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January 12, 1989

Let τ be a linear functional on $T_2(A) = (A * A)^+$. Then τ is equivalent to the sequence of multilinear functionals

$$\psi_{2n+1}(a_0, \ldots, a_{2n}) = \tau(a_0^+ a_1^- \cdots a_{2n}^-) \qquad n \geq 0$$

which can be completely arbitrary subject to the normalization condition:

$$\psi_{2n+1}(a_0, \ldots, a_{2n}) = 0 \qquad \text{if} \quad a_i = [illegible] \quad \text{some } i \geq 1.$$

Calculate:

$$(b\psi_{2n-1})(a_1, \ldots, a_{2n}) =$$

$$\tau(\underbrace{(a_1 a_2)^+}_{\cancel{a_1^+ a_2^+} + a_1^- a_2^-} a_3^- \cdots a_{2n}^-)$$

$$- \tau(a_1^+ \underbrace{(a_2 a_3)^-}_{\cancel{a_2^+ a_3^-} + \cancel{a_2^- a_3^+}} a_4^- \cdots)$$

$$+ \tau(a_1^+ a_2^- \underbrace{(a_3 a_4)^-}_{\cancel{a_3^+ a_4^-} + a_3^- a_4^+} \cdots)$$

$$+ \tau(a_1^+ a_2^- \cdots \underbrace{(a_{2n-1} a_{2n})^-}_{\cancel{a_{2n-1}^+ a_{2n}^-} + a_{2n-1}^- a_{2n}^+})$$

$$- \tau(\underbrace{(a_{2n} a_1)^+}_{a_{2n}^+ a_1^+ + a_{2n}^- a_1^-} a_2^- \cdots a_{2n-1}^-)$$

✓

$$= \left\{ \tau(a_1^- a_2^- \cdots a_{2n}^-) - \tau(a_{2n}^- a_1^- \cdots a_{2n-1}^-) \right\}$$

$$\left\{ \tau(a_1^+ a_2^- \cdots a_{2n-1}^- a_{2n}^+) - \tau(a_{2n}^+ a_1^+ a_2^- \cdots a_{2n-1}^-) \right\}$$

Formula

$$\boxed{\begin{aligned}&(b\psi_{2n-1})(a_1,\dots,a_{2n})\\&=\{\tau(a_1^-\cdots a_{2n}^-)-\tau(a_{2n}^-a_1^-\cdots a_{2n-1}^-)\}\\&\quad+\{\tau((a_1^+a_2^-\cdots a_{2n-1}^-)a_{2n}^+)-\tau(a_{2n}^+(a_1^+a_2^-\cdots a_{2n-1}^-))\}\end{aligned}}$$

Suppose τ is a trace on $(A*A)^+$. Then the second term in braces vanishes. Also we have that $\tau(a_1^-\cdots a_{2n}^-)$ is invariant under the 2-step cyclic shift $\sigma^2=\lambda^2$. Thus the first term, which is $(1+\lambda)\,\tau(a_1^-\cdots a_{2n}^-)$, is the same as $\frac{1}{n}N\tau(a_1^-\cdots a_{2n}^-)$. So we have

$$* \qquad \boxed{b\psi_{2n-1}=\frac{1}{n}B\psi_{2n+1}}$$

Conversely assume that $\psi_1,\psi_3,\psi_5,\dots$ are such that $\psi_{2n+1}(1,a_1,\dots,a_{2n})$ is invariant under σ^2 and also that $*$ holds. Then if τ is the corresponding linear functional on $T_n(A)=(A*A)^+$

$$\begin{aligned}\frac{1}{n}(B\psi_{2n+1})(a_1,\dots,a_{2n}) &= (1+\lambda)\,\psi_{2n+1}(1,a_1,\dots,a_{2n})\\&=\tau(a_1^-\cdots a_{2n}^-)-\tau(a_{2n}^-a_1^-\cdots a_{2n-1}^-)\end{aligned}$$

From the formula at the top we conclude

$$\tau((a_1^+a_2^-\cdots a_{2n-1}^-)a_{2n}^+)=\tau(a_{2n}^+(a_1^+a_2^-\cdots a_{2n-1}^-))$$

But the products $a_1^+a_2^-\cdots a_{2n-1}^-$ for $n\geq 1$ span $(A*A)^+$, so we have

$$\tau(a^+\xi)=\tau(\xi a^+)$$

for all $a\in A$, $\xi\in(A*A)^+$. Since $(A*A)^+$ is generated by $\{a^+\mid a\in A\}$, this implies τ is a trace on $(A*A)^+$.

January 13, 1989

The problem is to understand the difference between traces on $(A*A)^+$ and even supertraces on $A*A$. The latter satisfy the extra condition $\tau(\xi\eta) = -\tau(\eta\xi)$ for $\xi, \eta \in (A*A)^-$.

Let us restrict our attention to linear functionals on $A*A/J^3 \simeq A \oplus \Omega_A^1 \oplus \Omega_A^2$ which are supported on even elements. Such a linear functional τ is completely described by two cochains

$$\psi_1(a) = \tau(a^+)$$

$$\psi_3(a_0, a_1, a_2) = \tau(a_0^+ a_1^- a_2^-)$$

Yesterday's calculations give

$$b\psi_3 = 0 \iff \tau((a_1^+ a_2^- a_3^-)a^+) = \tau(a^+(a_1^+ a_2^- a_3^-))$$

$$b\psi_1 = B\psi_3 \iff \tau(a_1^+ a^+) = \tau(a^+ a_1^+)$$

so that τ is a trace on $(A*A)^+ \iff b\psi_3 = 0$ and $b\psi_1 = B\psi_3$.

The extra condition that τ be an even supertrace on $A*A$ is that

$$\tau(\xi\eta) = -\tau(\eta\xi)$$

where ξ, η are of the form $a_1^+ a_2^-$. Since we can move a^+ around, the condition becomes

$$\tau(a^- b^+ c^-) \overset{?}{=} -\tau(b^+ c^- a^-)$$

for $a, b, c \in A$. Equivalently

$$\text{or}\quad\boxed{\begin{aligned} \tau((ab)^- c^-) &= \tau(a^+ b^- c^-) - \tau(b^+ c^- a^-) \\ \psi_3(1, ab, c) &= \psi_3(a,b,c) - \psi_3(b,c,a) \end{aligned}}$$

From $b\psi_3 = 0$ we obtain

$$\psi_3(1, ab, c) - \psi_3(1, a, bc) = \psi_3(a,b,c) - \psi_3(c,a,b)$$

or $b'\varphi_2 = (1-T)\psi_3$. So we can transform the condition above into different forms.

The point to understand is how Connes replaces the (b,B) cocycle $(\psi_1, \psi_3, 0, \dots)$ by a cohomologous one satisfying this extra condition.

January 14, 1989

Trace on GNS:

Let τ be a trace on B, let $\rho : A \to B$, $\rho(1) = 1$ be as usual, and let $R = GNS(\rho)$. Thus $R = A \oplus A\otimes B\otimes A$, and the elements (a', b, a'') of $A\otimes B\otimes A$ will be written $a' \iota b \iota^* a''$ where $\iota^* a \iota = \rho(a)$, so that we understand its meaning better. We [illegible] define a linear functional on $A\otimes B\otimes A$, which is an ideal in R, ~~by~~

$$\tilde{\tau}\{(a'\iota b \iota^* a'')\} = \tau(b\rho(a''a'))$$

We claim this is a trace on this ideal considered as a bimodule over R.

$$\tilde{\tau}\{a(a'\iota b\iota^* a'')\} = \tau(b\rho(a''aa')\}$$

$$\tilde{\tau}\{(a'\iota b\iota^* a'')a\}$$

~~$\tilde{\tau}\{(a_1', b_1, a_1'')(a_2', b_2, a_2'')\} = \tilde{\tau}\{(a'$~~

$$\tilde{\tau}\{ a_1' \iota b_1 \iota^* a_1'' \cdot a_2' i b_2 \iota^* a_2''\}$$

$$= \tilde{\tau}\{ a_1' i (b_1 \rho(a_1''a_2') b_2) \iota^* a_2''\}$$

$$= \tau\left(b_1\rho(a_1''a_2')b_2\rho(a_2''a_1')\right)$$

$$= \tau\left(b_2\rho(a_2''a_1')b_1\rho(a_1''a_2')\right)$$

) trace property of τ

$$= \tilde{\tau}(a_2' \iota b_2 \iota^* a_2'' \cdot a_1' \iota b_1 \iota^* a_1'')$$

Obvious extension: If τ is a trace on the ideal $I \subset B$ considered as a bimodule, then $\tilde{\tau}$ is a trace on the ideal $A \otimes I \otimes A$ in R.

Note that

$$\tilde{\tau}\{\iota b \iota^*\} = \tau(b)$$

so $\tilde{\tau}$ extends τ relative to the non-unital embedding $B \longrightarrow R$. We note that $\tilde{\tau}$ is the unique extension of τ which is a trace since if φ is a trace on $A \otimes I \otimes A$ extending τ, then

$$\varphi(a' \iota b \iota^* a'') = \varphi(\iota b \iota^* a'' a')$$

$$= \varphi(\iota 1 \iota^* \iota b \iota^* a'' a') \qquad \iota 1 \iota^* = \hat{e}$$

$$= \varphi(\iota b \iota^* a'' a' \iota 1 \iota^*)$$

$$= \varphi(i\, b \rho(a''a') \iota^*) = \tau\{b \rho(a''a')\}$$

Recall from work in the spring the block decomposition

$$R = \begin{pmatrix} eRe & eRe' \\ e'Re & e'Re' \end{pmatrix} \qquad e' = 1-e$$

$$A \otimes B \otimes A \doteq \begin{pmatrix} 1 \otimes B \otimes 1 & 1 \otimes B \otimes \bar{A} \\ \bar{A} \otimes B \otimes 1 & \bar{A} \otimes B \otimes \bar{A} \end{pmatrix}$$

Here one is not assuming a splitting of $0 \to k \to A \to \bar{A} \to 0$ but rather using the splitting

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, "1\otimes ?"'] & A \otimes B \arrow[l, bend right, "m(\rho\otimes id)"'] \arrow[r] & \bar{A} \otimes B \arrow[r] & 0 \end{tikzcd}$$

Thus $\bar{A}\otimes B \hookrightarrow A\otimes B$ is the map

$$\bar{a}\otimes b \longmapsto a\otimes b - 1\otimes\rho(a)b$$

The pairing with K_0. Suppose we have $A \longrightarrow L/I$ where $I^{m+1}=0$, and we τ a trace on L. Then we have

$$K_0(A)\longrightarrow K_0(L/I)\xleftarrow{\sim}K_0(L)\xrightarrow{\tau_*}k$$

Suppose $A = L/I$, and let ρ be a lifting.

Let e be an idempotent matrix over A. Here's how to lift it to an idempotent $\tilde{e}$ over L. Lift e up to L to $\rho(e)$. Then $\rho(e)^2-\rho(e)\in I$. Thus we want to find a polynomial $f(x)$ so that $f(x)^2\equiv f(x)$ modulo $(x^2-x)^{m+1}$. Clearly

$$f(x) = \int_0^x (t-t^2)^n dt \Big/ \int_0^1 (t-t^2)^n dt$$

will do for any $n \geq m$. We have

$$f(x) = \int_0^1 x\,(tx-t^2x^2)^n\, dt \Big/ \frac{n!n!}{(2n+1)!}$$

so

$$\tau_*([\tilde{e}]) = \int_0^1 \tau\{\rho(e)(t\rho(e)-t^2\rho(e)^2)^n\}dt \Big/ \frac{n!n!}{(2n+1)!}$$

Another idea is to work with the involution $\varepsilon = 2e-1$. Then its lift is

$$\tilde{\varepsilon} = \rho(\varepsilon)\left(\rho(\varepsilon)^2\right)^{-1/2}$$

where the square root is defined by the binomial series $(1-z)^{-1/2} = 1+\frac{1}{2}z+\cdots$. This shows there is a formula *independent* of *n*, which might be useful

Let's discuss this in more detail. Consider the polynomial

$$f(x) = \int_0^x (1-t^2)^n dt \Big/ \int_0^1 (1-t^2)^n dt$$

$$= \int_0^1 x(1-t^2x^2)^n dt \Big/ \underbrace{\int_0^1 (1-t^2)^n dt}_{c_n}$$

$$= \int_0^1 x\left(1-t^2+t^2(1-x^2)\right)^n dt \,/\, c_n$$

$$= x\sum_{k=0}^{n}\left[\binom{n}{k}\int_0^1 (1-t^2)^{n-k}t^{2k}\, dt/c_n\right](1-x^2)^k$$

which is characterized by the fact that it is of degree $2n+1$ and satisfies $f(\pm 1) = \pm 1$

$f^{(k)}(\pm 1) = 0 \qquad 1 \leq k \leq n$. It follows that if we increase n, the lower coefficients don't change. This is checked by calculation which gives

$$[k\text{ th coeff}] = \frac{1\cdot 3 \cdots (2k-1)}{2^k\, k!}$$

But I think the interesting question is whether there is a natural integral formula which is $n \to \infty$ version of the above integral formula for $f(x)$. There is something reminiscent about Gaussian integrals in the expression $(1-z)^{-1/2}$.

We have seen that a trace on B extends uniquely to a trace on the ideal $M = A \otimes B \otimes A$ in $R = GNS(\rho)$. Thus the next question is whether a trace on the ideal M extends to a trace in R. But associated to the extension

$$0 \longrightarrow M \longrightarrow R \longrightarrow A \longrightarrow 0$$

is a six term exact sequence

$$H_1(R,M) \longrightarrow HC_1(R) \longrightarrow HC_1(A) \longrightarrow M/[R,M] \longrightarrow R/[R,R] \rightarrow A/[A,A] \rightarrow 0$$

which splits because [redacted] R is the semi-direct product. Thus a trace $\tau : M/[R,M] \longrightarrow V$ on the ideal M extends to R and the extension is unique if we prescribe it on A.

So in the case of $R = A * k[F]$ [redacted] we seem to have proved

$$R/[R,R] \simeq T_r(A)/[T_r(A), T_r(A)] \oplus A/[A,A]$$

which we ought to be able to check. In general one has I think for two algebras A, B

$$\overline{HC}_0(A * B) = \overline{HC}_0(A) \oplus \overline{HC}_0(B) \oplus \bigoplus_{n \geq 1} (\bar{A} \otimes \bar{B})^{\otimes n} / \text{cyclic permutations}$$

and so it checks.

January 16, 1989

Bar construction of a DGA; formulas.

Consider the tensor coalgebra $T(V)$, and let $p_n : T(V) \to V^{\otimes n}$ be the projections. Then we have $p_n = p_1^{\otimes n} \Delta^{(n)}$:

$$\begin{tikzcd}
T(V) \arrow[d, "\Delta^{(n)}"] \arrow[rr, "p_n"] & & V^{\otimes n} \\
T(V)^{\otimes n} \arrow[rr, "p_1^{\otimes n}"] & & V^{\otimes n}
\end{tikzcd}$$

hence if $u : C \to T(V)$ is a coalgebra morphism one has $p_n u = (p_1 u)^{\otimes n} \Delta^{(n)}$:

$$\begin{tikzcd}
C \arrow[r, "u"] \arrow[d, "\Delta^{(n)}"] & T(V) \arrow[d, "\Delta^{(n)}"] \arrow[dr, "p_n"] & \\
C^{\otimes n} \arrow[r, "u^{\otimes n}"] & T(V)^{\otimes n} \arrow[r, "p_1^{\otimes n}"] & V^{\otimes n}
\end{tikzcd}$$

Conversely this formula ~~allows one to~~ constructs the coalgebra morphism (co)extending a linear map $C \to V$ (assuming connectivity of C).

Next suppose we have a derivation D of $T(V)$, i.e. setting $C = T(V)$, suppose

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "D"] & C \otimes C \arrow[d, "D\otimes 1 + 1 \otimes D"] \\
C \arrow[r, "\Delta"] & C \otimes C
\end{tikzcd}$$

commutes. Here can be of arbitrary degree.

By induction

$$\Delta^{(n)} D = \sum_{i=1}^{n} \left(1^{\otimes i-1} \otimes D \otimes 1^{\otimes n-i} \right) \Delta^{(n)}$$

so for $C = T(V)$, we have from $p_n = p_1^{\otimes n} \Delta^{(n)}$

$$\boxed{p_n D = \sum_{i=1}^{n} \left(p_1^{\otimes i-1} \otimes p_1 D \otimes p_1^{\otimes i-1} \right) \Delta^{(n)}}$$

Now take a DGA L, let ΣL be the suspension of the complex L. Recall there is a canonical map of degree 1

$$\sigma : L \longrightarrow \Sigma L \qquad x \longmapsto \sigma x$$

which is an isomorphism

Define $\tilde{m} : \Sigma L \otimes \Sigma L \longrightarrow \Sigma L$ to be the degree -1 map such that

$$\tilde{m} (\sigma \otimes \sigma) = \sigma m$$

i.e.

$$\begin{tikzcd}
L \otimes L \arrow[r, "m"] \arrow[d, "\sigma\otimes\sigma"] & L \arrow[d, "\sigma"] \\
\Sigma L \otimes \Sigma L \arrow[r, "\tilde{m}"] & \Sigma L
\end{tikzcd}$$

commutes. Then define b' on $T(\Sigma L)$ to be the coderivation of degree -1 such that $\boxed{p_1 b' = \tilde{m} p_2}$.

Let's see what this does when $L = A[0]$.

Then ~~$\tilde{m}(\sigma\otimes\sigma)(a_1 \otimes a_2) = \tilde{m}(a_1, \sigma a_2)$~~

~~$\sigma m (a_1 \otimes a_2) =$~~

we have to keep straight $A[0]$ and $A[1]$. So we put in σ's to do this.

$$\tilde{m}(\sigma a_1, \sigma a_2) = \tilde{m}(\sigma\otimes\sigma)(a_1, a_2) \qquad \text{since } a_1 \text{ even}$$

$$= \sigma\, m(a_1, a_2) = \sigma(a_1 a_2)$$

Then b' is determined

$$p_n b' = \sum_{i=1}^{n} p_1^{\otimes i-1} \otimes \underbrace{p_1 b'}_{\tilde{m} p_2} \otimes\, p_1^{\otimes n-i}$$

So

$$b'(\sigma a_1, \dots, \sigma a_{n+1}) = \sum_{i=1}^{n} (-1)^{i-1} (\sigma a_1, \dots, \sigma a_{i-1}, \sigma(a_i a_{i+1}), \dots, \sigma a_{n+1})$$

the sign being due to the fact that the odd elements $\sigma a_1, \dots, \sigma a_{i-1}$ are moved past the odd map $\tilde{m} p_2$.

Let's check now that $(b')^2 = 0$ in general. For general reasons it should be a derivation of degree -2, and hence determined by $p_1 (b')^2$.

$$p_1 (b')^2 = \tilde{m} p_2 b' = \tilde{m}(p_1 b' \otimes p_1 + p_1 \otimes p_1 b')\Delta^{(2)}$$

$$= \tilde{m}(\tilde{m} p_2 \otimes p_1 + p_1 \otimes \tilde{m} p_2)\Delta^{(2)}$$

$$= \tilde{m}(\tilde{m}\otimes 1 + 1 \otimes \tilde{m})\, p_3$$

But

$$\tilde{m}(\tilde{m}\otimes 1)(\sigma\otimes\sigma\otimes\sigma) = \tilde{m}((\tilde{m}(\sigma\otimes\sigma))\otimes\sigma) = \tilde{m}(\sigma m \otimes \sigma)$$

$$= \tilde{m}(\sigma\otimes\sigma)(m\otimes 1) = \sigma\, m(m \otimes 1)$$

$$\tilde{m}(1\otimes\tilde{m})(\sigma\otimes\sigma\otimes\sigma) = -\tilde{m}(\sigma\otimes\tilde{m}(\sigma\otimes\sigma)) = -\tilde{m}(\sigma\otimes\sigma m)$$

$$= -\tilde{m}(\sigma\otimes\sigma)(1\otimes m) = -\sigma\, m(1\otimes m)$$

So when added we get $p_1(b')^2=0$ using the associativity of m.

Next the differential d on L determines a differential d on ΣL such that $d\sigma+\sigma d=0$. This d extends to a coderivation on $T(\Sigma L)$ which we again denote d. It is such that $p_1 d = dp_1$. Omit verification that $b'+d$ is a differential, thereby making $T(\Sigma L)$ with $b'+d$ into a DG coalagebra.

