July 15, 1988

Ideas: (1) The fundamental mystery in cyclic theory is how to reconcile the approaches given by index theory and Lie algebra cohomology of matrices. For example, consider the restricted Grassmannian of involutions congruent to a fixed involution modulo a Schatten ideal. If we fix a basepoint we obtain ~~left~~-invariant even degree differential forms on the restricted unitary group. So we obtain odd-degree cyclic cocycles φ_{2n+1} for $n \geq 0$. Given an element g of the restricted unitary group, it has an index. A basic result is that the index is essentially

$$\varphi_{2n+1}(g, g^{-1}, \ldots, g, g^{-1})$$

The problem is what has this to do with φ being related to the character form on the Grassmannian? Notice one is using that the unitary group can be embedded in its complexified Lie algebra. So what is the significance of the fact that GL_n can be embedded in its Lie algebra?

(2) Let $\not{D}_0 = X_0$ be an ungraded Dirac and g a gauge transformation, so there is an index defined. An analytic formula for the index is

$$(*) \qquad \text{Index} = tr\,(g^{-1}Fg - F)$$

where $F = \tau(X_0)$ and $\tau(p)$ approximates $sgn(p)$. What's the simplest τ and how is the formula $(*)$ related to the others you have found recently by using a path joining $\not{D}_0$ to $g^{-1}\not{D}_0 g$, or the family

$$\begin{pmatrix} g^{-1}X_0 g & 0 \\ 0 & -X_0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and the superconnection type 1-form?

July 16, 1988

Let's start with the index formula

$$\text{Index} =$$

$$= \frac{1}{2} \operatorname{tr}\left(g^{-1}Fg - F\right)$$

where $F = \varphi(\frac{1}{i}\not{D})$ and $\varphi(x)$ is a smooth approximation to $\operatorname{sgn}(x)$. On the circle this formula holds because the Hilbert involution commutes with functions modulo smooth kernel operators. In general there will be problems with the existence of the trace.

Let $\not{D}_t$ be a path joining $\not{D}$ to $g^{-1}\not{D}g$. Then $\varphi(\frac{1}{i}\not{D}_t)$ joins $\varphi(\frac{1}{i}\not{D}) = F$ to $\varphi(\frac{1}{i}g^{-1}\not{D}g) = g^{-1}Fg$.

Assuming that

$$\operatorname{tr}\left\{\varphi(\tfrac{1}{i}\not{D}_t) - \varphi(\tfrac{1}{i}\not{D})\right\}$$

exists and the usual ~~differentiation~~ formulas work, namely

$$\partial_t \operatorname{tr}\left\{\varphi(\tfrac{1}{i}\not{D}_t) - \varphi(\tfrac{1}{i}\not{D})\right\} = \operatorname{tr}\left\{\varphi'(\tfrac{1}{i}\not{D}_t)\tfrac{1}{i}\partial_t\not{D}_t\right\}$$

we obtain

$$\text{Index} = \frac{1}{2}\int_0^1 \frac{1}{i}\operatorname{tr}\left(\varphi'(\tfrac{1}{i}\not{D}_t)\,\partial_t\not{D}_t\right) dt$$

So if we take

$$\varphi(x) = \int_0^x e^{-ux^2} \frac{2\sqrt{u}}{\sqrt{\pi}}\, dx$$

we obtain our previous formula

$$\text{Index} = \int_0^1 \frac{\sqrt{u}}{i\sqrt{\pi}} \operatorname{tr}\left(e^{u\not{D}_t^2}\, \partial_t \not{D}_t\right) dt$$

derived <u>using</u> superconnection theory.

Relation with the η-invariant. Set $A = \frac{1}{i}\not{D}$ so that A is self-adjoint. [redacted] Recall the formulas for A such that $\operatorname{Ker}(A) \neq 0$.

$$\eta(A,s) = \operatorname{Tr}\left(\frac{A}{|A|}|A|^{-s}\right) \qquad \eta(A) = \eta(A,0)$$

$$\delta\eta(A,s) = -s\operatorname{Tr}\left(\delta A\,(A^2)^{-\left(\frac{s+1}{2}\right)}\right)$$

$$= -s\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty \operatorname{Tr}\left(\delta A e^{-tA^2}\right) t^{\frac{s+1}{2}}\frac{dt}{t}$$

$$= -s\frac{1}{\Gamma\left(\frac{s+1}{2}\right)}\int_0^\infty t^{\frac{1}{2}}\operatorname{Tr}\left(e^{-tA^2}\delta A\right) t^{\frac{s}{2}}\frac{dt}{t}$$

Recall that

$$\frac{i(2i)^{1/2} t \operatorname{Tr}\left(e^{-tA^2}\delta A\right)}{(-2\pi i t)^{1/2}} = \frac{t^{1/2}}{\sqrt{\pi}}\operatorname{Tr}\left(e^{-tA^2}\delta A\right)$$

is supposed to have integral periods. [redacted] Let's assume we have an asymptotic expansion (true for Dirac operators on odd manifolds)

$$t^{1/2}\operatorname{Tr}\left(e^{-tA^2}\delta A\right) = c_{-m}t^{-m} + \cdots + c_{-1}t^{-1} + c_0 + \cdots$$

and put $c_0 = \left[t^{1/2}\operatorname{Tr}\left(e^{-tA^2}\delta A\right)\right]_{(0)}$. Then we have

$$\delta\eta(A) = (-2)\left[\frac{t^{1/2}}{\sqrt{\pi}}\operatorname{Tr}\left(e^{-tA^2}\delta A\right)\right]_{(0)}$$

The moral appears to be that the η-invariant is essentially an integral for the supercon. 1-form

$$\frac{u^{1/2}}{\sqrt{\pi}} \operatorname{Tr}\left(e^{-uA^2}\delta A\right)$$

in the "limit" $u \downarrow 0$.

July 18, 1988

Let's return to the proof of the fact that Connes cocycles are related by the S-operation. Recall first the Connes cocycles in the ungraded case.

We suppose given a homomorphism of the algebra A to an algebra C which contains an involution F, an ideal K such that $[F,a] \in I$ for all $a \in A$, and such that there is a linear functional tr given on $K^{2n}/[K, K^{2n-1}]$. Let $\theta \in C^1(A,C)$ be the homomorphism, whence we have the "flat connection" $\delta + \theta$ which can be split into parts commuting and anti-commuting with respect to F

$$\delta + \theta = \underbrace{\left(\delta + \frac{\theta + F\theta F}{2}\right)}_{\nabla} + \underbrace{\left(\frac{\theta - F\theta F}{2}\right)}_{\alpha}$$

One has $[\nabla, \alpha] = 0$ and $\nabla^2 = -\alpha^2$.
Then $\alpha = \frac{1}{2}(\theta - F\theta F) = \frac{1}{2}F[F,\theta] \in C^1(A,K)$, and using the trace given on K^{2n}, one has a cyclic cochain

$$tr\,(F\alpha^{2n}) \in C_\lambda^{2n-1}(A)$$

defined. This is a cocycle because

$$\delta\, tr\,(F\alpha^{2n}) = tr\,[\nabla, F\alpha^{2n}] = 0$$

If we have a family of involutions $F = F_t$ depending on t, then

$$\partial_t\, tr\,(F\alpha^{2n}) = \cancel{tr\,(\underbrace{\dot{F}}_{\text{anti with } F}\alpha^{2n})} + tr\Big(F \underbrace{\sum_{i=0}^{2n-1} \alpha^i \dot{\alpha}\, \alpha^{2n-1-i}}_{\text{sign } (-1)^i}\Big) \qquad \text{sign } (-1)^{i(2n-i)} = (-1)^i$$

$$= 2n\; tr\,(F\dot{\alpha}\,\alpha^{2n-1})$$

$$= 2n\; tr\Big(F\; \frac{\dot{\alpha} - F\dot{\alpha}F}{2}\; \alpha^{2n-1}\Big)$$

But

$$\dot{\alpha} = \left(-\tfrac{1}{2}\right)\left(\dot{F}DF + FD\dot{F}\right)$$

$$= \left(-\tfrac{1}{2}\right)\Big(\underbrace{\dot{F}\nabla F + F\nabla\dot{F}}_{\text{antic. w. } F} + \underbrace{\dot{F}\alpha F + F\alpha\dot{F}}_{\text{comm. w. } F}\Big)$$

so $$\frac{\dot{\alpha} - F\dot{\alpha}F}{2} = \left(-\tfrac{1}{2}\right)\left(\dot{F}\nabla F + F\nabla\dot{F}\right) = \left(-\tfrac{1}{2}\right)[\nabla, F\dot{F}]$$

so $$\partial_t \, tr(F\alpha^{2n}) = 2n\left(-\tfrac{1}{2}\right) tr\Big(\underbrace{F[\nabla, F\dot{F}]\,\alpha^{2n-1}}_{[\nabla \text{[redacted]}, FF\dot{F}\alpha^{2n-1}]}\Big)$$

or $$\boxed{\partial_t \, tr(F\alpha^{2n}) = \delta(-n)\, tr(\dot{F}\alpha^{2n-1})}$$

Now that we have reviewed the formulas for Connes cocycles let's consider the deformation which leads to S. Given F we consider the deformation

$$\tilde{F} = \frac{1}{\sqrt{1+t^2}}\left(\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix} + \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}\right)$$

$$= \frac{1}{\sqrt{1+t^2}} F\varepsilon + \frac{t}{\sqrt{1+t^2}}\gamma^1$$

and we use the homomorphism from A to 2×2 matrices over C given by

$$\tilde{a} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} = a\left(\frac{1+\varepsilon}{2}\right)$$

Then $$[\tilde{F}, \tilde{a}] = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} [F, a] & -ta \\ ta & 0 \end{pmatrix}$$

so the cocycle associated to $\tilde{F}, \tilde{\Theta}$ is up to normalization factors

$$tr\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\frac{1}{(1+t^2)^{n-1/2}}$$

The class is independent of t; the coefficients of powers of t of the trace are S-transforms of lower Connes cocycles attached to F, θ. This gives the desired relations.

There's a problem with the existence of traces which means one must work with t an infinitesimal variable. The goal is now to set this up in a rigorous fashion.

What do we adjoin to C in order to have $\tilde{F}$ and $\tilde{\theta}$? We adjoin ε and $t\gamma'$. It is better to introduce the sine and cosine

$$s = \frac{t}{\sqrt{1+t^2}} \qquad c = \frac{1}{\sqrt{1+t^2}}$$

so that

$$\tilde{F} = cF\varepsilon + s\gamma'.$$

Thus we consider the algebra generated by $c, \varepsilon, s\gamma'$.

Put

$$z = c + is\gamma'$$

Then

$$z^{-1} = c - is\gamma' \qquad \text{and} \qquad \varepsilon z \varepsilon = z^{-1}.$$

Thus the algebra generated by $c, \varepsilon, s\gamma'$ can be identified with the group algebra of the infinite dihedral group.

Better formula:

$$z = c + s\gamma'\varepsilon = \begin{pmatrix} c & -s\\ s & c\end{pmatrix}$$

$$z^{-1} = c - s\gamma'\varepsilon$$

So $\tilde{F}, \tilde{\theta}$ are defined over

$$C \otimes \mathbb{C}[z, z^{-1}, \varepsilon]$$

where $\mathbb{C}[z, z^{-1}]$ = cross product of $\mathbb{C}[z, z^{-1}]$ with $\mathbb{C}[\mathbb{Z}/2] = \mathbb{C} + \mathbb{C}\varepsilon$ with $\varepsilon z \varepsilon = z^{-1}$. We have

$$\tilde{\theta} = \theta\left(\frac{1+\varepsilon}{2}\right) \qquad \tilde{F} = cF\varepsilon + s\gamma^1$$

and

$$[\tilde{F}, \tilde{\theta}] = \left[cF\varepsilon + s\gamma^1, \theta\left(\frac{1+\varepsilon}{2}\right)\right]$$

$$= c[F, \theta]\left(\frac{1+\varepsilon}{2}\right) + \theta s \gamma^1 \varepsilon$$

We next need an ideal $\tilde{K}$ in $\tilde{C} = C \otimes \mathbb{C}[z, z^{-1}, \varepsilon]$ containing $[\tilde{F} \tilde{\theta}]$. We take

$$\tilde{K} = K \otimes \mathbb{C}[z, z^{-1}, \varepsilon] + C \otimes (z-1)$$

Clearly we need to have K in $\tilde{K}$ and also $s\gamma^1$ in $\tilde{K}$, which means that $z \equiv z^{-1} \equiv c \mod \tilde{K}$. But in the example $c = \frac{1}{\sqrt{1+t^2}} = 1$ when $t=0$, so we ought to take $z - 1 \in \tilde{K}$.

We need a notation for $\mathbb{C}[z, z^{-1}, \varepsilon]$; denote it Λ, and let Λ_0 be the ideal generated by $z-1$. We to find a trace on $\tilde{K}^{2n}$. We have

$$\tilde{K} = K \otimes \Lambda + C \otimes \Lambda_0$$

so

$$\tilde{K}^n = \sum_{p+q=n} K^p \otimes \Lambda_0^q$$

We propose to use the (partially-defined) trace we have on C times a suitable trace on Λ. It is then clear that the trace on Λ

we use must vanish on Λ_0^8 for some q in order that the trace be defined on $\tilde{K}^n$ for n large.

So we might as well complete Λ adically wrt. Λ_0. Let's now try describe the completion $\hat{\Lambda}$.

