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<u>September 3, 1989</u>

Let R be a unital alg generated by A, a non unital subalg., and an element z. Let $I \subset R$ be an ideal containing $[z, A]$, $1-z^2$. Let τ be a trace on the ideal I^m. Using dilation:

$$F = \begin{pmatrix} z & 1-z^2 \\ 1 & -z \end{pmatrix} \qquad F^2 = 1$$

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad [F, \theta e] = \begin{pmatrix} [z,\theta] & \theta(z^2-1) \\ \theta & 0 \end{pmatrix}$$

one can construct cyclic cocycles

$$(tr \otimes \tau)\left(\theta e [F, \theta e]^{2n-1}\right) = (tr \otimes \tau)\left\{ \begin{pmatrix} \theta & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} [z,\theta] & \theta(z^2-1) \\ \theta & 0 \end{pmatrix}^{2n-1} \right\}$$

$$= \tau\left\{ \theta P^{(2n-1)}\left([z,\theta],\ \theta(z^2-1)\theta\right)\right\}$$

which are defined for n large ($n \geq m$ it seems).

We've seen a similar situation in the case where A is $\Omega^0(M, \operatorname{End} E)$, E a vector bundle with connection ∇. Choosing $\tilde{V} = E \oplus E^\perp$,

$$E \underset{i}{\overset{i^*}{\leftrightarrows}} \tilde{V} \underset{j}{\overset{j^*}{\leftrightarrows}} E^\perp$$

with $\nabla = i^* d i$, we have

$$i \theta i^* = \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix} \qquad d = \begin{pmatrix} i^* d i & i^* d j \\ j^* d i & j^* d j \end{pmatrix}$$

$$tr_{\tilde{V}}\, (i\theta i^*)[d, i\theta i^*]^n = tr_{\tilde{V}} \begin{pmatrix} \theta & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} [\nabla,\theta] & -\theta i^* d j \\ j^* d i \theta & 0 \end{pmatrix}^n$$

$$= tr_E\left\{\theta P^{(n)}\left([\nabla,\theta],\ \theta\nabla^2\theta\right)\right\}$$

However in this vector bundle ~~situation~~ we have seen that many other cyclic cocycles that can be constructed, for example by joining the connections [redacted] [redacted] $\delta + \nabla$, $\delta + \theta + \nabla$ by a linear path:

$$(\delta + \nabla + t\theta)^2 = \nabla^2 + t[\nabla, \theta] + (t^2 - t)\theta^2$$

or by superconnection methods:

$$\left(\begin{pmatrix} \delta + \nabla + \theta & 0 \\ 0 & \delta + \nabla \end{pmatrix} + t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sigma \right)^2$$

$$= \begin{pmatrix} \nabla^2 + [\nabla, \theta] & 0 \\ 0 & \nabla^2 \end{pmatrix} + t \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \sigma - t^2$$

[redacted] Presumably all these [redacted] cyclic cocycles are cohomologous, at least after S transforming. [redacted] There seems to be nothing canonical on the level of cyclic cocycles.

On the other hand there is the [redacted] periodic cocycle

$$\psi = tr_E^q\left(\partial\theta\, e^{\nabla^2 + [\nabla, \theta]}\right) \qquad \varphi = tr_E\left(e^{\nabla^2 + [\nabla, \theta]}\right)$$

which seems to be fairly canonical and also seems to be the simplest since it is built up out of ∇^2, $[\nabla, \theta]$ and at most a single θ.

So it's natural to ask for something similar [redacted] in the $\mathbb{Z}$ setup.

We adjoin σ as usual to $Hom(B(A), R)$. Thus we work in the superalgebra $Hom(B(A), R) \hat{\otimes} C_1 = Hom(B(A), R \otimes C_1)$, and we [redacted] use the supertrace $\tau_s(x+y\sigma) = \tau(y)$ which is defined on $I^m \otimes C_1$.

We have the 'superconnection' $\delta + \theta + z\sigma$ with [redacted] curvature

$$K = (\delta + \theta + z\sigma)^2 = [z\sigma, \theta] + z^2$$

This doesn't have values in I, but

$$K - 1 = [z\sigma, \theta] + (z^2 - 1)$$

is an I-valued cochain. Thus

$$\varphi^{(n)} = \tau_s\{(K-1)^n\} \qquad \psi^{(n)} = \tau_s^{4}\{\partial\theta(K-1)^n\}$$

are defined for $n \geq m$. We have the usual relations

$$\delta\varphi^{(n)} = \tau_s\{[\cancel{\delta + \theta + z\sigma}, (K-1)^n] - [\theta + \cancel{z\sigma}, (K-1)^n]\}$$

$$= \tau_s\{-[\theta, (K-1)^n]\} = \beta\{\tau_s^{4}\{\partial\theta(K-1)^n\}\} = \beta\psi^{(n)}$$

$$\delta\psi^{(n)} = \tau_s^{4}\{[\delta + \theta + z\sigma, \partial\theta(K-1)^n]\}$$

$$= \tau_s^{4}\{[\cancel{\delta + \theta} + z\sigma, \partial\theta](K-1)^n\}$$

$$= \tau_s^{4}\{\partial[z\sigma, \theta](K-1)^n\}$$

$$= \tau_s^{4}\{\partial([z\sigma, \theta] + z^2 - 1)(K-1)^n\}$$

$$= \tau_s^{4}\{\partial(K-1)(K-1)^n\} = \bar\partial\left\{\tau_s\left(\frac{(K-1)^{n+1}}{n+1}\right)\right\} = \frac{1}{n+1}\bar\partial\varphi^{(n+1)}$$

To simplify the notation write $K - 1 = A + B$ where $A = [z\sigma, \theta]$, $B = z^2 - 1$ are respectively the covariant derivative and curvature terms [redacted] respectively. We [redacted] would like to assemble the $\varphi^{(n)}$ and $\psi^{(n)}$ in some way so as to obtain cyclic cocycles.

Some questions. In the z setup, the curvature term z^2-1 is not nilpotent, nor is $[z,\theta]$, so there is a question as to whether one has periodic cocycles, i.e. big cocycles of finite degree, with the nice form built out of $\partial\theta$, A, B.

Suppose $z^2=1$. Is the parity correct? This case should be checked carefully.

$$[z\sigma,\theta] = \sigma[z,\theta]$$

$$[z\sigma,\theta]^2 = -[z,\theta]^2$$

$$\varphi^{(2n-1)} = \tau_s\left(\sigma[z,\theta]\right)^{2n-1} = (-1)^{n-1}\,\tau_s\left(\sigma[z,\theta]^{2n-1}\right)$$

Now $\tau_s = \left(\begin{smallmatrix}\text{trace}\\ \text{on } C_1\end{smallmatrix}\right)\otimes\tau$ on $C_1\otimes I^m$, where τ is even. So the only nonzero possible cochains are

$$\varphi^{(2n-1)} = (-1)^{n-1}\,\tau\left([z,\theta]^{2n-1}\right)$$

$$\psi^{(2n-1)} = (-1)^{n}\,\tau^{\natural}\left(\partial\theta\,[z,\theta]^{2n-1}\right)$$

But the former is zero because $[z,\theta]$ anti-commutes with z when $z^2=1$. So we are left with the Hochschild cochains

$$\psi_{2n} = \tau^{\natural}\left(\partial\theta\,[z,\theta]^{2n-1}\right)$$

which are Hochschild cocycles (in general) and in fact cyclic cocycles, since

$$\beta\,\psi^{(2n-1)} = \delta\,\varphi^{(2n-1)} = 0.$$

Here's why $\tau^4(\partial\theta[\tau,\theta]^n)$ is always a Hochschild cocycle:
In general if $D: A \longrightarrow M$ is a derivation we have

$$\delta(D\theta) = D(-\theta^2) = -D\theta\,\theta - \theta\, D\theta$$

or

$$[\delta + \theta, D\theta] = 0$$

So

$$\delta\, \tau^4(\partial\theta(D\theta)^n) = \tau^4([\delta+\theta, \partial\theta(D\theta)^n]) = 0$$

as $\delta + \theta$ kills both $\partial\theta$ and $D\theta$. What we are dealing with is a cup product of 1-dimensional Hochschild cocycles.

September 4, 1989

Let $Q = Q^+ \oplus Q^-$ be a superalgebra and form the superalgebra tensor product with C_1. As an algebra

$$Q \hat{\otimes} C_1 = Q \tilde{\otimes} \mathbb{C}[\gamma]$$

where $\gamma x \gamma = (-1)^{|x|} x$ for $x \in Q$.

We want to describe traces and supertraces on $R = Q \hat{\otimes} C_1$.

First note that the grading $x \longmapsto (-1)^{|x|} x$ is an automorphism of any superalgebra. Thus there is an action of $\mathbb{Z}/2 \times \mathbb{Z}/2$ on [redacted] R. We can decompose traces and supertraces with respect to this action into 4 types.

Let ε_Q ε_C be the gradings of Q and C_1 respectively, so that $\varepsilon_Q \otimes \varepsilon_C$ is the grading for R. As usual we write ε_Q for $\varepsilon_Q \otimes 1$ and $\varepsilon_Q \varepsilon_C$ for $\varepsilon_Q \otimes \varepsilon_C$.

It is easy to describe supertraces on R since we know that (assuming Q unital)

$$Q \hat{\otimes} C_1 / [\ ,\ ]_s = Q/[Q,Q]_s \otimes \underbrace{C_1/[C_1, C_1]}_{\parallel \atop \mathbb{C}F}$$

More precisely if τ_s is a supertrace on R, then for $x \in Q$ we have

$$\tau_s(x) = (-1)^{|x|} \tau_s(\gamma x \gamma) = (-1)^{|x|} (-1)^{|\gamma x|} \tau_x(x \gamma \gamma)$$
$$= -\tau_s(x)$$

so $\tau_s(Q) = 0$. Thus $\tau_s(x + y\gamma) = \varphi(y)$ where φ is a linear functional on Q which is a supertrace on Q:

$$\varphi(xy) = \tau_s(xy\gamma) = (-1)^{|y|}\tau_s(x\gamma y)$$
$$= (-1)^{|y|}(-1)^{|y|(|x|+1)}\tau_s(yx\gamma) = (-1)^{|x||y|}\varphi(yx)$$

Conversely given a supertrace φ on Q we check that

$$\tau(x + y\gamma) = \varphi(y)$$

defines a supertrace on R.

$$\tau([x,x']_s) = \tau(xx') \mp \tau(x'x) = 0$$
$$\tau([y\gamma, y'\gamma]_s) = \tau(\pm yy') \mp \tau(\pm y'y) = 0$$
$$\tau([x, y\gamma]_s) = \tau(xy\gamma) - (-1)^{|x|(|y|+1)}\tau(y\gamma x)$$
$$= \varphi(xy) - (-1)^{|x|(|y|+1)}(-1)^{|x|}\tau(yx\gamma)$$
$$= \varphi(xy) - (-1)^{|x||y|}\varphi(yx) = \varphi([x,y]_s) = 0.$$

Summary. Supertraces on $Q \hat{\otimes} C_1$ are supported on QF (better: vanish on Q) and are in one-one correspondence with supertraces on Q. The space of supertraces decomposes into two eigenspace for the action of the group $\mathbb{Z}/2 \times \mathbb{Z}/2$ with generators $\varepsilon_Q, \varepsilon_C$, namely where $\varepsilon_Q = +1$, $\varepsilon_C = -1$ and where $\varepsilon_Q = -1$, $\varepsilon_C = -1$. These are respectively the spaces of odd and even supertraces on $Q \hat{\otimes} C_1$, and they correspond to even and odd supertraces on Q.

Next we consider R as an algebra only and we describe traces on this algebra. Note that ε_Q is the inner automorphism given by γ, hence ε_Q acts trivially on the space of traces and therefore the space of traces decomposes into two eigenspaces where $\varepsilon_Q = 1, \varepsilon_C = 1$ and where $\varepsilon_Q = 1, \varepsilon_C = -1$.

Notice that a trace in the second space is odd relative to the total grading $\varepsilon_Q \varepsilon_C$ of $Q \hat{\otimes} C_1$, and conversely. So the second type of traces (those vanishing on Q) are just the odd (ordinary) traces on the superalg $Q \hat{\otimes} C_1$. But odd traces = odd supertraces on a superalgebra, so we conclude that traces of the second type are in one-one correspondence with even supertraces on Q.

Let τ be a trace of the first type, i.e. vanishing on $Q\gamma$. Then

$$\tau(x + y\gamma) = \varphi(x)$$

where φ is a linear functional on Q. Clearly φ is a trace on Q. Also

$$\varphi(x) = \tau(x) = \tau(\gamma^2 x) = \tau(\gamma x \gamma)$$
$$= (-1)^{|x|}\tau(x) = (-1)^{|x|}\varphi(x)$$

so φ is a trace on Q vanishing on Q^-. Conversely given such a φ, we check that $\tau(x+y\gamma) = \varphi(x)$ is a trace on R.

$$\tau([x, x']) = \tau(xx' - x'x) = \varphi(xx') - \varphi(x'x) = 0$$

$$\tau([x, y\gamma]) = \tau(xy\gamma - y\gamma x) = 0 \quad \text{as } \tau(Q\gamma) = 0$$

$$\tau([y\gamma, y'\gamma]) = \tau(y\gamma y'\gamma - y'\gamma y \gamma)$$
$$= \tau\left((-1)^{|y'|} yy' - (-1)^{|y|} y'y\right)$$
$$= (-1)^{|y'|}\varphi(yy') - (-1)^{|y|}\varphi(y'y)$$

Because φ is a trace on Q this vanishes when y, y' have the same parity. When they have opposite parity then yy' and $y'y$ are odd and so $\varphi(yy') = \varphi(y'y) = 0$ as $\varphi(Q^-) = 0$.

<u>Summary</u>: (Ordinary) traces on $Q \hat{\otimes} C_1$ vanish on $Q^- + Q^-\gamma$ and decompose into traces where $\varepsilon_Q = 1$, $\varepsilon_C = 1$ and where $\varepsilon_Q = 1$, $\varepsilon_C = -1$. The former are in one-one correspondence with even traces on Q via the formula $\tau(x + y\gamma) = \varphi(x)$. The latter are in 1-1 correspondence with even supertraces on Q via the formula $\tau(x+y\gamma) = \varphi(y)$.

Recall that the category of right supermodules over C_1 is equivalent to the category of vector spaces. In effect a supermodule E is of the form $E = H \otimes C_1$, where $H = E^+$. Put another way, there is a unique identification $E^+ = E^-$ such that right multiplication by γ is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

~~However~~ the category of right supermodules over C_1 is more than just a category, because not only do we have even degree maps $E_1 \to E_2$ but we also have odd degree maps. For example Right mult. by γ is an odd endom. of E as right supermodule over C_1.

Thus for two objects E_1, E_2 of the

category we have the full $\mathrm{Hom}(E_1, E_2)$ which is a supervector space. In fact

$$\underset{C_1\text{-rt}}{\mathrm{Hom}}(H_1 \otimes C_1,\ H_2 \otimes C_1) = \mathrm{Hom}(H_1, H_2) \otimes C_1$$

and

$$\underset{C_1\text{-rt}}{\mathrm{End}^{\cdot}}(H \otimes C_1) = \mathrm{End}_{\mathbb{C}}(H) \otimes C_1$$

Consider next supertraces. Recall that the usual trace formally comes out of duality considerations. It's possible that similar things hold in the supercategory, and there is a canonical supertrace on $\underset{C_1\text{-rt}}{\mathrm{End}^{\cdot}}(E)$ at least when E is finite dimensional. Such a thing would have to be the usual trace on $\mathrm{End}_{\mathbb{C}}(H)$ combined with $C_1 \to C_1/[C_1, C_1]_s = \mathbb{C}\varepsilon$. The interesting point is that this trace is supported on odd degree endomorphisms.

