Aug 21, 1988 – Nov 11, 1988

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Most of the important stuff here ended up in paper finished Jan 2, 1989. Exceptions:

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I adic filtration of $\Omega^1_R \otimes_R \rightleftarrows R$ p 27

cyclic homology $A \otimes B$ p 58

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$\overline{HC}_{2n+1}(A) \longrightarrow HC_{2n}(k)$ described by Chern-Simons 72

August 21, 1988

Let return to DG algebras and DG coalgebras and twisting cochains. In the presence of a good duality, say where things are highly connected and finite dimensional in each degree, then we can replace the DG coalgebra $C$ by the cochain algebra $C^*$. A ~~connection~~ twisting cochain $\tau$ becomes an element $\tau$ of degree 1 in the DG algebra $A \hat{\otimes} C^* = \operatorname{Hom}(C, A)$ which is a flat connection form:

$$[d,\tau] + \tau^2 = 0$$

We can use $\tau$ to construct odd degree closed forms $\operatorname{tr}(\tau^{2n+1})$. Here $\operatorname{tr}$ denotes the image in the commutator quotient space, or what is very close to this

$$A/[A,A] \hat{\otimes} C^*/[C^*,C^*] = \operatorname{Hom}\left(\begin{smallmatrix}\text{cocomm}\\ \text{subspace}\\ \text{of } C\end{smallmatrix}, A/[A,A]\right)$$

Now suppose $A, C$ are free and that $\tau$ yields an equivalence of $A, C$ i.e. $C \otimes_\tau A$ is acyclic. Then we expect $\tau$ to induce a quis ~~[illegible]~~ from the cocomm subspace of $C$ to $A/[A,A]$. (Denote the cocomm. subspace by $C^{[\,,\,]}$). Then $\tau^{2n+1} : C \to A$ is of degree $2n+1$, hence it lowers degrees by $2n+1$ in the homology grading. So it is natural to ask whether $\operatorname{tr}(\tau^{2n+1}) : C^{[\,,\,]} \longrightarrow A/[\,,\,]$ is the operator $S^n$ in the cyclic theory.

It shouldn't be hard to calculate $\operatorname{tr}(\tau^{2n+1})$ when $C = \bar{B}(A)$.

The DG alg, coalg, twisting cochain theory suggests an approach to the link between cyclic theory and equivariant homology of the free loop space. Suppose we have a space $X$ which is 1-connected, and we choose a cocomm. cofree DG algebra $C$ representing the chains on $X$ and a free DG Hopf alg $A$ representing the chains on $\Omega X$. Then the twisting cochain $\tau : C \longrightarrow A$ is supposed to identify $C \backslash [\,,]$ with $A/[\,,]$.

Actually we might as well start with a $C$ not required to be cofree and then take $A = \text{cobar}(C)$, whence $\bar{A}/[A,A]$ is by defn. the cyclic complex of $C$. Because $A$ is free we have a periodic resolution

$$\longrightarrow \bar{A} \longrightarrow \Omega^1_A \otimes_A \longrightarrow \bar{A} \longrightarrow \bar{A}/[A,A] \longrightarrow 0$$

Now $\bar{A}/[A,A]$ is supposed to give the equivariant homology of $\Lambda X$. This should essentially be the theorem that the cyclic homology of $\Omega^\bullet(M)$ is the equivariant cohomology of $\Lambda M$. To understand this better it would be nice to identify the homology of

$$\text{Cone}\left\{ \Omega^1_A \otimes_A \longrightarrow \bar{A} \right\}$$

with $\bar{H}_*(\Lambda X)$ somehow. This is not so far-fetched because one has a fibration

$$\Omega X \longrightarrow \Lambda X \longrightarrow X$$

and so $C(\Lambda X)$ is approximately $A \otimes C$. On the other hand when $A = \text{Cobar}(C)$ one has

$$\Omega^1_A \otimes_A = A \otimes \Sigma^{-1}(\bar{C})$$

So one has a triangle

$$\bar{A} \longrightarrow \mathrm{Cone}\{\Omega^1_A \otimes_A \longrightarrow \bar{A}\} \longrightarrow \Sigma(\Omega^1_A \otimes_A)$$

$$\| $$

$$A \otimes \bar{C}$$

which is quite consistent with an identification

$$\mathrm{Cone}\{\Omega^1_A \otimes_A \longrightarrow A\}$$

$$\|?$$

$$C(\Lambda X) \overset{\text{approx}}{=} A \otimes C.$$

So the <u>conjecture</u> is that

$C(\Lambda X)$ is quis to $\mathrm{Cone}\{\Omega^1_A \otimes_A \longrightarrow A\}$

where $A = C(\Omega^1 X)$ in the rational homotopy setting

Let's examine this from a different direction, namely, let's represent $\Omega X$ by a free simplicial group $G$. Recall the free loop space is the homotopy pull-back

$$\begin{tikzcd} \Lambda X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

of $\Delta$ by itself. When $X = BG$ we can replace $\Delta$ by the fibration $P(G \times G)^{G \times G}(G)_{left+right} \longrightarrow B(G \times G)$, so we have a heq

$$\Lambda X \sim P(G) \times^G (G)_{conj}$$

<u>August 23, 1988</u>

Let's review the cyclic homology of $A = k[G]$ where $G$ is a free group. For a general group $G$ we have

$$\mathrm{Hom}_{A\otimes A^{o}}(\Omega^1_A, M) = \left\{ \begin{array}{c} \text{alg. homs } A \xrightarrow{u} A \oplus M \\ \text{with } \pi u = id \end{array} \right\}$$

$$\| \qquad\qquad = \left\{ \begin{array}{c} \text{grp homs. } G \xrightarrow{u} (1+M)^{\times} \times G \\ \text{such that } \pi u = id \end{array} \right\}$$

$$\mathrm{Der}(A, M) = \mathrm{Der}(G, M) = \mathrm{Hom}_{G\text{-mod}}(I[G], M)$$

To be more precise given a derivation $D : A \to M$ we set $\delta(g) = Dg \cdot g^{-1}$. Then

$$\delta(g_1 g_2) = (Dg_1 \cdot g_2 + g_1 \cdot Dg_2) g_2^{-1} g_1^{-1}$$

$$= \underset{\uparrow}{g_1 *} \delta g_2 + \delta g_1$$

conjugation action of $G$ on $M$.

It follows that there is a bimodule isomorphism

$$\Omega^1_A = k[G \times G] \otimes_{k[\Delta G]} I[\Delta G]$$

Thus for a free group $G$, $\Omega^1_A$ is a free [redacted] $A$-bimodules. If [redacted] $x_i$ are the generators of $G$, then the $dx_i$ are a basis for $\Omega^1_A$ as a bimodule.

Because the bimodule $\Omega^1_A$ is free we have that the higher Hochschild homology of $A$ is trivial:

$$H_n(A, A) = 0 \qquad n \geq 2.$$

Consequently the Connes exact sequence

$$\begin{tikzcd} HC_3(A) \arrow[r] & HC_1(A) \arrow[r] & \cancel{H_2(A,A)} \arrow[r] & HC_2(A) \arrow[r] & HC_0(A) \arrow[r] & H_1(A,A) \arrow[d] \\ \cancel{H_3(A,A)} \arrow[u] & & & & & HC_1(A) \arrow[d] \\ & & & & & 0 \end{tikzcd}$$

shows us that the cyclic homology is

$$HC_{2n}(A) \xrightarrow{\sim} \mathrm{Ker}\{HC_0(A) \xrightarrow{d} H_1(A,A)\}$$

$$HC_{2n-1}(A) \xrightarrow{\sim} \mathrm{Coker}\{HC_0(A) \xrightarrow{d} H_1(A,A)\}$$

for $n \geq 1$. Thus we have to understand this map $d$, in fact just as in the free algebra case we want to compute the two maps

$$\Omega^1_A \otimes_A \underset{b}{\overset{d}{\rightleftarrows}} A$$

Let's first look at the complex of length 1

$$\circledast \qquad \Omega^1_A \otimes_A \xrightarrow{b} A$$

which computes $H_*(A,A)$. $\Omega^1_A$ is a free bimodule with basis $dx_i$, where the $x_i$ are the generators of the free group $G$. $A = k[G]$ has the k-basis given by the elements of $G$. The map $b$ is induced by the embedding $\Omega^1_A \longrightarrow A \otimes A$, $a_0 da_1 a_2 \longmapsto a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2$

Thus $b(a_0 da_1 a_2 \otimes) = [a_2 a_0, a_1]$. Let's use the basis $dx_i x_i^{-1} g$ for $\Omega^1_A \otimes_A$; This is legitimate since $\Omega^1_A \otimes_A = \left(A \otimes (\bigoplus_i \mathbb{C} dx_i) \otimes A\right) \otimes_A = \bigoplus_i \mathbb{C} dx_i \otimes A$.

Then

$$b(dx_i x_i^{-1} g) = [x_i^{-1} g, x_i] = x_i^{-1} g x_i - g$$

so we can identify the complex $\circledast$ with the complex of chains on the directed graph having the elements of $G$ for its vertices and an edge from $g$ to $x_i^{-1} g x_i$ for each generator $x_i$. So the Hochschild homology is the homology of this graph.

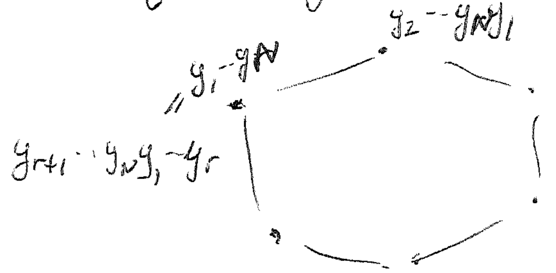
Up to homotopy a graph is a disjoint union of wedges of circles. The components the above graph are the different conjugacy classes of the free group $G$. Each conjugacy class is represented by a word $y_1 \cdots y_N$ where each $y_j$ is a generator $x_i$ or its inverse $x_i^{-1}$ and where $y_i y_{i+1} \neq 1$ for $i = 1, \ldots, N$ (interpreting $y_{N+1} = y_1$). This representative is unique up to cyclic permutations. In fact this representative and its cyclic perms. form a cycle in the conjugacy class which is a strong deformation retract (provided we are not dealing with the identity 1 in $G$). Thus each conjugacy class (viewed as a component of the graph) is $\sim S^1$ except for the identity which is a wedge of circles, one for each $x_i$.

Thus the identity conjugacy class contributes $G_{ab}$ to $H_1(\text{graph}, \mathbb{Z})$ and a $\mathbb{Z}$ to $H_0$, whereas the other conjugacy classes contribute $\mathbb{Z}$'s to both $H_1$ and $H_0$.

Next we recall what $d$ does on the homology. For a non-identity class represented by $y_1 \cdots y_N$ let $r$ be least so that

$$y_1 \cdots y_N = (y_1 \cdots y_r)^{N/r}$$

Then the generator of $H_1$ is represented by a cycle of length r




On the other hand d applied to $y_1 \cdots y_N$ will "cover" this $N/r$ times. Rationally then d is an isomorphism from $H_0$ to $H_1$ for the component.

For the identity class d is zero. So if we put everything together we find

$$HC_{2n}(A) = \mathbb{C} \qquad n \geq 1$$

$$HC_{2n-1}(A) = G_{ab} \otimes_{\mathbb{Z}} \mathbb{C} \qquad \text{"}$$

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Digression: Given an arbitrary G we can write it as $F/N$ where F is a free group. Then $A = k[G]$ is a quotient of $R = k[F]$ modulo the ideal I generated by the augmentation ideal of $k[N]$. Since $\Omega^1_R$ is a free R-bimodule it follows that your spectral sequences for the extension $R/I = A$ collapse in the same way as in ~~the~~ case where R is free. NO The cyclic homology of R is non-trivial, so this will interfere with the spectral sequence having the "odd" Connes homomorphism. However the other spectral sequence should be all right.

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<u>Problem:</u> In analogy to the cyclic homology of free algebras is there a simple non-computational way to obtain the <u>cyclic homology of a free group algebra</u>?

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Problem: Construct the isomorphism

$$\overline{HC}_{2n-1}(A) \longrightarrow G_{ab} \otimes \mathbb{C}$$

using curvature and connection methods. This should ~~[illegible]~~ result by constructing the map $\overline{HC}_1(A) \longrightarrow G_{ab} \otimes \mathbb{C}$ and then using $S$.

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Let us now consider a general group $G$ and let's study the cyclic homology of $A = \mathbb{C}[G]$ using a free simplicial group resolution.

More generally let us consider a free simplicial group $G_\cdot$ and study ~~the~~ reduced cyclic homology of the simplicial group algebra $A_\cdot = \mathbb{C}[G_\cdot]$. Formally it follows that the cyclic homology is given by the double complex

$$\longrightarrow \Omega^1_A \otimes_A \longrightarrow \bar{A} \longrightarrow \Omega^1_A \otimes_A \longrightarrow \bar{A}$$

The idea here is to apply the double complexes of the L-Q paper dimension-wise. For a free algebra or free group algebra, where we know $\Omega^1_A$ is free as an $A$-bimodule, we can recognize

$$\Omega^1_A \otimes_A \longrightarrow \bar{A}$$

as a quasimorphic quotient of the reduced Hochschild complex. So the simplicial double complex obtained by applying the $(b, B)$-double complex dimension-wise should be gives to the simplicial (or double) complex

$$\longrightarrow \bar{A}_\bullet \longrightarrow \Omega^1_{A_\bullet} \otimes_{A_\bullet} \longrightarrow \bar{A}_\bullet \qquad (*)$$

Unfortunately any attempt to use this double complex seems to lead to difficulties because we don't know anything about the homology of $\bar{A}_\bullet/[A_\bullet, A_\bullet]$. This is what happens if we first take homology horizontally

On the other hand suppose we take the homology vertically first. We know

$$\{\Omega^1_{A_\bullet} \otimes_{A_\bullet} \longrightarrow \bar{A}_\bullet\}$$

gives the reduced Hochschild homology of $k[G]$. I suppose $G_\bullet$ resolves the group $G$ whence $A_\bullet$ resolves $k[G]$. Thus $\bar{A}_\bullet$ has homology only in degree 0; it's a resolution of $\bar{k}[G]$. Thus it seems that the homology of $\Omega^1_{A_\bullet} \otimes_{A_\bullet}$ should coincide with the Hochschild homology of $k[G]$ in degrees $\geq 2$

$$0 \longrightarrow H_2(A,A) \xrightarrow{\sim} H_1(\Omega^1_{A_\bullet} \otimes_{A_\bullet}) \longrightarrow 0$$

$$\hookrightarrow H_1(A,A) \longrightarrow H_0(\Omega^1_{A_\bullet} \otimes_{A_\bullet}) \longrightarrow \bar{A} \longrightarrow \bar{H}_0(A,A) \longrightarrow 0$$

$$H_0(\Omega^1_{A_\bullet} \otimes_{A_\bullet}) = \Omega^1_A \otimes_A$$

This apparently doesn't lead anywhere either because all we get is the map $HC_n(A) \to H_{n+1}(A,A)$ which gives a map $HC_n(A) \longrightarrow H_{n+1}(G)$ instead of the kind of map ??

September 1, 1988

Problem: Consider $U(V)$ acting on $Gr(V)$. The canonical subbundle on $Gr(V)$ is an equivariant bundle equipped with an invariant connection. Hence ~~it has~~ not only are its characteristic classes represented by invariant forms, but also these invariant forms can be refined to equivariant forms representing the equivariant characteristic classes. The problem is to compute the equivariant which represent the character classes.

General discussion. The equivariant cohomology of $G$ acting on $G/H$ is the cohomology of $BH$. One has

$$\Omega_G(G/H) = \{S(\mathfrak{g}^*) \otimes \underbrace{\Omega(G/H)}\}^G$$

sections of an equivariant bundle over $G/H$, so its an induced module assoc. to a repn of $H$

$$= \{S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}/\mathfrak{h})^*\}^H$$

and this is supposed to give the cohomology of $BH$ which is given by

$$\Omega_H(pt) = \{S(\mathfrak{h}^*)\}^H$$

Thus we expect maybe to see a contracting homotopy, i.e. Chern-Simons type construction, in the $\mathfrak{g}/\mathfrak{h}$ directions.

Let's now review how to do equivariant characteristic classes. Let $P$ be a principal $H$-bundle over $M$, and let $G$ act on $P \to M$. Let $\theta \in \Omega^1(P) \otimes \mathfrak{h}$ be a connection in $P$ which

is $G$-invariant: $\theta \in \Omega^1(P)^G \otimes h$.
For $\theta$ to be a connection means that $\iota_Y \theta = Y$ for $Y \in h$ and that $\theta$ is $H$-invariant. We can view $\theta$ in

$$\Omega_G(P) \otimes h = \{S(g^*) \otimes \Omega(P)\}^G \otimes h$$

and it still is $H$-invariant and satisfies $\iota_Y \theta = Y$. Thus by the universal property of the Weil algebra we obtain a map

$$W(h) \longrightarrow \Omega_G(P) = \{S(g^*) \otimes \Omega(P)\}^G$$

and hence a map

$$S(h^*)^H \longrightarrow \Omega_G(M).$$

We note that the equivariant curvature of $\theta$ is $(d - \iota_X)\theta + \theta^2 = (d\theta + \theta^2) - \iota_X \theta$; this is the sum of the curvature of $\theta$ and the "momentum".

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Consider now $Gr(V)$ and the canonical subbundle $S$. We have the isometric embedding $i: S \longrightarrow \tilde{V}$ and equip $S$ with the induced connection $\iota^* \cdot d \cdot i$. Fix a component of $Gr(V)$, say the component consisting of subspaces isom. to $W$. Then over this component we have the principal $U(W)$-bundle $P$ consisting of isometric embeddings $W \to V$. Over $P$, $S$ becomes isom to $\tilde{W}$ and $i$ lifts to an isometric embedding

$$\iota: \tilde{W} \longrightarrow \tilde{V}$$

and the connection on $S$ becomes

$$\iota^* \cdot d \cdot \iota = d + \iota^*[d, i].$$

Thus the connection form is $\theta = \iota^*(di) \in \Omega^1(P) \otimes End(W)$, and the curvature is

$$i^* \cdot d \cdot i \cdot i^* \cdot d \cdot i = -\, i^*(dj)\, j^*(di)$$

If $X \in \mathrm{End}(W) = \mathrm{Lie}(U(W)) \otimes \mathbb{C}$, then the momentum map applied to $X$ is

$$\varphi_X = \iota_X \theta = \iota_X(i^*(di)) = i^* X i$$

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<u>September 2, 1988:</u>

We continue with the refining of the character forms on the Grassmannian to equivariant forms. More generally let $G$ act on $(M,E)$ and suppose given an invariant connection $D$ on $E$. Let $P$ be the principal bundle of $E$; say $P$ consists of isomorphisms of $W$ with fibres of $E$, so the group is $H = U(W)$. Then $E$ lifted to $P$ becomes isomorphic to $\tilde{W}$, and so $D = d + \theta$ with $\theta \in \Omega^1(P) \otimes \mathrm{End}(W)$ a connection form invariant under $G$. Then we can work in $\Omega_G(P)$ and take $H$-basic elements to end up with $\Omega_G(M)$. In particular we have the Chern-Weil map to equivariant forms:

$$\begin{tikzcd}
\Omega_G(P) & W(\mathfrak{h}) \arrow[l, "\text{assoc to } \theta"'] \\
\Omega_G(M) \arrow[u, hook] & S(\mathfrak{h}^*)^H \arrow[l] \arrow[u, hook]
\end{tikzcd}$$

In $\Omega_G(P) = \{S(\mathfrak{g}^*) \otimes \Omega(P)\}^G$ the differential is $d - \iota_X$, where $\iota_X$ applied to a polynomial function $\omega(X)$ on $\mathfrak{g}$ with values in $\Omega(P)$ gives the poly. function $X \mapsto \iota_X \omega(X)$. If we

like we can change the signs 
to $d + \iota_X$ using $-1$ on $\mathfrak{g}$. So let's do this and use the differential

$$d_{tot} = d + \partial \qquad \partial = \iota_X$$

To get the character forms we use the equivariant curvature

$$(d+\partial)\theta + \theta^2 = \underbrace{(d\theta + \theta^2)}_{K} + \underbrace{\partial\theta}_{\varphi}$$

where $\varphi \in \mathfrak{g}^* \otimes \Omega^0(P) \otimes \mathrm{End}(W)$ is given by

$$\varphi(X) = i_X \theta$$

The equivariant character form of degree $2n$ is

$$\mathrm{tr}\,(K+\varphi)^n = \sum_{k=0}^{n} \mathrm{tr}\left\{ P(\overset{n-k}{K}, \overset{k}{\varphi}) \right\}$$

This is equivariantly closed by the usual proof resting on the Bianchi identity

$$\left[(d+\theta+\partial), (K+\varphi)\right] = 0$$

$$\cancelto{0}{[d+\theta, K]} + \partial K + [d+\theta, \varphi] + \cancelto{0}{\partial\varphi} = 0$$

One can transcribe this proof into component form to show that

$$[d+\theta,\, P(\overset{l}{K}, \overset{k}{\varphi})] + \partial P(\overset{l+1}{K}, \overset{k-1}{\varphi}) = 0$$

if desired.

Notice that $\mathrm{tr}\,(K+\varphi)^n$ is a cocycle for $d+\partial$ which links the n-th

character form $tr(K^n)$ to the polynomial function
$$tr(\varphi_X^n) = tr\,(i^*Xi)^n.$$

Thus one has a symmetric tensor rather than the family skew-symmetric forms of odd degree. It's rather a mystery as to what these equivariant forms are good for.

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Let's review the Leray spectral sequence for a principal bundle $G \to P \to M$, where we work with differential forms and we suppose $G$ compact connected. We know that there is a quis $\Omega(P)^G \to \Omega(P)$. Now locally $P \simeq M \times G$, so

$$\begin{aligned}\Omega(P)^G &\simeq \left(\Omega(M) \otimes \Omega(G)\right)^G \\ &\simeq \left(\Omega(M) \otimes \Omega(G)^{G\ \text{right}}\right) \\ &\simeq \left(\Omega(M) \otimes \Lambda \mathfrak{g}^*\right).\end{aligned}$$

But we have to be careful not to confuse the isomorphism $\Omega(G)^{G\ \text{left}} = \Lambda \mathfrak{g}^*$, which is obvious as $\mathfrak{g}$ is by definition the space of left-invariant vector fields, with the isomorphism above, which depends on the trivialization of $P$. To sort this out consider $\Omega^1(P)^G$ which fits into an exact sequence

$$0 \longrightarrow \Omega^1(M) \longrightarrow \Omega^1(P)^G \longrightarrow \Gamma(M, P \times^G \mathfrak{g}^*) \longrightarrow 0$$

In effect $E = P \times^G \mathfrak{g}$ can be viewed as the bundle of $G$-invariant vertical vector fields on $P$. A connection splits this exact sequence and yields an algebra isomorphism

$$\Omega^{\cdot}(P)^G \xleftarrow{\ \sim\ } \Omega^{\cdot}(M, \Lambda^{\cdot} E^*)$$

We get a canonical isomorphism by filtering by powers of the ideal in $\Omega(P)^G$ generated by $\Omega^1(M)$:

$$gr_p\{\Omega^{\cdot}(P)^G\} = \Omega^p(M, \Lambda^{\cdot} E^*)$$

Associated to this filtration is a spectral sequence.

Here is what I think happens in 

this spectral sequence. The differential

$$d_0 : \Omega^p(M, \underbrace{\Lambda^q E^*}_{\text{bundle } P\times^G(\Lambda^q\mathfrak{g}^*)}) \longrightarrow \Omega^p(M, \Lambda^{q+1}E^*)$$

is given by the Lie algebra cochain differential on $\Lambda^{\cdot}\mathfrak{g}^*$. Hence

$$E_1^{pq} = \Omega^p(M)\otimes H^q(\mathfrak{g})$$

Now if we choose a connection, then we have a homomorphism

$$W(\mathfrak{g}) \longrightarrow \Omega^{\cdot}(P)^G$$

which shows that the primitive generators of $H^*(\mathfrak{g})$ are transgressive. So we should get then a spectral sequence

$$E_2^{pq} = H^p(M)\otimes H^q(\mathfrak{g}) \Longrightarrow H^{p+q}(P)$$

where the primitive generators of $H^*(\mathfrak{g})$ are transgressive.