Suppose $\theta : C \longrightarrow L$ is a twisting cochain, where C is a DG coalgebra. Thus θ is of degree -1 satisfying

$$d_L\theta + \theta d_C + m_L(\theta\otimes\theta)\Delta_C = 0$$

Then

$$\underbrace{\sigma d_L\theta}_{-d_{\Sigma L}\sigma\theta} + \sigma\theta d_C + \underbrace{\sigma m}_{\tilde m(\sigma\otimes\sigma)(\theta\otimes\theta) = -\tilde m(\sigma\theta\otimes\sigma\theta)}(\theta\otimes\theta)\Delta_C = 0$$

or

$$\boxed{(\sigma\theta)d_C = d(\sigma\theta) + \tilde m(\sigma\theta\otimes\sigma\theta)\Delta}$$

Let U be the extension of $\sigma\theta$ to a coalgebra map. To check ~~commutativity~~ of

$$\begin{tikzcd} C \arrow[r, "U"] \arrow[d, "d_C"] & T(\Sigma L) \arrow[d, "b'+d"] \\ C \arrow[r, "U"] & T(\Sigma L) \end{tikzcd}$$

it should be enough (since Ud_C and $(b'+d)U$ are coderivations, etc.) to check they agree after applying p_1. But

[redacted]

$$p_1 \mathcal{U} d_C = \sigma\theta d_C$$

$$p_1 b' \mathcal{U} = \tilde{m} p_2 \mathcal{U} = \tilde{m} (p_1 \mathcal{U})^{\otimes 2} \Delta$$

$$= \tilde{m} (\sigma\theta \otimes \sigma\theta) \Delta$$

$$p_1 d \mathcal{U} = d p_1 \mathcal{U} = d(\sigma\theta)$$

so this checks, and so $\mathcal{U}$ is a DG coalgebra map.

Next we [redacted] want to carry use this [redacted] discussion in the case of extensions. Thus we suppose $\rho : A \longrightarrow R$ is a homomorphism modulo I, and let $\omega = \delta\rho + \rho^2 : A^{\otimes 2} \longrightarrow I$ be its curvature. We [redacted] let L be the DGA $0 \longrightarrow I \longrightarrow R$ concentrated in degrees 0, 1. Thus $L_1 = I$, $L_0 = R$, and if we want to shift to the upper indexed $I = L^{-1}$, $R = L^0$.

Twisting cochain:

$$\begin{tikzcd} {} \arrow[r, "b'"] & A^{\otimes 2} \arrow[r, "m"] \arrow[dr, "\gamma"'] & A \arrow[r] \arrow[dr, "\rho"] & k \\ & & I \arrow[r, "i"] & R \end{tikzcd}$$

When is $\theta = \rho + \gamma$ a twisting cochain?

$$d_L \theta + \theta b' + m(\theta \otimes \theta)\Delta \overset{?}{=} 0$$

$$i\gamma + \rho b' + m(\rho \otimes \rho)\Delta)(a_1, a_2)$$

$$i\gamma(a_1, a_2) + \rho(a_1, a_2) - \rho(a_1)\rho(a_2) = 0$$

Thus $i\gamma = -\omega$

$$0 = \left(\gamma b' + m(\rho\otimes\underline{\gamma} + \gamma\otimes\rho)\Delta\right)(a_1,a_2,a_3)$$

$$= \gamma(a_1a_2,a_3) - \gamma(a_1,a_2a_3)$$

$$-\rho(a_1)\gamma(a_2,a_3) + \gamma(a_1,a_2)\rho(a_3)$$

which follows from the Bianchi identity for ω.

Now I propose, in order to simplify the signs, to redefine L so that the differential is $-i : I \to R$. The point is that we want to work with ΣL, and so it should be as simple as possible. So

$$\begin{tikzcd}
\Sigma L: & {} \arrow[r] & 0 \arrow[r] & \overset{2}{I} \arrow[r, hook, "i"] & \overset{1}{R} \arrow[r] \arrow[dr, "\sigma", "="'] & 0 \arrow[r] & {} \\
L: & & & & I \arrow[r, "-i"] \arrow[ul, "\sigma", "="'] & R &
\end{tikzcd}$$

Then we can take $i\gamma = \omega$ in the twisting cochain Θ. Thus $\sigma\Theta$ is the map

$$\begin{tikzcd}
{} \arrow[r, "b'"] & A^{\otimes 2} \arrow[r, "b'"] \arrow[d, "\omega"] & A \arrow[r] \arrow[d, "\rho"] & k \\
& I \arrow[r, "i"] & R. &
\end{tikzcd}$$

And its coalg extension $U: B(A) \longrightarrow B(L)$ is a DG coalgebra map.

So now what I need are the pictures and a rough check that it works.

$$\begin{tikzcd}
A^{\otimes 3} & R^{\otimes 3} \arrow[d, "b'"] & R^{\otimes 2}\otimes I + R\otimes I\otimes R + I\otimes R^{\otimes 2} \arrow[l] \arrow[d] & {} \arrow[d] \\
A^{\otimes 2} & R^{\otimes 2} \arrow[d, "m"] & R\otimes I \oplus I\otimes R \arrow[l, "{(-1\otimes i,\, i\otimes 1)}"'] \arrow[d, "{(m,-m)}"] & I^{\otimes 2} \arrow[l, "{(i\otimes 1,\, 1\otimes i)}"'] \\
A & R & I \arrow[l, "i"'] & \\
k & k & &
\end{tikzcd}$$

Here's how to obtain $\tilde{m} : \Sigma L \otimes \Sigma L \longrightarrow \Sigma L$.
Let $r \in R$, $z \in I$. Then

$$\tilde{m}(\sigma r \otimes \sigma z) = \tilde{m}(\sigma \otimes \sigma)(r \otimes z) = \sigma m(rz) = \sigma(rz)$$

$$\tilde{m}(\sigma z \otimes \sigma r) = -\tilde{m}(\sigma \otimes \sigma)(z \otimes r) = -\sigma m(zr) = -\sigma(zr)$$

For example suppose we want $b' : R\otimes I \otimes R \to R\otimes I \oplus I \otimes R$
We have $p_2 b' = (\tilde{m} \otimes 1 + 1 \otimes \tilde{m}) p_3$

$$\therefore\ b'(\sigma r_1, \sigma z, \sigma r_2) = (\tilde{m} \otimes 1 + 1 \otimes \tilde{m})(\underset{odd}{\sigma r_1}, \sigma z, \sigma r_2)$$

$$= (\sigma(r_1 z), \sigma r_2) \oplus (\sigma r_1, -\sigma(z r_2))$$

$$\text{so}\ \underbrace{\ }_{\text{deleting the } \sigma\text{'s}}\ b'(r_1, z, r_2) = (r_1 z, r_2) + (r_1, z r_2)$$

Thus [illegible] after deleting σ's, $\tilde{m} : \Sigma L \otimes \Sigma L \to \Sigma L$
is given by

$$\tilde{m}(r_1 \otimes r_2) = r_1 r_2 \qquad \tilde{m}(r, z) = rz$$

$$\tilde{m}(z, r) = -zr$$

and then b' is the coderivation with $p_1 b' = \tilde{m} p_2$

February 4, 1989

There's a formalism to be understood concerning $M_r A$ and A. There is a basic trace map on the cyclic complexes

$$tr : CC(M_r A) \longrightarrow CC(A)$$

given by

$$tr(\alpha^1, \dots, \alpha^n)_\lambda = \sum_{i_1, \dots, i_n} (\alpha^1_{i_1 i_2}, \alpha^2_{i_2 i_3}, \dots, \alpha^n_{i_n i_1})_\lambda$$

where the subscript λ denotes image in the cyclic complex: $A^{\otimes n}_\lambda$.

~~To understand this better, let S be an algebra~~ [illegible] ~~with a trace τ_S~~ [illegible] ~~Think of S as $M_r k$. We have~~ [illegible] ~~maps~~

~~$B(A \otimes S) \longrightarrow B(A) \otimes S \xrightarrow{1 \otimes \tau_S} B(A) \otimes S_\natural$~~

~~$(a_1 \otimes s_1, \dots, a_n \otimes s_n) \longmapsto (a_1, \dots, a_n) \otimes (s_1 s_2 \dots s_n)$~~

To understand this better let S be an algebra and think of S as $M_r k$. We have maps

$$B(A \otimes S) \longrightarrow B(A) \otimes S \longrightarrow B(A) \otimes S_\natural$$

$$(a_1 \otimes s_1, \dots, a_n \otimes s_n) \longmapsto (a_1, \dots, a_n) \otimes (s_1 \dots s_n) \longmapsto (a_1, \dots, a_n) \otimes \tau_S(s_1 \dots s_n)$$

which induce a map of complexes

$$B(A \otimes S)^\natural \longrightarrow B(A)^\natural \otimes S_\natural$$

which can be [illegible] identified with the above trace map when $S = M_r k$.

It would be nicer ~~to~~ explain the map $B(A \otimes S) \longrightarrow B(A) \otimes S$ from the universal property of the bar construction. At least when S is finite dimensional, one can proceed as follows. We have the coalgebra S^* and so $B(A\otimes S)\otimes S^*$ is a DG coalgebra. Moreover

$$B_1(A\otimes S)\otimes S^* = A\otimes S\otimes S^* \xrightarrow{1\otimes\langle\,,\rangle} A$$

should be a <u>twisting cochain</u>, whence we have a morphism of DGC's

$$\begin{tikzcd}
B(A\otimes S)\otimes S^* \arrow[r] & B(A) \\
B(A\otimes S)^\natural\otimes\underbrace{(S^*)^\natural}_{(S_\natural)^*} \arrow[r] \arrow[u, phantom, "\cup" description] & B(A)^\natural \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

which induces

and gives the map

$$B(A\otimes S)^\natural \longrightarrow B(A)^\natural\otimes S_\natural .$$

On the cochain level we have a map of DGA's:

$$\circledast \qquad \operatorname{Hom}(B(A),R) \longrightarrow \operatorname{Hom}(B(A\otimes S), R\otimes S)$$

$$f(a_1,\dots,a_n) \longmapsto \tilde{f}(a_1\otimes s_1,\dots,a_n\otimes s_n) = f(a_1,\dots,a_n)\otimes(s_1\cdots s_n)$$

which is the composition

$$\operatorname{Hom}(B(A),R) \xrightarrow{?\otimes 1} \operatorname{Hom}(B(A)\otimes S, R\otimes S) \xrightarrow{\text{transpose of } B(A\otimes S)\to B(A)\otimes S} \operatorname{Hom}(B(A\otimes S), R\otimes S)$$

We want the behavior with the trace, that is, a map

$$\operatorname{Hom}(B(A)^\natural, R_\natural) \longrightarrow \operatorname{Hom}(B(A\otimes S)^\natural, R_\natural\otimes S_\natural)$$

~~compatible~~ compatible with $\circledast$.

$$\begin{tikzcd}
\mathrm{Hom}(B(A),R) \arrow[rr, "?\otimes 1"] \arrow[ddd] & & \mathrm{Hom}(B(A)\otimes S, R\otimes S) \arrow[rr] \arrow[dd] & & \mathrm{Hom}(B(A\otimes S), R\otimes S) \arrow[ddd] \\
 & \mathrm{Hom}(B(A)\otimes S_\natural, R_\natural\otimes S_\natural) \arrow[ur, hook] \arrow[ddr] & & & \\
 & & \mathrm{Hom}(B(A)^\natural\otimes S, R_\natural\otimes S_\natural) & & \\
\mathrm{Hom}(B(A)^\natural, R_\natural) \arrow[rr, "?\otimes 1"] & & \mathrm{Hom}(B(A)^\natural\otimes S_\natural, R_\natural\otimes S_\natural) \arrow[u, hook] \arrow[rr] & & \mathrm{Hom}(B(A\otimes S)^\natural, R_\natural\otimes S_\natural)
\end{tikzcd}$$

$$\begin{tikzcd}
 & B(A)\otimes S \arrow[dl, phantom, sloped, "\subset"] \arrow[dr, two heads] & & B(A\otimes S) \arrow[ll] \arrow[dd, phantom, "\cup"] \\
B(A)^\natural\otimes S \arrow[dr, two heads] & & B(A)\otimes S_\natural \arrow[dl, phantom, sloped, "\subset"] & \\
 & B(A)^\natural\otimes S_\natural & & B(A\otimes S)^\natural \arrow[ll]
\end{tikzcd}$$

Thus we want to understand why the above square commutes. So let us begin with $f(a_1,\ldots,a_n) \in \mathrm{Hom}^n(B(A),R)$, and $\xi \in B(A\otimes S)^\natural_n \simeq (A\otimes S)^{\otimes n,\natural}$

To be specific suppose $\xi = N(a_1\otimes s_1, \ldots, a_n\otimes s_n)$. Then the upper right path applied to f and paired with ξ gives

$$\tau_R\otimes\tau_S\left\{\sum_{j\in\mathbb{Z}/n} (-1)^{(n-1)j}\, \tilde{f}\left(a_{1+j}\otimes s_{1+j}, \ldots, a_{n+j}\otimes s_{n+j}\right)\right\}$$

$$= (\tau_R\otimes\tau_S)\left\{\sum_{j\in\mathbb{Z}/n} (-1)^{(n-1)j} f(a_{1+j},\ldots,a_{n+j})\otimes(s_{1+j}\cdots s_{n+j})\right\}$$

$$= \underbrace{\tau_R\left\{\sum_{j\in\mathbb{Z}/n}(-1)^{(n-1)j} f(a_{1+j},\ldots,a_{n+j})\right\}} \otimes \tau_S(s_1\cdots s_n)$$

$$= \tau_R f N(a_1,\ldots,a_n) \otimes \tau_S(s_1\cdots s_n)$$

On the other hand the lower left path first sends f to $\tau_R^\natural(f) = \tau_R f \natural$ $\in \mathrm{Hom}^n(B(A)^\natural, R_\natural)$ and then to

$$\tau_R f \natural \otimes 1 \in \mathrm{Hom}(B(A)^\natural \otimes S_\natural, R_\natural \otimes S_\natural)$$

where it is to be applied to ~~paired with~~ the image of $\xi = N(a_1 \otimes s_1, \dots, a_n \otimes s_n)$ under the map $B(A\otimes S)^\natural \longrightarrow B(A)^\natural \otimes S_\natural$, which is induced by $B(A\otimes S) \longrightarrow B(A)\otimes S$.

$$N(a_1\otimes s_1, \dots, a_n \otimes s_n) = \sum_{j \in \mathbb{Z}/n} (-1)^{(n-1)j} (a_{1+j} \otimes s_{1+j}, \dots, a_{n+j} \otimes s_{n+j}) \quad \in B(A\otimes S)$$

$$\longmapsto \sum (-1)^{(n-1)j} (a_{1+j}, \dots, a_{n+j}) \otimes (s_{1+j} \cdots s_{n+j}) \in B(A)\otimes S$$

$$\longmapsto N(a_1, \dots, a_n) \otimes \tau_S(s_1 \cdots s_n) \in B(A)^\natural \otimes S_\natural$$

So the lower left route gives

$$(\tau_R f \natural \otimes 1)(N(a_1, \dots, a_n) \otimes \tau_S(s_1 \cdots s_n))$$

which is exactly the same.

February 10, 1989

Let C be a coalgebra, let R be an algebra, let X be a right C-comodule and let M be a left R-module. Then we claim that $X \otimes M$ is naturally a left $\mathrm{Hom}(C,R)$ module. We define a map

$$\mathrm{Hom}(C,R) \longrightarrow \mathrm{End}(X \otimes M)$$

by sending f to the operator $Op(f)$:

$$X\otimes M \xrightarrow{\Delta_X \otimes 1} X \otimes C \otimes M \xrightarrow{1\otimes f \otimes 1} X \otimes R \otimes M \xrightarrow{1 \otimes m} X \otimes M$$

The reason this is true is that X is naturally a left C^*-module, and so $X \otimes M$ is a left $(C^* \otimes R)$-module. Let's check this. Suppose $f,g \in C^*$. Then we have.

$$\begin{tikzcd}
X \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"'] \arrow[rr, bend left, "Op(f)"] \arrow[ddrr, bend right=60, "Op(gf)"'] & X\otimes C \arrow[r, "1\otimes f"] \arrow[d, "\Delta_X\otimes 1"] & X\otimes k \arrow[d, "\Delta_X\otimes 1"] \arrow[dd, bend left=60, "Op(g)"] \\
X\otimes C \arrow[r, "1\otimes\Delta_C"] \arrow[drr, "1\otimes gf"'] & X\otimes C\otimes C \arrow[r, "1\otimes 1\otimes f"] \arrow[dr, "1\otimes g\otimes f"] & X\otimes C\otimes k \arrow[d, "1\otimes g\otimes 1"] \\
 & & X\otimes k\otimes k
\end{tikzcd}$$

Thus $Op(gf) = Op(g)\, Op(f)$.

The same proof should work with $f,g \in \mathrm{Hom}(C,R)$. So take the above diagram and tensor with M

$$\begin{tikzcd}
X\otimes M \arrow[r, "\Delta_X\otimes 1"] \arrow[d, "\Delta_X\otimes 1"] & X\otimes C\otimes M \arrow[r, "1\otimes f\otimes 1"] \arrow[d, "\Delta_X\otimes 1\otimes 1"] & X\otimes R\otimes M \arrow[r, "1\otimes m"] \arrow[d] & X\otimes M \arrow[d, "\Delta_X\otimes 1"] \\
X\otimes C\otimes M \arrow[r, "1\otimes\Delta\otimes 1"] \arrow[rrdd, "1\otimes gf\otimes 1"] & X\otimes C\otimes C\otimes M \arrow[r, "1\otimes 1\otimes f\otimes 1"] \arrow[rd, "1\otimes g\otimes f\otimes 1"] & X\otimes C\otimes R\otimes M \arrow[r, "1\otimes 1\otimes m"] \arrow[d, "1\otimes g\otimes 1\otimes 1"] & X\otimes C\otimes M \arrow[d, "1\otimes g\otimes 1"] \\
 & & X\otimes R\otimes R\otimes M \arrow[r, "1\otimes 1\otimes m"] \arrow[d, "1\otimes m\otimes 1"] & X\otimes R\otimes M \arrow[d, "1\otimes m"] \\
 & & X\otimes R\otimes M \arrow[r, "1\otimes m"] & X\otimes M
\end{tikzcd}$$

So it works.

It is also clear that in the DG situation $\mathrm{Hom}(C,R) \longrightarrow \mathrm{End}(X\otimes M)$ is a map of DGA's. Thinking of bracketing with d as the differential on operators, then

$$[d, Op(f)] = [d, (1\otimes m)(1\otimes f\otimes 1)(\Delta\otimes 1)]$$
$$= (1\otimes m)[d, 1\otimes f\otimes 1](\Delta\otimes 1)$$
$$= (1\otimes m)(1\otimes [d,f]\otimes 1)(\Delta\otimes 1) = Op(df).$$

Thus it's clear that if θ is twisting cochain from C to R, then

$$d_{X\otimes M} + Op(\theta)$$

is a differential on $X\otimes M$.

Suppose next that Y is a left C-comodule and N is a right R-module. Then $N\otimes Y$ should be a right module over $\mathrm{Hom}(C,R)$. I checked that Y a left C-comodule is also a right C^*-module: $Op(fg) = Op(g)\,Op(f)$.