The preceding discussion is motivated by Fredholm modules in the odd case. As defined by Connes one considers a $\mathbb{Z}/2$-graded Hilbert space E, which is a right C_1-supermodule. We have A acting on the left by operators of degree 0 (assuming A is an ordinary algebra), and we have an odd operator Z on E such that $[Z, A]$, $Z^2 - 1$ are compact. Z, A commute with the right C_1-action.

Then assuming p-summability we can use a, $[Z, a]$, $z^2 - 1$ to form trace class operators on E commuting with the right C_1-action. Thus we have operators in $\mathcal{L}^1(H) \otimes C_1$ and we take the natural supertraces.

I think these constructions explain my device of introducing $z\sigma$ and a supertrace which sees only odd powers of σ. The whole thing would be clearer if one wrote γ instead of σ and identified $x + y\sigma$ for $x, y \in \mathcal{L}(H)$ with the operator $\begin{pmatrix} x & y \\ y & x \end{pmatrix} = x \cdot I + y \cdot \gamma$ on $H \otimes C_1$.

Program: To unify the graded + ungraded versions of the ~~z~~-setup.

Graded case: Here we form $R = \tilde{A} * \mathbb{C}[z]$ and regard it as a superalgebra, where $\tilde{A}$ is even and z is odd. In a Fredholm module situation we have R acting on $H = H^+ \oplus H^-$ and we obtain a supertrace on $I^m \subset R$ from tr ε on $\mathcal{L}^1(H)$.

Ungraded case: Let R be the same. In an ungraded Fredholm module ~~situation~~ with ungraded Hilbert space H, let R act on $H \otimes C_1 = H^{\oplus 2}$ by sending
$$a \longmapsto a \otimes 1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad \text{and}$$
$$z \longmapsto z \otimes \gamma = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$$
Then we get a superalgebra map $R \longrightarrow \mathrm{End}_{C_1 \, rt}(H \otimes C_1) = \mathcal{L}(H) \otimes C_1$, and we pull back the canonical

supertrace.

When we come to the cochain formalism we must use the fact z is odd, and for this reason it might be useful to put back $\sigma^{(\alpha\gamma)}$ into the formalism. This is very natural in the ungraded case as we have seen, and we've already used it in the even case with superconnections. Recall that if L is a superalgebra we have a superalgebra embedding

$$L = L^+ \oplus L^- \simeq L^+ \oplus L^-\sigma \subset \hat{L} \otimes_g C_1$$

where $\hat{L}$ signifies L as an algebra and $\otimes$ is the ordinary tensor product.

It should be possible to remember that z is odd and avoid introducing σ. We are therefore working with the superalgebra $\tilde{A} * \mathbb{C}[z]$ with z odd, $\tilde{A}$ even, the ideal I generated by $[z, A]$ and $z^2 - 1$, and supertraces τ either odd or even defined on a power I^m. We use the trick of letting τ have values in $\mathbb{C}[1]$ in the case of a supertrace supported on R^-, so as to make τ of even degree.

We now want to construct cyclic cocycles on A using τ and the I-valued cochains $[z, \theta]$ and $z^2 - 1$.

Put $X = [z, \theta]$, $Y = z^2 - 1$ and recall

that the "superconnection" $\delta+\theta+z$ has
curvature

$$K=(\delta+\theta+z)^2 = [z,\theta]+z^2$$

so that $K-1 = [z,\theta]+(z^2-1) = X+Y$

Recall that the sequence

$$\varphi^{(n)} = \tau\{(K-1)^n\} \qquad \psi^{(n)} = \tau^4\{\partial\theta\,(K-1)^n\}$$

satisfy
$$\begin{cases} \delta\varphi^{(n)} = \beta\,\psi^{(n)} \\ \delta\psi^{(n)} = \frac{1}{n+1}\,\bar\partial\varphi^{(n+1)} \end{cases}$$

The proof of these relations is based on the "Bianchi identity"

$$[\delta+\theta+z,\ K-1] = 0$$

i.e
$$[\delta+\theta+z,\ [z,\theta]+(z^2-1)] = 0$$

In fact one has the stronger relations

$$[\delta+\theta,\ [z,\theta]] = 0$$

$$(*)\qquad [\delta+\theta,\ z^2-1] + [z,[z,\theta]] = 0$$

$$[z,\ z^2-1] = 0$$

Check of the second relation

$$[z,[z,\theta]] = [z,\ z\theta+\theta z] = z(z\theta+\theta z)-(z\theta+\theta z)z$$

$$= z^2\theta-\theta z^2 = [z^2,\theta] \quad \left(= -[\delta+\theta,z^2-1]\right)$$

(or using Jacobi $\quad [z,[z,\theta]] = [\underbrace{[z,z]}_{2z^2},\theta]-[z,[z,\theta]]$).

These relations $(*)$ are formally similar to what we encounter with connections

$$[\delta+\theta,\ [\nabla,\theta]] = 0$$

$$[\delta+\theta,\ \nabla^2] + [\nabla,[\nabla,\theta]] = 0.$$

$$[\nabla,\nabla^2] = 0.$$

So we have reached a setup very similar to the sort of things we saw with connections or Dirac operators.

If a connection is flat or more generally its curvature is a scalar, then we have a bigraded differential algebra

$$(\delta+\theta+d)^2 = d\theta$$

encountered in the study of Chern-Simons forms. Here $d \leftrightarrow \nabla$.

September 6, 1989

[redacted] Our earlier discussion about traces and supertraces on $Q\hat{\otimes}C_1$ is somewhat beside the point, as it doesn't apply [redacted] directly to Fredholm modules. The point is that although supertraces on QA are relevant to cyclic theory, and supertraces on QA are equivalent to supertraces on $QA\hat{\otimes}C_1$, what actually occurs is the different superalgebra structure on $QA\hat{\otimes}C_1$, where QA is even. [redacted] One sees this clearly in the graded case, where one has $QA\hat{\otimes}C_1 \longrightarrow \mathcal{L}(H)$ [redacted] compatible with this grading and the supertrace $\mathrm{tr}\,\varepsilon$.

[redacted]

Here's the good approach

<u>Graded case:</u> Take $\mathrm{tr}\cdot\varepsilon$ and restrict it to QA, more precisely the appropriate ideal. [redacted] This gives a trace on QA which is the difference of the traces of the operators on H^+ and H^-. It is an odd trace because

$$\mathrm{tr}(\varepsilon x) = \mathrm{tr}(\varepsilon F^2 x) = -\mathrm{tr}(\varepsilon F x F) = (-1)^{|x|+1}\mathrm{tr}(\varepsilon x)$$

But an odd trace is the same as an odd supertrace.

<u>Ungraded case:</u> Take $\mathrm{tr}\cdot F$ and restrict to QA. This gives a linear functional satisfying

$$\mathrm{tr}(Fxy) = \mathrm{tr}(yFx) = (-1)^{|y|}\mathrm{tr}(Fyx) \qquad x,y\in QA$$

Taking $y=1$ we see $\tau(x)=\mathrm{tr}(Fx)$ [redacted] vanishes on QA^-, whence it is an even supertrace on QA.

Next we have to discuss the big cyclic cocycles associated to these supertraces.

This is all confused and perhaps it the

wrong viewpoint. It seems that there is no significance to the $\mathbb{Z}_2$-grading on QA on the operator level. What is the generalization to the z-setup?

So let's return to the z-setup. Consider the <u>ungraded case</u> where initially we have A, z operating on H. Then we extend these to operators $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ $\quad z \mapsto z\gamma = \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}$ on $H \otimes C_1$ obtaining a map

$$\tilde{A} * \mathbb{C}[z] \longrightarrow \mathcal{L}(H) \otimes C_1$$

compatible with the $\mathbb{Z}/2$ grading, where z is considered to be odd. On $\mathcal{L}^1(H) \otimes C_1$ we have the odd trace $=$ **odd supertrace**

$$x + y\gamma \longmapsto tr(y)$$

which we can pull back to get an <u>odd trace</u> defined on an ideal in $\tilde{A} * \mathbb{C}[z]$.

Consider next the <u>graded case</u>, where we have a map

$$\tilde{A} * \mathbb{C}[z] \longrightarrow \mathcal{L}(H)$$

compatible with $\mathbb{Z}/2$-grading. We can then pull back $tr\,\varepsilon$ to get <u>an even supertrace</u> defined on an ideal in $\tilde{A} * \mathbb{C}[z]$. Notice that if we let $R = \tilde{A} * \mathbb{C}[z] = R^+ \oplus R^-$, then $R^- = zR^+ = R^+z$, and an even supertrace is simply a trace τ on R^+ vanishing on $[z, R^-]_+$. In effect if $x, y \in R^+$, then

$$\tau(zx \cdot zy) = -\tau(xzyz) \qquad \text{as } \tau([z, xzy]_+) = 0$$

$$= -\tau(zy \cdot zx) \qquad \text{as } \tau \text{ is a trace on } R^+$$

Next we consider cocycles in the case where $z^2 = 1$. We know already that the sequence of cochains

$$\psi_{2n} = \tau^{4}\left(\partial\theta\,[z,\theta]^{2n-1}\right)$$

$$\varphi_{2n-1} = \tau\left([z,\theta]^{2n-1}\right)$$

in the ungraded case satisfy the big cocycle conditions

$$\delta\varphi_{2n-1} = \beta\,\psi_{2n} \qquad \delta\psi_{2n} = \bar\partial\,\varphi_{2n+1}$$

However more is true when $z^2=1$, namely the φ's are zero and the ψ's are cyclic cocycles.

$$\varphi_{2n-1} = \tau\left([z,\,\theta[z,\theta]^{2n-2}]\right) = 0$$

We now show that the [redacted] cyclic cocycles ψ_{2n} are related by the S-operation. ~~Starting~~ with ψ_{2n} we lift it via $\bar\partial$ to a φ_{2n} etc. [redacted] Since $z^2=1$ we have the identity

$$\tau\left(z[z,\theta]^{2n}\right) = \tau\left(\underbrace{(\theta + z\theta z)[z,\theta]}_{}{}^{2n-1}\right)$$
$$= 2\tau\left(\theta[z,\theta]^{2n-1}\right)$$

where we use $[z,[z,\theta]] = [z^2,\theta] = 0$ and the fact that $[z,\theta]$ is even. Let

$$\varphi_{2n} = \tau\left(z[z,\theta]^{2n}\right)$$

Then

$$\bar\partial\varphi_{2n} = \tau^{4}\left(z\sum_{i=1}^{2n}[z,\theta]^{i-1}[z,\partial\theta][z,\theta]^{2n-i}\right)$$

$$= 2n \quad \tau^\natural\big(\underbrace{z[z,\partial\theta]}_{\partial\theta + z\partial\theta z}\,[z,\theta]^{2n-1}\big)$$

$$= 4n \quad \tau^\natural\big(\partial\theta\,[z,\theta]^{2n-1}\big) = 4n\ \psi_{2n}$$

also

$$\delta\varphi_{2n} = \tau\big(\delta(z[z,\theta]^{2n})\big)$$

$$= \tau\big(-z(-[\theta,[z,\theta]^{2n}])\big)$$

$$= \tau\big(z[\theta,[z,\theta]^{2n}]\big) = \tau\big([z\theta,[z,\theta]^{2n}]\big)$$

$$= -\beta\ \underbrace{\tau^\natural\big(\partial(z\theta)\,[z,\theta]^{2n}\big)}_{\psi_{2n+1}}$$

Finally $\delta\psi_{2n+1} = \tau^\natural\big([\delta+\theta, z\partial\theta[z,\theta]^{2n}]\big)$

$$= \tau^\natural\big(\cdot[\theta,z]\,\partial\theta\,[z,\theta]^{2n}\big) = \tau^\natural\big(\partial\theta[z,\theta]^{2n+1}\big) = \psi_{2n+2}$$

This calculation shows that

$$\boxed{S[\psi_{2n}] = \frac{-1}{4n}[\psi_{2n+2}]}$$

[redacted]

In the above calculation we [redacted] uses the fact that $z^2 = 1$. Suppose we try to isolate this hypothesis. Instead let's look at the weaker condition that z^2 [redacted] is central, and ask what happens.

The [redacted] point is that the cochains

$$\psi_{n+1} = \tau^\natural\big(\partial\theta\,[z,\theta]^n\big) \qquad \varphi_n = \tau\big([z,\theta]^n\big)$$

are such that the ψ's are cyclic cocycles and the φ's are zero. (Here I don't make assumptions about the parity of τ).

To understand this better let's put $d = \mathrm{ad}(z)$ and note that

$$\psi_{n+1} = \tau^{\natural}(\partial\theta \, d\theta^n) \qquad \varphi_n = \tau(d\theta)^n = \tau(d(\theta d\theta^{n-1})) = 0$$

in this notation. [redacted] Since ψ_{n+1} is a cyclic cocycle we can calculate $S[\psi_{n+1}]$ by diagram-chasing, and this will give a different cocycle representing this [redacted] class. The new cocycle will exist without the assumption that $z^2 = 1$. And of course it [redacted] ought to be $\tau^{\natural}(\partial\theta \cdot P_{n,1}(d\theta, -\theta^2))$ up to a constant.

Here's how to do the diagram chasing. We lift $\psi_{n+1} = \tau^{\natural}(\partial\theta \, d\theta^n)$ to a φ_{n+1}, lift via $\bar\partial$. Try

$$\varphi_{n+1} = \tau(\theta \, d\theta^n)$$

Then

$$\bar\partial \varphi_{n+1} = \tau^{\natural}\Big(\partial\theta \, d\theta^n + \theta \sum_{i=1}^{n} d\theta^{i-1} d\partial\theta \, d\theta^{n-i}\Big)$$

$$= \tau^{\natural}(\partial\theta d\theta^n) + \sum_{i=1}^{n} \tau^{\natural}\Big(\cancel{-d(\theta d\theta^{i-1}\partial\theta \, d\theta^{n-i})} + d\theta^i \partial\theta \, d\theta^{n-i}\Big)$$

$$= (n+1)\,\tau^{\natural}(\partial\theta \, d\theta^n) = (n+1)\,\psi_{n+1}$$

Next

$$\delta \varphi_{n+1} = \tau(-\theta^2 d\theta^n + \theta[\theta, d\theta^n])$$

$$= -\tau(\theta \, d\theta^n \theta)$$

and we have to lift this via β to a ψ_{n+2}. Set

$$\psi_{n+2} = \tau^{\natural}\Big(\partial\theta \sum_{i=0}^{n} d\theta^{i}\,\theta\, d\theta^{n-i}\Big)$$

Then

$$-\beta\psi_{n+2} = \tau\Big([\theta, \textstyle\sum d\theta^{i}\theta d\theta^{n-i}]\Big)$$

$$= \sum_{i+j=n} \tau\big(\theta d\theta^{i}\theta d\theta^{j} + d\theta^{i}\theta d\theta^{j}\theta\big)$$

Next

$$d\big(\theta d\theta^{a}\theta\, d\theta^{b}\theta\big) = \left\{\begin{array}{l} d\theta^{a+1}\theta\, d\theta^{b}\theta \\ +\ \theta d\theta^{a}\theta\, d\theta^{b+1} \\ -\ \theta\, d\theta^{n}\theta \end{array}\right\}$$

allows us to replace

$$\tau(\theta d\theta^{i}\theta\, d\theta^{n-i}) + \tau(d\theta^{i+1}\theta\, d\theta^{n-i-1}\theta)$$

for $0 \le i \le n-1$ by $\tau(\theta d\theta^{n}\theta)$. This leaves out $\tau(\theta d\theta^{i}\theta d\theta^{j})$ for $i=n, j=0$ and $\tau(d\theta^{i}\theta\, d\theta^{j}\theta)$ for $i=0, j=n$, so we find

$$-\beta\psi_{n+2} = (n+2)\,\tau(\theta d\theta^{n}\theta) = -(n+2)\,\delta\varphi_{n+1}$$