Recall Λ is generated by $c, s\vartheta', \varepsilon$ where $s\vartheta' = \frac{t}{\sqrt{1+t^2}}$, $c = \frac{1}{\sqrt{1+t^2}}$ if we work formally near $t=0$. Then $\hat{\Lambda} = \mathbb{C}[[t]] \tilde{\otimes} \mathbb{C}[\varepsilon]$ with $\varepsilon t \varepsilon = t^{-1}$. If we use the matrix representation

$$t \longmapsto \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} \qquad \varepsilon \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\mathbb{C}[[t]] \longmapsto \left\{ \begin{pmatrix} f(t^2) & tg(t^2) \\ tg(t^2) & f(t^2) \end{pmatrix} \right\}$$

and

$$\hat{\Lambda} = \mathbb{C}[[t]] \tilde{\otimes} \mathbb{C}[\varepsilon] \longmapsto \begin{pmatrix} \mathbb{C}[[t^2]] & t\mathbb{C}[[t^2]] \\ t\mathbb{C}[[t^2]] & \mathbb{C}[[t^2]] \end{pmatrix}$$

$$\hat{\Lambda}_0 = t\mathbb{C}[[t]] \tilde{\otimes} \mathbb{C}[\varepsilon] \longmapsto \begin{pmatrix} t^2\mathbb{C}[[t^2]] & t\mathbb{C}[[t^2]] \\ t\mathbb{C}[[t^2]] & t^2\mathbb{C}[[t^2]] \end{pmatrix}$$

$$\hat{\Lambda}_0^{2n} \longmapsto \begin{pmatrix} t^{2n}\mathbb{C}[[t^2]] & t^{2n+1}\mathbb{C}[[t^2]] \\ t^{2n+1}\mathbb{C}[[t^2]] & t^{2n}\mathbb{C}[[t^2]] \end{pmatrix}$$

$$\hat{\Lambda}_0^{2n+1} \longmapsto \begin{pmatrix} t^{2n+2}\mathbb{C}[[t^2]] & t^{2n+1}\mathbb{C}[[t^2]] \\ t^{2n+1}\mathbb{C}[[t^2]] & t^{2n+2}\mathbb{C}[[t^2]] \end{pmatrix}$$

In particular $\hat{\Lambda}/\hat{\Lambda}_0^3 = \begin{pmatrix} \mathbb{C}+\mathbb{C}t^2 & \mathbb{C}t \\ \mathbb{C}t & \mathbb{C}+\mathbb{C}t^2 \end{pmatrix}$

[redacted] What are the traces on $\hat{\Lambda}$? Note that for $n>0$

$$\operatorname{tr}(t^{2n}\varepsilon) = \operatorname{tr}(t\,t^{2n-1}\varepsilon) = \operatorname{tr}(t^{2n-1}\varepsilon t)$$
$$= -\operatorname{tr}(t^{2n}\varepsilon)$$

so $\operatorname{tr}(t^{2n}\varepsilon) = 0 \quad$ for $\quad n \geq 1$.

Also $\operatorname{tr}(t^{2n+1}\varepsilon) = \operatorname{tr}(\varepsilon\, t^{2n+1}) = -\operatorname{tr}(t^{2n+1}\varepsilon)$, so [redacted] $\operatorname{tr}(f(t)\varepsilon) = f(0)\operatorname{tr}(\varepsilon)$. So there's nothing interesting from $\operatorname{tr}(f(t)\varepsilon)$. However any linear fnl on $\mathbb{C}[[t]]$ invariant under conjugation by ε will give a trace on the cross-product Λ. Thus the traces on Λ to look at take the trace of the 2×2 matrix, which is a series in t^2, [redacted] followed by a linear functional on $\mathbb{C}[[t^2]]$.

The next stage is to try to obtain the S-relation. We suppose the trace on C is defined on K^{2n} and we consider

$$\tilde{F} = \left(1-\frac{t^2}{2}\right)F_\varepsilon + t\gamma'$$

over $C \otimes \Lambda/\Lambda_0^3$. We have

$$\tilde{K} = K \otimes \Lambda/\Lambda_0^3 + C \otimes \Lambda_0/\Lambda_0^3$$

$$\tilde{K}^{2n+2} = K^{2n+2}\otimes \Lambda/\Lambda_0^3 + K^{2n+1}\otimes \Lambda_0/\Lambda_0^3 + K^{2n}\otimes \Lambda_0^2/\Lambda_0^3$$

We have a [redacted] trace defined on this with values in $\mathbb{C}+\mathbb{C}t^2$. Let's take the cocycle

$$\operatorname{tr}\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n+1}\right\}\left(1-\frac{t^2}{2}\right)^{2n+1} \qquad \text{values in } \mathbb{C}+\mathbb{C}t^2$$

The important thing is that its class [redacted] is independent of t.

7 July 19, 1988 (Erica is 10)

It is desirable to understand better the structure of arguments involving the deformation

$$\tilde F = \frac{1}{\sqrt{1+t^2}}\left(\begin{pmatrix} F & 0 \\ 0 & -F\end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}\right)$$

$$\tilde\theta = \begin{pmatrix} \theta & 0 \\ 0 & 0\end{pmatrix}$$

We are considering a bundle E with two flat connections $D_0 = \delta$, $D_1 = \delta + \theta$ and with a splitting F. We then form $E \oplus E$ with the flat connection $\begin{pmatrix} D_1 & 0 \\ 0 & D_0\end{pmatrix}$ and the splitting $\begin{pmatrix} F & 0 \\ 0 & -F\end{pmatrix}$.

This splitting $\begin{pmatrix} F & 0 \\ 0 & -F\end{pmatrix}$ has the deformation $\tilde F$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$ and the forms $tr(F\alpha^{2n})$ associated to $\begin{pmatrix} D_1 & 0 \\ 0 & D_0\end{pmatrix} = \bar D$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} = \tilde F$ vanish. In effect $\bar\alpha = \dfrac{\bar D - F\bar D F}{2}$ commutes with $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}$ and F anti commutes so $tr(\tilde F\bar\alpha^{2n}) = 0$.

From this deformation we obtain then a formula

$$tr(F\alpha_1^{2n}) - tr(F\alpha_0^{2n}) = d(\text{something})$$

$$\alpha_j = \tfrac{1}{2}(D_j - FD_jF)$$

Consider the case $D_0 = \delta$, $D_1 = \delta + \theta$. Then $tr(F\alpha_1^{2n}) = tr\left(F\left(\tfrac{1}{2}F[F,\theta]\right)^{2n}\right) = \frac{(-1)^n}{2^{2n}}\, tr(F[F,\theta]^{2n})$ and $tr(F\alpha_0^{2n}) = 0$, so that we have written the Connes form as $d(\text{something})$. It should be the case that the "something" is the Chern-Simons form.

All this should be connected with our earlier work comparing curvature forms associated to two connections via the linear path and the superconnection path.

Let D_1, D_0 be two connections on E compatible with a given inner product. Then we can find two isometric embeddings $j_k : E \longrightarrow \tilde{V}$, $k = 1, 2$, such that $j_k^* \cdot d \cdot j_k = D_k$. Now for $\dim(\tilde{V})$ large we know the space of isometric embeddings is highly connected. In fact we should have that any two isometric embeddings are isotopic, which means that we can find a path of unitary autos g_t of $\tilde{V}$ such that $g_1 j_0 = j_1$. It follows that if we set $g = g_1$ and $i = j_0$ that

$$i^* \cdot d \cdot i = D_0$$

$$i^* g^{-1} \cdot d \cdot g \cdot i = j_1^* \cdot d \cdot j_1 = D_1$$

In other words we have simultaneously "flattened" the two connections D_1, D_0.

July 20, 1988

Let's review the formulas. Consider a vector bundle E over M with a flat connection D and a splitting F. Let

$$D = \underbrace{\frac{D + FDF}{2}}_{\nabla} + \underbrace{\frac{D - FDF}{2}}_{\alpha}$$

Then $\nabla^2 = -\alpha^2$, $[\nabla, \alpha] = 0$ and

$$d\,\mathrm{tr}(F\alpha^{2n}) = \mathrm{tr}\,[\nabla, F\alpha^{2n}] = 0.$$

Suppose $F = F_t$ varies with respect to t. Then for $\tilde{D} = dt\,\partial_t + D$ on $pr_2^*(E)$ over $R \times M$ we have

$$\tilde{\alpha} = -\tfrac{1}{2}F[\tilde{D}, F] = \underbrace{-\tfrac{1}{2}F[D,F]}_{\alpha} \underbrace{-\tfrac{1}{2}F\,dt\,\partial_t F}_{-\frac{1}{2}dt\,F\dot{F}}$$

Then

$$\mathrm{tr}(F\tilde{\alpha}^{2n}) = \mathrm{tr}(F\alpha^{2n}) + dt\left(-\tfrac{1}{2}\right)(2n)\,\mathrm{tr}\left(\dot{F}\alpha^{2n-1}\right)$$

is closed over $R \times M$ so

$$\boxed{\partial_t\,\mathrm{tr}(F\alpha^{2n}) = d\left(-\tfrac{1}{2}\right)(2n)\,\mathrm{tr}\left(\dot{F}\alpha^{2n-1}\right)}$$

Next suppose $E = \tilde{V}$, $D = d + \theta$, and F is constant over M, i.e. $dF = 0$. Then

$$\alpha = \frac{(d+\theta) - F(d+\theta)F}{2} = \frac{\theta - F\theta F}{2} = \tfrac{1}{2}F[F,\theta]$$

and

$$\boxed{\mathrm{tr}(F\alpha^{2n}) = \frac{(-1)^n}{2^{2n}}\,\mathrm{tr}\left(F[F,\theta]^{2n}\right) = \frac{(-1)^n}{2^{2n-1}}\,\mathrm{tr}\left(\theta[F,\theta]^{2n-1}\right)}$$

Now suppose $F = F_t$ varies with respect to t but not over M. Then

$$\partial_t \, tr\left(F\alpha^{2n}\right) = d\left(-\tfrac{1}{2}\right)(2n)\, tr\left(\dot{F}\alpha^{2n-1}\right)$$

$$\parallel \qquad\qquad\qquad \parallel$$

$$\partial_t \frac{(-1)^n}{2^{2n-1}}\, tr\left(\theta[F,\theta]^{2n-1}\right) = d\left(-\frac{1}{2}\right)(2n)\frac{(-1)^{n-1}}{2^{2n-1}}\, tr\left(\dot{F}F[F,\theta]^{2n-1}\right)$$

$$\boxed{\partial_t\left\{tr\left(\theta[F,\theta]^{2n-1}\right)\right\} = d\left\{n\, tr\left(\dot{F}F[F,\theta]^{2n-1}\right)\right\}}$$

Now we want to apply this in the case where we start with $D = d+\theta$, F_0 on $\tilde{V}$ and we consider $\tilde{V}\oplus\tilde{V}$ with connection

$$d+\tilde{\theta} = \begin{pmatrix} d+\theta & 0 \\ 0 & d \end{pmatrix}$$

and family of splittings

$$\tilde{F} = \begin{pmatrix} F_0 & +t \\ t & -F_0 \end{pmatrix}\frac{1}{\sqrt{1+t^2}} = \frac{1}{\sqrt{1+t^2}}\left(F_0\varepsilon + t\gamma^1\right)$$

$$[\tilde{F},\tilde{\theta}] = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} [F_0,\theta] & -t\theta \\ t\theta & 0 \end{pmatrix}$$

Now we have

$$\dot{\tilde{F}}\tilde{F} = \left[\partial_t\left(\frac{1}{\sqrt{1+t^2}}\right)\left(F_0\varepsilon + t\gamma^1\right) + \frac{1}{\sqrt{1+t^2}}\gamma^1\right]\frac{1}{\sqrt{1+t^2}}\left(F_0\varepsilon + t\gamma^1\right)$$

$$= \partial_t\left(\frac{1}{\sqrt{1+t^2}}\right)\sqrt{1+t^2} + \frac{1}{1+t^2}\gamma^1\left(F_0\varepsilon + t\gamma^1\right)$$

$$= \left(-\frac{1}{2}\right)\frac{2t}{1+t^2} + \frac{t}{1+t^2} + \frac{1}{1+t^2}F_0\left(\gamma^1\varepsilon\right)$$

Thus

$$\boxed{\mathring{\tilde F}\tilde F = \frac{1}{1+t^2} F_0\, \gamma'_\varepsilon = \frac{1}{1+t^2}\begin{pmatrix}0 & -F_0\\ F_0 & 0\end{pmatrix}}$$

[redacted] Let's drop the $\circ$ from $F_\circ$.

We have the identity

$$\partial_t\, tr\left(\tilde\theta[\tilde F,\tilde\theta]^{2n-1}\right) = d\, n\, tr\left(\mathring{\tilde F}\tilde F[\tilde F,\tilde\theta]^{2n-1}\right).$$

$$\partial_t\left[(1+t^2)^{-n+\frac12}\, tr\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & \theta\end{pmatrix}^{2n-1}\right\}\right]$$

$$= d\, n\, \frac{1}{1+t^2}\,\frac{1}{(1+t^2)^{n-\frac12}}\, tr\left\{\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}$$

or

$$\boxed{\begin{aligned}&\left[(1+t^2)^{n+\frac12}\,\partial_t\,(1+t^2)^{-n+\frac12}\right] tr\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\\ &\qquad = d\, n\, tr\left\{\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\end{aligned}}$$

Now

$$(1+t^2)(1+t^2)^{n-\frac12}\cdot\partial_t\cdot(1+t^2)^{-n+\frac12} = (1+t^2)\left(\partial_t + \partial_t\left(\log(1+t^2)^{-n+\frac12}\right)\right)$$

$$= (1+t^2)\partial_t + \left(-n+\tfrac12\right)2t$$

Let's begin by working mod Ind.