So let us suppose

$$\mathrm{Hom}(C,R) \xrightarrow{\Phi} \mathrm{End}(N\otimes Y)$$

is an anti-homomorphism but compatible with differentials. This means for a DG maps $S \longrightarrow S'$ to be an anti-hom that

$$\begin{tikzcd}
S\otimes S \arrow[r, "\Phi\otimes\Phi"] \arrow[d, "\sigma"] & S'\otimes S' \arrow[dd, "m_{S'}"] \\
S\otimes S \arrow[d, "m"] & \\
S \arrow[r, "\Phi"] & S'
\end{tikzcd}$$

commutes, i.e.

$$(-1)^{|g||f|}\Phi(gf) = \Phi(f)\Phi(g)$$

Thus if θ is a twisting cochain in $S = \mathrm{Hom}(C,R)$ so that $d\theta = -\theta^2$, we have

$$d\,\Phi(\theta) = \Phi(d\theta) = -\Phi(\theta^2) = \Phi(\theta)^2$$

so $-\Phi(\theta)$ is a twisting cochain.

$$\begin{tikzcd}
N\otimes Y \arrow[r, "1\otimes\Delta"] \arrow[d, "1\otimes\Delta"] & N\otimes C\otimes Y \arrow[r, "1\otimes f\otimes 1"] \arrow[d, "1\otimes 1\otimes\Delta"] & N\otimes R\otimes Y \arrow[r, "m_r\otimes 1"] \arrow[d, "1\otimes 1\otimes\Delta"] & N\otimes Y \arrow[d, "1\otimes\Delta"] \\
N\otimes C\otimes Y \arrow[r, "1\otimes\Delta\otimes 1"] \arrow[ddrr, "(-1)^{|f||g|}fg"'] & N\otimes C\otimes C\otimes Y \arrow[r, "1\otimes f\otimes 1\otimes 1"] \arrow[dr, "(-1)^{|f||g|}(1\otimes f\otimes g\otimes 1)"'] & N\otimes R\otimes C\otimes Y \arrow[r, "m_r\otimes 1\otimes 1"] \arrow[d, "1\otimes 1\otimes g\otimes 1"] & N\otimes C\otimes Y \arrow[d, "1\otimes g\otimes 1"] \\
 & & N\otimes R\otimes R\otimes Y \arrow[r, "m_r\otimes 1\otimes 1"] \arrow[d, "1\otimes m\otimes 1"] & N\otimes R\otimes Y \arrow[d, "m_r\otimes 1"] \\
 & & N\otimes R\otimes Y \arrow[r, "m_r\otimes 1"] & N\otimes Y
\end{tikzcd}$$

$\therefore$ ~~$\Phi_p(g)\,\Phi_p(f) = (-1)^{|f||g|}\Phi_p$~~

$$\boxed{g_r f_r = (-1)^{|f||g|}(fg)_r}$$

February 22, 1989

$R \supset I$ ideal $\qquad L = (\overset{0}{R} \xleftarrow{-1} \overset{1}{I} \leftarrow 0)$

Then we have an increasing algebra filtration on L

$$F_{-1}L = 0 \ , \quad F_0 L = R \ , \quad F_pL = L \quad p \geq 1.$$

with
$$gr(L) = (R \xleftarrow{0} I \leftarrow 0 \leftarrow \cdots)$$
the semi-direct product $R \oplus \Sigma I$. I recall that $B(L)$ is a bigraded differential coalgebra with qth row $L^{\otimes q}$. $B(gr L)$ is the same bigraded coalgebra with the same vertical differential b', but with horizontal differential set 0.

Let's review a good way to deal with increasing algebra filtrations

$$\begin{tikzcd} gr(A) & \bigoplus_p \hat{h}^p F_p A \arrow[l, "0 \mapsfrom h"'] \arrow[r, "\hat{h} \mapsto 1"'] \arrow[d, phantom, "\cap"] & A \\ & k[\hat{h}] \otimes A & \end{tikzcd}$$

($\hat{h}$ is the variable to be specialized to h) By specializing $\hat{h}$ one gets a family of algebras A_h.

Now I need to check that if I split the filtration then I can identify all the algebras A_h with $gr(A)$ as vector spaces. This ought to hold just for filtered vector spaces.

So suppose we have V with increasing

filtration F_pV and we consider $\bigoplus_p \hat{h}^p F_p V \subset k[\hat{h}]\otimes V$.

Suppose $F_nV = \bigoplus_{p\leq n} V_p$. Then

$$\bigoplus_n \hat{h}^n F_n V = \bigoplus_{p\leq n} \hat{h}^n V_p = \bigoplus_p \hat{h}^p k[\hat{h}]\otimes V_p$$

so when we specialize we get $\bigoplus V_p$. But also we see that if we have a map

$$\hat{h}^p k[\hat{h}]\otimes V_p \longrightarrow \hat{h}^q k[\hat{h}]\otimes V_q$$

over $k[\hat{h}]$, i.e. a map $V_p \longrightarrow \hat{h}^{q-p} k[\hat{h}]\otimes V_q$ then this becomes a polynomial map depending on h.

Lets apply this to L.

$$\bigoplus \hat{h}^n F_n L = \quad k[\hat{h}]\otimes R \;\oplus\; \hat{h}k[\hat{h}]\otimes \Sigma I$$

Better:

$$\begin{tikzcd} \bigoplus \hat{h}^n F_n L = & \Big(k[\hat{h}]\otimes R \arrow[d, equal] & \hat{h}k[\hat{h}]\otimes I \Big) \arrow[l, "-(\mathrm{incl})"'] \\ \cong & \Big(k[\hat{h}]\otimes R & k[\hat{h}]\otimes I \Big) \arrow[l, "-\hat{h}\otimes in"'] \arrow[u, "\hat{h}\otimes 1"'] \end{tikzcd}$$

so when we specialize we get $L_h : R \xleftarrow{-h} I$. This means when we go to the bar construction that we get the family of DG coalgebras all with the same graded coalgebra, but with differential $hd + b'$.

Next one should look at the bar construction of $\tilde{L} = \bigoplus \hat{h}^n F_n L$ relative to the ground ring $k[\hat{h}]$.

$$B(\tilde{L} \text{ rel } k[\hat{h}]) \xrightarrow[\hat{h}\mapsto h]{} B(L_h)$$

This is clear because

$$\Big(\tilde{L}\otimes_{k[\hat{h}]} \overset{n \text{ times}}{\cdots} \otimes_{k[\hat{h}]} \tilde{L}\Big)\otimes_{k[\hat{h}]} k_h = L_h^{\otimes n}$$

I guess one should also notice that in general for filtered v.s. $\{F_nV\}$, $\{F_nW\}$ one has

$$\Big(\bigoplus_n \hat{h}^n F_n V\Big)\otimes_{k[\hat{h}]}\Big(\bigoplus_n \hat{h}^n F_n W\Big)$$

$$= \bigoplus_n \hat{h}^n\Big(\sum_{p+q=n} F_pV\otimes F_qW\Big) \subset k[\hat{h}]\otimes(V\otimes W)$$

Thus an increasing filtration on an algebra A induces one on its bar construction, whose associated graded DG coalg is the bar construction of gr A.

Next what does this mean in terms of twisting cochains.

Let's consider the category of vector spaces with increasing filtration $\{F_nV\}$. Given such a thing we associate the graded $k[h]$-module

$$\bigoplus h^n F_n V \subset k[h]\otimes V$$

(set $\hat{h} = h$ to simplify writing). Assume $V = \bigcup F_nV$ and then we get an equivalence with flat graded $k[h]$ modules. ∃ Obvious tensor product on latter category which we have seen corresponds to

$$F_n(V\otimes W) = \sum_{p+q=n} F_pV\otimes F_qW \subset V\otimes W.$$

Since $V \longrightarrow gr\,V$ is the same as ? $\otimes_{k[h]} k$ one gets $gr(V\otimes W) = gr(V)\otimes gr(W)$.

If we are in characteristic zero then any of the tensor functors with symmetry conditions is given by an idempotent in the group ring of the permutation group. So applying this idempotent to $(\oplus h^n F_n V)^{\otimes_{k[h]} n}$ gives a filtration to the idempotent applied to $V^{\otimes n}$. So it's clear that things like $V^{\otimes n,\sigma}$ have canonical filtrations compatible with gr.

Now take an algebra A with increasing filtration $\{F_pA\}$. (Notice that $F_pA\cdot F_qA \subset F_{p+q}A$ implies that F_pA is an ideal in F_0A for $p<0$.) This simply means that we have a map $A\otimes A \longrightarrow A$ in our category making A an algebra in our category.

When we form $B(A)$ we get a DG coalgebra in this tensor category. I guess we should think of having

$$B(\oplus h^n F_n A \text{ rel } k[h])$$

This is going to be a coalgebra over $k[h]$ with a $\mathbb{Z}$-grading compatible with the grading on $k[h]$.

Now what does this mean?

Suppose then we have a filtered coalgebra $C = \bigcup F_n C$. Then $\Delta : C \longrightarrow C \otimes C$ is a map of filtered vector spaces which means that $\Delta(F_n C) \subset \sum_{p+q=n} F_p C \otimes F_q C$. This implies that when we consider cochains we have a decreasing filtration F^p = annihilator of $F_{p-1}C$ satisfying $F^p \cdot F^q \subset F^{p+q}$. In effect given $f(F_p C) = 0$, $g(F_q C) = 0$ then

$$(fg)(FC_{p+q-1}) = m(f \otimes g) \sum_{p'+q'=p+q-1} F_{p'}C \otimes F_{q'}C$$

$$= \sum_{p'+q'=p+q} f(F_{p'}C)\, g(F_{q'}C) = 0$$

since either $p' \lneq p$ or $q' \leq q$

March 8, 1989

Review ideas relevant to the Noivkov conjecture. Let Γ be a discrete group and $P \longrightarrow M$ a principal Γ-bundle. We have a map

$$\mathrm{Repn}(\Gamma) \longrightarrow \mathrm{Vect}(M)$$

The classes in $K^0(M)$ obtained from representations of Γ are very special, since $ch(\text{flat bundle}) = 0$.

Lusztig's idea: Consider a family of repns. of Γ parametrized by a manifold Y. This is a vector bundle E over Y on which Γ operates. There is an induced v.b. over $Y \times M$

better notation:

$$\begin{tikzcd} E \times P \arrow[r] \arrow[d] & P \times^{\Gamma} E \arrow[d] \\ Y \times P \arrow[r] & Y \times M \end{tikzcd} \qquad \begin{tikzcd} P \times E \arrow[r, "\Gamma"] \arrow[d] & P \times^{\Gamma} E \arrow[d] \\ P \times Y \arrow[r, "\Gamma"] & M \times Y \end{tikzcd}$$

So we have an element of $K^0(Y \times M)$ which then can be "contracted" against elements in $K_0(Y)$ to give elements of $K^0(M)$.

Lusztig uses this in the case $\Gamma = \mathbb{Z}^n$ where Y = torus of characters of Γ. M is the torus $\mathbb{R}^n/\mathbb{Z}^n$ and Y is the dual torus, and E is the Poincare line bundle over $M \times Y$. One knows I think that $ch(E) \in H^*(M \times Y) = H^*(M) \otimes H^*(Y)$ gives an isomorphism of $H_*(Y)$ with $H^*(M)$.

Miscenko idea: Instead of a representation of Γ consider a pair of Hilbert representations

$H^{\pm}$ and a Fredholm operator $H^{+} \rightleftarrows H^{-}$ invariants under compacts modulo Γ. Then we have Hilbert bundles $P \times^{\Gamma} H^{\pm}$ over M. Consider the coset C of the Fred op modulo compacts; this is contractible, so the fibre bundle $P \times^{\Gamma} C$ over M has a section. This gives a Fredholm operator between the Hilbert bundles, so we obtain a class in $K^0(M)$.

There is a "map" Lusztig $\longrightarrow$ Miscenko as follows. Suppose the element of $K_0(Y)$ represented by a Dirac type operator $\not{D} : S^{+} \rightleftarrows S^{-}$. Given the family of repns of Γ, i.e. a vector bundle E over Y with Γ operating, we choose a connection ∇ on E and form the Dirac operator $\not{D} \otimes 1 + 1 \otimes \nabla$ on $S \otimes E$. Then "the" associated Fredholm will not be Γ-invariant, since ∇ is not, but it is invariant modulo compacts. ("The" associated Fredholm is a ΨDO of order zero whose symbol is Γ-invariant.)

What is the Novikov conjecture? Given a compact oriented manifold M and a principal Γ-bundle P over M, we take the homology class $\mathcal{L}(M) \cap [M] \in H(M)$ and push forward under the classifying map $M \longrightarrow B\Gamma$ to get a homology class

$$\operatorname{Im}\{\mathcal{L}(M) \cap [M]\} \in H_{\cdot}(B\Gamma)$$

The NC says this is a homotopy invariant of M. This means that given a h.eq. $M' \xrightarrow{f} M$ of compact oriented manifolds, that even though $f^* \mathcal{L}(M) \neq \mathcal{L}(M')$ in general, nevertheless the homology classes in $H_\cdot(B\Gamma)$ are the same.

An equivalent version using K-theory goes as follows. If we choose a Riemannian metric on M, then we obtain a signature operator. This represents a class in $K_0(M)$; I guess I am tacitly assuming M even-dimensional. Then given any class in $K^0(M)$ it can be paired with the signature operator class to give an index $\in \mathbb{Z}$. The NC asserts that for virtual bundles coming from $K^0(B\Gamma)$ this index is homotopy invariant.

Thus the central issue in the NC is to get control of virtual bundles over $B\Gamma$. (I should have added that to get started one can suppose $B\Gamma$ finite dimensional.)

There appear to be two approaches. The first which starts with Lusztig's idea and then Miscenko + Kasparov proposes to find analytical ways to represent elements of $K(B\Gamma)$. Thus one is exploring generalized representations of Γ in this approach. The second approach is that of Connes, where ~~one tries to~~ finds a group into which $K^0(B\Gamma)$ maps.

Thus I think Connes first applies the Chern character, [redacted] which converts to cohomology. Then a group cocycle is apparently viewed as a cyclic [redacted] cocycle. The Gromov machine is used to find bounded cocycles representing the cohomology classes.

Let's try to visualize things as follows. A key thing to keep track of is what one can prove homotopy invariance for, but unfortunately I don't know these arguments so we guess. The K-picture is

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Miscenko}\\ \text{gadgets}\end{matrix}\right\} \arrow[r] \arrow[dr, bend right, "\text{known to be h.inv.}"'] & K^0(B\Gamma) \arrow[d, "\cap[\sigma_M]"] \\
 & \mathbb{Z}
\end{tikzcd}$$

σ_M signature op on M.

The coh picture is

$$\begin{tikzcd}
\left\{\begin{matrix}\text{cyclic cocycles}\\ \text{bounded?}\end{matrix}\right\} \arrow[r] \arrow[dr, bend right, "\text{known to}"'] & H^{\cdot}(B\Gamma) \arrow[d, "\cap(\mathcal{L}(M)[M])"] \\
 & \mathbb{Q}
\end{tikzcd}$$

be homotopy inv
as these cocycles extension
to $\mathcal{S}(\Gamma)$ which has same K-theory
as $C_r(\Gamma)$ where one has h. invariance

March 9, 1989

Comments on NC. The NC by itself is a good problem, but should not be the whole story. The striking point about Connes-Moscovici is the use of group cohomology. There is a direct link between group cocycles and analysis that somehow occurs via cyclic theory.

The first idea must be that differential operators can be twisted by flat bundles. This must somehow be the ultimate reason why the NC is true. However it is too vague to be really useful.

There is another idea that given $P \xrightarrow{\Gamma} M$ with M compact one can carry out the analysis upstairs where things are "projective" over the group ring of Γ. Here group algebra means $C_r(\Gamma)$ because in the ℓ^2 situation one has $L^2(P) = L^2(M) \otimes \ell^2(\Gamma)$; this is Atiyah's observation.

Use of a cohomology class on Γ, a group cocycle on Γ, is the most striking aspect. Why should it be possible to assign analytical meaning to a group cocycle?

Alexander-Spanier cohomology. Where are the motivating examples?

March 13, 1989

Observation: There is a canonical additive isomorphism $A * A \simeq \Omega_A$.

Let's first describe this on the even subalgebra $B = (A*A)^+ = T_r(A)$. We have the filtration (decreasing algebra filtration)

$$B \supset I \supset I^2 \supset \cdots$$

and we have the increasing filtration

$$k \subset \rho(A) \subset \rho(A)^2 \subset \rho(A)^3 \subset \cdots$$

(increasing algebra filtration). We have the graded algebras

$$\bigoplus_{n\geq 0} I^n/I^{n+1} = \Omega_A^{ev}$$

$$\bigoplus_{n\geq 0} \rho(A)^n/\rho(A)^{n-1} = T(\bar{A})$$

The point is that the filtration

$$\rho(A) \subset \rho(A)^3 \subset \rho(A)^5 \subset \cdots$$

is complementary to the I-adic filtration:

$$\rho(A)^{2n-1} \oplus I^n = B$$

so we get a canonical additive isomorphism of B with $gr^I(B) = \Omega_A^{ev}$.

To find it let's consider $a_0 da_1 da_2$. Then observe that

$$\begin{aligned} a_0^+ a_1^- a_2^- &= a_0^+\left((a_1a_2)^+ - a_1^+a_2^+\right) \\ &= a_0^+(a_1a_2)^+ - a_0^+a_1^+a_2^+ \end{aligned}$$

belongs to $I \cap \rho(A)^3$ so the canonical isom.

identifies $a_0^+ a_1^- a_2^- \longleftrightarrow a_0\,da_1\,da_2$
similarly

$$a_0^+ a_1^- \cdots a_{2n}^- \in \text{[redacted]}\ \rho(A)(\rho(A)^2)^n \cap I^n = \rho(A)^{2n+1} \cap I^n$$

and so our isomorphism, which gives

$$a_0^+ a_1^- \cdots a_{2n}^- \longleftrightarrow a_0\,da_1 \cdots da_{2n},$$

is the canonical isomorphism. [redacted] Note that

$$a_1^- \cdots a_{2p}^- \, a_{2p+1}^+ \, a_{2p+2}^- \cdots a_{2n}^- \mapsto da_1 \cdots da_{2p}\,(a_{2p+1} - da_{2p+1}d)\,da_{2p+2} \cdots da_{2n} 1$$

$$= da_1 \cdots da_{2p}\, a_{2p+1}\, da_{2p+2} \cdots da_{2n}$$

So we get [redacted] various descriptions of the subspace of B corresponding to Ω_A^{2n} as the image of any of the cochains

[redacted] $\alpha^{2p} \rho\, \alpha^{2n-2p}$ $\qquad \alpha = \theta^- \qquad \rho = \theta^+$

Let's turn next to $A*A + \Omega_A$. We consider the decreasing algebra filtration

$$A*A \supset J \supset J^2 \supset \cdots\cdots$$

where J is the kernel of the folding map $A*A \longrightarrow A$. We need an increasing filtration. $A*A$ is generated by the elts. a^+, a^- $a \in A$. $a^- \in J$. Let's consider products of $n+1$ elements

$$\circledast \qquad a_1^- \cdots a_p^- \, \alpha^+ \, a_{p+1}^- \cdots a_n^-$$

where n are from [redacted] $\alpha(\bar{A})$ and one is from $\rho(A)$. Such a product corresponds under our additive isomorphism $A*A \simeq \Omega_A$ with

$$da_1 \cdots da_p(\alpha - d\alpha d)da_{p+1} \cdots da_n 1 = da_1 \cdots da_p\, \alpha\, da_{p+1} \cdots da_n$$

Thus the span of ⊛ for a fixed p is isom. to

Ω_A^n. One gets a rather funny increasing filtration. The operator $a^+ = a - dad$ has order 2, $a^- = da$ has order 1. Yet $a_1^+ a_2^+ = (a_1a_2)^+ - a_1^- a_2^-$ has order 2 instead of 4.

So we have learned that there is a canonical additive isomorphism of $A * A$ with Ω_A. It is the one we have been using, however we now know it is independent of writing differential forms in the form $a_0 da_1 - da_n$. This observation should be of use in calculating traces.

Let's discuss traces. I have yet to [redacted] understand Connes [redacted] link between $b+B$ cocycles and traces on the Cuntz + CZ algebras. I have not yet proved that his theorem is a description of supertraces on $A*A$.

[redacted] Let $L = L^+ \oplus L^-$ be a superalgebra. Then we have the trace and supertrace groups

$$L/[L,L]_{super} = \left(L^+/[L^+,L^+] + \{L^-,L^-\}\right) \oplus \left(L^-/[L^+,L^-]\right)$$

$$L/[L,L]_{ord} = \left(L^+/[L^+,L^+] + [L^-,L^-]\right) \oplus \left(L^-/[L^+,L^-]\right)$$

reflecting the fact that odd traces coincide with odd supertraces - these are both traces on L^- considered as an L^+ bimodule. Also even traces and even supertraces are special kinds of traces on the algebra L^+.

Note that

$$[L^-,L^-] + \{L^-,L^-\} = (L^-)^2$$

When $L = A * A$, then $(L^-)^2 = I \subset B$ since I is spanned by $a_0 a_1^- \cdots a_{2n}^-$, $n \geq 1$.