Now I want to change notation slightly and consider

$$H \longrightarrow G \longrightarrow G/H$$

where $H$ is compact connected. We will be interested in left-$G$-invariant forms.

$$\Omega(G)^{G\text{ left}} = \Lambda\mathfrak{g}^*$$

$$\Omega(G/H)^{G\text{ left}} = \{\Lambda(\mathfrak{g}/\mathfrak{h})^*\}^H = \{\Lambda\mathfrak{g}^*\}_{H\text{-basic}}$$

We apply the preceding discussion to the principal $H$-bundle $G \to G/H$ and get

$$gr_p\{\Omega^{\cdot}(G)^H\} = \Omega^p(G/H, \Lambda^{\cdot}E^*) \qquad E = G\times^H\mathfrak{h}$$

then pass to $G$-left-invariants to get

$$gr_p\,\{\Lambda^\cdot \mathfrak{g}^*\}^H = \{\Lambda^p(\mathfrak{g}/\mathfrak{h})^* \otimes \Lambda^\cdot \mathfrak{h}^*\}^H$$

The filtration in question on [redacted] $\Lambda\mathfrak{g}^*$ is the adic filtration for the ideal generated by $(\mathfrak{g}/\mathfrak{h})^* \subset \Lambda\mathfrak{g}^*$. Notice that this ideal is closed under $d$ on $\Lambda^\cdot\mathfrak{g}^*$. In effect $d: \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^*$ is the transpose of the bracket, and the composition

$$\Lambda^2\mathfrak{h} \hookrightarrow \Lambda^2\mathfrak{g} \xrightarrow{[\,,\,]} \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h}$$

is zero. Thus $d(\mathfrak{g}/\mathfrak{h})^* \subset \mathrm{Ker}\{\Lambda^2\mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{h}^*\} = \mathfrak{g}^* \wedge (\mathfrak{g}/\mathfrak{h})^*$.

[redacted]

Let's start again with a Lie algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$. Then in $\Lambda\mathfrak{g}^*$ we have the ideal $\mathrm{Ker}\{\Lambda\mathfrak{g}^* \longrightarrow \Lambda\mathfrak{h}^*\} = (\mathfrak{g}/\mathfrak{h})^*\Lambda\mathfrak{g}^*$ which is closed under $d$, and hence it gives rise to a spectral sequence with

$$E_0^{pq} = \Lambda^p(\mathfrak{g}/\mathfrak{h})^* \otimes \Lambda^q(\mathfrak{h}^*)$$

(I suppose $\mathfrak{h}$ finite dimensional reductive). What is $d_0$? The obvious conjecture is that it should be the Lie cohomology differential for cochains on $\mathfrak{h}$ with values in the $\mathfrak{h}$ module $\Lambda^p(\mathfrak{g}/\mathfrak{h})^*$.

Let's check this for $p=1$. We want the map

$$\begin{tikzcd}
(\mathfrak{g}/\mathfrak{h})^* \arrow[r, hook] \arrow[drr] & \mathfrak{g}^* \arrow[r, "\delta"] & \Lambda^2\mathfrak{g}^* \\
& & (\mathfrak{g}/\mathfrak{h})^* \wedge \mathfrak{g}^* \arrow[u, phantom, "\cup"] \arrow[r] & (\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{h}^*
\end{tikzcd}$$

which is the transpose of the map

$$\mathfrak{h} \otimes \mathfrak{g}/\mathfrak{h} \hookrightarrow \Lambda^2\mathfrak{g}/\Lambda^2\mathfrak{h} \xrightarrow{[\,,\,]} \mathfrak{g}/\mathfrak{h}.$$

Thus $d: (\mathfrak{g}/\mathfrak{h})^* \longrightarrow (\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{h}^*$ is the Lie coboundary in $C^\cdot(\mathfrak{h}, (\mathfrak{g}/\mathfrak{h})^*)$ and so the rest should be clear.

Thus we obtain

$$E_1^{pq} = H^q(h, \Lambda^p(g/h)^*)$$

Now assuming $h$ reductive and $g$ a sum of finite dimensional representations of $h$ we know

$$H^q(h, \Lambda^p(g/h)^*) = \Lambda^p(g/h)^{*h} \otimes H^q(h)$$

and so we end up with the desired spectral sequence

$$E_2^{pq} = H^p\{(\Lambda^\bullet(g/h)^*)^h\} \otimes H^q(h) \Longrightarrow H^{p+q}(g)$$

Summary: The interesting point is that for an arbitrary Lie algebra $g$ and subalgebra $h$ there appears to be (at least for $\dim(g) < \infty$) a spectral sequence

$$E_1^{pq} = H^q(h, (\Lambda^p(g/h))^*) \Longrightarrow H^{p+q}(g).$$

---

Next I want to apply this to the case where $g = gl(A)$ and $h = gl(\mathbb{C})$. The goal is to derive the fact that one has a triangle of complexes

$$CC(\mathbb{C}) \longrightarrow CC(A) \longrightarrow \overline{C}C(A).$$

One has

$$\{(\Lambda g)^*\}^h = \mathrm{Sym}\{CC(A)\}^*$$

$$\{(\Lambda g/h)^*\}^h = \mathrm{Sym}\{\overline{C}C(A)\}^*$$

$$\{(\Lambda h)^*\}^h = \mathrm{Sym}\{CC(\mathbb{C})\}^*$$

Maybe it would be better to forget the dualizing and write

$$(\Lambda \mathfrak{g})_{\mathfrak{h}} = \mathrm{Sym}\{\Sigma CC(A)\}$$

$$(\Lambda \mathfrak{g}/\mathfrak{h})_{\mathfrak{h}} = \mathrm{Sym}\{\Sigma \bar{C}C(A)\}$$

$$(\Lambda \mathfrak{h})_{\mathfrak{h}} = \mathrm{Sym}\{\Sigma CC(k)\}$$

However it is more or less clear that what I am after has to result from a filtration on $CC(A)$.

We can consider $A$ as a filtered ring; I should say algebra with an increasing filtration, by setting

$$F_0 A = \mathbb{C}$$

$$F_p A = A \qquad \text{for } p \geq 1.$$

Then $F_p A \cdot F_q A \subset F_{p+q} A$, and one has

$$\mathrm{gr}(A) = \mathbb{C} \oplus \bar{A} \oplus 0 \oplus \cdots$$

where $\bar{A} \cdot \bar{A} = 0$ in $\mathrm{gr}(A)$. This filtration on $A$ induces a filtration on the cyclic complex $CC(A)$ by setting

$$F_p\left(A_\lambda^{\otimes n}\right) = \sum_{i_1 + \cdots + i_n \leq p} \mathrm{Im}\left\{F_{i_1} A \otimes \cdots \otimes F_{i_n} A \longrightarrow A_\lambda^{\otimes n}\right\}$$

Thus $F_p(A_\lambda^{\otimes n})$ is spanned by symbols $(a_1, \dots, a_n)$ where $\leq p$ of the $a_i$ are in $A - \mathbb{C}$.
For this filtration of the cyclic complex we have

$$\mathrm{gr}\ CC(A) = CC(\mathrm{gr}\,A) = CC(\mathbb{C} \oplus \bar{A})$$

Indeed the filtration on $A$ induces a filtration on $A^{\otimes n}$ such that

$$\mathrm{gr}\, A^{\otimes n} = (\mathrm{gr}\, A)^{\otimes n}$$

and then we can take quotient by the cyclic groups to obtain the desired formula.

Next let's apply the Goodwillie et al. result on semi-direct products. This identifies $CC(\mathbb{C} \oplus \bar{A})_{(p)}$ with the complex

$$(\bar{A} \otimes K \otimes)^{p}_{\lambda}\,[p-1] \qquad p > 0$$

where $K$ is the standard resolution of $\mathbb{C}$ as a $\mathbb{C}$-bimodule

$$\xrightarrow{0} \mathbb{C}^{\otimes 4} \xrightarrow{1} \mathbb{C}^{\otimes 3} \xrightarrow{0} \mathbb{C}^{\otimes 2} \dashrightarrow \mathbb{C} \dashrightarrow 0$$

Put another way the [redacted] degree $p$ in $\bar{A}$ part of $CC(\mathbb{C} \oplus \bar{A})$ is spanned by symbols

$$a_1, 1, \ldots, 1, a_2, 1, \ldots, 1, \ldots\ldots, a_p, 1, \ldots, 1$$

modulo the action of $\mathbb{Z}_p$. The part of the differential applied to the commas around a sequence of $1$'s gives zero when there are an odd number of $1$'s and $1$ when there are an even number. So the cohomology is trivial except for $\bar{A}^{\otimes p}_{\lambda}$ in the bottom degree $p-1$.

It should now be clear that $CC(A)/CC(\mathbb{C})$ is quis to $\overline{CC}(A)$. [redacted]

Next we look at the transgression for the "principal fibring" $U_n \longrightarrow Gl_n(A) \longrightarrow U_n(A)/U_n$

What this means is that if I choose a "connection", which amounts to a splitting of

$$\textcircled{*} \qquad 0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0,$$

then I should be able to construct a [redacted] cyclic $2n$-cochain on $A$, whose coboundary [redacted] is a reduced cyclic cochain and whose restriction to $\mathbb{C}$ is the generator for [redacted] $HC^{2n}(\mathbb{C})$.

Let the splitting of $\textcircled{*}$ be given by a linear map $\rho : A \longrightarrow \mathbb{C}$ such that $\rho(1) = 1$. Then we can treat $\rho \in C^1(A)$ as a connection form and form the curvature $\gamma = \delta\rho + \rho^2$:

$$\gamma(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$$

which is a reduced cochain. Then we can form the Chern-Simons form and the character form

$$tr\,(\gamma^{n+1}) \in \bar{C}_\lambda^{2n+1}(A)$$

$$\int_0^1 dt\,(n+1)\, tr\,\{\rho(t\gamma + (t^2-t)\rho^2)^n\} \in C_\lambda^{2n}(A).$$

The Chern-Simons form is the desired transgression form.

~~[illegible]~~

September 6, 1988

Let's return to the problem of showing that the Connes homomorphisms

$$HC_{2n-1}(A) \longrightarrow I^n/[I, I^{n-1}]$$

are compatible with the S-operation. The S-operation will be defined by diagram chasing in the exact sequence of complexes

$$\begin{tikzcd}
0 & A_\lambda^{\otimes 2n+2} \arrow[l] & A^{\otimes 2n+2} \arrow[l] \arrow[d, "b"] & {} \arrow[l] \arrow[d] & & \\
& & A^{\otimes 2n+1} \arrow[d, "b"] & A^{\otimes 2n+1} \arrow[l, "1-t"'] \arrow[d, "b'"] & & \\
& & {} & A^{\otimes 2n} \arrow[l] & A_\lambda^{\otimes 2n} \arrow[l, "N"'] & 0 \arrow[l]
\end{tikzcd}$$

Let's start with a suitable "trace" $\tau : I^n \longrightarrow \mathbb{C}$. We choose $\rho : A \longrightarrow R$ a lifting. Then the curvature:

$$(\delta\rho + \rho^2)(a_1, a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$$

is a 2-cochain with values in $I$; put

$$\gamma = \delta\rho + \rho^2 \quad \in C^2(A, I).$$

Then $\quad \gamma^n \in C^{2n}(A, I^n) \quad$ and we can put

$$tr(\gamma^n) \overset{def}{=} \tau\gamma^n N : A_\lambda^{\otimes 2n} \longrightarrow \mathbb{C}.$$

We know this is a cyclic $(2n-1)$-cocycle on $A$.

Now we want to carry through the diagram chasing. We begin by extending $\tau\gamma^n N$

to $\tau\gamma^n$ on $A^{\otimes n}$, then we compose with $b'$. For even cochains $\delta f = -fb'$ so $\tau\gamma^n b'$ is

$$-\delta(\tau\gamma^n) = -\tau(\gamma^n\rho - \rho\gamma^n)$$
$$= \tau(\rho\gamma^n - \gamma^n\rho)$$

As a cochain this sends $(a_0, \dots, a_{2n})$ to

$$\tau\{\rho(a_0)\gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n}) - \gamma(a_0,a_1)\cdots\gamma(a_{2n-2},a_{2n-1})\rho(a_{2n})\}$$

$$= \tau\{\rho(a_0)\gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n}) - \rho(a_{2n})\gamma(a_0,a_1)\cdots\gamma(a_{2n-2},a_{2n-1})\}$$

$$= \tau(\rho\gamma^n)(1-t)(a_0,\dots,a_{2n})$$

This we lift to $\tau(\rho\gamma^n) : A^{\otimes(2n+1)} \longrightarrow \mathbb{C}$. Now we have to compute $\tau(\rho\gamma^n)b : A^{\otimes(2n+2)} \longrightarrow \mathbb{C}$ which will automatically factor through $A_\lambda^{\otimes(2n+2)}$.

At this point let's [redacted] assume we can identify [redacted] the Hochschild complex [redacted] with the coalgebra analogue of $\Omega^1_A \otimes_A$. To be more precise we are working in the cochain algebra $C(A,B) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{B}(A), B)$ which is roughly the tensor product algebra

$$T(A^*) \otimes B$$

Set $R = C(A)$. Trace theory over the free alg $R$ is somehow contained in the periodic sequence

$$\longrightarrow \bar{R} \longrightarrow \Omega^1_R \otimes_R \longrightarrow \bar{R} \longrightarrow \bar{R}/[R,R].$$

So we propose to tensor this with $B$. Because $\Omega^1_R = R \otimes A^* \otimes R$ we have $\Omega^1_R \otimes_R \simeq A^* \otimes R$, so this is a complex of cochains on $A$, but the differential is different. We should think of the $A^*$ as consisting of $d\lambda$ in $\Omega^1_R$ for $\lambda \in A^*$ = the generators of $R$. Then if $\delta(\lambda) = \lambda_i' \otimes \lambda_i'' \in (A^{\otimes 2})^*$, then

$$\delta\, d(\lambda) = d(\lambda_i' \otimes \lambda_i'')$$
$$= d\lambda_i' \otimes \lambda_i'' \pm \lambda_i' \otimes d\lambda_i''$$

which ought to lead to the usual Hochschild coboundary for cochains with values in $A^*$.

Let's adopt this formalism. We want to apply this to $\rho\gamma^n : A \otimes A^{\otimes 2n} \longrightarrow I^n$. I am viewing this as $d\rho \cdot \gamma^n$ in $(\Omega^1_R \otimes_R) \otimes I^n$. If we apply $\delta$ we get

$$\delta(d\rho\, \gamma^n) = d(\delta\rho)\gamma^n - d\rho(\gamma^n\rho - \rho\gamma^n)$$

Somehow if we ~~and~~ project into $I^n/[R, I^n]$, because we are using $\Omega^1_R \otimes_R$, we have $-d\rho\,\gamma^n\rho$ is equivalent to $\rho\, d\rho\, \gamma^n$. Thus we have on the right $(d(\delta\rho) + d\rho\,\rho + \rho\, d\rho)\gamma^n = (d\gamma)\gamma^n$, ~~and~~

Yesterday we encountered a simple formalism which seems to explain why the character forms are linked by the S-operator. Let $\rho \in C^1(A,R)$ $\gamma = \delta\rho + \rho^2 \in C^2(A,I)$ be as usual.

In addition to cochains on $A$, that is, linear maps from $\mathrm{Bar}(A)$ we will need Hochschild cochains, that is, linear maps from the Hochschild complex $(A^{\otimes(*+1)}, b)$. These two complexes are related by maps

$$\longrightarrow C^{\cdot}(A) \xrightarrow{\partial} C^{\cdot}_{Hoch}(A) \xrightarrow{\beta} C^{\cdot}(A) \xrightarrow{\partial}$$

which are of the form

$$\longrightarrow R \xrightarrow{\partial} \Omega^1_R \otimes_R \xrightarrow{\beta} R \xrightarrow{\partial}$$

The formal calculation runs as follows. In the calculation we work with cochains and Hochschild cochains with values in $I^n$, and the congruent sign means the cochains have the same values in $I^n/[R,I^n]$. The formulas are:

$$\delta(\gamma^n) = \gamma^n\rho - \rho\gamma^n$$

$$\beta(\gamma^n\partial\rho) = \gamma^n\rho - \left(\begin{smallmatrix}\text{left}\\ \text{mult}\end{smallmatrix}\ I^n\otimes R \to I^n\right)(\gamma^n\otimes\rho)$$

$$\equiv \gamma^n\rho - \rho\gamma^n$$

$$\partial(\gamma^{n+1}) = \sum_{i=0}^{n} \gamma^i\,\partial\gamma\,\gamma^{n-i} \equiv (n+1)\gamma^n\partial\gamma$$

$$\delta(\gamma^n\partial\rho) = (\gamma^n\rho - \rho\gamma^n)\partial\rho + \gamma^n\partial(\delta\rho)$$

$$\equiv \gamma^n(\rho\,\partial\rho + \partial\rho\,\rho + \partial(\delta\rho))$$

$$= \gamma^n\partial\gamma$$

Recall however that we had the same formulas for the map of complexes

$$\begin{tikzcd}
\arrow[r] & A_\lambda^{\otimes(2n+2)} \arrow[r] \arrow[d, "\frac{\gamma^{n+1}}{(n+1)!} N"] & A_\lambda^{\otimes(2n+1)} \arrow[r] \arrow[d, "\frac{\gamma^n}{n!} \partial\rho\, N"] & A_\lambda^{\otimes 2n} \arrow[r] \arrow[d, "\frac{\gamma^n}{n!} N"] & {} \\
\arrow[r] & I^{n+1} \arrow[r, "\partial"] & I^n \Omega^1_R \otimes_R \arrow[r, "\beta"] & I^n \arrow[r] & {}
\end{tikzcd}$$

where $\equiv$ means having the same restriction after restricting to the cyclic subcomplex, i.e. composing with $N$.

Thus in both cases we deal with a cochain algebra

$$C^\cdot(A, R) = \mathrm{Hom}_{\mathbb{C}}(\mathrm{Bar}(A), R) \sim T(A^*) \otimes R$$

In the first case we use the $\beta, \partial$ complex for $T(A^*)$, and in the second case we use the $\beta, \partial$ complex for $R$.

September 8, 1988

Let's consider $CC(R \leftarrow I)$. This is a double complex whose $p$-th column, $p \geq 1$, is the complex
$$\left(I \otimes_R B(R) \otimes_R\right)^p_\sigma = \left(I \overset{1}{\otimes}_R\right)^p_\sigma$$
where $B(R)$ is the standard $R\otimes R^\circ$-resolution of $R$. Now we have a map of complexes

$$(*) \quad \left(I \otimes_R B(R) \otimes_R\right)^n_\sigma \hookrightarrow \left(I \otimes_R B(R) \otimes_R\right)^n \longrightarrow \big(\overbrace{I \otimes_R \cdots \otimes_R I}^{I^{(n)}}_{n\text{-factors}}\big) \otimes_R B(R) \otimes_R$$
$$\to I^{(n)} \otimes_R \{ \Omega^1_R \longrightarrow R\otimes R\} \otimes_R$$

where the last complex is the two step complex

$$(**) \qquad I^{(n)} \otimes_R \Omega^1_R \otimes_R \xrightarrow{\ \beta\ } I^{(n)} \qquad \beta(\alpha\otimes dr\otimes) = \alpha r - r\alpha$$

which gives the 0-th and 1-th groups $H_*(R, I^{(n)})$. Let's work out the formula for $(*)$ in degree 1. One has
$$\left(I \otimes_R B(R) \otimes_R\right)^n_{(1)} = \bigoplus_{i=1}^n I^{\otimes i} \otimes R \otimes I^{\otimes n-i}$$

so
$$\left(I \otimes_R B(R) \otimes_R\right)^n_{\sigma,(1)} = \left( \qquad\qquad \right)_{\mathbb{Z}/n} \xleftarrow{\ \sim\ } I^{\otimes n} \otimes R$$

The norm embedding of $\sigma$-invariants then in degree 1 is
$$\begin{array}{ccc} I^{\otimes n} \otimes R & & \bigoplus_{i=1}^n I^{\otimes i} \otimes R \otimes I^{\otimes(n-i)} \\ \cup & & \cup \\ (x_1, \ldots, x_n, r) & \longmapsto & \sum_{j=1}^n (x_j, \ldots, x_n, r, x_1, \ldots, x_{j-1}) \end{array}$$

Upon using $B(R) \xrightarrow{\varepsilon} R$ for the first $n$-factors, this goes to
$$(x_1, \ldots, x_n, r) \in I^{(n)} \otimes_R B(R)_1 \otimes_R = I^{(n)} \otimes R$$
which then maps to $\quad (x_1, \ldots, x_n) \otimes dr \in I^{(n)} \otimes_R \Omega^1_R \otimes_R$

Next a formula for $(*)$ in degree 0.

$$\begin{array}{ccc} \left(I\otimes_R B(R)\otimes_R\right)^n_{\sigma(0)} & = & \left(I\otimes\right)^n_\sigma \\ \downarrow & & \downarrow \\ \left(I\otimes_R B(R)\otimes_R\right)^n & = & I^{\otimes n} \end{array}$$

We need the differential

$$\begin{array}{ccccc} \left(I\otimes_R B(R)\otimes_R\right)^n_{\sigma(1)} & = & I^{\otimes n}\otimes R & \hookrightarrow & \bigoplus_{1}^{n} I^{\otimes i}\otimes R\otimes I^{\otimes(n-i)} \\ \downarrow & & \downarrow & & \downarrow \\ \left(I\otimes_R B(R)\otimes_R\right)^n_{\sigma(0)} & = & I^{\otimes n}_\sigma & \hookrightarrow & I^{\otimes n} \end{array}$$

$$(x_1,\dots,x_n,r) \longmapsto \sum_{j=1}^{n}(x_j,\dots,x_n,r,x_1,\dots,x_{j-1})$$

$$\downarrow$$

$$\begin{array}{l}(x_1,\dots,x_n r)\\ -(rx_1,\dots,x_n)\end{array} \longmapsto \begin{array}{l}\sum_{j=1}^{n}(x_j,\dots,x_n r,x_1,\dots,x_{j-1})\\ \qquad -(x_j,\dots,x_n,rx_1,\dots x_{j-1})\end{array}$$

Note that there are no signs here because $I$ is of degree 0 and $R$ is of degree 1 when we work with $\left(I\otimes_R B(R)\otimes_R\right)^{\otimes n}_\sigma$.

So the conclusion is that the map

$$\left(I\otimes_R B(R)\otimes_R\right)^n_\sigma \longrightarrow \left\{ I^{(n)}\otimes_R \Omega^1_R\otimes_R \xrightarrow{\beta} I^{(n)}\right\}$$

is in degrees 0, 1 (in higher degree it is zero)

$$(x_1,\dots,x_n,r) \longmapsto (x_1,\dots,x_n)\,dr$$

$$\begin{array}{ccc} I^{\otimes n}\otimes R & \longrightarrow & I^{(n)}\otimes_R\Omega^1_R\otimes_R \\ \downarrow{\scriptstyle (x_1,\dots,x_n r)-(rx_1,\dots x_n)} & & \downarrow\beta \\ I^{\otimes n}_\sigma & \xrightarrow[\text{by } I^{\otimes n}\longrightarrow I^{(n)}]{\text{norm, followed}} & I^{(n)} \end{array}$$

Now the next stage will be to consider the horizontal arrows in $\mathbb{C}(R \leftarrow I)$. In particular these give maps of complexes

$$\circledast \qquad \Sigma\left(I \overset{!}{\otimes}_R\right)^{n+1}_\sigma \longrightarrow \left(I \overset{!}{\otimes}_R\right)^{n}_\sigma$$

which we would like to understand much better. As a first step let's show

$$\begin{tikzcd}
\Sigma\left(I \overset{!}{\otimes}_R\right)^{n+1}_\sigma \arrow[rr, "\circledast"] \arrow[d] & & \left(I \overset{!}{\otimes}_R\right)^{n} \arrow[d] \\
\Sigma\left\{I^{(n+1)} \otimes_R \Omega^1_R \otimes_R \longrightarrow I^{(n+1)}\right\} \arrow[dr, two heads] & & \left\{I^{(n)} \otimes_R \Omega^1_R \otimes_R \longrightarrow I^{(n)}\right\} \\
& \Sigma\left(I \otimes_R\right)^{n+1} \arrow[ur, dashed, "\partial"] &
\end{tikzcd}$$

commutes. In other words I have the map $\circledast$ above and the map from $\left(I \overset{!}{\otimes}_R\right)^n$ to a 2-step complex, and the composite map factors through $H_0\left(\left(I \overset{!}{\otimes}_R\right)^{n+1}_\sigma\right) = \left(I \otimes_R\right)^{n+1}_\sigma$, and I want to identify the result with $\partial$.