Finally as $\delta + ad\,\theta$ kills $\partial\theta$, $d\theta$

$$\delta\psi_{n+2} = -\tau^{\natural}\Big(\partial\theta \sum_{i=0}^{n} d\theta^{i}\,\underbrace{[\delta+\theta,\theta]}_{2\theta^{2}-\theta^{2}=\theta^{2}}\,d\theta^{n-i}\Big)$$

we get

$$\delta\psi_{n+2} = \tau^{\natural}\big(\partial\theta P_{n1}(d\theta, -\theta^{2})\big)$$

We observe that the constants work: The properly normalized Hochschild cocycle is $\tau^{\natural}(\partial\theta \frac{d\theta^{n}}{n!})$ and we have

$$\tau^{\natural}\left(\partial\theta \,\frac{d\theta^n}{n!}\right) = \bar\partial\, \tau\left(\theta\,\frac{d\theta^n}{(n+1)!}\right)$$

$$\delta\,\tau\left(\theta\,\frac{d\theta^n}{(n+1)!}\right) = -\tau\left(\theta\,\frac{d\theta^n}{(n+1)!}\,\theta\right)$$

$$= \beta\left\{\tau^{\natural}\left(\partial\theta\, P_{n1}(d\theta,\theta)\right)/(n+2)!\right\}$$

$$\delta\,\tau^{\natural}\left(\partial\theta\, P_{n1}(d\theta,\theta)/(n+2)!\right) = \tau^{\natural}\left(\partial\theta\,\frac{P_{n1}(d\theta,-\theta^2)}{(n+2)!}\right)$$

$$= \frac{1!}{(n+2\cdot 1)!}\,\tau^{\natural}\left(\partial\theta\, P_{n1}(d\theta,-\theta^2)\right)$$

General S-relations for $\boxed{\psi_{n+1} = \tau^{\natural}(\partial\theta[F,\theta]^n)}$ with n the parity of τ:

$$\tau^{\natural}(\partial\theta[F,\theta]^n) = \bar\partial\left\{\frac{1}{2(n+1)}\,\tau\left(F[F,\theta]^{n+1}\right)\right\}$$

$$\delta\left\{\tau\left(F[F,\theta]^{n+1}\right)\right\} = \tau\left([F\theta,\,[F,\theta]^{n+1}]\right)$$

$$= -\beta\left\{\tau^{\natural}\left(F\partial\theta\,[F,\theta]^{n+1}\right)\right\}$$

$$\delta\left\{\tau^{\natural}\left(F\partial\theta[F,\theta]^{n+1}\right)\right\} = \tau^{\natural}\left(\partial\theta[F,\theta]^{n+2}\right)$$

so $$\boxed{S[\psi_{n+1}] = -\frac{1}{2(n+1)}[\psi_{n+3}]}$$

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September 10, 1989

Let's consider a p-summable Fredholm module situation where we have A, F acting on H with $[F,a] \in \mathcal{L}^p(H)$ for all $a \in A$. In the ungraded case we get a homomorphism

$$\begin{array}{ccc} A * \mathbb{C}[F] & \longrightarrow & \mathcal{L}(H) \\ \| & & \\ QA \tilde{\otimes} \mathbb{C}[F] & & \end{array}$$

and the trace on $\mathcal{L}^1(H)$ pulls back to give a trace defined on some power of the ideal generated by the $[F,a]$.

Any trace on $A * \mathbb{C}[F]$ can be split into even and odd traces relative to the $\mathbb{Z}/2$-grading where F is odd and A is even. An even trace is the same as an even trace on QA. An odd trace is of the form

$$\tau(x + yF) = f(y)$$

where f is an even supertrace on QA.

Specifically let τ be a trace on $QA \tilde{\otimes} \mathbb{C}[F]$ and write it $\tau_0 + \tau_1$, where τ_0 ~~[illegible]~~ is even (vanishes on $Q \cdot F$, $Q = QA$) τ_1 is odd (vanishes on Q). Then

$$\tau(x + yF) = \tau_0(x + yF) + \tau_1(x + yF)$$
$$= \tau_0(x) + \tau_1(yF)$$

so that if we put $\tau_1(yF) = f(y)$, we have

$$f(y) = \tau_1(yF) = \tau(yF) = \tau(Fy)$$

<u>Summary</u>: If τ is a trace on $QA \tilde{\otimes} \mathbb{C}[F]$, then the even supertrace f on QA corresponding to the odd part of τ is $f(x) = \tau(Fx)$, $x \in QA$.

This even supertrace f on QA is what determines the big cyclic cocycle attached to τ:

$$\varphi_{2n} = \tau\Big(F\,\big(\overbrace{-(\theta^-)^2}^{\omega}\big)^n\Big)\Big/n! = (-1)^n\,\tau\big(F(\theta^-)^{2n}\big)\big/n!$$

$$\psi_{2n} = \tau^\natural\big(F\,\partial\theta\,\omega^n\big)\big/n! = (-1)^n\,\tau^\natural\big(F\,\partial\theta\,(\theta^-)^{2n}\big)\big/n!$$

Next consider the graded case. We have an alg. homomorphism

$$\begin{array}{ccc} (QA \tilde{\otimes} \mathbb{C}[F]) \tilde{\otimes} \mathbb{C}[\varepsilon] & \longrightarrow & \mathcal{L}(H) \\ \wr\Vert \ \text{(duality)} & & \\ QA \otimes M_2\mathbb{C} & & \end{array}$$

which is equivalent to an alg. homom.

$$QA \longrightarrow \mathcal{L}(H^+)$$

Thus the trace on $\mathcal{L}^1(H^+)$ pulls back to give a (partially-defined) trace on QA. This can be split into even and odd traces. The odd trace is the same as an odd supertrace and it gives rise to the cyclic cocycles.

Starting with tr_{H^+} the corresponding odd trace is $\boxed{f(x) = \frac{1}{2}\,tr_H(\varepsilon x)}$

Deferring the study of the graded case cocycles, let's return to the ungraded case and compare the big cocycle

$$\varphi_{2n} = \tau(F\omega^n)/n!$$

$$\psi_{2n+1} = \tau\varphi(F\partial\theta\,\omega^n)/n!$$

with the sort of thing obtained by Connes using non-commutative differential forms.

Let's go back to A, F acting on H. We form the graded right C_1-module $H \otimes C_1$ whose endomorphism super algebra is the superalgebra $\mathcal{L}(H) \otimes C_1$, where $\mathcal{L}(H)$ is even. We have a homomorphism

$$\theta : a \longmapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a \otimes 1, \quad A \longrightarrow \mathcal{L}(H) \otimes C_1$$

and the odd involution $F \otimes \gamma = \begin{pmatrix} 0 & F \\ F & 0 \end{pmatrix}$.

The operator $\mathrm{ad}(F \otimes \gamma)$ is an odd derivation of $\mathcal{L}(H) \otimes C_1$ whose square is zero; it makes $\mathcal{L}(H) \otimes C_1$ into a $\mathbb{Z}/2$-graded differential algebra. Then we have by the univ. prop of Ω_A a map of DGA ($\mathbb{Z}/2$ graded)

$$\Omega_A \longrightarrow \mathcal{L}(H) \otimes C_1$$

$$a_0\, da_1 \cdots da_n \longmapsto a_0[F, a_1] \cdots [F, a_n] \otimes \gamma^n$$

Finally on $\mathcal{L}^1(H) \otimes C_1$ we have the supertrace

$$\mathrm{tr}_s(x + y\gamma) = \mathrm{tr}_H(y)$$

Thus by working in $\mathrm{Hom}(B(A), \mathcal{L}(H) \otimes C_1)$ with $\delta, \theta, d = \mathrm{ad}(F \otimes \gamma)$ we can construct various cyclic cocycles and prove S-relations. The basic cyclic cocycles are

$$\mathrm{tr}_s^{\natural}\left(\partial\theta\,\frac{d\theta^{k}}{k!}\right) \qquad \text{for } n \text{ odd}$$

which by the definition above for tr_s can be written

$$(*) \qquad \mathrm{tr}_H^{\natural}\left(\partial\theta\,\frac{d\theta^{2n-1}}{(2n-1)!}\right)$$

Now the cyclic cocycles attached to the family φ_{2n}, ψ_{2n} above are

$$\bar\partial\varphi_{2n} = \bar\partial\{\tau(F\omega^{n})/n!\} \qquad \tau = \mathrm{tr}_H$$

Translation: $\rho = \frac{\theta + F\theta F}{2} \; (=\theta^{+})$ $\qquad \alpha = \frac{\theta - F\theta F}{2} \; (=\theta^{-}) = \frac{1}{2}F[F,\theta]$

$\omega = -\alpha^2 = \frac{1}{4}[F,\theta]^2$. Thus

$$\bar\partial\varphi_{2n} = \frac{1}{2^{2n}n!}\,\partial\tau\left(F[F,\theta]^{2n}\right) = \frac{1}{2^{2n}n!}\,\tau^{\natural}\Big(F\sum_{1}^{2n}\overbrace{[F,\theta]^{i-1}[F,\partial\theta][F,\theta]^{2n-i}}^{(-1)^{i-1}(-1)^{(i-1)(2n-i-1)}}\Big)$$

$$= \frac{2n}{2^{2n}n!}\,\tau^{\natural}\left(F[F,\partial\theta][F,\theta]^{2n-1}\right) = \frac{1}{2^{2(n-1)}(n-1)!}\,\tau^{\natural}\left(\partial\theta[F,\theta]^{2n-1}\right)$$

Thus we get the same cyclic cocycles except for normalization, i.e. the constant $(n-1)!$ vs. $(2n-1)!$

This calculation seems to show that using the supertrace on Ω_A obtained from the trace τ_H gives only the cyclic cocycles, and these come with wrong constants.

It seems that there is no way to establish S-relations among the cyclic cocycles (*) within the δ, d, Θ framework with DGA's. Thus using QA is better.

<u>September 16, 1989</u>

It seems our goal is to find an algebra R associated to A such that traces on R give the cyclic cohomology classes on A. Moreover we want traces on R to be essentially equivalent to big cocycles. We have studied the cases where R is the [redacted]iversal extension of A and where $R = QA$, but in these cases we obtain only big cocycles satisfying symmetry conditions.

A possible approach is suggested by Tomita-Takesaki-KMS theory. [redacted] In this theory one has a weight on a nice $*$ algebra (von Neumann algebra?). A weight is a ~~state~~ φ with nice properties such as being "faithful". Attached to φ is a 1-parameter group of automorphisms of A related to φ by a KMS condition. According to Connes the cross product of A and $\mathbb{R}$ defined by this autom. group carries a trace.

[redacted] In entire cyclic theory [redacted] one doesn't have traces around, rather things like $\operatorname{tr}(e^{uX^2} [redacted] \alpha)$. So it seems [redacted] important to study the TT-KMS-Connes theory, and especially keep in mind this idea that there should be some sort of canonical trace after making a cross product.

The prototype for a state is $\mathrm{tr}(Tx)$. Let's analyze this.

Example: Let $A = M_n$ and let $\rho : A \longrightarrow k$ be a linear functional such that $A \xrightarrow{\hat{\rho}} \mathrm{Hom}(A, k)$, where $\hat{\rho}(a) = (\alpha \longmapsto \rho(\alpha a))$, is injective. This is what faithful means. We can write $\rho(x) = \mathrm{tr}(Tx)$ for a unique T. Then ρ faithful means for any $0 \neq a \in A$, that there is an α with $\rho(\alpha a) = \mathrm{tr}(T\alpha a) \neq 0$. Hence $a \neq 0 \Rightarrow \exists\, \alpha$ with $\mathrm{tr}(aT\alpha) \neq 0$ $\Longleftrightarrow$ $aT \neq 0$. For matrices $a \neq 0 \Rightarrow aT \neq 0$ if and only if $\underline{T \text{ is invertible}}$. So we conclude the faithful states are ~~of~~ the form $x \mapsto \mathrm{tr}(Tx)$ with T invertible.

If $\varphi(x) = \mathrm{tr}(\gamma^{-1}x)$, then

$$\varphi(xy) = \mathrm{tr}(\gamma^{-1}xy) = \mathrm{tr}(y\gamma^{-1}x)$$

$$= \mathrm{tr}(\gamma^{-1}\gamma y \gamma^{-1} x) = \varphi(\gamma(y)x)$$

where $\gamma(y) = \gamma y \gamma^{-1}$. Let's write this in J+O's form

$$\varphi(xy) - \varphi(y^{\gamma}x) = 0$$

This is a kind of modified trace condition we can write down given an automorphism $x \longmapsto x^{\gamma}$ of A. Apparently

there is a modified [redacted]
Hochschild complex with B operator in this situation.

We can consider the A-bimodule, denote it ${}_\gamma A$ which is given by A with the usual right multiplication and with left multiplication twisted by γ. Thus

$$a * a' = a^\gamma a'$$

Then ${}_\gamma A \otimes_A \mathcal{B}(A) \otimes_A$ is the complex which in degree n consists of $A^{\otimes(n+1)}$ with differential

$$\tilde{b}(a_0, \dots, a_n) = b'(a_0, \dots, a_n) + (-1)^n (a_n^\gamma a_0, a_1, \dots, a_{n-1})$$

If $a \mapsto a^\gamma$ is an inner automorphism $a^\gamma = \gamma a \gamma^{-1}$, then we have a bimodule isomorphism

$$u : A \xrightarrow[\sim]{x \mapsto \gamma x} {}_\gamma A$$

since $u(ax) = \gamma a x = a^\gamma \gamma x = a^\gamma u(x)$. Thus the twisted Hochschild complex is isomorphic to the usual Hochschild complex.

Next let's consider the B operator in the normalized Hochschild complex.

$$B(a_0, \dots, a_n) = \sum_{i=0}^{n} (-1)^{in} (1, a_i, \dots, a_n, a_0, \dots, a_{i-1})$$

We would like to find a corresponding operator $\tilde{B}$ to go with $\tilde{b}$. What should $\tilde{B}(a)$ be?

$$\tilde{b}(\gamma, a) = \gamma a - a^\gamma \gamma = \gamma a - \gamma a \gamma^{-1} \gamma = 0$$

$$\tilde{b}(1,a) = a - a^{\gamma}$$

$$\tilde{b}(a,1) = a - a = 0$$

$$\tilde{b}(a,\gamma) = a\gamma - \gamma^{\gamma} a = a\gamma - \gamma a$$

It seems the only thing one can write down that will be killed by $\tilde{b}$ is (γ, a) or $(\gamma,a) - (a,1)$, the later being appropriate to the unnormalized complex, where $B = (1-\lambda)sN$. Thus although the twisted Hochschild complex is defined, it seems that B is not defined unless the automorphism γ is inner.

TT theory in the simplest case. Let φ be a faithful state on A. Then we have two inner products on A given by

$$\langle x|y\rangle = \varphi(x^* y) \qquad (x|y) = \varphi(y x^*)$$

The former gives upon completion a Hilbert space on which acts by a $*$ repn given by left multiplication

$$\langle ax|y\rangle = \varphi((ax)^* y) = \varphi(x^* a^* y) = \langle x|a^* y\rangle$$

The latter gives a right $*$ representation of A;

$$(xa|y) = \varphi(y(xa)^*) = \varphi(y a^* x^*) = (x|ya^*).$$

Call these Hilbert spaces H_ℓ and H_r. Note that because A sits inside both as a dense subspace we have densely defined maps

$$H_\ell \rightleftarrows H_r$$

I suspect the TT analysis shows these densely defined operators can be closed (recall that the closure of the graph of a densely defined operator is not necessarily a graph — one usually exhibits an adjoint to prove this.)

Once we have a closed densely defined operator $H_\ell \longrightarrow H_r$ we can take its polar decomposition to get a positive self-adjoint operator on H_ℓ. This perhaps generates the modular automorphism group.

Let's see how this works when $A = M_n$ and $\varphi(x) = tr(Tx)$ with $T > 0$. Note that $\varphi(x^*x) = tr(Tx^*x) = tr(xTx^*) > 0$ for $x \neq 0$, since $xTx^* > 0$.

