebra we take the dual of the bar construction on $\{\mathbf{R} \xrightarrow{1} \mathbf{R}\}$ considered as a DGA. [redacted] Dually we have the tensor coalgebra generated by R in degree 1 and R in degree 2. By analogy we expect to add Ω^1_R in degree 0 and Ω^1_R in degree 1.

Let's consider the DG algebra traditionally encountered in anomalies (e.g. Bonora Cotta-Ramasino), namely

$$\Omega^\circ(\mathcal{A}) \otimes \Lambda \mathfrak{g}_X^*$$

Lie cochains on the infinitesimal gauge transformations with values in functionals on connections. The differential is given by

$$\delta A = -D_A(X) = -dX - [A, X]$$

$$\delta X + X^2 = 0$$

Let us take $\mathfrak{g} = \Omega^\circ(M) \otimes M_N$, $\mathcal{A} = \Omega^1(M) \otimes M_N$ and consider

$$\left(S(\mathcal{A}^*) \otimes \Lambda \mathfrak{g}^*\right)^{gl_N}$$

which should be a Hopf algebra. (Here we consider poly functionals on $\mathcal{A}$). Proceeding in known fashion we consider

$$\left\{ S(\Omega^1(M) \otimes M_N)^* \otimes \Lambda(\Omega^\circ(M) \otimes M_N)^* \otimes M_N \right\}^{gl_N}$$

By invariant theory this should contain a tensor algebra on $\Omega^1(M)^*$ in degree 0 and $\Omega^\circ(M)^*$ in degree 1, and there is a differential δ. This free noncommutative DG algebra is the "noncommutative" version of $\Omega^\circ(\mathcal{A}) \otimes \Lambda \mathfrak{g}_X^*$. To calculate δ we look at the canonical twisting cochain

$$\left(\Lambda(\Omega^\circ \otimes M_N)^* \otimes S(\Omega^1 \otimes M_N)^* \otimes (\Omega^\circ \oplus \Omega^1) \otimes M_N\right)^{gl_N}$$

$$\cup$$

$$T\left[(\Omega^\circ)^* + (\Omega^1)^*\right] \otimes (\Omega^\circ \oplus \Omega^1)$$

$$\Psi$$

$$\Theta = X + A$$

Then the twisting cochain condition gives

$$(\delta+d)(x+A)\text{[redacted]} + (x+A)^2 = 0$$

$$\underbrace{(\delta x + x^2)}_{\text{values in } \Omega^0} + \underbrace{(\delta x + dA + [x,A])}_{\text{values in } \Omega^1} + \underbrace{(dA + A^2)}_{\text{values in } \Omega^2 = 0}$$

$$\therefore \quad \delta x + x^2 = 0 = \delta x + dA + [x,A]$$

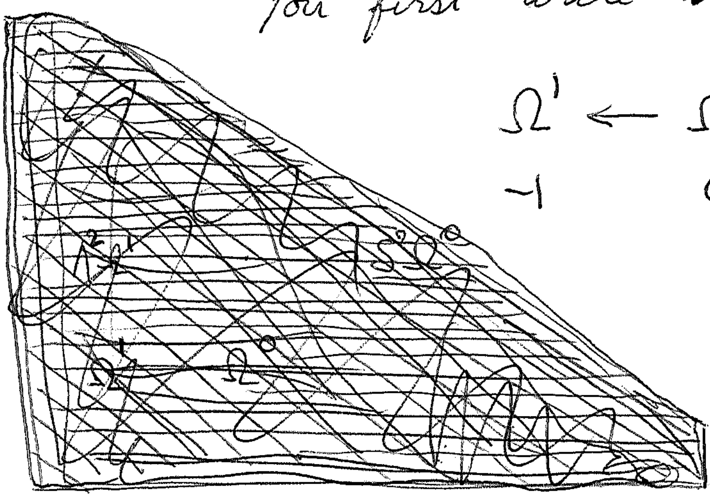
Therefore if our procedure is OKAY we should have that

$$\left(S(a)^* \otimes \Lambda \mathfrak{g}_x^*\right)^{\mathfrak{gl}_N}$$

is a Hopf [redacted] algebra whose primitive part is the cyclic cochain complex on the DG algebra

$$\begin{array}{cccccc} 0 & 0 & \Omega^0 \xrightarrow{d} & \Omega^1 & 0 & 0 \\ & & \uparrow & \uparrow & & \\ & & 0 & 1 & & \end{array}$$