Return to $\mathbb{C}(R \leftarrow I)$, $n$th row:

$$\begin{array}{ccc}
\longleftarrow \left\{\bigoplus_{i=0}^{n} I^{\otimes i} \otimes R \otimes I^{\otimes(n-i)}\right\}_{\mathbb{Z}/(n+1)} & \longleftarrow & I_\sigma^{\otimes(n+1)} \\
\uparrow \wr & & \\
I^{\otimes n} \otimes R & &
\end{array}$$

Under the boundary in the complex $(R \leftarrow I)^{\otimes(n+1)}_\sigma$, the element $(x_0, \dots, x_{n+1}) \in I_\sigma^{\otimes n+1}$ goes to

$$\sum_{i=0}^{n}(-1)^i\,(x_0,\dots,x_{i-1},u(x_i),x_{i+1},\dots,x_n)$$

in $\left(\bigoplus_{i=0}^{n} I^{\otimes i}\otimes R\otimes I^{\otimes(n-i)}\right)_{\mathbb{Z}/n+1}$

~~which corresponds to~~ ~~$(-1)^n(x_0,\dots,x_{n-1},u(x_n))$~~ ~~$\otimes R$~~ ~~under Goodwillie's isom.~~ where $u: I\to R$ is the embedding.

Under the Goodwillie isomorphism this corresponds to

$$\sum_{i=0}^{n}(-1)^i\Big(\underset{\text{sign in }\lambda}{(-1)^n}\ \underset{\text{e.g. move }x_n\text{ past }x_0\cdots u(x_i)\cdots x_{n-1}}{(-1)^{n-1}}\Big)^{n-i}(x_{i+1},\dots,x_n,x_0,\dots,x_{i-1})\otimes u(x_i)$$

$$=(-1)^n\sum_{i=0}^{n}(x_{i+1},\dots,x_n,x_0,\dots,x_{i-1})\otimes u(x_i)\ \in I^{\otimes n}\otimes R$$

which gets mapped to

$$(-1)^n\sum_{i=0}^{n}(x_{i+1},\dots,x_{i-1})\otimes dx_i\ \in I^{(n)}\otimes_R\Omega^1_R\otimes_R$$

which is $(-1)^n$ times $\partial(x_0,\dots,x_n)$, where

$$\partial:(I\otimes_R)^{n+1}_{\sigma}\longrightarrow I^{(n)}\otimes_R\Omega^1_R\otimes_R$$

is defined by

$$\partial(x_0,\dots,x_n)=\sum_0^n(x_{i+1},\dots,x_n,x_0,\dots,x_{i-1})\otimes dx_i$$

The $(-1)^n$ probably results from the fact that if we wish to view a double complex a complex of columns, then we put in this sign.

The purpose of this calculation was to check that we really have an "edge" map from $CC(R\leftarrow I)$ to the complex

$$\xrightarrow{\ \beta\ } I^{(n+1)}\xrightarrow{\ \partial\ } I^{(n)}\otimes_R\Omega^1_R\otimes_R\xrightarrow{\ \beta\ } I^{(n)}\xrightarrow{\ \partial\ }$$

Let's return to the puzzle of how to understand the maps

$$\sum \left(I \overset{L}{\otimes}_R\right)^{n+1}_\sigma \longrightarrow \left(I \overset{L}{\otimes}_R\right)^n_\sigma$$

from the viewpoint of derived categories.

The first thing to do is to [redacted] go over the situation without the derived functors. We assume more generally that $I$ is an $R$-bimodule equipped with a bimodule morphism $u : I \to R$ satisfying

$$u(x)y = x\,u(y) \qquad \text{for all } x, y \in I.$$

Consider $I \otimes_R I \otimes_R$. We have two maps from this to $I \otimes_R$ namely

$$u \otimes 1 : \quad x \otimes y \longmapsto u(x)y$$
$$1 \otimes u : \quad x \otimes y \longmapsto x\,u(y)$$

To see these are well-defined, note that the first is the composite

$$I \otimes_R I \otimes_R \xrightarrow{u \otimes \mathrm{id}} R \otimes_R I \otimes_R \xrightarrow{\substack{\text{left mult}\\ \text{of } R \text{ on } I.}} I \otimes_R$$

and similarly for the second. In fact I guess we learn that the two maps of $R$-bimodules

$$\begin{array}{ccccc} & \xrightarrow{\text{[redacted]}\,1\otimes u} & I \otimes_R R & \longrightarrow & \\ I \otimes_R I & & & & I \\ & \xrightarrow{u \otimes 1} & R \otimes_R I & \longrightarrow & \end{array}$$

coincide, and then the two maps $(I \otimes_R)^2 \rightrightarrows I \otimes_R$ coincide by applying $\otimes_R$.

This generalizes to ~~show~~ that the $n+1$ maps

$$\underbrace{I \otimes_R \cdots \otimes_R I}_{n+1} \longrightarrow \underbrace{I \otimes_R \cdots \otimes_R I}_{n}$$

obtained by applying $u$ to the different factors of $I^{(n+1)}$, coincide.

Next we observe that in the case of the cyclic tensor product the unique map obtained using $u$ actually factors

$$\begin{tikzcd}
(I\otimes_R)^{n+1} \arrow[rr] \arrow[dr] & & (I\otimes_R)^{n} \\
& (I\otimes_R)^{n+1}_{\sigma} \arrow[ur] &
\end{tikzcd}$$

through the cyclic quotient. In effect

$$\begin{aligned}
(u(x_0)x_1, \dots, x_n) &= (x_0 u(x_1), x_2, \dots, x_n) \\
&= (x_0, u(x_1)x_2, \dots, x_n) = (x_0, x_1 u(x_2), \dots) \\
&= \dots = (x_0, \dots, x_{n-1}u(x_n)) = (u(x_n)x_0, x_1, \dots, x_{n-1})
\end{aligned}$$

Now let's use derived tensor products. We have two maps of complexes

$$\begin{tikzcd}
I \overset{L}{\otimes}_R I \arrow[r, "1\otimes u"] \arrow[dr, "u\otimes 1"'] & I \overset{L}{\otimes}_R R \arrow[r, phantom, "\cong"] & I \\
& R \overset{L}{\otimes}_R I \arrow[ur, phantom, "\cong"] &
\end{tikzcd}$$

which are equal because $I$ is concentrated in degree zero and $H_0(I \overset{L}{\otimes}_R I) = I \otimes_R I$. Next we have possibly four maps

$$\begin{tikzcd}
I \overset{L}{\otimes}_R I \overset{L}{\otimes}_R I \arrow[rr] \arrow[d] \arrow[dr] & & I \overset{L}{\otimes}_R I \overset{L}{\otimes}_R R \arrow[dd] \\
R \overset{L}{\otimes}_R I \overset{L}{\otimes}_R I \arrow[drr] & I \overset{L}{\otimes}_R R \overset{L}{\otimes}_R I \arrow[dr, shift left] \arrow[dr, shift right] & \\
& & I \overset{L}{\otimes}_R I
\end{tikzcd}$$

although there are really only two different ones. What matters is what $\overset{L}{\otimes}_R$ is removed. Thus

$$I \overset{L}{\otimes}_R R \overset{L}{\otimes}_R I = I \otimes_R B \otimes_R B \otimes_R I$$

and there are two bimodule maps

$$B \otimes_R B \xrightarrow{\varepsilon\otimes 1,\ 1\otimes\varepsilon} B$$

which are homotopic
Similarly we have $(n)$ different maps

$$\underbrace{I \overset{L}{\otimes}_R \cdots \overset{L}{\otimes}_R I}_{(n+1)} \longrightarrow \underbrace{I \overset{L}{\otimes}_R \cdots \overset{L}{\otimes}_R I}_{n}$$

and homotopies joining each map to its "neighbor" (One can think of the map as obtained by using the unique map $I \overset{L}{\otimes}_R I \longrightarrow I$ at each of the $n$ $\overset{L}{\otimes}_R$-signs.)

Thus there are two maps

$$I \overset{L}{\otimes}_R I \overset{L}{\otimes}_R \longrightarrow I \overset{L}{\otimes}_R$$

interchanged by the symmetric group of order 2, and there is a homotopy joining them. If we compose with the norm inclusion

$$\left(I \overset{L}{\otimes}_R\right)^2_\sigma \hookrightarrow \left(I \overset{L}{\otimes}_R\right)^2$$

then we have a map from $\left(I \overset{L}{\otimes}_R\right)^2_\sigma$ to $I \overset{L}{\otimes}_R$ and a self-homotopy of it. This self-homotopy can be viewed as a map

$$\Sigma\left(I \overset{L}{\otimes}_R\right)^2_\sigma \longrightarrow I \overset{L}{\otimes}_R$$

Similarly we should have $(n+1)$-maps

$$\left(I \overset{L}{\otimes}_R\right)^{n+1} \longrightarrow \left(I \overset{L}{\otimes}_R\right)^n_\sigma$$

permuted by $\mathbb{Z}(n+1)$ acting on the left. Also there [redacted] should be a homotopy joining one map to its cyclic transform and so on restricting to invariants gives a self-homotopy.

September 12, 1988

One problem worth solving is to find a nice formula for the natural map from $CC(A)$ to the Connes $(b,B)$-complex. (Think of the latter as having columns which are copies of the reduced Hochschild complex $(A^+\otimes A^{\otimes n}, b)$ of $A^+$; and this complex as being the sum of the $b$ and $b'$ complexes of $A$. The natural map comes from the contracting homotopies of the $1-t$, $N$ rows.) One way to proceed is to use the resolution of $CC(A)$ provided by the columns of $CC(A \xrightarrow{1} A)$. Take $\rho_t(a) = ta$; this gives a family of maps $CC(A) \longrightarrow CC(A \xrightarrow{1} A)$ deforming the inclusion of $CC(A)$ as the 0th column to the zero map. It yields a chain homotopy joining the inclusion of $CC(A)$ to the zero map, and hence a map

$$\Sigma\, CC(A) \longrightarrow CC(A \xrightarrow{1} A)/CC(A)$$

from $CC(A)$ to the positive degree columns of $CC(A \xrightarrow{1} A)$. We know each positive degree column of $CC(A \to A)$ has the homotopy type of $A \overset{L}{\otimes}_A$; the problem is therefore reduced to mapping $\Sigma_{hor}^{-1} CC(A \to A)/CC(A)$ to the Connes $(b,B)$-bicomplex.

I perceive a slight difficulty, namely, the identification of the columns of $CC(A \xrightarrow{1} A)$ with $A \overset{L}{\otimes}_A$ I have done only when $A$ is unital, and in the present situation $A$ can be non-unital. However $CC(A \xrightarrow{1} A) = \overline{CC}(A \longrightarrow A^+)$ and we know the $p$-th column of the latter is (put $R = A^+$, $I = A$)

$$\left(I \otimes_R B_N(R) \otimes_R\right)_\sigma^p = \left(A \overset{L}{\otimes}_{A^+}\right)_\sigma^p$$

up to a suspension, where $B_N(R)_n = R \otimes \bar{R}^{\otimes n} \otimes R$.

In any case, let's put this difficulty aside and consider the simpler problem of mapping $CC(A)$ to the $(R,I)$-complex where $R = A^+$ and $I = A$.

Let's review what happens for a linear map $\rho : A \to R$ which is a homomorphism modulo $I$. We then have a map from $CC(A)$ to the $(R,I)$ complex.

$$\begin{tikzcd}
{} \arrow[r] & A_\lambda^{\otimes(2n+2)} \arrow[r, "b"] \arrow[d, "\left(\frac{\gamma^{n+1}}{(n+1)!}\right)N"] & A_\lambda^{\otimes(2n+1)} \arrow[r, "-b"] \arrow[d, "\left(\frac{\gamma^n}{n!}\partial\rho\right)N"] & A_\lambda^{\otimes 2n} \arrow[r] \arrow[d, "\left(\frac{\gamma^n}{n!}\right)N"] & {} \\
{} \arrow[r] & I^{(n+1)} \arrow[r, "\partial"] & I^{(n)} \otimes_R \Omega^1_R \otimes_R \arrow[r, "\beta"] & I^{(n)} \arrow[r, dash] & {}
\end{tikzcd}$$

Recall the calculations; the sign in $-b$ comes from the fact that $\delta$ on $C^p(A, ?) = \mathrm{Hom}(A^{\otimes p}, ?)$ is

$$\delta(f) = -(-1)^p f b' = \begin{cases} f b' & p \text{ odd} \\ -f b' & p \text{ even} \end{cases}$$

We have

$$(\gamma^n) N(-b) = \gamma^n(-b')N = \delta(\gamma^n)N$$

and denoting $\equiv$ to mean equality after composing with $N : CC(A) \hookrightarrow$ $Bar(A)$, we have the formulas

$$\delta(\gamma^n) = \gamma^n \rho - \rho\gamma^n$$

$$\beta(\gamma^n \partial\rho) = \gamma^n\rho - \rho\gamma^n\sigma$$

$$\equiv \gamma^n\rho - \rho\gamma^n$$

$\sigma$ = cyclic perm without sign as $\sigma = t$ for odd cochains

$$\delta(\gamma^n\partial\rho) = (\gamma^n\rho - \rho\gamma^n)\partial\rho + \gamma^n\partial\delta\rho$$

$$\equiv \gamma^n(\rho\partial\rho + \partial\rho\,\rho + \partial\delta\rho) = \gamma^n\partial\gamma$$

$$\partial(\gamma^{n+1}) = \sum_{i=0}^{n} \gamma^i\partial\gamma\,\gamma^{n-i} \equiv (n+1)\gamma^n\partial\gamma.$$

Next suppose we have a family $\rho = \rho_t : A \to R$ of homomorphisms mod $I$ so that $\mathring{\rho} : A \to I$. Then we have a contracting homotopy for the derivative map

$$
\begin{tikzcd}
A_\lambda^{\otimes(2n+2)} \arrow[r, "b"] & A_\lambda^{\otimes(2n+1)} \arrow[r, "-b'"] \arrow[dl, dashed, "\frac{\mu_{n+1}}{(n+1)!}N"'] \arrow[d, "\left(\frac{\gamma^n}{n!}\partial\rho\right)^\circ N"] & A_\lambda^{\otimes(2n)} \arrow[r, "b"] \arrow[dl, dashed, "\frac{1}{n!}\mu_n\partial\rho N"'] \arrow[d, "\left(\frac{\gamma^n}{n!}\right)^\circ N"] & A_\lambda^{\otimes(2n-1)} \arrow[dl, dashed, "\frac{\mu_n}{n!}N"'] & \\
I^{(n+1)} \arrow[r, "\partial"] & I^{(n)} \otimes_R \Omega^1_R \otimes_R \arrow[r, "\beta"] & I^{(n)} \arrow[r] & {}
\end{tikzcd}
$$

where $\mu = \sum_{i=0}^{n-1} \gamma^i \mathring{\rho}\, \gamma^{n-1-i}$. In effect

$$
\begin{aligned}
(\gamma^n)^\circ &= \sum_0^{n-1} \gamma^i (\delta\mathring{\rho} + \mathring{\rho}\rho + \rho\mathring{\rho})\gamma^{n-1-i} \\
&= [\delta + \rho, \mu_n] = \delta(\mu_n) + \rho\mu_n + \mu_n\rho \\
&\equiv \delta(\mu_n) + \beta(\mu_n \partial\rho)
\end{aligned}
$$

$$
\begin{aligned}
(\gamma^n \partial\rho)^\circ &= \big(\delta(\mu_n) + \rho\mu_n + \mu_n\rho\big)\partial\rho + \gamma^n \partial\mathring{\rho} \\
&= \delta(\mu_n\partial\rho) + \mu_n\,\delta(\partial\rho) + (\rho\mu_n + \mu_n\rho)\partial\rho + \gamma^n\partial\mathring{\rho} \\
&\equiv \delta(\mu_n\partial\rho) + \mu_n\,\partial\gamma + \gamma^n\partial\mathring{\rho} \\
&= \delta(\mu_n\partial\rho) + \sum_{i=0}^{n-1} \gamma^i\mathring{\rho}\,\gamma^{n-1-i}\partial\gamma + \gamma^n\partial\mathring{\rho} \\
&\equiv \delta(\mu_n\partial\rho) + \sum_{i=0}^{n-1} \gamma^{n-1-i}\partial\gamma\,\gamma^i\mathring{\rho} + \gamma^n\partial\mathring{\rho} \\
&= \delta(\mu_n\partial\rho) + \partial(\gamma^n\mathring{\rho}) \\
&\equiv \delta(\mu_n\partial\rho) + \partial\Big(\frac{1}{n+1}\underbrace{\sum_{i=0}^{n}\gamma^i\mathring{\rho}\,\gamma^{n-i}}_{\mu_{n+1}}\Big)
\end{aligned}
$$

In the last step we use the fact that $\partial : I^{(n+1)} \to I^{(n)} \otimes_R \Omega^1_R \otimes_R$

factors through $(I \otimes_R)^{n+1}_\sigma$.

Now let us take $R = A^+$, $I = A$ and $\rho(a) = ta$. Thus $\rho = t\theta$ and

$$\gamma = \delta\rho + \rho^2 = (t^2 - t)\theta^2$$

better $\gamma(a_1, a_2) = (t - t^2)(a_1 a_2)$. Then

$$\cancel{\int_0^1 dt\, \frac{\gamma^n \rho}{n!} = \frac{(-1)^n}{n!}\theta^{2n+1} \underbrace{\int_0^1 (t-t^2)^n dt}_{\beta(n+1,n+1) = \frac{n!n!}{(2n+1)!}} = (-1)^n \frac{n!}{(2n+1)!}}$$

$$\int_0^1 dt\, \frac{\mu_{n+1}}{(n+1)!} = \int_0^1 dt\, \frac{1}{(n+1)!}\sum_{i=0}^n \gamma^i \dot\rho \gamma^{n-i} \qquad \dot\rho = \theta$$

$$= \int_0^1 dt\, \frac{n+1}{(n+1)!}(-1)^n \theta^{2n+1}(t-t^2)^n = \frac{(-1)^n}{n!}\underbrace{\int_0^1 (t-t^2)^n dt}_{\beta(n+1,n+1) = \frac{n!n!}{(2n+1)!}}\, \theta^{2n+1}$$

$$= (-1)^n \frac{n!}{(2n+1)!}\theta^{2n+1}$$

$$\int_0^1 dt\, \frac{\mu_n \partial\rho}{n!} = \int_0^1 dt\, \frac{1}{n!}\overbrace{\sum_{i=0}^{n-1}\gamma^i \dot\rho \gamma^{n-1-i}}^{n(-1)^{n-1}\theta^{2n-1}(t-t^2)^{n-1}}\, \overset{\partial(t\theta)}{\partial\rho}$$

$$= \frac{(-1)^{n-1}}{(n-1)!}\theta^{2n-1}\partial\theta \underbrace{\int_0^1 t^n(1-t)^{n-1}dt}_{\beta(n+1,n) = \frac{n!(n-1)!}{(2n)!}}$$

$$= (-1)^{n-1}\frac{n!}{(2n)!}\theta^{2n-1}\partial\theta$$

Thus we have a map of complexes

$$\begin{tikzcd}
\cdots \arrow[r] & A_\lambda^{\otimes(2n+1)} \arrow[r, "-b"] \arrow[d, "(-1)^n \frac{n!}{(2n+1)!}\theta^{2n+1}\cdot N"] & A_\lambda^{\otimes 2n} \arrow[r, "b"] \arrow[d, "(-1)^{n-1}\frac{n!}{2n!}\theta^{2n-1}\partial\theta\cdot N"] & A_\lambda^{\otimes(2n-1)} \arrow[r] \arrow[d, "(-1)^{n-1}\frac{(n-1)!}{(2n-1)!}\theta^{2n-1}\cdot N"] & \cdots \\
\cdots \arrow[r] & A \arrow[r, "\partial"] & \Omega^1_{A^+}\otimes_A \arrow[r, "\beta"] & A \arrow[r] & \cdots
\end{tikzcd}$$

Check:

$$\delta(\theta^{2n-1}) = \theta^{2n-2}(-\theta^2) = -\theta^{2n}$$

$$\beta(\theta^{2n-1}\partial\theta) = \theta^{2n-1}\theta - \theta\theta^{2n-1}\sigma$$
$$= \theta^{2n}(1+t) \equiv 2\theta^{2n}$$

$$\delta(\theta^{2n-1}\partial\theta) = -\theta^{2n}\partial\theta + \theta^{2n-1}\partial(+\theta^2)$$
$$= -\theta^{2n}\partial\theta + \theta^{2n-1}(\partial\theta\,\theta + \partial\theta\,\theta)$$
$$\equiv \theta^{2n}\partial\theta$$

$$(-1)^{(i+1)(2n-i)} = 1.$$

$$\partial(\theta^{2n+1}) = \sum_{i=0}^{2n} \theta^i\,\partial\theta\,\theta^{2n-i} \equiv (2n+1)\theta^{2n}\partial\theta$$

Signs slightly off. That's because we have obtained a map $h$ with $dh + hd = 0$.