I think another way to explain what happens is to consider the sesquilear form $(x|y) = \varphi(yx^*)$ relative to the inner product $\langle x|y\rangle = \varphi(x^*y)$. The analysis ought to show that $(x|y) = \langle x|P(y)\rangle$ for some $P > 0$

In the example

$$(x|y) = tr(Tyx^*) = tr(x^*Ty) = tr(x^*(TyT^{-1})T)$$

$$\cancel{= tr(TT^{-1}x^*Ty)} \qquad = tr(Tx^*(TyT^{-1}))$$

$$\cancel{tr(T(TxT^{-1})^*y)} \qquad = \langle x \mid TyT^{-1}\rangle$$

Thus $P(y) = TyT^{-1}$ and $P^{1/2}(y) = T^{1/2}yT^{-1/2}$. It would appear that the modular autom. group is $P^{it}(y) = T^{it}yT^{-it}$. As a check observe that

$$\varphi(xy) = tr(Txy) = tr(TxT^{-1}Ty)$$
$$= tr(Ty(TxT^{-1})) = \varphi(y(TxT^{-1})).$$

Let A be an algebra with an automorphism γ and let $A_\gamma[z]$ be the twisted Laurent polynomial ring with

$$za = a^\gamma z$$

Notice that the multiplication group acts because $A_\gamma[z]$ is graded. Thus we have the automorphism $u_\lambda(\sum a_n z^n) = \sum a_n \lambda^n z^n$.

~~Given a trace τ on $A_\gamma(z)$, we can decompose into homogeneous traces~~

$$\cancel{\tau = \sum_{n \in \mathbb{Z}} \tau_n}$$

~~where~~ $\cancel{\tau_n u_\lambda = \lambda^n \tau_n},$

~~and then~~ $\cancel{\tau_n(\sum a_k z^k) = \tau(a_n z^n)}$?

Let τ be a linear functional on $A_\gamma[z]$. We have

$$\tau\left(\sum a_n z^n\right) = \sum_{n \in \mathbb{Z}} f_n(a_n)$$

where f_n is a sequence of linear functionals on A. Suppose τ is a trace. Since

$$[a, a_n z^n] = (a a_n - a_n \gamma^n(a)) z^n$$

$$[z, a_{n-1} z^{n-1}] = (\gamma(a_{n-1}) - a_{n-1}) z^n$$

it follows that f_n satisfies

$$f_n(ab - b\gamma^n(a)) = 0$$
$$f_n(\gamma(a) - a) = 0$$

and conversely if f_n satisfies these conditions for all n, then τ is a trace.

Thus we see that if φ is a linear fnl satisfying

$$\varphi(ab) = \varphi(b\gamma(a))$$

$$\varphi(\gamma(a)) = \varphi(a)$$

Then we get a canonical trace on $A_\gamma[z]$ given by

$$\tau\left(\sum a_n z^n\right) = \varphi(a_1)$$

Notice that this is a residue-like trace, the sort of thing which might be relevant later.

September 17, 1989

Let φ be a state on a $*$ algebra A, and assume φ is faithful in the sense that the pairing $x, y \longmapsto \varphi(xy)$ is nondegenerate. (This implies injectivity of the maps $A \rightrightarrows A^*$, $a \longmapsto \left(\alpha \mapsto \begin{cases}\varphi(\alpha a)\\ \varphi(a\alpha)\end{cases}\right)$)

Then we have two inner products on A given by

$$\langle x | y \rangle = \varphi(x^* y) \qquad (x|y) = \varphi(y x^*)$$

Assume the latter can be represented in terms of the former in the sense that there is an operator P on A such that

$$(x|y) = \langle x | P(y) \rangle$$

i.e. [redacted]

$$\varphi(y x^*) = \varphi(x^* P(y))$$

Notice that replacing x^* by x, we get

$$\boxed{\varphi(yx) = \varphi(x P(y))}$$

which makes no reference to the $*$ structure.

Example: $\varphi(x) = tr(Tx)$. Then

$$\varphi(yx) = tr(Tyx) = tr(TyT^{-1}Tx)$$
$$= tr(Tx\, TyT^{-1}) = \varphi(x(TyT^{-1}))$$

so $P(y) = TyT^{-1}$.

In the Tomita-Takesaki theory the remarkable fact, [redacted] besides the existence

of P, is that [redacted] conjugation by p^{it} preserves the algebra A. From the Hilbert space picture or examples, it appears silly to work with the unbounded operator P, which is [redacted] conjugation with the embounded operator T. My inclination is to regard P as generating an action of $\mathbb{Z}$ on A, but the von Neumann algebra viewpoint or setup tells one to use p^{it} and to try to recover P by analytic continuation.

Next let's consider Connes' idea of taking the cross product [redacted] of A with $\mathbb{R}$, and thereby obtain[redacted]ing an algebra with a canonical trace.

Actually this perhaps runs into difficulties because of the real + imaginary time business. [redacted] It seems best to work on $\mathbb{R}_{>0}$, i.e. take a Laplace transform viewpoint.

What is $A \times \mathbb{R}$ or $A \times \mathbb{R}_{>0}$? This should be like continuous power series, i.e.

$$\int dt\, f(t)\, z^t \quad \text{instead of} \quad \sum a_n z^n .$$

Then the product is

$$\int dt_1\, f(t_1)\, z^{t_1} \int dt_2\, g(t_2)\, z^{t_2}$$

$$= \iint dt_1\, dt_2\; f(t_1)\, \alpha^{t_1}(g(t_2))\; z^{t_1+t_2}$$

$$= \int dt \left[\int ds\, f(t-s)\, \alpha^{t-s}(g(s))\right] = \int dt \left[\int ds\, f(s)\, \alpha^{s}(g(t-s))\right]$$

Thus we have a twisted convolution product

$$(f*g)(t) = \int ds\, f(s)\, \alpha^s(g(t-s))$$

Next let's consider the sort of trace we have. In the physics we encounter things like

$$\mathrm{tr}\left(a_0 e^{-s_0 H} a_1 e^{-s_1 H} \cdots\cdots a_n e^{-s_n H}\right)$$

where the s_i are ≥ 0 and $s_0 + \cdots + s_n = 1$. We rewrite this as

$$\mathrm{tr}\Big(a_0 \left(e^{-s_0 H} a_1 e^{s_0 H}\right)\left(e^{-(s_0+s_1)H} a_2 e^{(s_0+s_1)H}\right)\cdots$$

$$\cdots\left(e^{-(s_0+s_1+\cdots+s_{n-1})H} a_n e^{(s_0+\cdots+s_{n-1})H}\right) \overbrace{e^{-(s_0+\cdots+s_n)H}}^{1}\Big)$$

$$= \mathrm{tr}\left(e^{-H} a_0\, a_1(t_1)\, a_2(t_2) \cdots a_n(t_n)\right)$$

where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n \leq 1$ $\qquad t_j = s_0 + \cdots + s_{j-1}$.

Here $\quad a(t) = e^{-tH} a\, e^{tH}$.

The way to think perhaps is the following. The cross product algebra is generated by elts. $a z^t$ with $a \in A$, $t \geq 0$. To az^t we associate the operator ae^{-tH}. Then

$$\begin{array}{ccc} z^t a & \longmapsto & e^{-tH} a \\ \| & & \| \\ \alpha^t(a)\, z^t & & \left(e^{-tH} a\, e^{tH}\right) e^{-tH} \end{array}$$

and so $\quad \boxed{\alpha^t(a) = e^{-tH} a\, e^{tH}}$

Finally the sort of trace on the cross product algebra is of the form

$$\int dt\, f(t)\, z^t \longmapsto \varphi(f(1))$$

where φ is a linear functional on A. The properties φ must have are easily found:

$$a_1 z^t a_2 z^{1-t} \longmapsto \varphi(a_1\, \alpha^t(a_2))$$

$$\parallel$$

$$a_2 z^{1-t} a_1 z^t \longmapsto \varphi(a_2\, \alpha^{1-t}(a_1))$$

Thus we need

$$\boxed{\varphi(a_1\, \alpha^t(a_2)) = \varphi(a_2\, \alpha^{1-t}(a_1))}$$

Special cases are

$$\begin{cases} \varphi(\alpha^t(a)) = \varphi(a) \\ \varphi(a_1 a_2) = \varphi(a_2\, \alpha(a_1)). \end{cases}$$

Conversely if these two conditions hold we have

$$\varphi(a_1\, \alpha^t(a_2)) = \varphi(\alpha^t(a_2)\, \alpha(a_1))$$

$$= \varphi(\alpha^t(a_2\, \alpha^{1-t}(a_1))) = \varphi(a_2\, \alpha^{1-t}(a_1))$$

September 23, 1989

Consider square zero extension with lifting

$$0 \longrightarrow M \longrightarrow E \underset{\rho}{\overset{\curvearrowleft}{\longrightarrow}} A \longrightarrow 0$$

Can we find a simple formula for $\rho(a)^n$ in terms of the Hochschild 2-cocycle associated to the extension and lifting.

We might as well consider the case where ρ is universal, whence $E = (QA)^+/I^2$. Lets calculate $\rho(a^n)$ in $(QA)^+$. Recall $\rho(a^n) = (a^n)^+$, better $\rho(a) = a^+$ for $a \in A$. But $(a^n)^+$ can be calculated from a^+, a^-

$$(a^n)^+ = (a^+ + a^-)^n$$

$$= \sum_{l \geq 0} P_{n-2l,2l}(a^+, a^-)$$

Let's calculate using the representation of QA^+ on Ω_A where $\rho(a) = a - dad$

$$(a-dad)1 = a$$

$$(a-dad)a = a^2 - dada$$

$$(a-dad)(a^2 - dada) = \underbrace{a^3 - adada - dad(a^2)}_{} + \text{higher}$$

$$(a-dad)(\quad) = a^4 - a^2dada - ada\,d(a^2) - dad(a^3)$$

Pattern is clear:

$$\boxed{(a-dad)^n 1 = a^n - \sum_{k=1}^{n-1} a^{n-1-k} da\,d(a^k) \quad \text{mod } \Omega^{\geq 2}}$$

Problem. Given $\rho : A \longrightarrow R$ we want to construct a map from the periodic cyclic complex of A (represented by the $\mathbb{Z}/2$-graded complex

$$B^c(A) \underset{\lambda - 1}{\overset{N}{\rightleftarrows}} \bar{B}(A)$$

to the periodic complex $R \rightleftarrows \Omega^1_{R\natural}$.

Consider the periodic cyclic complex of cochains

$$\begin{tikzcd}
\vdots & \vdots \\
\cdot \arrow[u] \arrow[r, shift left] & \cdot \arrow[u] \arrow[l, shift left] \\
\cdot \arrow[u, "-b"] \arrow[r, shift right, "1-\lambda"'] & \cdot \arrow[u, "b'"'] \arrow[l, shift right, "N"'] \\
\cdot \arrow[u, "-b"] \arrow[r, shift left] & \cdot \arrow[u, "b'"'] \arrow[l, shift left]
\end{tikzcd}$$

on one hand, and the complex of continuous linear functionals on the periodic complex of R for the I-adic filtration. We want like a map from the latter to the former:

$$\begin{tikzcd}
R^* \arrow[r] \arrow[d, "c^t"'] & \text{even cochains} \arrow[d, "\text{differential}"] \\
(\Omega^1_{R\natural})^* \arrow[r] \arrow[d, "d^t"'] & \text{odd cochains} \arrow[d, "\text{differential}"] \\
R^* \arrow[r] & \text{even cochains}
\end{tikzcd}$$

candidates

$f \longmapsto f(\omega^n)/n!\,,\ f^\natural(\partial\rho\,\omega^n)/n!$

$g \longmapsto g(\omega^{n-1}d\rho),\ -g(\rho\omega^n d\rho)/n!$ or $g(\mu_{2n-1}),\ g(\rho\mu_{2n-1})/n!$

$c(x\,dy) = -[x,y]$

The various identities show this almost works, e.g.

$$\begin{cases} b'(\omega^n) + (\lambda - 1)(\rho\omega^n) = c(\omega^n d\rho) \\ -b(\rho\omega^n) + \underbrace{(1+\lambda)}_{\text{would need } \frac{1}{n+1}N \text{ instead}}(\omega^{n+1}) = c(-\rho\omega^n d\rho) \end{cases}$$

Similarly if we use the g^μ cochain to go from $g \in (\Omega^1_{R\natural})^*$ to odd cochain then we get a commutative square by the key identity

$$\begin{cases} d(\omega^n) = b'\mu_{2n-1} + (\lambda-1)(-\rho\mu_{2n-1}) \\ d(\rho\omega^n) = (-b)(-\rho\mu_{2n-1}) + \frac{1}{n+1}N(\mu_{2n+1}) \end{cases}$$

In order to find the good map (which ought to exist) from the periodic cyclic complex of A to the periodic complex for R, let's return to old ideas. The map ρ belongs to the DGA $\mathrm{Hom}(B(A), R)$ which is close to $R \otimes S$ where $S = B(A)^*$. The cyclic theory of this tensor product when R, S are free can be understood in terms of

$$\circledast \qquad \left(\Omega^1_{R\natural} \rightleftarrows R\right) \otimes \left(\Omega^1_{S\natural} \rightleftarrows S\right)$$

this being a tensor product of mixed complexes. So we are led to the problem of generalizing S-relations associated to $\rho \in R \otimes S$ to take place in $\circledast$. Recall that we have

$$(**) \quad 0 \longrightarrow \Omega^1_R \otimes \Omega^1_S \longrightarrow \Omega^1_R \otimes (S \otimes S) \oplus (R \otimes R) \otimes \Omega^1_S \to \Omega^1_{R\otimes S} \to 0$$

so the usual S-relations holding for the element $\omega^n/n!$, $d\rho\,\omega^n/n!$ in $R \otimes S$, $\Omega^1_{R\otimes S,\natural}$ have to be refined.

We are supposing R, S unital. We then have an extension $R \otimes S = R * S / J$ where J is generated by $[R, S]$. We have an exact sequence $(F = R * S)$

$$0 \to J/J^2 \to (R\otimes S)\otimes_F \Omega^1_F \otimes_F (R\otimes S) \to \Omega^1_{R\otimes S} \to 0$$

and
$$\Omega^1_F = F\otimes_R \Omega^1_R \otimes_R F \oplus F\otimes_S \Omega^1_S \otimes_S F$$

from which one can identify the above exact sequence with (**). In particular we have
$$J/J^2 = \Omega^1_R \otimes \Omega^1_S .$$

Another fact we need to recall is that the periodic complex for an alg. R with I has an I-adic filtration by periodic subcomplexes:

$$\begin{tikzcd}
{} \arrow[r] & I^m\Omega^1_R\otimes_R \arrow[r] \arrow[d] & {[R,I^m]+I^{m+1}} \arrow[r, "d"] \arrow[d] & I^m\Omega^1_R\otimes_R \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \Omega^1_{R\natural} \arrow[r] \arrow[d] & R \arrow[r, "d"] \arrow[d] & \Omega^1_{R\natural} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & R/I^m\otimes\Omega^1_R\otimes_R \arrow[r] \arrow[d] & R/[R,I^m]+I^{m+1} \arrow[r] \arrow[d] & R/I^m\otimes_R\Omega^1_R\otimes_R \arrow[r] \arrow[d] & {} \\
& 0 & 0 & 0 &
\end{tikzcd}$$

I apply this to the alg $F = R\otimes S$ and the ideal J with $m=1$, to get a periodic complex

(+)
$$\begin{tikzcd}
{} \arrow[r] & (R\otimes S)\otimes_F \Omega^1_F \otimes_F \arrow[r] \arrow[d, equal] & F/[F,J]+J^2 \arrow[r] \arrow[d, equal, "\text{using lifting } R\otimes S \to F"] & {} \\
& (\Omega^1_{R,\natural}\otimes S)\oplus(R\otimes\Omega^1_{S,\natural}) & F/J \oplus J/[F,J]+J^2 \arrow[d, equal] & \\
& & (R\otimes S)\oplus(\Omega^1_{R,\natural}\otimes\Omega^1_{S,\natural}) &
\end{tikzcd}$$

Thus it appears that the preceding periodic cx $\oplus$ is equivalent to $\circledast$. In any case we can take $\rho \in R\otimes S$ and lift to $\tilde\rho \in F$ and then try to work out the S-relations for $\tilde\rho$ in the ~~quotient~~ complex $\oplus$.