We have

$$tr\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & \theta\end{pmatrix}^{2n-1}\right\} = \sum_{k=0}^{n-1}(-t^2)^k\, tr\left\{\theta\, P\left(\overset{2n-1-2k}{[F,\theta]}, \overset{k}{\theta^2}\right)\right\}$$

where $P(\overset{k}{X}, \overset{\ell}{Y})$ = sum of (non-commuting) monomials with k X-factors and ℓ Y-factors

Let's review earlier calculations with Chern-Simons components. Recall the identity

$$\boxed{k\,tr\{P(\overset{k}{X},\overset{l}{Y})\} = (k+l)\,tr\{X\,P(\overset{k-1}{X},\overset{l}{Y})\} \qquad \text{if } k>0}$$

which we established by a "Goodwillie" style argument using the action of cyclic permutations on the non-commutative monomials in X, Y. Here's a generating function proof:

$$\sum_{\substack{l\geq 0\\ k>0}} s^{k-1}t^{l}\,k\,tr\left(P(\overset{k}{X},\overset{l}{Y})\right) = tr\,\partial_s \sum_{k,l\geq 0} s^k t^l P(\overset{k}{X},\overset{l}{Y})$$

$$= tr\,\partial_s\left(\frac{1}{1-sX-tY}\right) = tr\left(\frac{1}{1-sX-tY}X\frac{1}{1-sX-tY}\right)$$

$$\sum_{\substack{k\geq 1\\ l\geq 0}} s^{k-1}t^{l}(k-1+l+1)\,tr\{X\,P(\overset{k-1}{X},\overset{l}{Y})\}$$

$$= tr\left\{X(s\partial_s + t\partial_t + 1)\frac{1}{1-sX-tY}\right\}$$

$$= tr\left\{X\frac{1}{1-sX-tY}\left((sX+tY)\frac{1}{1-sX-tY}+1\right)\right\}$$

$$= tr\left\{X\frac{1}{(1-sX-tY)^2}\right\}.$$

Recall $\delta(d\theta) = -d\,\delta\theta = d(\theta^2) = [d\theta,\theta]$

so $[\delta+\theta, d\theta] = [\delta+\theta,\theta^2] = 0$ $\qquad \delta\theta^2 = \delta^2\theta = 0$

So

$$\delta\,tr\{\theta\,P(\overset{l}{d\theta},\overset{k}{\theta^2})\} = tr\,[\delta+\theta, \theta P(\overset{l}{d\theta},\overset{k}{\theta^2})]$$

$$= tr\left([\delta+\theta,\theta]\,P(\overset{l}{d\theta},\overset{k}{\theta^2})\right) = tr\{\theta^2 P(\overset{l}{d\theta},\overset{k}{\theta^2})\}$$

$$= \frac{k+1}{k+l+1}\,tr\{P(\overset{l}{d\theta},\overset{k+1}{\theta^2})\}$$

Next $\quad (d+t\theta)^2 = t\,d\theta + t^2\theta^2 = t\,(d\theta + t\theta^2)$

so

$$d\,\mathrm{tr}\,\{\theta\,(d\theta+t\theta^2)^n\} = \mathrm{tr}\,[d+t\theta,\ \theta(d\theta+t\theta^2)^n]$$

$$= \mathrm{tr}\left\{[d+t\theta,\theta]\,(d\theta+t\theta^2)^n\right\}$$

$$= \mathrm{tr}\,\{d\theta\,(d\theta+t\theta^2)^n\} + 2t\,\mathrm{tr}\,\{\theta^2\,(d\theta+t\theta^2)^n\}$$

So

$$d\,\mathrm{tr}\,\{\theta\,P(\overset{l-1}{d\theta},\overset{k+1}{\theta^2})\} = \text{coeff } t^{k+1} \text{ in } d\,\mathrm{tr}\,\{\theta\,(d\theta+t\theta^2)^{k+l}\}$$

$$= \mathrm{tr}\,\{d\theta\,P(\overset{l-1}{d\theta},\overset{k+1}{\theta^2})\} + 2\,\mathrm{tr}\,\{\theta^2\,P(\overset{l}{d\theta},\overset{k}{\theta^2})\}$$

$$= \left(\frac{l}{k+l+1} + \frac{2(k+1)}{l+k+1}\right)\mathrm{tr}\,\{P(\overset{l}{d\theta},\overset{k+1}{\theta^2})\}$$

or

$$\boxed{\delta\,\mathrm{tr}\{\theta\,P(\overset{l}{d\theta},\overset{k}{\theta^2})\} = \frac{k+1}{k+l+1}\ \mathrm{tr}\,\{P(\overset{l}{d\theta},\overset{k+1}{\theta^2})\}}$$

$$= \frac{k+1}{k+l+1}\,\frac{k+l+1}{l+2k+2}\ d\,\mathrm{tr}\{\theta P(\overset{l-1}{d\theta},\overset{k+1}{\theta^2})\}$$

or

$$\boxed{\delta\,\mathrm{tr}\,\{\theta\,P(\overset{l}{d\theta},\overset{k}{\theta^2})\} = \frac{k+1}{l+2k+2}\ d\,\mathrm{tr}\{\theta\,P(\overset{l-1}{d\theta},\overset{k+1}{\theta^2})\}}$$

<u>July 21, 1988</u>

Recall from yesterday the formula

$$\boxed{\begin{aligned}&\Big((1+t^2)\partial_t + (-2n+1)t\Big)\, tr\left\{\begin{pmatrix}\theta & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\\ &= d\ n\ tr\left\{\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-1}\right\}\end{aligned}}$$

Let's calculate the last trace. It is the sum of two terms; the first is

$$tr\ \underbrace{\underbrace{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}}_{\begin{pmatrix}0 & -F\end{pmatrix}}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}}_{\begin{pmatrix}-tF\theta & 0\end{pmatrix}}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-2}\begin{pmatrix}1\\ 0\end{pmatrix}$$

$$= -t\ tr\sum_{k=0}^{n-1} F\theta\, P\big(\overset{2n-2-2k}{[F,\theta]}, \overset{k}{(-t^2)\theta^2}\big)$$

and the second is

$$tr\ \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0 & -F\\ F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-2}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}$$

$$= tr\ \begin{pmatrix}F & 0\end{pmatrix}\begin{pmatrix}[F,\theta] & -t\theta\\ t\theta & 0\end{pmatrix}^{2n-2}\begin{pmatrix}-t\theta\\ 0\end{pmatrix}$$

$$= (-t)\ tr\sum_{k=0}^{n-1} F\, P\big(\overset{2n-2-2k}{[F,\theta]}, \overset{k}{(-t^2)\theta^2}\big)\,\theta$$

Since the P factor is even as a diff'l form we can move θ around without sign. Thus we find

$$\boxed{\begin{aligned}&\mathrm{tr}\left\{\begin{pmatrix}0&-F\\F&0\end{pmatrix}\begin{pmatrix}[F,\theta]&-t\theta\\t\theta&0\end{pmatrix}^{2n-1}\right\} = (-2t)\sum_{k=0}^{n-1}\mathrm{tr}\left((F\theta+\theta F)\,P\big(\overset{2n-2-2k}{[F,\theta]},\overset{k}{-t\theta^2}\big)\right)\\ &= (-2t)\sum_{k=0}^{n-1}(-t^2)^k\,\mathrm{tr}\left\{(F\theta+\theta F)\,P\big(\overset{2n-2-2k}{[F,\theta]},\overset{k}{\theta^2}\big)\right\}\end{aligned}}$$

Next stage: Let's derive relations among S-transforms of the Connes cocycles. Set

$$\alpha_k^{(2n)} = \mathrm{tr}\left\{\theta\,P\big(\overset{2n-1-2k}{[F,\theta]},\overset{k}{-\theta^2}\big)\right\} \qquad 0\le k\le n-1$$

$$\beta_k^{(2n)} = \mathrm{tr}\left\{(F\theta+\theta F)\,P\big(\overset{2n-2-2k}{[F,\theta]},\overset{k}{-\theta^2}\big)\right\} \qquad 〃$$

Then we have

$$\mathrm{tr}\begin{pmatrix}\theta&0\\0&0\end{pmatrix}\begin{pmatrix}[F,\theta]&-t\theta\\t\theta&0\end{pmatrix}^{2n-1} = \sum_{k=0}^{n-1}t^{2k}\,\alpha_k^{(2n)}$$

$$\mathrm{tr}\begin{pmatrix}0&-F\\F&0\end{pmatrix}\begin{pmatrix}[F,\theta]&-t\theta\\t\theta&0\end{pmatrix}^{2n-1} = (-2t)\sum_{k=0}^{n-1}t^{2k}\,\beta_k^{(2n)}$$

$$\left(\partial_t + t^2\partial_t + (-2n+1)t\right)\sum_{k=0}^{n-1}t^{2k}\alpha_k^{(2n)} = d\,n\,(-2t)\sum_{k=0}^{n-1}t^{2k}\beta_k^{(2n)}$$

Look at the coefficient of t^{2k+1}

$$(2k+2)\,\alpha_{k+1}^{(2n)} + (2k-2n+1)\,\alpha_k^{(2n)} = (-2n)\,d\beta_k^{(2n)}$$

or

$$\boxed{\begin{aligned}&(2n-1-2k)\,\mathrm{tr}\left\{\theta\,P\big(\overset{2n-1-2k}{[F,\theta]},\overset{k}{-\theta^2}\big)\right\}\\ &= (2k+2)\,\mathrm{tr}\left\{\theta\,P\big(\overset{2n-3-2k}{[F,\theta]},\overset{k+1}{-\theta^2}\big)\right\} + 2n\,d\,\mathrm{tr}\left\{(F\theta+\theta F)\,P\big(\overset{2n-2-2k}{[F,\theta]},\overset{k}{-\theta^2}\big)\right\}\end{aligned}}$$

or

$$(2n-1-2k)\,\alpha_k^{(2n)} = (2k+1)\,\alpha_{k+1}^{(2n)} + 2n\, d\beta_k^{(2n)}$$

Iterating this formula allows us to write $\alpha_k^{(2n)}$ as the boundary of a linear combination with rational coefficients of $\beta_k^{(2n)}, \beta_{k+1}^{(2n)}, \dots, \beta_{n-1}^{(2n)}$. In particular $\alpha_0^{(2n)} = tr\left(\theta[F,\theta]^{2n-1}\right)$ is d of a linear combination of $\beta_0^{(n)}, \dots, \beta_{n-1}^{(2n)}$. This should be a version of the Chern–Simons form.

So the next project will be to try to find formulas for the Chern–Simons form.

July 24, 1988

$\tilde F = \tilde F_t = \frac{1}{\sqrt{1+t^2}}\begin{pmatrix} F & t \\ t & -F\end{pmatrix}$. Solving $\dot g_t g_t^{-1} = \frac{1}{2}\dot{\tilde F}_t \tilde F_t$ gives

$$g_t = \begin{pmatrix}1&0\\0&F\end{pmatrix} e^{\frac{\theta}{2}\begin{pmatrix}0&-1\\1&0\end{pmatrix}}\begin{pmatrix}1&0\\0&F\end{pmatrix} \qquad t = \tan\theta$$

$$= \begin{pmatrix} c & -sF \\ Fs & c\end{pmatrix} \qquad \begin{array}{l} c = \cos\theta/2 \\ s = \sin\theta/2 \end{array}$$

We have

$$g_t\begin{pmatrix}F&0\\0&-F\end{pmatrix}g_t^{-1} = \begin{pmatrix} c & -sF \\ Fs & c\end{pmatrix}\begin{pmatrix} Fc & s \\ s & -Fc\end{pmatrix}$$

$$= \begin{pmatrix} c^2-s^2 & +F(2sc) \\ F(2sc) & s^2-c^2\end{pmatrix} = \begin{pmatrix} \cos\theta & +F\sin\theta \\ F\sin\theta & -\cos\theta\end{pmatrix} = \tilde F_t$$

$$g_t^{-1}\begin{pmatrix}a&0\\0&0\end{pmatrix}g_t = \begin{pmatrix}1&0\\0&F\end{pmatrix}\begin{pmatrix}c&+s\\-s&c\end{pmatrix}\begin{pmatrix}a&0\\0&0\end{pmatrix}\begin{pmatrix}c&-s\\+s&c\end{pmatrix}\begin{pmatrix}1&0\\0&F\end{pmatrix}$$

$$= \begin{pmatrix}1&0\\0&F\end{pmatrix}\begin{pmatrix}ac^2&-asc\\-asc&as^2\end{pmatrix}\begin{pmatrix}1&0\\0&F\end{pmatrix} = \begin{pmatrix}ac^2&-aFsc\\-Fasc&s^2\end{pmatrix}$$

~~Thus conjugation by~~ $\cancel{\begin{pmatrix}1&0\\0&F\end{pmatrix}g_t^{-1}}$ ~~changes~~ $\cancel{a\mapsto\begin{pmatrix}a&0\\0&0\end{pmatrix}}$ ~~into the alg homom.~~ $\cancel{a\begin{pmatrix}c^2&-sc\\-sc&s^2\end{pmatrix}}$ ~~and it changes~~ $\cancel{\tilde F_t}$ ~~into~~ $\cancel{\begin{pmatrix}F&0\\0&-F\end{pmatrix}}$ ~~If we reduce to the +1 eigenspace of~~

Thus conjugation by g_t^{-1} changes $\tilde F_t$ to $\begin{pmatrix}F&0\\0&-F\end{pmatrix}$ and it changes the homomorphism $a\mapsto\begin{pmatrix}a&0\\0&0\end{pmatrix}$ to $a\mapsto\begin{pmatrix}ac^2&-aFsc\\-Fasc&s^2\end{pmatrix}$. Reducing to the +1 eigenspace of $\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$ gives

$$\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}\begin{pmatrix} ac^2 & \cancel{aF}sc \\ -Fasc & as^2 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix} = \begin{pmatrix} eaec^2 & ea\bar{e}cs \\ \bar{e}aecs & \bar{e}a\bar{e}s^2 \end{pmatrix}$$

Thus we have deformed $\rho(a) = eae$ to

$$* \quad \tilde{\rho}_t(a) = \begin{pmatrix} eaec^2 & ea\bar{e}cs \\ \bar{e}aecs & \bar{e}a\bar{e}s^2 \end{pmatrix} \quad *$$

Now we know that it should be possible working formally in t to obtain the S-relation between the cyclic cocycles $tr\,(\delta\rho+\rho^2)^{2n}$ associated to ρ. ~~[illegible]~~

Let's review some of the steps involved in my present "proof" of the S-relation for the Connes cocycles. One considers the flat connection

$$\tilde{D} = dt\,\partial_t + \delta + \tilde{\theta}$$

and the splitting $\tilde{F} = \tilde{F}_t$. This means that we work with differential forms over $\mathcal{G}\times R$. Algebraically we extend our ~~[illegible]~~ DGA $C^\cdot(A,\mathbb{C})$ by tensoring with diffl forms over $\mathbb{C}[[t]]$. The key relation is the fact that the Connes form $tr\left(\tilde{F}(\tilde{\nabla})^{2n}\right)$ is closed, and this, when resolved into components relative to the product $\mathcal{G}\times R$, gives an infinitesimal homotopy relation

$$\partial_t\, tr\left(F_t\alpha_t^{2n}\right) = (-n)\,\delta\, tr\left(\dot{F}_t\alpha_t^{2n-1}\right).$$

~~[illegible]~~ There seem to be various ways to interpret $\tilde{\rho}_t(a)$ above $*$. First we have the way this formula was obtained ~~by~~ reducing $g_t^{-1}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}g_t$ via $\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}$. A variant of this is to reduce the family of homomorphisms

$$a \longmapsto a\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$$

via the idempotent $\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}$. ~~On the other hand we have~~

$$\cancel{(ce + s\bar{e})^2 = c^2 e + s^2 \bar{e}}$$

~~doesn't work~~

Analogy: $t \mapsto \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$ is one way [redacted] of parametrizing the canonical 2x2 idempotent matrix over $\mathbb{P}^1(\mathbb{R})$. Hence $a \mapsto a\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix}$ is similar to tensoring with the Hopf bundle over [redacted] $\mathbb{P}^1(\mathbb{C})$, so we have something analogous to our ~~geometric~~ picture for S for cyclic cocycles. The ~~only~~ main [redacted] puzzle is why one uses $\begin{pmatrix} F & 0 \\ 0 & -F \end{pmatrix}$ instead of $F\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This is related to the "super" transformation of $\mathbb{P}^1(\mathbb{C})$ into the superalgebra C_2.