---

Return to our program involving $CC(A \Leftarrow A)$. The $p$-th column of this is up to qis + susp. $(A \overset{L}{\otimes}_{A^+})^p_\sigma$. For $p=1$ we have in fact the complex $n \longmapsto A \otimes_{A^+} (A^+ \otimes A^{\otimes n} \otimes A^+) \otimes_{A^+} = A^{\otimes(n+1)}$ with differential $b$. The assertion about $b$ can be seen from the map

$$CC(A \xleftarrow{1} A) \simeq \overline{CC}(A^+ \Leftrightarrow A) \hookrightarrow \overline{CC}(A^+ \xleftarrow{1} A^+)$$

and the fact that the $n$-th column of the latter is $\Sigma^{n-1}(A^+ \otimes_{A^+} B_N(A^+) \otimes_{A^+})^n_\sigma$. So I learn that I don't want to use the complex $CC(A \leftarrow A)$, because e.g. its first column is $A \otimes^L_{A^+}$ = b-complex instead of $A^+ \otimes^L_{A^+}$ = normalized Hochschild complex of $A^+$. Note we have an exact sequence

$$0 \to \underset{\substack{\| \\ b \text{ complex}}}{A \otimes^L_{A^+}} \longrightarrow \underset{\substack{\text{first two columns} \\ \text{of the } b, b', 1-t, N \\ \text{bicomplex}}}{A^+ \otimes^L_{A^+}} \longrightarrow \underset{\substack{\| \\ b' \text{ complex} + k \\ \text{at bottom}}}{k \otimes^L_{A^+}} \longrightarrow 0$$

So the bicomplex with the good columns is $\overline{CC}(A^+ \xleftarrow{1} A^+)/\overline{CC}(A^+)$. There is a map to this from the bicomplex $CC(A^+ \xleftarrow{1} A^+)/CC(A^+)$, which is columnwise a quis. Thus

$$\Sigma\, CC(A^+) \overset{heq}{\sim} \overline{CC}(A^+ \leftarrow A^+)/\overline{CC}(A^+)$$

## September 13, 1988

Problem: Suppose $A$ unital, find a formula for the $S$-operator

$$\bar{H}C_{2n}(A) \xrightarrow{S^n} \bar{H}C_0(A) = \bar{A}/[A,A]$$

Let's go over the non reduced case first. Recall that we have the map of complexes

$$\begin{tikzcd}
\arrow[r] & A_\lambda^{\otimes(2n+2)} \arrow[r, "b"] \arrow[d, "\frac{\gamma^{n+1}}{(n+1)!}"] & A_\lambda^{\otimes(2n+1)} \arrow[r, "-b"] \arrow[d, "\frac{\gamma^n}{n!}d\rho\, N"] & A_\lambda^{\otimes 2n} \arrow[r] \arrow[d, "\frac{\gamma^n}{n!}N"] & {} \\
\arrow[r] & I^{(n+1)} \arrow[r, "\partial"] & I^{(n)}\otimes_R \Omega^1_R \otimes_R \arrow[r, "\beta"] & I^{(n)} \arrow[r] & {}
\end{tikzcd}$$

associated to a linear map $\rho : A \to R$ which is a homomorphism modulo $I$. If $\rho(1)=1$, then $\gamma(a_1,a_2) = \rho(a_1a_2) - \rho(a_1)\rho(a_2)$ is reduced, so that the above map is actually defined on $\bar{C}C(A)$. Next if $\rho$ varies then the derivative of this map of complexes is null-homotopic, the homotopy operator being

$$\begin{tikzcd}
{} \arrow[d, "\left(\frac{\gamma^n}{n!}d\rho\right)^{\circ} N"] \arrow[dl, dashed, "\frac{\mu_{n+1}}{(n+1)!}N"'] & {} \arrow[dl, dashed, "\frac{\mu_n \partial\rho}{n!}N"'] \arrow[d, "\left(\frac{\gamma^n}{n!}\right)^{\circ} N"] & {} \arrow[dl, dashed, "\frac{\mu_n}{n!}N"'] \\
{} \arrow[r] & I^{(n)}\otimes_R \Omega^1_R \otimes \arrow[r, "\beta"] & I^{(n)}
\end{tikzcd}$$

where
$$\mu_n = \sum_{i=0}^{n-1} \gamma^i \mathring{\rho}\, \gamma^{n-1-i}.$$

We saw yesterday that if we take the non-unital family $\rho_t(a) = ta$, where $R=I=A$, then $\gamma = \delta(t\theta) + (t\theta)^2 = (t^2-t)\theta^2$, $\mathring{\rho} = \theta$, so

$$\int_0^1 \frac{\mu_n}{n!}\,dt = \frac{1}{n!}\, n\, (-1)^{n-1}\theta^{2n-1}\int_0^1 (t-t^2)^{n-1}dt = \frac{(-1)^{n-1}(n-1)!}{(2n-1)!}\,\theta^{2n-1}$$

gives a cyclic $(2n-2)$-cocycle with values in $A/[A,A]$. This is the familiar formula for $\delta^{n-1} : HC_{2n-2}(A) \longrightarrow A/[A,A]$

Now let's turn to the reduced theory. We choose a splitting of the exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow A \longrightarrow \bar{A} \longrightarrow 0 \qquad (\varepsilon : A \to \mathbb{C})$$

and consider, again with $R = I = A$, the family $\rho_t(a) = \varepsilon a + t(a - \varepsilon a)$ of maps $R \to R$ satisfying $\rho(1) = 1$. Note that

$$\rho_1(a) = a$$

is a homomorphism (so that it's curvature $\gamma_1 = 0$) while

$$\rho_0(a) = \varepsilon a$$

has values in $\mathbb{C}$. We have the cyclic $(2n-2)$-cochain with values in $A/[A,A]$

$$\circledast \qquad \left(\int_0^1 \frac{\mu_n}{n!} dt\right) N = \left(\int_0^1 \frac{1}{(n-1)!} \gamma_t^{n-1} \dot{\rho}_t \, dt\right) N$$

which is reduced ($\dot{\rho}(a) = a - \varepsilon a$ vanishes at $a = 1$), which satisfies

$$\delta \left(\int_0^1 \frac{\mu_n}{n!} dt\right) N = -\frac{1}{n!} \gamma_0^n N$$

since $\gamma_1 = 0$. But the last cochain has values in $\mathbb{C}$ so $\circledast$ is a reduced cyclic $(2n-2)$-cocycle when regarded as having values in $\bar{A}/[A,A]$.

Remark: Because $\rho_t$ is a linear path joining a flat connection to another connection, the cochain ⊛ is a Chern-Simon's form in some sense.

Recall

$$\big(\underbrace{(1-t)D_0 + tD_1}_{D_t}\big)^2 = (1-t)D_0^2 + tD_1^2 + (t^2-t)(D_1-D_0)^2$$

and in the present case $D_0 = \delta + \theta$, $D_1 = \delta + \varepsilon\theta$. Thus

$$(D_t)^2 = \gamma_t = t\underbrace{\gamma_1^{\,2}}_{(\varepsilon\theta)^2-\varepsilon(\theta^2)} + (t^2-t)(\varepsilon\theta-\theta)^2$$

$$\dot{\delta}_t = \dot{D}_t = \varepsilon\theta - \theta, \text{ so}$$

$$\int_0^1 \frac{\mu_n}{n!}\, dt\, N = \int_0^1 dt \frac{1}{(n-1)!}\left(t\gamma_1^{\,2} + (t^2-t)(\varepsilon\theta-\theta)^2\right)^{n-1}(\varepsilon\theta-\theta)\, N$$

(except we have changed $t$ and $1-t$.)

<u>September 15, 1988</u>

Program: I eventually want to prove the S-link between different cyclic cocycles with values in the $(R, I)$-complex. For this reason I have to understand the cyclic theory of

$$C^{\cdot}(A, R) = \mathrm{Hom}_k(\mathrm{Bar}(A), R)$$

Roughly this is the tensor product of two free algebras $T(A^*)$ and $R$. (I can suppose $R$ free in handling extensions.) Thus I am led to investigate the cyclic theory of a tensor product of free algebras. But Jones and the Russians have a Kunneth formula for the cyclic homology of a tensor product in general, so I now try to work this out.

The appropriate framework is Kassel's mixed complexes. These are complexes $(K_n, b: K_n \to K_{n-1})$ equipped with a map of degree $+1$

$$\begin{tikzcd}
\arrow[r] & K_{n+1} \arrow[r, "b"] & K_n \arrow[r, "b"] \arrow[dl, "B"'] & K_{n-1} \arrow[r] \arrow[dl, "B"'] & {} \\
\arrow[r] & K_{n+1} \arrow[r] & K_n \arrow[r] & {} &
\end{tikzcd}$$

such that $bB + Bb = 0$ and $B^2 = 0$. The first condition says $B$ is a morphism of complexes from $\Sigma K$ to $K$.

Alternatively a mixed complex is a DG module over the DG algebra $k[\varepsilon]$ where $\deg(\varepsilon) = 1$ and $d(\varepsilon) = 0$. Since $k[\varepsilon]$ is a DG Hopf algebra with $\Delta\varepsilon = \varepsilon \otimes 1 + 1 \otimes \varepsilon$ which is comm. + cocomm., we have a tensor product operation on mixed complex, namely we take the tensor product of the complexes: $K \otimes K'$ with $B = B \hat{\otimes} 1 + 1 \hat{\otimes} B$ (usual signs).

Note that there is a symmetry between $b, B$ for a mixed complex which has been ignored. This is because in cyclic theory one is interested in the associated Connes bicomplex:

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] \\
\arrow[r, "B"] & K_n \arrow[r, "B"] \arrow[d, "b"] & K_{n+1} \arrow[d, "b"] \\
\arrow[r, "B"] & K_{n-1} \arrow[r, "B"] \arrow[d] & K_n \arrow[d] \\
 & \phantom{.} & \phantom{.}
\end{tikzcd}$$

which is truncated in the $B$ direction, indicating we are concerned with a filtration in the $B$-direction.

Here is how to interpret this bicomplex. Set $\Lambda = k[\varepsilon]$ and form the normalized "bar" resolution

$$\mathcal{B}_N(A)_n = \Lambda \otimes \bar{\Lambda}^{\otimes n} \otimes \Lambda$$

of $\Lambda$ over $\Lambda \otimes \Lambda$. Here $\bar{\Lambda} = k\varepsilon$, but the differential treats $\varepsilon$ of total degree 2. So put $u =$ this $\varepsilon$ and consider the complex

$$k \otimes_\Lambda \mathcal{B}_N(\Lambda) \otimes_\Lambda K \quad \text{in degree } n \quad = \quad \bar{\Lambda}^{\otimes n} \otimes K = u^{\otimes n} \otimes K$$

The differential is

$$d(u^{\otimes n} \otimes x) = d\big[(\overbrace{u, \ldots, u}^{n}, 1) \otimes_\Lambda x\big]$$

$$= d(\overbrace{u, \ldots, u}^{n}, 1) \otimes_\Lambda x + (u, \ldots, u, 1) \otimes_\Lambda dx$$

$$= (\overbrace{u, \ldots, u}^{n-1}, \varepsilon) \otimes_\Lambda x + u^{\otimes n} \otimes dx$$

$$= u^{\otimes n-1} \otimes Bx + u^{\otimes n} \otimes bx$$

where we have used $\varepsilon^2 = 0$ and the fact that $u$ is even. Thus we see that the Connes bicomplex is just the complex

$$k \overset{L}{\otimes}_\Lambda K$$

up to quis.

Let's try to understand this construction a bit more generally. Let $C$ be a DG alg., let $A$ be a DGA and let $\tau$ be a twisting cochain from $C$ to $A$ such that $C \otimes_\tau A$ is acyclic. Given a DG-module $M$ over $A$ we can form

$$(C \otimes_\tau A) \otimes_A M$$

and hopefully this turns out to be a DG comodule over $C$. In the same way that $C \otimes_\tau A$ is an $A$-module, it should also be a $C$-comodule.

There are all kinds of questions arising in this game. For example if $C = \mathrm{Bar}(A)$, then in addition to [redacted] being able to form a right $A$-module complex

$$C \overset{\tau}{\otimes} A$$

we can also form a left $A$-module complex

$$A \overset{\tau}{\otimes} C$$

and we can [redacted] form the A-bimodule

$$A \overset{\tau}{\otimes} C \overset{\tau}{\otimes} A$$

which we know is quis A.

The formalism suggests that we have functors

$$\mathrm{Mod}(A) \longrightarrow \mathrm{Comod}(C)$$

$$M \longmapsto (C \overset{\tau}{\otimes} A) \otimes_A M = C \overset{\tau}{\otimes} M$$

$$\mathrm{Comod}(C) \longrightarrow \mathrm{Mod}(A)$$

$$N \longmapsto (A \overset{\tau}{\otimes} C) \overset{C}{\otimes} N = A \overset{\tau}{\otimes} N.$$

which set up equivalences on the homotopy categories, at least under suitable convergence hypotheses.

Let's see what this means when $A = \Lambda = k[\varepsilon]$ $\deg(\varepsilon) = 1$, [redacted] and when $C =$ [redacted] $\mathrm{Bar}(\Lambda) = k[u]$ with degree $u = 2$. What does it means for $N$ to be a $C$-comodule. It means $N$ is a complex equipped with an endomorphism of degree $-2$, i.e.

$$S : N \longrightarrow \Sigma^2 N$$

Supposing $N$ is cofree means that ignoring the differential $N = k[u] \otimes V$, where $S$ operates by $S(u^n) = u^{n-1}$ or $0$ if $n = 0$. Thus we can recover $V$ as $\mathrm{Ker}(S)$. And we have an increasing filtration of $N$ given by $\mathrm{Ker}\, S^n$ such that

$$gr(N) = k[u] \otimes V$$

as complexes. It doesn't seem likely that we can find a $B$ on $V$ such that $N$ is isom.

to the Connes bicomplex. But the important point to keep track of is the filtration with

$$gr(N) = k[u] \otimes \mathrm{Ker}(S)$$

as complexes, as this suffices to yield the long exact sequence from

$$0 \to \mathrm{Ker}\, S \longrightarrow N \xrightarrow{S} \Sigma^2 N \longrightarrow 0.$$

---

Next let turn to the cyclic homology of $A \otimes B$. The first point is that the Hochschild homology of $A \otimes B$ is the tensor product of the Hoch. homologies of $A$ and $B$. <u>Pf.</u> ① Let $P$ be a free $A \otimes A^0$ module resolution of $A$ and $Q$ a free $B \otimes B^0$-module resolution of $B$. Then the complex $P_\cdot \otimes Q_\cdot$ resolves $A \otimes B$ and it's a complex of free $A \otimes B$-bimodules. So

$$
\begin{aligned}
(P \otimes Q) \otimes_{(A \otimes B)} &= P \otimes Q \big/ [A \otimes B, P \otimes Q] \\
&= P \otimes Q \big/ [A, P] \otimes Q + P \otimes [B, Q] \\
&= P/[A, P] \otimes Q/[B, Q]
\end{aligned}
$$

gives the Hochschild homology of $A \otimes B$, and the rest is clear. ② Use the simplicial structure on the Hochschild complex $n \mapsto A^{\otimes(n+1)}$. The Hochschild complex of $A \otimes B$ will be the tensor product of the simplicial vector spaces $n \mapsto A^{\otimes(n+1)}$ and $n \mapsto B^{\otimes(n+1)}$, so the ordinary Kunneth formula yields the result.

In order to handle cyclic homology we need to know the Hochschild complex for $A \otimes B$ as a mixed complex is equivalent to the tensor product of Hoch cxs. This is not so clear.

Let's discuss <u>examples.</u> Recall the mixed complex given by the complex of length 1

$$\Omega^1_R \otimes_R \xrightarrow{b} R$$

with $B$ equal to $d : R \to \Omega^1_R \otimes_R$. Then the correspondinge $k[u]$ comodule is the complex

$$\xrightarrow{B} \Omega^1_R \otimes_R \xrightarrow{b} R \xrightarrow{B} \Omega^1_R \otimes_R \xrightarrow{b} R$$

and this gives the cyclic homology when $\Omega^1_R$ is projective.

Let's now consider two free algebras $A, B$. Then we can take the tensor product of the mixed complexes

$$\left(\Omega^1_A \otimes_A \rightleftarrows A\right) \otimes \left(\Omega^1_B \otimes_B \rightleftarrows B\right).$$

This will give us a mixed complex. Let's assume that it gives the cyclic homology of $A \otimes B$, after taking the Connes bicomplex. Can we then computing the cyclic homology of $A \otimes B$?

First note that as mixed complexes

$$\left(\Omega^1_A \otimes_A \rightleftarrows A\right) = \left(\Omega^1_A \otimes_A \rightleftarrows \bar{A}\right) \otimes \left(0 \rightleftarrows k\right)$$

whence we will have a direct sum decomp.

$$HC(A \otimes B) = HC(k) \oplus \overline{HC}(A) \oplus \overline{HC}(B) \oplus \; ?$$

where ? is the homology of the bicomplex associated to the mixed complex

$$\left(\Omega^1_A \otimes_A \rightleftarrows \bar{A}\right) \otimes \left(\Omega^1_B \otimes_B \rightleftarrows \bar{B}\right)$$

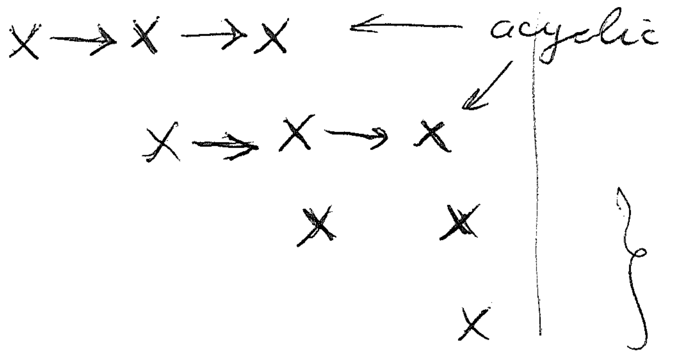
Let's now decompose $\Omega^1_{A} \otimes_{A} \overset{1}{\longleftarrow\!\!\longrightarrow} \bar{A}$ into irreducible pieces. The irreducibles are

$$k \underset{1}{\overset{0}{\rightleftarrows}} k \qquad \text{and} \qquad k \underset{0}{\overset{1}{\rightleftarrows}} k$$

The former complex has trivial $b$-homology and this will be true if it is tensored with any other complex, so it will lead to trivial homology for the bicomplex. Thus we have to compute the bicomplex homology for the latter type, i.e. for

$$\left(\bar{A}/[\,,\,] \underset{0}{\overset{1}{\rightleftarrows}} \bar{A}/[\,,\,]\right) \otimes \left(\bar{B}/[\,,\,] \underset{0}{\overset{1}{\rightleftarrows}} \bar{B}/[\,,\,]\right)$$

The $b$ differentials are zero, so in the bicomplex only the $B$ differentials appear, and these $B$ complexes are acyclic except where truncated.




these contribute $\bar{A}/[\,,\,] \otimes \bar{B}/[\,,\,]$

so the answer should be

$$\overline{HC}_n(A \otimes B) = \begin{cases} \bar{A}/[\,,\,] \oplus \bar{B}/[\,,\,] \oplus \bar{A}/[\,,\,] \otimes \bar{B}/[\,,\,] & n=0 \\ \bar{A}/[\,,\,] \otimes \bar{B}/[\,,\,] & n=1 \\ 0 & n>1 \end{cases}$$

Let's consider the bicomplex associated to the tensor product mixed complex

$$\left(\Omega^1_A \otimes_A \rightleftarrows A\right) \otimes \left(\Omega^1_B \otimes_B \rightleftarrows B\right).$$

It appears

$$\begin{tikzcd}
{} \arrow[r] & (\Omega^1_A\otimes_A)\otimes(\Omega^1_B\otimes_B) \arrow[d] & \\
{} \arrow[r] & (\Omega^1_A\otimes_A)\otimes B\oplus A\otimes(\Omega^1_B\otimes_B) \arrow[r] \arrow[d] & (\Omega^1_A\otimes_A)\otimes(\Omega^1_B\otimes_B) \arrow[d] \\
& A\otimes B \arrow[r] & (\Omega^1_A\otimes_A)\otimes B\oplus A\otimes(\Omega^1_B\otimes_B) \arrow[d] \\
& & A\otimes B
\end{tikzcd}$$

and so the total complex is

$$\longrightarrow A\otimes B \oplus (\Omega^1_A\otimes_A)\otimes(\Omega^1_B\otimes_B) \longrightarrow (\Omega^1_A\otimes_A)\otimes B \oplus A\otimes(\Omega^1_B\otimes_B) \longrightarrow A\otimes B$$

which repeats periodically. What one has apparently done is to take the tensor product of the periodic complexes for $A$, $B$ and then made some identifications to obtain a single periodic complex.

This suggests in general that if $M, N$ are mixed complexes with associated bicomplexes

$$C\overset{\tau}{\otimes} M \qquad C\overset{\tau}{\otimes} N$$

where $C = \mathrm{Bar}(\Lambda)$, then the bimod. complex associated to $M\otimes N$ is given by

$$C\overset{\tau}{\otimes}(M\otimes N) = (C\overset{\tau}{\otimes} M)\overset{C}{\otimes}(C\overset{\tau}{\otimes} N).$$

(This seems to be meaningful because $C$ is commutative — corresponds to $\Lambda$ being a DG Hopf algebra.)

Let us now return to the algebra

$$C^\bullet(A, R) = \mathrm{Hom}(\mathrm{Bar}(A), R)$$

which we want to view as analogous to $T(A^*) \otimes R$, and then we use the preceding cyclic theory. So we are led to the periodic complex

$$\begin{array}{c} \bar{C}^\bullet(A,\bar{R}) \\ \oplus \\ C^\bullet_H(A,\Omega^1_R\otimes_R) \end{array} \xleftarrow{\begin{pmatrix} 1-t & b \\ -B & N \end{pmatrix}} \begin{array}{c} C^\bullet_H(A,\bar{R}) \\ \oplus \\ \bar{C}^\bullet(A,\Omega^1_R\otimes_R) \end{array} \xleftarrow{\begin{pmatrix} N & -b \\ B & 1-t \end{pmatrix}} \begin{array}{c} \bar{C}^\bullet(A,\bar{R}) \\ \oplus \\ C^\bullet_H(A,\Omega^1_R\otimes_R) \end{array} \longleftarrow$$

which is truncated by killing $C_H(A,\Omega^1_R\otimes_R)$.

The hope is that the above calculation will help prove that the various cyclic cocycles with values in the R-I complex are related by S. The cocycles result from the map of complexes

$$\begin{tikzcd} \arrow[r] & A_\lambda^{\otimes(2n+2)} \arrow[r] \arrow[d, "\frac{\gamma^{n+1}}{(n+1)!}N"] & A_\lambda^{\otimes(2n+1)} \arrow[r] \arrow[d, "\frac{\gamma^n}{n!}d\rho\, N"] & A_\lambda^{\otimes 2n} \arrow[r] \arrow[d, "\frac{\gamma^n}{n!}N"] & {} \\ \arrow[r] & I^{(n+1)} \arrow[r] & I^{(n)}\otimes_R\Omega^1_R\otimes_R \arrow[r] & I^{(n)} \arrow[r] & {} \end{tikzcd}$$

together with the fact that the bottom complex has an obvious S-map lowering degree by 2.

September 16, 1988

Let $(R^\cdot, \delta)$ be a DGA, let $\rho \in R^1$ and set $\gamma = \delta\rho + \rho^2$. Then we can consider the "character" forms or cocycles

$$tr\left(\frac{\gamma^n}{n!}\right) \in R^{2n}/[\,,\,]$$

where $tr$ denotes the map $R \to R/[R,R]$.

Consider the double complex

$$\longrightarrow R \xrightarrow{\partial} \Omega^1_R \otimes_R \xrightarrow{\beta} R \longrightarrow$$

Then we have the following "cocycle"

$$\begin{array}{ccccc}
 & & & & \dfrac{\gamma^n}{n!} \longrightarrow \\
 & & \dfrac{1}{(n+1)!}\sum_0^n \gamma^i \partial\rho\, \gamma^{n-i} & & \downarrow \delta \\
 & & \| & & \\
 & & \dfrac{\gamma^n}{n!}\partial\rho & \stackrel{\beta}{\longmapsto} & \left[\dfrac{\gamma^n}{n!}, \rho\right] \\
 & & \downarrow \delta & & \\
\dfrac{\gamma^{n+1}}{(n+1)!} & \stackrel{\partial}{\longmapsto} & \dfrac{\gamma^n}{n!}\partial\gamma = \dfrac{1}{(n+1)!}\sum_{i=0}^n \gamma^i \partial\gamma\, \gamma^{n-i} & & \\
\downarrow \delta & & & &
\end{array}$$

Proof:

$$\delta(\gamma^n) = \sum_{i=0}^{n-1} \gamma^i \underbrace{\delta\gamma}_{[\gamma,\rho]} \gamma^{n-i} = [\gamma^n, \rho]$$

$$\partial(\gamma^{n+1}) = \sum_0^n \gamma^i \underbrace{\partial\gamma}_{[\delta+\rho,\, \partial\rho]} \gamma^{n-i} = \left[\delta+\rho, \sum_0^n \gamma^i \partial\rho\, \gamma^{n-i}\right]$$

$$= \delta\left((n+1)\gamma^n \partial\rho\right) \quad \text{in } \Omega^1_R \otimes_R$$

Now let us take $R = A \otimes B$. If we take the tensor product of the exact sequence $0 \to \Omega^1_A \to A \otimes A \to A \to 0$ and $0 \to \Omega^1_B \to B \otimes B \to B$, we get a length two resolution of $A \otimes B$ as an $A \otimes B$-bimod. This gives an exact sequence

$$0 \to \Omega^1_A \otimes \Omega^1_B \xrightarrow{(-,+)} \Omega^1_A \otimes (B \otimes B) \oplus (A \otimes A) \otimes \Omega^1_B \to \Omega^1_R \to 0$$

which then gives an exact sequence

$$(\Omega^1_A \otimes_A) \otimes (\Omega^1_B \otimes_B) \xrightarrow{(-,+)} (\Omega^1_A \otimes_A) \otimes B \oplus A \otimes (\Omega^1_B \otimes_B) \to (\Omega^1_R \otimes_R) \to 0$$

It follows that we have a map of complexes

$$\begin{tikzcd}
{} \arrow[r] & R \arrow[r] \arrow[d] & \Omega^1_R \otimes_R \arrow[r] \arrow[d, "\varphi"] & R \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & A \otimes (B/[B,B]) \arrow[r] & (\Omega^1_A \otimes_A) \otimes B/[B,B] \arrow[r] & A \otimes (B/[B,B]) \arrow[r] & {}
\end{tikzcd}$$

where $\varphi(a_0 \otimes b_0, a_1 \otimes b_1) = a_0 da_1 \otimes b_0 b_1$.