Take $\rho \in R\otimes S$ lift it to $\tilde\rho \in F$ form $\tilde\omega^{n} = d\tilde\rho + \tilde\rho^2$ in F, and then $\partial\tilde\rho\, \tilde\omega^n \in \Omega^1_F$. Next project this into

$$F/J \otimes_F \Omega^1_F \otimes_F F/J = (R\otimes S)\otimes_F \left(F\otimes_R \Omega^1_R \otimes_R F \oplus F\otimes_S \Omega^1_S \otimes_S F\right)\otimes_F (R\otimes S)$$

$$= (R\otimes S)\otimes_R \Omega^1_R \otimes_R (R\otimes S) \oplus (R\otimes S)\otimes_S \Omega^1_S \otimes_S (R\otimes S)$$

$$= S\otimes \Omega^1_R \otimes S \oplus R\otimes \Omega^1_S \otimes R$$

and finally into

$$F/J\otimes_F \Omega^1_F \otimes_F = \Omega^1_{R\natural}\otimes S \oplus R\otimes \Omega^1_{S\natural}$$

I claim that the first component is obtained as follows. Put $\partial' =$

$$R\otimes S \xrightarrow{\partial' = \partial\otimes 1} \Omega^1_R \otimes S$$

Then we can take ρ to $\partial'\rho$ and multiply by $\omega^n \in R\otimes S$, using the obvious right $R\otimes S$ module structure on $\Omega^1_R\otimes S$. Then project to $\Omega^1_{R\natural}\otimes S$.

To check this claim we can take $\rho = r\otimes s$, $\omega^n = r'\otimes s'$ to be arbitrary decomposable elts of $R\otimes S$. Take lifting $\tilde\rho = rs$, $\tilde\omega^n = r's'$. Then

$$\partial\tilde\rho\,\tilde\omega^n = \partial(rs)\,r's' = \underset{\displaystyle\downarrow \atop \displaystyle 1\otimes\partial r\otimes sr's'}{\partial r\, s r's'} + r\,\partial s\, r's' \in \underset{\displaystyle\downarrow \atop \displaystyle F\otimes_R\Omega^1_R\otimes F}{\Omega^1_F}$$

$$\begin{array}{ccc}
\downarrow & & \downarrow \\
1\otimes \partial r\otimes(r'\otimes ss') & \in & (R\otimes S)\otimes_R \Omega^1_R \otimes_R (R\otimes S) \\
\downarrow & & \downarrow \\
1\otimes \partial rr'\otimes ss' & \in & S\otimes\Omega^1_R\otimes S \\
\downarrow & & \downarrow \\
q(\partial rr')\otimes ss' & \in & \Omega^1_{R_q}\otimes S
\end{array}$$

But this is the same as

$$\partial'(r\otimes s) = \partial r\otimes s$$

multiplied by $r'\otimes s'$, which gives $\partial rr'\otimes ss'$, followed by projection to $\Omega^1_{R_q}\otimes S$.

Similarly on the other side

$$\begin{array}{ccc}
\partial_{\tilde\rho}\tilde{\omega}'' = \partial r\, s r's' + r\partial s\, r's' & \in & \Omega^1_F \\
\downarrow & & \downarrow \\
r\otimes \partial s\otimes r's' & \in & F\otimes_S\Omega^1_S\otimes F \\
\downarrow & & \downarrow \\
(r\otimes 1)\otimes \partial s\otimes(r'\otimes s') & \in & (R\otimes S)\otimes_S\Omega^1_S\otimes_S(R\otimes S) \\
\downarrow & & \downarrow \\
r\otimes \partial ss'\otimes r' & \in & R\otimes\Omega^1_S\otimes R \\
\downarrow & & \downarrow \\
r'r\otimes \partial ss' & \in & R\otimes\Omega^1_{S_q}
\end{array}$$

which means we take f and apply

$$R\otimes S \xrightarrow{\partial''=1\otimes\partial} R\otimes\Omega^1_S \qquad \partial''(r\otimes s) = r\otimes\partial s$$

then multiply by $\omega''=r'\otimes s'$ on the right, which gives $rr'\otimes\partial ss'$. Thus we don't get the same answer

??

<u>September 24, 1989</u>

<u>Problem.</u> ~~Starting~~ from $\rho \in R \otimes S$ to produce elements

$$e^{\tilde{\omega}} \in R\otimes S + \Omega^1_{R\natural} \otimes \Omega^1_{S\natural}$$

$$\partial_{\tilde{\rho}} e^{\tilde{\omega}} \in \Omega^1_{R\natural}\otimes S + R \otimes \Omega^1_{S\natural}$$

which constitute a cocycle wrt the differentials in $R\otimes S$ and the mixed complex differential $b+B$ for the tensor product of mixed complexes

$$(\Omega^1_{R\natural} \rightleftarrows R) \otimes (\Omega^1_{S\natural} \rightleftarrows S)$$

~~We consider the extension $R\otimes S = F/J$ where $F = R * S$ and~~

By taking the tensor product of the resolutions $(0 \to \Omega^1_R \to R\otimes R \to R \to 0)$ and similarly for S, we obtain an exact sequence

$$1)\quad 0 \to \Omega^1_R \otimes \Omega^1_S \longrightarrow \Omega^1_R \otimes (S\otimes S) \oplus (R\otimes R)\otimes \Omega^1_S \longrightarrow \Omega^1_{R\otimes S} \to 0$$

of $R\otimes S$-bimodules.

We consider the extension $R\otimes S = F/J$ with $F = R * S$. One has an exact sequence and identification

$$2)\quad 0 \longrightarrow J/J^2 \longrightarrow (R\otimes S)\otimes_F \Omega^1_F \otimes_F (R\otimes S) \longrightarrow \Omega^1_{R\otimes S} \longrightarrow 0$$

$$\| $$

$$(R\otimes S)\otimes_F \left(F\otimes_R \Omega^1_R \otimes_R F \oplus F \otimes_S \Omega^1_S \otimes_S F\right) \otimes_F (R\otimes S)$$

$$\|$$

$$S\otimes \Omega^1_R \otimes S \oplus R\otimes \Omega^1_S \otimes R$$

from which we can obtain an identification of the sequences 1) and 2).

We have

$$\begin{tikzcd}
0 \arrow[r] & J/J^2 \arrow[r] \arrow[d, equal] & F/J^2 \arrow[r] \arrow[d, "\partial_F"] & F/J \arrow[r] \arrow[d, "\partial"] \arrow[l, dotted, bend right] & 0 \\
0 \arrow[r] & J/J^2 \arrow[r] \arrow[d, equal] & F/J \otimes_F \Omega^1_F \otimes_F F/J \arrow[r] \arrow[d, equal] & \Omega^1_{F/J} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^1_R \otimes \Omega^1_S \arrow[r] & \underbrace{(\Omega^1_R \otimes S \otimes S) \oplus (R \otimes R \otimes \Omega^1_S)}_{E} \arrow[r] & \Omega^1_{R\otimes S} \arrow[r] & 0
\end{tikzcd}$$

We take the derivation $\partial : R\otimes S \longrightarrow \Omega^1_{R\otimes S}$ and use the lifting (dotted arrow) to lift it to a linear map $\tilde{\partial} : R\otimes S \longrightarrow E$. $\tilde{\partial}$ is not a 1-cocycle; its coboundary is a 2-cocycle on $R\otimes S$ with values in $\Omega^1_R \otimes \Omega^1_S$.

Let's calculate

$$\tilde{\partial}(r\otimes s) = \partial_F(rs) = \partial_F r \cdot s + r \cdot \partial_F s$$

$$= 1\otimes \partial r \otimes s + r \otimes \partial s \otimes 1 \in S\otimes \Omega^1_R \otimes S \oplus R \otimes \Omega^1_S \otimes R$$

$$= \partial r \otimes (1\otimes s) + (r\otimes 1)\otimes \partial s \in (\Omega^1_R \otimes S\otimes S) \oplus (R\otimes R\otimes \Omega^1_S)$$

The ~~associate~~ coboundary of $\tilde{\partial}$:

$$(r_1\otimes s_1)\tilde{\partial}(r_2\otimes s_2) - \tilde{\partial}(r_1r_2\otimes s_1s_2) + \tilde{\partial}(r_1\otimes s_1)(r_2\otimes s_2)$$

$$= (r_1\otimes s_1)\big(\partial r_2 \otimes (1\otimes s_2) + (r_2\otimes 1)\otimes \partial s_2\big)$$
$$-\partial(r_1r_2)\otimes(1\otimes s_1s_2) - (r_1r_2\otimes 1)\otimes \partial(s_1s_2)$$
$$+\big(\partial r_1 \otimes (1\otimes s_1) + (r_1\otimes 1)\otimes \partial s_1\big)(r_2\otimes s_2)$$

$$\cancel{= r_1\partial r_2 \otimes s_1 \otimes s_2 + r_1r_2\otimes \cdots \otimes s_1\partial s_2}$$
$$\cancel{-\partial r_1 r_2 \otimes 1\otimes s_1s_2 - r_1r_2\otimes 1\otimes \partial s_1 s_2} \quad \cancel{= r_1\partial r_2 \otimes \partial s_1 s_2}$$
$$\cancel{-r_1\partial r_2\otimes 1\otimes s_1s_2 - r_1r_2\otimes 1\otimes s_1\partial s_2} \quad \cancel{-r_1\partial r_2 \otimes}$$
$$\cancel{+ \cdots + r_1r_2\otimes 1\otimes \partial s_1 s_2}$$
$$\cancel{\partial r_1 r_2 \otimes 1\otimes s_1s_2}$$

$$
\begin{aligned}
& r_1 \partial r_2 \otimes (s_1 \otimes s_2) + \cancel{(r_1 r_2 \otimes 1)} \otimes s_1 \partial s_2 \\
& - \cancel{\partial r_1 r_2} \otimes (1 \otimes s_1 s_2) - \cancel{(r_1 r_2 \otimes 1) \otimes} \partial s_1 s_2 \\
& - r_1 \partial r_2 \otimes (1 \otimes s_1 s_2) - (r_1 r_2 \otimes 1) \otimes s_1 \partial s_2 \\
& + \cancel{\partial r_1 r_2 \otimes} (1 \otimes s_1 s_2) + (r_1 \otimes r_2) \cancel{\otimes} \partial s_1 s_2 \\
& \qquad (s_1 \otimes 1 - 1 \otimes s_1) \qquad (r_2 \otimes 1 - 1 \otimes r_2) \\
= \; & r_1 \partial r_2 \otimes \partial s_1 s_2 - r_1 \partial r_2 \otimes \partial s_1 s_2
\end{aligned}
$$

Thus the 2-cocycle on $R \otimes S$ with values in $\Omega^1_R \otimes \Omega^1_S$ is

$$\boxed{f(r_1 \otimes s_1, r_2 \otimes s_2) = -r_1 \partial r_2 \otimes \partial s_1 s_2}$$

On the other hand we can compute the 2-cocycle using the extension

$$0 \to J/J^2 \to F/J^2 \rightleftarrows F/J \to 0$$

and lifting $r \otimes s \longmapsto rs$. This gives

$$r_1 s_1 r_2 s_2 - r_1 r_2 s_1 s_2 = r_1 [s_1, r_2] s_2 = -r_1 [r_2, s_1] s_2$$

(and actually the curvature $\delta \rho + \rho^2$ for the lifting is $r_1 [r_2, s_1] s_2$). Now we need the identification $J/J^2 \to \Omega^1_R \otimes \Omega^1_S$ which is induced by ∂_F

$$
\begin{aligned}
\partial_F [r, s] &= \partial_F (rs) - \partial_F (sr) \\
&= [redacted] \; \partial r \otimes (1 \otimes s) - (1 \otimes r) \otimes \partial s \\
&\qquad + (r \otimes 1) \otimes \partial s - \partial r \otimes (s \otimes 1) \\
&= [redacted] \; -\partial r \otimes (s \otimes 1 - 1 \otimes s) + (r \otimes 1 - 1 \otimes r) \otimes \partial s
\end{aligned}
$$

which up to sign [redacted] comes from $\partial r \otimes \partial s$.

Actually the map

$$\Omega^1_R \otimes \Omega^1_S \longrightarrow \Omega^1_R \otimes (S \otimes S) \oplus (R \otimes R) \otimes \Omega^1_R$$

because it is the differential in a tensor product of complexes, must be

$$\partial r \otimes \partial s \longrightarrow (r\otimes 1 - 1\otimes r)\otimes \partial s - \partial r \otimes (s\otimes 1 - 1 \otimes s)$$

Thus the 2-cocycle is

$$f(r_1 \otimes s_1,\ r_2 \otimes s_2) = - r_1 \partial r_2 \otimes \partial s_1 s_2$$

computed as the coboundary of $\tilde{\partial}$.

The identification to use is

$$J/J^2 \xrightarrow{\sim} \Omega^1_R \otimes \Omega^1_S \qquad [r,s] \longleftrightarrow \partial r \otimes \partial s$$

Note: Given square zero ext.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & E & \underset{\xleftarrow{s}}{\longrightarrow} & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \partial_E} & & \downarrow & & \\ 0 & \longrightarrow & M & \xrightarrow[j]{} & R \otimes_E \Omega^1_E \otimes_E R & \longrightarrow & \Omega^1_R & \longrightarrow & 0 \end{array}$$

the coboundary of the cochain $x \longmapsto \partial_E s(x)$ is

$$x\, \partial_E s(y) - \partial_E s(xy) + \partial_E s(x) \cdot y$$

$$= \partial_E \big(s(x) s(y) - s(xy) \big) = j\big(f(x,y)\big)$$

$$\begin{aligned} &(r_1 \otimes s_1)\, \tilde{\partial}(r_2 \otimes s_2) \\ &- \tilde{\partial}(r_1 r_2 \otimes s_1 s_2) \\ &+ \tilde{\partial}(r_1 \otimes s_1)\, r_2 \otimes s_2 \end{aligned} = \begin{aligned} & r_1 \partial r_2 \otimes (s_1 \otimes s_2) + \cancel{(r_1 r_2 \otimes 1) \otimes s_1 \partial s_2} \\ & - \cancel{\partial r_1 r_2 \otimes (1 \otimes s_1 s_2)} - (r_1 r_2 \otimes 1) \otimes \partial s_1 s_2 \\ & - r_1 \partial r_2 \otimes (1 \otimes s_1 s_2) - \cancel{(r_1 r_2 \otimes 1) \otimes s_1 \partial s_2} \\ & + \cancel{\partial r_1 r_2 \otimes (1 \otimes s_1 s_2)} + (r_1 \otimes r_2) \otimes \partial s_1 s_2 \end{aligned}$$

$$= r_1 \partial r_2 \otimes (s_1 \otimes 1 - 1 \otimes s_1) s_2 - r_1 (r_2 \otimes 1 - 1 \otimes r_2) \otimes \partial s_1 s_2$$

$$= j \{ \underbrace{- r_1 \partial r_2 \otimes \partial s_1 s_2}_{f(r_1 \otimes s_1,\ r_2 \otimes s_2)} \}$$

Recall our goal is to produce a refinement of the elements

$$e^{\omega} \in R \otimes S$$

$$\natural(\partial_\rho e^{\omega}) \in \Omega^1_{R\otimes S, \natural}$$

but lying respectively in $(R\otimes S) \oplus (\Omega^1_{R\natural} \otimes \Omega^1_{S\natural})$ and

$$(\Omega^1_{R\natural} \otimes S) \oplus (R \otimes \Omega^1_{S\natural}) = (R\otimes S) \otimes_F \Omega^1_F \otimes_F .$$

Recall that we propose to lift $\rho \in R\otimes S$ up to $\tilde\rho \in F/J^2$, then take

$$e^{\tilde\omega} \quad \text{in} \quad F/J^2 + [F,J] \simeq R\otimes S + \Omega^1_{R\natural} \otimes \Omega^1_{S\natural}$$

$$\natural(\partial_{\tilde\rho} e^{\tilde\omega}) \quad \text{in} \quad F/J \otimes_F \Omega^1_F \otimes_F \simeq \Omega^1_{R\natural} \otimes S + R \otimes \Omega^1_{S\natural}$$

The latter can be calculated by taking $\partial\tilde\rho$ in $\Gamma = F/J \otimes_F \Omega^1_F \otimes_F F/J = \Omega^1_R \otimes (S\otimes S) \oplus (R \otimes R) \otimes \Omega^1_S$ multiplying by $e^{\tilde\omega}$ and then taking the image in $\Omega^1_{R\natural} \otimes S \oplus R \otimes \Omega^1_{S\natural}$. However $\partial\tilde\rho = \tilde\partial\rho$ in Γ and Γ is a $R\otimes S = F/J$- module, so

$$\partial\tilde\rho \cdot e^{\tilde\omega} = \tilde\partial\rho \cdot e^{\omega} \quad \text{in } \Gamma$$

Next we want to calculate $e^{\tilde\omega} \in$

$$F/J^2 = (R\otimes S) \oplus (\Omega^1_R \otimes \Omega^1_S)$$ with twisted mult.