The main mystery appears to be why the S-relation between the different cyclic cocycles should come out of an infinitesimal analysis [redacted] using 2×2 matrices. One would like to find a more intelligent way to proceed.

It seems to be desirable to be able to treat a family of homomorphisms $u_t : A \to C$ in an efficient way. Thus we seem to be after a continuous version of $A * A$.

So what we want to look for is is a sort of adjoint to taking the path algebra: $B \mapsto B^I$, or $B \otimes \mathbb{C}[t]$, etc. This is analogous to $A \mapsto A * A$ being adjoint to $B \mapsto B \times B$. Another idea is that we should be able to obtain an nth order path as a composition of first order paths.

To be more specific let us define an n-th order path of homomorphisms from A to B to be a homom

$$A \longrightarrow B \otimes \left(\mathbb{C}[t]/t^{n+1}\right)$$

Then is it possible to compose first order paths?

July 26, 1988

Given an algebra A we seek an algebra $R^{(n)}$ which is a kind of n-th order path algebra in the sense that

$$\mathrm{Hom}_{alg}(R^{(n)}, B) = \mathrm{Hom}_{alg}(A, B \otimes \mathbb{C}[t]/t^{n+1}).$$

Let $F_n(B)$ be the set on the right ↑. An element of $F_n(B)$ is a sequence $\varphi_0, \ldots, \varphi_n$ in $\mathrm{Hom}_{\mathbb{C}}(A, B)$ such that

$$\varphi_0(xy) = \varphi_0(x)\varphi_0(y) \qquad \varphi_0(1) = 1$$

$$\varphi_k(xy) = \sum_{i+j=k} \varphi_i(x)\varphi_j(y) \qquad k \leq n$$

Note that $F_n(B)$ has a natural action of the monoid $(\mathbb{C}, \times)$ namely $\lambda(\varphi_0, \ldots, \varphi_n) = (\ldots, \lambda^i\varphi_i, \ldots)$. This translates into a natural $\mathbb{N}$-grading on $R^{(n)}$ such that the universal $\varphi = (\varphi_0, \ldots, \varphi_n)$ is such that φ_i map A to $R^{(n)}_i$. We have

$$\mathrm{Hom}_{\mathbb{N}\text{-gr-alg.}}(R^{(n)}, B) = \left\{ (\varphi_0, \ldots, \varphi_n) \,\middle|\, \varphi_i \in \mathrm{Hom}_{\mathbb{C}}(A, B_i) \text{ satisfying above identities} \right\}$$

Call the right side $F_n'(B)$. Then we have

$$F_n'(B) = F_n'(B/B_{>n})$$

From this it should follow that $R^{(n)}$ can be found on the category of graded algebras which are zero for degrees $> n$ and then applying the left-adjoint functor to $B \longmapsto B/B_{>n}$.

For example, take $n = 1$. Then $F'(B)$ consists of pairs (φ_0, φ_1) where $\varphi_0 : A \to B$ is an alg. hom. and $\varphi_1 : A \to B$ is a derivation with values in B considered as a bimodule over A via φ_1. Thus

$$R^{(1)} / R^{(1)}_{>1} = A \oplus \Omega^1_A$$

and so

$$R^{(1)} = A \oplus \Omega^1_A \oplus \Omega^2_A \oplus \cdots\cdots$$

Next let's look at $R^{(2)}$. The identity

$$\varphi_2(xy) = \varphi_0(x)\varphi_2(y) + \varphi_1(x)\varphi_1(y) + \varphi_2(x)\varphi_0(y)$$

can be written

$$-(\delta\varphi_2)(x, y) = \varphi_1(x)\varphi_1(y).$$

In other words the fibre of

$$F_2'(B) \longrightarrow F_1'(B)$$

over φ_0, φ_1 consists of $\varphi_2 \in C^1(A, B_2)$ such that $-\delta\varphi_2 = \varphi_1 \cdot \varphi_1$. Thus if $\varphi_1 = 0$, φ_2 is an arbitrary 1-cocycle, or equivalently, an A-bimodule map $\Omega^1_A \longrightarrow B_2$. Thus we see that we have an exact sequence

$$\begin{array}{c} \Omega^1_A \otimes_A \Omega^1_A \\ \| \\ \Omega^2_A \end{array} \longrightarrow R^{(2)}_2 \longrightarrow \Omega^1_A \longrightarrow 0$$

It shouldn't be hard to use the universal φ_2 to produce a map $A \otimes \bar{A} \otimes A \longrightarrow R^{(2)}_2$ and to show it's an isomorphism. Thus we have

$$R^{(2)}/R^{(2)}_{>2} = A \oplus \Omega^1_A \oplus A\otimes\bar{A}\otimes A$$

$$R^{(2)} = A \oplus \Omega^1_A \oplus A\otimes\bar{A}\otimes A \oplus \begin{pmatrix} \Omega^1_A \otimes_A (A\otimes\bar{A}\otimes A) \\ \oplus \\ (A\otimes\bar{A}\otimes A)\otimes_A \Omega^1_A \end{pmatrix} \oplus \cdots$$

Now it seems clear that if we consider $R^{(n)}/(R^{(n)}_1, \ldots, R^{(n)}_{n-1})$, then this represents $(n+1)$-tuples $(\varphi_0, 0, \ldots, 0, \varphi_n)$, and so we should have

$$R^{(n)}/(R^{(n)}_1, \ldots, R^{(n)}_{n-1}) = A \oplus \underset{(n)}{\Omega^1_A} \oplus \underset{(2n)}{\Omega^2_A} \oplus \cdots$$

Thus $R^{(n)}$ appears as a rather gigantic non-commutative algebra whose <u>indecomposable space</u> is $\underset{(1)}{\Omega^1_A} \oplus \underset{(2)}{\Omega^1_A} \oplus \cdots \oplus \underset{(n)}{\Omega^1_A}$. Thus it roughly appears as if we take the A-bimodule $\Omega^1_A \otimes \left(t\mathbb{C}[t]/t^{n+1}\mathbb{C}[t]\right)$ and we form its tensor <u>algebra</u> over A.

Let's now go back to the original problem. We are considering $C = A * \mathbb{C}[\mathbb{Z}/2]$ generated by A, F as usual. We consider $M_2(C)$; this contains the idempotent $\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}$, and if we reduce via this idempotent we obtain

$$\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix} M_2(C) \begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix} = \begin{pmatrix} eCe & eC\bar{e} \\ \bar{e}Ce & \bar{e}C\bar{e} \end{pmatrix}$$

which is isomorphic to C. Next take the 1-par. family of homom. from A to $M_2(C)$ given by

$$a \longmapsto a\begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix} \qquad \begin{aligned} c &= \cos\theta/2 \\ s &= \sin\theta/2 \end{aligned}$$

and reduce via the idempotent $\begin{pmatrix} e & 0 \\ 0 & \bar{e} \end{pmatrix}$ to obtain

$$\rho_t(a) = \begin{pmatrix} c^2\, eae & cs\, ea\bar{e} \\ cs\, \bar{e}ae & s^2\, \bar{e}a\bar{e} \end{pmatrix}$$

which is a deformation of $\rho_0(a) = eae$. Here ρ_0 is to be viewed as a universal linear map from A to an alg with $\rho(1) = 1$. ρ_t is a deformation of ρ_0 not preserving this last condition.

The reason ρ_t is interesting is because one can obtain the S-operator relation between the cyclic cocycles

$$\mathrm{tr}\,(\delta\rho + \rho^2)^n$$

from this deformation!

July 27, 1988

Let $A = R/I$, where R is a free algebra. Assume $A \neq 0$ so that I^n injects into $\bar{R} = R/\mathbb{C}$ for $n > 0$. We consider the short exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\arrow[r] & I^{n+1} \arrow[r, "d"] \arrow[d] & I^n \otimes_R \Omega^1_R \otimes_R \arrow[r, "b"] \arrow[d] & \boxed{I^n} \arrow[r] \arrow[d] & I^{n-1} \otimes_R \Omega^1 \otimes_R \arrow[r] \arrow[d] & {} \\
\arrow[r] & \bar{R} \arrow[r, "d"] \arrow[d] & R \otimes_R \Omega^1 \otimes_R \arrow[r, "b"] \arrow[d] & \bar{R} \arrow[r, "d"] \arrow[d] & R \otimes_R \Omega^1 \otimes_R \arrow[r] \arrow[d] & {} \\
\arrow[r, "b"] & \bar{R}/I^{n+1} \arrow[r] \arrow[d] & \boxed{R/I^n \otimes_R \Omega^1 \otimes_R} \arrow[r, "b"] \arrow[d] & \bar{R}/I^n \arrow[r] \arrow[d] & R/I^{n-1} \otimes_R \Omega^1 \otimes_R \arrow[r] \arrow[d] & {} \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Because R is a free algebra the middle complex is exact. One has $\Omega^1_R = R \otimes V \otimes R$ if $R = T(V)$, so $R \otimes_R \Omega^1_R \otimes_R = R \otimes V \cong \bigoplus_{n>0} V^{\otimes n}$. The map b (resp. d) can be identified with $1 - \sigma$ ([redacted] resp. $N = \sum \sigma^i$).

Thus the homology of the top & bottom complexes are the same except for a shift. Look at the homology of the bottom complex at $\bar{R}/I^{n+1}$. It is

$$* \qquad \operatorname{Ker}\left\{ \overline{HC}_0(R/I^{n+1}) \xrightarrow{d} H_1(R, R/I^n) \right\}$$

This has to be isomorphic to the homology of the top complex at $I^n \otimes_R \Omega^1_R \otimes_R$ which is

[redacted] $\operatorname{Coker}\left\{ I^{n+1}/[R, I^{n+1}] \longrightarrow H_1(R, I^n) \right\}$

and which we know is isomorphic to $\overline{HC}_{2n}(A)$.

On the other hand the homology of the top complex at $\boxed{I^n}$ is

$$\operatorname{Ker}\left\{ I^n/[R,I^n] \longrightarrow H_1(R,I^{n-1}) \right\}$$

and the homology of the bottom complex at $\boxed{R/I^n \otimes_R \Omega^1 \otimes_R}$ is

$$\operatorname{Coker}\left\{ HC_0(\overline{R}/I^{n+1}) \xrightarrow{d} H_1(R, R/I^n) \right\}$$

The isomorphism between the two that we get from ~~the~~ exact sequence of complexes on the previous page is just the coboundary in

$$\longrightarrow H_1(R,R) \longrightarrow H_1(R,R/I^n) \xrightarrow{\delta} H_0(R,I^n) \longrightarrow$$

~~[illegible]~~ Now we know there is an action of $\mathbb{Z}/n$ on this exact sequence such that δ induces an isom. on the nontrivial parts $(1-\sigma)H_1(R,R/I^n) \xrightarrow{\sim} (1-\sigma)H_0(R,I^n)$

Thus

$$\begin{tikzcd}
\overline{HC}_0(R/I^{n+1}) \arrow[r, "d"] \arrow[dr, "0"'] & H_1(R,R/I^n) \arrow[r] \arrow[d, "\delta"] & H_1(R,R/I^n)/H_1(R,R/I^n)^\sigma \arrow[d, "\wr"] \\
& H_0(R,I^n) \arrow[r] & H_0(R,I^n)/H_1(R,I^n)^\sigma
\end{tikzcd}$$

implies that $d\,\overline{HC}_0(R/I^{n+1}) \subset H_1(R,R/I^n)^\sigma$. From this we ~~[illegible]~~ can see that the exact sequences

$$0 \longrightarrow \overline{HC}_{2n}(A) \longrightarrow \overline{HC}_0(R/I^{n+1}) \longrightarrow H_1(R,R/I^n)_\sigma \longrightarrow \overline{HC}_{2n-1}(A) \to 0$$

$$0 \longrightarrow \overline{HC}_{2n+1}(A) \longrightarrow I^{n+1}/[I,I^n] \longrightarrow H_1(R,I^n) \longrightarrow \overline{HC}_{2n}(A) \to 0$$

are equivalent.

The next step will be to compare [redacted] successive terms in the filtration.

$\bar{H}C_{2n+2}(A)$ $\qquad$ $\bar{H}C_{2n+1}(A) \oplus (1-\sigma) I^{n+1}/[R,I^{n+1}]$

$\bar{H}C_{2n}(A)$ $\qquad$ $\bar{H}C_{2n-1}(A) \oplus (1-\sigma) I^{n}/[R,I^{n}]$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
{} \arrow[r] & I^{n+2} \arrow[r, "d"] \arrow[d] & I^{n+1}\otimes_R \Omega^1 \otimes_R \arrow[r, "b"] \arrow[d] & \boxed{I^{n+1}} \arrow[r, "d"] \arrow[d] & I^n \otimes_R \Omega^1_R \otimes_R \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & I^{n+1} \arrow[r, "d"] \arrow[d] & I^{n}\otimes_R \Omega^1 \otimes_R \arrow[r, "b"] \arrow[d] & \boxed{I^{n}} \arrow[r, "d"] \arrow[d] & I^{n-1} \otimes_R \Omega^1 \otimes_R \arrow[r] \arrow[d] & \cdots \\
0 \arrow[r] & I^{n+1}/I^{n+2} \arrow[r, "d"] \arrow[d] & I^n/I^{n+1} \otimes_R \Omega^1 \otimes_R \arrow[r, "b"] \arrow[d] & I^n/I^{n+1} \arrow[r, "d"] \arrow[d] & I^{n-1}/I^n \otimes_R \Omega^1 \otimes_R \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

[redacted] Recall that the sequence

$[I,I^{n-1}]/[R,I^n]$

$0 \longrightarrow$ [redacted] $\longrightarrow I^n/[R,I^n] \longrightarrow I^n/[I,I^{n-1}] \longrightarrow 0$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{(I\otimes_R)^n}$ [redacted] $\qquad\qquad \overset{\shortparallel}{(I\otimes_R)^n_\sigma}$

splits canonically:

$$I^n/[R,I^n] = \left(I^n/[R,I^n]\right)^\sigma \oplus (1-\sigma)\left(I^n/[R,I^n]\right)$$

Moreover the non-trivial part $(1-\sigma)I^{n+1}/[R,I^{n+1}]$ goes to zero in $I^n/[R,I^n]$, since

$$\begin{tikzcd}
I^{n+1}/[R,I^{n+1}] \arrow[rr] \arrow[dr] & & I^n/[R,I^n] \\
 & I^{n+1}/[I,I^n] \arrow[ur] &
\end{tikzcd}$$

So in order to derive the Connes long exact sequence, we want to see that the homology of the bottom complex is the Hochschild homology plus pieces $\cong (1-\sigma)I^n/[R,I^n]$.