Diagram:

$$\begin{tikzcd}
L_+/[L_+,L_+] \arrow[r] & L_+/[L_+,L_+] + [L_-,L_-] \\
& \oplus \\
& L_+/[L_+,L_+] + \{L_-,L_-\} \arrow[d, bend left] \\
& L_+/[L_+,L_+] + (L_-)^2 \ (\simeq A/[A,A]) \arrow[r] & 0
\end{tikzcd}$$

If the first arrow is injective, then any trace on L^+ is the sum of an even trace and even supertrace on L.

This is not so unreasonable to expect, since we know any trace on $L^+ = B$ extends to a trace on $(A * A) \tilde{\otimes} \mathbb{C}[F]$, which we can then act on ~~by~~ the "dual" automorphism.

Remark: $A^+ A^- = A^- A^+$ in $A * A$, since clearly these two spaces contain A^- and we have $a_0^+ a_1^- = (a_0 a_1)^- - a_0^- a_1^+$.

The notation $A^\pm$ is in unfortunate conflict with the notation for a superalgebra. Suggest A^S and A^α?

March 14, 1989

Consider the GNS construction in the case of $\hat{\rho} : A \longrightarrow T_{rd}(A) = (A * A)^{+}$. In this case we want to show that the map

$$\begin{aligned} \Phi : A \otimes B &\longrightarrow Hom(A, B) \\ a \otimes b &\longmapsto (\alpha \mapsto \rho(\alpha a) b) \end{aligned}$$

is injective. I recall that the possible (E, ι, ι^*) are factorizations of this map. Thus $A \otimes B$ is the smallest possibility and it maps to any other.

Consider the direct sum decomposition

[redacted] [redacted]

$$\begin{tikzcd} \bar{a} \otimes b & a \otimes b \arrow[l, mapsto] \arrow[r, mapsto] & \rho(a) b \\ \bar{A} \otimes B \arrow[r, shift left] & A \otimes B \arrow[l, shift left] \arrow[r, shift left] & B \arrow[l, shift left] \\ & 1 \otimes b & b \arrow[l, mapsto] \end{tikzcd}$$

$$\bar{a} \otimes b \longmapsto a \otimes b - 1 \otimes \rho(a) b$$

We have diagram

$$\begin{tikzcd} 0 \arrow[r] & \bar{A} \otimes B \arrow[r] \arrow[d, "\Phi'"] & A \otimes B \arrow[r] \arrow[d, "\Phi"] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Hom(\bar{A} \otimes B) \arrow[r] & Hom(A, B) \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Then $\bar{a} \otimes b \longmapsto a \otimes b - 1 \otimes \rho(a) b \xmapsto{\Phi} (\alpha \mapsto \rho(\alpha a) b - \rho(\alpha) \rho(a) b)$

so $\Phi'(\bar{a} \otimes b) = (\alpha \longmapsto (\rho(\alpha a) - \rho(\alpha)\rho(a)) b)$. By the diagram Φ' has the same kernel and cokernel as Φ. But Φ' is clearly injective, because [redacted] if α is a non-zero element of $\bar{A}$, then [redacted] $\Phi'(\bar{a}\otimes b)(\alpha)$

$$= (\rho(\alpha a)-\rho(\alpha)\rho(a))b = \alpha^- a^- b \in (A*A)^+$$

and multiplication by α^- is injective on $A*A$. [redacted]

I guess the real lesson of all this is that the **GNS** module E is $A*A$, where A acts [redacted] via the embedding in the first factor, where ι, ι^* are the embedding and projection onto $B=(A*A)^+$, and where B acts by right multiplication. Let's check this

$$\begin{array}{ccc} A\otimes B & \longrightarrow & A*A \\ a\otimes b & \longmapsto & ab = a^+b + a^-b \end{array}$$

Then $\iota^*(aib) = (ab)^+ = a^+b = \rho(a)b$, so the above map is compatible with the ι, ι^* maps. Then you want to see it is bijective. $A\otimes B = 1\otimes B \oplus \mathrm{Im}(\bar{A}\otimes B)$, where the second factor is isom to $\bar{A}\otimes B$ via $\bar{a}\otimes b \longrightarrow a\otimes b - 1\otimes\rho(a)b$. This gives map $\bar{A}\otimes B \longrightarrow A*A$

$$\bar{a}\otimes b \longmapsto a\otimes b - 1\otimes\rho(a)b \longmapsto ab - a^+b = a^-b$$

which we know gives an isom $\bar{A}\otimes B = (A*A)^-$.

General discussion. Let e be an idempotent in a unital alg. R. Recall the functors

$$\mathrm{Mod}(eRe) \rightleftarrows \mathrm{Mod}(R)$$

which are obtained by

$$M \longmapsto eM = eR\otimes_R M$$

and

$$N \longmapsto Re\otimes_{eRe} N.$$

Since $eR\otimes_R Re = eRe$, we have a retraction of Mod(R) onto Mod(eRe). The composition the other way is an idempotent operation on Mod(R) given by the R-bimodule

$$Re\otimes_{eRe} eR$$

which we know is a "universal cover" of the ideal ReR, whose square is itself.

Note that when $R = A\oplus A\otimes B\otimes A$ is a GNS algebra, then

$$Re = A\otimes B \qquad eR = B\otimes A$$

and

$$Re\otimes_{eRe} eR = (A\otimes B)\otimes_B (B\otimes A) = A\otimes B\otimes A = ReR.$$

Hence

$$ReR\otimes_R \;=\; Re\otimes_{eRe} eR\otimes_R \;=\; eR\otimes_R Re\otimes_{eRe}$$

$$= eRe\otimes_{eRe}$$

as we have noted already (trace on $A\otimes B\otimes A$ considered as R-bimodule is same as a trace on B.)

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Traces. Let τ be a linear ful on $A*A$ and put

$$\psi_{(n)}(a_0, a_1, \ldots, a_n) = \tau(a_0^+ a_1^- \cdots a_n^-)$$
$$\parallel$$
$$\psi_{n+1}(a_0, \ldots, a_n)$$

General notation on subscripts: (n) = Connes indexing
n = degree as multilinear ful.

$$(b\psi_n)(a_0, \ldots, a_n) = \tau\big((a_0a_1)^+ a_2^- \cdots a_n^-\big) \qquad \cancel{a_0^+a_1^+} + a_0^-a_1^-$$
$$-\tau\big(a_0^+ (a_1a_2)^- a_3^- \cdots\big) \qquad \cancel{a_1^+a_2^-} + a_1^-a_2^+$$
$$\cdots\cdots$$
$$+(-1)^{n-1}\tau\big(a_0^+ a_1^- \cdots (a_{n-1}a_n)^-\big) \qquad \cancel{a_{n-1}^+a_n^-} + a_{n-1}^-a_n^+$$
$$+(-1)^n \tau\big((a_na_0)^+ a_2^- \cdots a_{n-1}^-\big) \qquad a_n^+a_0^+ + a_n^-a_0^-$$

$$\boxed{\begin{aligned}(b\psi_n)(a_0, \ldots, a_n) &= (-1)^n \tau\big([a_n^+, a_0^+ a_1^- \cdots a_{n-1}^-]\big)\\ &\quad + \tau(a_0^- \cdots a_n^-) + (-1)^n \tau(a_n^- a_0^- \cdots a_{n-1}^-)\end{aligned}}$$

Prop. 1) Assume τ linear functional on $(A*A)^-$. Then τ is a trace on $(A*A)^-$ as $(A*A)^+$-bimodule iff

a) $\tau(a_0^- a_1^- \cdots a_{2n}^-)$ cyclically symmetric $n \geq$ [illegible]

b) $b\psi_{(2n-1)} = \frac{2}{2n+1} B\psi_{(2n+1)}$ $n \geq 1.$

2) Assume τ linear ful. on $(A*A)^+$. Then it is a trace on the alg. $(A*A)^+$ iff

a) $\tau(a_0^- a_1^- \cdots a_{2n-1}^-)$ σ^2-invariant for $n \geq 1$

b) $b\psi_{\underset{(2n-2)}{[illegible]}} = \frac{1}{n\,[illegible]} B\psi_{(2n[illegible])}$ $n \geq 1$

Remark: Because $a_1^- a_2^- = (a_1 a_2)^+ - a_1^+ a_2^+$ $\tau(a_0^- a_1^- \cdots a_n^-)$ is invariant under cyclic shifting by two steps. So if n is odd, it is $\sigma = \lambda$-invariant.

So let us start with a trace on τ on the algebra $(A*A)^+$. Then we know that it extends to a trace on the CZ algebra $(A*A) \tilde{\otimes} k[F]$ unique up to a trace on A. In fact we know that τ on $\underline{I}$ will determine a "complementary" trace on I, and the pair gives the trace on the CZ algebra. Now I need to work out the formulas.

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March 15, 1989

Let τ be a linear map defined on $A * A$ and set

$$\psi_{n+1}(a_0, a_1, \dots, a_n) = \tau(a_0^+ a_1^- \cdots a_n^-)$$

$$\varphi_n(a_1, \dots, a_n) = \tau(a_1^- \cdots a_n^-)$$

Then we have the following identities.

$$(b\psi_n)(a_0, \dots, a_n) = \overbrace{\tau(a_0^- a_1^- \cdots a_n^-) + (-1)^{n-1}\tau(a_0^+ a_1^- \cdots a_{n-1}^- a_n^+)}^{b'\psi_{n-1}}$$
$$+ (-1)^n \tau(a_n^+ a_0^+ a_1^- \cdots a_{n-1}^-) + (-1)^n \tau(a_n^- a_0^- \cdots a_{n-1}^-)$$

$$\boxed{\begin{aligned}(b\psi_n)_1^{(a_0,\dots,a_n)} &= (-1)^n \tau([a_n^+, a_0^+ a_1^- \cdots a_{n-1}^-]) + (-1)^n \tau([a_n^-, a_0^- \cdots a_{n-1}^-]_{sup}) \\ &\quad + 2\tau(a_0^- \cdots a_n^-)\end{aligned}}$$

$$(b\varphi_n)(a_0, \dots, a_n) = \overbrace{\tau(a_0^+ a_1^- \cdots a_n^-) + (-1)^n \tau(a_0^- a_1^- \cdots a_{n-1}^- a_n^+)}^{b'\varphi_n}$$
$$+ (-1)^n \tau(a_n^+ a_0^- \cdots a_{n-1}^-) + (-1)^n \tau(a_n^- a_0^+ a_1^- \cdots a_{n-1}^-)$$

$$\boxed{(b\varphi_n)(a_0, \dots, a_n) = \tau([a_0^+ a_1^- \cdots a_{n-1}^-, a_n^-]_{sup}) + (-1)^n \tau([a_n^+, a_0^- \cdots a_{n-1}^-])}$$

If τ is a supertrace on $A * A$, then we see that φ_n is a cyclic cocycle. This is consistent with our past picture

$$\begin{tikzcd}
\psi \arrow[r, mapsto, "1-\lambda"] & {} \arrow[r, mapsto, "N"] & 0 \\
 & \varphi \arrow[u, "b'"] \arrow[r, mapsto, "N"] & {} \arrow[u, "b"]
\end{tikzcd}$$

Prop. τ is a supertrace on $A*A$ iff

$$\begin{cases} b\psi_n = \frac{2}{n+1} B\psi_{n+2} & \text{for } n \geq 1 \\ b\varphi_{n+1} = 0 & \text{for } n \geq 0 \end{cases}$$

Proof. Let's write the identities

$$(b\psi_n)(a_0,..,a_n) = (-1)^n \tau([a_n^+, a_0^+ a_1^- .. a_{n-1}^-]) + \underbrace{\tau(a_0^- .. a_n^-) + (-1)^n \tau(a_n^- a_0^- .. a_{n-1}^-)}_{(1+\lambda)\varphi_{n+1}}$$

$$(b\varphi_{n+1})(a_0,..,a_{n+1}) = \tau([a_0^+ a_1^- .. a_n^-, a_{n+1}^-]_{sup}) + (-1)^{n+1} \tau([a_{n+1}^+, a_0^- .. a_n^-])$$

Assume τ is a supertrace on $A*A$. Then φ_{n+1} is λ-invariant. The above identities give

$$b\varphi_{n+1} = 0, \quad b\psi_n = (1+\lambda)\varphi_{n+1} = \frac{2}{n+1} N\varphi_{n+1} = \frac{2}{n+1} B\psi_{n+2}$$

Conversely assume these ~~identities~~ equations satisfied by φ, ψ. Then $b\varphi_{n+1} = 0$, taking $a_0 = 1$ in this equation, yields $\tau([a_1^- .. a_n^-, a_{n+1}^-]_{sup}) = 0$, so φ_{n+1} is λ-invariant. Then $B\psi_{n+2} = N\varphi_{n+1} = (n+1)\varphi_{n+1}$, so $b\psi_n = 2\varphi_{n+1} = (1+\lambda)\varphi_{n+1}$, and we find $\tau([a_n^+, a_0^+ a_1^- .. a_{n-1}^-]) = 0$. Now consider this eqn. for the next level $n+2$, whence $\tau([a_{n+1}^+, a_0^- .. a_n^-]) = 0$. Then $b\varphi_{n+1} = 0 \Longrightarrow \tau[a_0^+ a_1^- .. a_n^-, a_{n+1}^-]_{sup} = 0$

Notice however that if you know that $\varphi_{n+1}(a_1,..,a_{n+1}) = \tau(a_1^- .. a_{n+1}^-)$ is λ-invariant, then

$$b\psi_n = \frac{2}{n+1} B\psi_{n+2} = \frac{2}{n+1} N\varphi_{n+1} = 2\varphi_{n+1}$$

and so you can conclude $b\varphi_{n+1} = 0$ since $b^2 = 0$. Thus we obtain Connes theorem describing the supertraces on $A * A$ as sequences of [redacted] normalized Hochschild cochains $\psi_{(n)}(a_0, \dots, a_n) = \tau(a_0^+ a_1^- \dots a_n^-)$ $n \geq 0$ satisfying the conditions

$$b\psi_{(n)} = \frac{2}{n+2} B\psi_{(n+2)} \quad ; \quad \psi_{(n)}(1, a_1, \dots, a_n) \ \lambda\text{-inv.}$$

Now that Connes theorem is under control we should tackle the relation with traces on B. The rough idea which has to be made more precise is that [redacted] there is a simple relation between traces on B and traces on the CZ algebra, and since $\mathbb{Z}/2$ acts on the latter, there is an action of $\mathbb{Z}/2$ on the former.

Structure on CZ algebra: $(A * A) \hat{\otimes} k[F]$. It is bigraded with respect to $\mathbb{Z}/2$, i.e. graded wrt $\mathbb{Z}_2 \times \mathbb{Z}/_2$.

Discussion of the general issues. [redacted] It seems that the good starting point for cyclic cohomology is with traces on extensions, since this leads most rapidly to the S-operation and periodic cyclic cohomology. Now if we adopt this viewpoint the use of cyclic [redacted] cochains is of secondary importance.

Now it would seem the choice of a linear lifting $\rho : A \longrightarrow R$ constitutes a rigidification sufficient to define [redacted] whatever cocycle one might use to represent the cyclic class of the higher trace. This seems fairly obvious because such a lifting ρ determines a map from $B = T_r(A)$ to R. So the problem becomes one of describing the equivalence classes.

A natural question is how are we to handle things in this picture. For example a trace defined on I^m for some m, how is this to be converted to something like a periodic cyclic cocycle. It seems like a trace on B/I^m might give rise to a periodic cyclic cocycle, but the other case is a mystery.

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March 21, 1989

Program: The original idea was to discuss GNS and the Cuntz algebra $A * A$, then use these as tools to study cyclic cohomology. There are two ideas which can be used. The first is the NR idea. There should be a way of doing connection + curvature + ch + cs [redacted] [redacted] in this picture which would appear very natural with extensions. The second idea is Connes description of supertraces on $A * A$ in terms of $b+B$ cocycles. I have the feeling that there's a whole theory to be [redacted] [redacted] developed which ties these ideas all together. [redacted]

Concrete questions: What is the analogue of CS forms in the NR picture? Given a trace on R/I^m do[redacted] the associated $b+B$ cocycle and the CS form represent the same periodic cyclic class?

It is important I think to begin on the extension side and not on the Connes-Cuntz side, since ~~you~~ I have a feeling for extensions and some idea of what's intrinsic.

Let $A = R/I$, nonunital setting, R free. Ultimately $R = \bar{T}(A)$. Recall from the past summer that we have two complexes which are quasi-isomorphic to the cyclic complex of A. One is a subcomplex of the periodic complex

$$\xrightarrow{\beta} R \xrightarrow{\bar{\partial}} \Omega^1_{\tilde{R}} \otimes_R \xrightarrow{\beta} R \xrightarrow{\bar{\partial}}$$

and the other is the quotient complex. Moreover given a lifting $\rho : A \to R$ we have explicit quasi-isoms from $CC(A)$ to these complexes. Review the construction. We start from the exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & \Omega^1_{\tilde{R}} \otimes_R I^n \otimes_R \arrow[r, "\beta"] \arrow[d] & I^n \arrow[r] \arrow[d] & \Omega^1_{\tilde{R}} \otimes_R I^{n-1} \otimes_R \arrow[r] \arrow[d] & I^{n-1} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \Omega^1_{\tilde{R}} \otimes_R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & \Omega^1_{\tilde{R}} \otimes_R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \Omega^1_{\tilde{R}} \otimes_R R/I^n \otimes_R \arrow[r] \arrow[d] & R/I^n \arrow[r] \arrow[d] & \Omega^1_{\tilde{R}} \otimes_R (R/I^{n-1}) \otimes_R \arrow[r] \arrow[d] & R/I^{n-1} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

At the circled points the homology coincides with the cyclic homology but at the other places, i.e. the I^n and $\Omega^1_{\tilde{R}} \otimes_R R/I^n \otimes_R$, it is too big. We have to cut I^n down a bit.

$$\begin{tikzcd}
I^{n+1}/[R, I^{n+1}] \arrow[r, phantom, "="] \arrow[d] & (I \otimes_R)^{n+1} \arrow[d] \arrow[dr, dashed, "\text{exists}"] & \\
I^n/[R, I^n] \arrow[r, phantom, "="] \arrow[d] & (I \otimes_R)^n \arrow[r, phantom, "\supset"] \arrow[d] \arrow[dr, phantom, "\text{bicart}"] & (I \otimes_R)^{n,\sigma} \arrow[d] \\
I^n/I^{n+1} + [R, I^n] \arrow[r, phantom, "="] \arrow[d] & (I/I^2 \otimes_A)^n \arrow[r, phantom, "\supset"] \arrow[d] & (I/I^2 \otimes_A)^{n,\sigma} \arrow[d] \\
0 & 0 & 0
\end{tikzcd}$$

This shows that the lower right square is bicartesian

The point is that there is a subspace between $I^{n+1}+[R,I^n]$ and I^n, which I will denote $I^{n,\sigma}$, although there is no action of σ on I^n, such that

$$I^{n,\sigma}/[R,I^n] = (I\otimes_R)^{n,\sigma}$$

$$I^{n,\sigma}/I^{n+1}+[R,I^n] = (I/I^2\otimes_A)^{n,\sigma}$$

With this definition we get the exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & \Omega^1_{\tilde{R}}\otimes_R I^n\otimes_R \arrow[r, "\beta"] \arrow[d] & I^{n,\sigma} \arrow[r] \arrow[d] & \Omega^1_{\tilde{R}}\otimes_R I^{n-1}\otimes_R \arrow[r] \arrow[d] & I^{n-1,\sigma} \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \Omega^1_{\tilde{R}}\otimes_R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & \Omega^1_R\otimes_R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \Omega^1_{\tilde{R}}\otimes_R R/I^n\otimes_R \arrow[r] \arrow[d] & R/I^{n,\sigma} \arrow[r] \arrow[d] & \Omega^1_{\tilde{R}}\otimes_R R/I^{n-1}\otimes_R \arrow[r] \arrow[d] & R/I^{n-1,\sigma} \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where the sub + quotient complex are quis $CC(A)$ up to a shift.

It ought to be true that the [redacted] subcomplex is a quotient of the cyclic complex of $\{I\to R\}$.

Here is a way to see that this is the case. Recall that on the bar construction of $\{I\longrightarrow R\}$ there are canonical cochains $\hat{\rho}, \hat{\omega}$. Therefore in the same way that we construct a cocycle on $CC(A)$ with values in the "I adic[redacted] version" of the periodic complex of R we obtain[redacted] a similar cocycle [redacted] on $CC(I \to R)$.