($\varphi$ is a particular case of the external cup product on Hochschild homology:

$$\bigoplus_{p+q=n} H_p(A,A) \otimes H_q(B,B) \xrightarrow{\sim} H_n(A \otimes B, A \otimes B)$$

Let $C^H_\bullet(A)$ be the Hochschild cx of $A$; it's the chain complex associated to a semi-simplicial vector space. Using the shuffle map one obtains an associative commutative map of complexes

$$C^H_\bullet(A) \otimes C^H_\bullet(B) \longrightarrow C^H_\bullet(A \otimes B)$$

This also applies to the normalized Hochschild complexes since the shuffle map goes $N(K) \otimes N(L) \to N(K \otimes L)$, where $N$ = normalization.)

September 17, 1988

Let $R = A\otimes B$, where $A^\cdot$ is a DGA and $B$ is an algebra with ideal $I$. Let $\rho\in R^1$ be such that $\gamma = \partial\rho + \rho^2 \in A\otimes I$; write $J = A\otimes I$. We have seen how to link the cocycles $tr\left(\frac{\gamma^n}{n!}\right) \in J^{(n)}\otimes_R = (J\otimes_R)^n$ $(A\otimes_A)\otimes(I\otimes_B)^n$ by using the double complex

$$\longrightarrow (A\otimes_A)\otimes I^{(n+1)} \longrightarrow (A\otimes_A)\otimes\left(I^{(n)}\otimes_B \Omega^1_B\otimes_B\right) \longrightarrow (A\otimes_A)\otimes I^{(n)} \longrightarrow$$

In effect, the classes

$$tr_A\left(\frac{\gamma^n}{n!}\right) \in (A\otimes_A)\otimes I^{(n)}$$

$$tr_A\left(\frac{\gamma^n}{n!}\partial\rho\right) \in (A\otimes_A)\otimes\left(I^{(n)}\otimes_B \Omega^1_B\otimes_B\right)$$

give a cocycle in the double complex. On the other hand, we similarly can link $tr\left(\frac{\gamma^n}{n!}\right) \in (A\otimes_A)\otimes(I\otimes_B)^n$ using the double complex

$$\cdots\cdots \longrightarrow A\otimes\left(I^{(n)}\otimes_B\right) \longrightarrow (\Omega^1_A\otimes_A)\otimes\left(I^{(n)}\otimes_B\right) \longrightarrow A\otimes\left(I^{(n)}\otimes_B\right) \longrightarrow \cdots\cdots$$

and the classes

$$tr_B\left(\frac{\gamma^n}{n!}\right) \in A\otimes\left(I^{(n)}\otimes_B\right)$$

$$tr_B\left(\frac{\gamma^n}{n!}\partial\rho\right) \in \left(\Omega^1_A\otimes_A\right)\otimes\left(I^{(n)}\otimes_B\right).$$

In order to see that the classes involving $\partial\rho$ are well-defined I use the canonical maps

$$(A\otimes_A)\otimes(\Omega^1_B\otimes_B) \longleftarrow \Omega^1_R\otimes_R \longrightarrow (\Omega^1_A\otimes_A)\otimes(B\otimes_B)$$

(Let's put aside for the moment the case of the ideal $I$, and suppose $B = I$.)

The problem is to see ~~that~~ the two linkings are compatible. Specifically I have a cocycle in

$$\circledast \qquad \cdots \longrightarrow (A\otimes_A)\otimes B \longrightarrow (A\otimes_A)\otimes(\Omega^1_B\otimes_B) \longrightarrow (A\otimes_A)\otimes B \longrightarrow$$

and a periodicity of this complex due to

$$0 \longrightarrow A\otimes_A \longrightarrow \Omega^1_A\otimes_A \longrightarrow A \longrightarrow A\otimes_A \longrightarrow 0$$

(which is exact after reducing when $A$ is free.) I want to see that the cocycle in $\circledast$ is compatible with this periodicity.

It's clear that I want to produce a cocycle in the tensor product of mixed complexes

$$(\Omega^1_A\otimes_A \rightleftarrows A)\otimes(\Omega^1_B\otimes_B \rightleftarrows B),$$

whereas at the moment all I can do is to produce one in

$$(\Omega^1_R\otimes_R \rightleftarrows R)$$

The point here I think is that this mixed complex for $R$ doesn't give the cyclic theory for $R$; the cyclic theory for $R$ is given by the tensor product of the complexes for $A, B$. Thus I have to refine my character calculations.

---

Let's try to understand $\Omega^1_R$ where $R = A\otimes B$. First of all by tensoring $0 \to \Omega^1_A \longrightarrow A\otimes A \longrightarrow A \to 0$ with the similar sequence for $B$ we obtain an exact sequence of $A\otimes B$ bimodules

$$0 \to \Omega^1_A \otimes \Omega^1_B \longrightarrow \Omega^1_A \otimes (B \otimes B) \oplus (A \otimes A) \otimes \Omega^1_B \longrightarrow \Omega^1_R \longrightarrow 0$$

Secondly we can use the extension

$$0 \to I \longrightarrow A * B \longrightarrow A \otimes B \longrightarrow 0$$

where $I$ is the ideal generated by $[A,B]$ in $A*B$ to obtain the exact sequence

$$0 \to I/I^2 \longrightarrow (A\otimes B) \otimes_{A*B} \Omega^1_{A*B} \otimes_{A*B} (A \otimes B) \longrightarrow \Omega^1_R \longrightarrow 0$$

$$\|$$

$$(A\otimes B) \otimes_{A*B} \left( A*B \otimes_A \Omega^1_A \otimes_A A*B + A*B \otimes_B \Omega^1_B \otimes_B A*B \right) \otimes_{A*B} (A \otimes B)$$

$$\|$$

$$(A\otimes B) \otimes_A \Omega^1_A \otimes (A\otimes B) \oplus (A \otimes B) \otimes_B \Omega^1_B \otimes_B (A \otimes B)$$

$$\|$$

$$B \otimes \Omega^1_A \otimes B \oplus A \otimes \Omega^1_B \otimes A$$

From this we conclude that

$$I/I^2 = \Omega^1_A \otimes \Omega^1_B .$$

~~[illegible]~~

It's clear that if we lift $\rho \in A \otimes B$ to $\tilde{\rho} \in A * B$, then we obtain a cycle in the complex

$$\longrightarrow A*B \longrightarrow \Omega^1_{A*B} \otimes_{(A*B)} \longrightarrow A*B \longrightarrow$$

$$\|$$

$$\cancel{\Omega_{A*B} \otimes_{A*B}}$$

$$\Omega^1_A \otimes_A (A*B) \otimes_A \oplus (A*B) \otimes_B \Omega^1_B \otimes_B$$

But $A*B \longrightarrow A\otimes B$ is a map of $A$-bimodules and also $B$-bimodules, so we have a map of complexes

$$\begin{tikzcd}
{} \arrow[r] & A*B \arrow[r] \arrow[dd] & \Omega^1_{A*B}\otimes_{(A*B)} \arrow[r] \arrow[d] & A*B \arrow[r] \arrow[dd] & {} \\
 & & {\left[\Omega^1_A\otimes_A(A\otimes B)\otimes_A\right]\oplus\left[(A\otimes B)\otimes_B\Omega^1_B\otimes_B\right]} \arrow[d, equal] & & \\
{} \arrow[r] & A\otimes B \arrow[r] & (\Omega^1_A\otimes_A)\otimes B \oplus A\otimes(\Omega^1_B\otimes_B) \arrow[r] & A\otimes B \arrow[r] & {}
\end{tikzcd}$$

But this doesn't seem to be enough since we don't get an element of $(\Omega^1_A\otimes_A)\otimes(\Omega^1_B\otimes_B)$ this way.

(The problem is that the first square maybe doesn't commute? Yes. **The arrow** doesn't exist. See below.)

<u>September 18, 1988</u>



Let's put $\hat{A} = A\otimes_A$, $\hat{\Omega}_A = \Omega^1_A \otimes_A$, and $\hat{A} = \bar{A}/[A,A]$, and similarly for $B$. Assuming $A$ free we have a resolution

$$\longrightarrow \bar{A} \longrightarrow \hat{\Omega}_{\bar A} \longrightarrow \bar{A} \longrightarrow \hat{\bar{A}} \to 0$$

Better to say that we have the truncated periodic complex

$$\circledast_A \qquad \cdots \longrightarrow A \longrightarrow \hat{\Omega}_A \longrightarrow A \dashrightarrow \hat{A} \dashrightarrow 0$$

which if we replace $A$ by $\bar{A}$ becomes a resolution of $\hat{\bar{A}}$ when $A$ is free. Now take the tensor product of $\circledast_A$ and $\circledast_B$ to obtain a double complex with augmentations, which will become acyclic in both directions when $A, B$ are free upon replacing $A, B$ by $\bar{A}, \bar{B}$.

$$\begin{tikzcd}
 & \arrow[d] & \arrow[d] & \\
\arrow[r] & \hat{\Omega}_A\otimes B \arrow[r]\arrow[d] & A\otimes B \arrow[r]\arrow[d] & \hat{A}\otimes B \arrow[r]\arrow[d] & 0 \\
\arrow[r] & \hat{\Omega}_A\otimes\hat{\Omega}_B \arrow[r]\arrow[d] & A\otimes\hat{\Omega}_B \arrow[r]\arrow[d] & \hat{A}\otimes\hat{\Omega}_B \arrow[r]\arrow[d] & 0 \\
\arrow[r] & \hat{\Omega}_A\otimes B \arrow[r]\arrow[d] & A\otimes B \arrow[r]\arrow[d] & \hat{A}\otimes B \arrow[r]\arrow[d] & 0 \\
\arrow[r] & \hat{\Omega}_A\otimes\hat{B} \arrow[r]\arrow[d] & A\otimes\hat{B} \arrow[r]\arrow[d] & \hat{A}\otimes\hat{B} \arrow[r]\arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Recall that $A$ is a DGA and we are given $\rho \in (A\otimes B)^1$, $\gamma = \delta\rho + \rho^2$. We have a cycle

$$\operatorname{tr}\left(\frac{\gamma^n}{n!}\right) \in \hat{A}\otimes\hat{B}$$

In the exact case ($A$ free replaced by $\bar{A}$ etc.) we know that this cycle may be lifted by diagram-chasing to cycles in both edge complexes $(\to \hat{\Omega}_A \to A)\otimes\hat{B}$ and $\hat{A}\otimes\begin{pmatrix}\hat{\Omega}_B\\ \downarrow \\ B\end{pmatrix}$ as well as the double complex. In fact without placing oneself in the exact case we know concretely how to construct the cycles in the edge complexes. The problem is to construct the cycle in the double complex, and of course one wants it to be "periodic" in both directions.

Set $S = A * B$, $S/I = A\otimes B$. We saw yesterday that

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, equal] & (S/I)\otimes_S \Omega^1_S \otimes_S (S/I) \arrow[r] \arrow[d, equal] & \Omega^1_{S/I} \arrow[r] & 0 \\
& \Omega^1_A\otimes\Omega^1_B & \Omega^1_A\otimes(B\otimes B)\oplus(A\otimes A)\otimes\Omega^1_B & &
\end{tikzcd}$$

exact seq. of $S/I$ bimod.

$$\begin{tikzcd}
& (I/I^2) \arrow[r] \arrow[d] & (S/I)\otimes_S \Omega^1_S \otimes_S \arrow[r] \arrow[d, equal] & \Omega^1_{S/I}\otimes_{S/I} \arrow[r] \arrow[d, "\beta"] & 0 \\
{} \arrow[r] & S/I^2 \arrow[r, "\partial"] & (S/I)\otimes_S \Omega^1_S \otimes_S \arrow[r, "\beta"] \arrow[d, equal] & S/I \arrow[r] \arrow[d, equal] & {} \\
& & \hat{\Omega}_A\otimes B \oplus A\otimes\hat{\Omega}_B & A\otimes B &
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, equal] & S/I^2 \arrow[r] & S/I \arrow[r] \arrow[d, equal] & 0 \\
& \Omega^1_A\otimes\Omega^1_B & & A\otimes B &
\end{tikzcd}$$

Let's go over the $S$ relation for Chern-Simons classes. Suppose $R$ a DGA with ideal $J$, $\rho \in R^1$, $\delta\rho + \rho^2 \in J$. We have the maps of complexes (actually double complexes)

$$\begin{tikzcd}
\arrow[r] & J^{(n+1)} \arrow[r] \arrow[d] & J^{(n)} \otimes_R \Omega^1_R \otimes_R \arrow[r] \arrow[d] & J^{(n)} \arrow[r] \arrow[d] & {} \\
\arrow[r] & R \arrow[r] \arrow[d] & \Omega^1_R \otimes_R \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & {} \\
\arrow[r] & R/J^{n+1} \arrow[r] & R/J^n \otimes_R \Omega^1_R \otimes_R \arrow[r] & R/J^n \arrow[r] & {}
\end{tikzcd}$$

which would be a short exact sequence of cxs. if $R$ is free. Recall we have a "cocycle" in the upper double complex given by the family

$$\frac{\gamma^n}{n!}\partial\rho \in J^{(n)} \otimes_R \Omega^1_R \otimes_R \qquad \frac{\gamma^n}{n!} \in J^{(n)}$$

the proof being

$$\delta\left(\frac{\gamma^n}{n!}\right) = \frac{1}{n!}(\gamma^n\rho - \rho\gamma^n) = \beta\left(\frac{\gamma^n}{n!}\partial\rho\right)$$

$$\partial\left(\frac{\gamma^{n+1}}{(n+1)!}\right) = \frac{1}{(n+1)!}\sum_0^n \gamma^i\,\partial\gamma\,\gamma^{n-i} = \frac{1}{n!}\gamma^n\,\partial\gamma$$

$$\|$$

$$\delta\left(\frac{\gamma^n}{n!}\partial\rho\right) = \frac{1}{n!}\left[(\gamma^n\rho - \rho\gamma^n)\partial\rho + \partial\delta\rho\right] = \frac{1}{n!}\gamma^n\left(\partial\delta\rho + \rho\partial\rho + \partial\rho\rho\right)$$

Suppose we have a family $\rho = \rho_t$. Then we can form the family

$$\frac{\mu_n}{n!}\partial\rho \in J^{(n)} \otimes_R \Omega^1_R \otimes_R \qquad \frac{\mu_n}{n!} \in J^{(n)}$$

$$\mu_n = \sum_{i=0}^{n-1} \gamma^i\,\dot\rho\,\gamma^{n-1-i}$$

provided $\dot\rho \in J$. We have from p. 38

$$(\gamma^n)^\circ = \sum_{i=0}^{n-1} \gamma^i \overset{[\delta+\rho,\mathring{\rho}]}{\mathring{\gamma}} \gamma^{n-1-i} = \left[\delta+\rho, \sum_0^{n-1} \gamma^i \mathring{\rho}\gamma^{n-1-i}\right]$$

$$= \delta(\mu_n) + \beta(\mu_n \partial\rho)$$

$$(\gamma^n \partial\rho)^\circ = (\delta\mu_n + \rho\mu_n + \mu_n\rho)\partial\rho + \gamma^n\partial\mathring{\rho}$$

$$= \delta(\mu_n\partial\rho) + \mu_n(\underbrace{\partial\delta\rho + \partial\rho\rho + \rho\partial\rho}_{\partial\gamma}) + \gamma^n\partial\mathring{\rho}$$

$$= \delta(\mu_n\partial\rho) + \underbrace{\sum_{i=0}^{n-1} \gamma^i\mathring{\rho}\gamma^{n-1-i}\partial\gamma}_{\sum_0^{n-1}\gamma^{n-1-i}\partial\gamma\,\gamma^i\mathring{\rho}} + \gamma^n\partial\mathring{\rho}$$

$$= \delta(\mu_n\partial\rho) + \partial(\gamma^n\mathring{\rho})$$

$$\partial\left(\frac{\mu_{n+1}}{n+1}\right) = \partial\left(\frac{1}{n+1}\sum_0^n \gamma^i\mathring{\rho}\gamma^{n-i}\right) = \partial(\gamma^n\mathring{\rho})$$

where in the last step we use the fact that $\partial$ factors through $J^{(n+1)} \longrightarrow (J\otimes_R)^{n+1}_\sigma$.

Now we actually apply these formulas when $I = R$ and we take $\rho_t = t(\text{original } \rho)$ call this $\rho_1$ so that $\gamma_t = t\delta\rho_1 + t^2\rho_1^2$ starts at $0$ and ends with $\gamma_1 \in J$. Then we have the Chern-Simons classes

$$\nu_n = \int_0^1 \frac{1}{n!}\left(\sum_0^{n-1} \gamma_t^i \rho_1 \gamma_t^{n-1-i}\right) dt \in R$$

$$\nu_n' = \int_0^1 \frac{1}{n!}\left(\sum_0^{n-1} \gamma_i^i \rho_1 \gamma_t^{n-1-i}\partial\rho_1\right) t\,dt \in \Omega^1_R \otimes_R$$

satisfying

$$\frac{\gamma^n}{n!} = \delta(\nu_n) + \beta(\nu_n')$$

$$\frac{\gamma^n}{n!}\partial\rho_1 = \delta(\nu_n') + \partial(\nu_{n+1})$$

since $\gamma_1 \in J$, this implies that the classes $\nu_n \in R/J^{n}$, $\nu_n' \in R/J^n \otimes_R \Omega^1_R \otimes_R$ satisfy

$$0 = \delta(\nu_n) + \beta(\nu_n')$$
$$0 = \delta(\nu_n') + \partial(\nu_{n+1})$$

Thus the $\left(\frac{\gamma_1^n}{n!}, \frac{\gamma_1^n}{n!}\partial\rho_1\right)$ cocycle in the $(R, J)$-subcomplex becomes the coboundary in the $(R,R)$ complex [redacted] of the $(\nu_n, \nu_n')$ cochain, and so we get a <u>cocycle</u> in the $(R,J)$-quotient complex.

Here is the missing point in the cyclic theory of $A \otimes B$. Let $S = A * B / I^2_{[A,B]}$ so that we have an extension

$$0 \longrightarrow I[redacted] \longrightarrow S \longrightarrow A \otimes B \longrightarrow 0$$
$$\| $$
$$\Omega^1_A \otimes \Omega^1_B .$$

It's the extension belonging to the external cup product of the canonical derivations $\partial : A \to \Omega^1_B$ and also for $B$. Note that we have an exact sequence

$$0 \longrightarrow I/[S,I] \longrightarrow S/[S,I] \longrightarrow A \otimes B \longrightarrow 0$$
$$\|$$
$$(\Omega^1_A \otimes_A) \otimes (\Omega^1_B \otimes_B)$$

hence using the lifting $A \otimes B \longrightarrow A * B$, $a \otimes b \mapsto ab$ gives an isomorphism (additive

$$S/[S,I] = (A \otimes B) \oplus (\Omega^1_A \otimes_A) \otimes (\Omega^1_B \otimes_B)$$

We've already seen that

$$(A\otimes B) \otimes_S \Omega^1_S \otimes_S (A \otimes B) = (A \otimes B) \otimes_{A*B} \Omega^1_{A*B} \otimes_{A*B} (A \otimes B)$$

$= \big(B\otimes\Omega^1_A\otimes$ [redacted] $B\big) \oplus \big(A\otimes\Omega^1_B\otimes A\big)$

hence $\quad S/I\otimes_S\Omega^1_S\otimes_S = (\Omega^1_A\otimes_A)\otimes B \oplus A\otimes(\Omega^1_B\otimes_B)$

Thus the periodic complex I want can be obtained as a quotient complex of $S, \Omega^1_S$ complex

$$
\begin{tikzcd}
 & & I\otimes_S\Omega^1_S\otimes_S \arrow[r] \arrow[d] & {[I,S]} \arrow[d] & \\
{} \arrow[r] & S \arrow[r, "\partial"] \arrow[d] & \Omega^1_S\otimes_S \arrow[r, "\beta"] \arrow[d] & S \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & S/[S,I] \arrow[r, "\text{[redacted]}"] & S/I\otimes_S\Omega^1_S\otimes_S \arrow[r] \arrow[d, equal] & S/[S,I] \arrow[r] \arrow[d, equal] & \text{[redacted]} \\
 & & \hat\Omega_A\otimes B\oplus A\otimes\hat\Omega_B & (A\otimes B)\oplus(\hat\Omega_A\otimes\hat\Omega_B) &
\end{tikzcd}
$$

Note that $\partial$ vanishes on $[S,S]$, hence on $[S,I]$. (Better treatment: take $S = A * B$)

# September 20, 1988

Let's now prove the S-relation between Connes' cocycles attached to a Fredholm module. We suppose given a homom. of algebras $A \longrightarrow B$, an involution $F \in B$ an ideal $I \subset B$ such that $[F, A] \subset B$, and a trace $\tau$ defined on $I^n$ for ~~large~~ $n$. Then we have $\theta \in C^1(A, B)$, which is a flat "connection form", and also the "~~splitting~~" $F$. Recall:

$$D = \delta + \theta \qquad D^2 = 0$$

$$D = \underbrace{\frac{D + FDF}{2}}_{\nabla} + \underbrace{\frac{D - FDF}{2}}_{\alpha}$$

$$[\nabla, F] = 0 \qquad F\alpha F = -F \qquad \nabla^2 = -\alpha^2 \qquad [\nabla, \alpha] = 0$$

$$\delta \operatorname{trace}(F\alpha^{2n}) = \operatorname{trace}[\nabla, F\alpha^{2n}] = 0$$

Also
$$\alpha = \frac{\theta - F\theta F}{2} = \frac{1}{2} F[F,\theta], \quad \text{so}$$

$$\nabla^{2n} = (-\alpha^2)^n = \frac{1}{2^{2n}} [F,\theta]^{2n}$$

We propose to relate the cyclic cocycles $\operatorname{trace}(F\alpha^{2n})$ for different $n$ via the $S$-operator, where trace means applying $N$ and $\tau$ to $F\alpha^{2n} \in C^{2n}(A, I^{2n})$. We lift $\operatorname{trace}(F\alpha^{2n})$ or rather $\operatorname{trace}(F[F,\theta]^{2n})$ to the cochain

$$\tau(F[F,\theta]^{2n}) \in C^{2n}(A)$$

Next we need a Hochschild cochain analogous to $\gamma^n \partial\rho$ and we take

$$\tau(F[F,\theta]^{2n}\partial\theta) \in C^{2n}(A, A^*)$$

Then we have
$$\delta[F,\theta] = [F, -\theta^2] = -\theta[F,\theta] - [F,\theta]\theta$$
$$\text{i.e.} \quad [\delta + \theta, [F,\theta]] = 0$$

$$\delta\{\tau(F[F,\theta]^{2n})\} = \tau(F([F,\theta]^{2n}\theta - \theta[F,\theta]^{2n}))$$
$$= \tau(F[F,\theta]^{2n}\theta - \theta[F,\theta]^{2n}F)$$
$$\|$$
$$\beta\{\tau(F[F,\theta]^{2n}\partial\theta)\} = \tau(F[F,\theta]^{2n}\theta - \theta F[F,\theta]^{2n})$$

$$\partial\{\tau(F[F,\theta]^{2n+2})\} = \tau(F\sum_{i=0}^{2n+1}[F,\theta]^i \overset{(-1)^i}{[F,\partial\theta]}[F,\theta]^{2n+1-i})$$
$$(-1)^{(i+1)(2n+1-i)} = (-1)^{i+1}$$
$$= -(2n+2)\,\tau([F,\theta]^{2n+1}F[F,\partial\theta])$$
$$= -(2n+2)\,\tau([F,\theta]^{2n+1}(\partial\theta - F\partial\theta F))$$
$$= -4(n+1)\,\tau([F,\theta]^{2n+1}\partial\theta)$$

$$\delta\{\tau(F[F,\theta]^{2n}\partial\theta)\} = \tau(F([F,\theta]^{2n}\theta - \theta[F,\theta]^{2n})\partial\theta + F[F,\theta]^{2n}\partial(-\theta^2))$$
$$= \tau\begin{pmatrix} \cancel{F[F,\theta]^{2n}\theta\partial\theta} - F\theta[F,\theta]^{2n}\partial\theta \\ -\underbrace{F[F,\theta]^{2n}\partial\theta\,\theta}_{-1} - \cancel{F[F,\theta]^{2n}\theta\partial\theta} \end{pmatrix}$$
$$= -\tau([F,\theta]^{2n+1}\partial\theta)$$

showing the S-relation desired.