<u>July 28, 1988</u>

$A = R/I$ where R is free. We have a complex

$$\xrightarrow{b} I^{n+1}/I^{n+2} \xrightarrow{\delta} I^n/I^{n+1} \otimes_R \Omega^1_A \otimes_R \xrightarrow{b} I^n/I^{n+1} \xrightarrow{\delta}$$

and we wish to determine its homology.

The map b is defined for any R-bimodule

$$M \otimes_R \Omega^1_R \otimes_R \longrightarrow M \quad , \quad m\otimes x\,dy\,z \mapsto [zmx, y]$$

and its cokernel and kernel are $H_i(R,M)$, $i = 1,2$. Thus

$$\mathrm{Coker}\{\xrightarrow{b} I^n/I^{n+1}\} = (I/I^2 \otimes_A)^n$$

Next we have an exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A\otimes_R \Omega^1_R \otimes_R A \longrightarrow \Omega^1_A \longrightarrow 0$$

and tensoring with I^n/I^{n+1} gives the first row in

[redacted]

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
 & & & & \mathrm{Tor}_1^{A\otimes A^o}(I^n/I^{n+1}, A) \arrow[d] & \\
0 \arrow[r] & \mathrm{Tor}_1^{A\otimes A^o}(I^n/I^{n+1}, \Omega^1_A) \arrow[r] & (I/I^2\otimes_A)^{n+1} \arrow[r] & (I^n/I^{n+1})\otimes_R \Omega^1_R \otimes_R \arrow[r] \arrow[dr, "b"] & I^n/I^{n+1}\otimes_A \Omega^1_A \otimes_A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_{2n+2}(A,A) \arrow[r] \arrow[u, "s"] & (I/I^2\otimes_A)^{n+1} \arrow[r] \arrow[u, "N"] & (I^n/I^{n+1})\otimes_R \Omega^1_R \otimes_R \arrow[u, equal] & I^n/I^{n+1} \arrow[d] & \\
 & & \mathrm{Coker}\{\xrightarrow{b} I^{n+1}/I^{n+2}\} = (I/I^2\otimes_A)^{n+1} \arrow[u] \arrow[uur, "\delta"] & & (I/I^2\otimes_A)^n \arrow[d] & \\
 & & & & 0 &
\end{tikzcd}$$

I recall that the second row results from using [redacted] the complex $K\otimes_A \cdots \otimes_A K$ ($n+1$ times) where K is the 2-step resolution

$$0 \longrightarrow I/I^2 \longrightarrow A\otimes_R \Omega^1_R \otimes_R A \longrightarrow A\otimes A \dashrightarrow A \dashrightarrow 0$$

of A as an A-bimodule, to approximate $A \overset{L}{\otimes}_A \cdots \overset{L}{\otimes}_A A$. The complex $(K\otimes_A)^n K$ is of length $2n+2$ and its a free bimodule except in the top degree. One can then continue and argue that $(K\otimes_A)^{n+1}_\sigma$ represents $(A\overset{L}{\otimes}_A)^{n+1}_\sigma \sim A\overset{L}{\otimes}_A$ up through degree $2n+2$, and this yields the exactness of the second row.

Finally from this diagram we can obtain the homology desired:

$$0 \longrightarrow (1-\sigma)(I/I^2\otimes_A)^{n+1} \longrightarrow \frac{\mathrm{Ker}(\delta \text{ on } I^{n+1}/I^{n+2})}{\mathrm{Im}\, b} \longrightarrow H_{2n+2}(A,A) \to 0$$

$$0 \longrightarrow (1-\sigma)(I/I^2\otimes_A)^{n+1} \longrightarrow \mathrm{Ker}\{b \text{ on } I^n/I^{n+1}\otimes_R \Omega^1_R \otimes_R\} \to H_{2n+1}(A,A) \to 0$$

Let's try to interpret the above calculation. Let's replace δ by the map

$$I^{n+1}/I^{n+2} \xrightarrow{d} I^n/I^{n+1} \otimes_R \Omega^1_R \otimes_R$$

which is the composition

$$I^{n+1}/I^{n+2} = I^n/I^{n+1}\otimes_A I/I^2 \longrightarrow (I^n/I^{n+1})\otimes_A (A\otimes_R \Omega^1_R \otimes_R A)$$

$$\twoheadrightarrow (I^n/I^{n+1})\otimes_A (A\otimes_R \Omega^1_R \otimes_R A)\otimes_A = (I^n/I^{n+1})\otimes_R \Omega^1_R \otimes_R .$$

Then I claim we get a complex giving the Hochschild homology:

$$\xrightarrow{b} I^{n+1}/I^{n+2} \xrightarrow{u} I^n/I^{n+1} \otimes_R \Omega^1_R \otimes_R \xrightarrow{b} I^n/I^{n+1} \xrightarrow{u}$$

This is clear from earlier diagram-chasing. In effect

$$\frac{\operatorname{Ker} u}{\operatorname{Im} b} \text{ at } I^{n+1}/I^{n+2} = \operatorname{Ker}\{(I/I^2 \otimes_A)^{n+1} \to (I^n/I^{n+1}) \otimes_R \Omega^1_R \otimes_R\}$$
$$= \operatorname{Tor}_1^{A\otimes A^\circ}(I^n/I^{n+1}, \Omega^1_A) = H_{2n+2}(A,A)$$

and

$$\frac{\operatorname{Ker} b}{\operatorname{Im} u} \text{ at } I^n/I^{n+1} \otimes_R \Omega^1_R \otimes_R = \operatorname{Ker}\{b : I^n/I^{n+1} \otimes_A \Omega^1_A \otimes_A \to I^n/I^{n+1}\}$$
$$= \operatorname{Tor}_1^{A\otimes A^\circ}(I^n/I^{n+1}, A) = H_{2n+1}(A,A)$$

[redacted]

Thus what we look at for the Connes exact sequence is a twisted version of a complex giving the Hochschild homology where the twisting amounts to a cyclic averaging of a differential. Formally it's similar to taking the d operator on Ω_A and replacing it by B [redacted].

Let's now consider the case where

$$R = \tilde{T}(A) = T(A)/(1 = \rho(1_A))$$

In this case we have a canonical isomorphism

$$gr^I(R) = \text{[redacted]} \ \Omega_A^{ev}$$

[redacted] i.e. $I^n/I^{n+1} = \Omega_A^{2n}$.

Then $I^n/I^{n+1} \otimes_R \Omega^1_R \otimes_R = I^n/I^{n+1} \otimes_R (R \otimes \bar{A} \otimes R) \otimes_R = \Omega^{2n}_A \otimes \bar{A} \simeq \Omega^{2n+1}_A$

Thus the u, b complex at the top of the preceding page appears

$$\xrightarrow{b} \Omega^{2n+2}_A \xrightarrow{u} \Omega^{2n+1}_A \xrightarrow{b} \Omega^{2n}_A \xrightarrow{u}$$

Let's check now that it is the Hochschild complex up to signs on the differentials. Let's begin with b. This is defined as follows:

$$* \quad \Omega^{2n+1}_A = \Omega^{2n}_A \otimes_A \Omega^1_A \longrightarrow \Omega^{2n}_A \otimes_A (A \otimes A) \twoheadrightarrow \underset{\Omega^{2n}_A}{\overset{}{\Omega^{2n}_A \otimes_A (A\otimes A) \otimes_A}}$$

$$\Omega^{2n}_A \otimes_A (A\otimes A) \otimes_A \;=\; \Omega^{2n}_A$$

where the middle map is induced by

$$\Omega^1_A \longrightarrow A \otimes A$$
$$x\,dy\,z \longmapsto (x\otimes 1)(y\otimes 1 - 1 \otimes y)(1\otimes z)$$

(Recall this map is ~~[illegible]~~ obtained from b':

$$\begin{tikzcd}
A\otimes \bar{A} \otimes A \arrow[rr, "b'"] \arrow[dr, two heads] & & A\otimes A \\
& \Omega^1_A \arrow[ur, hook] &
\end{tikzcd}$$

$$x\otimes y \otimes z \longmapsto x\,dy\,z \longmapsto b'(x\otimes y\otimes z) = xy\otimes z - x\otimes yz.$$
)

Calculate $*$ for any degree not just $2n$.

$$\Omega^{k+1}_A \ni a_0 da_1 \cdots da_{k+1} \longmapsto (a_0 da_1 \cdots da_k) \otimes (a_{k+1} \otimes 1 - 1 \otimes a_{k+1})$$

$$\longmapsto \underbrace{a_0 da_1 \cdots da_k\, a_{k+1}} - a_{k+1} a_0 da_1 \cdots da_k \in \Omega^k_A$$

$$a_0 da_1 \cdots d(a_k a_{k+1}) - a_0 da_1 \cdots d(a_{k-1} a_k) da_{k+1} + \cdots$$
$$+ (-1)^{k-1} a_0\, d(a_1 a_2) \cdots da_{k+1} + (-1)^k a_0 a_1 da_2 \cdots da_{k+1}$$

$$= (-1)^k \Big\{ a_0 a_1 da_2 \cdots da_{k+1} - a_0 d(a_1 a_2) \cdots da_{k+1} + \cdots$$
$$+ (-1)^k a_0 da_1 \cdots da_{k+1} + (-1)^{k+1} a_{k+1} a_0 da_1 \cdots da_k \Big\} \in \Omega^k_A$$

Thus we obtain up to sign the b in the Hochschild cx.

So next consider u. For this we need the isomorphism $I/I^2 = \Omega^2_A$ and the map $I/I^2 \longrightarrow A\otimes_R\Omega^1_R\otimes_R A = A\otimes\bar{A}\otimes A$. We have

$$da_1da_2 \mapsto b'(1\otimes a_1\otimes a_2\otimes 1) = a_1\otimes a_2\otimes 1 - 1\otimes a_1a_2\otimes 1 + 1\otimes a_1\otimes a_2$$

$$\begin{tikzcd}
0 \arrow[r] & \Omega^2_A \arrow[r] \arrow[d] & A\otimes\bar{A}\otimes A \arrow[r] \arrow[d, "1\otimes d\rho"] & \Omega^1_A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I/I^2 \arrow[r] & A\otimes_R\Omega^1_R\otimes_R A \arrow[r] & \Omega^1_A \arrow[r] & 0
\end{tikzcd}$$

$$a_1\otimes d\rho(a_2)\otimes 1 - 1\otimes d\rho(a_1a_2)\otimes 1 + 1\otimes d\rho(a_1)\otimes a_2$$
$$= 1\otimes d[\rho(a_1)\rho(a_2) - \rho(a_1a_2)]\otimes 1.$$

Thus we want to use the isomorphism

$$\Omega^2_A \longrightarrow I/I^2 \qquad da_1da_2 \longmapsto \rho(a_1)\rho(a_2)-\rho(a_1a_2)$$

in which case $I/I^2 \longrightarrow A\otimes_R\Omega^1_R\otimes_R A$ becomes the embedding $\Omega^2_A \longrightarrow A\otimes\bar{A}\otimes A$ induced by b'. Now return to the defn. of u three pages back.

$$I^{n+1}/I^{n+2} = \Omega^{2n+2}_A \longrightarrow I^n/I^{n+1}\otimes_A(A\otimes\bar{A}\otimes A)_A = \Omega^{2n+1}_A$$

$$\cup$$

$$a_0da_1\cdots da_{2n+2} \longmapsto a_0da_1\cdots da_{2n}\otimes\left[a_{2n+1}\otimes a_{2n+2}\otimes 1 - 1\otimes a_{2n+1}a_{2n+2}\otimes 1 + 1\otimes a_{2n+1}\otimes a_{2n+2}\right]_A$$

$$\longmapsto a_0da_1\cdots da_{2n}a_{2n+1}da_{2n+2} - a_0da_1\cdots da_{2n}d(a_{2n+1}a_{2n+2})$$
$$+ a_{2n+1}a_0da_1\cdots da_{2n}da_{2n+1} \qquad \in \Omega^{2n+1}_A$$

$$= a_{2n+1}a_0da_1\cdots da_{2n+1} - a_0da_1\cdots da_{2n}a_{2n+2}$$

and we have seen this is $\pm$ the boundary operator in the Hochschild complex.

Let's now summarize and try to explain the picture. By using the cyclic theory of extensions we were led to work out the cyclic cohomology of A by writing A as R/I where R is free, and looking at ~~trace~~ cyclic theory on the free algebra, especially how it links with the I-adic filtration.

Cyclic theory on a free algebra R seems to reduce to an acyclic periodic complex

$$\Omega^1_R \otimes_R R \underset{\delta}{\overset{b}{\rightleftarrows}} \bar{R}$$ or

$$\longrightarrow \bar{R} \xrightarrow{\delta} \Omega^1_R \otimes_R \xrightarrow{b} \bar{R} \xrightarrow{\delta} \Omega^1_R \otimes_R \xrightarrow{b}$$

We then consider the natural fiftration associated to the ideal I:

$$\longrightarrow I^{n+1} \xrightarrow{\delta} I^n \otimes_R \Omega^1_R \otimes_R \xrightarrow{b} I^n \xrightarrow{\delta}$$

The homology of this complex gives the cyclic homology of A up to the groups

$$[I, I^{n-1}]/[R, I^n] = (1-\sigma)(I\otimes_R)^n = (1-\sigma)(I/I^2 \otimes_A)^n$$

Thus from extension theory we obtain a filtered complex rather than the double complex of Connes.