[redacted]

At this point we have a very nice model for $CC(A)$ namely the chain complex

$$\overset{3}{\to \Omega^1_{\tilde R}\otimes_R R/I^2\otimes_R} \longrightarrow \overset{2}{R/I^{2,\natural}} \longrightarrow \overset{1}{\Omega^1_{\tilde R}\otimes_R R/I\otimes_R} \longrightarrow \overset{0}{R/I}$$

on which we can easily see the S-operation. I am still missing an argument which will identify the ~~associated graded~~ third complex with the Hochschild complex.

Taking the inverse [redacted] system given by this complex linked by the S-operation we get the periodic cyclic homology. Its a periodic pro complex. The corresponding cochains are cochains on the periodic complex for R which are continuous in the I-adic topology. [redacted] So we now have a nice model for periodic cyclic cohomology derived from our extension viewpoint.

[redacted] Next I want to try to bring in the [redacted] Connes model for periodic cyclic cohomology.

Discussion. Suppose $R = \bar{T}(A)$. Then the basic periodic complex we consider is [redacted]

$$R \cong \Omega^{ev}_{\tilde A}/k \qquad \Omega^1_R\otimes_R \cong \Omega^{odd}_{\tilde A}$$

so it is additively isomorphic to $\Omega_{\tilde A}/k$. I think moreover that these isomorphism respect the basic filtrations. So the real mystery is how to understand all of these cyclic symmetry conditions taking place at each level. The Connes' b, B description is remarkably free of these symmetry conditions.

What might be the correct working principle?

March 23, 1989 (Carl 24 yesterday)

Review program: To understand well GNS-NR ideas as well as Connes b+B complex. The b+B complex explains traces both for extensions and for $A*A$ except for mysterious cyclic symmetry conditions. Concrete questions: Find the analogue of CS forms in the NR picture; show that the CS forms associated to an even higher trace are in the same class as the associated b+B cocycle.

Progress. We reviewed the sub and quotient complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \Omega^1_{\tilde{R}} \otimes_R I^n \otimes_R \arrow[r] \arrow[d] & I^{n,\sigma} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Omega^1_{\tilde{R}} \otimes_R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \Omega^1_{\tilde{R}} \otimes_R (R/I^n) \otimes_R \arrow[r] \arrow[d] & R/I^{n,\sigma} \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 &
\end{tikzcd}$$

We showed that the sub complex is an "edge" quotient complex of $CC(I \to R)$, and a similar thing probably holds for the quotient complex.

The symmetry conditions inherent in using $I^{n,\sigma}$ remain perplexing. Somewhat related is the problem of identifying up to quis the complex

$$\longrightarrow \Omega^1_{\tilde{R}} \otimes_R (I^n/I^{n+1}) \otimes_R \longrightarrow I^{n,\sigma}/I^{n+1,\sigma} \longrightarrow$$

with the Hochschild complex of A.

We looked at the Hochschild complex of a semi-direct product $R\oplus I$ of DGA's. Let $C = B(R\oplus I) = T^B(E)$ where $B = B(R)$, $E = B\otimes_{\hat{\theta}} \Sigma I \otimes_{\widehat{-\theta}} B$. The Hochschild complex (in the nonunital setting) is the mapping cone of $C \xrightarrow{\beta} \Omega^C \otimes^C$. In the case of $C = T^B(E)$ we have

$$\Omega^C = C\otimes^B \Omega^B \otimes C \oplus C\otimes^B E \otimes^B C$$

so we have the mapping cone on

$$C \longrightarrow \Omega^B \otimes^B C \otimes^B \oplus E\otimes^B C \otimes^B$$

The process of taking the kernel of the first component map should give $C\otimes^B$ and then the kernel of the second component should give the cyclic invariants. Specifically in degree $I = n \geq 1$ we have a map

$$(E\otimes^B)^{n-1}E \longrightarrow \Omega^B \overset{B}{\otimes}(E\otimes^B)^n \oplus (E\otimes^B)^n$$

and if we pass to the kernel of the first component we obtain a quasi-isomorphic mapping cone for a map

⊛ $$(E\otimes^B)^n \xrightarrow{1-\sigma} (E\otimes^B)^n$$

which should be $1-\sigma$.

This should be a perfectly general result namely that the $\deg I = n$ part of the Hochschild complex of $R\oplus I$ is quis the mapping cone on ⊛, and hence to

$$(E\otimes^B)^{n,\sigma} \oplus \Sigma(E\otimes^B)^{n,\sigma}$$

When we consider $\{I \to R\}$ in the case of extensions, then its ~~cyclic~~ Hochschild complex will be filtered (the filtration is associated to $R \subset \{I \to R\}$) and the associated graded is the Hoch cx of $R \oplus \Sigma I$. Unfortunately it doesn't seem possible to use this.

The impression we get is that there might be a cleaner picture of the I-adically filtered periodic complex where the Hochschild complex appears in a simple form.

Next project is to look at the periodic cyclic homology where all these mysterious cyclic symmetry conditions seem not to matter. They appear as artifacts of the filtrations chosen.

Let us consider the complex

$$(*) \qquad \longrightarrow R \longrightarrow \Omega^1_{\tilde{R}} \otimes_R \longrightarrow$$

when R is free. When we filter this I-adically and consider continuous cochains, then we get a model for the periodic cyclic cohomology of A.

We next consider the problem of showing the above sequence is exact. This is roughly the old problem of why cyclic homology is trivial for free algebras. Except that to obtain Chern-Simons forms we use I think an explicit contracting homotopy.

The first approach is to use the derivation D on R, which gives the degree in V if we write $R = T(V)$, and to show that D acts trivially on the homology of ⊛. At the spot R this works in general:

$$\begin{tikzcd} R/[R,R] \arrow[r, "\bar\partial"] \arrow[d, "D"] & \Omega^1_R \tilde{\otimes}_R \arrow[dl, "(\tilde{D})\otimes_R"] \\ R/[R,R] & \end{tikzcd}$$

but [redacted] there are problems [redacted] at the other point. [redacted] We are asking in general whether a derivation acts trivially on

$$\text{Cokernel}\left\{HC_0(R) \longrightarrow \tilde{H}_1(R)\right\} = HC_1(R)$$

and even in the case, where R is a graded [redacted] alg $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdot$, where $R_0 = k$, this won't be true in general.

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March 27, 1989

Suppose $R = T(A)$. We have the exact periodic complex

$$\longrightarrow \Omega^1_{\tilde{R}} \otimes_R \longrightarrow R \longrightarrow \Omega^1_{\tilde{R}} \otimes_R \longrightarrow$$

which resembles the b, B complex. The problem is to understand the link between the two.

I like to think of the b, B complex in terms of bar and Hochschild cochains. We have a bijective map from linear maps τ defined on R to sequences of bar and Hochschild cochains

$$\varphi_{2n}(a_1, \ldots, a_{2n}) = \tau(a_1^- \cdots a_{2n}^-)$$

$$\psi_{2n+1}(a_0, \ldots, a_{2n}) = \tau(a_0^+ a_1^- \cdots a_{2n}^-)$$

which we have used to identify traces on R with b, B cocycles such that φ_{2n} is λ^2 invariant. This result is based on certain identities, which we might as well give in the universal case when $\tau = id$ on R. Then

$$\begin{cases} \varphi_{2n} = \omega^n \in Hom^{2n}(B(A), R) \\ \psi_{2n+1} = \partial\rho\,\omega^n \in Hom^{2n+1}(\Omega^{B(A),\natural}, R) \end{cases}$$

Let's derive the identities

$$\delta \omega^n = -\rho\omega^n + \omega^n\rho$$

$$(b'\varphi_{2n})(a_0, \ldots, a_{2n}) = a_0^+ a_1^- \cdots a_{2n}^- - a_0^- \cdots a_{2n-1}^- a_{2n}^+$$

$$+ \left(-a_{2n}^+ a_0^- \cdots a_{2n-1}^- + a_{2n}^+ a_0^- \cdots a_{2n-1}^- \right)$$

which gives the first formula

$$\boxed{\begin{aligned}&\left(b'\varphi_{2n}-(1-\lambda)\psi_{2n+1}\right)(a_0,\dots,a_{2n})\\ &=\left[a_{2n}^{+},a_0^{-}\cdots a_{2n-1}^{-}\right]\end{aligned}}$$

$$\boxed{\begin{aligned}&\left(b\psi_{2n+1}-(1+\lambda)\varphi_{2n+2}\right)(a_0,\dots,a_{2n+1})\\ &=\left[a_0^{+}a_1^{-}\cdots a_{2n}^{-},a_{2n+1}^{+}\right]\end{aligned}}$$

Proof of 2nd fmla.

$$\begin{aligned}\delta(\partial\rho\,\omega^n)&=\partial(\delta\rho)\,\omega^n+\partial\rho[\rho,\omega^n]\\ &=\partial(\delta\rho+\rho^2)\,\omega^n-\rho\,\partial\rho\,\omega^n-\partial\rho\,\omega^n\rho\end{aligned}$$

$$\left(\overset{\partial\omega\,\omega^n}{[illegible]}\right)(a_0,\dots,a_{2n+1})=\omega(a_0,a_1)\,\omega^n(a_2,\dots,a_{2n+1})+(-1)^{(2n+2)-1}\omega(a_{2n+1},a_0)\,\omega^n(a_1,\dots,a_{2n})$$

$$(\rho\,\partial\rho\,\omega^n)(a_0,\dots,a_{2n+1})=+a_{2n+1}^{+}a_0^{+}a_1^{-}\cdots a_{2n}^{-}$$

$$(\partial\rho\,\omega^n\rho)(a_0,\dots,a_{2n+1})=-a_0^{+}a_1^{-}\cdots a_{2n}^{-}a_{2n+1}^{+}$$

from §5 of cochain paper

$$\begin{aligned}(b\psi_{2n+1})(a_0,\dots,a_{2n+1})&=a_0^{-}\cdots a_{2n+1}^{-}-a_{2n+1}^{-}a_0^{-}\cdots a_{2n}^{-}\\ &\quad+\left[a_0^{+}a_1^{-}\cdots a_{2n}^{-},a_{2n+1}^{+}\right]\end{aligned}$$

The next ~~[illegible]~~ step is to discuss derivatives. ~~the method~~ The issue here is to see that the b, B cocycle attached to a trace on R changes by coboundaries as ρ is varied. In my cochain paper I considered a polynomial family $\rho_t : A \longrightarrow R_1$. This gives a family of homomorphisms $u_t : R \longrightarrow R_1$. Hence ~~from~~ a trace τ on R_1 we get a family of traces $\tau_t = \tau u_t$. ~~Differentiating~~ gives a derivation $\dot{u}_t : R \longrightarrow R_1$ considered as R-bimodule via u_t. This extends to a map $\tilde{\dot{u}}_t : \Omega^1_R \longrightarrow R_1$ such that $\tilde{\dot{u}}_t \, d = \dot{u}_t$. Then

$$\dot{\tau}_t = \tau \dot{u}_t = (\tau \tilde{\dot{u}}_t) d \qquad d : R \to \Omega^1_R$$

so we see that $\dot{\tau}_t$ comes from the trace $\tau \tilde{\dot{u}}_t$ on Ω^1_R.

We learn from this discussion that our previous discussion of homotopy, especially the infinitesimal homotopy formula (2.1 of cochain paper, also 6.11) can best be done by [illegible] a first order calculation interpreting the derivative • as the map $d : R \longrightarrow \Omega^1_{\tilde{R}}$.

Consequently we consider our old infinitesimal homotopy proof with this in mind.

$$\omega^n \in \mathrm{Hom}^{2n}(B(A), I^n)$$

$$d\omega^n = \sum_1^n \omega^{i-1} \overset{\dot{\omega}}{\boxed{d\omega}} \omega^{n-i} = \sum_1^n \omega^{i-1} [\delta + \rho, d\omega] \omega^{n-i}$$

$$= [\delta + \rho, \underbrace{\sum_1^n \omega^{i-1} d\rho \, \omega^{n-i}}_{\mu_n \in \mathrm{Hom}^{2n-1}(B(A), \Omega^1_{\tilde{R}})}]$$

I guess we want to consider μ_n to have values modulo brackets: whence really $\mu_n : A^{\otimes(2n-1)} \longrightarrow \Omega^1_{\tilde{R}} \otimes_R I^{n-1} \otimes_R$ is given by

$$\boxed{\mu_n(a_\phi, \dots, a_{2n-1}) = \sum_1^n a_1^- \cdot\cdot a_{2i-2}^- \, da_{2i-1}^+ \, a_{2i}^- \cdots a_{2n-1}^-}$$

The basic identities are

$$\tau(\omega^n)^\circ = \delta\, \tau(\mu_n) - \beta\, \tau^\natural(\partial\rho\, \mu_n)$$

$$\tau^\natural(\partial\rho\, \omega^n)^\circ = -\delta\, \tau^\natural(\partial\rho\, \mu_n) + \bar{\partial}\, \tau\left(\frac{\mu_{n+1}}{n+1}\right)$$

where τ is say a linear map defined on $\Omega^1_{\tilde{R}} \otimes_R$. The "natural" means we have Hochschild cochains.

Now take $\tau = id$ on $\Omega^1_{\tilde{R}} \otimes_R$ and we should get very simple identities.

$$\boxed{\begin{aligned} d(\omega^n) &= b'(\mu_n) + (1-\lambda)(\partial\rho\, \mu_n) \\ d(\partial\rho\,\omega^n) &= b(\partial\rho\, \mu_n) + \frac{1}{n+1} N(\mu_{n+1}) \end{aligned}}$$

Check for $n=1$. $(\omega)(a_1, a_2) = a_1^- a_2^-$

$$d(\omega)(a_1, a_2) = d(a_1^- a_2^-) = d(a_1 a_2)^+ - da_1^+ a_2^+ - a_1^+ da_2^+$$

$$b'(\mu_1)(a_1, a_2) = (\mu_1)(a_1 a_2) = (d\rho)(a_1 a_2) = d(a_1 a_2)^+$$

$$(\partial\rho\, \mu_1)(a_1, a_2) = -a_1^+ da_2^+$$

$$\big(\lambda(\partial\rho\, \mu_1)\big)(a_1, a_2) = -(\partial\rho\, \mu_1)(a_2, a_1) = +a_2^+ da_1^+ = da_1^+ a_2^+$$

Second identity

$$\begin{aligned} d(\partial\rho\omega)(a_0, a_1, a_2) &= d(a_0^+ a_1^- a_2^-) \\ &= da_0^+ a_1^- a_2^- + a_0^+ d(a_1 a_2)^+ - a_0^+ d(a_1^+ a_2^+) \end{aligned}$$

$$d(\partial_\rho\omega)(a_0,a_1,a_2) = da_0^+ a_1^- a_2^- + a_0^+ \, d(a_1 a_2)^+ - a_0^+ \, da_1^+ \, a_2^+ - a_0^+ a_1^+ da_2^+$$

$$\big(b(\partial_\rho\mathring{\rho})\big)(a_0,a_1,a_2) = -(a_0a_1)^+ da_2^+ + a_0^+ d(a_1a_2)^+ - (a_2a_0)^+ da_1^+$$

$$\frac{1}{2} N(\mathring{\rho}\omega + \omega\mathring{\rho})(a_0,a_1,a_2) = \cancel{\tfrac{1}{2}} \left\{ \begin{matrix} \xcancel{\begin{matrix} da_0^+ a_1^- a_2^- \\ + da_1^+ a_2^- a_0^- \\ + da_2^+ a_0^- a_1^- \end{matrix}} & \begin{matrix} + a_0^- a_1^- da_2^+ \\ + a_1^- a_2^- da_0^+ \\ + a_2^- a_0^- da_1^+ \end{matrix} \end{matrix} \right\}$$

which checks

March 29, 1989

I want to write out a proof along the lines in Cuntz's letter of the exact sequences

$$0 \longrightarrow HC_{2n-1}(A) \longrightarrow I^n/[I,I^{n-1}] \xrightarrow{d} \Omega^1_{\tilde{R}} \otimes_R I^{n-1} \otimes_R$$

$$0 \longrightarrow HC_{2n}(A) \longrightarrow HC_0(R/I^{n+1}) \xrightarrow{d} \Omega^1_{\tilde{R}} \otimes_R (R/I^n) \otimes_R$$

in the case of the universal extension $R = T(A)$. Let's begin with the injectivity at the left.

Recall that we have described traces on R as follows. Given a linear [redacted] map $\tau : R \longrightarrow V$ one defines bar and Hochschild cochains

φ_{2n}: $\tau(\omega^n) \in \mathrm{Hom}^{2n}(B, V)$ $\qquad B = B(A)$

ψ_{2n+1}: $\tau(\partial\rho\, \omega^n) \in \mathrm{Hom}^{2n+1}(\Omega^{B,A}, V)$

Thus

$$\tau(\omega^n)(a_1, \ldots, a_{2n}) = \tau(a_1^- \cdots a_{2n}^-)$$

$$\tau(\partial\rho\,\omega^n)(a_0, \ldots, a_{2n}) = \tau(a_0^+ a_1^- \cdots a_{2n}^-)$$

Then we have that τ is a trace iff

$$\begin{cases} b'\{\tau(\omega^n)\} = (1-\lambda)\tau(\partial\rho\,\omega^n) \\ b\{\tau(\partial\rho\,\omega^n)\} = \frac{1}{n+1} N \tau(\omega^{n+1}) \\ \tau(\omega^n) \text{ is } \lambda^2\text{-invariant} \end{cases}$$

I want now to go over the proof so as to see what happens when τ is a linear [redacted] map defined on I^m. One might as well take $\tau = id$. We have the identities from two days ago

$$\{b'\omega^n - (1-\lambda)(\rho\omega^n)\}(a_0, \dots, a_{2n})$$
$$= [a_{2n}^+, a_0^- \cdots a_{2n-1}^-]$$
$$\{b(\rho\omega^n) - (1+\lambda)(\omega^{n+1})\}(a_0, \dots, a_{2n+1})$$
$$= [a_0^+ a_1^- \cdots a_{2n}^-, a_{2n+1}^+]$$

Suppose τ defined on I^m vanishes on $[R, I^m]$. Then $\{\varphi_{2n} = \tau(\omega^n),\ \varphi_{2n+1} = \tau(\partial\rho\,\omega^n) \quad n \geq m\}$ satisfy

$$\begin{cases} b'\varphi_{2n} = (1-\lambda)\varphi_{2n+1} & n \geq m \\ b\varphi_{2n+1} = (1+\lambda)\varphi_{2n+2} & n \geq m \end{cases}$$

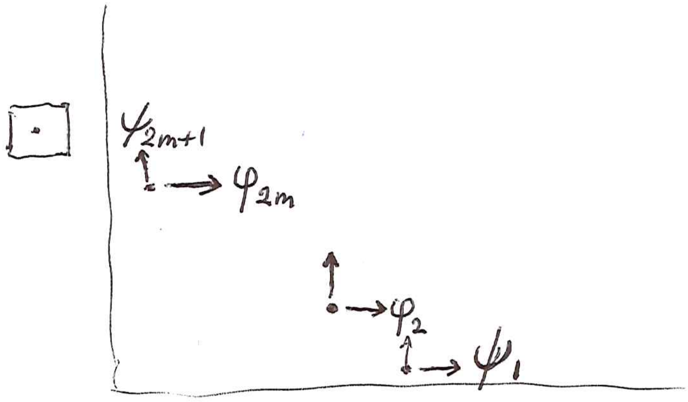
$\varphi_{2n}(a_1, \dots, a_{2n}) = \tau(a_1^- \cdots a_{2n}^-)$ [redacted] [redacted] is λ^2 symm. for $n > m$.

[redacted] These conditions are equivalent to

$$\left.\begin{array}{l} b'\varphi_{2n} = (1-\lambda)\varphi_{2n+1} \\ b\varphi_{2n} = \frac{1}{n+1} N\varphi_{2n+2} \\ \varphi_{2n+2} \text{ is } \lambda^2\text{-symm.} \end{array}\right\} \quad n \geq m$$

Thus a linear map τ on $I^m/[R, I^m]$ (a weak trace on I^m in Cuntz's terminology) is the same as a cocycle in the cyclic double cochain complex starting with φ_{2m}, satisfying the [redacted] λ^2-symmetry condition. (Actually I forgot to check the converse, but this is clear). Clearly also τ vanishes on $[I, I^{m-1}]$ iff in addition φ_{2m} is λ^2-symmetric. NOT CLEAR but true (Sept. 89)

Now let's prove injectivity of the map $HC_{2m}(A) \longrightarrow HC_0(R/I^{m+1})$, i.e. surjectivity on the dual spaces. [redacted] The map is defined by taking a trace on R/I^{m+1} into the associated "beg" cocycle $\{\varphi_{2n}, \psi_{2n+1}\}$ which satisfies $\varphi_{2n} = \psi_{2n+1} = 0$ for $n > m$. By diagram chasing in the double cochain complex this big cocycle is cohomologous to a cyclic 2m-cocyle



and someday I hope to show that this diagram chasing is done precisely by the Chern-Simons deformation. But for the moment all I need [redacted] is that [redacted] the above double complex gives the cyclic cohomology of A.