The above proof amounts to the following calculations. The cocycles are

$$\varphi_{(n)}(a_0, \ldots, a_{2n-1}) = \tau(F[F,a_0]\cdots[F,a_{2n-1}]) \in C^{2n}(A)$$

$$\psi_{(n)}(a_0, \ldots, a_{2n}) = \tau(Fa_0[F,a_1]\cdots[F,a_{2n}]) \in C^{2n}(A,A^*)$$

and the proof consists of checking the formulas

$$\boxed{(1-t)\psi_{(n)} = b'\varphi_{(n)}}$$

i.e.

$$(b'\varphi_{(n)})(a_0,\dots,a_{2n}) = \tau\big(F(a_0[F,a_1]+[F,a_0]a_1)[F,a_2]\cdots\big)$$
$$-\,\tau\big(F[F,a_0](a_1[F,a_2]+[F,a_1]a_2)[F,a_3]\cdots\big)$$
$$\cdots$$
$$+(-1)^{2n-1}\tau\big(F[F,a_0]\cdots(a_{2n-1}[F,a_{2n}]+[F,a_{2n-1}]a_{2n})\big)$$
$$= \tau(Fa_0[F,a_1]\cdots[F,a_{2n}])$$
$$-\,\tau(a_{2n}F[F,a_0]\cdots[F,a_{2n-1}])$$
$$= \tau(Fa_0[F,a_1]\cdots[F,a_{2n}]) - \tau(Fa_{2n}[F,a_0]\cdots[F,a_{2n-1}])$$
$$= ((1-t)\psi_{(n)})(a_0,\dots,a_{2n})$$

Next

$$(b\psi_{(n)})(a_0,\dots,a_{2n+1}) = \tau(Fa_0a_1\cancel{[F,a_2]}\cdots[F,a_{2n+1}])$$
$$-\,\tau\big(Fa_0(\cancel{a_1[F,a_2]}+[F,a_1]a_2)[F,a_3]\cdots\big)$$
$$\cdots$$
$$+(-1)^{2n}\tau\big(Fa_0[F,a_1]\cdots(\cancel{a_{2n}[F,a_{2n+1}]}+[F,a_{2n}]a_{2n+1})\big)$$
$$+(-1)^{2n+1}\tau(Fa_{2n+1}a_0[F,a_1]\cdots[F,a_{2n}])$$
$$= -\,\tau(a_0[F,a_1]\cdots[F,a_{2n+1}])$$

Observe that $\varphi_{(n+1)}$ is already cyclically invariant:

$$(N\varphi_{(n+1)})(a_0,\dots,a_{2n+1}) = (2n+2)\,\tau(F[F,a_0]\cdots[F,a_{2n+1}])$$
$$= 4(n+1)\,\tau(a_0[F,a_1]\cdots[F,a_{2n+1}])$$

Thus we have

$$\boxed{N\varphi_{(n+1)} = -4(n+1)\,b\psi_{(n)}}$$

establishing the S-relation.

Notice that although

$$\varphi_{(n)}(a_0,\dots,a_{2n-1}) = \tau\big(F[F,a_0]\cdots[F,a_{2n-1}]\big)$$

is a cyclic (2n-1)-cocycle, in the above proof $\varphi_{(n)}$ is regarded as a 2n-cochain $\varphi_{(n)} \in C^{2n}(A)$ lifting this cyclic (2n-1)-cocycle.

Thus the fact that $b\varphi_{(n)} = 0$ means $\varphi_{(n)}$ is a cocycle in $C_\lambda^{2n-1}$, but in the above proof we use $b'\varphi_{(n)} = (1-t)\psi_{(n)}$.

## September 22, 1988

Let's see if we can understand the S-relation between the canonical maps from the cyclic complex to de Rham cohomology.

First of all certain formulas become simpler if we use symmetrized products. These are defined by

$$S(X_1, \dots, X_n) \overset{\text{def}}{=} \frac{1}{n!} \sum_{\sigma \in \Sigma_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

$$S(\overset{k}{X}, \overset{l}{Y}) \overset{\text{def}}{=} S(\underbrace{X, \dots, X}_{k}, \underbrace{Y, \dots, Y}_{l})$$

$$= \frac{k!\,l!}{(k+l)!} P(\overset{k}{X}, \overset{l}{Y})$$

in my old notation. Thus our old identity

$$(k+1)\, \mathrm{tr}\{P(\overset{k+1}{X}, \overset{l}{Y})\} = (k+l+1)\, \mathrm{tr}\{P(\overset{k}{X}, \overset{l}{Y})X\}$$

becomes on multiplying by $\frac{k!\,l!}{(k+l+1)!}$

$$\boxed{\mathrm{tr}\{S(\overset{k+1}{X}, \overset{l}{Y})\} = \mathrm{tr}\{S(\overset{k}{X}, \overset{l}{Y})X\}}$$

Also

$$e^{uX+vY} = \sum_{n\geq 0} \frac{(uX+vY)^n}{n!}$$

$$= \sum_{k,l\geq 0} \frac{u^k v^l}{(k+l)!} P(\overset{k}{X}, \overset{l}{Y})$$

$$\boxed{e^{uX+vY} = \sum \frac{u^k v^l}{k!\,l!} S(\overset{k}{X}, \overset{l}{Y})}$$

Here $X, Y$ are elements in an algebra and $u, v$ are scalars.

This formula generalizes to several non-commuting variables $X_1, \ldots, X_m$

$$\boxed{e^{\sum_1^m u_i X_i} = \sum_{k_1, \ldots, k_m} \frac{u_1^{k_1} \ldots u_m^{k_m}}{k_1! \ldots k_m!} S(\overset{k_1}{X_1}, \ldots, \overset{k_m}{X_m})}$$

Consider a variation $X \mapsto X + \delta X$. Then

$$\delta(e^{uX+vY}) = e^{u(X+\delta X)+vY} - e^{uX+vY}$$

$$= \sum_{k,\ell \geq 0} \frac{u^{k+1} v^\ell}{k!\, 1!\, \ell!} S(\overset{k}{X}, \overset{1}{\delta X}, \overset{\ell}{Y})$$

Looking at the coefficient of $u^k v^\ell$ gives

$$\delta\, S(\overset{k}{X}, \overset{\ell}{Y}) = k\, S(\overset{k-1}{X}, \delta X, \overset{\ell}{Y}).$$

Similarly when $X, Y$ are both varied

$$\boxed{\delta\, S(\overset{k}{X}, \overset{\ell}{Y}) = k\, S(\overset{k-1}{X}, \overset{1}{\delta X}, \overset{\ell}{Y}) + \ell\, S(\overset{k}{X}, \overset{\ell-1}{Y}, \overset{1}{\delta Y})}$$

Now let us take $\delta X = \varepsilon X$, $\delta Y = 0$ where $\varepsilon^2 = 0$. Then

$$\delta\, S(\overset{k}{X}, \overset{\ell}{Y}) = \varepsilon k\, S(\overset{k}{X}, \overset{\ell}{Y}) \quad \text{by Euler}$$

$$\delta\{\mathrm{tr}\, S(\overset{k}{X}, \overset{\ell}{Y})\} = \varepsilon k\, \mathrm{tr}\, S(\overset{k}{X}, \overset{\ell}{Y})$$

$$\parallel$$

$$\mathrm{tr}\, \{k\, S(\overset{k-1}{X}, \varepsilon X, Y)\} = \mathrm{tr}\, \{k\, S(\overset{k-1}{X}, Y)\varepsilon X\}$$

(In general $\mathrm{tr}\{S(\overset{k}{X}, \overset{\ell}{Y}, Z)\} = \mathrm{tr}\{S(\overset{k}{X}, \overset{\ell}{Y}) Z\}$)

and so one again obtains

$$\mathrm{tr}\,\{S(\overset{k}{X}, \overset{\ell}{Y})\} = \mathrm{tr}\,\{X\, S(\overset{k-1}{X}, Y)\} \qquad k > 0.$$

Now let us turn to the Chern-Simons form

$$\int_0^1 \frac{\mu_n}{n!}\,dt = \int_0^1 \frac{1}{n!}\sum_{i=0}^{n-1} \gamma^i \mathring{\rho}\, \gamma^{n-1-i}\,dt$$

where $\gamma = t\,d\theta + (t^2-t)\theta^2$ and $\rho = t\theta$.

Formally if we have the inf. variation $\gamma \mapsto \gamma + \mathring{\rho}$, so that $\delta\gamma = \mathring{\rho}$ as a variation, then

$$\delta(\gamma^n) = \sum_0^{n-1} \gamma^i \mathring{\rho}\, \gamma^{n-1-i}.$$

Thus

$$\sum_{n\geq 1} \frac{u^n \mu_n}{n!} = \delta\left(e^{u\gamma}\right) \quad \theta \to \mathring{\rho}$$

$$= e^{u(t\,d\theta + (t^2-t)\theta^2 + \mathring{\rho})} - e^{u(t\,d\theta + (t^2-t)\theta^2)}$$

$$= \sum \frac{u^{k+l+1}}{k!\,l!}\, S(\overset{k}{(t^2-t)\theta^2}, \overset{l}{t\,d\theta}, \overset{1}{\theta})$$

$$\therefore \int_0^1 \frac{\mu_n}{n!}\,dt = \sum_{k+l=n-1} \frac{1}{k!\,l!} S(\overset{k}{\theta^2}, \overset{l}{d\theta}, \overset{1}{\theta}) \underbrace{\int_0^1 (t^2-t)^k t^l\,dt}_{(-1)^k \beta(k+l+1,\,k+1) \,=\, (-1)^k \frac{k!\,(k+l)!}{(2k+l+1)!}}$$

or

$$\boxed{\begin{aligned} \int_0^1 \frac{\mu_n}{n!}\,dt &= \sum_{k+l=n-1} S(\overset{k}{\theta^2}, \overset{l}{d\theta}, \overset{1}{\theta})\,(-1)^k \frac{(k+l)!}{l!\,(2k+l+1)!} \\ \int_0^1 \left(\frac{\mu_n}{n!}\partial\rho\right) dt &= \sum_{k+l=n-1} S(\overset{k}{\theta^2}, \overset{l}{d\theta}, \overset{1}{\theta})\,\partial\theta\,(-1)^k \frac{(k+l+1)!}{(l)!\,(2k+l+2)!} \end{aligned}}$$

<u>September 23, 1988:</u>

On signs: Recall $\Omega^1_R$ is

$$\Omega^1_R = \operatorname{Coker}\{R^{\otimes 3} \xrightarrow{b'} R^{\otimes 2}\} = \operatorname{Ker}\{R^{\otimes 2} \longrightarrow R\}$$

So we have the diagram

$$\begin{tikzcd}
x\otimes y\otimes z & x\,dy\,z & & \\
R^{\otimes 3} \arrow[u, mapsto] \arrow[r, two heads] \arrow[d] \arrow[rr, bend left, "b'"] & \Omega^1_R \arrow[u, mapsto] \arrow[r, hook] \arrow[d] & R^{\otimes 2} \arrow[d] & xy\otimes z - x\otimes yz \\
 & zx\,dy & & \\
R^{\otimes 2} \arrow[r] \arrow[rr, bend right, "b"'] & \Omega^1_R \otimes_R \arrow[r, "\beta"] & (R^{\otimes 2})\otimes_R = R & (zx)y - y(zx) \\
zx\otimes y & & &
\end{tikzcd}$$

Thus the identification $(R^{\otimes 2})\otimes_R \xrightarrow{\sim} R$ moves the last factor to the left. In particular we have

$$\beta(x\,dy\,z) = (zx)y - y(zx)$$

Now when we comes to a DGA we have to proceed from these conventions, so that

$$\beta(\xi\,\partial\eta\,\zeta) = (-1)^{[\deg(\xi)+\deg(\eta)]\deg(\zeta)}\beta(\zeta\,\xi\,\partial\eta)$$

$$= (-1)^{[\deg\xi+\deg\eta]\deg\zeta}\left\{\zeta\xi\eta - (-1)^{(\deg\zeta+\deg\xi)\deg\eta}\eta\zeta\xi\right\}$$

$$= (-1)^{|\xi\eta|\cdot|\zeta|}\zeta\xi\eta - (-1)^{|\xi|\cdot|\eta\zeta|}\eta\zeta\xi$$

where $|?| = \deg ?$

---

Problem: How do we relate the above discussion of the S-operator to Connes' device of tensoring $A$ with $\mathbb{C}\oplus\mathbb{C}e$, $e^2=e$ and the non-unital embedding

$$\begin{array}{ccc} A & \longrightarrow & Ae \subset A\otimes(\mathbb{C}\oplus\mathbb{C}e) \quad ? \\ a & \longmapsto & ae \end{array}$$

Let's review the formulas. Given $i: E \to \tilde{V}$ we have $a \mapsto \iota a \iota^*$ from $\Omega^0(M, \operatorname{End} E)$ to $\Omega^0(M)\otimes \operatorname{End}(V)$ and we can pull back $tr \frac{\theta(d\theta)^n}{(n+1)!}$ and we obtain

$$\frac{1}{(n+1)!}\, tr \begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[\nabla,\theta] & -\theta(i^*dj)\\ (j^*di)\theta & 0\end{pmatrix}^n$$

$$= \frac{1}{(n+1)!}\sum_{2k+\ell=n} tr\ \theta P(\overset{\ell}{[\nabla,\theta]}, \overset{k}{\theta\nabla^2\theta})$$

If we then restrict to $\Omega^0(M)$ via $a \mapsto a\, id_E$ we get

$$\frac{1}{(n+1)!}\sum_{\ell+2k=n} tr\{\theta P(\overset{\ell}{d\theta}, \overset{k}{\theta^2})\}\ tr_E(\nabla^2)^k$$

$$= \sum_{\ell+2k=n} tr\{\theta S(\overset{\ell}{d\theta}, \overset{k}{\theta^2})\}\frac{(\ell+k)!\,(-1)^k}{\ell!\,(2k+\ell+1)!}\ tr\frac{(-\nabla^2)^k}{k!}$$

which shows that we get exactly the components of the CS forms.

However ~~this~~ only ~~constitutes~~ a verification that Connes device works. One hopes for a better understanding.

<u>September 28, 1988</u>

The problem is to obtain the bicomplex made from the reduced Hochschild complex with the $(b, B)$-operators. One idea is to consider $R = T(A)/(1-\rho(1_A))$ and the maps

$$\Omega^1_R \otimes_R \underset{\beta}{\overset{\partial}{\rightleftarrows}} R$$

Now $R$ is canonically isomorphic to $\Omega_A^{ev}$ with $*$ multiplication, and $\Omega^1_R \otimes_R \simeq R \otimes \bar{A} \simeq \Omega_A^{odd}$. If we keep track of the two filtrations on $R$, then we see that $\beta$ takes $\Omega_A^{2n+1}$ on the left into $\Omega_A^{2n+2} \oplus \Omega_A^{2n}$ on the right, and a similar statement holds for $\partial$. The hope would be that the two maps $\Omega_A^{2n+1} \rightrightarrows \Omega_A^{2n}, \Omega_A^{2n+2}$ could be identified with $b$ and $B$ respectively.

Let's calculate. Take $\omega = a_0\, da_1 \cdots da_{2n} \in \Omega_A^{2n}$ and apply $\beta$ to the element of $\Omega^1_R \otimes_R$ corresponding to $\omega\, da_{n+1} \in \Omega_A^{2n+1}$. This element is $\rho(a_0)\gamma(a_1, a_2) \cdots \gamma(a_{2n-1}, a_{2n})\, \partial\rho(a_{2n+1})$, and the result of applying $\beta$ is

$$\omega * a_{2n+1} - a_{2n+1} * \omega$$

$$= \omega a_{2n+1} - a_{2n+1}\omega + d\omega\, da_{2n+1} - da_{2n+1}\, d\omega$$

The first term indeed corresponds to $b(a_0, \ldots, a_{2n+1})$, since

$$\left.\begin{array}{l} a_0\, da_1 \cdots da_{2n}\, a_{2n+1} \\ - a_{2n+1} a_0\, da_1 \cdots da_{2n} \end{array}\right\} = \left\{\begin{array}{l} a_0\, da_1 \cdots d(a_{2n}a_{2n+1}) - a_0\, da_1 \cdots d(a_{2n-1}a_{2n})\, da_{2n+1} \\ + \cdots - a_0\, d(a_1 a_2) \cdots da_{2n+1} + a_0\, da_1 \cdots da_{2n+1} \\ - a_{2n+1} a_0\, da_1 \cdots da_{2n} \end{array}\right.$$

However the second term does not correspond to

$B(a_0, \ldots, a_{2n+1})$ which is the sum over cyclic permutations 

$$\left(\sum_{i=0}^{2n+1} t^i\right) da_0 \cdots da_{2n+1}$$

Instead one only gets $(1+t)\, da_0 \cdots da_{2n+1}$.

Similarly suppose we apply $\partial$ to $\Omega_A^{2n} \subset R$. Here $\Omega_A^{2n} = I^n \cap \rho(A)^{2n+1}$ and if we want the component of $\partial | \Omega_A^{2n}$ in $\Omega_A^{2n-1}$, we can identify this component map with

$$\partial : I^n / I^{n+1} \longrightarrow I^{n-1}/I^n \otimes_R \Omega_R^1 \otimes_R$$

Let's calculate. First take $n=1$. Then

$$\partial\, \rho(a_0)\, \gamma(a_1, a_2) = \partial\rho(a_0)\, \gamma(a_1,a_2) + \rho(a_0)\big[\partial\rho(a_1)\, \rho(a_2) + \rho(a_1)\, \partial\rho(a_2) - \partial\rho(a_1 a_2)\big]$$

$$\underset{\text{mod } I}{\equiv} a_0\, \partial(a_1)\, a_2 + a_0 a_1\, \partial(a_2) - a_0\, \partial(a_1 a_2)$$

More precisely I am calculating the map $\partial$ in the exact sequence

$$0 \longrightarrow I/I^2 \xrightarrow{\ \partial\ } A \otimes_R \Omega_R^1 \otimes_R A \longrightarrow \Omega_A^1 \longrightarrow 0$$

Next let's calculate

$$\Omega_A^{2n} = I^n/I^{n+1} = I^{n-1}/I^n \otimes_A I/I^2 \xrightarrow{\ 1\otimes\partial\ } I^{n-1}/I^n \otimes_R \Omega_R^1 \otimes_R A$$

$$\downarrow$$

$$I^{n-1}/I^n \otimes_R \Omega_R^1 \otimes_R \;=\; \Omega_A^{2n-1}$$

It takes $a_0\, da_1 \ldots da_{2n-2}\, da_{2n-1}\, da_{2n}$ into

$$a_0\, da_1 \ldots da_{2n-2}\, \partial\rho(a_{2n-1})\, a_{2n} + a_0\, da_1 \ldots da_{2n-2}\, a_{2n-1}\, \partial\rho(a_{2n})$$
$$- a_0\, da_1 \ldots da_{2n-2}\, \partial\rho(a_{2n-1} a_{2n})$$

which is the same as in $\Omega_A^{2n-1}$

$$a_{2n}a_0 da_1 \cdots da_{2n-1} - a_0 da_1 \cdots da_{2n-2}\, d(a_{2n-1}a_{2n})$$
$$+ a_0 da_1 \cdots d(a_{2n-2}a_{2n-1})\, da_{2n} - \cdots$$
$$- a_0 d(a_1a_2)\, da_3 \cdots da_{2n} + a_0a_1\, da_2 \cdots da_{2n}$$

and this we recognize as $b(a_0,\dots,a_{2n})$.

But $\partial : I^n/I^{n+1} \to I^{n-1}/I^n \otimes_R \Omega^1_R \otimes_R$ involves cyclic symmetrization over the group $\mathbb{Z}/n$ of the above map $id_{I^{n-1}/I^n} \otimes \partial_{I/I^2}$.

---

The conclusion of the above discussion is that although the maps

$$\Omega^1_R \otimes_R \underset{\beta}{\overset{\partial}{\rightleftarrows}} R$$

do not yield the $b, B$ arrows on the nose, they seem to yield them up to cyclic symmetrization.

I feel that it ought to be possible to understand well the link between the mixed complex $\Omega^1_R \otimes_R \underset{\beta}{\overset{\partial}{\rightleftarrows}} R$ with its $I$-adic filtration and the mixed complex given by the reduced Hochschild complex with $(b, B)$ operators.

The difficulties are related to the fact that neither of the complexes

$$\longrightarrow I^{n+1} \xrightarrow{\partial} I^n \otimes_R \Omega^1_R \otimes_R \longrightarrow I^n \longrightarrow$$

$$\longrightarrow \overline{R/I^{n+1}} \xrightarrow{\partial} R/I^n \otimes_R \Omega^1_R \otimes_R \longrightarrow \overline{R/I^n} \longrightarrow$$

are quasi-isomorphic to the cyclic complex of $A$, precisely because of the necessity of taking cyclic coinvariants in the exact sequences

$$0 \to \overline{HC}_{2n+1}(A) \longrightarrow H_0(R, I^{n+1})_\sigma \xrightarrow{\partial} H_1(R, I^n) \to \overline{HC}_{2n}(A) \to 0$$

$$0 \to \overline{HC}_{2n}(A) \longrightarrow H_0(R, R/I^{n+1}) \xrightarrow{\partial} H_1(R, R/I^n)_\sigma \to \overline{HC}_{2n-1}(A) \to 0$$

I feel also that a satisfactory resolution of this problem will show how to explicitly map the cyclic complex $CC(A)$ to the $(b,B)$-bicomplex. At the moment I have a kind of Chern-Simon map

$$CC(A) \longrightarrow \Sigma^{-1}_{hor} CC(A \xleftarrow{1} A) / CC(A)$$

where the latter is a double complex whose $p$-th column is $\Sigma^{p}_{1}(A \overset{L}{\otimes}_A)^p_\sigma$, which we know [redacted] is quasi-isomorphic to $\Sigma^{p}(A \overset{L}{\otimes}_A)$. However I don't have yet a suitable map.

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To gain insight let's review the periodicity result for the cocycles attached to a Fredholm module. Take the ungraded case $(A, H, F)$, [redacted] and let $\tau$ be the trace on $H$. Then we have the cochains

$$[redacted] \quad \tau(F[F,\theta]^{2n}) \in C^{2n}(A)$$

and the Hochschild cochains

$$\tau(F[F,\theta]^{2n}\partial\theta) \in C^{2n}(A; A^*)$$

satisfying the identities

$$\delta\, \tau(F[F,\theta]^{2n}) = \beta(F[F,\theta]^{2n}\partial\theta)$$
$$\partial\, \tau(F[F,\theta]^{2n+2}) = (4n+4)\ \delta\, \tau(F[F,\theta]^{2n}\partial\theta)$$

A striking fact is that the cochain $\tau(F[F,\theta]^{2n})$ is already cyclic. In order to put this in perspective, let $R = C^{\cdot}(A) \sim T(A^*)$. Then we write our basic arrows 

$$R \xrightarrow{\partial} \Omega^1_R \otimes_R \xrightarrow{\beta} R$$

To say that $\tau(F[F,\theta]^{2n}) \in R^{2n} \simeq (A^*)^{\otimes 2n}$ is cyclic means that it lies in $[(A^*)^{\otimes 2n}]^{\lambda}$. This is analogous to working with

$$\begin{array}{ccccc}
R & \xrightarrow{\partial} & \Omega^1_R \otimes_R & \xrightarrow{\beta} & R \\
\cup & & \cup & & \cup \\
I^{n+1} & \longrightarrow & I^n \Omega^1_R \otimes_R & \longrightarrow & I^n
\end{array}$$

and cutting $I^n$ back to some sort of cyclic invariants. Does this have any meaning?