Next the associated graded complexes

$$\xrightarrow{b} I^{n+1}/I^{n+2} \xrightarrow{\delta} I^n/I^{n+1} \otimes_R \Omega^1_R \otimes_R \xrightarrow{b} I^n/I^{n+1} \longrightarrow$$

turn out to be a twisted version of a natural complex giving the Hochschild homology. The twisting takes the natural operator like b which works at one end, and it cyclically sums this b to make δ.

If we take $R = \tilde{T}(A)$, then our periodic complex

$$\Omega^1_R \otimes_R \rightleftarrows \bar{R}$$

is additively isomorphic to Ω_A. Thus

$$R = \tilde{T}(A) \simeq \Omega_A^{ev}$$

$$\Omega^1_R \otimes_R = R \otimes \bar{A} \simeq \Omega_A^{odd}$$

Furthermore the I-adic filtration corresponds to the filtration by degree on forms.

A natural question is whether the algebra structure we have on Ω_A, which gives the algebra $A * A$, is relevant. It seems unlikely since one doesn't view $\Omega^1_R \otimes_R$ as having even a natural bimodule structure over R.

Problems:

1) Cyclic theory for free algebras reduces to certain basic calculations

Ω^1_R projective over $R \otimes R^o$

$$\circledast \qquad \longrightarrow \bar{R} \xrightarrow{\delta} \Omega^1_R \otimes_R \xrightarrow{b} \bar{R} \xrightarrow{\delta} \Omega^1_R \otimes_R \longrightarrow$$

exact

The problem is how to develop the theory so that this is done early. A correct foundation for cyclic theory should make the triviality of red. cyclic homology for free algebras trivial.

The exactness of $\circledast$ results easily using Goodwillie's homotopy property - derivations composed with S give zero, and of course Connes exact sequence.

2) Does b^t result from $-b' = \delta$ on $C^\cdot(A)$ relative to the interpretation $R = C^\cdot(A)$, $\Omega^1_R \otimes_R = "C^{\cdot+1}(A)"$? YES

July 29, 1988

Here's a proof that free algebras have trivial cyclic homology starting from the formulas

$$\bar{HC}_{2n-1}(A) = \varprojlim_{R/I=A} [redacted]\ I^n/[I,I^{n-1}]$$

$$\bar{HC}_{2n}(A) = \varprojlim_{R/I=A} \bar{HC}_0(R/I^{n+1}).$$

When A is free we [redacted] know that the trivial extension $A \xrightarrow{id} A$ maps to any other. This shows immediately that $\bar{HC}_{2n-1}(A) = 0$ for $n>0$ and that

$$\bar{HC}_{2n}(A) \hookrightarrow \bar{HC}_0(A)$$

Recall from your paper on extension the fact that the composition

$$\bar{HC}_{2n}(A) \longrightarrow \bar{HC}_0(R/I^{n+1}) \xrightarrow{\delta} [redacted]\ R/I^n \otimes_R \Omega^1_R \otimes_R$$

is zero. (This is based upon the two maps $R \rightrightarrows R\oplus\Omega^1_R$ between extensions $R/I = R\oplus\Omega^1_R / I\oplus\Omega^1_R = A$.) [redacted]

[redacted]

Let D be a derivation of R such that $D(I)\subset I$. Then there is a unique R-bimodule map $u:\Omega^1_R \to R$ such that $ud = D$. Thus we have a commutative diagram

$$\begin{tikzcd}
\bar{HC}_{2n}(A) \arrow[r, "S"] & \bar{HC}_0(R/I^{n+1}) \arrow[r, "\delta"] \arrow[d, "\tilde{D}"] & R/I^n \otimes_R \Omega^1_R \otimes_R \arrow[dl, "\tilde{u}"] \\
& \bar{HC}_0(R/I^n) &
\end{tikzcd}$$

which shows that $\tilde{D}$ vanishes on the image of the Connes homomorphism S. ([redacted] strictly speaking in the present context γ is the canonical map from the inverse limit.)

Suppose $R = A$ and $I = 0$. Then we see that the image of $\overline{HC}_{2n}(A) \to \overline{HC}_0(A)$ is killed by all derivations of A. When A is free, say $A = T(V)$, there is a derivation giving the grading. This derivation induces the grading on $\overline{HC}_0(A)$ which is concentrated in positive (>0) degrees. Thus $\overline{HC}_{2n}(A) \to \overline{HC}_0(A)$ is zero and since it's <u>injective</u> we see $\overline{HC}_{2n}(A) = 0$ for $n > 0$.

Another consequence of the diagram arguments on the previous page is the following. Let us assume known (or take as defn.) the commutativity of

$$\begin{tikzcd}
\overline{HC}_{2n}(A) \arrow[r, "\gamma"] \arrow[d, "S"] & \overline{HC}_0(R/I^{n+1}) \arrow[d, "\widetilde{id}_R"] \\
\overline{HC}_{2n-2}(A) \arrow[r, "\gamma"] & \overline{HC}_0(R/I^n)
\end{tikzcd}$$

Then we have

$$\begin{tikzcd}
\overline{HC}_{2n}(A) \arrow[r, "\gamma"] \arrow[d, "S"] & \overline{HC}_0(R/I^{n+1}) \arrow[r, "\delta"] \arrow[d, "\widetilde{id}_R"] & R/I^n \otimes_R \Omega^1_R \otimes_R \arrow[dddl] \\
\overline{HC}_{2n-2}(A) \arrow[r, "\gamma"] \arrow[d, "\tilde{D}"] & \overline{HC}_0(R/I^n) \arrow[d, "\tilde{D}"] & \\
\overline{HC}_{2n-2}(A) \arrow[r, "\gamma"] & HC_0(R/I^n) &
\end{tikzcd}$$

which proves Goodwillie's thm. $\boxed{\tilde{D}S = 0.}$ (Have to lift Der A to Der R)

In previous work I think I gave a proof of Goodwillie's theorem along the following lines. The idea is to view D as producing an endomorphism of the Connes exact sequence

$$\begin{tikzcd}
 & {} \arrow[r, "I"] \arrow[d] & \tilde{H}C_n(A) \arrow[r, "S"] \arrow[d, "\tilde{D}"] \arrow[dl, dashed] & \tilde{H}C_{n-2}(A) \arrow[r, "B"] \arrow[d, "\tilde{D}"] & \cdot \\
{} \arrow[r] & \bar{H}_n(A,A) \arrow[r, "I"] & \tilde{H}C_n(A) \arrow[r, "S"] & \tilde{H}C_{n-2}(A) \arrow[r, "B"] & \tilde{H}_{n-1}(A,A)
\end{tikzcd}$$

and to lift $\tilde{D}$ by the dotted arrow. One has

$$HC_n(A) \rightrightarrows HC_n(A \oplus \Omega^1_A) = HC_n(A) \oplus H_n(A; \Omega^1_A) \oplus \cdots$$

which give a canonical map

$$HC_n(A) \longrightarrow H_n(A, \Omega^1_A) \underset{\text{for } n > 0}{\overset{\cong}{\longleftarrow}} H_{n+1}(A,A)$$

This is essentially the third map B in the Connes exact sequence. On the other hand the derivation D induces $\Omega^1_A \longrightarrow A$ whence we have a map $HC_n(A) \longrightarrow H_n(A,A)$ which should be our dotted arrow. The formulas are clear - when one differentiates, one automatically does something like a <u>cyclic</u> sum.

Let's take R to be $C^{\cdot}(A)$ with differentials δ. This is dual to the acyclic Hochschild complex. Formally R is $T(A^*)$ is a free algebra and $\Omega^1_R \otimes_R$ is the dual of the Hochschild complex. Thus Connes complex is roughly of the form

$$\longrightarrow R \longrightarrow \Omega^1_R \otimes_R \longrightarrow R \longrightarrow \Omega^1_R \otimes_R \longrightarrow \cdots$$

If we have a derivation of A it extends to a derivation of R and we get a map $\Omega^1_R \otimes_R \longrightarrow R/[R,R]$ which we used before. In this example $R = \dot{C}(A)$, we get a map from the dual of the Hochschild complex to the dual of the cyclic complex, or equivalently a map from the cyclic complex to the Hochschild. This ought to be the dotted arrow discussed above.

We now have two examples of free algebras R where we look at the periodic complex

$$\longrightarrow R \longrightarrow \Omega^1_R \otimes_R \longrightarrow \cdots$$

One is filtered I-adically and is related to the cyclic homology of A. The other is a bicomplex and is related to the cyclic cohomology of A. The problem is perhaps to fit them together to give the <u>bivariant</u> theory.

August 1, 1988

Suppose $A = R/I$ and recall that there is a quis

$$CC(R \leftarrow I) \longrightarrow CC(A)$$

It seems that we can construct a natural map of complexes in the opposite direction starting from a lifting $\rho : A \to R$.

The idea is to use the DG algebra $(C^\cdot(A), \delta)$ whose universal trace complex is dual to $CC(A)$ up to a shift. (Q: Is there any point for regarding $CC_\cdot(A)$ as the complex of traces on $C^\cdot(A)$?)

Since $R \leftarrow I$ is a DGA, the cochain algebra $C^\cdot(R \leftarrow I)$ will be a double DGA and there will be a map of DGA's

$$C^\cdot(A) \longrightarrow \text{[redacted]} \quad C^\cdot(R \leftarrow I)$$

But as algebras these are free algebras. Also the above map is a quis, so unless something strange happens one expects a map backwards. This is easy to produce on the algebra level, because one only has to map $(R \oplus I)^\vee$ to $C^1(A) \oplus C^2(A) = (A \oplus A^{\otimes 2})^\vee$, and one has the canonical elements

$$\rho \in C^1(A, R) \qquad \delta\rho + \rho^2 \in C^2(A, I).$$

The problem thus becomes one of checking compatibility with differentials.

Let's recall that $C^\cdot(A)$ is the [redacted] cochain algebra which is dual to the DG coalgebra which is the bar construction on A: $\bar{B}(A) =$

$$\longrightarrow A^{\otimes 3} \xrightarrow{-b'} A^{\otimes 2} \xrightarrow{-b'} A \xrightarrow{0} \mathbb{C}$$

So the map $C^{\bullet}(A) \longleftarrow C^{\bullet}(R \leftarrow I)$ we want should be the transpose of a DG coalgebra map

$$\circledast \qquad \bar{B}(A) \longrightarrow \bar{B}(R \leftarrow I)$$

But the bar construction has a universal property with respect to twisting cochains, so the map $\circledast$ should [redacted] result from a twisting cochain

$$\oplus \qquad \tau : \bar{B}(A) \longrightarrow \{I \to R\}$$

which is a map of degree -1 satisfying the identity

$$\boxed{\tau d + d\tau + \mu(\tau \otimes \tau)\Delta = 0}$$

(Note the formal similarity with the condition for a flat connection: $d\theta + \theta^2 = 0$ [redacted]. In fact if C is a DG coalgebra and A is a DG algebra, then $\mathrm{Hom}_{\mathbb{C}}(C, A)$ is a DG algebra and a twisting cochain is an element [redacted] $\tau \in \mathrm{Hom}^1_{\mathbb{C}}(C, A)$ such that $d\tau + \tau^2 = 0$.)

It remains to show that the map $\oplus$ given by

$$\begin{tikzcd}
{} \arrow[r] & A^{\otimes 3} \arrow[r, "+b'"] \arrow[d] & A^{\otimes 2} \arrow[r, "+b'"] \arrow[d, "\delta\rho+\rho^2"] & A \arrow[r, "+b'"] \arrow[d, "\rho"] & k \arrow[d, "0"] \\
{} \arrow[r] & 0 \arrow[r] & I \arrow[r, "i"] & R \arrow[r] & 0
\end{tikzcd}$$

is in fact a twisting cochain.

$$(d\tau + \tau d)(a_1 \otimes a_2) = i(\delta\rho + \rho^2)(a_1 \otimes a_2) + \rho(+b')(a_1 \otimes a_2)$$

$$= [redacted]\ \rho(a_1)\rho(a_2) - \cancel{\rho(a_1 a_2)} + \cancel{\rho(a_1 a_2)}$$

$$(d\tau + \tau d)(a_1 \otimes a_2 \otimes a_3) = (\delta\rho + \rho^2)(+a_1 a_2 \otimes a_3 \quad a_1 \otimes a_2 a_3)$$

[redacted]

In general

$$\Delta : T(A) \longrightarrow T(A) \otimes T(A)$$

$$A^{\otimes n} \longmapsto \sum_{k+l=n} A^{\otimes k} \otimes A^{\otimes l}$$

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{k=0}^{n} (a_1 \otimes \cdots \otimes a_k) \otimes (a_{k+1} \otimes \cdots \otimes a_n)$$

for the free coalgebra generated by A. Thus

$$\mu(\tau \otimes \tau)\Delta(a_1 \otimes a_2) = \mu(\tau\otimes\tau)(a_1 \otimes a_2) \qquad \text{as } \tau(1)=0$$

$$= -\mu\, \tau(a_1) \otimes \tau(a_2)$$

$$= -\mu\, \rho(a_1) \otimes \rho(a_2) = -\rho(a_1)\rho(a_2)$$

$$\mu(\tau\otimes\tau)\Delta(a_1 \otimes a_2 \otimes a_3) = \mu(\tau\otimes\tau)\big((a_1\otimes a_2)\otimes a_3 + a_1 \otimes (a_2 \otimes a_3)\big)$$

$$= (\delta\rho + \rho^2)(a_1 \otimes a_2)\,\rho(a_3) - \rho(a_1)\,(\delta\rho+\rho^2)(a_2 \otimes a_3)$$

$$= [\rho(a_1)\rho(a_2) - \rho(a_1a_2)]\,\rho(a_3) - \rho(a_1)\,[\rho(a_2)\rho(a_3) - \rho(a_2a_3)]$$

$$= -\rho(a_1a_2)\rho(a_3) + \rho(a_1)\rho(a_2a_3)$$

$$(d\tau + \tau d)(a_1 \otimes a_2 \otimes a_3) = \rho(a_1a_2)\rho(a_3) - \cancel{\rho(a_1a_2a_3)}$$
$$\qquad - \rho(a_1)\rho(a_2a_3) + \cancel{\rho(a_1a_2a_3)}$$

So it works!