Then

$$\begin{aligned} \tilde\omega &= d\rho + \rho * \rho \\ &= d\rho + \rho^2 + f(\rho,\rho) \\ &= \omega + f(\rho,\rho) \end{aligned}$$

Review: set

$$\Gamma = (R\otimes S)\otimes_F \Omega^1_F \otimes_F (R\otimes S)$$
$$= \Omega^1_R \otimes (S\otimes S) \oplus (R\otimes R)\otimes \Omega^1_S$$

and $\Gamma_\natural = \Gamma \otimes_{R\otimes S} = \Omega^1_{R\natural}\otimes S + R\otimes \Omega^1_{S\natural}$

Notice that for the map $\Gamma \to \Gamma_\natural$ we have $r_1\partial r_2 \otimes (s_1\otimes s_2) \longmapsto \natural(r_1\partial r_2)\otimes s_2 s_1$ and similar on the other side. This is because

$$\Omega^1_R \otimes (S\otimes S) = S\otimes \Omega^1_R \otimes S \longrightarrow \Omega^1_F$$
$$r_1\partial r_2 \otimes (s_1\otimes s_2) \longrightarrow s_1\otimes r_1\partial r_2\otimes s_2 \longmapsto s_1 r_1 \partial r_2\, s_2$$

and so when we pass to commutator quotient we get $s_2 s_1$.

We now wish to compute the maps

$$\Gamma_\natural \xrightarrow{\ \beta\ } F/J^2+[F,J] \xrightarrow{\ \partial\ } \Gamma_\natural$$

in terms of our description of these groups. Let's begin with ∂. We have the description

$$F/J^2 \xleftarrow{\ \simeq\ } (R\otimes S) \oplus (\Omega^1_R \otimes \Omega^1_S)$$
$$rs \longleftarrow r\otimes s$$
$$r_1[r_2,s_1]s_2 \longleftarrow r_1\partial r_2 \otimes \partial s_1\, s_2$$

We have $\partial(rs) = \partial r\otimes 1\otimes s + r\otimes 1\otimes \partial s$ in Γ.

Next $\partial : F/J^2 \longrightarrow \Gamma$ is necessarily a $R\otimes S$-bimodule map when restricted to J/J^2, and

$$\partial(rs) = \partial r\otimes(1\otimes s) + (r\otimes 1)\otimes \partial s$$
$$\partial(sr) = (1\otimes r)\otimes\partial s + \partial r\otimes(s\otimes 1)$$
$$\partial[r,s] = -\partial r\otimes(s\otimes 1 - 1\otimes s) + (r\otimes 1 - 1\otimes r)\otimes \partial s$$

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$$= j(\partial r \otimes \partial s).$$

Conclude $\partial : F/J^2 \longrightarrow \Gamma$ is given by

$$r\otimes s \longmapsto \tilde{\partial}(r\otimes s) = \partial r \otimes (1\otimes s) + (r\otimes 1)\otimes \partial s$$

$$r_1 \partial r_2 \otimes \partial s_1 s_2 \longmapsto j(r_1 \partial r_2 \otimes \partial s_1 s_2)$$

Further $\bar{\partial} : F/J^2 + [F,J] \longrightarrow \Gamma_\natural$ is given by

$$r\otimes s \longmapsto \partial r \otimes s + r \otimes \partial s$$

$$r_1 \partial r_2 \otimes \partial s_1 s_2 \longmapsto \natural\big((r_1 r_2 \otimes 1 - r_1 \otimes r_2) \otimes \partial s_1 s_2\big)$$

$$- \natural\big(r_1 \partial r_2 \otimes (s_1 \otimes s_2 - 1 \otimes s_1 s_2)\big)$$

$$= [r_1, r_2] \otimes \partial s_1 s_2 + r_1 \partial r_2 \otimes [s_1, s_2]$$

$$\in R \otimes \Omega^1_{S\natural} \oplus \Omega^1_{R\natural} \otimes S$$

As a check notice that the last map is defined on $\Omega^1_{R\natural} \otimes \Omega^1_{S\natural}$.

Next we want the bracket map

$$\Gamma_\natural \xrightarrow{\beta} F/J^2 + [F,J]$$

Start with $r_1 \partial r_2 \otimes (s_1 \otimes s_2) \in \Omega^1_R \otimes (S\otimes S)$, lift it to Ω^1_F, say to $s_1 r_1 \partial r_2 s_2$, apply β

$$\beta(s_1 r_1 \partial r_2 s_2) = [s_2 s_1 r_1, r_2]$$

$$\equiv [r_1 s_2 s_1, r_2] \quad \text{mod } [F,J]$$

$$= [r_1, r_2] s_2 s_1 + r_1 [s_2 s_1, r_2]$$

$$\leftrightarrow [r_1, r_2] \otimes s_2 s_1 - r_1 \partial r_2 \otimes \partial(s_2 s_1)$$

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As a check suppose we lifted to the element $r_1 s_1 \partial r_2 s_2$. Then

$$\beta(r_1 s_1 \partial r_2 s_2) = [s_2 r_1 s_1, r_2]$$
$$\equiv [r_1 s_2 s_1, r_2] \quad \mod [F,J]$$

The formula at the end regarded as an element of $R\otimes S + \Omega^1_{R\natural}\otimes\Omega^1_{S\natural}$ clearly depends only on $r_1 \partial r_2 \otimes s_2 s_1 \in \Omega^1_{R\natural}\otimes S$.

Similarly take $(r_1\otimes r_2)\otimes \partial s_1 s_2$, lift to $r_1 \partial s_1 s_2 r_2$ and apply β.

$$\beta(r_1 \partial s_1 s_2 r_2) = [s_2 r_2 r_1, s_1] \equiv [r_2 r_1 s_2, s_1]$$
$$= r_2 r_1 [s_2, s_1] + [r_2 r_1, s_1] s_2$$
$$= -r_2 r_1 \otimes [s_1, s_2] + \partial(r_2 r_1)\partial s_1 s_2$$

So we have for $\beta : \Gamma_\natural \longrightarrow F/J^2+[R,J] = R\otimes S \oplus \Omega^1_{R\natural}\otimes\Omega^1_{S\natural}$ the formulas

$$\boxed{\begin{aligned} \beta(r_1\partial r_2 \otimes s) &= [r_1, r_2]\otimes s - r_1\partial r_2 \otimes \partial s \\ \beta(r\otimes \partial s_1 s_2) &= -r\otimes[s_1,s_2] + \partial r\otimes\partial s_1 s_2 \end{aligned}}$$

We also have from before for $\bar\partial : F/J^2+[R,J] \longrightarrow \Gamma_\natural$

$$\boxed{\begin{aligned} \bar\partial(r\otimes s) &= \partial r\otimes s + r\otimes\partial s \\ \bar\partial(r_1\partial r_2\otimes\partial s_1 s_2) &= [r_1,r_2]\otimes\partial s_1 s_2 + r_1\partial r_2\otimes[s_1,s_2] \end{aligned}}$$

Check:

$$\beta\bar\partial(r\otimes s) = \beta(\partial r\otimes s + r\otimes\partial s)$$
$$= \cancel{[1,r]}\otimes s - \partial r\otimes\partial s - r\otimes\cancel{[s,1]} + \partial r\otimes\partial s = 0$$

$$\beta\bar\partial\left(r_1\partial r_2\otimes\partial s_1 s_2\right) = \beta\left([r_1,r_2]\otimes\partial s_1 s_2 + r_1\partial r_2\otimes[s_1,s_2]\right)$$

$$\left.\begin{aligned} &= -\cancel{[r_1,r_2]\otimes[s_1,s_2]} + \underbrace{\partial[r_1,r_2]}_{0}\otimes\partial s_1 s_2 \\ &\quad + \cancel{[r_1,r_2]\otimes[s_1,s_2]} - r_1\partial r_2\otimes\underbrace{\partial[s_1,s_2]}_{0 \text{ in } \Omega^1_{S\natural}} \end{aligned}\right\} = 0$$

$$\bar\partial\beta(r_1\partial r_2\otimes s) = \bar\partial\left([r_1,r_2]\otimes s - r_1\partial r_2\otimes\partial s\right)$$

$$\left.\begin{aligned} &= \underbrace{\partial[r_1,r_2]}_{0}\otimes s + [r_1,r_2]\otimes\partial s \\ &\quad - \left([r_1,r_2]\otimes\partial s - r_1\partial r_2\otimes\cancel{[s,1]}\right) \end{aligned}\right\} = 0$$

$$\bar\partial\beta\left(r\otimes\partial s_1 s_2\right) = \bar\partial\left(-r\otimes[s_1,s_2] + \partial r\otimes\partial s_1 s_2\right)$$

$$\left.\begin{aligned} &= -\cancel{\partial r\otimes[s_1,s_2]} - r\otimes\overbrace{\partial[s_1,s_2]}^{0} \\ &\quad + \underbrace{[1,r]}_{0}\otimes\partial s_1 s_2 + \cancel{\partial r\otimes[s_1,s_2]} \end{aligned}\right\} = 0$$

Notice that the formulas for

$$\begin{array}{ccc} \Omega^1_{R\natural}\otimes S & & R\otimes S \\ \oplus & \underset{\bar\partial}{\overset{\beta}{\rightleftarrows}} & \oplus \\ R\otimes\Omega^1_{S\natural} & & \Omega^1_{R\natural}\otimes\Omega^1_{S\natural} \end{array}$$

are what one would expect upon taking a tensor product of mixed complexes. For example

$$\bar\partial\left(r_1\partial r_2\otimes\partial s_1 s_2\right) = \beta\left(r_1\partial r_2\right)\otimes\partial s_1 s_2 - \left(r_1\partial r_2\right)\otimes\beta\left(\partial s_1 s_2\right)$$

where the $-$ sign is due to the fact that $r_1\partial r_2$ is odd and β is odd.

This suggests that a single notation should be used for $\beta, \bar\partial$.

$$\beta(r_1\partial r_2) = \beta(\partial r_2\, r_1) = -[r_2,r_1] = [r_1,r_2] \qquad r_i \text{ even}$$

The product in F/J^2 is determined by

$$\boxed{(r_1\otimes s_1) \underset{\wedge}{*} (r_2\otimes s_2) = r_1 r_2 \otimes s_1 s_2 - r_1 \partial r_1 \otimes \partial s_1 s_2}$$

September 25, 1989

Given R, S [redacted] we form $\Omega_R \otimes \Omega_S$ which is a bigraded differential algebra. We denote [redacted] the two differentials by ∂_R and ∂_S. Then in this algebra we have

$$\partial_S(r_1\otimes s_1)\cdot \partial_R(r_2\otimes s_2) = (r_1\otimes \partial s_1)\cdot(\partial r_2 \otimes s_2)$$
$$= - r_1 \partial r_2 \otimes \partial s_1 s_2$$

whence the product in F/J^2 is given by the formula

$$\boxed{x * y = xy + \partial_S x\, \partial_R y}$$

Consider two derivations $X: A \longrightarrow M$, $Y: A \longrightarrow N$ [redacted].

[redacted] Then $(a_1, a_2) \longrightarrow Xa_1\, Ya_2 \in M \otimes_A N$ is a 2-cocycle, and we can form the associated square zero extension $A \oplus M \otimes_A N$ with multiplication determined by

[redacted] $$a_1 * a_2 = a_1 a_2 + Xa_1\, Ya_2$$

The universal case is where $X = d: A \to \Omega_A^1$ and $Y = d: A \longrightarrow \Omega_A^1$ in which case we obtain $A \oplus \Omega_A^2$ with product determined by

$$a_1 * a_2 = a_1 a_2 + da_1 da_2$$

which up to sign is the $*$ product on $\Omega_A^{ev} / \Omega_A^{ev, \geq 2}$ defined before.

Since $a_1, a_2 \longmapsto da_1 da_2 \in \Omega^2_A$ is a universal 2-cocycle, it follows that for the purpose of establishing identities, we can suppose a 2-cocycle is the cup product of two derivations.

We now calculate a^{*n}, the n-fold power of a for the $*$ product.

$$a^{*2} = a^2 + XaYa$$

$$a^{*3} = a^3 + aXaYa + XaY(a^2)$$

$$= a^3 + aXaYa + XaYaa + XaaYa$$

$$a^{*4} = a^4 + a(aXaYa + XaYaa + XaaYa) + XaY(a^3)$$

$$= a^4 + (a^2XaYa + aXaYaa + aXaaYa + XaYaa^2 + XaaYaa + Xaa^2Ya)$$

The general pattern is clear

$a^{*n} = a^n +$ sum of all monomials in a, Xa, Ya of degree $(n-2, 1, 1)$ such that Xa occurs before Ya

$$\boxed{a^{*n} = a^n + \sum_{i+j+k=n-2}{}' \, a^i (Xa) a^j (Ya) a^k}$$

Check this by induction

$$a * (a^n + \textstyle\sum' \cdots\cdots)$$

$$= a^{n+1} + \underbrace{\sum_{i+j+k=n-2} a^{i+1} Xa\, a^j\, Ya\, a^k}_{\sum_{i+j+k=n-1,\ i \geq 1} a^i Xa\, a^j\, Ya\, a^k} + \underbrace{XaY(a^n)}_{Xa \sum_{j+k=n-1} a^j\, Ya\, a^k}$$

OK.