The cyclic chain complex looks as follows. You first write the DGA



$$\begin{array}{ccc} \Omega^1 & \leftarrow & \Omega^0 \\ -1 & & 0 \end{array}$$

Then you take the cyclic complex which has rows the $\otimes_x^n$ powers:

$$\begin{tikzcd} \arrow[d] & \arrow[d] & \arrow[d] \\ S^2\Omega^1 & \Omega^0\otimes\Omega^1 \arrow[l] \arrow[d] & \Lambda^2\Omega^0 \arrow[l] \arrow[d] \\ & \Omega^1 & \Omega^0 \arrow[l] \arrow[d, dash, dotted] \\ & & 0 \end{tikzcd}$$

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In the study of anomalies one encounters $\Omega^0(\mathcal{a})\otimes\Lambda\mathfrak{g}^*$, the complex of Lie cochains for the infinitesimal gauge transformations acting on functions on the space of connections. Actually the physicists want $\Omega^0(\mathcal{a})$ replaced by Γ_{loc} = the "local" functionals, integrals of polynomials in A and its derivatives. Anomalies are elements of

$$H^1(\mathfrak{g},\Gamma_{loc})$$

For example

$$W(A) = \log\det\partial_A$$

defined by regularization is nonlocal yet

$$\delta W(A) = \text{"Tr"}\left(\partial_A^{-1}\delta\partial_A\right)$$

is local (here $\delta\partial_A = [\theta,\partial_A]$ and the above expression is a local expression related to the index theorem). Thus $\delta W(A)$ can represent a nontrivial element of $H^1(\mathfrak{g},\Gamma_{loc})$.

One has

$$H^q(\mathfrak{g},\Omega^0(\mathcal{a})) = \Omega^0(\mathcal{a}/\mathcal{G})\otimes H^q(\mathcal{G})$$

if we restrict to gauge transformations = 1 at a basepoint of M. This relates anomalies to $H^1(\mathcal{G})$:

$$H^1(\mathfrak{g},\Gamma_{loc})\longrightarrow H^1(\mathfrak{g},\Omega^0(\mathcal{a})) = \Omega^0(\mathcal{a}/\mathcal{G})\otimes H^1(\mathcal{G})$$

which in turn is related to $H^2(B\mathcal{G})$ and determinant line bundles.

<u>Point:</u> Lie algebra cohomology for the [redacted] action on $\Omega^\circ(a)$ is directly related to $H^\circ(\mathcal{G})$:

$$\underline{H^q(\mathfrak{g}, \Omega^\circ(a)) = \Omega^\circ(a/\mathcal{G}) \otimes H^q(\mathcal{G})}.$$

Yesterday I thought that in the case $\mathcal{G} = U_N(C^\infty(M))$, N large, I could relate [redacted] $\Omega^\circ(a) \otimes \Lambda^\circ \mathfrak{g}$ to the cyclic cochains for the DGA

$$\begin{array}{ccc} \Omega^\circ(M) & \xrightarrow{d} & \Omega^1(M) \\ 0 & & +1 \end{array}$$

(Actually I looked at $\Omega^\circ(a)$ replaced by the polynomial functions). [redacted] Let's consider the reduced complex which corresponds to the condition $g = 1$ at the basepoint. Then [redacted]

$$d : \overline{\Omega^\circ} \longrightarrow \Omega^1 \quad \text{is injective}$$

and so the d [redacted]homology in the cyclic chains

$$\begin{tikzcd}
\uparrow & \uparrow & \uparrow \\
(\Omega^1)^{\otimes 2}_\sigma & \Omega^1 \otimes \overline{\Omega^\circ} \arrow[l] \arrow[d] & (\overline{\Omega^\circ})^{\otimes 2}_\lambda \arrow[l] \arrow[d] \\
 & \Omega^1 & \overline{\Omega^\circ} \arrow[l] \arrow[d] \\
 & & 0
\end{tikzcd}$$

is concentrated on the $\setminus$ diagonal. It looks therefore as if one obtains only something in degree 0 like $\Omega^\circ(a/\mathcal{G})$ and nothing in positive degrees.

Thus it is necessary to be cautious in this case.