August 2, 1988

Let's work on signs. Recall that the "cofree" coalgebra $T(V) = \oplus V^{\otimes n}$ has the coproduct

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \sum_{k=0}^{n} (v_1 \otimes \cdots \otimes v_k) \otimes (v_{k+1} \otimes \cdots \otimes v_n)$$

We want to work out the differential in the bar construction $\bar{B}(A)$ of an algebra A. In general given a DG algebra A and a DG coalgebra C one has the notion of a twisting cochain $\tau : C \to A$. It is an element $\tau \in \operatorname{Hom}^1_{\mathbb{C}}(C, A)$ satisfying

$$\boxed{d\tau + \tau^2 = 0}$$

in the DG algebra $\operatorname{Hom}^{\bullet}_{\mathbb{C}}(C, A)$, i.e.

$$d_A \tau + \tau d_C + \mu(\tau \otimes \tau)\Delta = 0$$

The bar construction comes with a canonical twisting cochain $\tau : \bar{B}(A) \longrightarrow A$ which is universal. To construct such a DG coalgebra, note that a twisting cochain $\tau : C \to A$ gives a map of graded vector spaces $C \longrightarrow \Sigma A$, which coextends uniquely to a coalgebra map

$$C \longrightarrow T(\Sigma A)$$

It turns out that there is a unique diff'l on $T(\Sigma A)$ such that the canonical map $T(\Sigma A) \longrightarrow A$ is a twisting cochain.

Let's now work out the differential in $\bar{B}(A) = T(\Sigma A)$ when A is concentrated in degree zero. We have

$$\bar{B}(A): \qquad \longrightarrow A^{\otimes 3} \xrightarrow{d} A^{\otimes 2} \xrightarrow{d} A \xrightarrow{0} k$$

$$\begin{array}{c} A \\ \downarrow \tau = id \\ A \end{array}$$

The twisting cochain identity gives

$$\tau d(a_1 \otimes a_2) + \mu(\tau\otimes\tau)\Delta(a_1\otimes a_2) = 0$$

$$d(a_1\otimes a_2) + \mu(\tau\otimes\tau)\big((a_1\otimes a_2)\otimes 1 + (a_1)\otimes(a_2) + 1\otimes(a_1\otimes a_2)\big) = 0$$

$$d(a_1\otimes a_2) + \mu(-1)\big(\tau(a_1)\otimes\tau(a_2)\big) = 0 \qquad \text{so}$$

$$\boxed{d(a_1\otimes a_2) = a_1 a_2}$$

It seems better to use $(a_1|a_2)$ or (a_1, a_2) instead of $a_1\otimes a_2$ when you have to deal with Δ on $\bar{B}(A)$.

Next

$$\Delta(a_1,a_2,a_3) = (a_1,a_2,a_3)\otimes 1 + (a_1,a_2)\otimes a_3 + a_1\otimes(a_2,a_3) + 1\otimes(a_1,a_2,a_3)$$

so $(d\otimes 1 + 1\otimes d)(\Delta - ?\otimes 1 - 1\otimes ?)(a_1,a_2,a_3)$

$$= (d\otimes 1 + 1\otimes d)\big((a_1,a_2)\otimes a_3 + a_1\otimes(a_2,a_3)\big)$$

$$= a_1 a_2\otimes a_3 - a_1\otimes a_2 a_3$$

$$\parallel$$

$(\Delta - ?\otimes 1 - 1\otimes ?)\, d(a_1,a_2,a_3)$ so we see that

$$\boxed{d(a_1,a_2,a_3) = (a_1a_2,a_3) - (a_1,a_2a_3)}$$

which means that the differential in $\bar{B}(A)$ is just Connes b'.

Next we want to identify the dual algebra $\bar{B}(A)^*$ with $C^{\cdot}(A)$. The formula for the product of $f\in \bar{B}(A)^*_p = (A^{\otimes p})^*$ and $g\in\bar{B}(A)^*_q$ is

$$(f\cdot g)(\xi) = \mu\, f\otimes g\ \Delta\xi$$

so

$$(f\cdot g)(a_1\otimes\cdots\otimes a_{p+q}) = \mu(f\otimes g)((a_1\otimes\cdots\otimes a_p)\otimes(a_{p+1}\otimes\cdots\otimes a_{p+q}))$$
$$= (-1)^{pq} f(a_1\otimes\cdots\otimes a_p)\, g(a_{p+1}\otimes\cdots\otimes a_{p+q})$$

In particular if $\lambda_i \in A^*$, then

$$(\lambda_1\cdot\lambda_2\cdots\lambda_p)(a_1\otimes\cdots\otimes a_p) = (-1)^{\frac{p(p-1)}{2}}\lambda_1(a_1)\cdots\lambda_p(a_p)$$

So to get an isomorphism

$$C^\cdot(A) \longrightarrow \overline{B}(A)^*$$

we want to send a cochain $f(a_1,\ldots,a_p)\in C^p(A)$ to the linear functional

$$\tilde f(a_1\otimes\cdots\otimes a_p) = (-1)^{\frac{p(p-1)}{2}} f(a_1,\ldots,a_p)$$

Check: If $f\in C^p$, $g\in C^q$, then

$$(\tilde f\,\tilde g)(a_1\otimes\cdots\otimes a_{p+q}) = (-1)^{pq}\tilde f(a_1\otimes\cdots\otimes a_p)\,\tilde g(a_{p+1}\otimes\cdots\otimes a_{p+q})$$
$$= (-1)^{pq}(-1)^{\frac{p(p-1)}{2}}(-1)^{\frac{q(q-1)}{2}} \underbrace{f(a_1,\ldots,a_p)\,g(a_{p+1},\ldots,a_{p+q})}$$
$$(\widetilde{f\cup g})(a_1\otimes\cdots\otimes a_{p+q}) = (-1)^{\frac{(p+q)(p+q-1)}{2}} \overbrace{(f\cup g)(a_1,\ldots,a_{p+q})}.$$

Also

$$(d(\tilde f))(a_1\otimes\cdots\otimes a_{p+1}) = (d\circ\tilde f - (-1)^p\tilde f\circ d)(a_1\otimes\cdots\otimes a_{p+1})$$
$$= (-1)^{p+1}\tilde f\Big(\sum_{i=1}^{p}(-1)^{i+1}(\cdots\otimes a_i a_{i+1}\otimes\cdots)\Big)$$
$$= \sum_{i=1}^{p}(-1)^p(-1)^i(-1)^{\frac{p(p-1)}{2}} f(\ldots,a_ia_{i+1},\ldots)$$
$$= (-1)^{p+\frac{p(p-1)}{2}}(\delta f)(a_1,\ldots,a_{p+1}) \qquad \frac{(p+1)p}{2}$$
$$= \widetilde{\delta f}(a_1\otimes\cdots\otimes a_{p+1})$$

Simpler check: Take $\rho : A \to R$ a linear map; i.e. $\rho \in C^1(A, B)$. Then to ρ we associate $\tilde{\rho} \in \mathrm{Hom}^1_{\mathbb{C}}(\bar{B}(A), B)$ given by $\tilde{\rho}(a) = +\rho(a)$. Then

$$(d\tilde{\rho})(a_1 \otimes a_2) = (d \circ \tilde{\rho} + \tilde{\rho} \circ d)(a_1 \otimes a_2) = \tilde{\rho}(a_1 a_2)$$
$$= \rho(a_1 a_2)$$

So $(d\tilde{\rho} + \tilde{\rho}^2)(a_1 \otimes a_2) = \rho(a_1 a_2) - \rho(a_1)\rho(a_2)$ and so ρ is a homomorphism $\iff$ $d\tilde{\rho} + \tilde{\rho}^2 = 0$.

What is the next stage? I would like to work out the formulas for the map $CC(A) \longrightarrow CC(R \leftarrow I)$ associated to the lifting ρ and also to show that different liftings lead to homotopic maps of the cyclic complexes.

Recall that we have a twisting cochain $\tau = (\rho, \gamma)$ where $\gamma(a_1 \otimes a_2) = \rho(a_1)\rho(a_2) - \rho(a_1 a_2)$:

$$\begin{tikzcd}
{} \arrow[r, "b'"] & A^{\otimes 3} \arrow[r, "b'"] & A^{\otimes 2} \arrow[r, "b'"] \arrow[drr, "\gamma"] & A \arrow[r] \arrow[dr, "\rho"] & k \\
 & & & I \arrow[r, "i"'] & R
\end{tikzcd}$$

This is a map $\bar{B}(A) \longrightarrow \Sigma(I \to R)$ where the former is a coalgebra so it coextends uniquely to a coalgebra map

$$\bar{B}(A) \longrightarrow T(\Sigma(I \to R)).$$

Let's study this coextension business in a simpler situation. Suppose C is a coalgebra and $u : C \to V$ is a linear map. How do we find its coextension $\hat{u} : C \to T(V)$?

We have a commutative diagram

$$\begin{tikzcd} C \arrow[r, "\hat{u}"] \arrow[d, "\Delta^{(n)}"] & T(V) \arrow[d, "\Delta^{(n)}"] \arrow[dr, "p_n"] & \\ C^{\otimes n} \arrow[r, "u^{\otimes n}"] \arrow[rr, bend right, "u^{\otimes n}"'] & T(V)^{\otimes n} \arrow[r, "p_1^{\otimes n}"] & V^{\otimes n} \end{tikzcd}$$

so if we want $p_n \hat{u}$ we see it is just [redacted] $u^{\otimes n}\Delta^{(n)}$. So now let us start with $u = \tau : \bar{B}(A) \longrightarrow (I \to R)$ and compute

$$\bar{B}(A) \xrightarrow{\hat{\tau}\ [redacted]} T(\Sigma(I \to R)) \xrightarrow{p_n} (\Sigma(I \to R))^{\otimes n}$$

Now $\bar{B}(A) = T(\Sigma A)$ and the n-fold coproduct map for $T(V)$ is given by

$$\Delta^{(n)}(v_1 \otimes \cdots \otimes v_N) = \sum_{j_1 + \cdots + j_n = N} (v_1 \otimes \cdots \otimes v_{j_1}) \otimes \underbrace{(\quad)}_{\text{length } j_2} \otimes \cdots \otimes \underbrace{(\quad \otimes v_N)}_{\text{length } j_n}$$

Now we apply $\tau^{\otimes n}$ to this, and τ sees only $A^{\otimes 2}$ and A, which means that the j_i to consider are $j_i = 1$ or 2. Thus suppose suppose we want the component of $\hat{u}$ in the piece of $T(\Sigma(I \to R))$ having k I-factors and l R-factors. So $n =$ [redacted] $k + l$ and $N = 2k + l$ and we send $a_1 \otimes \cdots \otimes a_N$ into the sum of all terms such as

$$\rho(a_1)\,\gamma(a_2 \otimes a_3) \cdots$$

involving k γ's and l ρ's.

This is evidently some sort of cup product of the cochains ρ, γ on A with values in the algebra $T(R \oplus I)$.

August 3, 1988

Let $\rho : A \longrightarrow R$ be a linear map such that ρ mod I : $A \longrightarrow R/I$ is a homomorphism. Then we saw that the pair (ρ, γ), where $\gamma : A^{\otimes 2} \longrightarrow I$ is $\gamma(a_1, a_2) = \rho(a_1)\rho(a_2) - \rho(a_1 a_2)$, defines a twisting cochain $\tau : \bar{B}(A) \longrightarrow (I \to R)$. Hence there is a DG coalgebra map

$$u : \bar{B}(A) \longrightarrow \bar{B}(I \to R) = T(\Sigma(I \to R))$$

which induces a map from the cyclic complex of A to the cyclic complex of $I \to R$.

Yesterday we saw u could be described as follows. $u(a_1 \otimes \cdots \otimes a_n)$ is the sum over all ways of partitioning the sequence $\{a_1, \ldots, a_n\}$ into intervals of lengths 1 or 2 where to such a partition one applies ρ to the partitions of length 1 and γ to the partitions of length 2, thereby obtaining an element of $T(I \oplus R)$. Thus for example

$$(a_1) \otimes (a_2 \otimes a_3) \otimes (a_4) \longmapsto \rho(a_1) \otimes \gamma(a_2, a_3) \otimes \rho(a_4) \in R \otimes I \otimes R$$

Now this map u induces the inverse quis

$$CC(A) \longrightarrow CC(I \to R)$$

need to define the odd Connes homomorphism. It was the idea that one ought to be able to obtain a map from $CC(A)$ to the complex

$$\longrightarrow (I^{(n+1)})' \longrightarrow I^{(n)} \Omega^1_R \otimes_R \longrightarrow (I^{(n)})' \longrightarrow \cdots$$

by showing the latter is an "edge complex" for $CC(I \to R)$. Here $(I^{(n)})'$ denotes the subspace which is the inverse image of the cyclic invariant subspace of $(I/I^2 \otimes_{R/I})^n$.