Let's derive an exponential version

$$\exp^*(ta) = e^{ta} + \varphi(t,a)$$

$$\partial_t \exp^*(ta) = a * \exp^*(ta)$$

$$\cancel{a e^{ta}} + \partial_t \varphi(t,a) = \cancel{a e^{ta}} + a\varphi(t,a) + X_a Y(e^{ta})$$

$$(\partial_t - a)\,\varphi(t,a) = X_a Y(e^{ta})$$

$$\partial_t\left(e^{-ta}\varphi(t,a)\right) = e^{-ta} X_a Y(e^{ta})$$

$$\varphi(t,a) = \int_0^t e^{(t-s)a} X_a Y(e^{sa})\, ds$$

$$\exp^*(a) = e^a + \int_0^1 e^{(1-s)a} X_a Y(e^{sa})\, ds$$

$$\exp^*(a) = e^a + \iint_{t_1 \le t_2} e^{(1-t_2)a} X_a e^{(t_2-t_1)a} Y e^{t_1 a}\, dt_1 dt_2$$

$$\boxed{\exp^*(a) = e^a + \iint_{t_0+t_1+t_2=1,\ t_i \ge 0} e^{t_0 a} X_a e^{t_1 a} Y_a e^{t_2 a}\, dt_1 dt_2}$$

~~To check we need~~

$$\cancel{c_n = \sum_{i+j+k=n} \iint_{\Delta(2)} \frac{t_0^i}{i!}\frac{t_1^j}{j!}\frac{t_2^k}{k!}\, dt_1 dt_2}$$

$$\cancel{\sum u^n c_n = \sum_{i+j+k\ge 0} \iint \frac{(ut_0)^i}{i!}\frac{(ut_1)^j}{j!}\frac{(ut_2)^k}{k!}\, dt_1 dt_2}$$

$$\cancel{= \iint_{t_0+t_1+t_2=1} e^{u(t_0+t_1+t_2)}\, dt_1 dt_2 = e^u \iint_{t_0+t_1+t_2=1} dt_1 dt_2}$$

This can be checked using

$$\iint\limits_{\substack{t_0+t_1+t_2=1 \\ t_i\geq 0}} \frac{t_0^i}{i!}\frac{t_1^j}{j!}\frac{t_2^k}{k!}\,dt_1\,dt_2 = \frac{1}{(i+j+k+2)!}$$

which [redacted] can be derived using the convolution theorem for Laplace transforms.

We are now in a position perhaps to calculate the [redacted] elements

$$e^{\tilde\omega} \in F/J^2+[F,J] = R\otimes S \oplus \Omega^1_{R\natural}\otimes\Omega^1_{S\natural}$$

$$\natural\left(\partial\rho\cdot e^{\tilde\omega}\right) \in \Gamma_\natural = \Omega^1_{R\natural}\otimes S \oplus R\otimes\Omega^1_{S\natural}$$

[redacted] Here we start with $\rho\in R\otimes S$ and lift it to F/J^2, [redacted]

[redacted] then calculate the curvature of the lift:

$$\tilde\omega = d\rho+\rho*\rho = d\rho+\rho^2+\partial''\rho\,\partial'\rho$$

$$= \omega + \partial''\rho\,\partial'\rho$$

where

$$\partial''(r\otimes s) = r\otimes ds \in R\otimes\Omega^1_S \qquad \sum\Omega^i_R\otimes\Omega^j_S$$

$$\partial'(r\otimes s) = dr\otimes s \in [redacted]\ \Omega^1_R\otimes S$$

and the product is done in $\Omega_R\otimes\Omega_S$.

$$\tilde\omega^2 = \underbrace{\omega*\omega}_{} + \omega\,\partial''\rho\,\partial'\rho + \partial''\rho\,\partial'\rho\,\omega$$

$$\omega^2+\partial''\omega\,\partial'\omega$$

$$\boxed{\tilde\omega^n = \omega^n + \sum_{i+j+k=n-2}\omega^i\,\partial''\omega\,\omega^j\,\partial'\omega\,\omega^k + \sum_{i+j=n-1}\omega^i\,\partial''\rho\,\partial'\rho\,\omega^j}$$

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We next compute $\partial\tilde{\rho}\cdot\tilde{\omega}^n$ first in Γ, then in $\Gamma_\natural$.

Say $\rho = r\otimes s$ lift to $\tilde{\rho} = rs \in F$

$$\partial\tilde{\rho} = \partial r s + r\partial s \longmapsto \partial r\otimes(1\otimes s) + (r\otimes 1)\otimes\partial s \in \Gamma$$

We want to multiply by $\tilde{\omega}^n$, really ω^n as Γ is an $R\otimes S$-bimodule. More generally taken any elt $r_1\otimes s_1 \in R\otimes S$. Then

$$\partial(\widetilde{r\otimes s})\, r_1\otimes s_1 = \partial r r_1\otimes(1\otimes s s_1) + (r\otimes r_1)\otimes\partial s s_1 \in \Gamma$$

$$\downarrow$$

$$\partial r r_1\otimes s s_1 + r_1 r\otimes\partial s s_1 \in \Gamma_\natural$$

$$= \natural(\partial r r_1)\otimes s s_1 + r r_1\otimes\natural(\partial s s_1)$$
$$- [r, r_1]\otimes\natural(\partial s s_1)$$

Observe this is

$$\partial'(r\otimes s)(r_1\otimes s_1) \in \Omega^1_R\otimes S \xrightarrow{\natural\otimes 1} \Gamma_\natural$$
$$+\ \partial''(r\otimes s)(r_1\otimes s_1) \in R\otimes\Omega^1_S \xrightarrow{1\otimes\natural} \Gamma_\natural$$
$$+\ \underbrace{\partial''(r\otimes s)\,\partial'(r_1\otimes s_1)} \in \Omega^1_R\otimes\Omega^1_S \xrightarrow{\beta\otimes\natural} \Gamma_\natural$$

$$r\otimes\partial s\cdot\partial r_1\otimes s_1 = -r\partial r_1\otimes\partial s_1 s_2 \xmapsto{\beta\otimes\natural} -[r, r_1]\otimes\natural(\partial s_1 s_2)$$

Thus

$$\boxed{\begin{aligned}\natural(\partial\tilde{\rho}\,\tilde{\omega}^n) &= (\natural\partial')\rho\ \omega^n && \in \Omega^1_{R\natural}\otimes S\\ &+ (\natural\partial'')\rho\ \omega^n && \in R\otimes\Omega^1_{S\natural}\\ &+ (\beta\otimes\natural)\partial''\rho\ \partial'(\omega^n) && \in R\otimes\Omega^1_{S\natural}\end{aligned}}$$

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These formulas are discouraging because when we try to apply them to the situation $R\otimes S = \mathrm{Hom}(B(A), R)$ things are very messy. We recognize

$$\omega^n = \omega^n \in \mathrm{Hom}(B(A), R)$$

$$(\natural\partial')_\rho\, \omega^n = d\rho\, \omega^n \in \mathrm{Hom}(B(A), \Omega^1_{R\natural})$$

$$(\natural\partial'')_\rho\, \omega^n = \partial\rho\, \omega^n \in \mathrm{Hom}(B^\natural(A), R)$$

$$\sum_{i+j=n-1} \omega^i \partial''\rho\, \partial'\rho\, \omega^j = d\rho\, \omega^{n-1}\partial\rho \in \mathrm{Hom}(B^\natural(A), \Omega^1_{R\natural})$$

but we don't seem to find μ_{2n-1} and we have lots of messy terms.

It is clear that μ_{2n-1} should have occurred, because we did find satisfactory formulas in both of the quotients

$$(\Omega^1_{R\natural} \rightleftarrows R) \otimes S_\natural \qquad R_\natural \otimes (\Omega^1_{S\natural} \rightleftarrows S)$$

of our mixed complex $(\Omega^1_{R\natural} \rightleftarrows R) \otimes (\Omega^1_{S\natural} \rightleftarrows S)$.

September 26, 1989

Let $k[\varepsilon] = k[X]/(X^2)$ be the dual numbers, let A, R be unital algebras, and consider an alg. hom.

$$A \longrightarrow R \otimes k[\varepsilon] = R \oplus R\varepsilon$$

Such a homomorphism is of the form

$$u + v\varepsilon$$

where $u, v : A \longrightarrow R$ are linear maps satisfying

$$\big(u(a_1) + v(a_1)\varepsilon\big)\big(u(a_2) + v(a_2)\varepsilon\big) = u(a_1a_2) + v(a_1a_2)\varepsilon \qquad u(1) + v(1)\varepsilon = 1$$

i.e.

$$u(a_1)u(a_2) = u(a_1a_2) \qquad u(1) = 1$$

$$u(a_1)v(a_2) + v(a_1)u(a_2) = v(a_1a_2) \qquad v(1) = 0$$

Thus u is a homomorphism and v is a derivation with respect to the A-bimodule structure on R defined by u.

Is a homomorphism $A \longrightarrow R \otimes k[\varepsilon]$ the same as a map from some algebra associated to A to R? Yes. Such a homomorphism is equivalent to a homomorphism $u : A \to R$ and a bimodule map $\tilde{v} : \Omega^1_A \longrightarrow R$ with respect to it. Thus the universal algebra is the tensor algebra $T_A(\Omega^1_A)$.

Next consider homs $A \longrightarrow R \otimes k[\varepsilon] \otimes k[\eta]$, where $k[\eta]$ is another ring of dual numbers. Such a homomorphism is of the form

$$U(a) = u(a) + v(a)\varepsilon + w(a)\varepsilon + f(a)\varepsilon\eta$$

with $u, v, w, f \in \mathrm{Hom}(A, R)$ such that u is a alg. homomorphism, v, w are derivations relative to u, To find the condition satisfied by f, suppose $u = 1$ and put X, Y for the derivations v, w.

$$U(a) = a + Xa\varepsilon + Ya\eta + f(a)\varepsilon\eta$$

$$U(a_1)\,U(a_2) = (a_1 + Xa_1\varepsilon + Ya_1\eta + f(a_1)\varepsilon\eta)(a_2 + Xa_2\varepsilon + Ya_2\eta + f(a_2)\varepsilon\eta)$$

$$= a_1a_2 + (Xa_1\,a_2 + a_1Xa_2)\varepsilon + (Ya_1\,a_2 + a_1Ya_2)\eta$$

$$+ (a_1f(a_2) + Xa_1Ya_2 + Ya_1Xa_2 + f(a_1)a_2)\varepsilon\eta$$

So we that

$$f(a_1a_2) = a_1f(a_2) + Xa_1Ya_2 + Ya_1Xa_2 + f(a_1)a_2$$

which means that f is a 1-cochain with

$$-\delta f = X \cup Y + Y \circ X$$

We get the following picture for the universal algebra associated to A for homomorphisms $A \longrightarrow R \otimes k[\varepsilon] \otimes k[\eta]$. First of all it is graded wrt $\mathbb{Z} \times \mathbb{Z}$, and it A in degree $(0,0)$ and copies of Ω^1_A in degrees $(1,0)$ and $(0,1)$. This gives 2 copies of Ω^2_A in degree $(1,1)$ corresponding to the derivations Xa_1Ya_2, Ya_1Xa_2. For the cochain f we have a map $A \otimes \bar{A} \otimes A \longrightarrow R^{1,1}$ and the relation above gives a cocartesian square

$$\begin{tikzcd}
\Omega^2_A \arrow[r] \arrow[d, "\int"'] & \Omega^2_A \oplus \Omega^2_A \arrow[d, "\text{``}X\circ Y + Y\circ X\text{''}"] \\
A \otimes \bar{A} \otimes A \arrow[r, "\text{``}f\text{''}"] & R^{1,1}
\end{tikzcd}$$

Then the rest of the algebra is generated freely by what we have described in degrees (p,q) $0 \leq p,q \leq 1$.

Here's the problem of interest. Suppose we have a 1-parameter family of homomorphisms $u_t : A \to R$ and a trace τ on R, then the derivative of the family of traces

$$\partial_t \tau(u_t) = \tau(\dot{u}_t)$$

on A is a special kind of trace, namely one which comes from a trace on Ω^1_A. Specifically $\dot{u}_t = \partial_t u_t : A \to R$ is a derivation, so it induces a bimodule map

$$\tilde{\dot{u}}_t : \Omega^1_A \longrightarrow R$$

so τ pulls back to the trace $\tau \tilde{\dot{u}}_t$ on Ω^1_A such that $(\tau \tilde{\dot{u}}_t) \circ d = \tau(\dot{u}_t)$. Integrating shows that the family $\tau(u_t)$ is constant modulo traces coming from Ω^1_A.

The idea roughly is that the derivative trace $\partial_t \tau(\dot{u}_t)$ is refined to a trace on Ω^1_A, i.e. a Hochschild 1-cocycle on A. Now we want to see how or whether the derivative of a suitable family of Hochschild 1-cocycles can be further refined.

September 27, 1989

Review: If $u_t : A \to R$ is a 1-parameter family of homomorphisms, $\tau : R \to k$ is a trace, then the family of traces τu_t on A has a special property, namely the derivative $\tau \dot{u}_t$ lifts canonically to a Hochschild 1-cocycle

$$\tau\left(u_t(a_0)\,\dot{u}_t(a_1)\right) = \tau\,\tilde{\dot{u}}_t\,(a_0 da_1)$$

Let $u_{st} : A \to R$ be a 2-parameter family of homomorphisms and τ be a trace on R. Then for t fixed we have a family of Hochschild 1-cocycles

$$\tau\left(u_{st}(a_0)\,\dot{u}_{st}(a_1)\right)$$

on A as s varies. We would like to know if this family of Hoch. 1-cocycles is special, i.e. whether the derivative with respect to s (denoted $'$)

$$\tau\left(u'_{st}(a_0)\,\dot{u}_{st}(a_1) + u_{st}(a_0)\,\dot{u}'_{st}(a_1)\right)$$

can be refined to a higher degree cocycle.

Why? In the u_t case, we have

$$\tau\, u_t\Big]_{t=0}^{t=1} = \left(\int_0^1 \tau\,\tilde{\dot{u}}_t\, dt\right)\cdot d$$

This shows that it is reasonable to view a triple (τ_0, τ_1, f) consisting of two traces and a Hochschild 1-cocycle satisfying $\tau_1 - \tau_0 = fd$ as a kind of path joining τ_0 to τ_1.

It is then natural to look for the notion of a homotopy of paths. This should be represented by a [redacted] square

$$\begin{tikzcd}
\tau_{01} \arrow[r, dash, "g_1"] \arrow[d, dash, "f_0"'] & \tau_{11} \arrow[d, dash, "f_1"] \\
\tau_{00} \arrow[r, dash, "g_0"'] & \tau_{10}
\end{tikzcd}$$

where the edges are paths and where the Hochschild 1-cocycle

$$g_0 + f_1 - g_1 - f_0$$

is the boundary of some sort of 2-dimensional cocycle.

In the path situation one way to produce the Hochschild 1-cocycle from [redacted] u_t, τ is as follows. To handle $\dot{u}_t$ work with the dual numbers $k[\eta] = k + k\eta$, $\eta^2 = 0$. At fixed time t, we have a homomorphism

$$A \xrightarrow{u + \dot{u}\eta} R \otimes k[\eta] = R \oplus R\eta$$

Then u is a homom. and $\dot{u}$ is a derivation, so this map factors

$$A \xrightarrow{1 + d\eta} A \oplus \Omega^1_A \eta \xrightarrow{u \oplus \tilde{\dot{u}}} R \oplus R\eta$$

~~In the 2 parameter situation we have a homomorphism~~

$\cancel{A \xrightarrow{u + \dot{u}\epsilon + u'\eta + \dot{u}'\epsilon\eta}}$

Actually let's change notation

$$A \xrightarrow{u + \eta \dot{u}} k[\eta] \otimes R = R + \eta R$$

$$A \xrightarrow{1 + \eta d} A \oplus \eta \Omega^1_A \xrightarrow{u + \tilde{\dot{u}}} R + \eta R$$

In the two parameter ~~situation~~ we have an algebra homomorphism

$$\circledast \quad A \xrightarrow{u + \varepsilon u' + \eta \dot{u} + \varepsilon\eta \dot{u}'} R \oplus \varepsilon R \oplus \eta R \oplus \varepsilon\eta R$$

Here $u, u', \dot{u}, \dot{u}'$ are the values and derivatives of the family u_{st} at fixed s,t. It might be better to write the above homomorphism

$$u_{s+\varepsilon, t+\eta} = u_{st} + \varepsilon u'_{st} + \eta \dot{u}_{st} + \varepsilon\eta \dot{u}'_{st}$$

To avoid confusion we write the above homomorphism $\circledast$ as

$$a \longmapsto a + \varepsilon X a + \eta Y a + \varepsilon\eta f(a)$$

$$** \qquad A \longrightarrow R \oplus \varepsilon R \oplus \eta R \oplus \varepsilon\eta R$$

where the initial homomorphism $A \to R$ written as an inclusion.