The big goal in cyclic theory ought to be able to produce maps ~~of~~ cyclic complexes, that is, bivariant cyclic classes in $RHom_{\mathbb{Z}}(CC(A), CC(B))$. This should be analogous to what happens in KK-theory. [redacted] By considering the DG algebra $Hom_C(Bar(A), B) = C^{\bullet}(A,B)$, or more generally $C^{\bullet}(A,R)$ where $B = R/I$, I have been able to produce maps from $CC(A)$ to $B/[B,B]$, or perhaps more generally to the periodic complex belonging to $\Omega^1_B \otimes_B \rightleftarrows B$. I feel that at the moment I have good control over maps from $CC(A)$ via $Bar(A)$, but not good control over maps into $CC(A)$.

The best model for maps into $CC(A)$ so far is various $I$-adic filtered complexes associated to $\Omega^1_R \otimes_R \rightleftarrows R$, where $A = R/I$ with $R$ free.

A natural thing to look at is how to construct cyclic homology classes, [redacted] and a natural way to obtain cyclic homology classes is from K-classes.

Let's consider an element of $K_0(A)$ represented by an idempotent [redacted] matrix $e \in M_n(A)$. If $\rho: A \longrightarrow R$ is a homomorphism modulo $I$, then we can do connection and curvature calculations in $M_n(C^{\bullet}(A,R)) = C^{\bullet}(A,R) \otimes M_n(\mathbb{C})$

It is natural before going on to inquire about the effect of passing to matrix algebras on the models we have for cyclic theory.

Let $B = \mathrm{End}(V)$, $V = \mathbb{C}^n$. Then $B$ is a projective $B \otimes B^{op}$-module. In effect choose $v_0 \in V$, $\lambda_0 \in V^*$ with $\lambda_0(v_0) = 1$. Then we have an isomorphism

$$\begin{array}{ccc} B & \xleftarrow{\sim} & V \otimes V^* \\ v\lambda & \longleftarrow & v \otimes \lambda \end{array}$$

of $B$-bimodules, and maps

$$\begin{array}{ccc} V & \longrightarrow & B \\ v & \longmapsto & v\lambda_0 \end{array} \qquad \begin{array}{ccc} V^* & \longrightarrow & B \\ \lambda & \longmapsto & v_0\lambda \end{array}$$

of left (resp right) $B$-modules. Hence we get a $B$-bimodule map

$$\begin{array}{ccccc} B & \xleftarrow{\sim} & V \otimes V^* & \longrightarrow & B \otimes B \\ v\lambda & \longleftarrow & v \otimes \lambda & \longmapsto & v\lambda_0 \otimes v_0\lambda \end{array}$$

which when composed with $B \otimes B \xrightarrow{\text{mult}} B$ gives the identity.

In particular we have a $B$-bimodule splitting of

$$0 \longrightarrow \Omega^1_B \longrightarrow B \otimes B \longrightarrow B \longrightarrow 0$$

from which its clear that any derivation of $B$ with values in a bimodule is inner. Also we have that

$$0 \longrightarrow \Omega^1_B \otimes_B \longrightarrow B \longrightarrow B/[B,B] \longrightarrow 0$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\quad \Vert\ \text{trace}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\quad k$$

is exact.

Next let's return to our study of $A \otimes B$, in connection with the mixed complex $\Omega^1_R\otimes_R \rightleftarrows R$, but now let us suppose $B = \mathrm{End}(V)$. Recall that for $R = A\otimes B$ we established an exact sequence

$$(\Omega^1_A\otimes_A)\otimes(\Omega^1_B\otimes_B) \longrightarrow (\Omega^1_A\otimes_A)\otimes B \oplus A\otimes(\Omega^1_B\otimes_B) \longrightarrow \Omega^1_R\otimes_R \longrightarrow 0$$

From the exactness of $0\to\Omega^1_B\otimes_B \to B \to k \to 0$ we obtain an exact sequence on the top:

$$\cancel{\Omega^1_A\otimes_A \oplus A\otimes(\Omega^1_B\otimes_B) \quad \Omega^1_R\otimes_R}$$

$$\begin{tikzcd}
0 \arrow[r] & A\otimes(\Omega^1_B\otimes_B) \arrow[r] \arrow[d] & \Omega^1_R\otimes_R \arrow[r, "tr_V"'] \arrow[d] & \Omega^1_A\otimes_A \arrow[r] \arrow[l, dashed, bend right] & 0 \\
0 \arrow[r] & A\otimes B \arrow[r, phantom, "="] \arrow[d, "tr_V"] & R \arrow[r] & 0 \arrow[r] & 0 \\
& A & & &
\end{tikzcd}$$

We conclude that $(\Omega^1_R\otimes_R \to R) \overset{tr_V}{\text{quis}} (\Omega^1_A\otimes_A \to A)$ which is no surprise.

---

Tensor products: It seems to be hard to obtain the cyclic homology of $A\otimes B$ using $\mathrm{Bar}(\tilde{A})$-methods. Suppose $A, B$ are non-unital. Then we form the augmented algebras $\tilde{A}, \tilde{B}$ and their tensor product $\tilde{A}\otimes\tilde{B}$. It's clear that

$$\mathrm{Tor}_\bullet^{\tilde{A}\otimes\tilde{B}}(k,k) = \mathrm{Tor}_\bullet^{\tilde{A}}(k,k)\otimes\mathrm{Tor}_\bullet^{\tilde{B}}(k,k)$$

because a free $\tilde{A}\otimes\tilde{B}$ resolution of $k$ is obtained by tensoring $\tilde{A}$ and $\tilde{B}$ resolutions. It should also be clear that there is an obvious twisting

cochain from $Bar(\tilde{A}) \otimes Bar(\tilde{B})$ to $\tilde{A} \otimes \tilde{B}$ 
whence we have a map of DG coalgs.

$$Bar(\tilde{A}) \otimes Bar(\tilde{B}) \longrightarrow Bar(\tilde{A} \otimes \tilde{B})$$

October 1, 1988

Question: To what extent can we find analogues for $\mathcal{B}^N(A)$ of the coalgebra structure on $Bar(\tilde{A})$ and the identification of $CC(A)$ with the cocommutator subspace?

Let's recall that $\mathcal{B}^N(A)$ is the standard normalized bar resolution of $A$ as an $A$-bimodule:

$$\longrightarrow A\otimes\bar{A}^{\otimes 2}\otimes A \longrightarrow A\otimes\bar{A}\otimes A \longrightarrow A\otimes A \dashrightarrow A \dashrightarrow 0$$

and that for any $A$-bimodule $M$ we have

$$\mathrm{Hom}_{A\otimes A^o}\left(\mathcal{B}^N(A), M\right) = C_N^{\cdot}(A,M)$$

is the complex of normalized Hochschild cochains with values in $M$. On these cochains we have a cup product

$$C_N^{\cdot}(A,M)\otimes C_N^{\cdot}(A,N) \longrightarrow C_N^{\cdot}(A, M\otimes_A N)$$

which is associative. Corresponding to this cup product is a bimodule complex map

$$\mathcal{B}^N(A) \xrightarrow{\ \Delta\ } \mathcal{B}^N(A)\otimes_A \mathcal{B}^N(A)$$

$$\Delta(a_0,\dots,a_{n+1}) = \sum_{p=0}^{n} (a_0,\dots,a_p,1)\otimes_A(1,a_{p+1},\dots,a_{n+1})$$

---

The problem is to see whether one can use $\mathcal{B}^N(A)$ which is a sort of DG coalgebra over $A$ to do cyclic theory. For example if $B$ is an $A$-bimodule with a bimodule map $B\otimes_A B \longrightarrow B$ making it an algebra, then $\mathrm{Hom}_{A\otimes A^o}(\mathcal{B}^N(A), B) = C_N^{\cdot}(A,B)$

is a DG algebra. One would then like to do connection and curvature games in this algebra of cochains. Notice that these cochains are normalized (vanish if one of the arguments [redacted] = 1), so one might hope to obtain reduced cyclic cochains.

Also classes of operators can be defined by conditions on their kernels, so it seems natural to use bimodules when dealing with operators.

# October 2, 1988

Consider a Fredholm module situation $(A, H, F)$, ungraded case. Recall that we have the (bar) cochains and Hochschild cochains

$$\tau\left(F[F,\theta]^{2n}\right) \in C^{2n}(A)$$

$\tau =$ trace on $H$

$$\tau\left(F[F,\theta]^{2n}\partial\theta\right) \in C^{2n}(A, A^*)$$

related by the formulas

$$\delta\tau\left(F[F,\theta]^{2n}\right) = \beta\tau\left(F[F,\theta]^{2n}\partial\theta\right)$$

$$\delta\tau\left(F[F,\theta]^{2n}\partial\theta\right) = \frac{\pm 1}{4(n+1)}\,\partial\,\tau\left(F[F,\theta]^{2n+2}\right)$$

Concretely we have the cochains

$$\varphi_{(n)}(a_1,\dots,a_{2n}) = \tau\left(F[F,a_1]\cdots[F,a_{2n}]\right)$$

$$\psi_{(n)}(a_0,a_1,\dots,a_{2n}) = \tau\left(Fa_0[F,a_1]\cdots[F,a_{2n}]\right)$$

which satisfy

$$\left(b'\varphi_{(n)}\right)(a_0,\dots,a_{2n}) = (1-t)\psi_{(n)}(a_0,\dots,a_{2n})$$

$t =$ forward shift

$$\left(b\psi_{(n)}\right)(a_0,\dots,a_{2n+1}) = \left(-\tfrac{1}{2}\right)\tau\left(F[F,a_0]\cdots[F,a_{2n+1}]\right)$$

$$= \frac{-1}{4(n+1)}\left(N\varphi_{(n+1)}\right)(a_0,\dots,a_{2n+1})$$

In fact notice that $\psi_{(n)}$ is a normalized Hochschild cochain and

$$\varphi_{(n)}(a_1,\dots,a_{2n}) = \psi_{(n)}(1,a_1,\dots,a_{2n})$$

$$= \tau\left(F[F,a_1]\cdots[F,a_{2n}]\right)$$

is cyclically invariant

Thus

$$B\psi_{(n+1)}(a_1,\dots,a_{2n+2}) = \sum_{i=1}^{2n+2} (-1)^{i-1} \psi_{(n+1)}(1, a_i, \dots, a_{2n+2}, a_1, \dots, a_{i-1})$$

$$= (2n+2)\, \psi_{(n+1)}(1, a_1, \dots, a_{2n+2})$$

Consequently the sequence of normalized Hochschild cochains

$$\psi_{(n)}(a_0,\dots,a_{2n}) = \tau\left(F a_0 [F,a_1] \cdots [F,a_{2n}]\right)$$

satisfies

$$\boxed{\begin{aligned} b\psi_{(n)}(a_1,\dots,a_{2n+2}) &= \left(-\tfrac{1}{2}\right) \psi_{(n+1)}(1, a_1, \dots, a_{2n+2}) \\ B\psi_{(n+1)}(a_1,\dots,a_{2n+2}) &= (2n+2)\, \psi_{(n+1)}(1, a_1, \dots, a_{2n+2}) \end{aligned}}$$

which means that up to constants we obtain an even (odd?) cycle in the $(b,B)$ bicomplex made from the normalized Hochschild cochains.

---

Significance? It appears that what's important are certain Hochschild cochains which are normalized. Let's consider the analogous formulas for extensions

Suppose $\rho : A \longrightarrow R$, $\rho(1) = 1$, and $\rho(a_1)\rho(a_2) - \rho(a_1 a_2) \in I$. Basic Hochschild cochain is

$$\boxed{\psi_{(n)}(a_0,\dots,a_{2n}) = \tau\left(\rho(a_0)\, \gamma(a_1,a_2) \cdots \gamma(a_{2n-1}, a_{2n})\right)}$$

To do calculations I will go back to the system with $\delta = -b'$, so that $\gamma(a_1,a_2) = (\delta\rho + \rho^2)(a_1,a_2)$ $= \rho(a_1)\rho(a_2) - \rho(a_1 a_2)$. Then

$$\gamma(a_1a_2, a_3) - \gamma(a_1, a_2a_3)$$

$$= \rho(a_1a_2)\rho(a_3) - \cancel{\rho(a_1a_2a_3)} - \rho(a_1)\rho(a_2)\rho(a_3)$$
$$- \rho(a_1)\rho(a_2a_3) + \cancel{\rho(a_1a_2a_3)} + \underline{\qquad}$$

$$= \rho(a_1)\gamma(a_2, a_3) - \gamma(a_1, a_2)\rho(a_3)$$

Thus

$$b\psi_{(n)}(a_0, \ldots, a_{2n+1}) = \tau\{\rho(a_0a_1)\gamma(a_2,a_3)\cdots\gamma(a_{2n}, a_{2n+1})\}$$
$$-\tau\{\rho(a_0)[\rho(a_1)\gamma(a_2,a_3) - \cancel{\gamma(a_1,a_2)}\rho(a_3)]\gamma(a_4,a_5)\cdots\}$$
$$-\tau\{\rho(a_0)\gamma(a_1,a_2)[\rho(a_3)\cancel{\gamma(a_4,a_5)} - \gamma(a_3,a_4)\rho(a_5)]\cdots\}$$

$$\cdots$$

$$-\tau\{\rho(a_0)\gamma(a_1,a_2)\cdots\gamma(a_{2n-3}, a_{2n-2})[\rho(a_{2n-1})\cancel{\gamma(a_{2n}, a_{2n+1})} - \gamma(a_{2n-1}, a_{2n})\rho(a_{2n+1})]\}$$
$$-\tau\{\rho(a_{2n+1}a_0)\gamma(a_1,a_2)\cdots\gamma(a_{2n-1}, a_{2n})\}$$

$$\therefore \boxed{\begin{aligned} b\psi_{(n)}(a_0, \ldots, a_{2n+1}) = &-\tau\{\gamma(a_0,a_1)\cdots\gamma(a_{2n}, a_{2n+1})\} \\ &+\tau\{\gamma(a_{2n+1}, a_0)\cdots\gamma(a_{2n-1}, a_{2n})\}\end{aligned}}$$

also

$$\boxed{\begin{aligned} B\psi_{(n+1)}(a_0, \ldots, a_{2n+1}) &= \sum_{i=0}^{2n+1}(-1)^{i}\psi_{(n+1)}(1, a_i, \ldots, a_{2n+1}, a_0, \ldots, a_{i-1}) \\ &= (n+1)\big[\tau(\gamma(a_0,a_1)\cdots\gamma(a_{2n}, a_{2n+1})) \\ &\qquad - \tau(\gamma(a_{2n+1}, a_0)\cdots\gamma(a_{2n-1}, a_{2n}))\big]\end{aligned}}$$

Mystery: We start with the algebra homomorphism $A \to R/I$ and lift it to $\rho : A \to R$ with $\rho(1) = 1$. Then we can write the cochains

$$\psi_{(n)} \in C_N^{2n}(A, A^*)$$

$$\psi_{(n)}(a_0, \dots, a_{2n}) = \tau\big(\rho(a_0)\,\gamma(a_1, a_2)\cdots\gamma(a_{2n-1}, a_{2n})\big)$$

which turn out to be normalized Hochschild cochains. Now these are not Hochschild cocycles, so the problem is how to pin them down in a theoretical way so as to see their "classes" are independent of the choice of $\rho$, etc.

Now we have arrows

$$\overline{CC}_\cdot(A) \twoheadleftarrow \quad \underset{\text{(reduced Hochschild complex)}}{\square} \quad \hookleftarrow \Sigma\overline{CC}_\cdot(A)$$

which are part of a triangle. The injection is essentially the $B$ map. The cochain $\psi_{(n)}$ when restricted via this injection gives a cyclic cocycle, namely $tr(\gamma^n)$.

## October 3, 1988



The big mystery is how to interpret the Hochschild complex.

Suppose $A = k \oplus \mathfrak{a}$ is augmented. Then we have the DG coalgebra $Bar(A)$ with the canonical twisting cochain $\tau \in Hom_{\mathbb{C}}(Bar(A), A)$. We have the left DG-comodule

$$Bar(A) \otimes_\tau A$$

which is a free right-A-module resolution of $k$, and the right DG-comodule

$$A \otimes_\tau Bar(A)$$

which is a free left A-module resolution of $k$.

(We digress for the boundary formulas. Write $C = Bar(A)$. If $M$ is an A-module, then the DG comodule $C \otimes_{(-\tau)} M$ has the boundary

$$\tilde{d}(\omega \otimes m) = d\omega \otimes m + \sum_i (-1)^{\deg \omega_i'} \omega_i' \otimes \tau(\omega_i'') m$$

e.g.

$$\tilde{d}(a \otimes m) = 1 \otimes am \qquad \Delta a = a \otimes 1 + 1 \otimes a$$

$$\tilde{d}(a_1 \otimes a_2 \otimes m) = a_1 a_2 \otimes m - a_1 \otimes a_2 m \qquad \begin{aligned} &\Delta(a_1, a_2) \\ &= \begin{cases} (a_1, a_2) \otimes 1 \\ + a_1 \otimes a_2 \\ + 1 \otimes (a_1, a_2) \end{cases} \end{aligned}$$

Similarly if $M$ is a right A-module, then $M \otimes_{(-\tau)} C$ has the boundary

$$\tilde{d}(m \otimes \omega) = m \otimes d\omega - \sum_i m\tau(\omega_i') \otimes \omega_i''$$

e.g.

$$\tilde{d}(m \otimes a) = -ma$$

$$\tilde{d}(m \otimes a_1 \otimes a_2) = m \otimes a_1 a_2 - ma_1 \otimes a_2 \quad )$$

Then we obtain the standard normalized resolution of the $A$-bimodule $A$ as

$$B^N(A) = \left[A \otimes_{(-\tau)} Bar(A)\right] \boxtimes_{Bar(A)} \left[Bar(A) \otimes_{(+\tau)} A\right]$$

cotensor product.

The normalized Hochschild complex $B^N(A) \otimes_A$ is then (at least formally)

$$B^N(A) \otimes_A \cong \left[Bar(A) \otimes_\tau A \otimes_{(-\tau)} Bar(A)\right] \boxtimes_{Bar(A)}$$

$$\cong A \otimes_{(-\tau)} Bar(A) \otimes_\tau .$$

I claim this is the coalgebra analogue of $\mathrm{Cone}\{\Omega^1_R \otimes_R \to R\}$. In effect the coalg analogue of $\Omega^1_R$ is

$$0 \longrightarrow Bar(A) \xrightarrow{\Delta} Bar(A) \otimes Bar(A) \longrightarrow \mathrm{Coker}\,\Delta \longrightarrow 0$$

$$\mathrm{Coker}\,\Delta \cong Bar(A) \otimes \bar{A} \otimes Bar(A)$$

and one has a bimodule exact sequence

$$0 \longrightarrow \bar{A} \longrightarrow A \longrightarrow k \longrightarrow 0$$

which gives rise to an exact sequence

$$0 \to \underbrace{\bar{A} \otimes_{(-\tau)} Bar(A) \otimes_\tau}_{\substack{\text{Hoch cx,} \\ \text{of } \bar{A}}} \longrightarrow \underbrace{A \otimes_{(-\tau)} Bar(A) \otimes_\tau}_{B^N(A)} \longrightarrow \underbrace{k \otimes_{(-\tau)} Bar(A) \otimes_\tau}_{= Bar(A)} \longrightarrow 0$$

This means

$$B^N(A) = \mathrm{Fibre}\left\{ Bar(A) \longrightarrow \begin{matrix}\text{Hoch cx.} \\ \text{of } \bar{A}\end{matrix} \right\}$$

<u>Summary</u>: Starting from the bar construction of an augmented algebra, we can obtain the standard normaliz[redacted]ed resolution $B_{\otimes}^{N}(A)$ of $A$ as an $A\otimes A^{\circ}$-module by

$$B^N(A) = [A \otimes_{-\tau} \mathrm{Bar}(A)] \boxtimes_{\mathrm{Bar}(A)} [\mathrm{Bar}(A) \otimes_{\tau} A]$$

and this is consistent with viewing the Hochschild cx. of $\bar{A}$ as the coalg analogue of $\Omega^1_R \otimes_R$, where $R$ becomes $\mathrm{Bar}(A)$.

---

The difficulty to be overcome. We saw that the interesting cochains in the case of extensions, or Fredholm modules, are Hochschild cochains such as

$$\tau\{\rho(a_0)\gamma(a_1, a_2) \cdots \gamma(a_{2n-1}, a_{2n})\}$$

$$\tau\{Fa_0 [F, a_1] \cdots [F, a_{2n}]\}$$

These are not Hochschild cocycles, so we are faced with a problem of pinning down a cochain which is not a cocycle

Something similar is the Chern-Simons form, i.e. a transgression cochain, which seems to have a canonical nature, although it doesn't represent a cohomology class.

---

Let's discuss the [redacted] $(b, B)$ Hochschild bicomplex in cyclic theory with a view towards entire cyclic cohomology. Suppose we work with an augmented algebra + the reduced cyclic theory so that the $(b,B)$-Hochschild bicomplex can be identified with the bicomplex with exact rows:

$$\begin{tikzcd}
\cdot & \cdot & \cdot & \\
\cdot \arrow[u, "b"] \arrow[r, "1-t"] & \cdot \arrow[u, "b"] \arrow[r, "N"] & \cdot \arrow[u, "b"] \arrow[r] & {} \\
\cdot \arrow[u, "b"] \arrow[r, "1-t"] & \cdot \arrow[u, "b"] \arrow[r, "N"] & \cdot \arrow[u, "b"] \arrow[r] & {}
\end{tikzcd}$$

Given a cyclic $n$-cocycle $\alpha$, it gives rise to a family of cocycles in the double complex, namely, we can put $\alpha$ in the positions $(2k, n)$ for $k = 0, 1, \dots$ and put 0's at the other points of the same degree $2k+n$. Because the rows are exact $\alpha$ in position $(2k, n)$ is cohomologous to a cyclic cocycle of degree $2k+n$.

Let's next consider the completely periodic complex in the UHP. If we allow cochains with infinitely many components, the cohomology is zero, because using the exact rows we can write an cocycle as a coboundary. On the other hand if we consider only cochains with finitely many components, then we simply get the inductive limit of $HC^{*+2k}(A)$ under the S-maps.

Entire theory looks at cochains with certain growth conditions. We need some examples. I guess the place to start is with idempotents, where $A = \mathbb{C} + \mathbb{C}e$, because these have to pair with entire cyclic cocycles.

Let $R = \mathbb{C}[X] \longrightarrow A = \mathbb{C}[e]$, $X \mapsto e$ and on $A$ consider the trace $\tau$ with $\tau(1) = 0$ $\tau(e) = 1$. Thus $\tau(f(X)) = f(1) - f(0)$ for $f \in R/I^n$.