Suppose now that we are given a cyclic cohomology class in $HC^{2m}(A)$. We represent it by a cyclic cocycle $\psi_{2m+1}(a_0, a_1, \cdot, a_{2m})$. Then we get a trace on R/I^m by taking [redacted] $\varphi_2 = \cdots = \varphi_{2m} = \psi_1 = \cdots = \psi_{2m-1} = 0$.

I just learned something fantastically simple, namely that [redacted] cyclic cocycles of degree 2m are the same as traces on R/I^{m+1} vanishing on the very bottom $\omega^n(A^{\otimes 2n})$ and on [redacted] $\sum_{k \leq 2m-1} \rho(A)^k$.

Review after interruption: We have this map from linear maps defined on R [redacted] (or on I^m) to sequences of cochains $\varphi_{2n}, \psi_{2n+1}, \ldots$ and we have characterized traces as "big" cocycles with $\{\varphi_j\}$ being λ^2-symmetric. Now we are in a position to establish the injectivity of the maps

[redacted]

$$HC_{2m-1}(A) \longrightarrow I^m/[I, I^{m-1}]$$

$$HC_{2m}(A) \longrightarrow HC_0(R/I^{m+1})$$

i.e. surjectivity on the duals. So one starts with a cyclic cocycle and shows it comes from a trace.

In the even case a cyclic 2m-cocycle is a cochain $\psi_{2m+1}(a_0, \ldots, a_{2m})$ which together with $\varphi_2 = \varphi_4 = \cdots = \varphi_{2m} = 0$, $\psi_1 = \psi_3 = \cdots = \psi_{2m-1} = 0$ is a big cocycle satisfying the λ^2-symmetry condition, so [redacted] amazingly ψ_{2m+1} by itself gives one a trace on R/I^{m+1}.

In the odd case we take a cyclic (2m-1) cocycle $\varphi_{2m}(a_1, \ldots, a_{2m})$. Now do the diagram chasing

$$\begin{tikzcd}
{} & & \\
\psi_{2m+1} \arrow[u] \arrow[r] & {} & \\
& \varphi_{2m} \arrow[u] \arrow[r, "N"] & \varphi_{2m}
\end{tikzcd}$$

and note that if we use the obvious lifting of [redacted] a cyclic cochain to a bar cochain, this is λ symmetric, hence λ^2 symmetric. So one gets

more than a trace on I^m - one
gets an even supertrace on J^m in $A*A$.

Before going on let's discuss the "conjugation" action on traces defined on I^m say. The idea is that we have characterized these traces by big cocycles $\varphi_{2m}, \psi_{2m+1}, \varphi_{2m+2}, \psi_{2m+3}, \cdots$ such that the φ_{2j} are λ^2-invariant. By adding the coboundary of a suitable big cochain concentrated in a single degree ψ_{2n} we change only φ_{2n} and ψ_{2n+1}, that is, the cochains which "see" Ω^{2n}. Take $\psi'_{2n} = -\varphi_{2n}$. Then $\varphi_{2n} \longmapsto \varphi_{2n} - (1-\lambda)\varphi_{2n}$ $= \lambda\varphi_{2n}$ and ψ_{2n+1} ~~$\psi_{2n+1} - b\varphi_{2n} = \psi_{2n+1}$~~ goes to

$$\psi_{2n+1} - b\varphi_{2n} = \psi_{2n+1} - \underbrace{b'\varphi_{2n}}_{(1-\lambda)\psi_{2n+1}} - \text{crossover term in } b\varphi_{2n}$$

$$= \lambda\psi_{2n+1} - \text{crossover}$$

Thus the new trace τ' has

$$(\varphi'_{2n})(a_1, \ldots, a_{2n}) = -\tau(a_{2n}^- a_1^- \cdots a_{2n-1}^-)$$

$$(\psi'_{2n+1})(a_0, \ldots, a_{2n}) = \tau(a_{2n}^+ a_0^- \cdots a_{2n-1}^-) - \tau((a_{2n}a_0)^- a_1^- \cdots a_{2n-1}^-)$$

But we know that the conjugate trace should satisfy

$$\tau'(a_0^+ a_1^- \cdots a_{2n}^-) = -\tau(a_{2n}^- a_0^+ a_1^- \cdots a_{2n-1}^-)$$

$$= -\tau((a_{2n}a_0)^- a_1^- \cdots a_{2n-1}^-)$$

$$+ \tau(a_{2n}^+ a_0^- a_1^- \cdots a_{2n-1}^-)$$

so it checks.

Next we want to establish exactness of

$$HC_{2m-1}(A) \longrightarrow I^m/[I,I^{m-1}] \xrightarrow{d} \Omega^1_R \otimes_R I^{m-1} \otimes_R$$

$$HC_{2m}(A) \longrightarrow HC_0(R/I^{m+1}) \xrightarrow{d} \Omega^1_R \otimes_R R/I^m \otimes_R$$

Here we start with a trace whose cyclic class is trivial and we have to show it comes from a trace on the bimodule of differentials.

March 30, 1989

Let $R = T_r(A)$ (unital setup) and recall that we have an increasing algebra filtration $F_n = \rho(A)^n$ with $gr R = T(\bar{A})$, as well a decreasing filtration I^n with [redacted]

$$gr^I(R) = \bigoplus_n I^n/I^{n+1} = \bigoplus_{n \geq 0} \Omega_A^{2n}$$

[redacted] The odd part of the former filtration is complementary to the I-adic filtration:

$$\rho(A)^{2n+1} \oplus I^{n+1} = R.$$

Hence we have a canonical isomorphism of vector spaces $R \cong \Omega_A^{ev}$, precisely

$$\rho(A)^{2n+1} \cap I^n \simeq \Omega_A^{2n}$$

$$a_0^+ a_1^- \cdots a_{2n}^- \longleftrightarrow a_0 da_1 \cdots da_{2n}$$

[redacted] $$a_0^- \cdots a_{2i-1}^- a_{2i}^+ a_{2i-1}^- \cdots a_{2n}^- \longleftrightarrow da_0 \cdots da_{2i-1} \, a_{2i} \, da_{2i+1} \cdots da_{2n}$$

Notice that [redacted] in this way Ω_A^{2n} has a canonical subspace $\rho(A)^{2n} \cap I^n$. Clearly

$$\begin{tikzcd}
0 \arrow[r] & \rho(A)^{2n} \cap I^n \arrow[r] \arrow[d, equal] & \rho(A)^{2n+1} \cap I^n \arrow[r] \arrow[d, equal] & \rho(A)^{2n+1} \cap I^n / \rho(A)^{2n} \cap I^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \bar{A}^{\otimes 2n} \arrow[r] & \Omega_A^{2n} \arrow[r] & \bar{A}^{\otimes 2n+1} \arrow[r] & 0
\end{tikzcd}$$

and $\rho(A)^{2n} \cap I^n$ can be identified with the space spanned by [redacted] $1 \, a_1^- \cdots a_{2n}^-$. Thus

$$\rho(A)^{2n} \cap I^n \cong d\Omega_A^{2n-1} = dA^n$$

What I should be concerned with here is [redacted] what is canonical and what depends upon choices such as left and right.

It turns out I think that the operators d, B on Ω_A are canonical in this sense but that b is not.

Let's try to describe d intrinsically using the Cuntz algebra. We have the filtration

$$F_n(A*A) = A^+(A^-)^n = (A^-)^i A^+ (A^-)^{n-i} \qquad 0 \leq i \leq n$$

which is complementary to the J-adic filtration. (Note that F_n [redacted] is not an increasing algebra filtration, so we don't get a graded algebra). We have

$$F_n(A*A) \oplus J^{n+1} = A*A$$

$$F_n(A*A) \cap J^n \simeq \Omega^n_A$$

$$a_0^- \cdots a_{i-1}^- a_i^+ a_{i+1}^- \cdots a_n^- \longleftrightarrow da_0 \cdots da_{i-1}\, a_i\, da_{i+1} \cdots da_n$$

It appears that d is slightly non canonical up to sign. (?)

An important point is the following. When we associate a big cochain φ_{2n}, ψ_{2n} to a linear map τ on R:

$$\varphi_{2n}(a_1, \ldots, a_{2n}) = \tau(a_1^- \cdots a_{2n}^-)$$

$$\psi_{2n+1}(a_0, a_1, \ldots, a_{2n}) = \tau(a_0^+ a_1^- \cdots a_{2n}^-)$$

there's a reversal of ordering. Thus φ_{2n} which comes before ψ_{2n+1} sees the space $(A^-)^{2n}$ which is lower than the space $A^+(A^-)^{2n}$ seen by ψ_{2n+1}.

How to describe? The original idea I had was to identify the periodic complex

$$R \rightleftarrows \Omega^1_R \otimes_R$$

[redacted] suitably filtered with the periodic complex associated to the cyclic bicomplex.

Let's return to this later.

We return to describing the map $d: R \longrightarrow \Omega^1_{\tilde{R}} \otimes_R$ (nonunital setup). We have already seen how to associate a big cocycle $\{\cdots \varphi_{2n}, \psi_{2n+1} \cdots\}$ to a linear map defined on R. We now wish to see what cocycles are obtained from linear maps of the form τd with τ a linear map on $\Omega^1_{\tilde{R}} \otimes_R$. We have (with $\cdot$ denoting d) the formulas

$$(\omega^n)^{\cdot} = b'(\mu_n) + (1-\lambda)(\partial_\rho \mu_n)$$

$$(\partial_\rho \omega^n)^{\cdot} = b(\partial_\rho \mu_n) + \frac{1}{n+1} N(\mu_{n+1})$$

where as usual $\mu_n = \sum_1^n \omega^{i-1} \dot{\rho}\, \omega^{n-i}$. This shows that the big cochain associated to τd:

$$\varphi_{2n} = \tau(\omega^n)^{\cdot} \qquad \psi_{2n+1} = \tau((\partial_\rho \omega^n)^{\cdot})$$

is the coboundary of the big cochain

$$ⓧ \qquad \varphi_{2n-1} = \tau(\mu_n) \qquad \psi_{2n} = \tau(\partial_\rho \mu_n)$$

So we now propose to understand just what sort of big cochains are available in the form ⓧ. What should turn out is that we have the cochains needed to do the diagram chasing arguments within the class of big cocycles coming from traces.

Let us consider then special linear maps τ on $\Omega^1_R \otimes_R$, i.e. traces on $\Omega^1_{\tilde{R}}$. We have

$$\Omega^1_{\tilde{R}} \otimes_R \simeq \tilde{R} \otimes d(A^+)$$

~~Whereas the~~ and μ_n has values in $(A^-)^{2n-2} d(A^+)$ whereas $\partial_\rho \mu_n$ has values in $A^+ (A^-)^{2n-2} d(A^+)$. Thus provided τ is supported in the subspace $A^+(A^-)^{2n-2}d(A^+) = \Omega_A^{2n-2}d(A^+)$, this means it vanishes on all the others, we obtain a big cochain where only φ_{2n-1} and ψ_{2n} [redacted] can be non-zero.

Among such τ's let's see what possible φ_{2n-1} occur.

$$\mu_n(a_1,\dots,a_{2n-1}) = \sum_1^n (\omega^{i-1}\mathring{\rho}\,\omega^{n-i})(a_1,\dots,a_{2n-1})$$

$$= \sum_1^n a_1^- \cdots a_{2i-2}^- \, da_{2i-1}^+ \, a_{2i}^- \cdots a_{2n-1}^-$$

$$= \sum_1^n a_{2i}^- \cdots a_{2n-1}^- \, a_1^- \cdots a_{2i-2}^- \; \text{[redacted]} \; da_{2i-1}^+$$

$$= \sum_1^n (\omega^{n-1}d\rho)(a_{2i},\dots,a_{2n-1},a_1,\dots,a_{2i-1})$$

$$= (\omega^{n-1}d\rho)\underbrace{\left(\sum_{i=0}^{n-1}\lambda^{-2i}\right)}_{\sum_{j=0}^{n-1}\lambda^{2j+1}}(a_1,\dots,a_{2n-1})$$

We now want to show that $\sum_{j=0}^{n-1}\lambda^{2j+1}$ is invertible in the group ring of $\mathbb{Z}/(2n-1)$. It is enough to show that ~~[illegible]~~

$$f(x) = \sum_0^{n-1} x^{2j}$$

doesn't vanish at any of the $(2n-1)$th roots of unity. $f(1) = n \neq 0$. If $\zeta^{2n-1} = 1$ and $\zeta \neq 1$. Then

$$f(\zeta) + \zeta f(\zeta) = \sum_{j=0}^{n-1} \zeta^{2j} + \zeta^{2j+1} = \sum_{k=0}^{2n-1} \zeta^k + 1$$

$$\therefore\ (1+\zeta)f(\zeta) = 1, \quad \text{so } f(\zeta) \neq 0.$$

It follows then that any multilinear map $\varphi_{2n-1}(a_1, \dots, a_{2n-1})$ is of the form $\tau(\mu_n)$ for some τ a linear map on $\Omega_A^{2n-2} \cdot dA^+$.

Next let's consider τ's which see the top of $\Omega_A^{2n}\, d(A^+)$. Thus we consider a τ on $\Omega^1_R \otimes_R$ support in the piece $\Omega_A^{2n}\, dA^+$ and which vanishes on $(A^-)^{2n} dA^+$. Then the big cochain corresponding to τd has only one possible nonzero component namely

$$\psi_{2n+2}(a_0, \dots, a_{2n+1}) = \tau(\partial_\rho \mu_{n+1})(a_0, \dots, a_{2n+1})$$

Take τ be the projection onto $\Omega_A^{2n}\, dA^+$ followed by the map

$$\begin{aligned} \Omega_A^{2n}\, dA^+ &\longrightarrow A^{\otimes(2n+2)} \\ a_0^+ a_1^- \cdots a_{2n}^-\, da_{2n+1}^+ &\longmapsto (a_0, a_1, \dots, a_{2n+1}) \\ a_1^- \cdots a_{2n}^-\, da_{2n+1}^+ &\longmapsto 0 \end{aligned}$$

Then $\tau\left(\partial_\rho \sum_0^n \omega^i\, d\rho\, \omega^{n-i}\right)(a_0, \dots, a_{2n+1})$

$$= \sum_0^n \tau\left(a_0^+ a_1^- \cdots a_{2i}^-\, da_{2i+1}^+\, a_{2i+2}^- \cdots a_{2n+1}^-\right)$$

$$= \sum_0^n \tau\left(a_{2i+2}^- \cdots a_{2n+1}^-\, a_0^+ a_1^- \cdots a_{2i}^-\, da_{2i+1}^+\right)$$

$$= \sum_0^n \left(a_{2i+2}, \dots, a_{2n+1}, a_0, a_1, \dots, a_{2i+1}\right)$$

$$= \underbrace{\left(\sum_0^n \lambda^{-2i-2}\right)}(a_0, \dots, a_{2n+1})$$

$$= \sum_0^n \lambda^{2n+2-2i-2} = \sum_0^n \lambda^{2(n-i)} = \sum_0^n \lambda^{2i}$$

Thus we can obtain any λ^2-invariant cochain ψ_{2n+2}, and these are exactly the cochains obtained.

Next we consider proving exactness of

$$HC_{2m}(A) \longrightarrow HC_0(R/I^{m+1}) \longrightarrow \Omega^1_R \otimes_R R/I^m \otimes_R$$

(nonunital mode). Start with a trace on R/I^m whence we have a big cocycle

$$\begin{tikzcd}
0 & & & & & \\
\psi_{2m+1} \arrow[u] \arrow[r] & {} & & & & \\
& \varphi_{2m} \arrow[u] \arrow[r] & {} & & & \\
& & & {} & & \\
& & & \varphi_2 \arrow[u] \arrow[r] & {} & \\
& & & & \psi_1 \arrow[u] \arrow[r, "0"] & 0
\end{tikzcd}$$

Now modulo traces coming from linear maps on $\Omega^1_R \otimes_R R/I^m \otimes_R$ we should be able to replace this big cocycle by a single cyclic $2m$ cocycle.

Lets go through the process. Suppose that we have managed to "deform" our trace so that the first cochain which is $\neq 0$ is ψ_{2n-1}. Then ψ_{2n-1} is a cyclic $(2n-2)$-cocycle so we can write it as $N\varphi_{2n-1}$ where φ_{2n-1} can be assumed λ invariant. Any cochain of degree $2n-1$ can be obtained from a τ' on $\Omega_R \otimes_R$ supported on $\Omega_A^{2n-2} = I^{n-1}/I^n$, and such a τ' gives a big ~~cocycle~~ cochain having only the components $\varphi_{2n-1}, \psi_{2n-2}$.

Changing τ by $\tau' d$ gives a trace whose big cocycle begins with φ_{2n}. Thus $N\varphi_{2n} = 0$ and we can write $\varphi_{2n} = (1-\lambda)\psi_{2n}$ where $\lambda^2 \psi_{2n} = \psi_{2n}$. In fact we know already that $\lambda^2 \varphi_{2n} = \varphi_{2n}$ so $0 = N\varphi_{2n} = n(1+\lambda)\varphi_{2n}$ and so $\psi_{2n} = \frac{1}{2}\varphi_{2n}$.

Next we know ψ_{2n} comes from a [redacted] τ' on $\Omega^1_R \otimes_R$ supported on $\Omega_A^{2n-2} dA^+$ I^{n-1}/I^n and vanishing on $(A^-)^{2n-2} dA^+$.

Hence we can modify τ, [redacted] so that it's leading cochain is ψ_{2n+1}.

At this point we see the process can be continue for $n \geq m$ and ends with a [redacted] big cocycle whose leading cochain is ψ_{2m+1}. If the trace vanishes on I^{m+1}, then $\varphi_{2m+2} = 0$ and so we have a cyclic $2m$-cocycle equivalent to the original big cocycle.

~~Thus we have taken any trace on $HC_0(R/I^n)$~~

~~Now suppose the class of ψ_{2m+1} is trivial which means that $\psi_{2m+1} = b\psi_{2m}$, where ψ_{2m} is λ-invariant. Then we know ψ_{2m+1}~~

So we have started with a trace τ on R/I^{m+1} and shown that modulo traces of the form $\tau' d$ with τ' a linear map defined on $\Omega^1_R \otimes_R R/I^m \otimes_R$, that τ can be replaced by a trace τ_1 whose leading cochain is ψ_{2m+1} which is a cyclic cocycle representing the cyclic $(2m)$-class assoc. to τ. If this class is trivial we can lift ψ_{2m+1} to $b\psi_{2m}$ with ψ_{2m} cyclic and we know, then that τ_1 comes from $\Omega^1_R \otimes_R R/I^m \otimes_R$. This concludes the proof of the exactness.

Comments: This proof takes place the complex of linear forms on $R \rightleftarrows \Omega^1_R \otimes_R$. I feel there ought to be a direct proof completely independent of anything. [redacted] Secondly [redacted] it should be possible to use Chern-Simons to do the "diagram chasing."

March 31, 1989

I want now to try to prove the exact sequences

$$0 \to HC_{2n}(A) \longrightarrow HC_0(R/I^{n+1}) \xrightarrow{d} \Omega^1_R \otimes_R (R/I^n) \otimes_R$$

without using big cocycles and the cyclic bicomplex. It should be possible to proceed directly using the exactness of

(*) $\quad \Omega^1_R \otimes_R$ [redacted] $\longrightarrow R \longrightarrow \Omega^1_R \otimes_R$

I think.

In any case the exactness of (*) gives a way to write any trace τ on R [redacted] in the form $\tau' d$, where τ' is a trace on Ω^1_R. This then gives a way of writing the big cocycle attached to τ as a coboundary, and it would be nice to understand this operation, which should be closely related to Chern-Simons cyclic cocycles I think.