We propose now to construct a map of complexes

$$\begin{tikzcd} \arrow[r] & CC_2(A) \arrow[r] \arrow[d] & CC_1(A) \arrow[r] \arrow[d] & CC_0(A) \arrow[d] \\ \arrow[r] & I\Omega^1_R \otimes_R \arrow[r] & I \arrow[r] & \Omega^1_R \otimes_R \end{tikzcd}$$

which ought to be obtained from u by a kind of edge homomorphism from $CC(I \to R)$. We have the cochains $\rho \in C^1(A,R)$, $\gamma = \delta\rho + \rho^2 \in C^2(A,I)$ and hence we have cochains

$$\gamma^n \in C^{2n}(A, I^{(n)}) \qquad \gamma^n \otimes d\rho \in C^{2n+1}(A, I^{(n)} \otimes_R \Omega^1_R \otimes_R)$$

Let's work with R free so that we have $I^{(n)} = I^n$. We have

$$\delta\gamma^n = \gamma^n\rho - \rho\gamma^n$$

$$\delta(\gamma^n d\rho) = (\gamma^n\rho - \rho\gamma^n)d\rho + \gamma^n d(\delta\rho)$$

Let $\sim$ denote the effect of cyclically summing ~~the~~ the arguments of a cochain. From the first identity we have

$$-b\tilde{\gamma}^n = \widetilde{\delta\gamma^n} = \widetilde{\gamma^n\rho - \rho\gamma^n}$$

$$= \text{image of } \widetilde{\gamma^n d\rho} \in C_\lambda^{2n}(A, I^n\Omega^1_R \otimes_R)$$

under the map induced by

$$I^n\Omega^1_R \otimes_R \longrightarrow I^n \qquad z\,dx \mapsto [z,x]$$

The point is that
$$(\gamma^n\rho)(a_1,\dots,a_{2n+1}) = \gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n})\rho(a_{2n+1})$$
$$(\rho\gamma^n)(a_1,\dots,a_{2n+1}) = \rho(a_1)\gamma(a_2,a_3)\cdots\gamma(a_{2n},a_{2n+1})$$

and the image of $\gamma^n d\rho$ under the map $z\,dx \longmapsto [z,x]$ is the cocycle

$$(a_1,\dots,a_{2n+1}) \longmapsto \gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n})\,d\rho(a_{2n+1})$$

$$\longmapsto \gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n})\,\rho(a_{2n+1}) - \rho(a_{2n+1})\,\gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n})$$

This doesn't agree with $\gamma^n\rho - \rho\gamma^n$ but it does after cyclic summation.

Similarly we can consider the second identity which gives

$$-b\,\widetilde{\gamma^n d\rho} = \widetilde{\delta(\gamma^n d\rho)} = \widetilde{(\gamma^n\rho-\rho\gamma^n)\,d\rho + \gamma^n d(\delta\rho)}$$

$$= \widetilde{\gamma^n(\rho\,d\rho + d\rho\,\rho + d\delta\rho)} = \gamma^n d(\rho^2+\delta\rho)$$

$$= \widetilde{\gamma^n d\gamma}$$

Again the point seems to be that

$$(\rho\gamma^n d\rho)(a_0,a_1,\dots,a_{2n},a_{2n+1}) = \rho(a_0)\gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n})\,d\rho(a_{2n})$$

$$= \gamma(a_1,a_2)\cdots\gamma(a_{2n-1},a_{2n})\,(d\rho(a_{2n}))\,\rho(a_0)$$

$$(\gamma^n d\rho\,\rho)(a_0,\dots,a_{2n+1}) = \gamma(a_0,a_1)\cdots\gamma(a_{2n-2},a_{2n-1})\,d\rho(a_{2n})\,\rho(a_{2n+1})$$

and when one cyclically (skew-) symmetrizes, because the even cyclic perms have $-$ sign, one obtains

$$\widetilde{\rho\gamma^n d\rho} = -\widetilde{\gamma^n d\rho\,\rho}.$$

But under the map $I^{n+1} \longrightarrow I^n\Omega^1_R\otimes_R$ induced by d one has

$$d\gamma^{n+1} = \sum_{i=0}^{n} \gamma^i\,d\gamma\,\gamma^{n-i} \in C^{2n+2}(A, I^n\Omega^1_R\otimes_R)$$

so

$$\widetilde{d\gamma^{n+1}} = (n+1)\,\widetilde{\gamma^n d\gamma} \in C^{2n+1}_\lambda(A, I^n\Omega^1_R\otimes_R)$$

Thus up to the constant $n+1$, we see that we obtain a map of complexes

$$\begin{tikzcd}
\arrow[r, "-b'"] & CC_{2n+1}(A) \arrow[r, "-b'"] \arrow[d, "\tilde{\gamma}^{n+1}"] & CC_{2n}(A) \arrow[r, "-b'"] \arrow[d, "\widetilde{\gamma^n d\gamma}"] & CC_{2n-1}(A) \arrow[r] \arrow[d, "\tilde{\gamma}^n"] & {} \\
\arrow[r] & I^{n+1} \arrow[r] & I^n \Omega^1_R \otimes_R \arrow[r] & I^n \arrow[r] & {}
\end{tikzcd}$$

In fact it seems that we can make the constants work by using $\frac{\gamma^n}{n!}$.

August 4, 1988

Let's go over yesterday's discovery. We have $\rho \in C^1(A,R)$, $\gamma = \delta\rho + \rho^2 \in C^2(A,I)$. We saw that the cochains

$$\frac{1}{n!}\gamma^n \in C^{2n}(A,I^n) \qquad \frac{1}{n!}\gamma^n d\rho \in C^{2n+1}(A, I^n \otimes_R \Omega^1_R \otimes_R)$$

for different n induce a map of complexes

$$\begin{tikzcd}
\arrow[r] & CC_{2n+1}(A) \arrow[r] \arrow[d, "\overline{\frac{\gamma^{n+1}}{(n+1)!}}"] & CC_{2n}(A) \arrow[r] \arrow[d, "\overline{\frac{\gamma^n d\rho}{n!}}"] & CC_{2n-1}(A) \arrow[r] \arrow[d, "\overline{\frac{\gamma^n}{n!}}"] & {} \\
\arrow[r] & I^{n+1} \arrow[r, "d"] & I^n \Omega^1_R \otimes_R \arrow[r, "b"] & I^n \arrow[r] & {}
\end{tikzcd}$$

~~Let's go over the calculation~~ Here the $\overline{}$ over a cochain means taking the (skew) cyclic sum "N" so as to obtain a cyclic cochain.

Let's review the calculations and use $\equiv$ to indicate that two cochains give the same cyclic cochain. We have

$$\delta\gamma^n = \gamma^n\rho - \rho\gamma^n$$

$$b\,\gamma^n d\rho = \gamma^n\rho - \left(\substack{\text{left mult} \\ I^n\otimes R \to I^n}\right)_* \gamma^n\rho$$

$$\equiv \gamma^n\rho - \rho\gamma^n$$

and so $\delta\gamma^n \equiv b\,\gamma^n d\rho$. Also

$$d\gamma^{n+1} = \sum_{i=0}^{n} \gamma^i d\gamma\, \gamma^{n-i} \in \bigoplus C^{2n+2}(A, I^i \otimes_R \Omega^1_R \otimes_R I^{n-i})$$

$$\equiv (n+1)\gamma^n \underbrace{d(\delta\rho+\rho^2)}_{[\delta+\rho,\, d\rho]} \in C^{2n+2}(A, I^n\Omega^1_R)$$

$$\equiv (n+1)[\delta+\rho, \gamma^n d\rho]$$

$$\equiv (n+1)\,\delta(\gamma^n d\rho)$$

For clarity I assume R free so that the distinction between $I^{(n)}$ and I^n can be ignored. The last calculation might be clearer as follows.

$$d\gamma^{n+1} = \sum \gamma^i \, d\gamma \, \gamma^{n-i} \in C^{2n+2}(A, \sum I^i \Omega_R^1 I^{n-i})$$

$$= \sum \gamma^i [\delta+\rho, d\rho] \gamma^{n-i}$$

$$= [\delta+\rho, \sum \gamma^i d\rho \, \gamma^{n-i}]$$

Then we apply the map $\sum I^i \Omega_R^1 I^{n-i} \longrightarrow I^n \Omega_R^1 \otimes_R$ and obtain

$$d\gamma^{n+1} \equiv \delta(\sum \gamma^i d\rho \, \gamma^{n-i}) \equiv (n+1)\, \delta(\gamma^n d\rho)$$

as desired.

Next we want to discuss homotopy. Suppose $\rho = \rho_t$ depends on a parameter t and $\dot\rho \in C^1(A, I)$. We wish to show that the derivative of the map of complexes given by the family of cochains $\frac{\gamma^n}{n!}$ $\frac{\gamma^n d\rho}{n!}$ is null-homotopic.

We have

$$\dot\gamma = \delta\dot\rho + \rho\dot\rho + \dot\rho\rho = [\delta+\rho, \dot\rho]$$

$$(\gamma^{n+1})^\circ = \sum \gamma^i \, \dot\gamma \, \gamma^{n-i}$$

$$= [\delta+\rho, \underbrace{\sum_{i=0}^{n} \gamma^i \dot\rho \, \gamma^{n-i}}_{\text{call this } \mu_n \text{ or } \mu}]$$

$$= \delta\mu + \rho\mu + \mu\rho$$

$$\equiv \delta(\mu) + b(\mu \, d\rho)$$

Thus we want to see if the family $\mu_n, \mu_n d\rho$ gives rise to a null-homotopy of $\partial_t(\gamma^n, \gamma^n d\rho)$

Picture:

$$\begin{tikzcd}
{} \arrow[r] & CC_{2n+1}(A) \arrow[r] \arrow[d, "(\gamma^{n+1})^\circ"] \arrow[dl, dashed, "\overline{\mu d\rho}"'] & CC_{2n}(A) \arrow[r] \arrow[d] \arrow[dl, dashed, "\overline{\mu_n}"'] & {} \arrow[dl, dashed, "\overline{\mu_{n-1} d\rho}"] \\
\arrow[r] I^{n+1}\Omega^1_R \otimes_R \arrow[r, "b"] & I^{n+1} \arrow[r, "d"] & I^n \Omega^1_R \otimes_R \arrow[r] & {}
\end{tikzcd}$$

Now we have $\mu_n = \sum \gamma^i \mathring{\rho} \gamma^{n-i} \equiv (n+1)\gamma^n \mathring{\rho}$

Then

$$\left(\frac{1}{n+1}\right) d\mu_n \equiv d(\gamma^n \mathring{\rho}) = [\delta+\rho, \underbrace{\sum \gamma^i d\rho\, \gamma^{n-1-i}}_{\text{call this } \nu}]\mathring{\rho} + \gamma^n d\mathring{\rho}$$

$$= (\delta\nu)\mathring{\rho} + (\rho\nu + \nu\rho)\mathring{\rho} + \gamma^n d\mathring{\rho}$$

$$\equiv \delta(\nu\mathring{\rho}) + \nu\,\delta\mathring{\rho} + \nu(\rho\mathring{\rho} + \mathring{\rho}\rho) + \gamma^n d\mathring{\rho}$$

$$= \delta\left(\sum \gamma^i d\rho\, \gamma^{n-1-i}\mathring{\rho}\right) + \sum \gamma^i d\rho\, \gamma^{n-1-i}\mathring{\gamma} + \gamma^n d\mathring{\rho}$$

$$\equiv \delta\left(-\sum \gamma^{n-1-i}\mathring{\rho}\,\gamma^i d\rho\right) + \sum \gamma^{n-1-i}\mathring{\gamma}\gamma^i d\rho + \gamma^n d\mathring{\rho}$$

$$= -\delta(\mu_{n-1} d\rho) + (\gamma^n d\rho)^\circ$$

Here's how to adjust the constants

$$\left(\frac{\gamma^{n+1}}{(n+1)!}\right)^\circ \equiv \delta\left(\frac{\mu_n}{(n+1)!}\right) + b\left(\frac{\mu_n}{(n+1)!} d\rho\right)$$

$$\left(\frac{\gamma^n}{n!} d\rho\right)^\circ \equiv \delta\left(\frac{\mu_{n-1}}{n!} d\rho\right) + d\left(\frac{\mu_n}{(n+1)!}\right)$$

Project for the future: Derive from these formulas a map from $CC_\cdot(A)$ to the quotient complex

$$\longrightarrow R/I^{n+1} \longrightarrow R/I^n \otimes_R \Omega^1_R \otimes_R \longrightarrow R/I^n \longrightarrow$$

Let's discuss cyclic homology for DG algebras and coalgebras in generality. Recall the basic concept of twisting cochain leads to adjoint functors

$$(\text{DG algs}) \underset{\text{Bar}}{\overset{\text{Cobar}}{\rightleftarrows}} (\text{DG coalgs}),$$

at least under suitable connectedness assumptions. These assumptions imply the above adjoint functors set up an equivalence of "homotopy" categories. If there is a twisting cochain $\tau : C \to A$ such that the twisted complex $C \otimes_\tau A$ is acyclic, then C and A correspondence under the equivalence (I recall that there is a spectral sequence associated to $C \otimes_\tau A$ which can be used, when $C \otimes_\tau A$ is acyclic and when there is suitable connectivity to show that the maps associated to τ

$$C \longrightarrow \bar{B}(A) \qquad\qquad \text{Cobar}(C) \longrightarrow A$$

are quasi-isomorphisms.)

The cyclic homology of a DG alg A we know is given by the cocommutator subcomplex of $\bar{B}(A)$. Dually we can define the cyclic homology of a DG coalg. C by taking the commutator <u>quotient</u> complex $R./[R.,R.]$ where $R. = \text{Cobar}(C)$. To see these are consistent we want to see that if $\tau : C \to A$ is such that $C \otimes_\tau A$ is acyclic, and A is a free DG alg and C is a cofree DG coalgebra, then the map induced by τ

$$\tau : \bar{C}^{[\,,\,]} \longrightarrow \bar{A}/[\,,\,]$$

is a quis. I think the rest is then clear.

Let's check this. If $R \to R'$ is a map of DG algs which are free as algebras, and this map is a quis, then we can use the resolutions

$$\xrightarrow{b} R \otimes \bar{R}^{\otimes 2} \xrightarrow{b} R \otimes \bar{R} \longrightarrow \Omega^1_R \otimes_R \longrightarrow 0$$

$$\cdots \longrightarrow \bar{R} \longrightarrow \Omega^1_R \otimes_R \longrightarrow \bar{R} \longrightarrow \bar{R}/[\,,\,] \to 0$$

to conclude that $\bar{R}/[\,,\,] \longrightarrow \bar{R}'/[\,,\,]$ is a quis. There's a dual statement for a quis of cofree DG coalgebras.

This assertion tells us that if we have an equivalence $\tau : C \to A$, then the cyclic homology of C is given by the commutator quotient of any free DG algebra R mapping quis to A. In effect, suppose $R = \mathrm{Cob}(C)$; then the commutator quotient space of R is the complex giving the cyclic homology of C. The general R case then should reduce to this one.

Similarly in the case of an equivalence $\tau : C \to A$ the cyclic homology of A should be given by the cocommutator subspace of any cofree DGA quasi-isom to C.

But what we can't seem to do immediately is to link $\bar{C}^{[\,]}$ and $\bar{A}/[\,,\,]$.