Choose $\rho: A \to R$ with $\rho(1)=1$, $\rho(e)=X$ whence

$$\gamma(e,e) = \rho(e)^2 - \rho(e^2) = X^2 - X$$

The cyclic cocycles on $\bar{A} = \mathbb{C}e$ we are after are of Chern-Simons type. This means

$$\int_0^1 dt \ \mathrm{tr}\left(\dot{\rho}_t \, \gamma_t^n / n!\right)$$

where $\rho_t(e) = tX$, $\gamma_t(e,e) = t^2X^2 - tX$,

and trace means using $\tau(f(X)) = f(1) - f(0)$ on $R$ together with $N$. Thus we get the cochain sending $(e, \dots, e)$ $2n+1$ times into

$$(2n+1)\int_0^1 dt \left[\frac{X\,(t^2X - tX)^n}{n!}\right]_{X=0}^{X=1}$$

$$= (2n+1)\int_0^1 dt \ \frac{(t^2-t)^n}{n!} = \frac{(2n+1)}{n!}(-1)^n \beta(n+1, n+1)$$

$$= \frac{2n+1}{n!}(-1)^n \frac{n!\,n!}{(2n+1)!} = (-1)^n \frac{n!}{2n!}$$

Now the double complex for $A = \mathbb{C} + \mathbb{C}e$ for cochains is

$$\begin{tikzcd}
\mathbb{C} & \arrow[r, "0"] & {} \arrow[r, "5"] & \mathbb{C} \arrow[r, dash] & {} \\
0 & \mathbb{C} \arrow[u, "1"] \arrow[r, "2"] & \mathbb{C} \arrow[u, "0"] \arrow[r, "0"] & \mathbb{C} \arrow[u, "1"] \arrow[r, dash] & {} \\
\mathbb{C} & \mathbb{C} \arrow[u, "0"] \arrow[r, "0"] & \mathbb{C} \arrow[u, "1"] \arrow[r, "3"] & \mathbb{C} \arrow[u, "0"] \arrow[r, dash] & {} \\
0 & \mathbb{C} \arrow[u, "1"] \arrow[r, "2"] & \mathbb{C} \arrow[u, "0"] \arrow[r, "0"] & \mathbb{C} \arrow[u, "1"] \arrow[r, dash] & {} \\
\mathbb{C} & \mathbb{C} \arrow[u, "0"] \arrow[r, "0"] & \mathbb{C} \arrow[u, "1"] \arrow[r, "1"] & \mathbb{C} \arrow[u, "0"] \arrow[r, dash] & {}
\end{tikzcd}$$

where we identify a cochain with its values on $(e, \dots, e)$

Let's check consistency with $S$ (up to sign at least). Take the cyclic cocycle of degree $2n$, call it $\alpha_{2n}$. Then place it in position $(2, 2n)$ and use the exact rows to move it to a cyclic $(2n+2)$-cocycle.

$$\begin{tikzcd}
\frac{(n+1)!}{(2n+2)!} = \pm\alpha_{2n+2} \arrow[d, equal] & & \\
\frac{n!}{2(2n+1)!} \arrow[r, mapsto, "2"] & \phantom{x} & (-1)^n\alpha_{2n} \arrow[d, equal] \\
& \frac{n!}{(2n+1)!} \arrow[u, mapsto, "1"] \arrow[r, mapsto, "2n+1"] & \frac{n!}{2n!}
\end{tikzcd}$$

So it works. What is important for entire cyclic theory are families of normalized Hochschild cochains of the same parity which are cycles for $b+B$. This means the same thing as a $(\infty)$cycle in the infinite (i.e. UHP) bicomplex.

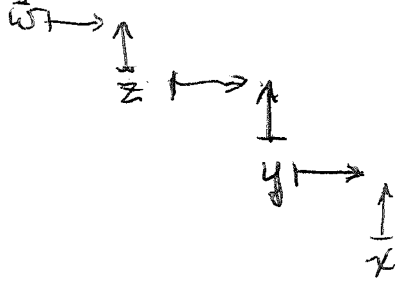
So let's discuss this further. Let's begin with the bicomplex in the UHP which has an obvious periodicity

$$\begin{tikzcd}
& \uparrow & \uparrow & \\
\arrow[r] & C^2 \arrow[r] & C^2 \arrow[r] & {} \\
& C^1 \arrow[u] \arrow[r, "1-t"] & C^2 \arrow[u] \arrow[r, "N"] & {} \\
\arrow[r] & C^0 \arrow[u] \arrow[r, "1-t"] & C^0 \arrow[u] \arrow[r, "N"] & {}
\end{tikzcd}$$

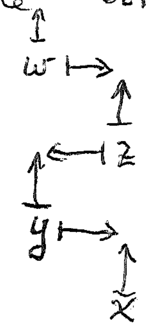
Sitting inside are the subcomplexes which are zero for column degrees $< k$ for different $k$.

In particular we have the 1st quadrant complex which is quis to the cyclic cochain complex.

Because of the periodicity, the complexes defined by a growth condition on the family of cochains have only odd & even cohomology. And we can obtain this cohomology by working in a $\mathbb{Z}_2$-graded complex consisting of two columns. Thus a cocycle in the UHP cx.



becomes



It's hard for me to see entire cyclic cocycles because up to now I've only dealt with cyclic co~~cycles~~. However a cyclic cocycle is a cycle in the double complex of the form:

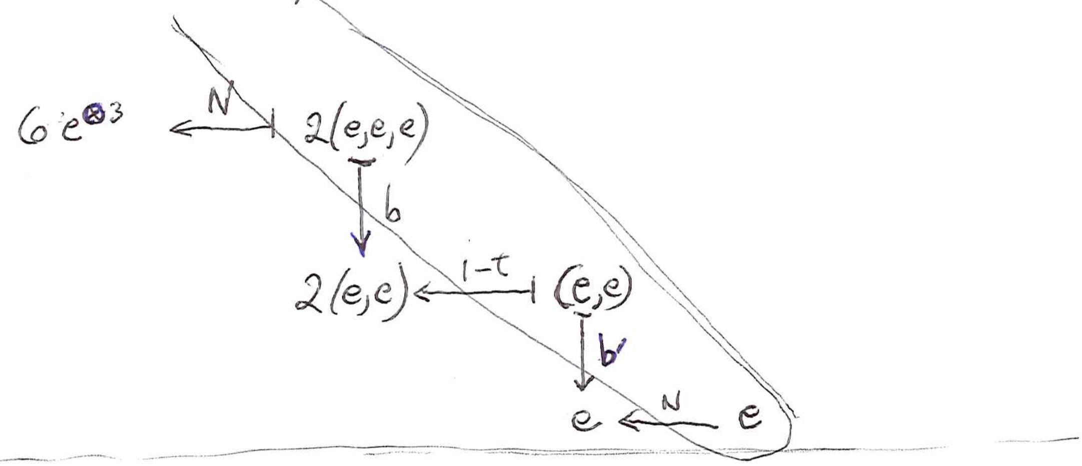
$$\begin{tikzcd} 0 \\ x \arrow[u] \arrow[r, mapsto] & 0 \end{tikzcd}$$

and we have seen that cycles in the first quadrant can be pushed into this form.

Let's try next to understand the pairing with idempotents. The cochain double complex is the dual of the double chain complex

$$\begin{tikzcd} & {} \arrow[d] & {} \arrow[d] & \\ {} & \bar{A}^{\otimes(n+1)} \arrow[l] \arrow[d, "b"] & \bar{A}^{\otimes(n+1)} \arrow[l, "1-t"'] \arrow[d, "b'"] & {} \arrow[l, "N"'] \\ {} & \bar{A}^{\otimes n} \arrow[l] \arrow[d] & \bar{A}^{\otimes(n)} \arrow[l, "1-t"'] \arrow[d] & {} \arrow[l, "N"'] \\ & {} & {} & \end{tikzcd}$$

Here the right quadrant complexes fit into an inverse system dual to the way the right-quadrant cochains form subcomplexes. An idempotent, being an algebra homomorphism $\mathbb{C}e \longrightarrow \bar{A}$, determines obvious infinite cycles in the chain double complex. These are




Thus we get

$$\begin{cases} \dfrac{(2n)!}{n!}\, e^{\otimes 2n+1} \in \bar{A}^{\otimes(2n+1)} & \text{Hoch cx.} \\[2ex] \dfrac{(2n)!}{2(n!)}\, e^{\otimes 2n} \in \bar{A}^{\otimes n} & \text{bar cx.} \end{cases}$$

Now it's clear that the only cochains which can be paired with the general infinite chain are finite cochains. However because the coefficients $\frac{2n!}{n!}$ grow roughly like $n!\,2^n$, the above cycle can be paired with cochains not growing too fast.

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Let's return to left-invariant forms on gauge transformation groups. Let's start with a Fredholm module situation $(A, H, F)$. Let

$$\mathcal{G} = U_n(A)\ , \quad \tilde{\mathfrak{g}} = \mathrm{Lie}(\mathcal{G})_c = \mathfrak{gl}_n(A).$$



These act on $V = H^n$; let $W = (eH)^n$ where $e = \frac{1+F}{2}$. Then we have a map

$$\mathcal{G} \longrightarrow Gr(V) \qquad g \longmapsto gW.$$

which is $\mathcal{G}$ equivariant, and pulling back the character forms on the Grassmannian leads to left-invariant forms on $\mathcal{G}$. I recall how we calculate these. The character forms are associated to the canonical subbundle with its Grassmannian connection. When pulled by to $\mathcal{G}$ the subbundle becomes isomorphic to $\tilde{W}$, the isomorphism of $W$ with the fibre of the subbundle at $g$, namely $gW < V$, being given by $g$. We have the embedding

$$i : \tilde{W} \longrightarrow \tilde{V}$$

$$i_g = g\iota_0 \qquad \iota_0 \text{ inclusion of } W \text{ in } V.$$

and the induced connection is

$$i^* \cdot d_V i = \iota_0^* g^{-1} \cdot dg\, \iota_0 = d_W + \iota_0^* g^{-1}(dg)\iota_0$$

where here $g$ denotes the tautolical $\mathrm{Aut}(V)$ function on $\mathcal{G}$. Thus the connection form on $\tilde{W}$ is obtained by taking the Maurer–Cartan form $g^{-1}dg$ $\in \Omega^1(\mathcal{G}, \mathrm{End}(V))^{\mathcal{G}}$ and projecting down to obtain a 1-form on $\mathcal{G}$ with values in $\mathrm{End}(W)$, which is left-invariant.

Therefore one works in the algebra of left-invariant differential forms on $\mathcal{G}$ with values in the algebra $\mathrm{End}(W)$ on which $\mathcal{G}$ operates trivially. But there appears to be an additional action of $U_n$ or $\mathfrak{gl}_n(\mathbb{C})$ on the whole picture

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Let $\mathcal{G}$ act on a Hilbert space $V$, let $W$ be a subspace and let $G$ be a subgroup of $\mathcal{G}$ fixing $W$. Then $\mathcal{G}$ acts on the Grassmannian $Gr(V)$ of subspaces isom. to $W$ and the canonical subbundle $S$ over $Gr(V)$ ~~is~~ an equivariant bundle for $\mathcal{G}$; the Grassmannian connection on $S$ is $\mathcal{G}$-invariant.

Using the $\mathcal{G}$-map

$$\mathcal{G}/G \xrightarrow{f} Gr(V)$$

sending the identity coset to the subspace $W$, we obtain over $\mathcal{G}/G$ a $\mathcal{G}$-bundle $f^*(S)$ equipped with a $\mathcal{G}$-invariant connection. The bundle $f^*S$ can be identified with the induced bundle $\mathcal{G}\times^G W$. The connection comes from the embedding

$$\mathcal{G}\times^G W \hookrightarrow \mathcal{G}/G \times V$$

(*)

$$(\tilde{g}, w) \longmapsto (\tilde{g}G, \tilde{g}w)$$

Since we have a $\mathcal{G}$-bundle $f^*(S)$ over $\mathcal{G}/G$ we can consider its algebra of endomorphism-valued forms. The connection $\nabla$ on $f^*(S)$ will give rise to a derivation $[\nabla, \ ]$ on End-valued forms, but there will be curvature. Finally, because $\mathcal{G}$ acts on $f^*(S)$ preserving $\nabla$, we can consider $\mathcal{G}$-left-invariant forms. This, because $\mathcal{G}$ acts transitively on $\mathcal{G}/G$, gives us the algebra

$$\mathrm{Hom}\left(\Lambda^{\cdot}(\tilde{\mathfrak{g}}/\mathfrak{g}), \mathrm{End}\, W\right)^{\mathfrak{g}}.$$

We can compute the effect of $[\nabla, \ ]$ as follows. First note that the induced bundle over $\mathcal{G}/G$ associated to the $G$-action on $V$ can be identified with the trivial bundle:

$$\mathcal{G} \times^G V \xrightarrow{\sim} (\mathcal{G}/G) \times V$$
$$(\hat{g}, v) \longmapsto (\hat{g}G, \hat{g}v)$$

and hence the embedding $(*)$ is just the result of inducing the embedding $W \hookrightarrow V$ of $G$-representations, and then using the above identification.

Thus we have

$$\mathrm{Hom}(\Lambda^\bullet(\tilde{\mathfrak{g}}/\mathfrak{g}), \mathrm{End}\, W)^{\mathfrak{g}} = i^* \,\mathrm{Hom}(\Lambda^\bullet(\tilde{\mathfrak{g}}/\mathfrak{g}), \mathrm{End}\, V)^{\mathfrak{g}} i$$

where $i$ is the inclusion of $W$ in $V$. I think this means we have the usual picture about end-valued forms for a bundle embedded in a trivial bundle.

Next let's consider "the" principal bundle of $f^*S = \mathcal{G} \times^G W$. "The" principal bundle for $Gr(V)$ can be viewed as the space of isometric embeddings of $W$ into $V$. $\mathcal{G}$ acts on the left and $\mathrm{Aut}(W)$ acts on the right. We have a $\mathcal{G}$-map

$$\mathcal{G} \longrightarrow \mathrm{Emb}(W, V)$$

sending $1$ to the inclusion $i : W \hookrightarrow V$, hence upon lifting $f^*S$ to $\mathcal{G}$ it becomes canonically isomorphic to the trivial bundle $\mathcal{G} \times W$. Since the purpose of the principal bundle is to represent trivializations, we might as well work with the principal $G$-bundle $\mathcal{G}$ over which $f^*S$ has a canonical trivialization.



Thus let $\pi : \mathcal{G} \to \mathcal{G}/G$ be the canonical map, whence we have

$$\pi^* f^* \mathcal{S} = \mathcal{G} \times W$$

with $\mathcal{G}$ acting by left multiplication on the first factor $\mathcal{G}$, and $G$ acting, so as to form $\mathcal{G} \times^G W$.

~~Now that~~

Let's recapitulate. We have over $\mathcal{G}/G$ the bundle $f^*(\mathcal{S})$ which is a $\mathcal{G}$-vector bundle and which has an invariant connection. We are interested in the $\mathcal{G}$-left-invariant character forms associated to this $\mathcal{G}$-bundle with invariant connection. Hence we are interested in the endo-valued forms for $f^*\mathcal{S}$ which are $\mathcal{G}$-invariants.

We have the $\mathcal{G}$ map $\pi : \mathcal{G} \to \mathcal{G}/G$ which is a [redacted] principal [redacted] $G$-bundle, and further we have the trivialization

$$\pi^* f^* \mathcal{S} = \mathcal{G} \times W.$$

Now the trivialization enables us to calculate characteristic forms upstairs. Specifically, end-valued forms for $f^*\mathcal{S}$ are $G$-basic End-valued forms for $\mathcal{G}\times W$ on $\mathcal{G}$. Basic forms are forms which are both horizontal and invariant. The $\mathcal{G}$-invariant End-valued forms for $\mathcal{G}\times W$ lie in the algebra

$$\mathrm{Hom}(\Lambda(\tilde{\mathfrak{g}}), \mathrm{End}\, W)$$

and this algebra is a DG cochain algebra. The $G$-invariant end-valued forms for $\mathcal{G} \times W$ lie in the subalgebra

$$\mathrm{Hom}\left(\Lambda^\circ(\tilde{\mathfrak{g}}), \mathrm{End}\, W\right)^G$$

which is also a DG cochain algebra and finally the basic forms lie in

$$\mathrm{Hom}\left(\Lambda^\circ(\tilde{\mathfrak{g}}/\mathfrak{g}), \mathrm{End}\, W\right)^G$$

which is an algebra with derivation coming from the connection.

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Summary: $\mathcal{G}$ acts on the Hilb space $V$, $G$ subgp of $\mathcal{G}$ leaving the subspace $W$ of $V$ invariant. Then over $\mathcal{G}/G$ we have the induced vector bundle $\mathcal{G}\times^G W$; it's a $\mathcal{G}$-vector bundle over $\mathcal{G}/G$. It also has a canonical embedding in the trivial bundle with fibre $V$:

$$\mathcal{G}\times^G W \hookrightarrow \mathcal{G}\times^G V \xrightarrow{\sim} (\mathcal{G}/G)\times V$$
$$(\hat{g}, v) \longmapsto (\hat{g}G, \hat{g}v)$$

so there is a $\mathcal{G}$-invariant Grassmannian connection in $\mathcal{G}\times^G W$.

When this connection on $\mathcal{G}\times^G W$ is pulled up to a connection on $\mathcal{G}\times W$, the connection form $\rho$ is $i^*\theta i$, where $\theta\in\Omega^1(\mathcal{G}, \mathrm{End}(V))$ is the M-C form of $\mathcal{G}$ as it acts on $V$, and where $i: W\to V$. Connection-curvature calculations for $\mathcal{G}\times^G W$ can be done in the $(\mathcal{G},G)$-invariant forms for $\mathcal{G}\times W$, which is

$$\mathrm{Hom}\left(\Lambda^\circ(\tilde{\mathfrak{g}}), \mathrm{End}\, W\right)^G$$

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Next let's consider the main example where we start with $\left(A, H, e = \frac{1+F}{2}\right)$ and take

$$\mathcal{G} = U_n(A), \quad V = \mathbb{C}^n \otimes H, \quad W = \mathbb{C}^n \times cH$$
$$G = U_n(\mathbb{C}), \quad \tilde{g} = gl_n \otimes A, \quad g = gl_n$$

We are then interested in the algebra

$$\mathrm{Hom}\Big(\Lambda^{\cdot}(g \otimes A), \overset{M_n}{\overset{\parallel}{g}} \otimes \mathrm{End}(cH)\Big)^{g}$$

$$= \mathrm{Hom}\Big(\big(\Lambda^{\cdot}(g \otimes A) \otimes M_n^*\big)_g, \mathrm{End}(cH)\Big)$$

of $\mathcal{G}, G$-invariant $\mathrm{End}(W)$-forms. This algebra contains the connection and curvature forms.

Notice that $M_n^*$ being the dual of an algebra is a coalgebra. So

$$\big(\Lambda^{\cdot}(g \otimes A) \otimes M_n^*\big)_g \quad \text{is a DG coalgebra.}$$

Now let's apply invariant theory as in L-Q+F-T

$$\big(\Lambda^p(g \otimes A) \otimes g^*\big)_g = \big(g^{\otimes p} \otimes A^{\otimes p} \otimes \overset{g}{\overset{\parallel}{g^*}}\big)_{g \times \Sigma_p}$$

Writing $g = V \otimes V^*$, $V = \mathbb{C}^n$, we take $g$-coinvariants by contracting $V$ against $V^*$. There are $(p+1)!$ possible ways of contracting, and these are permuted around by $\Sigma_p$.

One can think of having $p+1$ vertices one of which is marked (it corresponds to $g^*$) and the others are attached to a copy of $A$. ~~[illegible]~~ Each contraction scheme is a disjoint union of loops. Look at the loop with the marked vertex. Up to ~~symmetric~~ permutations it can be arranged ~~so~~ as ~~[illegible]~~ the consist of the marked vertex and the first group of $A$-vertices. The remaining loops have cyclic stabilizers, so they lead to the L-Q+F-T cyclic type chains ~~[illegible]~~.

The conclusion is that our coalgebra is the tensor product of the Hopf algebra (we are in stable range) [redacted] $\left(\mathring{\Lambda}(g \otimes A)\right)_{\mathfrak{g}}$ of LQFT and a tensor coalgebra on $A$. This should be the bar construction on $\tilde{\tilde{A}}$.

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New project. Can we link up the Chern-Simons cocycles which give the even Connes homomorphisms

$$HC_{2n}(A) \longrightarrow HC_0(R/I^{n+1})$$

to Connes graded $F$ cocycles.

Let's begin by reviewing the odd cocycles.

Consider the algebra [redacted] $A * \mathbb{C}[F] = (A*A)\tilde{\otimes}\mathbb{C}[F]$ and the DG algebra $C^{\cdot}(A, A*\mathbb{C}[F])$ of cochains. The embedding $A \longrightarrow A*\mathbb{C}[F]$ gives a 1-cochain $\theta$ which is flat: $\delta\theta + \theta^2 = 0$. Then over this DG algebra we have a "flat connection" $\delta + \theta$ and a splitting $F$ so we can associate character forms provided we have a trace. Let $\tau$ be a trace defined on some power of the ideal $K$ in $R = A * \mathbb{C}[F]$ generated by commutators $[F,a]$. Then for large $n$ we have cyclic cocycles

$$\frac{(-1)^n}{2 \cdot n!}\, tr\left(F\alpha^{2n}\right) = \frac{1}{n!\, 2^{2n+1}}\, tr\left(F[F,\theta]^{2n}\right)$$

where trace means [redacted] $\tau$ combined with $N$. Recall

$$\delta + \theta = \underbrace{\delta + \frac{\theta + F\theta F}{2}}_{\nabla} + \underbrace{\frac{\theta - F\theta F}{2}}_{\alpha}$$

$$\nabla^2 = -\alpha^2 = -\left(\tfrac{1}{2}F[F,\theta]\right)^2 = \tfrac{1}{4}[F,\theta]^2$$

Recall also that $R = A * \mathbb{C}[F]$ has the block decomposition

$$R = \begin{pmatrix} eRe & eR\bar{e} \\ \bar{e}Re & \bar{e}R\bar{e} \end{pmatrix} \quad \text{where} \quad F = \begin{pmatrix} e & 0 \\ 0 & -\bar{e} \end{pmatrix}$$

Then $$K = \begin{pmatrix} eR\bar{e}Re & eR\bar{e} \\ \bar{e}Re & \bar{e}ReR\bar{e} \end{pmatrix} \qquad K^2 = \begin{pmatrix} eR\bar{e}Re & eR\bar{e}ReR\bar{e} \\ \bar{e}ReR\bar{e}Re & \bar{e}ReR\bar{e} \end{pmatrix}$$

Let $B = eRe$; then $B \xleftarrow{\sim} T(A)/(1 = \rho(1_A))$ with $\rho : A \to B$ given by $a \mapsto eae$. Let $I = eR\bar{e}Re = \mathrm{Ker}\{B \twoheadrightarrow A\}$.

If we have a trace $\tau$ defined on $K^{2n}/[K, K^{2n-1}]$ then $\tau$ has to vanish on the off-diagonal blocks. Hence it's supported on the diagonal blocks $I^n$ and $\bar{I}^n$. I recall showing that

$$K^{2n}/[K, K^{2n-1}] \xleftarrow{\sim} I^n/[I, I^{n-1}]$$

so that traces on $K^{2n}/[\,,\,]$ & on $I^n/[\,,\,]$ coincide.

I think that our cyclic cocycle $\frac{1}{n!\, 2^{2n+1}} \mathrm{tr}(F[F,e]^{2n})$ coincide with the cocycle defined via $\rho : A \to B$ $\gamma = \delta\rho + \rho^2$ [redacted] by the formula

$$\mathrm{tr}\left(\frac{\gamma^n}{n!}\right)$$

The lesson to be learned is somehow that [redacted] the GNS construction is kind of a formal Narasimhan–Ramanan theorem, ~~so~~ it enables one to assume a connection is Grassmannian, hence the curvature is a quadratic effect due to interaction with the complement.

Next let's discuss the graded case. Here the algebra one initially expects to deal with is the algebra which would act in the case of a Fredholm module situation $(A, H, \varepsilon, F)$, namely $(A * \mathbb{C}[F]) \tilde{\otimes} \mathbb{C}[\varepsilon] = (A * A) \hat{\otimes} (\mathbb{C}[F] \tilde{\otimes} \mathbb{C}[\varepsilon])$. Moreover the kind of trace one deals with is $tr_H(\varepsilon\, ?)$, a supertrace on the superalgebra tensor product $(A*A) \hat{\otimes} (\mathbb{C}[F] \tilde{\otimes} \mathbb{C}[\varepsilon])$. Now by Morita invariance we are realing dealing with a supertrace on the superalgebra $A*A$, or really defined on a power of the aug. ideal. NO. This isn't correct, because as operators on $H$, $A*A$ is of even degree.

Let's begin again with $H = H^+ \oplus \overset{= H^-}{\widetilde{H^{+}}}$, $F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and with $A$ acting by even operators. Then $a \mapsto \begin{pmatrix} a^+ & 0 \\ 0 & a^- \end{pmatrix}$ and $FaF \longmapsto \begin{pmatrix} a^- & 0 \\ 0 & a^+ \end{pmatrix}$ so that we have two actions of $A$ on $H^+$. Thus we have $A*A$ acting on $H^+$, and the action on the odd copy $H^-$ of $H^+$ is the flip. Now what is $tr_H^{\varepsilon} = tr_{H^+} - tr_{H^-}$ on $A*A$. One expects in any normal situation that $tr_{H^+}(\alpha) = tr_{H^-}(F\alpha F)$. Thus we have a trace $\tau$ given on $A*A$ and our supertrace is $\tau - \tau^F$. Thus we are interested in traces on $A*A$ (really on powers of the ideal $\operatorname{Ker}\{A*A \longrightarrow A\}$) which are odd.

We are still no closer to bringing in Chern-Simons.