[redacted]

Basic philosophy. Suppose given [redacted] algebras R, R' and a family a homomorphisms $u_t : R \to R'$ and a trace τ on R'. Then we have a family of traces $\tau_t = \tau u_t$ on R. This is a special type of variation of traces. In effect the derivative $\dot{\tau}_t$ is not in general an arbitrary trace. To see this note $\dot{u}_t : R \to R'$ is a derivation with respect to u_t, [redacted] more precisely with values in R' considered as an R-bimodule via u_t. Thus we have an R-bimodule map

$\tilde{U}_t : \Omega^1_R \longrightarrow R'$ and a trace
$\tau'_t = \tau \tilde{\mathring{U}}_t$ on Ω^1_R such that $\mathring{\tau}_t = \tau'_t d$.

More generally given a derivation $D : R \to M$, where M is an R-bimodule, and a trace τ on M, then $\tau D = (\tau \tilde{D}) d$ is a trace on R coming from the trace $\tau \tilde{D}$ on Ω^1_R.

Thus we might introduce the terms "derivation trace" to [redacted] mean a trace on Ω^1_R and "principal derivation trace" to mean a trace on Ω^1_R coming from R via the bracket map $\beta : \Omega^1_R \longrightarrow R$. The deviation of traces being equivalent to derivation traces mod princ. deriv. traces is measured by the failure of the canonical map

[redacted]
$$HC_0(R) \xrightarrow{d} H_1(R,R)$$

to be an isomorphism.

Lesson: Two traces on R are "homotopic" when they differ by a trace coming from a trace on Ω^1_R. And a "nullhomotopy" of a trace is equivalent to a trace on Ω^1_R [redacted].

Next let's consider a free algebra $R = \bar{T}(A)$ where for the moment A is just a vector space, [redacted] and let τ be a trace on R. We consider the family $U_t : R \longrightarrow R$ with $U_t(a) = ta$. Then

$$\tilde{\mathring{U}}_t : \Omega^1_R \longrightarrow R$$

$$\tilde{\mathring{U}}_t (a_1 \cdots a_{i-1} da_i \, a_{i+1} \cdots a_n) = U_t(a_1 \cdots a_{i-1}) \mathring{U}_t(a_i) U_t(a_{i+1} \cdots a_n)$$
$$= t^{n-1} a_1 \cdots a_n$$

so $\tau'_t = \tau \dot{\tilde{\mathcal{U}}}_t$ is given by

$$\tau'_t(a_1\cdots a_{i-1}\,da_i\,a_{i+1}\cdots a_n) = t^{n-1}\tau(a_1\cdots a_n).$$

Now [illegible] we integrate from $t=0$ to $t=1$.

$$\underbrace{\left(\int_0^1 \tau'_t\,dt\right)}_{\tau'}(a_1\cdots a_{i-1}\,da_i\,a_{i+1}\cdots a_n) = \frac{1}{n}\tau(a_1\cdots a_n)$$

and indeed $\tau' d = \tau$.

This is the way we write any trace on R as coming from a trace on Ω^1_R. Now we know that this process ~~[illegible]~~ of going from τ to τ', better of writing τ in the form $\tau' d$, [illegible] can be described as a way of writing the big cocycle attached to τ as a coboundary. Thus τ can be described in terms of a big cocycle $\varphi_1, \varphi_2, \ldots$ (satisfying a symmetry condition) and τ' determines a big cochain $\psi_1, \psi_2, \psi_3, \ldots$

$$\begin{tikzcd}
 & & ch_4 \arrow[dl, dash] & \\
\varphi_4 \arrow[r, mapsto] & N\varphi_4 & & \\
\varphi_3 & \psi_3 \arrow[u] & & \\
 & \psi_2 & \varphi_2 & \\
 & & \varphi_1 & \psi_1
\end{tikzcd}$$

Natural question is whether the cyclic cocycle $\psi_3 - b\psi_2 = N\varphi_3$ (up to a constant $\frac{1}{3}$) is the same as CS_3. It seems this has to be true, since there are no choices made; we have used only the standard Chern-Simons deformation.

Thus we seem to have identified the CS forms.

It appears that we have a very simple proof of the exactness of

$$0 \longrightarrow HC_{2m}(A) \longrightarrow HC_0(R/I^{m+1}) \xrightarrow{d} \Omega^1_R \otimes_R (R/I^m) \otimes_R$$

as follows. Given a trace τ on R/I^{m+1} we have the Chern-Simons deformation of it which writes it coming from a trace τ' on Ω^1_R. Now approximate τ' by a trace τ'' vanishing on $\Omega^1_R \otimes_R I^m \otimes_R$ and remove $\tau'' d$ from τ. At this point you should have a trace equivalent to a cyclic (2m-) cocycle. In other words it appears that we have in effect an actual projection π of $HC_0(R/I^{m+1})$ back onto $HC_{2m}(A)$, that is an explicit contracting homotopy for the above sequence.

This should be checked carefully tomorrow.

However it is not immediately clear how to handle the corresponding sequence

$$0 \longrightarrow HC_{2m-1}(A) \longrightarrow I^m/[I, I^{m-1}] \longrightarrow \Omega^1_R \otimes_R I^{m-1} \otimes_R$$

since I don't have Chern-Simons deformation for a trace on $I^m/[I,I^{m-1}]$. I know how to prove this exactness by using big cochains and the "local" deformation which works on each level. In analyzing the process whereby a cyclic 2m-1 cocycle φ_{2m} is expressed as the coboundary of a big cochain $\varphi_{2m}, \varphi_{2m+1}, \varphi_{2m+2}, \cdots$ satisfying the symmetry conditions for it to be a trace on I^m we see that we would like a Chern-Simon process of opposite parity. Thus I really want a good reason for the exactness of $\Omega^1_R \otimes_R \to R \to \Omega^1_R \otimes_R \to R$ at the point $\Omega^1_R \otimes_R$. This can be done by calculation and is a key point in showing $HC_n(R) = 0$ for $n > 0$.

So we reach again the problem of understanding why free algebras have trivial cyclic homology, except now we have a very elegant proof of the exactness of the periodic sequence at the point R. We have a very explicit τ':

$$\begin{tikzcd}
\Omega^1_R \otimes_R \arrow[r] & R \arrow[r, "d"] \arrow[d, "\tau"] & \Omega^1_R \otimes_R \arrow[dl, "\tau'"] \\
& R/[R,R] &
\end{tikzcd}$$

So my feeling is that there might be a similar situation, in fact a canonical homotopy of this periodic sequence which perhaps can be nicely explained using the CS deformation of R.

It's not completely trivial. The Goodwillie theorem and the derivation on R given by its tensor grading together with

$$HC_2(R) \xrightarrow{S=0} HC_0(R) \xrightarrow{d} H_1(R,R) \longrightarrow HC_1(R) \longrightarrow 0$$

shows the injectivity of $R/[R,R] \longrightarrow H_1(R,R) \subset \Omega^1_R \otimes_R$. But the surjectivity of $HC_0(R) \to H_1(R,R)$ is equivalent to $HC_1(R) = 0$, and by the sequence

$$HC_3(R) \xrightarrow{S=0} HC_1(R) \xrightarrow{B} H_2(R,R) \longrightarrow HC_2(R) \xrightarrow{S=0} HC_0(R)$$

Any derivation style proof that $HC_1(R)=0$ would have to proof $H_2(R,R)=0$.

Problem: Starting from the idea that the cyclic complex is the cocommutator subspace of the bar construction, find a simple proof that for a free algebra A the cyclic complex is a resolution of $A/[A,A]$.

This is related to the question of proving that if one has a twisting cochain $\theta : C \longrightarrow A$ with C, A free DG coalg + alg resp, such that θ induces a quis $\mathrm{Cobar}(C) \longrightarrow C$ (or equivalently I think $C \longrightarrow B(A)$), then $\Sigma^{-1}C^{\natural} \longrightarrow A_{\natural}$ is a quis.

April 2, 1989

Ideas to write up & work on

1) variation map ~~[illegible]~~ cyclic $\xrightarrow{B}$ Hochschild induced by $B(A) \longrightarrow B(A \oplus \Omega^1_A)$. Proof that cyclic homology of a free algebra is trivial: If D is a derivation, then

$$\begin{tikzcd}
HC_n(A) \arrow[r, "B"] \arrow[d, "L_D"] & H_{n+1}(A,A) \arrow[d, "\iota_D"] \\
HC_n(A) & H_n(A,A) \arrow[l]
\end{tikzcd}$$

commutes. Applied to a graded algebra (grading wrt N), this gives $HC_n(A) \hookrightarrow H_{n+1}(A,A)$.

2) Analyze the variation map from the viewpoint of twisting cochains. Coincidence: Hochschild complex arises from $CC(A \oplus \Omega^1_A)$, i.e. from varying in the target of the twisting cochain, and also from $B \to \Omega^{B,4}$, i.e. from varying in the source.

3) Link with Karoubi approach, or more generally with flat connections. Karoubi takes a representation $\Gamma \longrightarrow GL_n(A)$ uses a Sullivan-style model for $\Omega^\circ(B\Gamma)$, gets a connection & curvature leading to character ~~[illegible]~~ forms in $\left(\Omega^\circ(B\Gamma) \otimes \Omega^\circ_A\right)/[\,,]$ $= \Omega^\circ(B\Gamma) \otimes \left(\Omega^\circ_A/[\,,]\right)$, whence maps $K_*(A) \to H^{DR}_*(A)$. Geometrically he considers $A = C^\infty(M)$ and a bundle over $X \times M$ with partial flat connection in the X-direction, then he extends it to a full connection and uses the character classes on the product to map $H_*(X) \longrightarrow H^*(M)$. ~~[illegible]~~.

There is a Lie alternative approach where one considers ~~[illegible]~~ a flat connection on the trivial bundle. In this case one considers the bundle ~~[illegible]~~

$pr_2^*(E)$ over $X \times M$ with

$$\Theta \in \Omega^{1,0}(X \times M, \mathrm{End}(pr_2^* E)) = \Omega^1(X, \Omega^0(M, \mathrm{End}\, E))$$

satisfying $d_X\Theta + \Theta^2 = 0$. This gives Lie or cyclic cohomology classes.

Two approaches: | alg K / Lie make sense for a Lie group G with Lie alg $\mathfrak{g}$.

$$\begin{tikzcd}
G_{space} \arrow[r] \arrow[d, equal] & \mathfrak{g} \arrow[r] & B_{cont}G \arrow[r] & BG \arrow[d, equal] \\
G_{space} \arrow[r] & G/G_\delta \arrow[r] \arrow[u] & BG_\delta \arrow[r] \arrow[u] & BG
\end{tikzcd}$$

In general a flat bundle is nontrivial as a bundle. ~~[illegible]~~ So flat bundles are not the same as trivial bundles with flat connection. There appear to be two reasons they are different: $\mathfrak{g} \xleftarrow{①} G/G_\delta \xrightarrow{②} BG_\delta$. However the deformation theory appears to be the same at least for GL_∞ – this is Goodwillie's theorem.

In the past we analyzed flat connections on the trivial bundle, better $\Theta \in \Omega^1(X) \otimes M_n(A)$ with $d\Theta + \Theta^2 = 0$ using $\Omega_A^\bullet$. Now we have a much better method based on the bar construction.

April 4, 1989

Consider an extension $A = R/I$, nonunital situation with R free. Take a trace τ on I considered as R-bimodule, extend it to a linear functional $\tilde{\tau}$ on R and form $(b\tilde{\tau})(x,y) = \tilde{\tau}([x,y])$. This is a cyclic 1-cocycle on R, and it vanishes if x or $y \in I$, so it yields a cyclic 1-cocycle on A.

Suppose we made a different choice of extension. Then the difference $\tilde{\tau}-\tilde{\tau}'$ is a linear functional on A, and the difference of the cyclic 1-cocycles associated is the 1-coboundary of this linear functional. It follows that we have well-defined maps

$$\left(I/[R,I]\right)^* \longrightarrow HC^1(A)$$

$$HC_1(A) \longrightarrow I/[R,I]$$

Next suppose ~~the~~ trace τ comes from a trace on R. Then we can choose $\tilde{\tau}$ to be this trace whence the associated cyclic 1-cocycle is zero. Thus the composite maps

$$\left(R/[R,R]\right)^* \longrightarrow \left(I/[R,I]\right)^* \longrightarrow HC^1(A)$$

$$HC_1(A) \longrightarrow I/[R,I] \longrightarrow R/[R,R]$$

are zero. (Here $*$ should be replaced by $\mathrm{Hom}(\ ,V)$ for an arbitrary vector space V).

Now suppose R is free. Then we have the periodic exact sequence

$$\xrightarrow{b} R \xrightarrow{d} \Omega^1_R \otimes_R \xrightarrow{b} R \xrightarrow{d}$$

which implies that any cyclic 1-cocycle ψ on R

is of the form $\psi(x,y) =$ [redacted] $f([x,y]) = (bf)(x,y)$ where f is a linear function on R.

(Here we use $R_\lambda^{\otimes 2}/bR_\lambda^{\otimes 3} = \Omega^1_R \otimes_R / dR$ i.e. that cyclic 1-cocycles are the same as Hochschild 1-cocycles $(\psi(x,y), \varphi(x))$ such that $\varphi = 0$.

Given a cyclic 1-cocycle ψ on A we lift it to R and write it in the form $(bf)(x,y) = f([x,y])$ with f a linear function on R. Then the restriction of f to I is a trace on I considered as R-bimodule. This shows that $(I/[R,I])^* \longrightarrow HC^1(A)$ is surjective, hence $HC_1(A) \hookrightarrow I/[R,I]$.

Next suppose that [redacted] we have $\tau \in (I/[R,I])^*$ such that $\tilde{\tau}([x,y]) = f([x,y])$ with $f \in A^*$, then using $\tilde{\tau} - f$ instead of $\tilde{\tau}$, we see that τ extends to a trace $\tilde{\tau} - f$ on R.

Conclusion: We have verified by hand the exact sequence

$$HC_1(R) \longrightarrow HC_1(A) \longrightarrow I/[R,I] \longrightarrow R/[R,R] \longrightarrow HC_0(A) \longrightarrow 0$$

and its consequence for R free.

What is important maybe about the above argument is that it shows clearly the extension process from odd cyclic cocycles to traces involves writing a cyclic 1-cocycle on R in the form $f([x,y])$.

Does this have anything to do with the moment map?

Let's now consider the problem of proving $HC_1(R)=0$ for R free, and more generally for the higher groups. We wish to give a simple proof as explicit as possible. In essense the proof uses the Connes exact sequence

$$HC_{n+2}(R) \xrightarrow{S} HC_n(R) \longrightarrow H_{n+1}(R,\tilde{R}) \longrightarrow$$

and the Goodwillie theorem. The latter tells us that S is zero for any positively graded alg. On the other hand we have $H_n(R,\tilde{R})=0$ for $n\geq 2$ for R free, so we win.

To be more explicit we bring in the proof of the Goodwillie theorem. Let D be a derivation of A, then we have a commutative diagram

$$\begin{tikzcd}
HC_n(A) \arrow[r, "B"] \arrow[d, "L_D"] & H_{n+1}(A,\tilde{A}) \arrow[r, "\partial", "\text{isom}\quad n>0"'] \arrow[d, "i_D"] & H_n(A,\Omega^1_A) \\
HC_n(A) & H_n(A,\tilde{A}) \arrow[l] &
\end{tikzcd}$$

where the top composition is induced by the map $A \xrightarrow{1+d} A\oplus\Omega^1_A$ together with projection of $HC_\cdot(A\oplus\Omega^1_A)$ onto its part of degree 1 wrt Ω^1_A, which is $H_\cdot(A,\Omega^1_A)$. i_D is cup (Yoneda) product with $D\in H^1(A,A)$. Since $BS=0$ it follows that $L_D S=0$ which is the Goodwillie thm.

Let's next describe things on the cochain level. Recall that $H^\cdot(A,M)$ is calculated with cochains on A with values in the bimodule M, and that the dual of $H_n(A,\tilde{A})$ is $H^n(A,\tilde{A}^*)$. Because of Connes the $\tilde{A}$ variable comes first. Thus an element of [redacted] $H^1(A,\tilde{A}^*)$ is represented by a cochain $f(x,y) = f_y(x)$ such that

$$(\delta f)_{y,z} = y f_z - f_{yz} + f_y z = 0$$

or

$$f(xy,z) - f(x,yz) + f(zx,y) = 0 .$$

If D is a derivation of A, then the cup product of $[D] \in H^1(A,A)$ and $[f] \in H^1(A,\tilde{A}^*)$ is represented by

$$(D \cup f)_{y,z}(x) = (Dy \cdot f_z)(x) = f(xDy, z)$$

This agrees with the formula for i_D on the Hochschild complex [redacted] from the Kassel-Husemoller notes:

$$\iota_D(a_0, a_1, \dots, a_n) = (a_0 Da_1, \dots, a_n)$$

$a_0 \in \tilde{A}$

Next consider

$$\begin{tikzcd}
HC^n(A) & H^{n+1}(A,\tilde{A}^*) \arrow[l, "B"'] \\
HC^n(A) \arrow[u, "L_D"'] \arrow[r, "I"] & H^n(A,\tilde{A}^*) \arrow[u, "\iota_D"']
\end{tikzcd}$$

for $n=1$. Given $\psi(x,y)$ a cyclic 1-cocycle I takes it [redacted] to the Hochschild 1-cocycle $(\psi(x,y), 0)$ then ι_D takes this to

$$(\psi(xDy, z), \ \psi(Dy, z))$$

and B takes this to

$$(x,y) \longmapsto \psi(Dx,y) - \psi(Dy,x)$$

$$= \psi(Dx,y) + \psi(x,Dy) \qquad \text{because } \psi \text{ cyclic}$$

$$= (L_D\psi)(x,y)$$

proving the commutativity of the above square.

Our next project will be to learn why $H^2(R,\tilde{R}^*)=0$ for a free algebra in an explicit way.

Let's recall that a Hochschild 2-cocycle $f: R\times R \longrightarrow M$, where M is an R-bimodule, can be identified with an algebra extension together with linear lifting:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & E \arrow[r] & R \arrow[r] \arrow[l, bend right, "\rho"'] & 0 \end{tikzcd}$$

where $f(x,y) = -\rho(xy) + \rho(x)\rho(y)$ is the "curvature". Changing the [redacted] lifting by a linear map alters the 2-cocycle f by a coboundary.