Because $**$ is a homomorphism we find X, Y are derivations with values in R considered as a bimodule over A via the homomorphism $A \to R$, and $f : A \to R$ satisfies

$$f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2 + X a_1 \, Y a_2 + Y a_1 \, X a_2$$

Next we look for a universal algebra constructed from A analogous to $A \oplus \Omega^1_A$ for a homomorphism of the form $**$. We want to replace $k[\varepsilon, \eta] \otimes R$ by a $\mathbb{Z}\times\mathbb{Z}$-graded algebra, which I will denote R, and which is supported in degrees (p,q), $0 \le p,q \le 1$:

$$R = R^{00} + R^{10} + R^{01} + R^{11}$$

We are looking at homomorphisms $A \to R$

 and such a homomorphism has 64

the form

$$a \longmapsto a + Xa + Ya + fa$$

where X, Y, a satisfy the above conditions. The universal such algebra + homomorphism are such that

$$R = A \oplus R^{10} \oplus R^{01} \oplus R^{11}$$

where

$$\Omega^1_A \xrightarrow{\sim} R^{10} \qquad a_1 da_2 \longmapsto a_1 X a_2$$

$$\Omega^1_A \xrightarrow{\sim} R^{01} \qquad a_1 da_2 \longmapsto a_1 Y a_2$$

and where R^{11} is given by a pushout square of A-bimodules

$$da_1 da_2 \longmapsto Xa_1 \otimes Ya_2 + Ya_1 \otimes Xa_2$$

$$\begin{tikzcd}
\Omega^2_A \arrow[r] \arrow[d] & R^{10} \otimes R^{01} \oplus R^{01} \otimes R^{10} \arrow[d] \\
A \otimes \bar{A} \otimes A \arrow[r] & R^{11}
\end{tikzcd}$$

$$a_0 \otimes a_1 \otimes a_2 \longmapsto a_0 f(a_1) a_2$$

$$da_1 da_2 \longmapsto b'(1, a_1, a_2, 1) = (a_1, a_2, 1) - (1, a_1 a_2, 1) + (1, a_1, a_2)$$

So we have the picture

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Omega^2_A \arrow[r, "\Delta"] \arrow[d] & \Omega^2_A \oplus \Omega^2_A \arrow[r] \arrow[d] & \Omega^2_A \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & A \otimes \bar{A} \otimes A \arrow[r] \arrow[d] & R^{11} \arrow[r] \arrow[d] & \Omega^2_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega^1_A \arrow[r, "\sim"] \arrow[d] & \Omega^1_A \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

In this graded setup we have a 2-parameter family of homomorphisms

$$u(a) = a + sXa + tYa + stfa$$

~~Maybe we should put in $\varepsilon, \varepsilon'$ to keep track of the grading but actually this~~

Then we have

$$\dot{u}(a) = Ya + sfa$$

$$u'(a) = Xa + tfa$$

$$\dot{u}'(a) = fa$$

If τ is a trace on R, then the family of Hochschild 1-cocycles is

$$\tau(u(a_0)\dot{u}(a_1)) = \tau((a_0 + sXa_0 + tYa_0 + stfa_0)(Ya_1 + sfa_1))$$

$$= \tau(a_0 Ya_1 + sXa_0 Ya_1 + s\, a_0 fa_1)$$

and its derivative is

$$\tau(u'(a_0)\dot{u}(a_1) + u(a_0)\dot{u}'(a_1)) = \tau(Xa_0 Ya_1 + a_0 fa_1)$$

Thus $$\boxed{\varphi(a_0, a_1) = \tau(Xa_0 Ya_1 + a_0 fa_1)}$$

is the Hochschild 1-cocycle we have to refined.

The obvious way to proceed now is to describe the traces on the universal algebra R and see what cocycles they lead to.

Much simpler description of R^{11}:
Note that Ya_1, Xa_2 can be calculated from f and Xa_1, Ya_2 via the formula

$$f(a_1 a_2) = a_1 f(a_2) + f(a_1) a_2 + Xa_1\, Ya_2 + Ya_1\, Xa_2$$

Consequently a homomorphism

$$A \longrightarrow R^{00} + R^{10} + R^{01} + R^{11}$$

is the same as quadruple consisting of a hom. $A \to R^{00}$, derivations $X : A \to R^{10}$, $Y : A \to R^{01}$ with respect to this hom., and a linear map $f : \bar{A} \longrightarrow R^{11}$. The universal such R has

$$A \xrightarrow{\sim} R^{00}, \qquad \Omega^1_A \xrightarrow{\sim} R^{10} \qquad a_1 da_2 \longrightarrow a_1 Xa_2$$
$$\Omega^1_A \xrightarrow{\sim} R^{01} \qquad a_1 da_2 \longrightarrow a_1 Ya_2$$

and

$$A \otimes \bar{A} \otimes A \oplus \Omega^2_A \xrightarrow{\sim} R^{11}$$
$$a_0 \otimes a_1 \otimes a_2 + a_0' da_1' da_2' \longrightarrow a_0 f(a_1) a_2 + a_0' Xa_1' Ya_2'$$

with obvious algebra structure defined using the above identity.

Now let's determine the traces on the universal R. Because R is graded, so is the space of traces. We are concerned with traces supported on R^{11}. Such a trace is first of all a trace on R^{11} considered as A-bimodule. As

$$R^{11} = A \otimes \bar{A} \otimes A \oplus \Omega^2_A$$

a trace on R^{11} as A-bimodule is the same as a Hochschild 1-cochain and a Hochschild 2-cocycle given by

$$\tau(a_0 f(a_1)) \qquad \tau(a_0 Xa_1 Ya_2)$$

To be a trace on R as an algebra we must have in addition

$$\tau(\alpha\beta) = \tau(\beta\alpha) \qquad \alpha \in R^{01},\ \beta \in R^{10}$$

$$\tau(a_0 Y a_1 \cdot X a_2 a_3) \overset{?}{=} \tau(X a_2 a_3 \cdot a_0 Y a_1)$$

$$\| $$

$$\tau(a_3 a_0 Y a_1 \cdot X a_2)$$

This condition depends only on $a_3 a_0$, so we can suppose $a_3 = 1$. Then we want

$$\tau(X a_2\, a_0\, Y a_1) \overset{?}{=} \tau(a_0 Y a_1 X a_2)$$

$$= \tau\left(a_0\left\{f(a_1 a_2) - a_1 f(a_2) - f(a_1)a_2 - X a_1 Y a_2\right\}\right)$$

or

$$\boxed{\begin{aligned}&\tau(X a_2 a_0 Y a_1) + \tau(a_0 X a_1 Y a_2)\\ &\quad = -\tau\left(a_0 a_1 f(a_2) - a_0 f(a_1 a_2) + a_2 a_0 f(a_1)\right)\end{aligned}}$$

The right side we recognize as the coboundary of $\tau(a_0 f(a_1))$ with minus sign.

This condition has the following interpretation. Recall that there is an operation of $\mathbb{Z}/n$ on Hochschild n cocycles which is transpose to the natural action of $\mathbb{Z}/n$ on

$$\Omega^n_A \otimes_A = (\Omega^1_A \otimes_A)^n$$

For $n = 2$, this gives the operation

$$\sigma : a_0\, da_1\, da_2 \longmapsto da_2\, a_0\, da_1$$

which can be put in more standard form

$$da_2 a_0 da_1 = d(a_2 a_0) da_1 - a_2 da_0 da_1$$

Thus given a Hochschild 2-cocycle $\psi(a_0, a_1, a_2)$ we interpret it as a linear functional on $\Omega_A^2 \otimes_A$

$$\langle \psi \mid a_0 da_1 da_2 \rangle = \psi(a_0, a_1, a_2)$$

and pull it back via the cyclic operation σ to get another Hochschild 2-cocycle ψ^σ:

$$\langle \psi^\sigma \mid a_0 da_1 da_2 \rangle = \langle \psi \mid \sigma(a_0 da_1 da_2) \rangle$$
$$= \langle \psi \mid da_2 a_0 da_1 \rangle$$
$$= \langle \psi \mid d(a_2 a_0) da_1 - a_2 da_0 da_1 \rangle$$
$$= \psi(1, a_2 a_0, a_1) - \psi(a_2, a_0, a_1)$$

If you apply this to $\psi(a_0, a_1, a_2) = \tau(a_0 X a_1 Y a_2)$ you get

$$\psi^\sigma(a_0, a_1, a_2) = \psi(1, a_2 a_0, a_1) - \psi(a_2, a_0, a_1)$$
$$= \tau(X(a_2 a_0) Y a_1) - \tau(a_2 X a_0 Y a_1)$$
$$= \tau(X a_2 a_0 Y a_1).$$

~~So~~ now we have reached a nice description of traces on the 2nd order universal algebra

$$A \oplus \Omega_A^1 \oplus \Omega_A^1 \oplus (\Omega_A^1 \oplus A \otimes \bar{A} \otimes A)$$

The traces supported on the (1,1) part can be identified with pairs (ψ, φ) where ψ is a Hochschild 2-cocycle and φ is a Hochschild 1-cochain such that $\boxed{\psi + \psi^{\sigma} + \delta\varphi = 0}$

General observations. We know there is a cyclic group action on Hochschild cocycles, so it is natural to ask if the action passes to cohomology. The induced action of $\mathbb{Z}/n$ on $H^n(A,A^*)$ or $H_n(A,A)$ should be zero if it exists.

This issue can be settled by formulas. Consider the usual operator s on normalized Hochschild cochains which satisfies $b's + sb' = \mathrm{id}$. but this time use it as a homotopy operator on the Hochschild complex. Recall that

$$(s\psi_{n+1})(a_1, \ldots, a_n) = \psi_{n+1}(1, a_1, \ldots, a_n)$$

Then compute

$$\begin{aligned}(bs\,\psi_{n+1})(a_0, \ldots, a_n) &= (b's\,\psi_{n+1})(a_0, \ldots, a_n) \\ &\quad + (-1)^n (s\psi_{n+1})(a_n a_0, \ldots, a_{n-1}) \\ &= (b's\,\psi_{n+1})(a_0, \ldots, a_n) + (-1)^n \psi_{n+1}(1, a_n a_0, \ldots, a_{n-1})\end{aligned}$$

$$(sb\,\psi_{n+1})(a_0, \ldots, a_n) = (sb'\psi_{n+1})(a_0, \ldots, a_n) + (-1)^{n+1}\psi_{n+1}(a_n, a_0, \ldots, a_{n-1})$$

We need s applied to the crossover term in $b'\psi_{n+1}$ which is crossover term in $b\psi_{n+1}$ applied to $(1, a_0, \ldots, a_n)$ which is $(-1)^{n+1}\psi_{n+1}(a_n, a_0, \ldots, a_{n-1})$.

Thus

$$\boxed{\begin{aligned}\big((bs+sb)\psi_{n+1}\big)(a_0, \ldots, a_n) = \;& \psi_{n+1}(a_0, \ldots, a_n) \\ & + (-1)^{n+1}\psi_{n+1}(1, a_n a_0, a_1, \ldots, a_{n-1}) \\ & + (-1)^n \psi_{n+1}(a_n, a_0, \ldots, a_{n-1})\end{aligned}}$$

Now consider the cyclic operator λ on $\Omega_A^n \otimes_A = (\Omega_A^1 \otimes_A)^n$ and its effect on Hochschild n-cocycles. We have

$$\langle \psi^\lambda | a_0 da_1 \cdots da_n \rangle = (-1)^{n-1} \langle \psi | da_n a_0 da_1 \cdots da_{n-1} \rangle$$

$$= (-1)^{n-1} \psi(1, a_n a_0, a_1, \ldots, a_{n-1})$$

$$+ (-1)^n \psi(a_n, a_0, \ldots, a_{n-1})$$

Thus we see that we have the identity

$$\boxed{(bs + sb)\psi = \psi - \psi^\lambda}$$

on the normalized Hochschild cochains. This implies that $\psi \longmapsto \psi^\lambda$ is an endomorphism of the Hochschild complex which is homotopic to the identity.

Now let's return to a trace τ on the universal algebra $R^{00} + R^{10} + R^{01} + R^{11}$ generated by A, and try to explain the Hochschild 1-cocycle $\tau(Xa_1 Ya_2 + a_1 f(a_2))$ associated to it. We have identified the 1,1 part of τ with a pair consisting of a Hoch. 1-cochain φ_2 and 2-cocycle ψ_3 satisfying $(1-\lambda)\psi_3 + b\varphi_2 = 0$. However we already have $(1-\lambda)\psi = bs\psi$ and so $s\psi_3 + \varphi_2$ is a Hochschild 1-cocycle. But recall

$$\psi_3(a_0, a_1, a_2) = \tau(a_0 X a_1 Y a_2)$$

$$\varphi_2(a_1, a_2) = \tau(a_1 f(a_2))$$

so

$$\boxed{(s\psi_3 + \varphi_2)(a_1, a_2) = \tau(Xa_1 Ya_2 + a_1 f(a_2))}$$

However not only do we have the Hochschild 1-cocycle

$$\tau\left(u'(a_0)\,\mathring{u}(a_1) + u(a_0)\,\mathring{u}'(a_1)\right)$$
$$= \tau\left(X a_0 Y a_1 + a_0 f a_1\right)$$

but we also have

$$\tau\left(\mathring{u}(a_0)u'(a_1) + u(a_0)\mathring{u}'(a_1)\right)$$
$$= \tau\left(Y a_0 X a_1 + a_0 f a_1\right)$$

This 1-cocycle should arise from using $s\psi_3^\lambda$ instead of $s\psi_3$. We have

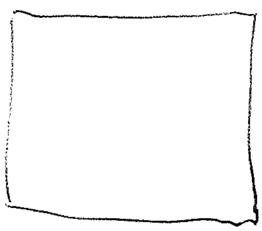
$$\psi_3^\lambda(a_0, a_1, a_2) = \psi_3(a_2, a_0, a_1) - \psi_3(1, a_2 a_0, a_1)$$

so
$$(s\psi_3^\lambda)(a_1, a_2) = -\psi_3(1, a_2, a_1) = -\tau(X a_2 Y a_1)$$
$$= -\tau(Y a_1 X a_2)$$

Check $\psi_3 - \psi_3^\lambda = \delta s\psi_3 = -\delta s\psi_3^\lambda = -\delta\varphi_2$, and we have

$$(s\psi_3 + \varphi_2)(a_1, a_2) = \tau(X a_1 Y a_2 + a_1 f a_2)$$
$$(-s\psi_3^\lambda + \varphi_2)(a_1, a_2) = \tau(Y a_1 X a_2 + a_1 f a_2)$$

Next let's return to the idea that the couple (ψ_3, φ_2) is supposed to be a square



where one pair of opposite sides is the 1-cocycle $s\psi_3 + \varphi_2$ and the other is $-s\psi_3^\lambda + \varphi_2$.

Then a square where nothing is changing along the horizontal sides is a pair with say $-s\psi_3^\lambda + \varphi_2 = 0$, so it is equivalent to the Hochschild 2 cocycle ψ_3. If we also ask that nothing is changing along the vertical sides, then we obtain $s\psi_3 + s\psi_3^\lambda = 0$. Now

$$(s\psi_3 + s\psi_3^\lambda)(a_1, a_2) = \psi_3(1, a_1, a_2) - \psi_3(1, a_2, a_1)$$
$$= (B\psi_3)(a_1, a_2)$$

so we are dealing with a Hochschild 2-cocycle ψ_3 with $B\psi_3 = 0$. But such a ψ_3 is more or less equivalent to a cyclic 2-cocycle. Think in terms of the double complex when A is augmented:

$$\begin{tikzcd} 0 & & \\ \bullet \arrow[u, "-b'"] \arrow[r, "\lambda-1"] & {} & \\ & \bullet \arrow[u, "b'"] \arrow[r, "N"] & 0 \end{tikzcd}$$