When R is a free algebra there is a canonical way to construct a splitting of the extension given a lifting ρ, namely you take the unique homomorphism $h: R \to E$ given by ρ on the generators of R. Thus if $R = \bar{T}(V)$

$$h(v_1 \cdots v_n) = \rho(v_1)\cdots\rho(v_n)$$

The difference $h-\rho$ is a 1-cochain vanishing on the generators. Hence any two cocycle is uniquely the coboundary of a 1-cochain vanishing on the generators. This corresponds to the splitting

$$\begin{tikzcd} & & & \tilde{R}\otimes V\otimes\tilde{R} \arrow[dl, dashed] \arrow[d, equal] & \\ 0 \arrow[r] & \Omega^2_R \arrow[r] & \tilde{R}\otimes R\otimes\tilde{R} \arrow[r] & \Omega^1_R \arrow[r] & 0 \end{tikzcd}$$

Suppose we have given a 2-cocycle $f(x,y)$ on $R=T(V)$ with values in M. Construct the extension E with lifting such that

$$\rho(x)\rho(y) = \rho(xy)+f(x,y)$$

and let h be the homom. $h: R\longrightarrow E$ with $h(v)=\rho(v)$ for $v\in V$. Then

$$\rho(v_1)\rho(v_2) = \rho(v_1v_2)+f(v_1,v_2)$$

$$\rho(v_1)\rho(v_2)\rho(v_3) = \rho(v_1v_2v_3)+f(v_1,v_2v_3)+v_1f(v_2,v_3)$$

$$\rho(v_1)\cdots\rho(v_4) = \rho(v_1\cdots v_4)+f(v_1,v_2v_3v_4)+v_1f(v_2,v_3v_4)+v_1v_2f(v_3,v_4)$$

Now set $g(x)=h(x)-\rho(x)$ so that

$$g(v_1\cdots v_n) = f(v_1,v_2\cdot\text{[redacted]}v_n)+v_1f(v_2,v_3\cdots v_n)+v_1v_2f(v_3,v_4\cdots v_n)+\cdots$$
$$\cdots+\text{[redacted]}\,v_1\cdots v_{n-2}f(v_{n-1},v_n)$$

Then $\quad \big(\rho(x)+g(x)\big)\big(\rho(y)+g(y)\big) = \rho(xy)+g(xy)$

i.e. $\quad f(x,y)+xg(y)-g(xy)+g(x)y=0$

A nice point is that

$$\Omega^2_R = \Omega^1_R\otimes_R\Omega^1_R = \tilde{R}\otimes V\otimes\tilde{R}\otimes V\otimes\tilde{R}$$

so that a 2-cocycle f is completely determined by its values $f(v,y)$ with $v\in V$, $y\in R$ and these can be arbitrary

April 5, 1989

From yesterday, given a Hochschild 2-cocycle $f : R\times R \longrightarrow M$ we have

$$f = -\delta K f$$

where

$$(Kf)(v_1\cdots v_n) = f(v_1, v_2\cdots v_n) + v_1 f(v_2, v_3\cdots v_n) + v_1 v_2 f(v_3, v_4\cdots v_n) + \cdots + v_1\cdots v_{n-2} f(v_{n-1}, v_n)$$

Check

$$\Omega^2_R \longrightarrow \tilde{R}\otimes R \otimes \tilde{R}$$

$$dx\,dy \longmapsto b'(1,x,y,1) = (x,y,1) - (1,xy,1) + (1,x,y)$$

Ω^2_R is the free $\tilde{R}$-bimodule spanned by $dV\cdot dR$

so

$$dv_1\, d(v_2\cdots v_n) \longmapsto (v_1, v_2\cdots v_n, 1) - (1, v_1\cdots v_n, 1) + (1, v_1, v_2\cdots v_n)$$

$$\xmapsto{K} v_1 \begin{bmatrix} dv_2\, d(v_3\cdots v_n) \\ v_2\, dv_3\, d(v_4\cdots v_n) \\ \cdots \\ v_2\cdots v_{n-2}\, dv_{n-1}\, dv_n \end{bmatrix} - \begin{pmatrix} dv_1\, d(v_2\cdots v_n) \\ v_1\, dv_2\, d(v_3\cdots v_n) \\ \cdots \\ v_1\cdots v_{n-2}\, dv_{n-1}\, dv_n \end{pmatrix} + 0$$

$$= -dv_1\, d(v_2\cdots v_n).$$

Next suppose $\varphi(x,y)$ is a cyclic 1-cocycle and let's try to write $L_D\varphi$ as the boundary of a cyclic 0-cochain. φ lifts to the Hoch 1-cocycle $\varphi(x,y)$ where here $x\in\tilde{R}$, which then goes under ι_D to the Hoch 2-cocycle ψ: $\varphi(xDy, z)$. Here $x\in\tilde{R}$ and when we use K above we only need $y\in V$, where $Dy = y$. Thus $K\psi$ is the Hoch 1-cochain

$$\theta'(x, v_1\cdots v_n) = \varphi(xv_1, v_2\cdots v_n) + \varphi(xv_1v_2, v_3\cdots v_n)$$
$$+ \varphi(xv_1\cdots v_{n-1}, v_n)$$

Apply B to θ' gives the cyclic 0-cochain

$$\boxed{\begin{aligned}\theta(v_1\cdots v_n) = \varphi(v_1, v_2\cdots v_n) + \varphi(v_1 v_2, v_3\cdots v_n) + \cdots \\ + \varphi(v_1\cdots v_{n-1}, v_n)\end{aligned}}$$

since $-b\theta' = \psi$ and $B\psi = L_D\varphi$ we have

$$b\theta = b(B\theta') = -Bb\theta' = +B\psi = L_D\varphi$$

Now because φ is a cyclic 1-cocycle, we have

$$\varphi(v_1, v_2\cdots v_n) = \sum_{i=2}^{n} \varphi(v_{i+1}\cdots v_n v_1\cdots v_{i-1}, v_i)$$

$$\varphi(v_1 v_2, v_3\cdots v_n) = \sum_{i=3}^{n} \underline{\hspace{5cm}}$$

which gives

$$\boxed{\theta(v_1\cdots v_n) = \sum_{i=1}^{n} (i-1)\,\varphi(v_{i+1}\cdots v_n v_1\cdots v_{i-1}, v_i)}$$

Check this has $b\theta = n\varphi$

$$\theta(v_1\cdots v_n) = \sum_{1}^{n} (i-1)\,\varphi(v_{i+1}\cdots v_n v_1\cdots v_{i-1}, v_i)$$

$$\theta(v_n v_1\cdots v_{n-1}) = \sum_{1}^{n-1} i\;\varphi(v_{i+1}\cdots v_n v_1\cdots v_{i-1}, v_i)$$

$$\begin{aligned}\therefore\ \theta([v_1\cdots v_{n-1}, v_n]) &= (n-1)\,\varphi(v_1\cdots v_{n-1}, v_n) \\ &\quad - \sum_{1}^{n-1} \varphi(v_{i+1}\cdots v_n v_1\cdots v_{i-1}, v_i) \\ &= (n-1)\,\varphi(v_1\cdots v_{n-1}, v_n) - \varphi(v_n, v_1\cdots v_{n-1}) \\ &= n\,\varphi(v_1\cdots v_{n-1}, v_n)\end{aligned}$$

Let's return to the sequence

$$\longrightarrow R \xrightarrow{d} \Omega^1_R \otimes_R \xrightarrow{b} R \xrightarrow{d}$$

when $R = T(V)$. There is an obvious contracting homotopy for this sequence which results from looking at a given tensor degree

$$\longrightarrow V^{\otimes n} \xrightarrow{N} V^{\otimes n} \xrightarrow{1-\sigma} V^{\otimes n} \longrightarrow \cdots$$

(Recall $d(v_1 \cdots v_n) = \sum_i v_{i+1} \cdots v_n v_1 \cdots v_{i-1} \otimes dv_i$ and

$b\{(v_1 \cdots v_{n-1}) dv_n\} = [v_1 \cdots v_{n-1}, v_n] = v_1 \cdots v_n - v_n v_1 \cdots v_{n-1}$)

This homotopy operator results from projectors in the group algebra of the cyclic group:

$$e = \frac{1}{n} N$$

$$1 - e = 1 - \frac{1}{n}\sum_0^{n-1} \sigma^i = \frac{1}{n}\sum_{i=0}^{n-1}(1-\sigma^i)$$

$$= \frac{1}{n}\sum_{i=1}^{n-1}(1+\sigma+\cdots+\sigma^{i-1}) \cdot (1-\sigma)$$

$$= \frac{1}{n}\sum_{j=0}^{n-2}(n-1-j)\sigma^j = \frac{1}{n}\sum_{j=0}^{n-1}(n-1-j)\sigma^j \, (1-\sigma)$$

$$= \underbrace{\frac{1}{n}\sum_{i=1}^{n}(i-1)\,\sigma^{n-i}}_{\text{not good Green's op. (3/20)}} \cdot (1-\sigma)$$

This is exactly the formula we encountered on the previous page. (This has to be the case by invariant theory, namely the only natural operators on $V^{\otimes n}$ come from the group ring of the symmetric group. ?)

At this point I have a fairly good understanding of cyclic 1-cocycles on a free algebra, traces too, and I would like to apply this to cyclic theory.

The idea here is to replace traces on I^n by cyclic 1-cocycles on R/I^n. Thus we have (when R is free) exact sequences

$$\begin{tikzcd}
 & & (I\otimes_R)^n \arrow[d, phantom, "\parallel"] & \\
0 \arrow[r] & HC_1(R/I^n) \arrow[r] \arrow[d, shift right] & I^n/[R,I^n] \arrow[r] \arrow[d, shift right] & R/[R,R] \arrow[d, phantom, "\parallel"] \\
0 \arrow[r] & HC_{2n-1}(R/I) \arrow[r] \arrow[u, shift right] & I^n/[I,I^{n-1}] \arrow[r] \arrow[u, shift right] \arrow[d, phantom, "\parallel"] & R/[R,R] \\
 & & (I\otimes_R)^n_\sigma &
\end{tikzcd}$$

The bottom sequence is a direct factor of the top. Actually the injections appear to be more basic.

It seems that for any extension $A=R/I$ there is a canonical map

$$\textcircled{*} \qquad HC_{2n-1}(A) \longrightarrow HC_1(R/I^n)$$

for $n \geq 1$ which makes the square

$$\begin{tikzcd}
HC_1(R/I^n) \arrow[r] & I^n/[R,I^n] \\
HC_{2n-1}(A) \arrow[r] \arrow[u] & (I\otimes_R)^{n,\sigma} \arrow[u]
\end{tikzcd}$$

commute. By naturality it suffices to define ⊛ when R is free, and this we saw above can be done

April 6-10, 1989

Time taken off to do income tax. Some ideas:

An interesting problem is to find a direct proof of the theorems on extensions. I know how to do this using the big cocycle description of traces on $R = \bar{T}(A)$, but it might be possible to proceed directly and use the Chern-Simons deformation.

Recall that we have a cyclic cocycle

$$\begin{tikzcd}
{} \arrow[r] & A^{2n+2,\lambda} \arrow[r] \arrow[d, "\frac{\omega^{n+1}}{(n+1)!}\natural"] & A^{\otimes 2n+1,\lambda} \arrow[r] \arrow[d, "\partial\rho\frac{\omega^n}{n!}\natural"] & A^{\otimes 2n,\lambda} \arrow[r] \arrow[d, "\frac{\omega^n}{n!}\natural"] & {} \\
{} \arrow[r] & I^{n+1,\sigma} \arrow[r] & \Omega^1_R \otimes_R I^n \otimes_R \arrow[r] & I^{n,\sigma} \arrow[r] & {}
\end{tikzcd}$$

When we push this cocycle into the periodic complex for R it becomes null-homotopic via the Chern-Simons deformation, and so one gets a Chern-Simons cocycle with values in the quotient complex

$$\begin{tikzcd}
{} \arrow[r] & A^{\otimes 2n+1,\lambda} \arrow[r] \arrow[d, "\int_0^1\frac{\mu_{n+1}}{(n+1)!}dt\,\natural"] & A^{\otimes 2n,\lambda} \arrow[r] \arrow[d, "\int_0^1\partial\rho\frac{\mu_n}{n!}dt\,\natural"] & A^{\otimes 2n-1,\lambda} \arrow[r] \arrow[d, "\int_0^1\frac{\mu_n}{n!}dt\,\natural"] & {} \\
{} \arrow[r] & R/I^{n+1,\sigma} \arrow[r] & \Omega^1_R \otimes_R R/I^n \otimes_R \arrow[r] & R/I^{n,\sigma} \arrow[r] & {}
\end{tikzcd}$$

Formulas: $\delta e^{\omega}\natural = \beta(\partial\rho e^{\omega})\natural \; ; \; \delta(\partial\rho e^{\omega})\natural = \partial e^{\omega}\natural$

$$(e^{\omega})^{\circ}\natural = \delta\mu\natural - \beta(\partial\rho\mu)\natural \qquad \mu = \int_0^1 e^{(1-t)\omega}\mathring{\rho}\, e^{t\omega}dt$$

$$(\partial\rho e^{\omega})^{\circ}\natural = -\delta(d\rho\mu)\natural + \partial\mu\natural$$

April 12, 1989

Working with extensions seems difficult and in the wrong direction from the entire theory. So we return to JLO. It seems we can link Connes approach in his entire paper, where he uses traces on the Cuntz algebra, with JLO.

Consider A acting on $\mathcal{H}$ and a skew adjoint operator X on $\mathcal{H}$. Let $u = \frac{1+X}{1-X} = -1 + \frac{2}{1-X}$. Conjugating by u gives another action of A on $\mathcal{H}$

$$a \longmapsto \tilde{a} = u^{-1} a u$$

hence we have an action of $A * A$ on $\mathcal{H}$ with

$$a^{\pm} = \frac{a \pm \tilde{a}}{2} = \frac{1}{2}\left\{ a \pm \frac{1-X}{1+X}\, a\, \frac{1+X}{1-X} \right\}$$

$$= \frac{1}{2}\frac{1}{1+X}\left\{ (1+X)a(1-X) \pm (1-X)a(1+X) \right\} \frac{1}{1-X}$$

so

$$\boxed{\begin{aligned} a^{+} &= \frac{1}{1+X}\left(a - XaX\right)\frac{1}{1-X} \\ a^{-} &= \frac{1}{1+X}\left[X, a\right]\frac{1}{1-X} \end{aligned}}$$

This is the first version, but a slightly better version (I think), at least more symmetrical and therefore more suited to the superalgebra viewpoint of $A * A$ is the following

$$\boxed{\begin{aligned} & a^{+} = \frac{1}{1-X^{2}}\left(a - XaX\right) \qquad a^{-} = \frac{1}{1-X^{2}}[X,a] \\ & \iota a \longmapsto a^{+} + a^{-} = \frac{1}{1-X}\, a(1-X) \\ & \tilde{\iota} a \longmapsto a^{+} - a^{-} = \frac{1}{1+X}\, a(1+X) \end{aligned}}$$

There is an obvious similarity with

$$a \longmapsto (1+d)a(1-d) = a + da - dad$$

Next we want to consider traces, and really, to fit with our discussion of $\sqrt{\Sigma}\Omega$, we want supertraces on $A*A$.

First note that, just as $A*A$ is the superalgebra generated by the algebra A, that is the universal superalgebra with a map from A to its underlying algebra, there is ~~the~~ following right adjoint version. If S is a superalgebra and L is an algebra, and $S \xrightarrow{u} L$ is an algebra map, then one has a superalgebra map

$$S \xrightarrow{(u,\, u\varepsilon)} L \times L$$

where the $\mathbb{Z}/2$ action on $L\times L$ flips the factors. We have $L \times L \simeq L \otimes (\mathbb{C}\times\mathbb{C}) = L \otimes \underbrace{\mathbb{C}[\sigma]}_{C_1} = L[\sigma]$

and $S \longrightarrow L[\sigma]$ is given by

$$x \longmapsto u(x^{+}) + \sigma u(x^{-})$$

We have constructed a map of algebras $A*A \longrightarrow L = \mathcal{L}(H)$ with $a^{\pm}$ given by the formulas above. This (co)extends to a map of superalgebras

$$A * A \longrightarrow L[\sigma]$$

$$a^+ \longmapsto \frac{1}{1-X^2}(a - XaX)$$

$$a^- \longmapsto \frac{1}{1-X^2}\sigma[X,a]$$

In the graded case $L = \mathcal{L}(\mathcal{H})$ is already a superalgebra and the map $A*A \to \mathcal{L}$ is a superalgebra homomorphism.

Now we use the appropriate supertrace on L or $L[\sigma]$ or really on the trace class ideal. In the graded case this means we get cochains

$$\psi_{2n+1}(a_0, \ldots, a_{2n}) = tr\left(\varepsilon\, a_0^+ a_1^- \cdots a_{2n}^-\right)$$

$$\varphi_{2n}(a_1, \ldots, a_{2n}) = tr\left(\varepsilon \frac{1}{1-X^2}[X,a_1] \cdots \frac{1}{1-X^2}[X,a_{2n}]\right)$$

defined for $n \geq m$ some m.

This is a big cocycle

$$\begin{tikzcd} {} & \\ \psi_{2m+1} \arrow[u, "b"] \arrow[r, "1-\lambda"] & {} \\ & \varphi_{2m} \arrow[u, "b'"] \end{tikzcd}$$

and the φ's are λ-invariant. Thus φ_{2m} is a cyclic cocycle of degree $2m-1$. This is the wrong parity, so it doesn't represent the cyclic cohomology class of the extension.

Question: Does the JLO big cocycle necessarily come from a supertrace on the Cuntz algebra?

April 15, 1989

Observation: The Gelfand-Feigin-Fuks variation map described in Kassel's notes is not the same as the variation map I was thinking about. Mine corresponds to B and theirs to I in cyclic theory.

Here's a version, entirely cyclic theory version, of their map. Recall from the extension paper the canonical map of complexes

$$\mathrm{Hom}(B(A), R) \longrightarrow \mathrm{Hem}(B(A\otimes S), R\otimes S)$$

$$f \longmapsto \tilde{f}(a_1\otimes s_1, \ldots, a_n\otimes s_n) = f(a_1, \ldots, a_n)\otimes s_1\cdots s_n$$

[redacted] where R can be a vector space, but S is an algebra. Taking $R = B(A)$ we get a canonical map

$$B(A\otimes S) \longrightarrow B(A)\otimes S$$

which induces a [redacted] trace map

$$\boxed{B(A\otimes S)^{\natural} \longrightarrow B(A)^{\natural}\otimes S_{\natural}}$$

on cyclic complexes.

To obtain the GFF variation map take $S = k[\varepsilon]$, $\varepsilon^2=0$, and we get

$$B(A\otimes k[\varepsilon])^{\natural} \longrightarrow B(A)^{\natural}\otimes k[\varepsilon]$$

But $A\otimes k[\varepsilon]$ [redacted] $= A\oplus\varepsilon A =$ the semi-direct product, where [redacted] $\varepsilon A = A$ considered as an A-bimodule. Thus by Goodwillie

$$B(A\otimes k[\varepsilon])^{\natural} = B(A)^{\natural}\oplus\varepsilon\underbrace{(\Sigma A)\otimes_{-\theta} B(A)\otimes_{\theta}}_{\substack{\text{cyclic bar construction} \\ \text{suspended.} \ = \{b\text{-complex}\}}} \oplus \cdots$$

and so we get a canonical map

$$\{\text{b-complex}\} \longrightarrow \mathbb{C}(A)$$

This map pretty much has to be the canonical surjection.

My variational map is obtained from the canonical map

$$A \longrightarrow A \oplus \Omega^1_A \qquad a \longmapsto a + da$$

This induces a map on cyclic complexes

$B(A)^\natural \longrightarrow B(A$ [redacted] $\oplus \Omega^1_A)^\natural = B(A)^\natural \oplus (\Sigma_i \Omega^1_A \otimes_{-\theta} B(A) \otimes_\theta) \oplus \cdots$

so we obtain a map of complexes

$$B(A)^\natural \longrightarrow \Sigma \Omega^1_A \otimes_{-\theta} B(A) \otimes_\theta$$

Now the latter complex [redacted] needs to be understood.

I claim we have an exact sequence of complexes

$$0 \to \tilde{A} \longrightarrow \tilde{A} \otimes_{-\theta} B(A) \otimes_\theta \longrightarrow \Sigma_i \Omega^1_A \otimes_{-\theta} B(A) \otimes_\theta \longrightarrow 0$$

In effect the differential in the latter is

supp

$$\begin{aligned} d\{a_0 \partial a_1 \otimes (a_2, \cdot, a_n)\} = (-1)\{ &- a_0 \overbrace{\partial a_1 \cdot a_2}^{\partial(a_1 a_2) - a_1 \partial a_2} \otimes (a_3, \cdot, a_n) \\ &+ a_0 \partial a_1 \otimes b'(a_2, \cdot, a_n) \\ &+ (-1)^{n-2} a_n a_0 \partial a_1 \otimes (a_2, \cdot, a_{n-1})\} \end{aligned}$$

$$\begin{aligned} = (-1)\{ & a_0 a_1 \partial a_2 \otimes (a_3, \cdot, a_n) - a_0 \partial(a_1 a_2) \otimes (a_3, \cdot, a_n) \\ &+ a_0 \partial a_1 \otimes b'(a_3, \cdot, a_n) + (-1)^{n} a_n a_0 \partial a_1 \otimes (a_2, \cdot, a_{n-1})\} \end{aligned}$$

Thus if we make the correspondence

$$a_0 \partial a_1 \otimes (a_3, \ldots, a_n) \longleftrightarrow (a_0, a_1, \ldots, a_n)$$

the differential corresponds to $-b$ which is the differential in $\tilde{A} \otimes_{-\theta} B(A) \otimes_{\theta}$.

Now one has to check that the map $B(A)^{\natural} \longrightarrow \Sigma \Omega_A^1 \otimes_{-\theta} B(A) \otimes_{\theta}$ when lifted back to a map $B(A)^{\natural} \longrightarrow \tilde{A} \otimes_{-\theta} B(A) \otimes_{\theta}$ becomes a map of complexes. This doesn't appear obvious from what we have done.

Summary: We have variational interpretations of the maps I, B in the Connes exact sequence, in which the Hochschild or cyclic bar complex appears via Goodwillie's thm. on semidirect products.

But there are various steps that are mysterious, and it is certainly the case that we do not have an understanding based on the universal property of the bar construction.

Suggestive point: Adjointness property of $A \oplus \Omega_A^1$ versus $A \oplus \varepsilon A$, which reminds me of Cuntz's approach to KK which involves $A * A$ versus $A \times A$.